A GENERALIZED HIGH PASS/LOW PASS AVERAGING PROCEDURE FOR DERIVING AND SOLVING TURBULENT FLOW EQUATIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Woon Kwang YEO, B.S., M.S.

*****

The Ohio State University

1987

Dissertation Committee:
Dr. Keith W. Bedford
Dr. Vincent T. Ricca
Dr. Odus R. Burggraf
Dr. John T. Scheick

Approved by

Keith W. Bedford, Adviser
Department of Civil Engineering
DEDICATION

To Hong K., Seung K., Hannah K., and Kyung H. YEO

My Beloved Family
ACKNOWLEDGEMENTS

For this research during the past several years, I am indebted to many people. I would first like to thank Prof. Keith W. Bedford, my adviser, for his encouragement and guidance throughout my graduate education. Without his endless supports it could not have been completed. Indeed, he has been more than enough academically and personally. Also, I gratefully acknowledge many discussions with Dr. Ricca, Dr. Burggraf and Dr. Scheick, who were always prepared to help me solve the many problems.

I wish to thank Dr. M.J. Lee who provides the data for use in this work. My grateful thanks are due to my colleagues who have helped a lot. A particular debt is owned to Mr. James Yen for his excellent typing of the manuscript and a special thank is directed to Mr. S.J. Hong and Dr. M.W. Ko for their personal concerns as a friend.

I must acknowledge the love and patience of my wife, sons and daughter for their ever-present encouragement and assistance. None of this would have been possible without understanding, devotion and sacrifices of my family, to whom I will always be grateful. I wish to express my appreciation as a grateful son to my loving mother and mother-in-law.

Finally, I offer my sincere gratitude to the late father and father-in-law of blessed memory.

God bless you all.

WKY
VITA

June 2, 1962 ................................ Born in Seoul, Korea

1975 ........................................ B.S. in Civil Engineering
........................................ Seoul National University
........................................ Seoul, Korea

1975-1976 .................................... Enlisted army service

1976-1978 ..................................... Junior Engineer
........................................ Hyundai Construction Co.
........................................ Seoul, Korea

1980 .......................................... M.S. in Civil Engineering
........................................ Seoul National University
........................................ Seoul, Korea

1981-1983 ..................................... Instructor
........................................ Department of Civil Engineering
........................................ Myong-Ji University
........................................ Seoul, Korea

1983-present ................................ Research Associate
........................................ Department of Civil Engineering
........................................ The Ohio State University
........................................ Columbus, Ohio

PUBLICATION

FIELDS OF STUDY

Major Field: Civil Engineering

Fluid Mechanics; Dr. K.W. Bedford
Dr. O.R. Burggraf

Hydraulics and Hydrology; Dr. V.T. Ricca

Coastal Engineering and Oceanography; Dr. K.W. Bedford

Viscous Flow and Heat Transfer; Dr. O.R. Burggraf

Minor Field: Applied Mathematics

Studies in Numerical Models; Dr. J.T. Scheick
TABLE OF CONTENTS

PAGE

DEDICATION .............................................................. ii

ACKNOWLEDGEMENTS ..................................................... iii

VITA ........................................................................ iv

LIST OF FIGURES ......................................................... ix

LIST OF TABLES .......................................................... xi

CHAPTER

I. INTRODUCTION ......................................................... 1

II. BASIC DEFINITIONS AND CONCEPTS OF THE FILTERING
    PROCEDURE ............................................................ 7

   2.1 Introduction ....................................................... 7
   2.2 Basic Definitions ................................................ 9
       2.2.1 Averaging Definition and Filter Function .......... 10
       2.2.2 Response Function and its Property ............... 11
   2.3 Low Pass Filter .................................................. 14
       2.3.1 Uniform Filter ............................................ 14
       2.3.2 Sub-Grid Scale Filter ................................. 16
       2.3.3 Gaussian Filter ......................................... 16
   2.4 High Pass Filter .................................................. 18

III. APPLICATION OF FILTERING OPERATIONS TO THE
    TURBULENT TRANSPORT EQUATIONS .............................. 22

   3.1 Application to the Navier-Stokes Equations ............... 22
3.2 Reynolds Average ........................................ 24
3.3 Volume Average ......................................... 26
  3.3.1 The Leonard Term .................................. 28
  3.3.2 The Cross Term .................................... 28
3.4 Space-Time Filter of Dakhoul and Bedford .......... 30
3.5 Sub-Grid Scale (SGS) Terms .......................... 31

IV. RATIONALE FOR AND DERIVATION OF A NEW
FILTERING APPROACH ........................................ 32

  4.1 Two Descriptions For The SGS Terms ................ 32
  4.2 Derivation Of The SGS Quantity From The High Pass
      Filter ................................................... 36
  4.3 Application Of YB-II Series Solution To The Navier-
      Stokes Equations ....................................... 40

V. DERIVATION OF THE TURBULENCE CORRELATIONS
   AND FINAL FORMS OF THE AVERAGED
   EQUATIONS ................................................ 43

  5.1 Derivation of the Double Correlation Terms .......... 43
  5.2 Derivation of the Triple correlation Terms .......... 49
  5.3 The Final Form of the Averaged Momentum and
      Transport Equation ................................... 51
  5.4 The Mean Flow Energy Equation ....................... 53
  5.5 The Averaged Vorticity Equation ..................... 55
  5.5.1 Deformation and Vorticity ........................ 55
  5.5.2 Vorticity Equation in the Mean Flow ............... 58
  5.5.3 Vorticity in the Momentum Equation ............... 59

VI. DERIVATION OF THE EQUATIONS GOVERNING THE
   TURBULENCE QUANTITIES ................................. 64

  6.1 Kinetic Energy of Turbulence ....................... 64
  6.2 Momentum Flux of Turbulence ........................ 66
  6.3 Scalar Variance of Turbulence ...................... 69
  6.4 Turbulence Scalar Flux Equation .................... 70

VII. ANALYSIS OF THE YB CLOSURE ......................... 73

  7.1 Some Mathematical Properties of the YB Closure .... 73
    7.1.1 Convergence Behavior Of The YB-I And YB-II
           Series .............................................. 73
    7.1.2 Convergence Behavior of the YB-III Series ....... 77
    7.1.3 The Effect of Filter Isotropy on $R_{ij}$ .......... 83
  7.2 Kinematical Aspects of $R_{ij}$ in the two Dimensional
      Case .................................................. 85
7.2.1 $R_{ij}$ for an Isotropic Filter ........................................ 85
7.2.2 $R_{ij}$ for a Non-Isotropic Filter .................................. 88
7.3 Kinematical Aspects of $R_{ij}$ in the Three Dimensional Case ................................................................. 89
   7.3.1 The $A_{ij}$ Term ...................................................... 91
   7.3.2 The $B_{ij}$ Term ...................................................... 92
   7.3.3 The $C_{ij}$ Term ...................................................... 93
7.4 Dynamical Aspects of $R_{ij}$ .............................................. 94
   7.4.1 Decomposition of $R_{ij}$ and Definitions ......................... 94
   7.4.2 Kinetic Energy Equations ........................................ 96
   7.4.3 Energy Production and Dissipation .............................. 99
   7.4.4 Equilibrium State ................................................. 101

VIII. COMPARISON OF $R_{ij}$ TO EXISTING CLOSURE MODELS ....... 103

   8.1 Closure models employing Reynolds Averages .................. 103
   8.2 Comparison of $R_{ij}$ To Reynolds Average Closures .......... 107
   8.3 Comparison of $R_{ij}$ to SGS Closures from Volume Averages .................. 113

IX. INITIAL TEST OF THE NEW SGS MODEL ............................ 120

   9.1 Data to be used ...................................................... 120
   9.2 The Averaging Procedure ........................................... 124
   9.3 Evaluating of the SGS Model ....................................... 131
       9.3.1 Testing of the Residual Turbulence Terms ................. 131
       9.3.2 Comparison of the $A_{ij}$, $B_{ij}$, and $C_{ij}$ terms .... 134

X. CONCLUSIONS AND RECOMMENDATIONS ............................... 139

BIBLIOGRAPHY ............................................................... 143
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>13</td>
</tr>
<tr>
<td>2.2</td>
<td>15</td>
</tr>
<tr>
<td>2.3</td>
<td>17</td>
</tr>
<tr>
<td>3.1</td>
<td>27</td>
</tr>
<tr>
<td>5.1</td>
<td>57</td>
</tr>
<tr>
<td>7.1</td>
<td>76</td>
</tr>
<tr>
<td>7.2</td>
<td>76</td>
</tr>
<tr>
<td>7.3</td>
<td>79</td>
</tr>
<tr>
<td>7.4</td>
<td>80</td>
</tr>
<tr>
<td>7.5</td>
<td>80</td>
</tr>
<tr>
<td>7.6</td>
<td>82</td>
</tr>
<tr>
<td>7.7</td>
<td>87</td>
</tr>
<tr>
<td>7.8</td>
<td>93</td>
</tr>
<tr>
<td>7.9</td>
<td>97</td>
</tr>
<tr>
<td>7.10</td>
<td>100</td>
</tr>
<tr>
<td>8.1</td>
<td>110</td>
</tr>
<tr>
<td>9.1</td>
<td>123</td>
</tr>
<tr>
<td>9.2</td>
<td>125</td>
</tr>
<tr>
<td>9.3</td>
<td>127</td>
</tr>
<tr>
<td>9.4</td>
<td>127</td>
</tr>
</tbody>
</table>
9.5 Unaveraged velocity field .................................. 128
9.6 Averaged velocity field (32x32x32) ...................... 129
9.7 Averaged velocity field (16x16x16) ...................... 130
## LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Filter functions used in the existing models</td>
</tr>
<tr>
<td>5.1</td>
<td>Turbulence equations in the existing models</td>
</tr>
<tr>
<td>7.1</td>
<td>Rate of Convergence for the YB-I and YB-II series</td>
</tr>
<tr>
<td>7.2</td>
<td>Rate of convergence at the cut-off wavenumber</td>
</tr>
<tr>
<td>7.3</td>
<td>Convergence rate and the number of terms</td>
</tr>
<tr>
<td>9.1</td>
<td>R-square values : Case (I)</td>
</tr>
<tr>
<td>9.2</td>
<td>R-square values : Case (II)</td>
</tr>
<tr>
<td>9.3</td>
<td>R-square values : Case (III)</td>
</tr>
<tr>
<td>9.4</td>
<td>R-square values : Case (A)</td>
</tr>
<tr>
<td>9.5</td>
<td>R-square values : Case (B)</td>
</tr>
<tr>
<td>9.6</td>
<td>R-square values : Case (C)</td>
</tr>
<tr>
<td>9.7</td>
<td>Comparison of the diagonal and off-diagonal terms</td>
</tr>
<tr>
<td>9.8</td>
<td>R-square values : Case (AB)</td>
</tr>
<tr>
<td>9.9</td>
<td>R-square values : Case (AC)</td>
</tr>
<tr>
<td>9.10</td>
<td>R-square values : Case (BC)</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

Turbulence has been investigated for more than a century, but no general solution to the problem is yet known. Most flows occurring in nature are turbulent, and are marked by an irregular, three-dimensional, rotational, and dissipative structures. Often large diffusion-like behavior causes rapid mixing in flows and increases the rate of momentum, heat and mass transfer.

A continuum concept is used to describe turbulent flows and is implemented such that the scale of the analyzed flows is usually larger than the molecular activities. Because turbulence is not a property of fluids but a feature of fluid flows, it is very difficult to give a precise definition for turbulence, therefore, various methods of preparing the continuum descriptions exist. Moreover, achieving a solution may not be easy; the primary difficulty being the randomness and non-linearity, which make the equations of turbulence nearly intractable (see Tennekes and Lumley 1972). An averaging approach is often used in the description of turbulent flow but leads to the closure problem, and the use of many assumptions is required to achieve closure. An imprecise solution which is not universal results. Successful solution of the turbulence problem depends strongly on the inspiration involved in making the crucial closure assumptions.
Numerical simulations have been used as an alternative to exact solution methods for solving complicated turbulent flows. However, using the computational approach gives rise to another difficulty which is the inconsistency between the governing equations and the solution techniques. Usually, turbulence is characterized by a vast range of scales in time and space, however, the Nyquist theorem (Blinchikoff and Zverev 1976) says that the maximum wavenumber which can be detected from data at the grid interval, $\Delta$, is equal to $1/2\Delta$, so that numerical simulations have the inherent limitation that the portion of the spectrum less than a certain wavenumber or frequency can not be resolved even though its effect on the resolved flow may be important. If the computer has the capability or size to deal with the smallest scale activities, the problems resulting from the Nyquist interval limitation can be reduced. However, it is well-known (Tennekes and Lumley 1972, Kwak et al. 1975, Clark et al. 1977, Rogallo and Moin 1984) that any existing or proposed computer can not yet handle the required number of calculations for resolving the above microscales, rationally and economically.

For the above reasons, it has been customary to remove the relatively unimportant small scale activity explicitly, but keep its effect implicitly. In view of the physical complexity induced by the coupling of all scales the separation into the resolveable large scales and the unresolveable small scales is used with the hope that the small scales may be parameterized. Such a separation approach is quite useful for an engineering solution where the investigation of the behavior at large scales would be enough to satisfy most problem solution requirements. Based on this separation hypothesis, the remaining problem is how to eliminate the small scale activities but reasonably retain the effect of them on the large scale.
Historically since Osborne Reynolds in 1893, much effort has been devoted to mean flow separation in the description of turbulence. He introduced the concept of separation in the turbulence, and derived his famous Reynolds equations for the mean flow, where time averaging of the whole variable (total variable or field variable) defined the mean. His derivation is based upon the idea that, because of the inherent randomness of the turbulence, it is not expected that any analytic solutions describing the detailed nature of flows will be found but, instead, it may be feasible to know the statistical characteristics. So a statistical treatment interpreted as a time averaging was done to derive the equations. Strictly speaking, his averaging operation is only valid for stationary turbulence, which is a requirement often violated in practice. By subdividing the average over some proper time period, quasi-stationarity is achieved and has been suggested as a resolution of this weakness (White 1974, and Rodi 1980).

Rather than simple arithmetic averaging, the ensemble average concept was introduced to define the mean flow variables by taking weighted averages in probability space. This procedure is certainly applicable to unsteady mean flows. However, it invokes the ergodic hypothesis (Monin and Yaglom 1971) for a stationary process, which implies that the ensemble and time averages are the same. Regardless of the averaging, the previous separation methods produce additional terms in the resulting governing equations, which are unknown correlations between the fluctuating components. For example, Reynolds stress or flux terms result from the second-order correlations or moments, while in the transport equations the third-order correlations appear (see Monin and Yaglom 1971, 1975). Consequently, closure terms must be formulated from the higher-order correlations in order to obtain the exact equations.
Efforts to determine the closure terms started with Boussinesq (1877). According to Rodi (1979) and Bradshaw (1978), who present excellent reviews of this approach, many successful models have been developed for specialized flow regions by defining the closure terms particularized to those regions. The search for a general model is still underway principally because energy-containing eddies are not independent of the flow geometry. Additionally, many closure models result in increased computational complexity, as the closure terms often have a number of empirical constants to be determined. Some constants can be determined by investigating simple flow simulations. Others however are not easily determined, and thus, some researchers have doubts about obtaining a truly general closure model.

One new possible separation approach is the Large Eddy Simulation (LES) method, which applies higher-order averaging to the equations of motion instead of introducing the ensemble averages. The development of the LES methods has been reviewed by Rogallo and Moin (1984), and Ferziger (1982a, 1982b). The LES procedure was developed initially by Deardorff (1970, 1971) and Lilly (1966, 1967, 1971) for use in meteorological calculations. In both papers, a uniform numerical cell or volume averaging was defined, and, as a consequence, the subgrid scale (SGS), or unresolvable scale effects were modeled by a Smagorinsky’s or Lilly’s (see Smagorinsky 1963 and Lilly 1966) method. Later, Leonard (1974) improved the LES procedure by use of a low pass filter function in the averaging definition. As a result of applying this higher-order filter, several correction terms, called the Leonard terms, result. The Stanford Ther-moscience group has continued to develop the LES method. While most of the LES research has concentrated on the turbulent flows, applications to geophysical flows, i.e., surface water and meteorological models are scarce. The application of the LES method to surface water models was initially done by Bedford and Babajimopoulos (1980),
Babajimopoulos and Bedford (1981) and Bedford (1981). Their models simulate wind driven homogeneous lake circulation and passive scalar transport, and both rigid lid and free surface formulations were used. Dakhoul and Bedford (1986a, 1986b) also developed a model using both the spatial and temporal filtering technique. Findikakis and Street (1982) presented a two-dimensional vertical plane wind driven stratified cavity model. Excellent agreement with available data was reported.

In any existing LES model for turbulent flows, the filtered non-linear term, $u_i u_j$, results from application of the filtering operation to the Navier-Stokes equation, and the individuality of each model results from the description of that term. For instance, Reynolds equations assume that $u_i u_j = \bar{u}_i \bar{u}_j + \bar{u}_i \bar{u}_j$ (i.e. the Reynolds stress terms), Leonard (1974) suggested the relation of $\bar{u}_i u_j = \bar{u}_i \bar{u}_j + \left( \Delta^2/24 \right) \nabla^2 (\bar{u}_i \bar{u}_j) + \bar{u}_i \bar{u}_j$, and Clark et al. (1977, 1979), by approximating the cross terms and combining them with Leonard's terms, obtained the product terms of the first derivatives such that $\bar{u}_i u_j = \bar{u}_i \bar{u}_j + \left( \Delta^2/12 \right) \left( \nabla \bar{u}_i \nabla \bar{u}_j \right) + \bar{u}_i \bar{u}_j$. These descriptions can be expanded with respect to time as well as both time and space, done by Dakhoul and Bedford (1986a, 1986b). However, the subgrid scale terms, $\bar{u}_i \bar{u}_j$, always appear in these procedures, and require closure.

In the previous discussion, the closure problem is considered to be essential and unavoidable in analyzing turbulence. But a small body of research work here is likely to provide a new method of closure. Two infinite series representations of the small scale quantities are formulated by the high pass filter operation. One is the general form used in the conventional LES approach, and the other is an advanced form directly describing the small scale activities in terms of the large scale components. In addition, combining these two series results in the third series representation, which is able
to define the averaged non-linear terms. Though not fully explored, they are initially attractive in that, by making use of these series representations, it becomes possible to discover another filtering procedure totally different from the conventional approaches. The new procedure reformulates the averaging procedure in such a way as to completely avoid the averaged closure terms thereby eliminating the closure problem in its present form. The objective of this dissertation is to derive, formulate and initially evaluate these new series representations.
CHAPTER II
BASIC DEFINITIONS AND CONCEPTS OF THE FILTERING PROCEDURE

2.1 INTRODUCTION

Filtering is a process in a system which eliminates or attenuates a particular wavenumber or frequency portion of a variable or function. A system provides one with the interrelationship between the excitation and the response, and is mathematically defined as a mapping of one set of variables or functions into another. A filter is a device for successfully carrying out those processes and, in general, a functional form is used for it. This is known as the filter function. The response function, defined as the Fourier transform of the filter function, gives a measure of the effects of the filter in wavenumber or frequency space. In general, the following different types of filters are obtained in either analog or digital filter form; high pass, low pass, band pass and band reject as determined by the response function. The analog or continuous filter is acting on a continuous system such that the elements in a set are composed of the continuous variables, while the elements containing a sequence of numbers are generated upon a digital or discrete filter. For details, see Blinchikoff et al. (1976), Papoulis (1977), Hamming (1972, 1983).

The analog filter was initially developed in the field of electrical engineering with later extensions in most other scientific fields. In electrical engineering, any unwanted
frequency or wavelength information is eliminated or damped out by passing a continuously varying voltage or current through the analog filter. As LES research has shown, the analog filter has also been used in hydrodynamics. All governing equations in turbulence are presumably established via the continuum concept, and are three-dimensional and time dependent so that their solutions should be valid over the whole flow field and at any time. In addition, the equations span all wavenumbers or frequencies. Therefore, as mentioned, the equations need to be simplified by removing the generally unimportant scales before attempting to solve them. The procedure for getting the simplified equations is called ‘preparing the equation’ and the resultant equation is known as the ‘prepared’ equation. This LES filtering procedure is completely in accordance with analog filtering.

The digital filter is a discrete version of the analog filter. It replaces the continuous variable or function by a discrete representation, in which the variable is defined only at grid points. Digital filtering concepts are found in the time series analysis or signal processing literature. In particular, the importance of digital representations has increased over the last three decades with the rapid development of the computer technology. Here, most data are measured by the electronic signals and stored as discrete sequences of numbers. For fluids computations, this digital filter is strongly related to the numerical scheme; Bedford et al. (1987a) and Dingman (1986) show that the numerical scheme can be generalized by making use of the discrete filter.
2.2 BASIC DEFINITIONS

Turbulence is active throughout all scales from the largest to the smallest in both time and space. The small scales one wishes to parameterize are non-linearly coupled to the large scales and, because of their role in energy dissipation, are dynamically too essential to be neglected.

According to Kolmogorov's arguments (1941), the length scale of the dissipative eddies, $\eta$, is written as

$$
\eta^4 = \frac{\nu^3}{\epsilon}
$$

(2.1)

where $\nu$ is the kinematic viscosity and $\epsilon$ denotes the rate of dissipation of turbulent kinetic energy per unit mass. $\epsilon$ is also represented as the ratio of the cube of the velocity scale ($q$) to the length scale ($L$) of the energy-containing eddies, i.e., $\epsilon = q^3/L$. From energy budget observations, the required number of grid points for resolving both the largest and the smallest scales is estimated as

$$
N = \left(\frac{L}{\eta}\right)^3 = R_L^{9/4}
$$

(2.2)

where $R_L$ is the Reynolds number. By the above estimation, more than $10^{13}$ nodal points will be required in the typical turbulent flow, in which $R_L = 10^7 - 10^8$ (Kwak et al. 1975). It is clearly impossible to handle them with today's largest and fastest computers.
2.2.1 Averaging Definition and Filter Function

Now, let \( f(x,t) \) be the total variable (field variable), which is three-dimensional and time dependent, and decompose into its averaged component \( \bar{f} \) and its deviation \( f' \) such that

\[
f(x,t) = \bar{f}(x,t) + f'(x,t) \tag{2.3}
\]

where \( x=(x_1,x_2,x_3)=(x,y,z) \). It is noted that neither \( \bar{f} \) nor \( f' \) are not necessarily a function of time and space. Of course, \( \bar{f} \) depends on the physics as well as on the averaging definition. Let's define the averaged variable (filtered variable) as

\[
\bar{f}(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x-x',t-t') f(x',t') \, dx' \, dt' \tag{2.4}
\]

where \( G(x,t) \) is a filter function (or averaged, weight function) which will be discussed in the next section.

Note that the above averaging procedure is different from the ensemble average. Eq.(2.4) says that the averaged variable (or function), \( \bar{f}(x,t) \), is totally affected by the filter function, \( G(x,t) \). The choice of \( G(x,t) \) particularly depends on what information is desired to remain as a result of the filtering procedure. Accordingly, various types of \( G(x,t) \) exist and are selected on a case by case basis. For example, when low wavenumber information is useful, a low pass filter function, able to eliminate high wavenumber effects, is designed.

The following condition for \( G(x,t) \) must be satisfied because \( \bar{f} \) is expected to be identical to \( f \) itself at a constant value.
\[
\int \int \int G(x,t) \, dx \, dt = 1
\]  

(2.5)

Eq. (2.5) may be also interpreted as a weight function constraint. In general, \(G(x,t)\) may be decomposed into two parts, i.e., spatial \(G_s\) and temporal \(G_t\) components, and is written as

\[
G(x,t) = G_t(x)G_s(t) = G_t(t) \prod_{i=1}^{3} G_i(x_i)
\]  

(2.6)

Here, \(G_i(x_i)\) denotes the component of \(G_s(x)\) in the i-th direction.

2.2.2 Response Function and its Property

The model for which the filtering procedure is used is characterized by the filter function. Thus, it is desirable to investigate its properties in order to know how \(\tilde{f}(x,t)\) is related to \(f(x,t)\) or how \(G(x,t)\) affects the filtering procedure. This can be demonstrated by taking the Fourier transform of the filtering definition in Eq.(2.4), which results in the following relation:

\[
\mathcal{F}[\tilde{f}] = \mathcal{F}[G] \mathcal{F}[f]
\]  

(2.7)

where \(\mathcal{F}[\cdot]\) denotes the Fourier transform of \((\cdot)\). The response function, \(R(k,\omega)\), is defined as the ratio:

\[
R(k,\omega) = \frac{\mathcal{F}[\tilde{f}]}{\mathcal{F}[f]} = \mathcal{F}[G]
\]  

(2.8)

where \(k\) and \(\omega\) are the wavenumber and frequency, respectively.
Eq.(28) shows that the averaged variable ($\overline{f}$) is directly proportional to the filter function in Fourier space. As a result, one can determine how much the amplitude of a particular signal's wavenumber or frequency spectrum is affected by the filtering procedure. Based upon the character of the response function, there exist four classes of filters: low pass, high pass, band pass and band stop (band reject), which are schematized in Fig.2.1. It shows that the low pass filter response function is close to one at the small values of $k$ or $\omega$, and goes to zero at the large values. It means that the low wavenumber (frequency) information is passed or resolved but the high wavenumber (frequency) information is eliminated. The high pass filter is the opposite. Band pass or band stop filters pass or reject the wavenumber (frequency) information between two particular values. The low pass filter is important because it yields the averaged variables which describe the large scale components used in computing on a coarse grid interval. The high pass filter is also useful because it is directly associated with the subgrid scale (SGS) activities.
Figure 2.1: Various filter types
2.3 LOW PASS FILTER

To date, three kinds of low pass filter functions have been employed in fluids modeling: uniform, subgrid scale and Gaussian filter types. High pass filter functions, the low pass counterpart, have never been used in hydrodynamic models.

2.3.1 Uniform Filter

As the simplest one, the uniform filter (also known as top-hat or box filter) has been the most frequently used, and is written as

$$G(x, t) = \frac{1}{\Delta t} \prod_{i=1}^{3} \frac{1}{\Delta_j} \quad \text{for} \quad -\frac{\Delta}{2} < t < \frac{\Delta}{2}, \quad -\frac{\Delta}{2} < x < \frac{\Delta}{2}$$

This leads to the analytic form of an equally weighted moving average within an interval. It has been the case that either the spatial or temporal part has been separately employed. For instance, the application of only temporal averaging in Eq.(2.9) gives the traditional Reynolds equation such that

$$\bar{f}(x, t) = \lim_{\Delta t \to \infty} \int_{t-\Delta t}^{t+\Delta t} \frac{1}{\Delta t} f(x, t') dt'$$

(2.10)

In this case, the response function, $R(\omega)$, will be
\[ R(\omega) = \frac{\sin(\frac{\Delta \lambda}{2})}{(\omega \frac{\Delta \lambda}{2})} \]  

(2.11)

Fig. 2.2 presents the uniform filter function and its response, and it shows that the low frequency (wavenumber) behavior is satisfactory, but not so satisfactory as far as the high frequency (wavenumber) response where the information is not fully eliminated. The negative value indicates not only a 180° phase shift but a serious change of amplitude which can introduce misleading information into the smooth averaged quantity. Moreover, the existence of the singular points (the zeroes) prevents one from taking the inverse transform in Fourier space. Although the uniform filter is commonly used to analyze experimental or field data, and largely employed in the time series analysis, it seems to be inappropriate for models because of such undesirable properties. It will be particularly poor in the case that correct spectral model results are of interest.

Figure 2.2: Uniform filter function and its response function
2.3.2 Sub-Grid Scale Filter

Another filter form is the sub-grid scale filter, which is the Fourier space version of the uniform filter. It results from a sharp cut-off of the wavenumber or frequency information at higher than a critical cut-off value, $k_c$. The spatial form is expressed as

$$G(x-x') = \prod_{i=1}^{3} \frac{\sin k_i(x_i-x'_i)}{\pi(x_i-x'_i)}$$

(2.12)

This filter completely eliminates high wavenumber (frequency) information, and the results, consequently, will be equivalent to those from a Fourier computation. Thus, it would be proper only if the subgrid scale term really represents the residual portion. Kwak et al. (1975) pointed out in detail the limitations of its use for hydrodynamic equations and, therefore, this filter is rarely used.

2.3.3 Gaussian Filter

In LES models, the Gaussian filter is most widely employed; the mathematical form is written in its most general form (Dakhoul and Bedford, 1986a) as

$$G(x,t) = \left(\frac{\gamma_t}{\pi}\right)^{\frac{1}{2}} \frac{1}{\Delta_t} \exp\left(-\frac{\gamma_t^{t^2}}{\Delta_t^2}\right) \prod_{i=1}^{3} \left(\frac{\gamma_i}{\pi}\right)^{\frac{1}{2}} \frac{1}{\Delta_i} \exp\left(-\frac{\gamma_i^{x^2}}{\Delta_i^2}\right)$$

(2.13)

where $\gamma_t$ and $\gamma_i$ are constants in time and space, respectively, and $\Delta_t$ and $\Delta_i$ are the averaging scales to be determined according to the problem under consideration. This filter function is already well-known in the statistical literature, it is easily converted into the spectral domain, and it is relatively free from differentiation or integration,
contrary to the prementioned two filters. For these reasons, this filter is frequently employed in recent hydrodynamic modeling research. Its response function in the temporal average (for convenience) is given by

\[ R(\omega) = \exp\left(-\frac{\Delta^2}{4\gamma_t} \omega^2\right) \] (2.14)

This filter function and its response are also presented in Fig. 2.3. It shows that the response, \( R \), is close to one for low frequencies (wavenumbers) and decreases exponentially to zero for large values. In addition, it has the desirable aspect that the response has neither a negative nor zero value. Due to these properties, the Gaussian filter is preferable to most other filters used in LES.

![Figure 2.3: Gaussian filter function and its response function](image-url)
It is worth noting that one should not exclude the possibility of many other different filters existing. The reason is that the filter function, \( G(x,t) \), is used to define a statistical average in the same way as the probability density function (pdf). In other words, the pdf is conceptually in accordance with the filter function above and satisfies the constraints in the filter function as well. Therefore, it is expected that any pdf may be used for the filter function and, for example, the Poisson, gamma, or exponential type pdf could also be employed as the filter function. However, one should remember here that the choice of the filter function depends on how well it describes the character of the turbulence as well as how conveniently it is realized in the equation. Thus, in order to select the most appropriate filter function, full understanding of the turbulence statistics is needed. The development of this concept remains for further research. In this research the Gaussian filter function is the only filter considered.

2.4 HIGH PASS FILTER

In addition to a low pass filter, a high pass filter can also be designed depending on what the particular interest is in a given problem. A high pass filter is used to eliminate the low frequency (small wavenumber) activities and to resolve the remaining high frequency values. This property indicates that the high pass filter may be used for investigating the subgrid scale activities which are most directly associated with the high frequency (wavenumber) turbulence. Indeed, this thesis shows that the small scale components can be simply obtained for the first time by a high pass filter realization, based on the observation that the overall signal is equivalent to the sum of large and small scales.
When the Fourier transform of the averaging definition is taken and both sides are divided by the resulting term on the left, the following relation is obtained:

$$1 = \frac{H[f]}{H[f]} + \frac{H[f']}{H[f]}$$  \hspace{1cm} (2.15)

By the definition, the first and second terms on the right hand side denote the low and high pass response functions, respectively. Thus, it gives the high pass transfer function as

$$H(k,\omega) = 1 - L(k,\omega)$$  \hspace{1cm} (2.16)

where $H(k,\omega)$ and $L(k,\omega)$ denote the high pass and low pass response function. This relation states that once a low pass filter function is given, the corresponding high pass filter is determined.

Using this definition, the uniform and Gaussian low pass response functions give, respectively, their high pass responses for a time filter as

$$H(\omega) = 1 - \frac{\sin \left( \frac{\omega \Delta}{2} \right)}{\left( \omega - \frac{\Delta}{2} \right)}$$  \hspace{1cm} (2.17)

$$H(\omega) = 1 - \exp \left( - \frac{\Delta^2}{4\gamma_t} \omega^2 \right)$$  \hspace{1cm} (2.18)

The corresponding high pass filter functions, $g(t)$, can be found by the inverse Fourier transform such that
\[ g(t) = F^{-1}[H(\omega)] = \delta(t) - G(t) \]  

(2.19)

where \( F^{-1} \) denotes the inverse Fourier transform, and \( \delta(t) \) is a Dirac delta function. Eq.(2.19) leads to an equation for the SGS components by applying the high pass filter of either the uniform or the Gaussian type to the governing equation:

\[ f'(t) = \int_{-\infty}^{\infty} [\delta(t-t') - G(t-t')] f(t') \, dt' \]

\[ = f(t) - \bar{f}(t) \]

\[ = CA_i^2 \frac{\partial^2 f}{\partial t^2} + O(\Delta_t^4) \]  

(2.20)

where \( C \) is the constant to be determined by the second moment of the filter function. Note that the equivalent result is also derived from a different approach, done by Kwak et al. (1975), and Clark et al. (1977, 1979). They formulated the SGS components by assuming that, if the function is fairly smooth, then its averaged function can be locally approximated by a Taylor series expansion. This hypothesis often leaves an open question of whether the use of the Taylor series expansion is valid for SGS quantities, which will be reviewed in Chapter III. Table 2.1 summarizes each filter used in the existing models.
Table 2.1
Filter functions used in the existing models

<table>
<thead>
<tr>
<th>Definition of the Large-scale component</th>
<th>Reynolds Temporal Averaging 1883 to Present</th>
<th>Uniform Spatial Averaging 1930 to Present</th>
<th>Leonard's Spatial Filtering 1974 to Present</th>
<th>Dakhoul and Bedford's STF 1985 to Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x,t) = \int_{-\frac{\Delta_t}{2}}^{\frac{\Delta_t}{2}} G(x-s)dx</td>
<td>f(x,t) = \int_{-\frac{\Delta_s}{2}}^{\frac{\Delta_s}{2}} G(x-s')dx'</td>
<td>f(x,t) = \int_{-\frac{\Delta_s}{2}}^{\frac{\Delta_s}{2}} G(x-s')dx'</td>
<td>f(x,t) = \int_{-\frac{\Delta_s}{2}}^{\frac{\Delta_s}{2}} G(x-s')dx'</td>
<td></td>
</tr>
<tr>
<td>g(x) = \frac{1}{\Delta_t} \text{ for } -\frac{\Delta_t}{2} \leq x \leq \frac{\Delta_t}{2}</td>
<td>G(x) = \prod_{i=1}^{n} G(x_i)</td>
<td>G(x) = \prod_{i=1}^{n} G(x_i)</td>
<td>G(x) = \prod_{i=1}^{n} G(x_i)</td>
<td></td>
</tr>
<tr>
<td>= 0 \text{ otherwise}</td>
<td>where</td>
<td>where</td>
<td>where</td>
<td></td>
</tr>
<tr>
<td>G(x) = \frac{1}{\Delta_s} \text{ for } -\frac{\Delta_s}{2} \leq x \leq \frac{\Delta_s}{2}</td>
<td>G(x_i) = \frac{1}{\Delta_s} e^{-\frac{x^2}{\Delta_s}}</td>
<td>G(x_i) = \frac{1}{\Delta_s} e^{-\frac{x^2}{\Delta_s}}</td>
<td>G(x_i) = \frac{1}{\Delta_s} e^{-\frac{x^2}{\Delta_s}}</td>
<td></td>
</tr>
<tr>
<td>= 0 \text{ otherwise}</td>
<td>where</td>
<td>where</td>
<td>where</td>
<td></td>
</tr>
<tr>
<td>G(x) = \left( \frac{x}{\Delta_s} \right)^{1/2} e^{-\frac{x^2}{\Delta_s}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td></td>
</tr>
<tr>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td>\tilde{\gamma}(\chi) = \frac{1}{\Delta_s} e^{-\frac{\chi^2}{2\Delta_s^2}}</td>
<td></td>
</tr>
</tbody>
</table>

Fourier Transform of the Large-scale component

- \tilde{F}(F(x,t)) = \mathcal{F}[G(x)] F(x,t)
- \tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)
- \tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)
- \tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)

where

- \tilde{G}(x) = \frac{1}{\Delta_s} \sin(\frac{\pi x}{\Delta_s})
- \tilde{G}(x) = \frac{1}{\Delta_s} \sin(\frac{\pi x}{\Delta_s})
- \tilde{G}(x) = \frac{1}{\Delta_s} \sin(\frac{\pi x}{\Delta_s})
- \tilde{G}(x) = \frac{1}{\Delta_s} \sin(\frac{\pi x}{\Delta_s})

\tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)

\tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)

\tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)

\tilde{F}(G(x)) = \prod_{i=1}^{n} \tilde{G}(x_i) F(x,t)
CHAPTER III
APPLICATION OF FILTERING OPERATIONS TO THE
TURBULENT TRANSPORT EQUATIONS

As mentioned in the previous Chapter, it is necessary to filter the equations first, before numerically solving the continuum equations of turbulent flow. This Chapter reviews the application of the filtering procedure to the Navier-Stokes equations and shows how filtering affects the turbulent transport equations.

3.1 APPLICATION TO THE NAVIER-STOKES EQUATIONS

The general averaging definition was already given in Chapter II. In this Section, the filtering operation is applied to the Navier-Stokes equations and the averaged equations are derived. Using the Cartesian tensor notation and the corresponding summation convention, the continuity equation, the equation of motion, and the general transport equation for an incompressible fluid are

\[
\begin{align*}
    u_{i,i} &= 0 \\
    \frac{\partial u_i}{\partial t} + (u_j u_j)_j &= -\frac{1}{\rho} p_{,i} + \nu u_{i,jj} \\
    \frac{\partial \phi}{\partial t} + (\phi u_j)_j &= D \phi_{,ij}
\end{align*}
\]
where \( \rho \) and \( \nu \) are the density and the kinematic viscosity of fluid, respectively, \( D \) denotes the diffusivity, \( \phi \) is a scalar such as concentration or temperature, and the comma represents the derivatives. These equations act in a domain consisting of the total variables. When the averaging definition in Eq.(2.4) is applied to these equations, the resulting equations are:

\[
\bar{u}_{i, i} = 0 \tag{3.4}
\]

\[
\frac{\partial \bar{u}_i}{\partial t} + (\bar{u}_i \bar{u}_j)_{,j} = -\frac{1}{\rho} \bar{p}_{,i} + \nu \bar{u}_{ij} \tag{3.5}
\]

\[
\frac{\partial \bar{\phi}}{\partial t} + (\bar{\phi} \bar{u}_j)_{,j} = D \bar{\phi}_{,ij} \tag{3.6}
\]

Subtracting the averaged equations in Eq.(3.4) - (3.6) from Eq.(3.1) - (3.3), respectively, one can show how the filtering operation acts on the governing equations, i.e.,

\[
u'_{i, i} = 0 \tag{3.7}
\]

\[
\frac{\partial \nu'}{\partial t} + (\nu' \nu_j)'_{,j} = -\frac{1}{\rho} \nu'_{,i} + \nu \nu'_{ij} \tag{3.8}
\]

\[
\frac{\partial \nu'}{\partial t} + (\nu' \nu_j)'_{,j} = D \nu'_{,ij} \tag{3.9}
\]

Eq.(3.7) - (3.9) indicate that application of the averaging rule or low pass filter operation is equivalent to removing the small scale quantities \((u', (u_i u_j)', p', \phi')\) from the governing equations. The absence of these highly fluctuating components is also interpreted as the elimination due to the linearity in each term. It implies that, while solutions of Eq.(3.1) - (3.3) consist of the actual quantities \((u, p, \phi)\) themselves, those of Eq.(3.4) - (3.6) are composed of the averaged values \((\bar{u}, \bar{p}, \bar{\phi})\). This discrepancy is not a bad situation in most engineering problems, if solutions are feasible and easily obtai-
nable. Remembering that the prepared equation must be described in terms of the averaged quantity of the single variable, the linear terms are in agreement with this requirement. However, the non-linear terms in Eq. (3.5) and (3.6) are different. They contain a most serious difficulty; the existence of the averaged non-linear terms \((\overline{u_j u_j}, \overline{\phi u_j})\). These terms should be reasonably replaced by the averaged quantities of the single variable \((\overline{u_j}, \overline{\phi})\). Their replacement is not easy but very important because all the complexity existing in the turbulence is concentrated on these terms. In other words, the success of turbulence models depends on how these terms are dealt with.

By the definition in Eq. (2.3), these averaged non-linear terms are decomposed into the following doubly averaged terms.

\[
\overline{u_j u_j} = \overline{u_i u_j} + \overline{u_i u_j} + \overline{u_i u_j} + \overline{u_i u_j} \tag{3.10}
\]

\[
\overline{\phi u_j} = \overline{\phi u_j} + \overline{\phi u_j} + \overline{\phi u_j} + \overline{\phi u_j} \tag{3.11}
\]

Most research has been done with regard to the four terms, separately, and many successful results for each term are reported. The following Sections will show how each term in Eq. (3.10) and (3.11) is defined and specified.

3.2 REYNOLDS AVERAGE

Reynolds average has been a root of the statistical turbulence theory for over a century. He developed his equations for the mean flow, based on the following averaging rules:

(A) \( \overline{T \pm g} = \overline{T} \pm \overline{g} \)
(B) \( \bar{a}^\prime = a^\prime \), \( a = \text{const.} \)

(C) \( \bar{a} = a \), \( a = \text{const.} \)

(D) \( \frac{\partial \bar{f}}{\partial s} = \frac{\partial \bar{g}}{\partial s} \), where \( s = x, y, z, t \).

(E) \( \bar{f} \bar{g} = \bar{f} \bar{g} \)

Mathematically, the averaging rules (A)-(D) are exactly derived by the averaging definiton with the arbitrary filter function satisfying the condition of Eq.(2.5), but (E) is not. Reynolds believed that (E) can be made satisfactory with a comparatively high degree of accuracy by choosing the averaging interval large in comparison with the characteristic periods of the fluctuating component, but small in comparison with the periods of the averaged quantity (Monin and Yaglom 1971). Assuming that (E) is allowed, the following important auxiliary axioms result;

(F) \( \bar{f} = \bar{f} \)

(G) \( \bar{f}^\prime = 0 \)

(H) \( \bar{f} \bar{g} = \bar{f} \bar{g} \)

(I) \( \bar{f} \bar{g} = \bar{f} \bar{g} = 0 \)

With these averaging axioms, the averaged non-linear terms in Eq.(3.10) and (3.11) are simplified as

\[
\bar{u}_i \bar{u}_j = \bar{u}_i \bar{u}_j + \bar{u}_i \bar{u}_j \quad (3.12)
\]

\[
\bar{\phi} \bar{u}_j = \bar{\phi} \bar{u}_j + \bar{\phi} \bar{u}_j \quad (3.13)
\]
Here, the last terms on the right hand side of Eq. (3.12) and (3.13) are known as the Reynolds stress terms and Reynolds flux terms, respectively. It is noted that Reynolds equations are described in terms of a time average but extensions to the whole probability space by the ensemble average have occurred. Therefore, it can be said that the ensemble average presumably follows the Reynolds axioms as well, i.e.,

\[
\begin{align*}
\bar{u}_i \bar{u}_j &= \bar{\bar{u}}_i \bar{u}_j \quad \text{or} \quad \bar{\phi} \bar{u}_j &= \bar{\phi \bar{u}}_j \\
\bar{u}^*_i \bar{u}_j &= \bar{\bar{u}}^*_i \bar{u}_j \quad \text{or} \quad \bar{\phi \bar{u}}^*_j &= \bar{\bar{\phi \bar{u}}}^*_j
\end{align*}
\] (3.14) (3.15)

Both relations are true from a statistical point of view.

3.3 VOLUME AVERAGE

The Reynolds average was implemented with averaging rule, (E). However, it is believed that this qualitative assumption is not satisfied in the general sense. For instance, the averaging definition in this dissertation, the volume average, denotes a moving average so that the above relations in Eq. (3.14) and (3.15) are no longer valid. Of course, the averaging rules (E)-(I) are not allowed, either. Thus, Reynolds average might be thought of as the limiting case; being valid only within a fixed interval. This difference is presented in Fig.3.1. The most significant difference between volume averages and ensemble averages or Reynolds averages is that all the turbulence is completely eliminated in the latter, but only the components of turbulence smaller than the averaging scale are removed in the former. Thus, both averages will be identical as the averaging volume becomes large enough. Wyngaard (1982) pointed out that it could be advantageous to solve the equation of the ensemble average, providing that the Reynolds stress terms are well specified. However, accurately specifying the Reynolds
terms is so difficult that the closure terms are usually employed for it. They will be discussed in Chapter VIII.

Figure 3.1: Reynolds average and volume average
(after Leonard 1974)

In particular, when the non-linear turbulence equations are integrated by the numerical method, it is generally admitted that Eq.(3.14) and (3.15) are poorly approximated assumptions. For better solutions, many attempts have been made and various models are proposed, and are presented below.
3.3.1 The Leonard Term

Leonard (1974) formulated the first term in Eq.(3.10) only in space. He insisted that the assumption of Eq.(3.14) is inappropriate to produce the large scale dissipation and, thus, the term $\overline{u_i u_j}$ should be re-evaluated. Assuming that $\overline{u_i u_j}$ is smooth enough to use the Taylor series expansion with respect to the scale of the filter width, the following equation is derived.

$$
\overline{u_i u_j}(x) = \int \int \int_{-\infty}^{+\infty} G(x-\zeta) [\overline{u_i u_j}](x) + (\zeta_k - x_k) \overline{\Delta_i u_j(x, \zeta_k)} + \cdots \cdot d\zeta
$$

$$
= \overline{u_i u_j} + C \Delta^2 (\overline{u_i u_j})_{x x}
$$

(3.16)

where $C$ is 1/24 for the uniform filter, and 1/4$\gamma$ for the Gaussian filter. According to Leonard's discussion, the last term on the right side of Eq.(3.16) plays a significant role in the dissipation due to the large scale motion while the remaining unresolved dissipation must be produced by the SGS terms, which are supposed to be modeled. This second derivative term is called the Leonard term.

3.3.2 The Cross Term

Clark et al. (1977) developed a decomposition for the cross term (for example, $\overline{u_i u_j}$), which is totally analogous to the derivation for the Leonard term. He suggested that the fluctuating components can also be defined, based on the derivation of Eq.(3.16). For example, if $u_i$ is replaced into $\overline{u_i u_j}$ in Eq.(3.16), the fluctuating component, $u'_i$, is written as follows.
\( u_i = u_i + C \Delta_s^2 u_{kk} \) or \( u'_i = u - u_i = -C \Delta_s^2 u_{kk} \) \hspace{1cm} (3.17)

With simple substitution of \( u'_i \) into the cross term \( (\tilde{u}'_i \tilde{u}'_j) \), one gets

\[ \tilde{u}'_i \tilde{u}'_j = (-C \Delta_s^2) \tilde{u}_{kk} \tilde{u}_j = -(C \Delta_s^2) \tilde{u}_{kk} \tilde{u}_j \] \hspace{1cm} (3.18)

Similarly, the cross term for the scalar fluxes is

\[ \tilde{\phi}' \tilde{u}'_j = (-C \Delta_s^2) \tilde{\phi}_{kk} \tilde{u}_j = -(C \Delta_s^2) \tilde{\phi}_{kk} \tilde{u}_j \] \hspace{1cm} (3.19)

After combining Leonard's term in Eq.(3.16), the following simplified decompositions result:

\[ \tilde{u}'_i \tilde{u}'_j + \tilde{u}'_i \tilde{u}'_j + \tilde{u}'_i \tilde{u}'_j = \tilde{u}_i \tilde{u}_j + 2C \Delta_s^2 \tilde{u}_{kk} \tilde{u}_{jk} \] \hspace{1cm} (3.20)

\[ \tilde{\phi}' \tilde{u}'_j + \tilde{\phi}' \tilde{u}'_j + \tilde{\phi}' \tilde{u}'_j = \tilde{\phi} \tilde{u}_j + 2C \Delta_s^2 \tilde{\phi}_{kk} \tilde{u}_{jk} \] \hspace{1cm} (3.21)

The last term on the right hand side of Eq.(3.20) or (3.21), which is the product of the first derivative terms, is called Clark's reduction. These terms are very useful in numerical applications because of their lower-order derivatives. But, these cross terms also generate the same question on the convergence validity as the Leonard term, i.e., whether or not it is feasible to represent the highly fluctuating nature of \( u'_i \) or \( \phi' \) by a Taylor series expansion.
3.4 SPACE-TIME FILTER OF DAKHOUL AND BEDFORD

Leonard (1974), Kwak et al. (1975) and Clark et al. (1979) developed their models by formulating only spatial filters. Dakhoul and Bedford (1986a, 1986b) employed the Gaussian filter in both time and space, and obtained improved numerical results with the model. For instance, the space-time filter averaging method with the Gaussian filter decomposes the non-linear terms as

\[
\overline{u_j u_j} = \overline{u_j u_j} + \left( \frac{\Delta_t^2}{4\gamma_t} \right) \frac{\partial^2}{\partial t^2} (\overline{u_j u_j}) + \frac{\Delta_s^2}{4\gamma_s} (\overline{u_j u_j})_{kk} \tag{3.22}
\]

\[
\overline{\phi u_j} = \overline{\phi u_j} + \left( \frac{\Delta_t^2}{4\gamma_t} \right) \frac{\partial^2}{\partial t^2} (\overline{\phi u_j}) + \frac{\Delta_s^2}{4\gamma_s} (\overline{\phi u_j})_{kk} \tag{3.23}
\]

This spatial and temporal filter can be extended to the Clark's version as well. The resultant forms for the non-linear terms are

\[
\overline{u_j u_j} = \overline{u_j u_j} + \frac{\Delta_t^2}{2\gamma_t} \left( \frac{\partial u_j}{\partial t} \right) \left( \frac{\partial u_j}{\partial t} \right) + \frac{\Delta_s^2}{2\gamma_s} (\overline{u_k u_k}) + \overline{u_j u_j} \tag{3.24}
\]

\[
\overline{\phi u_j} = \overline{\phi u_j} + \frac{\Delta_t^2}{2\gamma_t} \left( \frac{\partial \phi}{\partial t} \right) \left( \frac{\partial \phi}{\partial t} \right) + \frac{\Delta_s^2}{2\gamma_s} (\overline{\phi u_k}) + \overline{\phi u_j} \tag{3.25}
\]

The improvements resulting from this method have been initially verified by means of statistical and spectral comparisons with an exact or fully resolved turbulence solution for a time-space varying form of the Burgers equation.
3.5 SUB-GRID SCALE (SGS) TERM

The last terms in Eq.(3.10) and (3.11), $\overline{u_i' u_j'}$ and $\overline{\phi' u_j'}$, must be also parameterized by the large scale quantities to yield a closed set of equations. Many different ways have been proposed for closure models not only in Reynolds averages (ensemble averages) but in volume averages. In Reynolds averaging type models, closure models are often classified according to the number of equations used to describe the SGS quantities (Rodi 1980, Bedford et al. 1987b). To date, four groups are defined; zero-, one-, two- and turbulent stress/flux equation models. They are summarized and discussed in chapter VIII. In volume averaging (mainly the LES approach), Voke et al. (1983) classified four types of models according to the way determining the eddy viscosity; constant, stress, vorticity, and SGS energy models. The most commonly and extensively used one is the stress model, which is also known as the Smagorinsky-Lilly model. These models will be also discussed in Chapter VIII.

In this Chapter, the current state of applying the filtering procedure to the turbulence equations was reviewed. Indeed, it is worth noting that the Leonard or cross terms are derived on the basis of Taylor series expansion technique. A question arises as to the convergence of the series, i.e., how many terms are needed for obtaining the approximated values with the required accuracy. If many higher-order derivative terms are necessary, these modified terms might be useless, from a practical point of view. In particular, the derivation of the cross terms raises a serious question of the mathematical validity; whether or not it is feasible to represent the highly fluctuating nature of $u'_i$ or $\phi'$ by Taylor series expansion.
CHAPTER IV
RATIONALE FOR AND DERIVATION OF A NEW FILTERING APPROACH

In Chapter II and III, the conventional averaging procedure was reviewed as regards deriving the prepared equations for the turbulent flows. Large Eddy Simulation (LES) has been developed on the basis of the volume averaging and is thought of as a relatively new and very promising method for numerically analyzing turbulence structures with large Reynolds numbers. Regardless of the potential in this approach, it is not possible to say that the averaged non-linear terms are satisfactorily specified, to date. Accordingly, the unknown variables depend on the form of the SGS terms and the resulting closure. In this Chapter, an analysis and investigation of the basic averaging definitions used in the LES models are presented in order to answer the question as to what the fundamental character of the SGS terms is. Consequently, a new filtering approach to these problems is introduced.

4.1 TWO DESCRIPTIONS FOR THE SGS TERMS

Initially, all the complexity is caused by the existence of the non-linear terms. Two possible ways to describe the non-linear terms are found. When f and g denote arbitrary variables, two different forms for the non-linear term are written, respectively, as

\[ \bar{f}g + (fg)' \]

(4.1)
\begin{equation}
fg = (f + f')(g + g')
\end{equation}

Eq.(4.1) is the lumped form and Eq.(4.2) the separated form. By selection of these decomposed forms at the initial stage of the filtering procedure, the resulting equations are completely different. Because both Eq.(4.1) and (4.2) are identical when \( g \) is the constant, the linear terms are not affected by which form may be used. However, their use in the non-linear terms leads to the two different equation forms. For instance, the lumped description in Eq.(4.1) gives the equation of motion as

\begin{equation}
\frac{\partial}{\partial t}(u_i' + u_j') + [(u_i'u_j') + (u_iu_j)]_j = -\frac{1}{\rho}(\bar{g} + \rho), + \nu (u_i' + u_j')_j,
\end{equation}

while the separated form is

\begin{equation}
\frac{\partial}{\partial t}(u_i' + u_j') + [(u_i'u_j') + (u_iu_j)]_j = -\frac{1}{\rho}(\bar{g} + \rho), + \nu (u_i' + u_j')_j
\end{equation}

In the same way, the two types of transport equations are

\begin{equation}
\frac{\partial}{\partial t} (\bar{g} + \phi') + [(\bar{g}u_j') + (\phi u_j')] = D (\bar{g} + \phi')_j
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} (\bar{g} + \phi') + [(\bar{g} + \phi')(u_j + u_j')]_j = D (\bar{g} + \phi')_j
\end{equation}

Certainly, the difference between the two types of equations concerns the description of the non-linear terms, i.e.,

\begin{equation}
u_j = u_i'u_j' + u_i' \quad \text{or} \quad \phi u_j = \bar{g}u_j' + (\phi u_j')
\end{equation}

\begin{equation}u_i'u_j = (u_i + u_i')(u_j + u_j') \quad \text{or} \quad \phi u_j = (\bar{g} + \phi')(u_j + u_j')
\end{equation}
From this observation, it is termed that the conventional approach follows the lumped description at the initial averaging stage. By applying the filtering operation to the governing equations, the conventional approach provides simplified equations, where it is not necessary to specify the lumped SGS terms, \((u'u_j)\) and \((\phi u_j)\), because they are eliminated by the linearity in each term. This was already mentioned in Section 3.1, and these simplified forms are attractive. However, this approach concentrates all the detailed complexity on the averaged non-linear terms. Thus, unless these terms are well specified, this approach is not plausible. As discussed earlier, the conventional approach suffers from the lack of interpretation of the averaged non-linear terms. To define these terms, the separated description in Eq.(4.2) is used and, consequently, results in four terms at the doubly averaged level. On the other hand, the use of the separated form at the initial averaging step removes these averaged non-linear terms, and gives a quite different style for the SGS quantities. In short, both approaches have their good points; the lumped approach results in the simplified form and the separated method does not produce the averaged non-linear terms.

It is useful to look into the conventional approach and to find what the problem is with this approach. In order to analyze the averaged non-linear terms (\( \overline{u'u_j} \) or \( \overline{\phi u_j} \)), the existing LES models have been developed principally based on the following description of the total variable. When only the spatial filter is considered, for convenience, the total variable (for example, \( u_i \)) is written, from Eq.(3.17), as

\[
  u_i = \overline{u_i} + u'_i = \overline{u_i} - C\Delta^2 u_{ikk} \tag{4.9}
\]

where \( C \) is given in Eq.(3.16). The detailed derivation was already presented in Section 3.3.1. Eq.(4.9) shows that the SGS quantity \( (u'_i) \) is expressed in terms of the second
derivative of the total variable \( u_i \) itself. It means that the resultant description is the function of both \( \bar{u}_i \) and \( u_i \). In the LES approach, one wishes that all the variables be replaced by the averaged quantities and, thus, the 'prepared equations' should consist of the averaged components themselves. In this sense, the description in Eq.\((4.9)\) is certainly not appropriate. Accordingly, one more step is to replace \( u_i \) in the SGS term by \( \bar{u}_i \). To do this, an additional decomposition of \( u_i \) and \( u_j \) in the averaged non-linear term is performed by substituting them into the averaged and fluctuating components, which results in four terms at the double-averaging level, i.e.,

\[
\bar{u}_i \bar{u}_j = (\bar{u}_i + u_i)(\bar{u}_j + u_j)
\]  

(4.10)

The averaged transport flux terms are similar. It is awkward that all terms on the right should be replaced by the quantities at the single averaging level which implies that such a breaking up into the parts becomes rather complicated since these terms give three averaging levels; non-, single- and double-averaging. In this sense, this procedure follows a relatively longer path. Indeed, although it is reported these models produce very successful results, it is considered that they leave something more to be desired. The primary reason is that most of the research is devoted to specifying each term alone out of four separated terms at the doubly averaged level without the necessary physical understanding being available. For instance, even though the Leonard term is supposed to be exactly specified, it does not guarantee that the averaged non-linear term also provides the equivalent results.

These undesirable aspects result from the SGS description in terms of the total variable. As long as Eq.\((4.9)\) is maintained in specifying the non-linear terms, the decomposing process resulting in four terms at the double averaging level is an inescapable
product. This is why another description of the SGS terms is developed in this dissertation. One possible way might be to directly describe the SGS terms with the averaged variables instead of the total variables. If such a description is found, the four decomposed terms will disappear and another filtering approach making use of the second description in Eq.(4.2) at the initial stage will be obtained. In the following Section, a new description for the SGS quantities is derived by applying the basic filtering definitions shown in Chapter II.

4.2 DERIVATION OF THE SGS QUANTITY FROM THE HIGH PASS FILTER

As mentioned in Chapter II, the response function, \( R(k, \omega) \), is defined as the Fourier transform of the filter function, \( G(x,t) \), and is equivalent to the ratio of the averaged variable \( \bar{f} \) to the total variable \( f \) in the Fourier space. For convenience, it is rewritten as

\[
R(k, \omega) = \frac{\mathcal{F}[\bar{f}]}{\mathcal{F}[f]} = \mathcal{F}[G] \tag{4.11}
\]

where \( k \) and \( \omega \) are the wavenumber and frequency, respectively, and \( \mathcal{F}[\cdot] \) denotes the Fourier transform of \( \cdot \). For simplicity, consider only the spatial filter. The three-dimensional Gaussian filter, \( G(x) \), and its response function, \( L(k) \), are written, respectively, as

\[
G(x) = \prod_{i=1}^{3} \left( \frac{\gamma_i}{\pi} \right)^{\frac{1}{2}} \frac{1}{\Delta_i} \exp\left( -\frac{\gamma_i x_i^2}{\Delta_i^2} \right) \tag{4.12}
\]

\[
L(k) = \mathcal{F}[G] = \exp\left( -\sum_{i=1}^{3} \alpha_i k_i^2 \right) \tag{4.13}
\]
where \( k_i \) is the wavenumber in \( i \)-th direction, and \( a_i = \Delta_i^2/4\gamma_i \). By making use of the series expansion of the exponential function, Eq.(4.13) gives

\[
L(k) = 1 - (a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2) + \frac{1}{2!} (a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2)^2 + \cdots \tag{4.14}
\]

The corresponding high pass response function, \( H(k) \), is

\[
H(k) = 1 - L(k) = (a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2) - \frac{1}{2!} (a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2)^2 + \cdots \tag{4.15}
\]

On the other hand, Eq.(4.11) or (4.13) is equivalent to the following relations:

\[
F[f'] = L(k) F[f] \tag{4.16}
\]

or

\[
F[f'] = \frac{F[f]}{L(k)} \tag{4.17}
\]

Now, taking advantage of the above properties of the Gaussian filter as well as the general formula often used in Fourier space (Hildebrand 1976, Blinchikoff and Zverev 1976), the SGS components of the variable, \( f' \), are defined as follows.

\[
F[f'] = F[f] - F[f]
\]

\[
= H(k) F[f]
\]

\[
= [(a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2) - \frac{1}{2!} (a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2)^2 + \cdots ] F[f]
\]

\[
= - \frac{\partial^2}{\partial x^2} + a_2 \frac{\partial^2}{\partial y^2} + a_3 \frac{\partial^2}{\partial z^2} \chi + \frac{1}{2!} (a_1 \frac{\partial^4}{\partial x^4} + \cdots ) \chi + \cdots \tag{4.18}
\]
The inverse Fourier transform of Eq.(4.18) gives

\[ f' = f - \bar{f} \]

\[ = -\left[ (a_1 \frac{\partial^2}{\partial x^2} + a_2 \frac{\partial^2}{\partial y^2} + a_3 \frac{\partial^2}{\partial z^2}) f + \frac{1}{2!} (a_1^2 \frac{\partial^4}{\partial x^4} + \cdots) f + \cdots \right] \]

\[ = -(m_1 + m_2 + m_3) f - \frac{1}{2!} (m_1^2 + m_2^2 + m_3^2) f - \frac{1}{3!} (m_1 + m_2 + m_3)^3 f - \cdots \]

\[ = -M(f) - \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) - \cdots \quad (4.19) \]

where \( m_i \) and \( M \) are the differential operators, as shown in the above equation. With the assumption of the same filter width (\( \Delta = \Delta_1 = \Delta_2 = \Delta_3 \)) and the same filter constant (\( \gamma = \gamma_1 = \gamma_2 = \gamma_3 \)), Eq.(4.19) will be

\[ f' = f - \bar{f} \]

\[ = -a \nabla^2 f - \frac{1}{2!} a^2 \nabla^4 f - \frac{1}{3!} a^3 \nabla^6 f - \cdots \quad (4.20) \]

where \( a = \frac{\Delta^2}{4\gamma} \).

Eq.(4.19) and (4.20) show that the SGS quantities are expressed in terms of an infinite series, in which each term is a function of the total variable (\( f \)) and the high-order derivatives thereof. These are related to the constant (\( \gamma \)) and the filter width (\( \Delta \)). It is quite interesting that this series, to the second-order of filter width, is exactly identical to Leonard's or Clark's formulation, which was derived by the Taylor series expansion technique in Chapter III. Thus, those terms are considered to describe only the first leading term out of infinite series. Accordingly, Eq.(4.19) or (4.20) is believed, with confidence, to be the general description of the SGS terms used in the conventional approach.
From a different point of view, another form of the SGS components can be formulated with simple algebra:

\[ F[f'] = H(k) F[f] \]
\[ = \frac{H(k)}{L(k)} F[f] \]
\[ = F[-M(f) + \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) + \cdots] \quad (4.21) \]

In the same way, the inverse Fourier transform results in the following final form:

\[ f' = f - \bar{f} \]
\[ = -M(f) + \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) + \cdots \quad (4.22) \]

The use of the same filter width also gives

\[ f' = f - \bar{f} \]
\[ = -a \nabla^2 \bar{f} + \frac{1}{2!} a^2 \nabla^4 \bar{f} - \frac{1}{3!} a^3 \nabla^6 \bar{f} + \cdots \quad (4.23) \]

Although Eq.(4.19) and (4.22) are expressed by infinite series, but they are quite different in their implications. The most significant difference is that the series terms in the latter consist of the averaged variable ( \( \bar{f} \) ), while those in the former are a function of the total variable ( \( f \) ). Certainly, Eq.(4.22) or (4.23) would be more convenient forms for analyzing the non-linear terms, as discussed before, because they do not require any manipulation to replace the total variable quantity appearing in the derivative terms of the SGS activities by the averaged quantity. In other words, it implies only one step from non-averaging to the complete averaging level. As a result, taking advantage of
the above equations can eliminate all the unnecessary filtering step mentioned before, and illuminate the ambiguity in the non-linear terms. Meanwhile, both Eq.(4.19) and (4.22) are mathematically exact in describing an arbitrary variable so that they will give the same results whichever one may be used.

Up to now, two series representing the SGS components have been derived. Their averaging procedure, which makes use of the response function and the inverse Fourier transform for formulating the SGS components, has never been used to date and, therefore, is a totally new approach. For convenience, it will be called the Yeo and Bedford filtering method (YB filtering method), hereafter, and the newly derived series of Eq.(4.19) and (4.22) will be called the YB-I series (the first kind of the Yeo and Bedford series) and the YB-II series (the second kind of the Yeo and Bedford series), respectively. The filtering procedure with the YB-I series will certainly be the same as the conventional filtering approach used in the LES for over two decades. The YB-II series written in Eq.(4.22) or (4.23) is a new method with the apparent advantage that it is no longer necessary to directly apply the basic averaging or filtering operation to the governing equations. In the next Section, the two series' behavior will be investigated.

4.3 APPLICATION OF YB-II SERIES SOLUTION TO THE NAVIER-STOKES EQUATIONS

The objective of this Section is to apply the YB-II series representation to the derivation of the turbulence equations. The 'prepared' equations in the conventional approach were obtained by applying the filtering operations to the governing equations in Chapter III. Indeed, the new approach starts with simply separating the total variables into the averaged and the SGS components, and, after that, the resulting SGS
terms will be again replaced by the averaged variables from YB-II series. The resultant equation of motion will be

\[ \frac{\partial}{\partial t} [u_i - M(u_i) + \frac{1}{2l} M^2(u_i) + \cdots ] + \left[ (u_i - M(u_i) + \frac{1}{2l} M^2(u_i) + \cdots ) (u_j - M(u_j) + \frac{1}{2l} M^2(u_j) + \cdots ) \right]_{ij} = -\frac{1}{\rho} [\bar{p} - M(\bar{p}) + \frac{1}{2l} M^2(\bar{p}) + \cdots ]_{ij} + \nu [u_i - M(u_i) + \frac{1}{2l} M^2(u_i) + \cdots ]_{ij} \]  

(4.24)

With the assumption of the same filter width, Eq.(4.24) becomes

\[ \frac{\partial}{\partial t} [u_i - a \nabla^2 u_i + \frac{1}{2l} a^2 \nabla^4 u_i + \cdots ] + \left[ (u_i - a \nabla^2 u_i + \frac{1}{2l} a^2 \nabla^4 u_i + \cdots ) (u_j - a \nabla^2 u_j + \frac{1}{2l} a^2 \nabla^4 u_j + \cdots ) \right]_{ij} = -\frac{1}{\rho} [\bar{p} - a \nabla^2 \bar{p} + \frac{1}{2l} a^2 \nabla^4 \bar{p} + \cdots ]_{ij} + \nu [u_i - a \nabla^2 u_i + \frac{1}{2l} a^2 \nabla^4 u_i + \cdots ]_{ij} \]  

(4.25)

With the same procedure, the turbulence transport equations are

\[ \frac{\partial}{\partial t} [\bar{\phi} - M(\bar{\phi}) + \frac{1}{2l} M^2(\bar{\phi}) + \cdots ] + \left[ (\bar{\phi} - M(\bar{\phi}) + \frac{1}{2l} M^2(\bar{\phi}) + \cdots ) (u_j - M(u_j) + \frac{1}{2l} M^2(u_j) + \cdots ) \right]_{ij} = D [\bar{\phi} - M(\bar{\phi}) + \frac{1}{2l} M^2(\bar{\phi}) + \cdots ]_{ij} \]  

(4.26)

and

\[ \frac{\partial}{\partial t} [\bar{\phi} - a \nabla^2 \bar{\phi} + \frac{1}{2l} a^2 \nabla^4 \bar{\phi} + \cdots ] + \left[ (\bar{\phi} - a \nabla^2 \bar{\phi} + \frac{1}{2l} a^2 \nabla^4 \bar{\phi} + \cdots ) (u_j - a \nabla^2 u_j + \frac{1}{2l} a^2 \nabla^4 u_j + \cdots ) \right]_{ij} \]
\[ D(\phi - a \nabla^2 \phi + \frac{1}{2!} a^2 \nabla^4 \phi + \cdots )_{ij} \] (4.27)

These equations contain an infinite number of higher-order terms which reflect and replace the SGS terms; the implication being that, with the new averaging, the traditional closure problem no longer exists. Conceptually, the above equations must be a conceivable form since no major assumption has been employed in the derivation. The only apparent consideration becomes to which order one wishes to compute and the selection of a filter. The choice of higher order terms should be determined according to the degree of accuracy required in the solution and afforded by the numerical scheme. From a practical point of view, this problem might be more serious than what it is expected to be. It is not easy, in numerical models, to solve the SGS terms to higher than fourth order without being influenced by truncation errors (Roache 1972). The higher the derivative terms are involved, the greater the influence of these errors becomes. In particular, the higher order time-derivative terms appearing in the series above will give rise to a severe stability problem, so that the resulting equations would not be useful unless low frequency (wavenumber) activities are dominant in a flow. In this sense, the conventional approach would be rather recommended here because the use of the linearity can lead to a relatively simpler form and the higher-order time-derivative terms disappear. Regardless of the appearance of the higher-order terms in the new method, it is not denied that the series gives some important tools with which one can investigate the role of large wavenumber (frequency) activity in the equation. Thus, it will be interesting to compare the series solution with closure models.
CHAPTER V
DERIVATION OF THE TURBULENCE CORRELATIONS AND
FINAL FORMS OF THE AVERAGED EQUATIONS

Two approaches for solving the turbulence equations have been mentioned in
Chapter IV; the YB-I series being the general form used in the conventional averaging
procedure and the YB-II series being a new description able to provide another filtering
method. This Chapter shows how the YB filtering method allows the turbulence corre-
lation terms to be derived, i.e., the double correlations appearing in the equation of
motion and the triple correlations in the energy equation. Final forms of the momen-
tum, transport, energy and vorticity equations are summarized.

5.1 DERIVATION OF THE DOUBLE CORRELATION TERMS

It has been emphasized that an appropriate description of the averaged non-linear
terms ($\overline{u_i u_j}$ or $\overline{u_i \partial}$) is most important for analyzing the turbulence equations.
Although, for over a century, a great deal of research has been done, a generalized way
to specify these terms has not been found. A large number of turbulence models have
been developed on the basis of defining them but their applications are valid only for
specific flow situations. It is shown here that the YB-I and YB-II series descriptions for
the SGS quantities are very useful for specifying the averaged non-linear terms. In this
Section, the averaged non-linear terms are formulated by combining the two series rep-
resentations.
From the YB filtering approach shown in Chapter IV, the YB-I and YB-II series with respect to an arbitrary variable \( f \) are written as

\[
f = \bar{f} - M(f) - \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) - \cdots \tag{5.1}
\]

and

\[
f = \bar{f} - M(f) + \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) + \cdots \tag{5.2}
\]

where \( M \) denotes the differential operator already defined in Eq.(4.19).

Now, consider the averaged non-linear term formed from two arbitrary variables, \( f \) and \( g \). From the YB-I series in Eq.(5.1), their product is given by

\[
g_f = fg + M(fg) + \frac{1}{2!} M^2(fg) + \frac{1}{3!} M^3(fg) + \cdots \tag{5.3}
\]

Here, one wants to describe the product terms \( (fg) \) on the right hand side of Eq.(5.3) in terms of their averaged quantities \( (\bar{f} \text{ and } \bar{g}) \). To do that, the YB-II series with respect to the variable, \( g \), is written as

\[
g = \bar{g} - M(g) + \frac{1}{2!} M^2(g) - \frac{1}{3!} M^3(g) + \cdots \tag{5.4}
\]

Multiplying Eq.(5.4) by Eq.(5.2), one gets the product term of two variables \( (fg) \) as

\[
f_g = f \bar{g} - f M(g) - \bar{g} M(f) + \frac{1}{2!} f M^2(g) + M(f) M(g) + \frac{1}{2!} g M^2(f) + \cdots \tag{5.5}
\]

After substitution of Eq.(5.5) into Eq.(5.3), the following equation results:
\[ f_g = \tilde{f}_g - \tilde{f}M(\tilde{g}) - \tilde{g}M(\tilde{f}) + \frac{1}{2!} \tilde{f}M^2(\tilde{g}) + M(\tilde{f})M(\tilde{g}) + \frac{1}{2!} \tilde{g}M^2(\tilde{f}) + \cdots \]
\[ + M \left[ \tilde{f}_g - \tilde{f}M(\tilde{g}) - \tilde{g}M(\tilde{f}) + \frac{1}{2!} \tilde{f}M^2(\tilde{g}) + M(\tilde{f})M(\tilde{g}) + \frac{1}{2!} \tilde{g}M^2(\tilde{f}) + \cdots \right] \]
\[ + \frac{1}{2!} M^2 \left[ \tilde{f}_g - \tilde{f}M(\tilde{g}) - \tilde{g}M(\tilde{f}) + \frac{1}{2!} \tilde{f}M^2(\tilde{g}) + M(\tilde{f})M(\tilde{g}) + \frac{1}{2!} \tilde{g}M^2(\tilde{f}) + \cdots \right] \]
\[ + \cdots \] (5.6)

Certainly, all terms on the right hand side of Eq.(5.6) consist of the averaged variables \((\tilde{f} \text{ and } \tilde{g})\) or products of them. Therefore, "closure" has been achieved straightforwardly and without any empirical assumption or coefficient. Eq.(5.6) seems complicated but a compact form results from expanding the differential operator \((M)\), by applying various algebraic operations and rearranging each term by the order of the filter width. Before preceding the final compact form, it is noted that terms of \(M^n\) have the \(2n\)-th order of the filter width. For instance, the differential operator, \(M^2\) is expanded as

\[ M^2 = \left( a_1^2 \frac{\partial^4}{\partial x^4} + a_2^2 \frac{\partial^4}{\partial y^4} + a_3^2 \frac{\partial^4}{\partial z^4} + 2a_1a_2 \frac{\partial^4}{\partial x^2 \partial y^2} + 2a_2a_3 \frac{\partial^4}{\partial y^2 \partial z^2} + 2a_3a_1 \frac{\partial^4}{\partial z^2 \partial x^2} \right) \] (5.7)

where \(a_i = \frac{\Delta_i^2}{4\gamma_i}, i = 1, 2, 3.\)

Finally, after each term in Eq.(5.6) is rearranged by the order of \(M\), Eq.(5.6) becomes

\[ f_g = I_0 + I_1 + I_2 + \cdots \] (5.8)

where
\[ I_o = \bar{\Gamma} \bar{g} \]  
\[ I_1 = -\bar{\Gamma} M(\bar{g}) - \bar{g} M(\bar{\Gamma}) + M(\bar{\Gamma} \bar{g}) \]  
\[ I_2 = \frac{1}{2i} \bar{\Gamma} M^2(\bar{g}) + M(\bar{\Gamma}) M(\bar{g}) + \frac{1}{2i} \bar{g} M^2(\bar{\Gamma}) - M(\bar{\Gamma}) M(\bar{g}) + \bar{\Gamma} M(\bar{g}) + \frac{1}{2i} M^2(\bar{\Gamma} \bar{g}) \]  

Before expanding each term, it may be convenient to use the following relationships.

By definition of the differential operator \( M(\bar{\Gamma}) \) and the tensor notation,

\[ M(\bar{\Gamma}) = (a_1 \frac{\partial^2}{\partial x^2} + a_2 \frac{\partial^2}{\partial y^2} + a_3 \frac{\partial^2}{\partial z^2}) \bar{\Gamma} \]  
\[ = C(k) \bar{\Gamma}_{,kk} \]  
\[ M(\bar{\Gamma} \bar{g}) = C(k) \bar{\Gamma} \bar{g}_{,kk} \]  
\[ = C(k) \bar{\Gamma} \bar{g}_{,kk} + 2C(k) \bar{\Gamma}_{,k} g_{,k} + C(k) \bar{\Gamma}_{,kk} \bar{g} \]  
\[ = \bar{\Gamma} M(\bar{g}) + 2C(k) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  
\[ M(\bar{\Gamma}) M(\bar{g}) = \bar{\Gamma} M(\bar{g}) + 2C(k) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  
\[ M(\bar{\Gamma} \bar{g}) = \bar{\Gamma} M(\bar{g}) + 2C(k) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  
\[ M(\bar{\Gamma}) M(\bar{g}) = \bar{\Gamma} M(\bar{g}) + 2C(k) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  
\[ M^2(\bar{\Gamma}) = C(k) C(l) \bar{\Gamma}_{,kll} \]  
\[ M(\bar{\Gamma} \bar{g}) = \bar{\Gamma} M^2(\bar{g}) + 2C(k) C(l) \bar{\Gamma}_{,kk} g_{,k} + M(\bar{g}) M(\bar{\Gamma}) \]  
\[ M(\bar{\Gamma}) M(\bar{g}) = \bar{\Gamma} M^2(\bar{g}) + 2C(k) C(l) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M^2(\bar{\Gamma}) + 4C(k) C(l) \bar{\Gamma}_{,kk} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  

etc.

Eq.(5.14) - (5.16) are readily verified in the same way as Eq.(5.12) and (5.13). By making use of these formulae, the first-order terms are simplified as follows:

\[ I_1 = -\bar{\Gamma} M(\bar{g}) - \bar{g} M(\bar{\Gamma}) + M(\bar{\Gamma} \bar{g}) \]  
\[ = -\bar{\Gamma} M(\bar{g}) - \bar{g} M(\bar{\Gamma}) + \bar{\Gamma} M(\bar{g}) + 2C(k) \bar{\Gamma}_{,k} g_{,k} + \bar{g} M(\bar{\Gamma}) \]  
\[ = 2C(k) \bar{\Gamma}_{,k} g_{,k} \]  
\[ = 2C(k) \bar{\Gamma}_{,k} g_{,k} \]  

(5.17)
The second-order terms are also given by

\[
I_2 = \frac{1}{2l} \Gamma M^2(\Gamma) + M(\Gamma)M(\Gamma) + \frac{1}{2l} \overline{\Gamma} M^2(\overline{\Gamma}) - M(\overline{\Gamma}M(\overline{\Gamma}) + \frac{1}{2l} \overline{\Gamma} M^2(\overline{\Gamma})
\]

\[
= \frac{1}{2l} \Gamma M^2(\Gamma) + M(\Gamma)M(\Gamma) + \frac{1}{2l} \overline{\Gamma} M^2(\overline{\Gamma}) - \overline{\Gamma} M^2(\overline{\Gamma}) - 2M(\Gamma)M(\overline{\Gamma}) - \overline{\Gamma} M^2(\overline{\Gamma}) + \frac{1}{2l} \overline{\Gamma} M^2(\overline{\Gamma})
\]

\[
+ M(\Gamma)M(\overline{\Gamma}) + \frac{1}{2l} \overline{\Gamma} M^2(\overline{\Gamma}) - 2C(k)C(l) \left[ \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} - \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \right]
\]

\[
= \frac{1}{2l} 4C(k)C(l) \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk}
\]

(5.18)

The higher-order terms can be similarly compacted, but their derivations are so tedious that the detailed explanation is neglected here. The resulting third-order terms are

\[
I_3 = \frac{1}{3l} 8C(k)C(l)C(m) \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk}
\]

(5.19)

As a result, the averaged non-linear term in Eq.(5.8) is described as

\[
\overline{f_g} = \overline{f_g} + 2C(k)\overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \frac{1}{2l} 4C(k)C(l) \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \frac{1}{3l} 8C(k)C(l)C(m) \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \cdots
\]

(5.20)

where \( C(i) = \frac{\Delta^2}{4y_i} \), \( i = 1,2,3 \). When the isotropic filter width and the same filter constant are used, Eq.(5.20) becomes

\[
\overline{f_g} = \overline{f_g} + (2a) \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \frac{1}{2l} (2a)^2 \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \frac{1}{3l} (2a)^3 \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} \overline{\Gamma}_{kk} + \cdots
\]

(5.21)

where \( a = \frac{\Delta^2}{4\gamma} \).
As seen above, the resultant equations are quite compact and simplified. Indeed, these equations appear to contain a number of interesting implementations to be explored in subsequent Chapters. Hereafter, Eq. (5.20) is called the third kind of the Yeo and Bedford series; the YB-III series. By this YB-III series representation, the velocity correlations in the equation of motion and the scalar flux correlations in the transport equation are defined as follows. If the arbitrary variables \((f\) and \(g\)) are replaced by the velocity components \((u_i\) and \(u_j\)), the averaged velocity correlation terms are

\[
\bar{u}_i \bar{u}_j = \bar{u}_i \bar{u}_j + 2C(k) \bar{u}_{ik} \bar{u}_{jk} + \frac{1}{2l} 4C(k)C(1) \bar{u}_{ikl} \bar{u}_{jkl} + \cdots \quad (5.22)
\]

The averaged flux terms are obtained by substituting into the velocity component \((u_i)\) and the scalar quantity \((\phi)\), and are written as

\[
\bar{u}_i \bar{\phi} = \bar{u}_i \bar{\phi} + 2C(k) \bar{u}_{ik} \bar{\phi}_k + \frac{1}{2l} 4C(k)C(1) \bar{u}_{ikl} \bar{\phi}_{kl} + \cdots \quad (5.23)
\]

With the assumption of the same filter width, these two non-linear terms are given, respectively, as

\[
\bar{u}_i \bar{u}_j = \bar{u}_i \bar{u}_j + (2a) \bar{u}_{ik} \bar{u}_{jk} + \frac{1}{2l} (2a)^2 \bar{u}_{ikl} \bar{u}_{jkl} + \cdots \quad (5.24)
\]

\[
\bar{u}_i \bar{\phi} = \bar{u}_i \bar{\phi} + (2a) \bar{u}_{ik} \bar{\phi}_k + \frac{1}{2l} (2a)^2 \bar{u}_{ikl} \bar{\phi}_{kl} + \cdots \quad (5.25)
\]

The above equations are totally new and mathematically exact. They differ from the conventional approach in which the averaged non-linear term is separated into four terms at the doubly averaged level.
5.2 DERIVATION OF THE TRIPLE CORRELATION TERMS

The triple correlation terms usually appear when the filtering operation is applied to the energy equation. Thus, it is necessary to find the descriptive form of these triple correlations \( \langle u_i u_j u_k \rangle \) in order to investigate the energy structure in the mean flow. These terms can be derived by the YB-I and YB-II series representations in the same fashion as the double correlations. The YB-I series leads to the following triple correlation terms as regards three arbitrary variables \( f, g \) and \( h \);

\[
\tilde{fgh} = fgh + M(fgh) + \frac{1}{2!} M^2(fgh) + \cdots \tag{5.26}
\]

By the YB-II series, each variable is written as

\[
f = \tilde{f} - M(f) + \frac{1}{2!} M^2(f) - \frac{1}{3!} M^3(f) + \cdots \tag{5.27}
\]

\[
g = \tilde{g} - M(g) + \frac{1}{2!} M^2(g) - \frac{1}{3!} M^3(g) + \cdots \tag{5.28}
\]

\[
h = \tilde{h} - M(h) + \frac{1}{2!} M^2(h) - \frac{1}{3!} M^3(h) + \cdots \tag{5.29}
\]

Multiplying the above three equations and arranging the resultant terms by the order of \( M \), one gets

\[
fgh = \tilde{f} \tilde{g} \tilde{h} - [\tilde{f} \tilde{g} M(h) + \tilde{f} M(g) \tilde{h} + M(f) \tilde{g} \tilde{h}] + \frac{1}{2!} [\tilde{f} \tilde{g} M^2(h) + \tilde{f} M^2(g) \tilde{h} + M(f) M(g) \tilde{h}] + \cdots
\]

\[+ M^2(f) \tilde{g} \tilde{h} + 2M(f) \tilde{g} M(h) + 2M(f) M(g) \tilde{h} + 2M(f) M(g) \tilde{h}] + \cdots \tag{5.30}
\]

By substituting Eq.(5.30) into each term on the right hand side of Eq.(5.26) and rearranging by the order of the filter width, the following compact equation is formulated;
\[
\tilde{f}_{\text{gh}} = f_g \tilde{g}_h + 2C(k)[f_{g\text{g}_h} \tilde{f}_{\text{g}_h} + f_{g\tilde{g}_h} \tilde{f}_{\text{g}_h} + f_{\tilde{g}\tilde{g}_h} \tilde{f}_{\text{g}_h}] + \frac{1}{2}\text{i} 4C(k)C(1)[f_{\text{g}g_{\tilde{g}_h}} \tilde{f}_{\text{g}_h} \tilde{f}_{\text{g}_h}] \\
+ \tilde{f}_{g\text{g}_h} \tilde{g}_{\tilde{g}_h} \tilde{f}_{\text{g}_h} + 2f_{g\tilde{g}_h} \tilde{g}_{\tilde{g}_h} \tilde{f}_{\text{g}_h} + 2f_{\tilde{g}\tilde{g}_h} \tilde{g}_{\tilde{g}_h} \tilde{f}_{\text{g}_h} \tilde{f}_{\text{g}_h} \tilde{f}_{\text{g}_h} + \cdots
\] (5.31)

Assuming the same filter width and filter constant are used in all directions, Eq.(5.31) will be

\[
\tilde{f}_{\text{gh}} = f_g \tilde{g}_h + (2a)[f_{g\text{g}_h} \tilde{f}_{\text{g}_h} + f_{g\tilde{g}_h} \tilde{f}_{\text{g}_h} + f_{\tilde{g}\tilde{g}_h} \tilde{f}_{\text{g}_h}] + \cdots
\] (5.32)

As with the double correlation terms, very concise forms result. Now, consider the triple correlation terms appearing in the energy equation, where they are supposed to result from multiplying the velocity component by the advective term and applying the averaging operation to the resulting triple product term. By Eq.(5.32), these terms are written as

\[
u_i u_j \tilde{u}_k = u_i u_j \tilde{u}_k + (2a)[u_i \tilde{u}_j \tilde{u}_k + u_i u_j \tilde{u}_k + u_i u_j \tilde{u}_k] + \cdots
\] (5.33)

An alternative form may be more useful than Eq.(5.33) in view of the turbulent stresses. When \( R_{ij} \) is defined as the difference between \( \bar{u}_i \bar{u}_j \) and \( \bar{u}_i \bar{u}_j \), i.e.,

\[
R_{ij} = \bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j
\] (5.34)

then, Eq.(5.33) becomes

\[
u_i u_j \bar{u}_k = u_i (u_j \bar{u}_j) + (2a) \bar{u}_k (u_j \bar{u}_j) + \cdots
\]

\[= u_i (u_j \bar{u}_j + R_{ij}) + (2a) \bar{u}_k (u_j \bar{u}_j + R_{ij}) + \cdots
\]

\[= u_i u_j + u_i R_{ij} + (2a) \bar{u}_k u_j \bar{u}_j + \bar{u}_k u_j \bar{u}_j + \bar{u}_k R_{ij}) + \cdots
\] (5.35)
Ultimately, these relations will help in understanding the energy structure resulting from the turbulence stresses because $R_{ij}$ may be interpreted as the turbulence stress and is equivalent to Reynolds stress term from the Reynolds averaging procedure. In addition to the energy equation, the triple correlation terms are usually observed in the equation models for the closure quantities, for example, k-equation, $\epsilon -$ equation and turbulent stress/flux equation models (Rodi 1980, Bedford et al. 1987b). It will be shown in Chapter VI. Therefore, the general form of triple correlations written in Eq.(5.32) is also expected to be able to play a role in designing the closure models.

5.3 THE FINAL FORM OF THE AVERAGED MOMENTUM AND TRANSPORT EQUATIONS

In the previous two Sections, the double and the triple turbulence correlation terms are formulated by making use of the YB-I and YB-II series descriptions. Indeed, the YB-III series representation for the double correlations is quite important because it indicates the general form of $\bar{u}_i\bar{u}_j$ and $\bar{u}_j\bar{\phi}$ in the averaged momentum and transport equations, respectively, and the triple correlations may be useful in understanding the energy structure. Of course, the higher correlations can be also derived by the same procedure as the double or triple correlations. Following next is an application of these correlations to the derivation of the fundamental turbulence equations.

To aid in this presentation, it is useful to define $R_{ij}$ be the difference between $\bar{u}_i\bar{u}_j$ and $\bar{u}_i\bar{\phi}$ and $Q_j$ between $\bar{u}_j\bar{\phi}$ and $\bar{u}_j\bar{\phi}$, respectively. Assuming the same filter width is used in each direction, one gets $R_{ij}$ and $Q_j$ as follows:

$$R_{ij} = \bar{u}_i\bar{u}_j - \bar{u}_i \bar{u}_j$$
\begin{equation}
= (2a) \bar{u}_{ik} \bar{u}_{jk} + \frac{1}{2l} (2a)^2 \bar{u}_{ikl} \bar{u}_{jkl} + \cdots
\end{equation}
(5.36)

\[ Q_j = \bar{u}_j \phi - \bar{u}_j \overline{\phi} \]

\begin{equation}
= (2a) \bar{u}_{ik} \overline{\phi}_k + \frac{1}{2l} (2a)^2 \bar{u}_{ikl} \overline{\phi}_{kl} + \cdots
\end{equation}
(5.37)

As shown before, these double correlations appear in the averaged momentum and transport equations;

\begin{equation}
\frac{\partial \bar{u}_i}{\partial t} + (\bar{u}_i \bar{u}_j)_j = -\frac{1}{\rho} \bar{p}_i + \nu \bar{u}_{ijj}
\end{equation}
(5.38)

\begin{equation}
\frac{\partial \overline{\phi}}{\partial t} + (\bar{u}_i \overline{\phi})_j = D \overline{\phi}_{jj}
\end{equation}
(5.39)

Substituting \( R_{ij} \) and \( Q_j \) in Eq. (5.36) and (5.37) into Eq. (5.38) and (5.39), respectively, the averaged governing equations are written as

\begin{equation}
\frac{\partial \bar{u}_i}{\partial t} + (\bar{u}_i \bar{u}_j)_j = -\frac{1}{\rho} \bar{p}_i + \nu \bar{u}_{ijj} - R_{ij}
\end{equation}
(5.40)

\begin{equation}
\frac{\partial \overline{\phi}}{\partial t} + (\bar{u}_i \overline{\phi})_j = D \overline{\phi}_{jj} - Q_j
\end{equation}
(5.41)

It is clear that \( R_{ij} \) and \( Q_j \) denote the residual stress and the residual flux. It is also obvious that \( R_{ij} \) is not only strongly related to the Reynolds stress from the Reynolds averaging procedure but it is the general form of the residual stress in the LES. \( Q_j \) is also interpreted in the same manner. Indeed, these final equations are favorably suggestive because all terms are analytically specified without any unknown quantity.
5.4 THE MEAN FLOW ENERGY EQUATION

The kinetic energy equation of the mean flow is formulated by multiplying the averaged Navier-Stokes equation in Eq.(5.40) by the mean velocity component, i.e.,

$$
\bar{u}_i \frac{\partial}{\partial t} \bar{u}_i + \bar{u}_i (\bar{u}_j \bar{u}_j) = -\frac{1}{\rho} \bar{u}_i \bar{p}_i + \nu \bar{u}_i \bar{u}_i, \bar{u}_j - \bar{u}_i \bar{R}_{i,jj} \tag{5.42}
$$

By making use of the continuity relation, Eq.(5.42) is equivalent to

$$
\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{u}_i \bar{u}_i \right) + \bar{u}_j \left( \frac{1}{2} \bar{u}_i \bar{u}_i \right)_j = (T_{ij} \bar{u}_i)_j - \nu \bar{u}_i \bar{u}_i, \bar{u}_j + R_{ij} \bar{u}_i \tag{5.43}
$$

where $T_{ij} = -\frac{1}{\rho} \bar{p} \delta_{ij} + \nu \bar{u}_i \bar{u}_j - R_{ij}$. $T_{ij}$ is a tensor which relates pressure, viscosity and turbulence effects. Terms on the left hand side of Eq.(5.43) represent the rate of change of the mean flow energy and the spatial energy transport. The first term on the right denotes the transport of the mean flow energy due to the stress, $T_{ij}$. The second and third terms are the energy dissipation by the viscosity and the deformation work due to the turbulence, respectively. If the Reynolds number is large enough, the last term is expected to be much greater than the viscosity term.

The variation of total energy in a region is often determined by an ensemble average which is defined by integrating over an infinite spatial domain. It may also be determined by averaging over a large number of realizations in a bounded flow field (Meecham and Siegel 1964). Now, let $\left< \cdot \right>$ denote the ensemble average and define as

$$
\left< \cdot \right> = \int_V \cdot dV \tag{5.44}
$$
where $V$ is a control volume. When the ensemble average is taken over Eq. (5.43), the following equation results:

$$
\frac{\partial}{\partial t} \left< \frac{1}{2} \dot{u}_i \dot{u}_i \right> + \left< \frac{1}{2} \frac{\partial \dot{u}_i \dot{u}_j}{\partial x} \right> = \left< (T_{ij} \nu)_{,ij} \right> - \nu \left< u_j \frac{\partial u_i}{\partial x} \right> + \left< R_{ij} \nu_{,ij} \right> \tag{5.45}
$$

The first and second terms on the left represent the rate of change of energy over an entire mean flow field. By definition of the ensemble average and use of the divergence theorem, the second term becomes

$$
\left< u_j \left( \frac{1}{2} \frac{\partial \dot{u}_i}{\partial x} \right) \right> = \int \int_S \left( \frac{1}{2} \dot{u}_i \dot{u}_j n_j \right) dS \tag{5.46}
$$

where $n_j$ denote the unit vector normal to the control surface. Therefore, this term will vanish when there is neither a source/sink term nor energy passing through the control surface. In other words, this inertia term does not contribute to the energy variation in isotropic turbulence.

In the same fashion, the first term on the right becomes zero if it is integrated with an isotropic turbulence assumption;

$$
\int_V (T_{ij} \nu)_{,ij} dV = \int_S n_j T_{ij} dS \tag{5.47}
$$

After this observation, three terms remain and contribute to the energy variation:

$$
\frac{\partial}{\partial t} \left< \frac{1}{2} \dot{u}_i \dot{u}_i \right> = -\nu \left< u_j \frac{\partial \dot{u}_i}{\partial x} \right> + \left< R_{ij} \nu_{,ij} \right> \tag{5.48}
$$
Assuming that the viscous term is small enough to neglect for flows with high Reynolds numbers, the last term will play a significant role in the redistribution of energy within the control volume.

5.5 THE AVERAGED VORTICITY EQUATION

5.5.1 Deformation and Vorticity

Before deriving the vorticity equation, it may be useful to introduce several mathematical results which aid in specifying the physical character of vorticity and allow a convenient description of them. These mathematical notation will be also helpful in the investigation of the newly derived terms such as $R_{ij}$ and $Q_j$.

From a kinematical point of view, the velocity gradient tensor, $\bar{\omega}_{ij}$, is usually decomposed into its symmetric and skew symmetric parts:

$$\bar{\omega}_{ij} = S_{ij} + \Omega_{ij}$$  \hspace{1cm} (5.49)

in which an index following a comma denotes a derivative. The symmetric tensor, $S_{ij}$, is called the rate of deformation (rate of strain, stretching, strain rate) tensor, and is written as

$$S_{ij} = S_{ji} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji})$$  \hspace{1cm} (5.50)

Physically, the diagonal elements of $S_{ij}$ are known as the stretching or rate of extension components, while the off-diagonal terms are the rate of change between directions at right angles and are often called the shear rates.
The skew symmetric tensor, $\Omega_{ij}$ is known as the rotation (spin) tensor and is defined by

$$\Omega_{ij} = -\Omega_{ji} = \frac{1}{2} (\mathbf{u}_{ij} - \mathbf{u}_{ji})$$ \hspace{1cm} (5.51)

By definition, the diagonal terms of $\Omega_{ij}$ are equal to zero. The tensor, $2\Omega_{ij}$ is called the vorticity tensor, and the dual vector associated with the vorticity tensor is known as the vorticity vector defined as

$$\omega_i = \epsilon_{ijk} \Omega_{kj} = -\epsilon_{ijk} u_{jk}$$ \hspace{1cm} (5.52)

This vorticity vector is equivalent to the curl of the velocity field and one-half the vorticity vector is known as the angular velocity vector. Here the symbol, $\epsilon_{ijk}$, denotes the unit permutation tensor. With some tensor algebra, the following relationships between the rotation tensor and the vorticity vector are found;

$$\Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k$$ \hspace{1cm} (5.53)

$$\omega_i = -\epsilon_{ijk} \Omega_{jk}$$ \hspace{1cm} (5.54)

With these notations, the continuity equation for an incompressible flow is represented as:

$$\mathbf{u}_{i,i} = 0 \text{ or } S_{ii} = 0$$ \hspace{1cm} (5.55)

where the index notation follows the rule of the summation convention. The deformation tensor ($S_{ij}$) and the rotation tensor ($\Omega_{ij}$) on the x-y plane are illustrated in Fig.
5.1.

\[
\begin{align*}
\text{a) } S_{11} &= \frac{\partial \Pi}{\partial x} \quad & S_{22} &= \frac{\partial \nu}{\partial y} \\
\text{b) } S_{12} &= S_{21} = \frac{1}{2} \left( \frac{\partial \Pi}{\partial y} + \frac{\partial \nu}{\partial x} \right) \\
\text{c) } \Omega_{12} &= -\Omega_{21} = \frac{1}{2} \left( \frac{\partial \nu}{\partial x} - \frac{\partial \Pi}{\partial y} \right)
\end{align*}
\]

**Figure 5.1:** Deformation and rotation tensors
5.5.2 Vorticity Equation in the Mean Flow

Assuming that a fluid is incompressible and homogeneous, the vorticity equation in the mean flow is formulated by taking the curl of the averaged Navier-Stokes equation shown in Eq.(5.40), and the curl in the indicial notation is \( \epsilon_{mn} \frac{\partial \mathbf{U}}{\partial t} \). When this differential operator is applied to both sides of Eq.(5.40), each term is transformed as follows,

\[
\epsilon_{mn} \left( \frac{\partial \mathbf{U}}{\partial t} \right)_{n} = \frac{\partial}{\partial t} \left( \epsilon_{mnli} \mathbf{U}_{li} \right) = \frac{\partial \omega_{m}}{\partial t} 
\]

\[
\epsilon_{mn} \left( \mathbf{U}_{lj} \right)_{jn} = \epsilon_{mn} \left( \mathbf{U}_{lj} \mathbf{U}_{jn} \right) 
= \epsilon_{mn} \mathbf{U}_{jn} \mathbf{U}_{lj} + \mathbf{U}_{lj} \left( \epsilon_{mni} \mathbf{U}_{ni} \right) 
= \mathbf{U}_{lj} \omega_{mj} - \omega_{m} \mathbf{U}_{jn} 
\]

\[
\epsilon_{mn} \left( \frac{1}{\rho} \mathbf{U}_{lj} \right)_{n} = 0 
\]

\[
\nu \epsilon_{mn} \left( \mathbf{U}_{lj} \right)_{jn} = \nu \left( \epsilon_{mnli} \mathbf{U}_{ni} \right)_{lj} = \nu \omega_{mj} 
\]

and

\[
- \epsilon_{mn} \left( R_{ij} \right)_{n} = - \epsilon_{mn} R_{ijnj} 
\]

Therefore, the resulting vorticity equation is

\[
\frac{\partial \omega_{m}}{\partial t} + \mathbf{U}_{lj} \omega_{mj} = \omega_{lj} \mathbf{U}_{mj} + \nu \omega_{mj} - \epsilon_{mn} R_{ijmj} 
\]

By making use of the series expansion for \( R_{ij} \), one gets

\[
\epsilon_{mn} R_{ijnj} = [ \frac{1}{(2a)^{2}} \mathbf{U}_{ij} \omega_{mk} + \frac{1}{(2a)^{4}} \mathbf{U}_{ij} \omega_{mk} + \cdots ] 
\]

The final form of the vorticity equation in the mean flow is
\[ \frac{\partial \omega_m}{\partial t} + \bar{u}_j \omega_{m,j} = \omega_j \bar{u}_m - (2\alpha \bar{u}_m \omega_{m,k} + \frac{1}{2l} (2\alpha)^2 \bar{u}_{j,k} \omega_{m,k} + \cdots )_j \] (5.63)

The first term on the right-hand side will vanish in the two-dimensional case but show the vortex stretching in the three-dimensional, i.e.,

\[ \omega_j \bar{u}_{m,j} = \omega_j (S_{mj} + \Omega_{mj}) \]
\[ = \omega_j S_{mj} - \frac{1}{2} \varepsilon_{mjk} \omega_j \omega_k \] (5.64)

By the tensor property, the last term becomes zero since \( j \) and \( k \) are dummy indices and, thus, the only remaining term \( (\omega_j S_{mj}) \) represents the vortex stretching in the mean flow that the vorticity is changed by the strain rate. Therefore, the alternative form of Eq.(5.63) is

\[ \frac{\partial \omega_m}{\partial t} + \bar{u}_j \omega_{m,j} = \omega_j S_{mj} + \nu \omega_{m,ij} - [(2\alpha \bar{u}_m \omega_{m,k} + \frac{1}{2l} (2\alpha)^2 \bar{u}_{j,k} \omega_{m,k} + \cdots )_j \] (5.65)

According to Tennekes and Lumley (1972), vortex stretching may be very important in turbulence because it always involves a change of length scale.

5.5.3 Vorticity in the Momentum Equation

The relationship between vorticity and the Navier-Stokes equations is determined when the inertia and the viscous term are transformed such that

\[ (\bar{u}_j \bar{u})_j = (\bar{u}_j \bar{u} + R_{uj})_j \]
\[ \begin{align*}
\bar{u}_j (\bar{u}_i - \bar{u}_i') + \bar{u}_j' \bar{u}_i' + R_{ijj} \\
= 2\nu \Omega_{ij} + \left( \frac{1}{2} \bar{u}_j \bar{u}_i' \right)' + R_{ijj} \\
= -\epsilon_{ijk} \bar{u}_j \omega_k + \left( \frac{1}{2} \bar{u}_j \bar{u}_i' \right)' + R_{ijj}
\end{align*} \] (5.66)

and
\[ \begin{align*}
\nu \bar{u}_{ijj} &= \nu (\bar{u}_i - \bar{u}_i')_j + \nu \bar{u}_{ijj} \\
&= 2\nu \Omega_{ijj} + 0 \\
&= -\nu \epsilon_{ijk} \omega_{kj}
\end{align*} \] (5.67)

If these terms are substituted into the averaged Navier-Stokes equation in Eq.(5.38), one has

\[ \frac{\partial \bar{u}_i}{\partial t} = -\left( \frac{p}{\rho} + \frac{1}{2} \bar{u}_j \bar{u}_j' \right)' + \epsilon_{ijk} \bar{u}_i' \omega_k - R_{ijj} - \nu \epsilon_{ijk} \omega_{kj} \] (5.68)

If the flow is irrotational, the vorticity \( (\omega_k) \) becomes zero and the second and last terms vanish. But, Eq.(5.68) does not reduce to the Bernoulli equation because the residual stress term \( (R_{ijj}) \) remains. To get the form of the Bernoulli equation, \( R_{ijj} \) is decomposed into

\[ \begin{align*}
R_{ijj} &= (2a)\bar{u}_i \bar{u}_j \bar{u}_k \bar{u}_l + \frac{1}{2} (2a)^2 (\bar{u}_i \bar{u}_j \bar{u}_k) \bar{u}_l + \cdots \\
&= (2a)^2 \bar{u}_i \bar{u}_j \bar{u}_k + \frac{1}{2} (2a)^2 \bar{u}_j \bar{u}_k \bar{u}_l + \cdots \\
&= (2a)\bar{u}_i \bar{u}_j \bar{u}_k + \frac{1}{2} (2a)^2 \bar{u}_j \bar{u}_k \bar{u}_l + \cdots \\
&+ (2a)^2 \bar{u}_i \bar{u}_j \bar{u}_k + \frac{1}{2} (2a)^2 \bar{u}_j \bar{u}_k \bar{u}_l + \cdots
\end{align*} \]
\[ = 2[(2a)\xi_{jk} \omega_{jk} + \frac{1}{2l} (2a)^2 \omega_{jk} \omega_{jk} + \cdots ] + \frac{1}{2} [(2a)\xi_{jk} \omega_{jk} + \frac{1}{2l} (2a)^2 \omega_{jk} \omega_{jk} + \cdots ],_i \]

\[ = -\epsilon_{ijk} [(2a)\xi_{jk} \omega_{kl} + \frac{1}{2l} (2a)^2 \omega_{jk} \omega_{kl} + \cdots ] + \frac{1}{2} R_{ij} \quad (5.69) \]

Substituting Eq.(5.69) into Eq.(5.68), one gets

\[ \frac{\partial \xi_{ij}}{\partial t} = -\left( \frac{\overline{p}}{\rho} + \frac{1}{2} \overline{u}_i \overline{u}_j + \frac{1}{2} R_{ij} \right) + \epsilon_{ijk} \left[ \overline{u}_j \omega_k + (2a)\xi_{jk} \omega_{kj} + \cdots \right] - \nu \epsilon_{ijk} \omega_{ij} \quad (5.70) \]

It indicates that the turbulent stress gradients contain both a dynamical pressure gradient and an interactive term between the vorticity and the velocity fluctuations. In steady irrotational flow, Eq.(5.70) is derived to the following Bernoulli equation:

\[ \frac{\overline{p}}{\rho} + \frac{1}{2} \overline{u}_j \overline{u}_j + \frac{1}{2} R_{ij} = \text{Const} \quad (5.71) \]

In unsteady irrotational flow, a velocity potential \( \Phi ( u = \nabla \Phi ) \) exists and Eq.(5.71) then becomes

\[ \frac{\partial \Phi}{\partial t} + \frac{\overline{p}}{\rho} + \frac{1}{2} \overline{u}_j \overline{u}_j + \frac{1}{2} R_{ij} = \text{Const} \quad (5.72) \]

In addition, more detailed examination of the newly derived terms, \( R_{ij} \) and \( Q_{ij} \), will be done in Chapter VII. Table 5.1 summarizes the averaged equation forms resulting from the YB filtering method and also presents the comparison to the different forms used in the existing models.
### Table 5.1
Turbulence equations in the existing models

#### a) Momentum equations

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unaveraged</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} )</td>
</tr>
<tr>
<td>Reynolds average</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j') )</td>
</tr>
<tr>
<td>Leonard's average</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{u}_i \bar{u}_j)) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j') )</td>
</tr>
<tr>
<td>Clark's reduction</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{v}_i \bar{v}_j)) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{w}_j') )</td>
</tr>
<tr>
<td>Dakkouh and Bedford's STF</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{u}_i \bar{u}_j) + \frac{\lambda^2}{4} \nabla^2 (\bar{u}_i \bar{u}_j)) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{w}_j') )</td>
</tr>
<tr>
<td>STF with Clark's reduction</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{u}_i \bar{u}_j)) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{w}_j') )</td>
</tr>
<tr>
<td>YB filtering method</td>
<td>( \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j + \bar{R}_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} )</td>
</tr>
</tbody>
</table>

#### b) Scalar transport equations

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unaveraged</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} )</td>
</tr>
<tr>
<td>Reynolds average</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j') )</td>
</tr>
<tr>
<td>Leonard's average</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{\phi} \bar{u}_j)) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j') )</td>
</tr>
<tr>
<td>Clark's reduction</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{\phi} \bar{u}_j)) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j') )</td>
</tr>
<tr>
<td>Dakkouh and Bedford's STF</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{\phi} \bar{u}_j) + \frac{\lambda^2}{4} \nabla^2 (\bar{\phi} \bar{u}_j)) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j') )</td>
</tr>
<tr>
<td>STF with Clark's reduction</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j + \frac{\lambda^2}{4} \nabla^2 (\bar{\phi} \bar{u}_j)) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} - \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j') )</td>
</tr>
<tr>
<td>YB filtering method</td>
<td>( \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\phi} \bar{u}_j + \bar{Q}_j) = D \frac{\partial^2 \bar{\phi}}{\partial x_j^2} )</td>
</tr>
</tbody>
</table>
c) Energy equations

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unaveraged</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u \cdot p u \right) + u \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u \cdot p u \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu u \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} )</td>
</tr>
<tr>
<td>Reynolds average</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u \cdot p u \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u \cdot p u \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu u \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - u \cdot \frac{\partial}{\partial x_j} \left( \bar{u} \cdot \bar{u} \right) )</td>
</tr>
<tr>
<td>Leonard's average</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i u_i \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu u \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - u \cdot \frac{\partial}{\partial x_j} \left( \bar{u} u_j + \bar{u} u_j + \bar{u} u_j \right) )</td>
</tr>
<tr>
<td>Clark's reduction</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i u_i \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu u \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - u \cdot \frac{\partial}{\partial x_j} \left( \bar{u} \bar{u} + \bar{u} \bar{u} + \bar{u} \bar{u} \right) )</td>
</tr>
<tr>
<td>Dakhoul and Bedfor's STF</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i u_i + \frac{1}{4} \frac{\partial^2}{\partial x_j^2} \nabla \cdot \bar{u} \bar{u} \right) + \frac{1}{4} \frac{\partial^2}{\partial x_j^2} \nabla \cdot \bar{u} \right) )</td>
</tr>
<tr>
<td>STF with Clark's reduction</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i u_i + \frac{1}{2} \frac{\partial^2}{\partial x_j^2} \nabla \cdot \bar{u} \bar{u} \right) + \frac{1}{2} \frac{\partial^2}{\partial x_j^2} \nabla \cdot \bar{u} \right) )</td>
</tr>
<tr>
<td>YB filtering method</td>
<td>( \frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + u_j \cdot \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i u_i + R_{ij} \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu u \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} )</td>
</tr>
</tbody>
</table>

d) Vorticity equations

<table>
<thead>
<tr>
<th>Method</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unaveraged</td>
<td>( \frac{\partial}{\partial t} \omega_i + u_j \cdot \frac{\partial}{\partial x_j} \omega_i = \omega_j + \nu \frac{\partial^2 \omega_i}{\partial x_i \partial x_j} )</td>
</tr>
<tr>
<td>Reynolds average</td>
<td>( \frac{\partial}{\partial t} \omega_i + u_j \cdot \frac{\partial}{\partial x_j} \omega_i = \omega_j + \nu \frac{\partial^2 \omega_i}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_j} \left( \omega u_j - \bar{u} \bar{u}_j \right) )</td>
</tr>
<tr>
<td>YB filtering method</td>
<td>( \frac{\partial}{\partial t} \omega_i + u_j \cdot \frac{\partial}{\partial x_j} \omega_i = \omega_j + \nu \frac{\partial^2 \omega_i}{\partial x_i \partial x_j} - \epsilon_{ij} \frac{\partial^2 R_{ij}}{\partial x_i \partial x_j} )</td>
</tr>
</tbody>
</table>
CHAPTER VI

DERIVATION OF THE EQUATIONS GOVERNING THE
TURBULENCE QUANTITIES

In Section 5.1 and 5.2, the general forms of the double and triple turbulence correlations were already formulated from the YB-I and II series descriptions. These correlations make it possible to present the equations governing the turbulence quantities by subtracting the mean flow equations from the averaged governing equations. In this Chapter, these equations for various turbulence quantities of interest are derived.

6.1 KINETIC ENERGY OF TURBULENCE

The turbulent kinetic energy is defined as the difference between $\frac{1}{2} \bar{u}_i \bar{u}_i$ and $\frac{1}{2} \bar{u}_j \bar{u}_j$, and the governing equation for this quantity is formulated as follows. Taking the average over all terms of the energy equation in the unaveraged field, one gets

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{u}_i \bar{u}_i \right) + \left( \frac{1}{2} \bar{u}_i \bar{u}_j \right)_j = -\frac{1}{\rho} \left( \bar{u}_i \bar{p}_j \right)_j + \nu \left( \frac{1}{2} \bar{u}_i \bar{u}_j \right)_j - \nu \left( \bar{u}_i \bar{u}_j \right)_j \quad (6.1)$$

By making use of the double and triple correlations already shown in Eq.(5.24) and Eq.(5.35), Eq.(6.1) is expanded to

$$\frac{\partial}{\partial t} \left[ \frac{1}{2} \bar{u}_i \bar{u}_j + (2a) \bar{u}_k \bar{u}_k + \cdots \right] + \frac{1}{2} \left[ \bar{u}_i \bar{u}_j + (2a) (\bar{u}_i \bar{u}_k \bar{u}_k + \bar{u}_k \bar{u}_k \bar{u}_j + \cdots) + \cdots \right]$$
\[\frac{\partial}{\partial t} \left( \frac{1}{2} R_{ii} \right) + \mathbf{u}_j \cdot \frac{1}{2} \mathbf{R}_{ij} = \left( -\frac{1}{\rho} \delta_{ij} H_i - \Psi_j + \frac{\nu}{2} R_{ij} \right)_{ij} - \nu \Theta - R_{ij} u_{ij} \] (6.3)

where

\[R_{ii} = (2a) \mathbf{u}_i \mathbf{u}_i + \frac{1}{2l} (2a)^2 \mathbf{u}_{ik} \mathbf{u}_{ik} + \cdots\]

\[H_i = (2a) \mathbf{p}_i \mathbf{u}_i + \frac{1}{2l} (2a)^2 \mathbf{p}_{ai} \mathbf{u}_{ik} + \cdots\]

\[\Psi_j = \frac{1}{2l} (2a)^2 \mathbf{u}_{ik} (\mathbf{u}_i \mathbf{u}_j)_{ik} + \cdots\]

and

\[\Theta = (2a) \mathbf{u}_{ik} \mathbf{u}_{ik} + \cdots\]

It is clear that \(\frac{1}{2} R_{ii}\) represents the turbulent kinetic energy and it is equivalent to \(\frac{1}{2} \mathbf{u}_i \mathbf{u}_i\) for the special case of Reynolds averaging. Thus, the first and second terms on the left hand side of Eq.(6.3) denote the rate of change of the turbulent kinetic energy and its transport due to the mean flow, respectively. Terms in parenthesis on the right are the pressure gradient work, transport of turbulent velocity fluctuations and transport of the energy by viscosity. These terms are divergences of energy flux so that their ensemble average will vanish, together with the second term on the left, if there
is no net energy flux out of or into a closed control volume. The fourth term is always positive because of its quadratic form, so that it indicates the dissipation of energy by the viscosity. This viscous term will not be small in flows with the high Reynolds numbers. The last term represents the deformation work due to the turbulence stress, \( R_{ij} \), which has already appeared in the energy equation for the mean flow with an opposite sign. Indeed, this term is important in the energy budget since it is interpreted as the energy exchange between the mean flow and the turbulence. As a result, the ensemble average of Eq.(6.3) is

\[
\frac{\partial}{\partial t} \left< \frac{1}{2} R_{ii} \right> = -\nu \left< \Theta \right> - \left< R_{ij} u_i u_j \right>
\]  

(6.4)

and comparing to Eq.(5.48), one can see the interaction between two energy equations by the last term. It is extremely important to note that while all previous forms of closure based upon the eddy viscosity or diffusivity maintain positive definite values, \( R_{ij} u_i u_j \) can be either positive or negative, indicating that this form of closure permits the local transfer of turbulent energy to larger or smaller wavenumbers. This is the first closure to have this property without empirical remedies.

6.2 MOMENTUM FLUX OF TURBULENCE

The transport equation for momentum flux \((u_i u_j)\) in the total variable can be obtained by multiplying the Navier-Stokes equations for \(u_i\) and \(u_j\) by the two velocity components \(u_j\) and \(u_i\), respectively, and adding the two resulting equations, i.e.,

\[
u_j \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}
\]  

(6.5)
\[
\frac{\partial u_j}{\partial t} + u_i u_j u_{jk} = -\frac{1}{\rho} u_i p_j + \nu u_i u_{jk}
\] (6.6)

Summing up each term in Eq.(6.5) and (6.6) with the continuity relation, one has

\[
\frac{u_j}{\partial t} + u_i \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t}(u_i u_j)
\] (6.7)

\[
u u_i u_{jk} + u_i u_k u_{jk} = u_k (u_i u_j)_{kk} = (u_i u_j)_{kk}
\] (6.8)

\[-\frac{1}{\rho} (u_j p_i + u_i p_j) = -\frac{1}{\rho} [(p u_j)_{,i} + (p u_i)_{,j} - p (u_{ji} + u_{ij})]
\] (6.9)

and

\[
u (u_j u_{kk} + u_i u_{kk}) = \nu [(u_i u_j)_{kk} - 2 u_{ik} u_{jk}]
\] (6.10)

Therefore, the momentum flux equation becomes

\[
\frac{\partial}{\partial t}(u_i u_j) + (u_i u_j)_{,k} + \frac{1}{\rho} (u_j p_i + u_i p_j) + \nu (u_i u_j)_{kk} - 2 \nu u_{ik} u_{jk}
\] (6.11)

Now, taking the filtering operation on both sides of Eq.(6.11) gives

\[
\frac{\partial}{\partial t}(\overline{u_i u_j}) + (\overline{u_i u_j})_{,k} + \frac{1}{\rho} (\overline{p u_j}_{,i} + \overline{p u_i}_{,j}) + \nu (\overline{u_i u_j})_{kk} - 2 \nu \overline{u_{ik} u_{jk}}
\] (6.12)

In the same way, the mean flow momentum flux equation can be formulated by multiplying the averaged Navier-Stokes equation in Eq.(5.40) by the mean velocity component. The resulting form is given as
\[
\frac{\partial}{\partial t}(\overline{u_j u_j}) + (u_j u_j)_{,j} = -\frac{1}{\rho}[(\overline{p u_j})_j + (\overline{p u_j})_j] \\
+ \frac{\bar{\nu}}{\rho}(\overline{u_i u_j} + \overline{u_j u_i}) + \nu(u_i u_j)_{,kk} - 2\nu u_{,ik}u_{,jk} - (\overline{u_j R_{ik}} + \overline{u_i R_{jk}}) \tag{6.13}
\]

When Eq.(6.13) is subtracted from Eq.(6.12) after expanding the correlation terms, the turbulent momentum flux equation is obtained and each term will be

\[
\frac{\partial}{\partial t}(\overline{u_j u_j}) - \frac{\partial}{\partial t}(\overline{u_i u_j}) = \frac{\partial}{\partial t}(R_{ij}) \tag{6.14}
\]

\[
(\overline{u_j u_j u_k})_{,j} - (u_j u_j u_k)_{,j} \\
= [(2a)(\overline{u_j u_j u_k})_{,j} + \overline{u_j u_j u_k},_{,j} + \overline{u_i u_j u_k} + \nu u_{,ji}u_{,ik}) + \frac{1}{2}(2a)^2(\overline{u_j u_j u_k})_{,ji} + \cdots \cdots]_{,j} \\
= [\overline{u_j R_{ik}} + \overline{u_i R_{jk}} + \nu u_{,ji}R_{ij} + \cdots \cdots]_{,j} \\
= \overline{u_j R_{jk}} + R_{jk} \overline{u_i u_i} + \overline{u_i R_{ik}} + R_{ik} \overline{u_j u_j} + \overline{u_i R_{ik}} + \text{H.O.T.} \tag{6.15}
\]

\[
-\frac{1}{\rho}[(\overline{p u_j})_j + (\overline{p u_j})_j - (\overline{p u_i})_j] \\
= -\frac{1}{\rho}[(2a)(\overline{p u_j u_j}) + \frac{1}{2l}(2a)^2(\overline{p u_j u_j}) + \cdots \cdots]_j + ((2a)(\overline{p u_j u_j}) + \cdots \cdots)_j \\
= \Pi_1 \tag{6.16}
\]

\[
\frac{1}{\rho}[(\overline{p u_j u_j} + \overline{u_j u_j}) - \overline{p(u_j + u_j)}] \\
= \frac{1}{\rho}[(2a)(\overline{p u_j S_{j,i}}) + \frac{1}{2l}(2a)^2(\overline{p u_j S_{j,i}} + \cdots \cdots)] \\
= \Pi_2 \tag{6.17}
\]
\[ \nu[(\bar{u}_i\bar{u}_j)_{,kk} - (\bar{u}_i\bar{u}_j)_{,kl}] = \nu R_{ikkk} = \Pi_3 \]  

and

\[ -2\nu(\bar{u}_k\bar{u}_{jk} - \bar{u}_{jk}\bar{u}_{jk}) \]

\[ = -2\nu[(2a\bar{u}_{,kl}\bar{u}_{,kl}) + \frac{1}{2l}(2a)^2\bar{u}_{,klkm}\bar{u}_{,klm} + \cdots ] \]

\[ = \Pi_4 \]  

The final form is given by

\[ \frac{\partial \overline{R}_{ij}}{\partial t} + \bar{u}_k R_{ijk} = -(R_{ik}\bar{u}_{jk} + R_{jk}\bar{u}_{ik}) + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 \]  

(6.20)

Physically, the first and second term represent the rate of change of the turbulent momentum flux and the inertial transport, respectively. Terms on the right may be expressed, in order, as the stress production, pressure-strain transport, diffusive transport, and stress destruction by the viscosity.

6.3 SCALAR VARIANCE OF TURBULENCE

The equation for the turbulence scalar variance starts with the scalar transport equation. The initial governing equation is obtained by multiplying the scalar component \( \phi \), i.e.,

\[ \phi \frac{\partial \phi}{\partial t} + u_j \phi \phi_{,j} = D\phi \phi_{,j} \]  

(6.21)

or

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \phi \phi \right) + (\frac{1}{2} \phi \phi \phi_{,j}) = D(\frac{1}{2} \phi \phi)_{,j} - D\phi \phi_{,j} \]  

(6.22)
Taking average on both sides of Eq. (6.22), one gets

$$\frac{\partial}{\partial t} (\frac{1}{2} \bar{\phi} \bar{\phi}) + (\frac{1}{2} \bar{\phi} \bar{u}_j) = D (\frac{1}{2} \bar{\phi} \bar{\phi})_{ji} - D \bar{\phi} \bar{\phi}_j$$

(6.23)

The corresponding equation in the mean flow is given by

$$\frac{\partial}{\partial t} (\frac{1}{2} \bar{\phi} \bar{\phi}) + (\frac{1}{2} \bar{\phi} \bar{u}_j) = D (\frac{1}{2} \bar{\phi} \bar{\phi})_{ji} - D \bar{\phi} \bar{\phi}_j - \bar{Q}_j \bar{\phi}$$

(6.24)

where \(Q_j\) was defined in Eq. (5.37). After Eq. (6.23) is expanded by using the double and triple correlations and is subtracted by Eq. (6.24), the resulting equation is

$$\frac{\partial}{\partial t} (\frac{1}{2} H) + \alpha_j (\frac{1}{2} H) = M - N - Q_j \bar{\phi}_j$$

(6.25)

where

$$H = \bar{\phi} \bar{\phi} - \bar{\phi} \bar{\phi} = (2a) \bar{\phi} \bar{\phi} + \frac{1}{2l} (2a)^2 \bar{\phi} \bar{\phi} + \cdots$$

$$M = D (\frac{1}{2} H)_{ij} - \frac{1}{2l} (2a)^2 (4 \bar{\phi} \bar{\phi} \bar{u}_j)_{ij} + \cdots$$

and

$$N = D (2a) \bar{\phi} \bar{\phi} + \frac{1}{2l} (2a)^2 \bar{\phi} \bar{\phi} \bar{u}_j$$

6.4 TURBULENCE SCALAR FLUX EQUATION

The scalar flux equation due to the turbulence is obtained in the same manner with the turbulence momentum equation. First of all, the Navier-Stokes equation is multiplied by the scalar quantity \( \phi \) and the scalar transport equation by the velocity component \( u_j \), respectively.
\begin{align*}
\frac{\partial \phi_i}{\partial t} + \phi(\mathbf{u}_i \mathbf{u}_j)_j &= -\frac{1}{\rho} \phi \mathbf{p}_j + \nu \phi \mathbf{u}_{ij} \\
u_i \frac{\partial \phi}{\partial t} + \mathbf{u}_i (\phi \mathbf{u}_j)_j &= D \phi \mathbf{u}_{ij} 
\end{align*}

Adding these two equations and using the continuity property, one gets

\begin{align*}
\frac{\partial}{\partial t} (\mathbf{u}_i \phi) + (\mathbf{u}_i \mathbf{u}_j \phi)_j &= -\frac{1}{\rho} \phi \mathbf{p}_j + \nu \phi \mathbf{u}_{ij} + D \phi \mathbf{u}_{ij} 
\end{align*}

(6.28)

The application of filtering operation to Eq.(6.28) leads to

\begin{align*}
\frac{\partial}{\partial t} (\bar{\mathbf{u}}_i \bar{\phi}) + (\bar{\mathbf{u}}_i \bar{\mathbf{u}}_j \bar{\phi})_j &= -\frac{1}{\rho} \bar{\phi} \bar{\mathbf{p}}_j + \nu \bar{\phi} \bar{\mathbf{u}}_{ij} + D \bar{\phi} \mathbf{u}_{ij} 
\end{align*}

(6.29)

Meanwhile, the scalar flux in the mean flow is expressed as

\begin{align*}
\frac{\partial}{\partial t} (\bar{\mathbf{u}}_i \bar{\phi}) + (\bar{\mathbf{u}}_i \bar{\mathbf{u}}_j \bar{\phi})_j &= -\frac{1}{\rho} \bar{\phi} \bar{\mathbf{p}}_j + \nu \bar{\phi} \bar{\mathbf{u}}_{ij} + D \bar{\phi} \mathbf{u}_{ij} - \bar{\phi} \mathbf{R}_{ij} - \bar{\mathbf{u}}_i Q_{ij} 
\end{align*}

(6.30)

Therefore, the turbulence scalar flux equation results from subtracting Eq.(6.30) from Eq.(6.29), after Eq.(6.29) is expanded with the aid of the double and triple correlation formulae. The resultant equation is written as

\begin{align*}
\frac{\partial Q_i}{\partial t} + \bar{\mathbf{u}}_j Q_{ij} &= -\frac{1}{\rho} A_i + \nu B_i + D C_i - D_{ijj} 
\end{align*}

(6.31)

where

\begin{align*}
Q_i &= (2a) \bar{\mathbf{u}}_i \bar{\phi} + \frac{1}{2l} (2a)^2 \bar{\mathbf{u}}_{ik} \bar{\phi}_{jkl} + \cdots \\
A_i &= (2a) \bar{\phi} \bar{\mathbf{p}}_{ik} + \frac{1}{2l} (2a)^2 \bar{\phi}_{ik} \bar{\mathbf{p}}_{jkl} + \cdots
\end{align*}
\[ B_i = (2a)\phi_{,i} u_{,i} + \frac{1}{2l} (2a)^2 \phi_{,i} u_{,i} u_{,i;k} + \cdots \]

\[ C_i = (2a)\bar{\phi}_{,i} \bar{u}_{,i} + \frac{1}{2l} (2a)^2 \bar{\phi}_{,i} \bar{u}_{,i} \bar{u}_{,i;k} + \cdots \]

and

\[ D_{ij} = \frac{1}{2l} (2a)^2 (\bar{u}_{,i} \bar{u}_{,j;k} \phi_{,i} + \bar{u}_{,i} \bar{u}_{,j} \phi_{,k} + \bar{u}_{,i} \bar{u}_{,j} \phi_{,k} + \bar{u}_{,i} \bar{u}_{,j} \phi_{,k}) + \cdots \]
CHAPTER VII

ANALYSIS OF THE YB CLOSURE

In Chapter V, the YB-III series representation for the averaged non-linear terms was formulated. The derivation procedure is straightforward and the resulting equation is quite compact. No specific assumptions or hypotheses are involved and, in fact, such an exact form of closure has never been found. By using the YB-III series description, one can eliminate the intrinsic limitations in the conventional closure models. In addition, although the YB filtering approach has been initially developed with a generalized moving volume averaging concept, the resulting equations are also valid in the traditional Reynolds averaging form because, as mentioned earlier, both averages become identical as the averaging interval is adjusted properly. Therefore, the YB-III series description shows promise in understanding the non-linear structure in the turbulence equations.

7.1 SOME MATHEMATICAL PROPERTIES OF THE YB CLOSURE

7.1.1 Convergence Behavior of the YB-I and YB-II Series

The series expansion of the exponential function, used in the derivation of two series solutions, is valid with respect to all real number arguments. Although the YB-I and YB-II series are mathematically exact, it is necessary, in practice, to cut-off the unimportant terms, since an infinite number of terms in the series cannot be employed.
Therefore, knowledge of the convergence behavior is needed. To show the rate of series' convergence, it is convenient to non-dimensionalize their terms in such a way that both sides of the series are divided by the total variable. When only the one-dimensional case is considered, for simplicity, the non-dimensionalized YB-I series for the arbitrary function \( f \) is written as

\[
1 = \frac{f}{f} - a \frac{f_{xx}}{f} - \frac{a^2}{2!} \frac{f_{xxxx}}{f} - \cdots \quad (7.1)
\]

where the subscript denotes the derivatives.

The YB-II series also gives

\[
1 = \frac{f}{f} - a \frac{f_{xx}}{f} + \frac{a^2}{2!} \frac{f_{xxxx}}{f} - \cdots \quad (7.2)
\]

Now, suppose that \( f(x) \) is a trigonometrical function such as

\[
f(x) = C \cos kx \quad (7.3)
\]

where \( C \) and \( k \) denote the amplitude and the wavenumber, respectively.

Then, the averaged function, \( \overline{f}(x) \), is

\[
\overline{f}(x) = e^{-ak^2} C \cos kx \quad (7.4)
\]

Eq.(7.3) and (7.4) give the non-dimensional form of YB-I and YB-II series as

\[
1 = e^{-ak^2} + ak^2 - \frac{1}{2!} a^2 k^4 + \frac{1}{3!} a^3 k^6 + \cdots \quad (7.5)
\]
\[ 1 = e^{(-ak^2)}(1 + ak^2 + \frac{1}{2!}a^2k^4 + \frac{1}{3!}a^3k^6 + \cdots) \]  

(7.6)

Each term’s contribution is tabulated in Table 7.1, and is also illustrated in Fig. 7.1. It shows that these series rapidly converge to the exact value at small numbers of \(ak^2\), in which the first few terms are dominant and, thus, one can get the approximate value with the high accuracy even by choosing the first few terms. When \(ak^2\) has large values, the leading terms are moved to the higher derivative terms, and more terms are necessary for convergence. Table 7.1 indicates that five terms are necessary at \(ak^2 = 1.0\) to get the approximation within a one percent error, but only two terms are enough at 0.1. The convergence behavior of the two series is quite different; YB-I series converges oscillatory and YB-II series monotonically. Fig 7.2 shows their different behavior at \(ak^2 = 1.0\).

Table 7.1

Rate of Convergence for the YB-I and YB-II series

<table>
<thead>
<tr>
<th>(ak^2) term</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>YB-I</td>
<td>0.9048</td>
<td>0.1000</td>
<td>-0.0050</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>sum</td>
<td>0.9048</td>
<td>1.0048</td>
<td>0.9998</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>YB-II</td>
<td>0.9048</td>
<td>0.0904</td>
<td>0.0045</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>sum</td>
<td>0.9048</td>
<td>0.9953</td>
<td>0.9998</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>YB-I</td>
<td>0.3679</td>
<td>1.0000</td>
<td>-0.5000</td>
<td>0.1666</td>
<td>-0.0416</td>
<td>0.0083</td>
</tr>
<tr>
<td>sum</td>
<td>0.3679</td>
<td>1.3679</td>
<td>0.8679</td>
<td>1.0545</td>
<td>0.9929</td>
<td>1.0012</td>
</tr>
<tr>
<td>1.0</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>YB-II</td>
<td>0.3679</td>
<td>0.3679</td>
<td>0.1839</td>
<td>0.0613</td>
<td>0.0153</td>
<td>0.0031</td>
</tr>
<tr>
<td>sum</td>
<td>0.3679</td>
<td>0.7358</td>
<td>0.9197</td>
<td>0.9810</td>
<td>0.9963</td>
<td>0.9994</td>
</tr>
</tbody>
</table>
Figure 7.1: Comparison of the significant terms in the YB-I and YB-II series

Figure 7.2: Convergence behavior of the YB-I and YB-II series
Assuming that the cut-off wavenumber, \( k_c \), is \( \pi/\Delta \), the corresponding \( ak^2 \) is equal to \( \pi^2/4\gamma \). This value is closely related to determining the number of terms which are supposed to be used in the numerical models. If \( \gamma \) is selected to be large, only few terms are enough but the resultant accuracy from the models will decrease. When \( \gamma \) is chosen to be 6.0, each term's portion in magnitude is presented in Table 7.2.

**Table 7.2**

Rate of convergence at the cut-off wavenumber

<table>
<thead>
<tr>
<th></th>
<th>term 1</th>
<th>term 2</th>
<th>term 3</th>
<th>term 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ak^2 )</td>
<td>0.6628</td>
<td>0.4112</td>
<td>-0.0645</td>
<td>0.0116</td>
</tr>
<tr>
<td>( \pi^2/4\gamma )</td>
<td>0.6628</td>
<td>1.0740</td>
<td>0.9895</td>
<td>1.0011</td>
</tr>
<tr>
<td>( YB-I )</td>
<td>0.6628</td>
<td>0.2725</td>
<td>0.0560</td>
<td>0.0077</td>
</tr>
<tr>
<td>( \text{sum} )</td>
<td>0.6628</td>
<td>0.9353</td>
<td>0.9913</td>
<td>0.9990</td>
</tr>
</tbody>
</table>

If only the first term is used, a more than 35 percent error in amplitude is expected for the particular wavelength with \( 2\Delta \), the first two terms are more than 5 percent in error, three terms are less than 3 percent in error, and four terms are almost exact.

### 7.1.2 Convergence Behavior of the YB-III Series

The YB-III series has also an infinite number of terms which are composed of high-order derivatives. Of course, the number of terms required to get the sufficiently accurate values depends on the character of flow, grid interval, computation expense, expected model accuracy, etc. In practice, it will not be easy to employ higher than third-order derivative terms in the numerical computations. In other words, it is desirable to know whether a few of the lower order derivative terms in the series are dominant, and, if so, what are the necessary grid sizes and the filter constants for proper
lower-order convergence. It is believed that the convergence problem in the YB-III series is not nearly so serious as it is in the YB-I and YB-II series. This is because the YB-III series is formulated from the averaged non-linear terms where the small scale quantities have been removed out. Briefly to show series convergence in \( R_{ij} \), assume that two velocity components in the one-dimensional case are defined as

\[
u_i = c_i e^{ik_i x} \quad \text{(no sum)} \\
u_j = c_j e^{ik_j x} \quad \text{(no sum)}
\]

(7.7) (7.8)

By the averaging definition, their averaged values are

\[
\bar{v}_i = c_i e^{-\kappa k_i^2} e^{ik_i x} \\
\bar{v}_j = c_j e^{-\kappa k_j^2} e^{ik_j x}
\]

(7.9) (7.10)

and

\[
\bar{v}_i \bar{v}_j = c_i c_j e^{-\kappa (k_i + k_j)^2} e^{ik_i x + k_j x}
\]

(7.11)

where \( a = \frac{\Delta^2}{4\gamma} \). With Eq. (7.9) and (7.10), the YB-III series for \( R_{ij} \) leads to the following relation;

\[
c_i c_j e^{-\kappa (k_i + k_j)^2} = c_i c_j e^{-\kappa k_i^2} e^{-\kappa k_j^2} \left[ 1 - (2a)k_i k_j + \frac{3}{2} (2a)^2 k_i^2 k_j^2 + \cdots \right]
\]

(7.12)

Dividing both sides by the coefficient on the right, one gets
\[ e^{-2ak_jk_j} = 1 - (2a)k_jk_j + \frac{1}{2!}(2a)^2k_j^2k_j^2 + \cdots \]  

(7.13)

The term on the left, which is the exact value of the series, is presented in Fig. 7.3.

![Figure 7.3: Exact value of the series](image)

Fig. 7.3 shows that it is equal to 1.0 at both axes and becomes small where the argument, \((2a)k_jk_j\), becomes large. This implies that the first term on the right hand side of Eq.(7.13) is enough if either \(u_j\) or \(u_j\) is constant, otherwise, the large wavenumber effects are not significant. On the other hand, terms on the right hand side of Eq.(7.13) require the first few terms at which \((2a)k_jk_j\) is small but more terms are needed at large wavenumbers. Fig. 7.4 presents the approximate values when only the first two terms are considered. The employment of these two terms means that \(\overline{u_ju_j}\) is approximated by \(u_ju_j + (2a)\overline{u_ju_j}\). The error bound in this case is shown in Fig. 7.5. As anticipated, their difference is zero at both axes and grows as the wavenumbers become
large. The maximum difference occurs at the axis where \( k_i = k_j \).

Figure 7.4: Approximated values by the first order term

\[ e^{-2ak_i k_j} - \left( 1 - 2ak_i k_j \right) \]

Figure 7.5: Difference between the exact and the approximated value
It is necessary to investigate the convergence behavior at the cut-off wavelength. Assuming that the cut-off wavenumber is \( \pi/\Delta \) and \( \gamma \) is equal to 6.0, respectively, the resolvable portion within the cut-off wavenumber is obtained by integrating Eq.(7.13) from zero to the cut-off wavenumber, i.e.,

\[
\int \int_0^{\pi/\Delta} e^{-2\pi k_i k_j} \, dk_i \, dk_j = \int \int_0^{\pi/\Delta} \left[ 1 - (2a) k_i k_j + \frac{1}{2!} (2a)^2 k_i^2 k_j^2 + \cdots \right] dk_i \, dk_j \quad (7.14)
\]

In order to compare both sides of Eq.(7.14), it may be more convenient to non-dimensionalize the equation, by means of changing the wavenumber to a non-dimensional one such that

\[
k_i^0 = \frac{\Delta}{\pi} k_i \quad \text{and} \quad k_j^0 = \frac{\Delta}{\pi} k_j \quad (7.15)
\]

Then, Eq.(7.14) becomes

\[
\int \int_0^1 e^{-\frac{\pi^2}{12} k_i^0 k_j^0} \, dk_i^0 \, dk_j^0 = \int \int_0^1 \left[ 1 - \frac{\pi^2}{12} k_i^0 k_j^0 + \frac{\pi^4}{288} k_i^0 k_j^0 + \cdots \right] dk_i^0 \, dk_j^0 \quad (7.16)
\]

Eq.(7.16) can be integrated analytically. The calculated results as regards the number of terms are tabulated in Table 7.3 and are also presented in Fig.7.6.

As seen in Table 7.3 and Fig.7.6, less than 4% difference occurs at the first two terms and less than 1% at the 3rd terms. It indicates that the first few terms out of the series significantly contribute to the variation of the magnitude and, thus, low-order terms dominate. It is anticipated that accurate results can be obtained by employing the first or second leading terms although errors are expected near the cut-off wavenumber. Therefore, it is assumed, hereafter, that only the first few terms play a sig-
### Table 7.3
Convergence rate and the number of terms

<table>
<thead>
<tr>
<th>exact</th>
<th>first term</th>
<th>second term</th>
<th>third term</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.82615</td>
<td>1.0</td>
<td>-0.20562</td>
<td>+0.03758</td>
</tr>
<tr>
<td>sum</td>
<td>0.82615</td>
<td>0.79438</td>
<td>0.83196</td>
</tr>
<tr>
<td>ratio</td>
<td>1.0</td>
<td>1.21043</td>
<td>0.96154</td>
</tr>
<tr>
<td>error</td>
<td>0.0 %</td>
<td>21.0 %</td>
<td>3.85 %</td>
</tr>
</tbody>
</table>

Significant role in the series and subsequent interpretation and evaluation will be on these terms, i.e., \((2a)\overline{u}_k\overline{w}_k\).

![Graph showing convergence rate and the number of terms](image)

**Figure 7.6:** Convergence rate and the number of terms
7.1.3 The Effect of Filter Isotropy on $R_{ij}$

In Chapter V, the YB-III series representation for the double correlations was formulated and, then, the averaged non-linear terms as regards the velocity components were described by introducing the residual stress, $R_{ij}$, in Eq.(5.34). For convenience, the definition of $R_{ij}$ for the isotropic filter is rewritten as

$$R_{ij} = \overline{u_i u_j} - \overline{u_i} \overline{u_j} = R_{ij}^{(1)} + R_{ij}^{(2)} + \cdots$$  \hspace{1cm} (7.17)

where

$$R_{ij}^{(1)} = (2a)\overline{u_i u_j}, \quad R_{ij}^{(2)} = \frac{1}{2l} (2a)^2 \overline{u_k u_j u_k u_j}, \quad \cdots$$ etc, and $a = \frac{\Delta^2}{4\gamma}$.

It is obvious that $R_{ij}$ is a symmetric tensor as is the turbulent stress in the Reynolds averaging procedure, since

$$R_{ij}^{(m)} = R_{ji}^{(m)}, \quad m = 1, 2, 3, \cdots$$  \hspace{1cm} (7.18)

It is also shown that all diagonal terms in $R_{ij}$ consist of quadratic forms but off-diagonal terms are not necessary so. For instance, the components comprising of the first term in Eq.(7.17) are

$$R_{11}^{(1)} = (2a)(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2})$$  \hspace{1cm} (7.19)

$$R_{12}^{(2)} = (2a)(\overline{u_1 u_2} + \overline{u_2 u_3} + \overline{u_3 u_1})$$  \hspace{1cm} (7.20)

In the case that the non-isotropic filter is used, it is given by
\[ R_{ij} = 2C(k)U_{ik}U_{jk} + \frac{1}{2l}4C(k)C(1)U_{ikl}U_{jkl} + \cdots \]  
(7.21)

where \( C(i) = \frac{\Delta_i^2}{4}\gamma \), \( i = 1, 2, 3 \). Eq.(7.21) is also symmetric because

\[ R_{ji} = 2C(k)U_{jk}U_{ik} + \frac{1}{2l}4C(k)C(1)U_{jkl}U_{ikl} + \cdots \]
\[ = 2C(k)U_{ik}U_{jk} + \frac{1}{2l}4C(k)C(1)U_{ikl}U_{jkl} + \cdots \]
\[ = R_{ij} \]  
(7.22)

and the diagonal terms are quadratic as well. The expanded forms corresponding to Eq.(7.19) and (7.20) are

\[ R_{11}^{(1)} = (2a_x)\bar{u}_x^2 + (2a_y)\bar{u}_y^2 + (2a_z)\bar{u}_z^2 \]  
(7.23)
\[ R_{12}^{(1)} = (2a_x)\bar{u}_x\bar{v}_x + (2a_y)\bar{u}_y\bar{v}_y + (2a_z)\bar{u}_z\bar{v}_x \]  
(7.24)

where \( a_i = \frac{\Delta_i^2}{4}\gamma \), \( i = 1, 2, 3 \).

Compared to Eq.(7.19) and (7.20), the difference between isotropic and non-isotropic filter is found explicitly in the coefficients of sub-components which constitute each component in \( R_{ij} \). If the filter width in a particular direction becomes large, then the contribution to the velocity gradients in that direction will increase. For example, the coefficient of the second term in Eq.(7.24) becomes large as the large filter width in y-direction ( \( \Delta_y \) or \( \Delta_y \) ) is used, so that the portion of the second term is relatively significant. This is intuitive because the large filter width indicates the flat Gaussian filter and thus the residual stress grows up as the flat Gaussian filter function is employed. However, the velocity gradients ( \( \bar{u}_y, \bar{v}_y \) ) will simultaneously decrease,
such that the resulting effect does not become larger at the same rate the change of coefficient does. In other words, the net increase will not be proportional to the square of the filter width (Δy²). Meanwhile, if the filter width is equal to zero in a particular direction, the corresponding term in that direction vanishes. This is also anticipated because the zero filter width has the same meaning as no averaging.

7.2 KINEMATICAL ASPECTS OF R_ij IN THE TWO DIMENSIONAL CASE

7.2.1 R_ij for an Isotropic Filter

The turbulent stress, R_ij, with the isotropic filter width is given in the x-y plane as

\[
R_{ij} = R_{ij}^{(1)} + R_{ij}^{(2)} + \cdots
\]  

(7.25)

where

\[
R_{ij}^{(1)} = (2a) \begin{bmatrix}
\bar{u}_x \bar{u}_x + \bar{u}_y \bar{u}_y & \bar{u}_x \bar{v}_x + \bar{u}_y \bar{v}_y \\
\text{symm} & \bar{v}_x \bar{v}_x + \bar{v}_y \bar{v}_y
\end{bmatrix}
\]

\[
R_{ij}^{(2)} = \frac{1}{2} \begin{bmatrix}
\bar{u}_x^2 + 2\bar{u}_x \bar{u}_y + \bar{u}_y^2 & \bar{u}_x \bar{v}_x + 2\bar{u}_x \bar{v}_y + \bar{u}_y \bar{v}_y \\
\text{symm} & \bar{v}_x^2 + 2\bar{v}_x \bar{v}_y + \bar{v}_y^2
\end{bmatrix}
\]

In the above equation, the diagonal elements are always positive regardless of the differentiation order since they consist of the quadratic forms. It is readily shown that, by the definition of R_ij, half the sum of these diagonal terms becomes equal to the kinetic energy of the turbulence, i.e.,

\[
\frac{1}{2} R_{ii} = \frac{1}{2} \bar{u}_i \bar{u}_i - \frac{1}{2} \bar{u}_i \bar{u}_i
\]  

(7.26)
In the Reynolds or ensemble averaging concept, Eq.(7.26) is equivalent to

\[
k = \frac{1}{2} \bar{u}_i \bar{u}_i^2
= \frac{1}{2} (R_{ii}^{(1)} + R_{ii}^{(2)} + \cdots ) \tag{7.27}
\]

where \( k \) denotes the turbulent kinetic energy. These diagonal terms do not vanish either in the rotational flow or in the irrotational flow. In addition, as seen in the vorticity equation written in Eq.(5.68), they are associated with the dynamic pressure gradient and, thus, they directly contribute to the Bernoulli equation in the irrotational flow. The off-diagonal elements in \( R_{ij}^{(1)} \) are, with the aid of the continuity relation, written as

\[
R_{12}^{(1)} = (2a) \bar{u}_x \bar{v}_x + \bar{u}_y \bar{v}_y
= (2a) \bar{u}_x (\bar{v}_x - \bar{v}_y) = -(2a) \bar{u}_x \bar{\omega}_y = (2a) \bar{\omega}_x \bar{\omega}_y \tag{7.28}
\]

\[
R_{12}^{(2)} = \frac{1}{2l} (2a)^2 (\bar{u}_{xx} \bar{v}_x + 2 \bar{u}_{xy} \bar{v}_y \bar{v}_x + \bar{u}_{yy} \bar{v}_y)
= \frac{1}{2l} (2a)^2 [ -\bar{u}_{xx} (\bar{\omega}_x) + \bar{\omega}_y (\bar{\omega}_x) ] \tag{7.29}
\]

The interpretation of the off-diagonal terms is of particular interest since they are directly related to the residual shear stress. Unlike the diagonal terms, these elements vanish in the irrotational flow and, consequently, the residual shear stress does not appear. This non-existence of it shows the difference with most of the previous proposed models, such as the SGS model and the eddy viscosity model. Note that non-vanishing of the turbulent shear stress in the irrotational flow is inappropriate in real
physics. Certainly, the off-diagonal elements in $R_y$ represent the interaction between the velocity gradients and the vorticity. For a closer look at this interaction, consider the vorticity-velocity products, for example, $\vec{v} \omega_z$ illustrated in Fig. 7.7. This interactive term is observed as the vortex term in the equation of motion, already shown in Eq.(5.68). This vortex term appears in the Coriolis force as well, stating that the rotating coordinate system results in the force perpendicular to the vortex line and the velocity. Thus, in an analogous manner, it is interpreted that the vortex term, $\vec{v} \omega_z$, brings about the vortex force in the x-direction, $F_x$.

\[ F_x = -w \omega_y \]

\[ \Rightarrow F_x = v \omega_z \]

Figure 7.7: Interaction of vorticity and velocity
However, the off-diagonal components in $R_{ij}$ are composed of the velocity gradients ($\nabla v$), instead of the velocity itself ($\nabla \cdot v$). On the grounds that $u_1$ or $u_2$ represents the stretching or squeezing of the fluid elements, $u_1 \omega_2$ or $u_2 \omega_1$ in Eq.(7.28) may be explained as the resulting force from the distortion of the fluid particles due to the vortex and the stretching, i.e., the vortex stretching. According to Tennekes and Lumley (1972), it is believed that the vortex stretching may be one of the significant parameters in turbulence because it is associated with the change of size of eddies (i.e., change of length scales).

### 7.2.2 $R_{ij}$ for a non-Isotropic Filter

The two-dimensional $R_{ij}$ with the non-isotropic filter is written as

$$R_{ij} = R_{ij}^{(1)} + R_{ij}^{(2)} + \cdots$$  (7.30)

where

$$R_{ij}^{(1)} = 2a_1 u_2 \frac{u_x}{u_x} + a_2 u_y \frac{u_y}{u_y} = a_1 u_2 \frac{\nabla u_x}{\nabla u_x} + a_2 u_y \frac{\nabla u_y}{\nabla u_y}$$

$$R_{ij}^{(2)} = \frac{1}{2} \left[ a_1 u_2^{2} \frac{\nabla u_x}{\nabla u_x} + 2a_1 a_2 u_{xy}^{2} + 2a_2 u_{yy}^{2} - a_1 u_2 \frac{\nabla u_x}{\nabla u_x} + 2a_1 a_2 u_{xy} \frac{\nabla u_x}{\nabla u_x} + a_2 u_{yy} \frac{\nabla u_y}{\nabla u_y} \right]$$

As seen in the isotropic filter width, the diagonal terms are positive definite and are also related to the turbulent kinetic energy. Like the isotropic filter, these elements remain both in the rotational flow and in the irrotational flow. Meanwhile, the off-diagonal element, $R_{ij}^{(1)}$, with the continuity equation, is
\[ R_{12}^{(1)} = 2a_1 \bar{u}_x \bar{v}_x + 2a_2 \bar{u}_y \bar{v}_y \]

\[ = 2 \bar{u}_x \{ a_1 \bar{v}_z - a_2 \bar{v}_y \} \]  

(7.31)

Unless \( a_1 \) is equal to \( a_2 \), this term does not vanish even in the irrotational flow. The use of the different filter widths results in the weight variation to their directions. Therefore, the non-isotropic filter maintains in general the residual stress terms at off-diagonal terms, which disappear in the isotropic filter case.

7.3 KINEMATICAL ASPECTS OF \( R_{ij} \) IN THE THREE DIMENSIONAL CASE

In order to interpret the properties of \( R_{ij} \) in the three-dimensional case, it may be useful to describe \( R_{ij} \) in terms of the deformation tensor and the rotation tensor. These two tensors are already defined in Section 5.5.1, and are written as

\[ S_{ij} = \frac{1}{2} (\bar{u}_{ij} + \bar{u}_{ji}) \]  

(7.32)

\[ \Omega_{ij} = \frac{1}{2} (\bar{u}_{ij} - \bar{u}_{ji}) \]  

(7.33)

In matrix form, they are

\[
[S] = \begin{bmatrix}
\bar{u}_x & \frac{1}{2}(\bar{u}_y + \bar{v}_x) & \frac{1}{2}(\bar{u}_z + \bar{w}_x) \\
\frac{1}{2}(\bar{v}_y + \bar{u}_x) & \bar{v}_y & \frac{1}{2}(\bar{v}_z + \bar{w}_y) \\
\text{symm} & \frac{1}{2}(\bar{w}_x + \bar{w}_y) & \bar{w}_z
\end{bmatrix}
\]  

(7.34)
\[
\begin{bmatrix}
\Omega
\end{bmatrix} = \begin{bmatrix}
0 & \frac{1}{2}(u_y - v_x) & \frac{1}{2}(\omega_z - w_x) \\
-\frac{1}{2}(u_y - v_x) & 0 & \frac{1}{2}(v_z - w_y) \\
\frac{1}{2}(\omega_z - w_x) & -\frac{1}{2}(v_z - w_y) & 0
\end{bmatrix}
\] (7.35)

With the use of the above notations, \( R_{ij} \) is decomposed into four terms, i.e.,

\[
R_{ij} = R^{(1)}_{ij} + R^{(2)}_{ij} + \cdots
\] (7.36)

where

\[
R^{(1)}_{ij} = (2a) \bar{u}_{ik} \bar{u}_{jk}
\]

\[
= (2a) [S_{ik} S_{jk} + \Omega_{ik} \Omega_{jk} + \Omega_{ik} S_{jk} + \Omega_{ik} \Omega_{jk}]
\] (7.37)

\[
R^{(2)}_{ij} = \frac{1}{2l} (2a)^2 \bar{u}_{ikl} \bar{u}_{jkl}
\]

\[
= \frac{1}{2} (2a)^2 [S_{ikl} S_{jkl} + \Omega_{ikl} \Omega_{jkl} + \Omega_{ikl} S_{jkl} + \Omega_{ikl} \Omega_{jkl}]
\] (7.38)

In the four decomposed terms, one sees that the first term represents deformation, the second and the third terms are vortex stretching and the last denotes the interaction between the vorticities. In addition, the first and last terms are symmetric, while the second and third terms are not. However, one can show that two tensors in the second and third terms are connected with their transpose, i.e.,

\[
S_{ik} \Omega_{jk} = (\Omega_{ik} S_{jk})^T
\] (7.39)

From this relation, the sum of these two tensors will result in another symmetric tensor form and, consequently, three types of symmetric tensors are obtained and defined as follows:
\[ A_{ij} = (2a) A_{ij}^{(1)} + \frac{1}{2l}(2a)^2 A_{ij}^{(2)} + \cdots \quad (7.40) \]

where \[ A_{ij}^{(1)} = S_{ik}S_{jk} \quad A_{ij}^{(2)} = S_{ikl}S_{jkl} \quad \text{etc} \]

\[ B_{ij} = (2a) B_{ij}^{(1)} + \frac{1}{2l}(2a)^2 B_{ij}^{(2)} + \cdots \quad (7.41) \]

where \[ B_{ij}^{(1)} = S_{ik} \Omega_{jk} + \Omega_{ik}S_{jk} \quad B_{ij}^{(2)} = S_{ikl} \Omega_{jk,l} + \Omega_{ikl}S_{jkl} \quad \text{etc} \]

\[ C_{ij} = (2a) C_{ij}^{(1)} + \frac{1}{2l}(2a)^2 C_{ij}^{(2)} + \cdots \quad (7.42) \]

where \[ C_{ij}^{(1)} = \Omega_{ik} \Omega_{jk} \quad C_{ij}^{(2)} = \Omega_{ikl} \Omega_{jkl} \quad \text{etc} \]

With the above definitions, an alternative form of \( R_{ij} \) is given as

\[ R_{ij} = A_{ij} + B_{ij} + C_{ij} \quad (7.43) \]

These three terms contain an infinite number of terms which are composed of high-order derivatives but, as mentioned in Section 7.1, the first few terms are dominant in the series and therefore the interpretation subsequently will focus on these terms, i.e., \( A_{ij}^{(1)}, B_{ij}^{(1)} \) and \( C_{ij}^{(1)} \).

7.3.1 The \( A_{ij} \) Term

The first term, \( A_{ij} \), consists of the deformation tensor or its gradients, by showing their self-products. Thus, it represents the interaction of the deformation of the fluid elements. For instance, \( A_{ij}^{(1)} \) is given by the square matrix of \( S \) and is written as
\[
[A]^{(1)} = [S][S]
\]  \hspace{1cm} (7.44)

where \([S]\) is the symmetric deformation matrix, i.e.,

\[
[S] = \begin{bmatrix}
    s_{11} & s_{12} & s_{13} \\
    s_{12} & s_{22} & s_{23} \\
    \text{symm} & s_{33}
\end{bmatrix}
\]  \hspace{1cm} (7.45)

A model such that the residual stress is described in terms of the square of the deformation tensor is found in the Smagorinsky's closure model, which is generally used to resolve the SGS quantities with some success, to date (Clark et al. 1977). Detailed comparisons to various existing closure models will be done in Chapter VIII.

7.3.2 The \(B_{ij}\) Term

The cross term, \(B_{ij}\), is associated with both the vorticity and the deformation. According to Tennekes and Lumley (1972), the energy transfer structure from large eddies to small eddies may be explained in view of the vortex stretching because the change of the vorticity results in the change of the eddy size. As an example, one can observe this phenomena in the flow passing through a tunnel with a contraction, as shown in Fig. 7.8. In this flow, the fluid element is stretched and the sectional area becomes smaller at point B by the continuity relation. Then, the inertia moment will increase and, consequently, the angular velocity should be increased by the conservation law of angular momentum. In short, the scale of the eddies stretched by the strain rate becomes small as the vorticity is amplified. It implies that the length scale is changed by the vortex stretching. Certainly, this cross term describes the change of the
vorticity caused by the stretching of the fluid elements. It is quite strange that, although this cross term is possibly of great importance in turbulence, such a vortex stretching model has never been found in existing closure schemes.

Figure 7.8: Vortex stretching
(Tennekes and Lumley 1972)

7.3.3 The $C_{ij}$ Term

The last term, $C_{ij}$, consists of the rotation tensor, which vanishes in the irrotational flow. It is important in cases the vorticity becomes large such as in the jets or wakes. With the aid of the relation between the vorticity vector and the rotation tensor in Eq.(5.53) and (5.54), $C_{ij}^{(1)}$ is transformed as
\[
\Omega_{ik} \Omega_{jk} = (-\frac{1}{2}\varepsilon_{ikm}\omega_m)(-\frac{1}{2}\varepsilon_{jkn}\omega_n)
\]
\[
= \frac{1}{4}(\varepsilon_{ikm}\varepsilon_{jkn}\omega_m\omega_n)
\]
\[
= \frac{1}{4}(\delta_{mn}\delta_{ij} - \delta_{mj}\delta_{in})\omega_m\omega_n
\]
\[
= \frac{1}{4}(\delta_{ij}\omega_m\omega_n - \omega_i\omega_j)
\]  

(7.46)

Physically, \( \frac{1}{2}\omega_m\omega_m \) represents the kinetic energy of a rigid body rotation at which the angular deformation only exists. Thus, the off-diagonal terms in Eq.(7.46) denote the products of vorticities in two different directions and the sum of the diagonal elements is interpreted as the kinetic energy due to the pure rotation.

7.4 DYNAMICAL ASPECTS OF \( R_{ij} \)

7.4.1 Decomposition of \( R_{ij} \) and Definitions

One of the distinctive features in the turbulence is a dissipative nature. It is generally admitted that kinetic energy is transferred from the mean flow to the turbulence and it is finally converted into heat by the viscosity. By this energy transfer structure, a continuous supply of energy is necessary in order to continuously maintain the turbulence in a flow. The turbulent motion due to the energy transfer is also closely associated with a diffusion of fluid particles. Therefore, the knowledge of the energy transfer mechanism is required for a better understanding on the turbulence.

The kinetic energy in the averaged field is defined as \( \frac{1}{2} \langle u_i u_i \rangle \), the kinetic energy of
the mean flow is given by $\frac{1}{2} \overline{u_i u_i}$, and the turbulent kinetic energy, $k$, is expressed as their difference. With the series description of $R_{ij}$, this turbulent kinetic energy is written as

$$k = \frac{1}{2} (\overline{u_i u_i} - \overline{u_j u_j})$$

$$= \frac{1}{2} R_{ii}$$

$$= \frac{1}{2} [(2a) \overline{u_{ik} u_{ik}} + \frac{1}{2} (2a) \overline{u_{ik} u_{jk}} + \cdots ] \quad (7.47)$$

As mentioned before, all terms on the right hand side of Eq. (7.47) are positive and $k$ becomes equal to the sum of the diagonal terms in $R_{ij}$. It may be instructive to show the energy distribution in terms of the decomposed three tensors, $A_{ii}$, $B_{ii}$ and $C_{ii}$, i.e.,

$$k = \frac{1}{2} (A_{ii} + B_{ii} + C_{ii}) \quad (7.48)$$

It was shown in the previous Section that the tensor, $B_{ij}$, is composed of the symmetric and the skew symmetric tensor. Thus, by the tensor algebra, the trace of their products is equal to zero. For instance, the first-order term in $B_{ij}$ is given by $S_{ik} \Omega_{jk}$ and its trace is

$$\text{tr}(S \Omega) = \text{tr}(\Omega S)^T = 0 \quad (7.49)$$

in which $S$ is symmetric and $\Omega$ is skew-symmetric. Therefore, this cross term does not contribute to the turbulent kinetic energy, while $\frac{1}{2} A_{ii}$ denotes the energy due to the
deformation and $\frac{1}{2} C_{ij}$ due to the rotation.

### 7.4.2 Kinetic Energy Equations

Assuming that there is no external force and a fluid is incompressible, the kinetic energy equation is obtained by multiplying the velocity component with the equation of motion. With the continuity equation, it gives

$$\frac{\partial}{\partial t} \left( \frac{1}{2} u_i u_i \right) + \frac{1}{2} u_i u_j u_j = -\frac{1}{\rho} (\delta_{ij} p_{i,j}) + \nu \left( \frac{1}{2} u_i u_j \right)_{ii} - \nu (u_i u_i u_j)$$  \hspace{1cm} (7.50)

Taking the ensemble average over an entire region with the isotropic and homogeneous conditions and using the divergence theorem, one has

$$\frac{\partial}{\partial t} \left< \frac{1}{2} u_i u_i \right> = -\nu <u_i u_{ii}>$$  \hspace{1cm} (7.51)

where $< >$ was defined in Eq.(5.44).

Now, application of the filtering operation to Eq.(7.50) results in the following equation:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \overline{u_i u_i} \right) + \frac{1}{2} \overline{u_i u_j u_j} = -\frac{1}{\rho} (\delta_{ij} \overline{p_{i,j}}) + \nu \left( \frac{1}{2} \overline{u_i u_j} \right)_{ii} - \nu (\overline{u_i u_i u_j})$$  \hspace{1cm} (7.52)

It is stressed that, Eq.(7.52) is exact in the averaged field because no assumptions have been made so far. Although one may argue with this claim, it can be explained in the same fashion with the Fourier, Laplace, or Z-transform because the filtering operation is recognized as one of those transformations. For instance, if the Fourier transform is applied to a governing equation in the actual field, then the transformed equation
should be an exact form in the wavenumber or frequency space. Likewise, Eq.(7.52) is also a transformed equation with the averaging operation (temporarily saying the Gaussian transformation), so that it is exact in the averaged field. It is necessary to mention that whether the inverse Gaussian transform exists is worth little consideration since all that matters here is in the averaged field. These relations are illustrated in Fig. 7.9.

![Diagram](image)

**Figure 7.9:** Actual field and Averaged field

The kinetic energy equation of the mean flow is obtained by multiplying the averaged momentum equation by the mean velocity. It was already formulated in Eq.(5.43) and is written as

$$
\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{u}_i \bar{u}_i \right) + \bar{u}_j \left( \frac{1}{2} \bar{u}_i \bar{u}_i \right)_j = (T_{ij} \bar{v}_j) - \nu \bar{u}_i \bar{u}_i + R_{ij} \bar{u}_i 
$$

(7.53)
where \( T_{ij} = -\frac{1}{\rho} \bar{p}_{ij} + \nu \bar{u}_{ij} - R_{ij} \). The interpretation of each term was given in Section 5.4 as well. In view of the ensemble average, Eq.(7.53) becomes

\[
\frac{\partial}{\partial t} \left< \frac{1}{2} \bar{u}_i \bar{u}_j \right> = -\nu \left< \bar{u}_i \bar{u}_j \right> + \left< R_{ij} \right> \quad (7.54)
\]

Comparing Eq.(7.54) to (7.51), the last term appears as an extra term and represents the effects of the turbulent stress, \( R_{ij} \), over a region. This term will be larger as the Reynolds number increases. It is necessary to pay attention to this extra term because it is expected to provide a connection with the turbulent energy variation.

The turbulent kinetic energy equation is formulated by subtracting Eq.(7.53) from Eq.(7.52). With the aid of the series expansions for the double and triple correlations, the resulting equation is

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} R_{ii} \right) + \bar{u}_j \left( \frac{1}{2} R_{ij} \right) = \left( -\frac{1}{\rho} \delta_j - \Psi_j + \frac{\nu}{2} R_{ij} \right) - \nu \Theta - R_{ij} \bar{u}_{ij} \quad (7.55)
\]

where

\[
R_{ii} = (2a)\bar{u}_{ik} \bar{u}_{ik} + \frac{1}{2l} (2a)^2 \bar{u}_{ik} \bar{u}_{ik} + \ldots
\]

\[
H_i = (2a)\bar{u}_{ik} \bar{u}_{ik} + \frac{1}{2l} (2a)^2 \bar{u}_{ik} \bar{u}_{ik} + \ldots
\]

\[
\Psi_j = \frac{1}{2l} (2a)^2 \bar{u}_{ik} (\bar{u}_i \bar{u}_j)_{ik} + \ldots
\]

and

\[
\Theta = (2a)\bar{u}_{ik} \bar{u}_{ik} + \ldots
\]

The detailed derivation and explanation of terms were shown in Section 6.1. The ensemble average equation is
\[
\frac{\partial}{\partial t} \langle \frac{1}{2} R_{ii} \rangle = -\nu \langle \Theta \rangle - \langle R_{ij} \bar{u}_{ij} \rangle
\] (7.56)

7.4.3 Energy Production and Dissipation

In the previous Section, the kinetic energy equation in the mean flow and the turbulent kinetic equation are derived and their ensemble averages are rewritten, for comparison, as

\[
\frac{\partial}{\partial t} \langle \frac{1}{2} u_{i} \bar{u}_{i} \rangle = -\nu \langle u_{ij} \bar{u}_{ij} \rangle + \langle R_{ij} \bar{u}_{ij} \rangle
\] (7.57)

\[
\frac{\partial}{\partial t} \langle \frac{1}{2} R_{ii} \rangle = -\nu \langle \Theta \rangle - \langle R_{ij} \bar{u}_{ij} \rangle
\] (7.58)

As briefly mentioned before, \(\langle R_{ij} \bar{u}_{ij} \rangle\) is important in the energy budget because it appears in both equations with an opposite sign. Indeed, it is recognized that this term allows energy to be exchanged between the mean flow and the turbulence. A negative value for this term means a loss of energy in the mean flow and a gain of energy in the turbulence. On the contrary, if this term has a positive value, the mean flow receives energy and the turbulence loses energy. The former case is in agreement with the existing isotropic turbulence theory in which the energy is transferred from the large scales to the small. However, at this moment, one can not find any evidence that it is negative definite and, thus, it is hardly said whether the contribution of \(R_{ij}\) to energy variation is dissipative or creative in the mean flow, even though Tennekekes and Lumley (1972) mentioned that, in a viewpoint of Reynolds stresses, this term becomes dissipative in most flows. It may be helpful, for example, to see the dissipative situation as observed in a one-dimensional turbulence equation from the Burgers’ equation (Burgers 1939, 1948, 1964). This Burgers’ equation is one of the limited non-linear equa-
tions which have an exact solution (Benton 1966, 1967, Jeng 1969, 1971). From the non-linearity, the solution shows the appearance of the shock. The typical solution with a single symmetric pulse initial condition is illustrated in Fig. 7.10 (Zabusky, 1968).

\[ t_0 < t_1 < t_2 \]

**Figure 7.10:** Solution of Burgers' equation (after Zabusky 1968)

In the shock, the velocity gradients become quite large with the negative sign as the time increase and the maximum slope is determined by the viscosity. In this case, the first leading term of \( R_{ij} \bar{u}_{ij} \) is

\[ \langle R_{ij} \bar{u}_{ij} \rangle = (2a) \langle \bar{u}_x^3 \rangle \]

(7.59)

At the initial time, \( t = t_0 \), this ensemble average will be zero because of its symmetry. As time goes on, the positive gradients decrease but the negative portion increases, so that the balance is broken and Eq. (7.59) has a negative value for \( t > t_0 \). As a result,
this term is dissipative. However, it should be noted that the creative feature can be observed locally and, in particular, the absence of the pressure term and the ignorance of the continuity constraint in this Burgers' equation may bring about the dynamical differences between the resultant solutions and the real turbulence. Therefore, although the dissipative nature in the Burgers' equation is observed, the possibility of a creative nature in the mean flow should not be excluded.

7.4.4 Equilibrium State

It may be instructive to explore the equilibrium energy state where the production of turbulence is equal to its dissipation. Clearly, the term, \( R_{ij} \hat{u}_{ij} \), is a parameter relating the mean flow kinetic energy and the turbulence. Eq.(7.57) says that, if \( <R_{ij}\hat{u}_{ij}> \) has the negative value, a steady state is not maintained without any external source of energy because all terms on the right hand side are negative. It indicates that, in order to maintain the steady state, the energy should be supplied to the turbulence as much as the mean flow energy decreases through the term of \( <R_{ij}\hat{u}_{ij}> \). Meanwhile, in view of the turbulent energy equation written in Eq.(7.58), the negative value of this parameter term implies the gain of energy from the mean flow. The received energy is finally dissipated into heat by the viscous term, \( <\Theta> \). Thus, an equilibrium state occurs at which the energy production (\( P \)) is balanced by the dissipation (\( \epsilon \)) and is given by

\[
P = \epsilon \tag{7.60}
\]

where

\[
P = - <R_{ij}\hat{u}_{ij}>
= - <[(2a)\hat{u}_{ik}\hat{u}_{jk} + \frac{1}{2^l}(2a)^2\hat{u}_{ik}\hat{u}_{jk}\hat{u}_{lk} + \cdots ]\hat{u}_{ij}> \tag{7.61}
\]
and

\[ e = \nu \theta > \\
= \nu < (2a)\bar{u}_{ikl} \bar{u}_{ikl} + \frac{1}{2l} (2a)^2 \bar{u}_{iklm} \bar{u}_{iklm} + \cdots > \]  

(7.62)

In the other case that \( <R_{ij}\bar{u}_{ij} > \) is positive, there does not exist a steady state in the turbulent energy equation as written in Eq.(7.58). Instead, the mean flow gains energy from the turbulence. Although it is not known whether such a situation may occur in real flows, the equilibrium state in this case is written as

\[ \nu <\bar{u}_{ij} \bar{u}_{ij} > = <(2a)\bar{u}_{ik} \bar{u}_{jk} + \frac{1}{2l} (2a)^2 \bar{u}_{iklm} \bar{u}_{iklm} + \cdots > \bar{u}_{ij} > \]  

(7.63)
CHAPTER VIII
COMPARISON OF $R_{ij}$ TO EXISTING CLOSURE MODELS

To date, the essence of turbulence modeling is how to parameterize the turbulence correlation terms appearing in the momentum equation or the scalar transport equations. Theoretical, statistical and empirical tools are used for closure analysis. However, it is really unfortunate that most models depend on the closures with empirical assumptions or hypotheses. Here, the YB-III series representation for the double correlation terms seems to provide a closure form quite different from the existing models. As discussed earlier, the resultant terms can be applied to both volume averages and Reynolds averages, providing that the averaging interval is adjusted properly. This Chapter is devoted to the comparison of the newly derived terms with the existing closure models.

8.1 CLOSURE MODELS EMPLOYING REYNOLDS AVERAGES

It is customary to classify the turbulence models derived from Reynolds averaging according to the number of transport equations employed for the unknown turbulence quantities. From reviews presented by Launder and Spalding (1972), Rodi (1980) and Bedford et al. (1987b) four groups are classified; zero-, one-, two-, and stress/flux equation models.

The initial turbulence model of Boussinesq in 1877 introduced the turbulent eddy viscosity, suggesting that the cross-correlation of fluctuating velocities is analogous to stress and is proportional to the mean velocity gradients. The general form is given by
\[- \overline{u_i u_j} = \frac{\tau_{ij}}{\rho} = \nu' S_{ij} \quad \text{(8.1)}\]

where \( \tau, \rho \) and \( \nu' \) denote stress, density and the eddy viscosity, respectively. The eddy viscosity is usually too large in turbulent flows to be ignored and it is difficult to get accurate values because it depends on the structure of the flow and, therefore, varies over the whole flow field.

Prandtl (1925) proposed the mixing length hypothesis, stating that the eddy viscosity is determined by the product of the square of a characteristic length scale in the turbulent motion and the mean velocity gradients. This length scale is known as the mixing length. According to Launder and Spalding (1972), this mixing length model is successfully used in boundary layer flows where the mixing length is proportional to the distance from the wall. However, Rodi (1980) pointed out that this model is presumably developed by assuming the local equilibrium so that it is not suitable when the advective/diffusive transports of turbulence or their historical effects are important. In short, the mixing length model does not take into account the influence of neighboring points in time and space. In spite of many limitations in its application and performance, this model is still recommended in many ways particularly because it gives a quite simple form of the turbulent stress.

To make up for the shortcomings of mixing length models, a one-equation model has been developed where the transport or history effects can be considered by solving differential transport equations for turbulence quantities. Dimensional analysis says that the eddy viscosity consists of a velocity scale \( (V) \) and a length scale \( (L) \), i.e.,
\[ \nu_t \propto V L \]  

(8.2)

In most turbulence closure models, the square root of the kinetic turbulence energy is usually selected as the velocity scale because it is considered to be the most significant parameter representing the character of the turbulence. When \( k \) denotes the kinetic turbulence energy, Eq.(8.2) is

\[ \nu_t = C_1 \sqrt{k} L \]  

(8.3)

where \( C_1 \) is an empirical constant. The kinetic energy distribution is determined by the following equation (Rodi, 1980);

\[ \frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{1}{2} \left[ \left( \frac{1}{\rho} \frac{\partial}{\partial t} + \frac{1}{2} u'_j \frac{\partial}{\partial x_j} \right) u'_i \right]_{,i} - u'_i u'_j \frac{\partial \overline{u'_i u'_j}}{\partial x_j} - \nu \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} \]  

(8.4)

For a detailed explanation of each term, see Rodi (1980). In the one-equation model, the eddy viscosity is assumed to be determined by solving this \( k \)-equation but the length scale is not considered in an analytic manner. Launder and Spalding (1972) insists that the length scale is equally important. To add this length scale, the following combination form of two scales is used;

\[ \nu_t = C_2 \frac{k^2}{\varepsilon} \]  

(8.5)

where \( C_2 \) is an empirical constant and \( \varepsilon \) denotes the energy dissipation. A typical form of the \( \varepsilon \) - equation is also given by Rodi (1980). It is
\[
\frac{\partial \varepsilon}{\partial t} + \mathbf{u}_j \varepsilon_j = - (\mathbf{u}_i \varepsilon)_i - 2 \nu \mathbf{u}_l \mathbf{u}_k \mathbf{u}_l \mathbf{u}_k - 2 \nu \mathbf{u}_l \mathbf{u}_l \mathbf{u}_k \mathbf{u}_k \quad (8.6)
\]

The two-equation model determines the eddy viscosity by solving both the \(k\)-equation in Eq.(8.4) and the \(\varepsilon\)-equation in Eq.(8.6). Thus, it is often called the \(k\)-\(\varepsilon\) equation model or the \(k\)-\(\varepsilon\) model.

It is emphasized that either the one- or two-equation models employ the Boussinesq concept for the eddy viscosity or diffusivity. Boussinesq's eddy viscosity is developed with the isotropic turbulence assumption (Beiford et al. 1987b) so that the above models should not consequently be appropriate for non-isotropic flows. The turbulent stress/flux equation model makes an effort to directly obtain the turbulence correlations without involving the eddy viscosity concept. Thus, it does not require the specification of either the velocity or the length scale as in the eddy viscosity. Instead, the transport equations for \(\overline{u_iu_j}\) and \(\overline{u_i\phi}\) are analytically formulated from a mathematical viewpoint. By multiplying the velocity components of the governing equations and making use of the Reynolds averaging rules, the following equations result (Abbott, 1982):

\[
\frac{\partial \overline{u_iu_j}}{\partial t} + \overline{u_i(u_j^\prime u^\prime)}_j = - (\overline{u_iu_j})_j - \frac{1}{\rho} [(\overline{u_i^\prime u^\prime}_i) + (\overline{u_i^\prime u^\prime})]_j - \overline{u_iu_ju_i} + \frac{1}{\rho} \overline{p(u_i^\prime u^\prime_i + u^\prime u^\prime)}_j - 2 \nu \overline{u_iu_ju_k} \quad (8.7)
\]

\[
\frac{\partial \overline{u_i\phi}}{\partial t} + \overline{u_i(u_i\phi^\prime)}_j = - (\overline{u_iu_j\phi})_j + \frac{1}{\rho} (\overline{u_j\phi u_i} + \overline{u_i\phi u_j}) - \overline{u_iu_j\phi} - \nu \overline{u_iu_j\phi} \quad (8.8)
\]
For detailed explanations of the involved terms, see Rodi (1980), Abbott (1982) and Bedford et al. (1987b). By solving these equations, individual stresses and fluxes are directly calculated at every time and space point. Furthermore, these equations are analytic and exact. This model promises better results than the others, but it is also expected to be computationally quite expensive. Thus, unless the significant improvement is guaranteed compared to other models (in particular, the $k$-$\varepsilon$ model), the intensive use of this model is, in practice, awkward (Rodi 1980, Bedford et al. 1987).

8.2 COMPARISON OF $R_{ij}$ TO REYNOLDS AVERAGE CLOSURES

In the previous Section, the existing closure models are briefly reviewed. Indeed, various models specifying the turbulence correlations ($\overline{u_i u_j}$) exist, but are still far from being satisfactory. As shown earlier, all the models, except the turbulent stress/flux model, assume the Boussinesq's eddy viscosity concept which is not formulated from an analytic point of view, and therefore contain unknown correlations which must be specified by models of the correlations. It has been reported that several terms of these correlations have been successfully modeled with experimental data and/or hypotheses (Prandtl 1925, 1945, Kolmogorov 1941, Bradshaw et al. 1967, Rodi 1980). However, it is difficult to expect that all the terms can be always modeled with high accuracy, in particular, the triple correlation terms in all transport equations for the turbulence quantities. Unless these triple correlations are specified, the transport equations are not solvable. For example, the turbulence energy ($k$) equation is not closed until the triple correlation term ($\overline{u_i u_i u_j}$) in the $k$-equation is known. If $k$ is not exactly determined, then neither is the eddy viscosity. When one remembers that the purpose of the $k$-equation was to determine the double correlation terms ($\overline{u_i u_j}$), the appearance of the triple correlations is certainly an irony because the specification of
them is expected to be much more complicated. Of course, other transport equations for the triple correlations can be formulated in the same way but one can expect that the resulting equations contain correlations higher than the triple. The new method offered here suggests an alternate way to close the equation models and it is natural to ask how the method here relates these earlier turbulence model formulation.

Now, consider the relation between the $k$-equation and $R_{ij}$. In fact, the $k$-equation was developed for the purpose of obtaining the turbulence kinetic energy, which is written by

$$k = \frac{1}{2} \overline{u_i u_i},$$  \hspace{1cm} \text{(8.9)}

By the definition of $R_{ij}$, this quantity is equivalent to

$$\frac{1}{2} R_{ii} = \frac{1}{2} \left[ (2a) \overline{u_i u_i} + \frac{1}{2l} \overline{(2a)^2 u_{ki} u_{ki} + \cdots} \right]$$

$$= \frac{1}{2} [A_{ii} + C_{ii}]$$  \hspace{1cm} \text{(8.10)}

in which the tensors, $A_{ij}$ and $C_{ij}$ were already defined in Section 7.3. In the one- or two-equation model the distribution of $k$ is obtained by directly solving the $k$-equation, but Eq.(8.10) indicates that it can be simply defined by $R_{ij}$. Indeed, if it is valid, it would not be necessary any longer to solve the $k$-equation. In the same manner, the dissipation rate ($\epsilon$), turbulent stress ($\overline{u_i u_j}$) and turbulent flux ($\overline{u_i \Phi^t}$) can be directly obtained from the definition of $R_{ij}$ or $Q_i$ without any transport equation for them. They are written as
\[ \varepsilon = \nu \overline{u_i u_j} \]

\[ = \nu \left\{ (2a) \overline{u_{i,j,k}} \overline{u_{j,k}} + \frac{1}{2l} (2a)^2 \overline{u_{i,k,l}} \overline{u_{j,k,l}} + \cdots \right\} \]  \hspace{1cm} (8.11)

\[ \overline{u_i u_j} = (2a) \overline{u_{i,k}} \overline{u_{j,k}} + \frac{1}{2l} (2a)^2 \overline{u_{i,k,l}} \overline{u_{j,k,l}} + \cdots \]  \hspace{1cm} (8.12)

and

\[ \overline{u_i \Phi} = (2a) \overline{u_{i,k}} \overline{\Phi_{,k}} + \frac{1}{2l} (2a)^2 \overline{u_{i,k,l}} \overline{\Phi_{,k,l}} + \cdots \]  \hspace{1cm} (8.13)

At present, the relationship between the turbulence transport equations and Eq.(8.10) - (8.13) is not known, but it may be believed that these turbulence quantities are closely related to the solutions of those equations, providing that all terms in equations are accurately defined.

Substituting the turbulent energy in (8.10) into the eddy viscosity in Eq.(8.3), one gets the following eddy viscosity for the one-equation model.

\[ \nu_t = C_1 \left\{ \frac{1}{2} \left( (2a) \overline{u_{i,k}} \overline{u_{j,k}} + \frac{1}{2l} (2a)^2 \overline{u_{i,k,l}} \overline{u_{j,k,l}} + \cdots \right) \right\}^{1/2} L \]

\[ = C_1 \left\{ \frac{1}{2} \left( A_{ij} + C_{ij} \right) \right\}^{1/2} L \]  \hspace{1cm} (8.14)

where \( C_1 \) is a constant and \( L \) denotes the length scale. The turbulent stress resulting from Eq.(8.14) is written as

\[ \frac{\tau_{ij}}{\rho} = C_1 \left\{ \frac{1}{2} \left( (2a) \overline{u_{m,n}} \overline{u_{m,n}} + \frac{1}{2l} (2a)^2 \overline{u_{m,n,l}} \overline{u_{m,n,l}} + \cdots \right) \right\}^{1/2} L S_{ij} \]

\[ = C_1 \left\{ \frac{1}{2} \left( A_{mn} + C_{mn} \right) \right\}^{1/2} L S_{ij} \]  \hspace{1cm} (8.15)
where $S_{ij}$ denotes the strain rate tensor. To show what the physical meanings of the above representation are, a parallel flow, shown in Fig. 8.1, will be considered as an example.

![Figure 8.1: A parallel flow](image)

In this flow, Eq.(8.15) is given by

$$\frac{\tau_{12}}{\rho} = \frac{C_1}{\sqrt{2}}[(2a)x_1^2 + \frac{1}{2l}(2a)^2u_{xy}^2 + \cdots ]^{1/2} L u_y$$

(8.16)

Here, recall that the coefficient in the bracket consists of the filter width presumably to be determined by the character of the turbulence. This filter width is the same as the length scale $L$, so that one may combine these two length scales. Assuming that these two length scales are combined and the first leading term in the bracket of Eq.(8.16) is
only considered, one gets
\[ \frac{\tau_{12}}{\rho} = C l^2 \bar{u}_y \bar{u}_y \]  
(8.17)
in which \( C \) is a constant and \( l \) denotes the resulting length scale after combining the filter width \( (\Delta) \) with the turbulent length scale \( (L) \), i.e., \( l = (\Delta L)^{1/2} \). It is quite surprising that Eq.(8.17) is exactly identical to the turbulent shear stress resulting from the mixing length hypothesis, which is written as
\[ \frac{\tau_{xy}}{\rho} = l_m^2 \bar{u}_y \bar{u}_y \]  
(8.18)
where \( l_m \) is the mixing length. The mixing length theory was developed by adaptation of the kinetic theory of gases in which the shear stress is explained in terms of a molecular transport. It states that the velocity difference between two thin layers causes the instantaneous velocity component \( (v') \) and, consequently, the momentum in the layers is exchanged. By this change of momentum shearing stress occurs. In addition, the transverse velocity can be regarded as the turbulent velocity component and is assumed to be correlated with the fluctuating velocity along the stream line. Based on the correlation of two fluctuating velocities, Prandtl (1925) proposed the mixing length hypothesis written in Eq.(8.18). Clearly, Eq.(8.17) and (8.18) show the existence of similarity between the mixing length theory and the YB-III series. Of course, it cannot be said that the mixing length is described directly in terms of the filter width. However, both length scales may depend on the properties of the flow such as the turbulent intensity.
On the other hand, Eq. (8.18) shows that the turbulent stress vanishes wherever the velocity gradient becomes zero, e.g., at the middle of a pipe. Certainly, this is not the case because turbulent mixing can exist at points of the maximum velocity. To make up for such difficulties, the following semi-empirical formula is often used. (Le Mehaute 1976, Schlichting 1968).

\[
\frac{\tau_{xy}}{\rho} = l_m^2 \left[ \bar{u}_y^2 + l_1^2 \bar{u}_{yy}^2 \right]^{1/2} \bar{u}_y
\]  

(8.19)

where \( l_1 \) is another length scale different from the mixing length, \( l_m \). From Eq. (8.16) and including the second-order term, one gets

\[
\frac{\tau_{12}}{\rho} = C_1 l^2 \left[ \bar{u}_y^2 + l_1^2 \bar{u}_{yy}^2 \right]^{1/2} \bar{u}_y
\]  

(8.20)

where \( l \) is the combined length scale as in Eq. (8.17) and \( l_1 \) is proportional to the filter width. Comparing Eq. (8.20) to (8.19), both forms are also identical. Nevertheless, it is hardly believed that these similarities result by chance. The above observation also shows that there is not much difference between the mixing length hypothesis and the one-equation model if \( \frac{1}{2} R_{ii} \) is admitted to be the solution of the k-equation. They are considered to be closely related. In a similar way, the dissipation rate (\( \epsilon \)) used in the two-equation model is also described;

\[
\epsilon = \nu \left[ \bar{u}_{ij} \bar{u}_{ij} \right]
\]

\[
= \nu \left[ (2a) \bar{u}_{ik} \bar{u}_{jk} + \frac{1}{2l} (2a) \bar{u}_{ikl} \bar{u}_{ijl} + \cdots \right]
\]

(8.21)

The corresponding eddy viscosity resulting from the k-\( \epsilon \) model is
\[ \nu_t = \frac{C}{4} \left[ \frac{(2a)^2 \overline{u}_{ik} \overline{u}_{lk} + \frac{1}{2!}(2a)^2 \overline{u}_{kk} \overline{u}_{kk} + \cdots }{\nu \left( (2a)^2 \overline{u}_{jk} \overline{u}_{jk} + \frac{1}{2!}(2a)^2 \overline{u}_{kk} \overline{u}_{kk} + \cdots \right)} \right] \] (8.22)

In the parallel flow, the first leading term in Eq.(8.22) gives

\[ \nu_t \propto \frac{\overline{u}_y^4}{\overline{u}_{yy}^2} \] (8.23)

The ratio of the first to the second spatial derivatives of the mean velocity in Eq.(8.23) is identical to von Karman's similarity hypothesis (1934). Although his approach is totally different, this similar form is quite interesting and compelling.

The above similarities show that the YB-III series has a high probability of representing the turbulence effects. At least, it can not be said that the newly derived description is nothing to do with the existing closure models. The new series description makes the mixing length hypothesis more clear by providing a general form of the turbulent shear stress. Certainly, the YB filtering procedure produces the distinctive closure terms in which it is not necessary to solve the transport equation for the turbulence quantities.

### 8.3 COMPARISON OF R_{ij} TO SGS CLOSURES FROM VOLUME AVERAGES

Most of the SGS models in LES has been developed based on the closures in the Reynolds averaging. According to Voke and Collins (1983), the SGS models in LES are classified into four groups: constant, stress, vorticity and SGS energy models. Their mathematical forms are written, respectively, as
\begin{align*}
\nu_t &= c_o \quad (\text{constant model}) \quad (8.24) \\
\nu_t &= c_1 S L^2 \quad (\text{stress model}) \quad (8.25) \\
\nu_t &= c_2 \omega L^2 \quad (\text{vorticity model}) \quad (8.26) \\
\nu_t &= c_3 k^{1/2} L \quad (\text{SGS energy model}) \quad (8.27)
\end{align*}

in which \( S^2 = S_{ij} S_{ij}, \omega^2 = \omega_j \omega_j, \) \( k \) denotes the SGS energy and \( L \) is the length scale usually associated with the filter width or grid spacing. These models have all been derived from empirical or non-dimensional analysis. As the simplest one, the constant model assumes the same eddy viscosity over a whole flow domain and, thus, it will be valid in the homogeneous turbulence or in the flow with extremely low Reynolds numbers. The stress model in Eq.(8.25) was initially proposed by Smagorinsky (1963, 1965) and the vorticity model in Eq.(8.26) is the modified one of the stress models. The SGS energy model in Eq.(8.27) was suggested by Schumann (1975,1976), on the grounds that the turbulence energy is most significantly involved in the eddy viscosity, and it is in accordance with the turbulence kinetic energy concept in the \( k \)- or the \( k-\epsilon \) equation models for the closure terms in the Reynolds averaging.

Although all the models have been successfully employed in LES, it is known that the Smagorinsky's model is generally better than any others (Clark et al. 1977, Ferziger 1977). Smagorinsky (1965) suggested that, in a numerical model, the SGS eddy viscosity is proportional to the product of the magnitude of resolved stresses and the square of the grid interval. At the beginning, this formulation was intended to apply to the quasi two-dimensional large scale atmospheric general circulations and, later, it is extended to three-dimensional turbulent motions by Lilly (1966). He also estimated theoretically the proportionality coefficient by considering the turbulent energy within the inertial
 subrange. So it is often called the Smagorinsky-Lilly model. This model gives the SGS stress, \( \tau_{ij} \), such that

\[
\frac{\tau_{ij}}{\rho} = (c_1 \Delta^2) \sqrt{S_{mn}S_{mn}} S_{ij}
\]  

(8.28)

where \( \Delta \) is the grid spacing. From Eq.(8.28), the eddy viscosity is

\[
\nu_t = c_1 \Delta^2 \sqrt{S_{mn}S_{mn}}
\]

(8.29)

Here, it may be convenient to use the notation of the norm, which defines the magnitude of a function or matrix. By this notation, Eq.(8.29) is

\[
\nu_t = c_1 \Delta^2 \| S \|
\]

(8.30)

where \( \| S \| = \sqrt{S_{mn}S_{mn}} \). As shown in Section 7.3, \( R_{ij} \) consists of three symmetric tensors such as \( A_{ij}, B_{ij} \), and \( C_{ij} \). For convenience, \( R_{ij} \) in Eq.(7.43) is rewritten as

\[
R_{ij} = A_{ij} + B_{ij} + C_{ij}
\]

(8.31)

where

\[
A_{ij} = (2a) S_{ik} S_{jk} + \frac{1}{2i} (2a)^2 S_{ik,l} S_{jk,l} + \cdots
\]

\[
B_{ij} = (2a \Omega_{ik} S_{jk} + S_{ik} \Omega_{jk}) + \cdots
\]

and

\[
C_{ij} = (2a) \Omega_{ik} \Omega_{jk} + \cdots
\]

With the matrix form, the only leading term in \( A_{ij} \) leads to
\[ [A] = \frac{1}{2\gamma} \Delta^2 [S][S] \]  
(8.32)

By Eq.(8.28), the Smagorinsky model is also written as

\[ \frac{1}{\rho} [\tau]= c_j \Delta^2 ||S|| [S] \]  
(8.33)

Comparing Eq.(8.33) to Eq.(8.32), both equations are related to the deformation matrix, 
\([S]\), with the same coefficients by the square of the filter width. The only difference
is that the norm of the matrix is used in the Smagorinsky model, instead of matrix
itself. Physically, the norm of \([S]\) is considered to represent the scalar magnitude of
the resolved stress. At present, although one can not know what the substitution of the
norm means or how much their difference is, this similarity implies that \(A_{ij}\) is strongly
related to the Smagorinsky model.

Meanwhile, this model has the weakness that it does not contain the effect of rotation,
which is thought of as an indispensable feature in turbulence. To make up for
this shortcoming, a vorticity model quite similar to the Smagorinsky model is proposed
(Kwak et al. 1975, Mansour et al. 1977, Clark et al. 1979);

\[ \nu_t = c_2 \Delta^2 \sqrt{\omega_m \omega_m} \]  
(8.34)

where \(\omega\) is the vorticity. Eq.(8.34) is nothing but the modification of the Smagorinsky
model, by replacing the deformation tensor with the vorticity vector. In the same way,
this vorticity model gives the residual stress by
\[
\frac{1}{\rho} \tau = c_2 \Delta^2 \| \omega \| [S] \tag{8.35}
\]

where \( \| \omega \| \) denotes the norm representing the scalar magnitude of the vorticity. It is clear that Eq.(8.35) is associated with \( B_{ij} \) as showing the interaction between rotation and deformation of the local fluid elements;

\[
[B] = \frac{\Delta^2}{2\gamma} (\Omega[S] + [S][\Omega]) \tag{8.36}
\]

Of course, these relations are not exactly in agreement with from a mathematical point of view. However, when one recognizes either the Smagorinsky or vorticity models are not analytically formulated, it may be excessive to require the exact forms in making a comparison of these models. Moreover, if it is admitted that these models are not completely satisfied in the general flow situation and still need more modifications, it would be immoderate to deny and get rid of these qualitative similarities simply because they are not equivalent in a mathematical representation. Furthermore, not only the effects of the higher derivative terms in \( A_{ij} \) and \( B_{ij} \) are not known but the last term \( C_{ij} \) is not considered yet. In fact, while \( R_{ij} \) consists of the products of matrices, the Smagorinsky and the vorticity model have the product term of matrices and their norms.

As an alternative explanation, it is necessary to pay attention to what these norms represent in real physics; the norms in the Smagorinsky and the vorticity model may be interpreted as the local energy by the deformation and by the pure rotation of the fluid elements, respectively. As regards using the energy quantities for the eddy viscosity, the Smagorinsky and the vorticity model are similar to the SGS energy model written in Eq.(8.27). Thus, it would be helpful, for understanding their relationships, to
describe the turbulence kinetic energy in terms of the YB-III series representation. The turbulence kinetic energy is, by definition, given by

\[ k = \frac{1}{2} R_{ij} \]

\[ = \frac{1}{2} [(2a)\overline{\nu}_{ik}\overline{\nu}_{ik} + \frac{1}{2!}(2a)^2\overline{\nu}_{ikl}\overline{\nu}_{ikl} + \cdots ] \]  

(8.37)

Substituting Eq.(8.37) into Eq.(8.27), one can get

\[ \nu_t = \frac{c_3}{\sqrt{2}} \left[ (2a)\overline{\nu}_{ik}\overline{\nu}_{ik} + \frac{1}{2!}(2a)^2\overline{\nu}_{ikl}\overline{\nu}_{ikl} + \cdots \right]^{1/2} L \]  

(8.38)

Note that Eq.(8.38) is identical to the eddy viscosity description in Eq.(8.14) for one-equation model in the Reynolds averaging, in which the velocity scale is replaced by the square root of the turbulence kinetic energy which is assumed to be able to most significantly represent the characteristics of the turbulence. When only the first leading term in Eq.(8.38) is considered, the eddy viscosity is

\[ \nu_t = c' \Delta (\overline{\nu}_{ik}\overline{\nu}_{ik})^{1/2} L \]  

(8.39)

where \( c' \) is a constant. Then, the Boussinesq concept for the turbulent shear stress leads to

\[ \frac{\tau_{ij}}{\rho} = c' \Delta (\overline{\nu}_{mn}\overline{\nu}_{mn})^{1/2} L S_{ij} \]

(8.40)

With the decomposition of \( R_{ij} \) in Eq.(8.31), Eq.(8.40) becomes
\[ \frac{\tau_{ij}}{\rho} = c'\Delta \left[ S_{mn} S_{mn} + \Omega_{mn} S_{mn} + S_{mn} \Omega_{mn} + \Omega_{mn} \Omega_{mn} \right]^{1/2} L S_{ij} \]  
\hspace{10cm} (8.41)

As shown earlier, the trace of the second and third term in the parenthesis vanish by Eq. (7.49) and the last term can be rewritten in terms of the vorticity by Eq. (7.46). The resulting equation is

\[ \frac{\tau_{ij}}{\rho} = c(\Delta L) \sqrt{S_{mn} S_{mn} + 0.5 \omega_m \omega_n} S_{ij} \]  
\hspace{10cm} (8.42)

Assuming that the length scale, \( L \), is related to the filter width and these two length scales are combined, the final equation becomes

\[ \frac{\tau_{ij}}{\rho} = C \Delta^2 \sqrt{S_{mn} S_{mn} + 0.5 \omega_m \omega_n} S_{ij} \]  
\hspace{10cm} (8.43)

To surprise, Eq. (8.43) is completely identical to the Smagorinsky model if the vorticity term is neglected, and it is also identical to the vorticity model if deformation term is removed. Therefore, this procedure provides a full of theoretical derivations of both the Smagorinsky and the vorticity models, and Eq. (8.43) makes it clear what the Smagorinsky and the vorticity models are. It is obvious that Eq. (8.43) is the general form of turbulence energy models and, furthermore, three types of models ( Smagorinsky, vorticity and SGS energy models ) are not distinctive at all in view of the turbulence energy. As a result, it is expected that Eq. (8.43) would be much better than other models. Certainly, this complete similarity is enough to show the validity of the YB filtering procedure.
CHAPTER IX
INITIAL TEST OF THE NEW SGS MODEL

9.1 DATA TO BE USED

It is necessary to show the validity of the newly developed model for the turbulence. This is difficult as analytic solutions are not often found and, therefore, the SGS model must be verified by means of comparison with the available experimental information. However, there are often many difficulties in getting data suitable for evaluating the SGS model in this research since the smallest scale activities need to be measured with sufficient time and space accuracy. An alternative verification is to compare results computed with the method level to computational results from a direct simulation of turbulence as done with a sufficiently fine grid size and time step. Since all the scales in the fluid motion are resolved in the direct simulation, the resulting solution is regarded as being exact.

Initially, in many LES models, the one-dimensional analogy of the Navier-Stokes equations, the Burgers equation, has been often used to evaluate proposed models (Love and Leslie 1977, Love 1980, Dakhoul and Bedford 1986b, and Aldama 1986). Certainly, the allowable model resolution of the one- or two-dimensional model equations is much greater than that of the general three-dimensional flows simulated on present computers. However, these simplified equations are inappropriate for the present purpose particularly because they do not completely contain the vorticity effects which are considered to be significant in the SGS terms.
Recently, Lee and Reynolds (1985) completed a comprehensive database on homogeneous turbulence which is maintained at the NASA-Ames Research Center for the purpose of promoting research on turbulence. This three-dimensional computational data set is suitable for an initial test of the SGS model developed in this research and, in this Chapter, the SGS model resulting from the YB filtering approach is evaluated with these numerical data.

The data set used in this dissertation was computed numerically by solving the three-dimensional time dependent exact Navier-Stokes equations without any turbulence model. This full turbulence simulation was performed on a 128x128x128 mesh with the same grid interval in all directions. The pseudo-spectral method for spatial differentiation and the second-order Runge-Kutta method for time advancement were used along with periodic boundary conditions. The initial velocity distribution assumes that the energy spectrum function consists of the $k^2$ slope in a low wavenumber region and the $k^{-5/3}$ in the inertial subrange for equilibrium turbulence, where $k$ denotes the wavenumber. The computation was also carried out over 1200 time steps and statistical comparisons of these results to existing theory show that these data reproduce homogeneous turbulence characteristics quite well. Note that the computed velocity field is initially stored in the form of Fourier components but it has been inversely transformed to real space for use here. More details are formed in the report by Lee and Reynolds (1985).

To test the SGS model, one realization of velocity field was selected. According to the Batchelor's experimental observation (1953,1967), the derivative skewness of low Reynolds number wind tunnel turbulence is approximately equal to -0.4, i.e.,
\[ SK = -\frac{\langle u_3^3 \rangle_{u_i}}{\langle u_i^2 \rangle_{u_i}^{3/2}} = -0.4 \quad ( \text{no sum}) \quad (9.1) \]

where SK denotes the skewness. Based on this observation, the turbulence is considered fully developed when the skewness is -0.4 and, as a consequence, a velocity field with this skewness was selected for investigating the SGS quantities. The variation of skewness with respect to the time is presented in Fig. 9.1, together with the evolution of the Reynolds number. It shows that the skewness is close to -0.4 around time step 700 and, thus, the velocity field at \( N = 700 \) is selected. Note that the data set used here is termed HIE by Lee and Reynolds (1985) and the arrow in Fig. 9.1 indicates the skewness and the Reynolds number at \( N = 700 \).
Figure 9.1: Skewness vs time steps and time evolution of Reynolds numbers

\[ 1 = \frac{q^3}{\epsilon} \]

- ○, HIA; ▽, HIB; △, HIC; □, HID; ☐, HIE.

( after Lee and Reynolds 1985 )
9.2 THE AVERAGING PROCEDURE

The experimental velocity field is given on a $128 \times 128 \times 128$ mesh with a cube. The filtering computations are performed on two coarse grid cases with grids consisting of $32 \times 32 \times 32$ and $16 \times 16 \times 16$ nodes. The $16^3$ coarse mesh indicates that 8 times larger a grid interval of the original fine mesh is used and, thus, each cell of the coarse mesh contains the 512 fine grid points. The $32^3$ coarse mesh has 4 times larger a grid interval than the fine mesh. The fine mesh and the coarse mesh are illustrated in Fig. 9.2.

The filtered velocity quantity is calculated by using the Gaussian filter function with the isotropic filter width because the full simulation was performed with a $1:1:1$ mesh configuration. The filter constant, $\gamma$, is chosen to be 6.0 and the filter width ($\Delta$) by $2\delta x$ where $\delta x$ is the coarse grid interval. By the Gaussian filtering definition, the averaged value at a point $(i, j, k)$ is given by

$$
\bar{u}(i, j, k) = \left(2\pi\right)^{3/2} \frac{1}{\Delta^3} \sum_{i'} \sum_{j'} \sum_{k'} e^{-\frac{\gamma^2((i'-i)^2+(j'-j)^2+(k'-k)^2)}} u(i-i', j-j', k-k')
$$

(9.2)

Eq. (9.2) shows that one has to integrate over an entire domain. The variation that one experimental value ($u$) at a fine mesh location contributes to the averaged value ($\bar{u}$) is dependent upon the distance from the origin $(i, j, k)$. The degree of contribution by the points at the same radius from $(i, j, k)$ will be the same and decays exponentially as the radius becomes large. In short, the neighboring points around the core are significant but points remote from the origin contribute very little. This is illustrated in Fig. 9.3. Based on the degree of contribution, the finite integrating limits in Eq. (9.2) can be determined, instead of its infinity. Note that $u$ and $\bar{u}$ should be the same if the velocity field is constant and is integrated with the infinite integral limits in Eq. (9.2). Prelimi-
16x16x16 coarse grid case

Figure 9.2: Fine mesh and coarse mesh
nary calculations show that the averaged value can be obtained to more than 99.1% accuracy by integrating over all points within a sphere whose radius is the filter width (Δ). Fig.9.4 presents the integrating interval with regard to x-axis in a 16³ coarse mesh case. With this integrating interval, the averaged value in a 16³ mesh is approximated by

\[
\bar{u}(i,j,k) = \left( \frac{2}{\pi} \right)^{3/2} \frac{1}{\Delta^3} \sum_{i'=-16}^{16} \sum_{j'=-16}^{16} \sum_{k'=-16}^{16} e^{-\frac{2}{\Delta^2}(i'^2+j'^2+k'^2)} u(i-i',j-j',k-k') \quad (9.3)
\]

where \( i'^2 + j'^2 + k'^2 \leq 16^2 \)

The average in a 32³ node grid is calculated in the same manner. From Eq.(9.3), 17041 fine grid points are necessary for obtaining the averaged value at a coarse grid point. The periodic condition is used at the boundary. The averaged non-linear term (for example, \( \bar{uv} \)) is also obtained in the same way, i.e.,

\[
\bar{uv}(i,j,k) = \left( \frac{2}{\pi} \right)^{3/2} \frac{1}{\Delta^3} \sum_{i'=-16}^{16} \sum_{j'=-16}^{16} \sum_{k'=-16}^{16} e^{-\frac{2}{\Delta^2}(i'^2+j'^2+k'^2)} uv(i-i',j-j',k-k') \quad (9.4)
\]

where \( i'^2 + j'^2 + k'^2 \leq 16^2 \)

To test the SGS quantities, the three averaged velocity components (\( \bar{u}, \bar{v}, \text{and} \bar{w} \)) are calculated by Eq.(9.3) and the six averaged non-linear terms (\( \bar{uu}, \bar{uv}, \bar{uw}, \bar{vv}, \bar{vw}, \text{and} \bar{ww} \)) by Eq.(9.4). Fig. 9.5 - 9.7 present the unaveraged and the averaged velocity fields on an arbitrary x-y plane.
Figure 9.3: The contribution to the averaged value

Figure 9.4: Integrating interval
Figure 9.7: Averaged velocity field (16x16x16)
9.3 EVALUATING OF THE SGS MODEL

After completely calculating the averaged quantities, the SGS terms are investigated by showing how well the SGS model is correlated. In general, the correlation coefficient between two observations (X and Y) is defined as

\[
R(X,Y) = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum(X_i - \bar{X})^2 \sum(Y_i - \bar{Y})^2}} \tag{9.5}
\]

where \( \bar{X} \) and \( \bar{Y} \) denote their statistical mean values, respectively. If \( X \) is perfectly correlated to \( Y \), the absolute value of \( R(X,Y) \) is equal to 1.0 and, if \( X \) and \( Y \) are totally unrelated, \( R(X,Y) \) becomes zero. Frequently, it is more convenient to use the R-square form, instead of \( R \), because of the appearance of the negative sign in Eq.(9.5). To get the R-square values the Ohio State University Computer Center Statistical Analysis System (SAS) subroutines for the general linear model are used.

9.3.1 Testing of the Residual Turbulence Terms

The averaged non-linear terms resulting from the YB-III series are rewritten, for convenience, as

\[
u_i \hat{\nu}_j = \hat{\nu}_i \hat{\nu}_j + (2a)\hat{\nu}_{ik} \hat{\nu}_{jk} + \frac{1}{2l}(2a)^2 \hat{\nu}_{ik} \hat{\nu}_{jk} + \cdots \tag{9.6}
\]

The central difference fourth-order scheme has been used in evaluating all of the derivative terms since the second residual terms in Eq.(9.6) are fourth order. Recall that the filtering definition here is developed with the moving average and, thus, it can be applied to a coarse mesh as well as a fine mesh. However, the differentiation was done on the coarse mesh because the coarse mesh is used in a real simulation.
The R-square values are computed for both the $32^3$ and $16^3$ meshes for the following three cases:

Case(1) \[ \overline{u_i u_j} \leftrightarrow \overline{u_i u_j} \]

Case(II) \[ \overline{u_i u_j} \leftrightarrow \overline{u_i u_j} + (2a)\overline{u_{ik} u_{jk}} \]

Case(III) \[ \overline{u_i u_j} \leftrightarrow \overline{u_i u_j} + (2a)\overline{u_{ik} u_{jk}} + \frac{1}{2a}(2a)^2\overline{u_{ikl} u_{jkl}} \]

In each case, the six independent R-square values result because of their symmetry. The detailed results are tabulated in Table 9.1 - 9.3. In both coarse meshes, the R-square values significantly increase by employing the residual turbulence terms resulting from the YB-III series. The improvement from including the first terms of the series is most significant and the effect of the second terms in the series is not as great. However, the most improved model results from including both of them. It is not surprising because it was anticipated in Chapter VII where the first terms were shown to be dominant. When only the first terms of the series \((2a u_{ik} u_{jk})\) are employed, the mean R-square values increase by 0.12 and 0.16 in the $32^3$ mesh and the $16^3$ mesh, respectively, while the second terms enhance the correlations by 0.02 and 0.04. This indicates that the SGS terms are more important in the coarser mesh.
Table 9.1
R-square values: Case (I)

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.758 0.803 0.793</td>
<td>0.451 0.552 0.506</td>
</tr>
<tr>
<td>0.803 0.784 0.791</td>
<td>0.552 0.513 0.489</td>
</tr>
<tr>
<td>0.793 0.791 0.757</td>
<td>0.506 0.489 0.452</td>
</tr>
</tbody>
</table>

mean = 0.786

mean = 0.501

Table 9.2
R-square values: Case (II)

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.886 0.913 0.915</td>
<td>0.590 0.700 0.661</td>
</tr>
<tr>
<td>0.913 0.906 0.912</td>
<td>0.700 0.698 0.654</td>
</tr>
<tr>
<td>0.915 0.912 0.885</td>
<td>0.651 0.654 0.610</td>
</tr>
</tbody>
</table>

mean = 0.906

mean = 0.659

Table 9.3
R-square values: Case (III)

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.906 0.931 0.934</td>
<td>0.624 0.730 0.700</td>
</tr>
<tr>
<td>0.931 0.923 0.932</td>
<td>0.730 0.729 0.693</td>
</tr>
<tr>
<td>0.934 0.932 0.905</td>
<td>0.700 0.693 0.646</td>
</tr>
</tbody>
</table>

mean = 0.925

mean = 0.694
In each case, significant differences between the R-square values for the nine components or for the diagonal and the off-diagonal terms are not found. However, the R-square values on the $32^3$ mesh are, in general, higher than those on the $16^3$ mesh. This is probably because the small scale activities are eliminated more in the smoother velocity field produced by the flatter Gaussian filter function and these eliminated quantities are maintained in the SGS terms. This claim is partially supported by the observation above that the SGS terms in the $16^3$ mesh play a more important role than those in the $32^3$ mesh, although their differences are not significant. Certainly, the R-square values confirm that the significantly improved model can be obtained by considering the YB-III series terms.

9.3.2 Comparison of the $A_{ij}$, $B_{ij}$, and $C_{ij}$ terms

In the previous Section, it was shown that the first terms of the series dominate on the SGS model and that only the first two terms appear necessary for turbulence model improvements. As discussed in Chapter VII, the residual stress tensor $R_{ij}$ is decomposed into the three different tensors $A_{ij}, B_{ij}$ and $C_{ij}$. It is instructive to examine how significant these three terms are in the residual stress. Assuming that the high-order series terms are small enough to be neglected, the relationship between the three terms is given by

$$
\bar{u}_{ik} \bar{u}_{jk} = A_{ij} + B_{ij} + C_{ij}
$$

(9.7)

where $A_{ij} = S_{ik} S_{jk}, B_{ij} = \Omega_{ik} \Omega_{jk} + \Omega_{ik} S_{jk},$ and $C_{ij} = \Omega_{ik} \Omega_{jk}.$
The R-square values are calculated for the following three cases and their detailed results are presented in Tables 9.4 - 9.6.

Case (A): $\bar{u}_{ik} \bar{u}_{jk} \leftrightarrow A_{ij}$

Case (B): $\bar{u}_{ik} \bar{u}_{jk} \leftrightarrow B_{ij}$

Case (C): $\bar{u}_{ik} \bar{v}_{jk} \leftrightarrow C_{ij}$

---

**Table 9.4**

R-square values: Case (A)

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.240 0.167 0.157</td>
<td>0.224 0.131 0.146</td>
</tr>
<tr>
<td>0.167 0.277 0.159</td>
<td>0.131 0.338 0.168</td>
</tr>
<tr>
<td>0.157 0.159 0.255</td>
<td>0.146 0.168 0.248</td>
</tr>
</tbody>
</table>

mean = 0.193  mean = 0.189

---

**Table 9.5**

R-square values: Case (B)

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.387 0.655 0.663</td>
<td>0.362 0.668 0.657</td>
</tr>
<tr>
<td>0.655 0.449 0.681</td>
<td>0.668 0.491 0.700</td>
</tr>
<tr>
<td>0.663 0.681 0.565</td>
<td>0.657 0.700 0.331</td>
</tr>
</tbody>
</table>

mean = 0.578  mean = 0.581
The highest correlation is found in case (B), the lowest one in case (A) and case (C) is in the middle. However, the R-square values in (A) or (C) are considerably below those in (B). The significance or importance of case (B) is shown in both the $32^3$ and $16^3$ mesh cases. It indicates that the tensor $B_{ij}$ is the most important determinant of the residual stress out of the three terms. In Chapter VII, it was suggested that the tensor $B_{ij}$ is important in turbulence modeling since it represents the vortex stretching which is associated with the change of length scales. The above calculation suggests that the vortex stretching term is too essential in the SGS model to be neglected. In particular, the off-diagonal components have larger R-square values than the diagonal terms. This implies that the vortex stretching is more important to the turbulent shear stress than to the dynamic pressure. Further, it is recognized from Tennekes and Lumley's discussion (1972) that the turbulent stress is associated with eddies whose vorticity is aligned with the mean strain rate. The comparison of the diagonal and the off-diagonal terms is presented in Table 9.7.
Table 9.7

Comparison of the diagonal and off-diagonal terms

<table>
<thead>
<tr>
<th></th>
<th>mesh</th>
<th>diagonal</th>
<th>off-diagonal</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aij</td>
<td>32x32x32</td>
<td>0.257</td>
<td>0.161</td>
<td>0.193</td>
</tr>
<tr>
<td></td>
<td>16x16x16</td>
<td>0.270</td>
<td>0.148</td>
<td>0.189</td>
</tr>
<tr>
<td>Bij</td>
<td>32x32x32</td>
<td>0.400</td>
<td>0.666</td>
<td>0.578</td>
</tr>
<tr>
<td></td>
<td>16x16x16</td>
<td>0.395</td>
<td>0.675</td>
<td>0.581</td>
</tr>
<tr>
<td>Cij</td>
<td>32x32x32</td>
<td>0.386</td>
<td>0.189</td>
<td>0.255</td>
</tr>
<tr>
<td></td>
<td>16x16x16</td>
<td>0.367</td>
<td>0.169</td>
<td>0.235</td>
</tr>
</tbody>
</table>

As an auxiliary computation, Table 9.8 - 9.10 present the R-square values resulting from the following combinations out of the three tensors A_{ij}, B_{ij} and C_{ij}.

Case (AB) : $\bar{u}_k\bar{u}_l \leftrightarrow A_{ij} + B_{ij}$

Case (AC) : $\bar{u}_k\bar{u}_l \leftrightarrow A_{ij} + C_{ij}$

Case (BC) : $\bar{u}_l\bar{u}_k \leftrightarrow B_{ij} + C_{ij}$

The results show that, as expected, the lowest correlation appears where the term $B_{ij}$ is not included, while the highest is the case (BC). Indeed, this gives the conclusive evidence that the turbulent residual stress terms are largely controlled by the vortex stretching.
### Table 9.8

**R-square values: Case (AB)**

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.636  0.828  0.830</td>
<td>0.642  0.826  0.809</td>
</tr>
<tr>
<td>0.828  0.633  0.839</td>
<td>0.826  0.724  0.855</td>
</tr>
<tr>
<td>0.830  0.839  0.639</td>
<td>0.809  0.855  0.611</td>
</tr>
</tbody>
</table>

- diagonal = 0.653
- off-diag = 0.832
- mean = 0.772

- diagonal = 0.659
- off-diag = 0.830
- mean = 0.773

### Table 9.9

**R-square values: Case (AC)**

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.598  0.324  0.301</td>
<td>0.582  0.299  0.284</td>
</tr>
<tr>
<td>0.324  0.647  0.328</td>
<td>0.299  0.681  0.324</td>
</tr>
<tr>
<td>0.301  0.328  0.602</td>
<td>0.284  0.324  0.600</td>
</tr>
</tbody>
</table>

- diagonal = 0.616
- off-diag = 0.317
- mean = 0.417

- diagonal = 0.621
- off-diag = 0.302
- mean = 0.409

### Table 9.10

**R-square values: Case (BC)**

<table>
<thead>
<tr>
<th>(a) 32x32x32</th>
<th>(b) 16x16x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.769  0.860  0.870</td>
<td>0.734  0.853  0.859</td>
</tr>
<tr>
<td>0.860  0.793  0.874</td>
<td>0.853  0.792  0.867</td>
</tr>
<tr>
<td>0.870  0.874  0.759</td>
<td>0.859  0.867  0.714</td>
</tr>
</tbody>
</table>

- diagonal = 0.774
- off-diag = 0.868
- mean = 0.837

- diagonal = 0.747
- off-diag = 0.860
- mean = 0.822
CHAPTER X
CONCLUSIONS AND RECOMMENDATIONS

Two infinite series representations of the small scale turbulence quantities are formulated by making use of the Gaussian response function and its inverse Fourier transform. The YB-I series is a general form used in the conventional LES approach and the YB-II series is an advanced form directly describing the small scale activities in terms of the large scale or average component. When these two terms are combined, the third series description is formulated and is able to define the averaged non-linear terms. Indeed, this YB filtering approach provides a new method for deriving and analyzing the turbulence equations.

The following conclusions and recommendations are concerned with the fundamental filtering operation as applied to the governing turbulence equations.

1. Once a low pass filter function is given, the corresponding high pass filter is determined from their complementary property. Without any major assumption, the small scale components can be simply obtained by applying the high pass filter to the governing equations.

2. By the YB-III series, the averaged non-linear terms are directly represented in terms of the averaged variables. This YB filtering approach is superior to the conventional convolution averaging methods where all four terms at the doubly averaged level must be defined separately. Even though the Leonard's, cross and

- 139 -
the SGS terms are supposed to be exactly specified, the resulting averaged non-
linear term doesn't provide equivalent results.

3. In the same manner as the derivation of the YB-III series representation for the
double turbulence correlations, the triple correlations as well as any higher corre-
lations can be formulated straightforwardly.

4. The YB filtering approach can be applied to any turbulence non-linear equations.
In addition, by making use of the YB-III series description, the equations govern-
ing the turbulence quantities such as the turbulence kinetic energy, momentum
flux, scalar variance and scalar flux, can be easily derived.

5. The YB filtering approach demonstrates the possibility that any probability densi-
ty function (pdf) can be used for the filter function because the pdf is conceptu-
ally in accordance with the filter function applied in this research and satisfies
the constraints on the filter function as well. In order to find the most appropri-
ate filter function, full understanding of the turbulence statistics is required. The
development of this concept remains for further research.

6. The term $<R_{ij}>$ appearing in the ensemble energy equation acts as a parama-
ter connecting the mean flow and the turbulence. In view of the energy budget,
although it is recognized that energy is globally transferred from the mean flow
to the turbulence, the form here possibly permits for the first time local energy
transfer from small to large scales. More research on this is necessary.

By using the YB-III series description, one can eliminate the intrinsic limitations in
the conventional closure models. Although the YB filtering approach has been initially
developed with a generalized moving volume averaging concept, the resulting equations are also valid in the traditional Reynolds averaging form. The YB filtering method makes possible another closure totally different from the conventional approach in the Reynolds average as well as the LES, which for the first time explicitly accounts for vortex stretching.

The next set of conclusions and recommendations are concerned with the closure resulting from the YB filtering approach.

7. The YB filtering method suggests an alternate way to close the equation. The closure term resulting from the YB-III series description is derived without any empirical constants or hypothesis.

8. The YB filtering procedure produces distinctive closure terms not requiring any transport equation solution for the turbulent parameters. The turbulence kinetic energy representation resulting from the YB-III series explains the mixing length hypothesis more clearly by providing a general form of the turbulent shear stress. Accordingly, there is not much difference between the mixing length hypothesis and the one-equation model.

9. All SGS models in the LES have been derived from empirical and non-dimensional analyses but the YB filtering procedure provides full a theoretical derivation of them including the Smagorinsky and the vorticity models. The YB-III series description also gives a general form for turbulent energy models, by which it is recognized that the Smagorinsky, the vorticity and the SGS energy models are not distinctive.
10. The comparison of the correlation coefficients based on the experimental data shows that the vortex stretching acts most significantly on the turbulence residual stress, and its significance strongly supports the claim that the vortex stretching is essential in the transfer of turbulence. Therefore, it is suggested that the effects of the vortex stretching should be included in SGS models.

11. Detailed research on an anisotropic filter is necessary since it is expected to provide more knowledge of different length scale activity in existing turbulence theory and it may be very useful for analyzing turbulence problems near a boundary.
BIBLIOGRAPHY


75. Prandtl, L., "Bericht über Untersuchungen zur augebildeten Turbulenz," ZAMM 5, 136, 1925.


