NONLINEAR DYNAMICS OF ONE-WAY CLUTCHES AND
DRY FRICTION TENSIONERS IN BELT-PULLEY
SYSTEMS

DISSERTATION

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ABSTRACT

Serpentine belts are widely used for efficient power transmission in automobiles and heavy vehicles, but they suffer noise and belt wear problems that have their roots in system vibration. The primary goal of this work is to develop mathematical models for serpentine drives with nonlinear elements and implement vibration analysis to understand certain nonlinear dynamic behaviors, such as belt slip and premature belt fatigue, and provide guidance for practical design.

Nonlinear one-way clutches integrated with accessory pulleys are effective devices to decouple the motions of an accessory and its pulley during disengagements and mitigate the rotational pulley vibration problems. A mathematical model of a one-way clutch in belt-pulley systems is established, where a wrap-spring type of clutch is modeled as a nonlinear spring with discontinuous stiffness. Efficient methods such as multi-term harmonic balance, arclength continuation and numerical integration with attention to engagement/disengagement transition criteria are developed to analyze the clutch dynamics and clarify the behavior across important frequency ranges. Analysis of steady state and transient vibrations exhibits the mechanisms behind one-way clutches’ effectiveness, and identifies where the one-way clutch works most effectively to reduce vibration and noise.
To explain the numerical results and clarify how design parameters affect the system dynamics, the method of multiple scales is employed to approximate the steady-state periodic solutions. The discontinuous separation function is expanded as a Fourier series in the perturbation analysis. The closed-form frequency-response relation is determined at the first order, and an implicit expression is obtained for the second-order approximation. This study evaluates the validity of the perturbation method for such strong nonlinearity through comparison of analytical and numerical solutions.

A one-way clutch that functions based on the relative velocity of the driven pulley and its accessory is common in applications. This type of one-way clutch is included in a hybrid continuum-discrete model incorporating span vibration and pulley rotation. This model is more accurate than the traditional spring-belt model with only pulley rotational degrees of freedom. The engagement/disengagement status transitions lead to a piece-wise linear system with alternate locked clutch and disengaged clutch configurations. Use of the transition matrix to evaluate the system response in discrete time series at each linear configuration saves computation time. The dynamic response and dynamic tension fluctuations are examined for varying system parameters. Investigation of the effectiveness of the vibration reduction due to the one-way clutch provides design guidelines in practice.

A tensioning system, consisting of a rigid arm pivoting around a fixed point and an idler pulley rotating at the free end of the arm, is typically used in belt drives to maintain belt tension as operating conditions change. Dry friction is deliberately introduced at the tensioner arm pivot to control the arm movement. The interest of this work lies in the
influence of dry friction tensioner on the dynamics of the belt drives incorporating belt bending stiffness. The belt bending stiffness is a new modeling feature in this nonlinear problem and the nonlinear stick-slip motion of the arm brings new characteristics into the hybrid continuum-discrete gyroscopic system dynamics. The discretized formulations are divided into linear and nonlinear subsystems, which reduces the dimension of the iteration matrices dramatically when employing the harmonic balance method. This study offers guidance for designers by examining how to adjust the Coulomb torque at the tensioner arm pivot to mitigate the system vibration, reduce the span tension fluctuation, and dissipate the system energy.
Dedicated to my mother, Xiangzhen Zhu

and to my husband, Gang Liu
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CHAPTER 1

INTRODUCTION

1.1 Motivation and Objectives

Serpentine belts are widely used in automobiles and heavy vehicles to transmit crankshaft power to the accessories. Despite their advantages of efficient power transmission, simplified assembly, longevity and compactness, serpentine drives suffer noise and belt wear problems that have their roots in system vibration. Periodic engine pulsations from cylinder ignition generate dynamic excitation for the belt drive. This excitation can, especially near resonance, cause large pulley rotations that lead to “chirp” noise from belt-pulley friction, noise from transverse span vibration, belt slip, and premature belt fatigue or bearing failure [1,2].

Serpentine drives exhibit certain nonlinear dynamic behaviors stemming from the following sources, some of which are the major concern of this work. Tension fluctuation of the belt induced by engine pulsations can lead to nonlinear coupling of transverse
motion of the belt and rotational vibrations of the pulleys when the belt stretches finitely [3]. Tensioner arms are typically used in belt drives to maintain belt tension as operating conditions change and their dynamic assembly induces geometric nonlinearity [3-6]. Coulomb friction at the tensioner pivot provides the major source of energy dissipation in serpentine drives [7]. Nonlinear one-way clutches integrated with accessory pulleys are effective devices to decouple the motions of an accessory and its pulley during disengagements [1,2] and therefore mitigate the rotational pulley vibration problems. The mechanism of belt creep and belt slip also receives research attention [4,8-11].

This work addresses one-way clutches in belt-pulley systems. The evolution of modern vehicles has led to increased electrical needs resulting in increased alternator inertias. The alternator is usually the accessory with the largest inertia and is commonly at the heart of observed vibration problems. One-way clutches are effective devices to mitigate these vibration problems, though they have proved useful on other accessories as well [1,2]. There are several types of one-way clutches, and different names are sometimes used (e.g., one-way decouplers, over-running pulleys). These devices are mounted in the load path between the pulley and the accessory shaft that the pulley drives. Different types of one-way clutches perform essentially the same function, namely that of decoupling pulley rotations in the non-driven direction caused by pulley vibrations from rotations of the driven shaft. In essence, the pulley and driven shaft are allowed to rotate relative to each other when the clutch disengages.

Although applications of one-way clutches are extensive, literature on their dynamics is limited. The prototypical belt drives in previous works consist of a driving
pulley, driven pulleys and a dynamic tensioner. One-way clutches have not been included so far. Nonlinear phenomena during performance of one-way clutches are observed \cite{1,2,12,13}, but an effective mathematical model for one-way clutches in belt-pulley systems has not been put forward. The relevant knowledge is required in practical applications. This work seeks to build and analyze a mathematical model of a belt-pulley system equipped with a one-way clutch and to investigate the impact of the design parameters on the system dynamics. There exist a variety of mathematical techniques to analyze nonlinear systems. In the current research, analytical and numerical methods are used.

This work also examines the dynamics of a belt-pulley system with a dry friction tensioner. The connection between these two topics, in addition to the belt application, is that both systems involve nonlinear issues and similar methods may be adopted to implement the analyses. A tensioning system, known as a tensioner, in serpentine drives automatically adjusts the belt tension during operation. It consists of an idler pulley (tensioner pulley) at the end of a rigid arm (tensioner arm). The arm pivots around a fixed point and the pulley is pinned at the free end of the arm \cite{7}. Generally the tensioner arm pivot is modeled as a preloaded coil spring. Here, dry friction damping is added. Energy dissipation occurs when the two mating surfaces at the pivot rotate relative to each other. Dry friction is a controlled design input that manufacturers use to distinguish their products. It is one of the few practical ways to introduce system damping. The tensioner arm exhibits stick-slip motions because of Coulomb damping. Most literature ignores this damping to simplify the analysis. Leamy et al. \cite{7} and their subsequent study \cite{14}
introduce a tensioner Coulomb damper to study the influence of the dry friction on the dynamics of front end accessory drives (FEAD). It is based on a discrete pulley rotation model. Kong and Parker [11] develop a serpentine drive model that includes belt bending stiffness. This beam-like belt model exhibits new characteristics of the dynamics and extends the belt drive model to broader application conditions. In particular, all span motions are coupled to pulley rotations. This work will combine the above works to employ a belt drive model with belt bending stiffness and consider the nonlinear influence of tensioner dry friction on the coupled belt-pulley dynamics.

1.2 Literature Review

The work relevant to the current research is briefly reviewed in this section, which consists of four topics. The first topic focuses on the discussion of axially moving materials including comprehensive reviews and specialty in dynamic analysis. The second topic addresses the analysis of serpentine belt drives categorized according to different belt models. The third one discusses papers that address the nonlinear dynamics evolved in belt-pulley systems. The fourth one reviews work concentrating on the analytical and numerical techniques used for analysis of nonlinear systems.

1.2.1 Vibration of axially moving materials

Moving belts, chains, tapes, strings, filaments, etc. are widely used in applications, especially for power transmission in mechanical systems. These transmission media are classified as axially moving materials and have received much research attention with long history. Several reviews are conducted from different research angles. Fawcett [15]
presents a survey on the literature related to chain drives and belt drives, both of which consist of a continuous flexible element passing over two pulleys. Relevant topics extend to vibration, noise, material properties and pulley-belt interaction, etc. Wickert and Mote [16] review prior studies on vibration of axially moving materials encompassing traveling strings, cables, tapes, beams, and band saws, etc. The focus is on single span systems. The contents are categorized into second-order systems (strings) and fourth-order systems (beams). The modeling, vibration analyses, conclusions and pertinent issues are discussed. Later, Abrate [17] provides a complete review of prior works on power transmission belts with emphasis on single span systems. The mathematical model and/or eigensolutions for belt transverse and pulley rotational vibrations are summarized. The effects of different parameters and nonlinearity of the moving media and their relevant applications to belt drives are discussed.

Recent studies on the dynamics of single span moving media are extensive. Wickert and Mote [18] present a classical form for vibration analysis of axially moving media involving gyroscopic effects. The equations are expressed with a symmetric and a skew-symmetric operator acting on a state vector that leads to orthogonality of the eigenfunctions. This work extends the canonical form of the eigenvalue problem and modal analysis from discrete gyroscopic systems [19,20] to continuous ones and advances the following research on analytical solutions for moving media. Jha and Parker [21] systematically examine the spatial discretization of axially moving media eigenvalue problems. The tension and/or speed fluctuation of axially moving media induce parametric instability relevant to belt applications. Mockenstrum et al. [22] utilize
translating string eigenfunctions as Galerkin basis functions to develop an efficient one-term, \( n^{th} \)-mode discrete model, and evaluate the parametric instability and limit cycles of moving strings with tension fluctuation. Pakdemirli and Ulsoy [23] determine the stability boundaries resulting from speed fluctuation. Parker and Lin [24] consider parametric instability stemming from both tension and speed fluctuation, and the moving media is subjected to multi-frequency parametric excitation. Because of the geometric nonlinearity of moving media, distinct jump and hysteretic phenomena have been observed by experiments and confirmed by direct numerical integration of the non-linear model [25]. Diverse assumptions of pulley support models that generate different boundary conditions for the moving span are also proposed [26,27].

1.2.2 Serpentine belt drives

As a branch of moving media, power transmission belts play an important role. Since the late 1970s, serpentine belts have replaced V-belts in automotive accessory drives and heavy vehicles. Multiple belt model choices are available for belt-pulley systems. A discrete spring belt model yields a system with only pulley rotational vibrations. Based on this spirit, Barker et al. [28] establish a mathematical model for the entire serpentine belt drive with a dynamic tensioner. The relation for determining tensioner arm geometric configuration and span tension arising from belt stretching are addressed in detail. Transient responses to the engine firing pulsations are discussed. Hwang et al. [4] determine the equilibria for the system with a tensioner arm including geometric nonlinearity. The characteristics are captured from the eigenvalue problem for
the linearized equations about equilibrium. The onset of belt slip in both static and dynamic states is predicted. References [14,29] adopt a model similar to that in [4].

To investigate coupling of the transverse motions of the belt and rotational motions of the pulleys, continuum string models of belt spans are adopted in the following research. Ulsoy et al. [30] develop a subsystem model consisting of a tensioner and its adjacent belt spans. Although the motion of the tensioner arm is uncoupled from belt transverse motion, the potential parametric instability induced by tension variation is examined. Other researchers have paid attention to the coupling characteristics of the pulley rotational and span transverse vibrations. Beikmann [5] introduces a prototypical three-pulley system involving a driving, a driven and a tensioner pulley. Steady state, dynamic and experimental analyses are presented. Beikmann et al. [31] examine the linear coupling of the transverse span and pulley rotational motions for free vibration from infinitesimal belt stretching. Only the two spans adjacent to the tensioner couple with the pulley rotations. Their conclusion that for a string belt model no coupling exists between the pulley rotations and the transverse motion of spans between fixed-center pulleys guides subsequent works to omit this coupling for the string belt model system. Their subsequent study [3] investigates nonlinear coupling of the transverse and rotational motions resulting from finite belt stretching. Following the three-pulley typical system and the spirit discussed in [31], Zhang and Zu [32] implement modal analysis by dividing the entire system into two subsystems, and a closed form characteristic equation is obtained. Moreover, through investigation of one-to-one internal resonance, they use the direct multiple scales method to study nonlinear coupling, which is demonstrated to
generate better results than the discretization multiple scales method [33]. Parker [34] expresses the equations of motion in an extended operator form and presents an efficient method to calculate eigensolutions, dynamic response, and eigensensitivities to the system parameters. All the above models address constant operating speed. Jha [6] considers a three-pulley prototypical system and derives equations of motion under changing operating speed conditions. Uncoupled equations are formed by decoupling the longitudinal motion from the transverse motion with mathematically rigorous quasi-static assumption. The tensioner arm model has geometric nonlinearity from its dynamic orientation. Jha investigates the effect of this nonlinearity by Taylor series expansion about operating equilibrium up to third order. The jump phenomena, sub- and super-harmonic responses are observed from this cubic model.

Kong and Parker [11] establish a serpentine drive belt model for a prototypical three-pulley system considering belt bending stiffness. They develop a singular perturbation solution and a computational method based on boundary-value problem solvers to obtain the numerically exact equilibrium solutions and define a coupling indicator for each span. The equilibrium deflections and belt-pulley coupling are shown to be sensitive to the system geometry, belt bending stiffness and span tensions. In their subsequent work [11], this gyroscopic, discrete-continuous hybrid model is linearized about the equilibrium, and then is reformulated to make Galerkin discretization readily applicable. Through examining the eigensolutions from the discretized equations, the dynamic characteristics due to incorporating belt bending stiffness are discussed. Unlike string belt models, the spans between fixed-center pulleys are coupled with pulley
rotation by the non-trivial steady span deflection. The impacts of design parameters on
the system are investigated.

1.2.3 Nonlinear dynamics of serpentine belt drives

The aforementioned research concentrates mostly on linear analysis of the belt-pulley systems. Important nonlinear mechanisms in these systems need to be revealed. The Coulomb damping at the tensioner pivot is of the utmost interest. In [28], a Coulomb-damped tensioner arm is first introduced with employing Runge-Kutta numerical integration to solve the equations. Kraver et al. [35] develop a complex modal analysis procedure for a generalized FEAD with equivalent viscous damping to model the Coulomb damped tensioner arm. The analysis includes only pulley rotational vibration. They state that the belt viscous damping has significant impact on the rotational vibration amplitude while Coulomb damping affects it slightly. Leamy et al. [29] follow the model in [4] but add a Coulomb damper at the tensioner arm pivot. They describe in detail the simulation of the Coulomb torque acting on the tensioner arm as well as the belt constitutive relations. The fourth/fifth order Runge-Kutta method is employed to integrate the equations. The code is modified to surmount numerical difficulties induced by tensioner arm stick-slip motion. From the response, super- and sub-harmonic responses are found that are absent from the models without the Coulomb damper. Later, Leamy and Perkins [14] utilize the incremental harmonic balance method to efficiently yield the nonlinear periodic response. Again secondary resonances are predicted. This work adopts the serpentine drive model in [11] such that belt bending stiffness, belt-
pulley coupling, and gyroscopic effects are considered. Nonlinear dry friction is added at the tensioner pivot, and the impact of this element is examined.

With the extensive application of one-way clutches in FEAD, demands for modeling, analysis, and design tools have increased. However, literature on their dynamics is still sparse. Vernay et al. [12] present an experimental study of sprag-type clutches used in the air turbine starters of jet engines. The clutch is composed of sprags, mounted between two races, and springs that connect the sprags and ensure contact between the sprags and races. The goal is to identify sliding during engagement. King and Monahan [2] discuss a wrap-spring type clutch and elaborate on its functional details. Solfrank and Kelm [1] describe a model for an entire automobile accessory drive system. As an element of their system, a model for an “overrunning alternator pulley” is introduced. The model consists of a speed-dependent damping and a parallel stiffness element. Leamy and Wasfy [13] study belt creep on the pulley by considering the friction contact between the pulleys and the belt using the finite element method. A one-way clutch as an optional element is modeled using a torque proportional to relative pulley/accessory speed that is active only for torque transmission in a single direction. Tsangarides and Tobler [36] investigate the dynamic behavior of a torque converter with centrifugal bypass clutch, where the proper implementation of the clutch can reduce noise and vibration. The bypass clutch is modeled in terms of torque capacity, the clutch slip speed and the engaging pressure. The current work investigates the nonlinear dynamic analysis of one-way clutches in belt-pulley systems.
1.2.4 Techniques of nonlinear analysis

An effective analysis technique is required to deal with the nonlinear issues. Pilipchuk [37] formulates a saw-tooth time transformation technique to allow analytical calculation of free and forced response for strongly nonlinear system. Kim and Perkins [38] generalize the harmonic balance/Galerkin method for non-smooth dynamic systems to accelerate the convergence. Perturbation methods are used extensively for both linear and nonlinear systems [11,24,33,39-43] to analytically approximate the solutions and are applied to the nonlinear systems in this work.

The existing literature shows that the harmonic balance method (HBM) is widely used to seek periodic solutions of nonlinear systems, especially those having clearance nonlinearity. With the assumption of periodic motion, the variables are expanded by Fourier series so as to map the system from the time domain into the frequency domain to get periodic solutions. By taking a gear pair as an example and applying the standard HBM, Blankenship and Kahraman [44] study the behavior of a mechanical system exhibiting combined parametric excitation and clearance type nonlinearity. Padmanabhan and Singh [45] analyze periodically excited nonlinear systems with an example of a gear pair with discontinuous mesh stiffness by using a parametric continuation technique. Continuation techniques have been used by many researchers for nonlinear systems [46,47]. It is a path following procedure using arclength continuation to trace the bifurcation diagram. By HBM with arclength method, Raghothama and Narayanan [48] study the bifurcation and chaos in a geared rotor bearing system; Von Groll and Ewins
[49] examine rotor/stator interaction dynamics for varying shaft rotation speeds. In the current research, HBM with arclength continuation technique is adopted.

To avoid numerical difficulties arising from the discontinuity, smoothing functions are considered to approximate the original discontinuous nonlinear functions. For example, arc-tangent function [50], hyperbolic-tangent function, hyperbolic-cosine function and polynomial have all been utilized to approximate the \( \text{sgn} \) function. Ferri and Heck [51] use a line with finite slope to connect the disconnected points within a tiny interval to approximate the \( \text{sgn} \) function. Kim and Singh [52] summarize use of smoothing functions for a type of clearance nonlinearity and examine the effect on the frequency response of the oscillator. The discontinuous nonlinearities in this work are smoothed for some of the solution methods.

1.3 Scope of Investigation

The current research addresses nonlinear problems in serpentine belt drive vibrations that are of practical importance in industrial power transmission use. It aims to detect the dynamic behavior through establishing mathematical models for the belt-pulley system with nonlinear elements, and consequently to provide guidance for the practical design by examining the impact of the design parameters on the system vibrations. Because of sparse related literature, the work helps to advance the understanding of the broader field of nonlinearity occurring in hybrid belt-pulley systems.

Chapter 2 establishes a mathematical model of a nonlinear one-way clutch in belt-pulley systems, where a wrap-spring type of clutch is modeled as a nonlinear spring with
discontinuous stiffness. Efficient methods such as multi-term harmonic balance, arclength continuation and numerical integration with attention to engagement and disengagement transition criteria are developed to analyze the clutch dynamics and clarify the behavior across important frequency ranges. Analysis of steady state and transient vibrations exhibits mechanisms behind one-way clutches’ effectiveness, and identifies where the one-way clutch works most effectively to reduce vibration and noise. To explain the numerical results and clarify how design parameters affect the system dynamics, the method of multiple scales is employed in Chapter 3 to approximate the steady-state periodic solutions. The discontinuous separation function is expanded as a Fourier series in the perturbation analysis. The closed-form frequency-response relation is determined at the first order, and an implicit expression is obtained for the second-order approximation. This study evaluates the validity of the perturbation method for such strong nonlinearity through comparison of analytical and numerical solutions.

Chapter 4 includes a one-way clutch, which functions based on the relative velocity of the driven pulley and its accessory, in a hybrid continuum-discrete model incorporating span vibration and pulley rotation. This model is more accurate than the traditional spring-belt model with only pulley rotational degrees of freedom. The engagement/disengagement status transitions lead to a piece-wise linear system with alternate locked clutch and disengaged clutch configurations. Use of the transition matrix to evaluate the system response in discrete time series at each linear configuration significantly saves computation time. The dynamic response and dynamic tension fluctuations are examined for varying system parameters.
The interest of Chapter 5 lies in the influence of dry friction tensioner on the dynamics of the belt drives incorporating belt bending stiffness. The belt bending stiffness is a new modeling feature in this nonlinear problem and the nonlinear stick-slip motion of the arm brings new characteristics into the hybrid continuum-discrete gyroscopic system dynamics. The discretized formulations are divided into linear and nonlinear subsystems, which reduces the dimension of the iteration matrices dramatically when employing the harmonic balance method. The developed method demonstrates advantages relevant to the systems with similar nonlinearity. This study offers guidance for designers by examining how to adjust the Coulomb torque at the tensioner arm pivot to mitigate the system vibration, reduce the span tension fluctuation, and dissipate the system energy.

The contributions of the current research are summarized as follows.

1. Mathematical models and analysis tools for one-way clutches are established. Their nonlinear dynamics in belt-pulley systems are studied, and the impact of system parameters on the dynamics is investigated so as to reduce the vibrations and noise, and to identify and predict the characteristics of the system from the nonlinear model.

2. Efficient methods, harmonic balance with arc-length continuation and perturbation analysis, are developed to analyze clearance-type nonlinear systems. Transition matrix is used to efficiently address the steady state response of piece-wise linear dynamic systems. These methods can be generalized to systems with similar mechanism.
3. The impact of dry friction tensioners and one-way clutches on the serpentine drives with belt bending stiffness is investigated to bring new nonlinear characteristics into the hybrid continuous-discrete gyroscopic systems.

4. Linear systems are introduced to make comparison with corresponding nonlinear systems for better understanding the function of the nonlinear elements in belt drives. This spirit can be extended to other nonlinear system research.

5. Guidance for designers through the investigation of the design parameters on the system dynamics is provided.
A two-pulley system is analyzed in this chapter, where the driven pulley has a one-way clutch between the pulley and accessory shaft that engages only for positive relative displacement between these components. The belt is modeled with linear springs that transmit torque from the driving pulley to the accessory pulley. The one-way clutch is modeled as a piecewise linear spring with discontinuous stiffness that separates the driven pulley into two degrees of freedom (DOF). The harmonic balance method combined with arc-length continuation is employed to illustrate the nonlinear dynamic behavior of the one-way clutch and determine the stable and unstable periodic solutions for given parameters. Classical softening nonlinearity induced by clutch disengagement over portions of the response period is observed in the primary resonance of the system.
response. The dependence on clutch stiffness, excitation amplitude, and inertia ratio between the pulley and accessory is studied to characterize the nonlinear dynamics across a range of conditions.

2.1 Introduction

Serpentine belts are widely used in automobiles and heavy vehicles to transmit crankshaft power to all the accessories. Noise reduction and increased belt life are major industry concerns that have their roots in system vibration. Periodic engine pulsations from cylinder ignition generate dynamic excitation for the belt drive. This excitation can, especially near resonance, cause large pulley rotations that lead to “chirp” noise from belt-pulley friction, noise from transverse span vibration, belt slip, and premature belt fatigue or bearing failure [1,2]. Serpentine drives typically have a tensioner arm to maintain belt tension as operating conditions change. Several prior studies [4,6,11,31,34] address the dynamics of serpentine systems.

The evolution of modern vehicles has led to increased electrical needs resulting in increased alternator inertias. The alternator is usually the accessory with the largest inertia and is commonly at the heart of observed vibration problems. One-way clutches are effective devices to mitigate these vibration problems, though they have proved useful on other accessories as well [1,2]. There are several types of one-way clutches, and different names are sometimes used (e.g., one-way decouplers, over-running pulleys). These devices are mounted in the load path between the pulley and the accessory shaft that the pulley drives. In industry, fans in some mechanical systems are equipped with one-way (or over-running) clutches that consist of a ring, hub, rollers and springs. For
helicopters, sprag-type over-running clutches are used on the main rotor. Wrap-spring clutches are another design. Different types of one-way clutches perform essentially the same function, namely that of decoupling pulley rotations in the non-driven direction caused by pulley vibrations from rotations of the driven shaft. In essence, the pulley and driven shaft are allowed to rotate relative to each other when the clutch disengages.

One one-way clutch design engages or disengages based on the relative angular speed between the pulley and the accessory shaft. When the accessory pulley decelerates, the one-way clutch disengages, and the accessory rotates freely from the pulley. Because of its greater inertia, however, its deceleration is lower than that of the pulley. When the dynamics of the belt drive system cause the pulley to accelerate, the pulley and accessory remain disengaged until the pulley speed equals that of the accessory. At this point, the one-way clutch engages the accessory and the two rotate together (presuming infinite clutch stiffness) until the pulley again decelerates [1]. This decoupling of the accessory inertia during large vibrations limits belt slip on the pulley.

The clutch design studied in this chapter is based on relative displacement between the pulley and accessory. When pulley rotation exceeds accessory shaft rotation, the clutch is engaged with a finite rotational stiffness between the elements. This is the normal driving case. In the opposite case, the clutch disengages and there is no mechanical link between pulley and accessory. This operation is achieved, as one example, with a wrap-spring clutch design. The wrap-spring ends are connected to the pulley and accessory. For positive relative rotation, uncoiling of the spring expands its diameter, creating dry friction on a mating cylinder that couples the pulley and accessory.
Negative relative rotation contracts the diameter, releasing the friction and the link between elements.

Though applications of one-way clutches are extensive, literature on their dynamics is limited. Vernay et al. [12] presented an experimental study of sprag-type clutches used in the air turbine starters of jet engines. The clutch is composed of sprags, mounted between two races, and springs that connect the sprags and ensure contact between the sprags and races. The goal is to identify sliding during engagement. King and Monahan [2] discuss a wrap-spring type clutch and elaborate on its functional details. Solfrank and Kelm [1] describe a model for a whole automobile accessory drive system. As an element of their system, a model for an “overrunning alternator pulley” is introduced. The model consists of a speed-dependent damping and a parallel stiffness element. Leamy and Wasfy [13] studied belt creep on the pulley by considering the friction contact between the pulleys and the belt using finite element method. Their one-way clutch model is a torque proportional to relative pulley/accessory speed that is active only for torque transmission in a single direction.

This chapter concentrates on building and analyzing a mathematical model of a two-pulley belt system equipped with a one-way clutch. To focus on the one-way clutch dynamics, no tensioner is included. The belt is modeled as a linear spring transmitting specified motion of the driving pulley (crankshaft) to the driven (accessory) pulley. The one-way clutch is modeled as a nonlinear spring with discontinuous stiffness, that is, zero stiffness for the disengaged clutch and finite, linear stiffness for the engaged clutch. A multi-term harmonic balance method is developed based on Fourier expansion of the
response and discretization of the fundamental period into time increments. Stability of the calculated periodic solutions is assessed by Floquet analysis. Arclength continuation follows the solution branches to determine the stable and unstable solutions as a parameter changes. The results are confirmed by numerical integration and the bifurcation software AUTO [53]. Numerical integration results showing quasiperiodic and chaotic response are calculated for regions of aperiodic response. The system is analyzed for a range of different parameters (magnitude of the nonlinear stiffness, excitation amplitude, ratio of the inertia of the pulley to that of the accessory) to characterize where the one-way clutch works most effectively and when the system operates periodically or aperiodically.

2.2 Mathematical Model of One-Way Clutch

Based on the physical system of the wrap spring one-way clutch, the model can be described as follows. When the rotations of the wrap spring ends (that is, the pulley and the accessory shaft) are such that the pulley rotation exceeds the accessory shaft rotation (positive relative displacement) then the clutch is engaged and the clutch torque satisfies 

\[ g = K_d (\theta_p - \theta_a) \]

Power transmission occurs from driving to driven pulley. For the alternate case where pulley rotation is less than accessory rotation, the wrap spring diameter decreases and the clutch disengages; no torque is transmitted. In this work, we examine the impact of this one-way clutch on a two-pulley system (Figure 2.1a). The driving pulley represents the crankshaft, and its motion is specified as 

\[ \theta_{c/s} = A_m \cos \omega t \]
In vehicle applications, engine firing pulsations induce periodic fluctuations in crankshaft speed at the firing frequency $\omega$. The driven pulley connected to the accessory has inertia $J_p$. The one-way clutch is integrated between the accessory pulley with rotation $\theta_p$ and the accessory shaft with rotation $\theta_a$. Mathematically, the torque transmitted between the accessory pulley and shaft is (Figure 2.1b)

$$g(\delta \theta) = \begin{cases} K_d \delta \theta & \delta \theta > 0 \\ 0 & \delta \theta \leq 0 \end{cases}$$

(2.1)

where $\delta \theta = \theta_p - \theta_a$. The stiffness of the spring is $K_d$. The belt is modeled as a discrete spring with stiffness $K_b$. Steady belt tension and belt speed do not affect the system for this belt model. Energy dissipation is modeled as viscous damping using a modal damping model.

The equations of motion for the pulley and accessory shaft are

$$J_p \ddot{\theta}_p + C_{11} \dot{\theta}_p + C_{12} \dot{\theta}_a + 2K_b r_p^2 \dot{\theta}_p + g(\delta \theta) = M_0 + 2K_b r_{c/s} A_m \cos \omega T$$

$$J_a \ddot{\theta}_a + C_{21} \dot{\theta}_p + C_{22} \dot{\theta}_a - g(\delta \theta) = -M_0$$

(2.2)

where $r_p$, $r_{c/s}$ are the radii of the pulley and the crankshaft. Subscripts $p$ and $a$ refer to the pulley and the accessory, respectively. $M_0$ is pre-load.

Letting $t = \omega_0 T$, one obtains the dimensionless equations of motion

$$\theta_p'' + \bar{C}_{11} \theta_p' + \bar{C}_{12} \theta_a' + \bar{K}_p \theta_p + \bar{g}(\delta \theta) = \bar{M} + \beta \bar{K}_b A_m \cos \Omega t$$

$$\alpha \theta_a'' + \bar{C}_{21} \theta_p' + \bar{C}_{22} \theta_a' - \bar{g}(\delta \theta) = -\bar{M}$$

(2.3)
Figure 2.1. a) Two degree-of-freedom one-way clutch system; b) Clutch torque $g(\delta \theta)$ in (2.1) and the dimensional smoothed function $g_s(\delta \theta)$ according to (2.6) for different smoothing parameters $\varepsilon$.

$$g(\delta \theta) = \begin{cases} K_d \delta \theta & \delta \theta > 0 \\ 0 & \delta \theta \leq 0 \end{cases}$$

(2.4)

where the dimensionless parameters are
The frequency $\omega_0$ is chosen as the natural frequency for the linear system in which the accessory is not equipped with a one-way clutch. In this case it is a single degree of freedom system and the natural frequency is $\omega_0 = \sqrt{\frac{2K_b r_p^2}{J_p \omega_0^2}}$. 

\[
\Omega = \frac{\omega}{\omega_0}, \alpha = \frac{J_a}{J_p}, \beta = \frac{r_{clx}}{r_p}, K_b = \frac{2K_b r_p^2}{J_p \omega_0^2}, K_d = \frac{K_d}{J_p \omega_0^2}, \bar{M} = \frac{M_0}{J_p \omega_0^2} \tag{2.5}
\]

The damping matrix $\bar{C} = V^{-T} diag(2\zeta \Omega_i) V^{-1}$ is obtained by transforming the modal damping matrix $diag(2\zeta \Omega_i)$ where $\zeta$ is a specified modal damping ratio and $\Omega_i, \Omega_i$ are the eigensolutions for the two degree of freedom linear system with the clutch engaged. Corresponding to this linear system, the dimensionless natural frequencies are $\Omega_1 = 0.982$ and $\Omega_2 = 6.811$ for the values in Table 2.1. Figure 2.2 shows the mode shapes for the linear system. The pulley and accessory rotate in-phase in the first mode and out-of-phase in the second mode.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_p = 0.028575, m$</td>
<td></td>
<td>Radius of pulley</td>
</tr>
<tr>
<td>$r_{cfs} = 0.040625, m$</td>
<td></td>
<td>Radius of crankshaft</td>
</tr>
<tr>
<td>$J_p = 0.001607, kg \cdot m^2$</td>
<td></td>
<td>Pulley inertia</td>
</tr>
<tr>
<td>$\alpha = J_a / J_p = 1.620$</td>
<td></td>
<td>Inertia ratio</td>
</tr>
<tr>
<td>$\zeta = 3%$</td>
<td></td>
<td>Modal damping ratio</td>
</tr>
<tr>
<td>$K_b = 2.5 \times 10^5, N/m$</td>
<td></td>
<td>Belt stiffness</td>
</tr>
<tr>
<td>$K_d = 5000, N/rad$</td>
<td></td>
<td>Clutch spring stiffness</td>
</tr>
<tr>
<td>$A_m = 0.001, rad$</td>
<td></td>
<td>Excitation amplitude</td>
</tr>
<tr>
<td>$M_0 = 2.3, N \cdot m$</td>
<td></td>
<td>Preload</td>
</tr>
</tbody>
</table>

Table 2.1. Parameters for the nominal case.
Figure 2.2. Vibration modes for the parameters in Table 2.1. a) Mode 1, $\Omega_1 = 0.982$; b) Mode 2, $\Omega_2 = 6.81$.

Figure 2.3. RMS of $\delta\theta - \delta\theta_{\text{mean}}$ for modal damping ratios $\zeta = 3\%, 5\%, 8\%$ obtained by numerical integration with step function for the clutch torque $g(\delta\theta)$ (see equation (2.4)). Parameters are in Table 2.1.
Figure 2.3 shows the root mean square (RMS) of $\delta \theta - \delta \theta_{\text{mean}}$ versus excitation frequency for the values $\zeta = 3\%, 5\%, 8\%$ obtained by numerical integration for parameter values in Table 2.1. Increasing and decreasing frequency sweeps are shown. For $\zeta = 8\%$, the branches calculated for increasing and decreasing frequencies overlay each other; the behavior is linear. For $\zeta = 3\%$ and $\zeta = 5\%$, nonlinear jump phenomena occur. These results are sensitive to the excitation amplitude. For the nominal excitation amplitude used in this work (Table 2.1), we specify 3% modal damping to capture the relatively light damping internal to a one-way clutch and induce the nonlinear response exhibited by these systems in practice.

In this chapter, multiple methods are employed to address the discontinuous stiffness nonlinearity shown in Figure 2.1b. Among these are harmonic balance (HBM) and the bifurcation software AUTO, which requires continuous functions. To approximate $g(\delta \theta)$ in (2.4) by a smooth function, the hyperbolic tangent function is used according to

$$g_s(\delta \theta) = \frac{1}{2} \bar{K}_d (1 + \tanh(\epsilon \delta \theta)) \delta \theta$$

(2.6)

Figure 2.1b compares the dimensional clutch torque $g(\delta \theta)$ in (2.1) and the smoothed function $g_s(\delta \theta)$ corresponding to (2.6). The difference is indistinguishable for large $\epsilon > 100$. By numerical investigation, Figure 2.4 shows the effect of approximating the step function by (2.6) with given $\epsilon$. $\epsilon > 2000$ produces an acceptable approximation to the step function for this system. The value $\epsilon = 10,000$ yields a better approximation and
causes no numerical trouble for HBM and AUTO, so this value is used throughout. \( \overline{g}_s(\delta \theta) \) replaces \( \overline{g}(\delta \theta) \) in subsequent results except where indicated otherwise.

![Graph showing dynamic response](image)

Figure 2.4. Dynamic response, RMS of \( \delta \theta - \delta \theta_{\text{mean}} \), by numerical integration using the step function \( \overline{g}(\delta \theta) \) in (2.4) and different \( \epsilon \) in the approximation \( \overline{g}_s(\delta \theta) \) in (2.6).

### 2.3 HBM with Arclength Continuation and Stability

The harmonic balance method is widely used to seek periodic solutions of nonlinear systems, especially those having clearance nonlinearity. With the assumption of periodic motion, the variables are expanded by Fourier series so as to map the system from the
time domain into the frequency domain to get periodic solutions. By taking a gear pair as an example and applying the standard HBM, Blankenship and Kahraman [44] studied the behavior of a mechanical system exhibiting combined parametric excitation and clearance type nonlinearity. Leamy and Perkins [14] utilized the harmonic balance method to investigate the nonlinear periodic response of engine accessory drives with dry friction tensioners. Padmanabhan and Singh [45] analyzed periodically excited nonlinear systems with an example of a gear pair with discontinuous mesh stiffness by using a parametric continuation technique. Continuation techniques have been used by many researchers for nonlinear systems. It is a path following procedure using arclength continuation to trace the bifurcation diagram. By this method, Raghothama and Narayanan [48] studied the bifurcation and chaos in a geared rotor bearing system. Von Groll and Ewins [49] examined rotor/stator interaction dynamics for varying shaft rotation speeds.

Combined with arclength continuation, HBM is effective to examine how the system behavior varies with the system parameters, and this method is adopted in this study. A brief description of HBM with arclength continuation applied to (2.3) with smoothing function (2.6) is given below. First, consider the response to be periodic and expand the solution using Fourier series truncated to $R$ harmonics

$$
\theta_p(t) = u_{p,1} + \sum_{r=1}^{R} (u_{p,2r} \cos r\Omega t + u_{p,2r+1} \sin r\Omega t)
$$

$$
\theta_a(t) = u_{a,1} + \sum_{r=1}^{R} (u_{a,2r} \cos r\Omega t + u_{a,2r+1} \sin r\Omega t)
$$

(2.7)
Then, discretize the time domain into \( N \) intervals as \( t_0, \cdots, t_n, \cdots, t_{N-1} \) and introduce the operator \( L_0 \) such that the time-discretized response vector is

\[
\mathbf{x}(t) = \begin{bmatrix} \theta_p(t_0) & \cdots & \theta_p(t_{N-1}) & \theta_s(t_0) & \cdots & \theta_s(t_{N-1}) \end{bmatrix}^T = \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \mathbf{u} = \mathbf{L} \mathbf{u} \quad (2.8)
\]

where \( \mathbf{u} = \begin{bmatrix} u_{p,1} & \cdots & u_{p,2^{R+1}} & u_{a,1} & \cdots & u_{a,2^{R+1}} \end{bmatrix}^T \). Introducing a \((2R+1) \times (2R+1)\) operator \( A_0 = \text{diag}(0,1^2,1^2,2^2,2^2,\cdots,r^2,r^2,\cdots) \) and a similar operator \( B_0 \) and defining \( A = (-\Omega^2) \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix}, \quad B = \Omega \begin{bmatrix} B_0 & 0 \\ 0 & B_0 \end{bmatrix} \), one can express \( \ddot{x}(t) \) and \( \dddot{x}(t) \) as

\[
\ddot{x} = \mathbf{L} \mathbf{B} \mathbf{u}, \quad \dddot{x} = \mathbf{L} \mathbf{A} \mathbf{u} \quad (2.9)
\]

The nonlinear function \( \mathbf{h} = \begin{bmatrix} \bar{g}_s(\delta \theta) \quad -\bar{g}_s(\delta \theta) \end{bmatrix}^T \) and the forcing function \( \mathbf{f} = \begin{bmatrix} \tilde{M} + \beta \dot{K}_b A_m \cos \Omega t - \tilde{M} \end{bmatrix}^T \) are similarly expanded in Fourier series as

\[
\mathbf{h} = \mathbf{L} \mathbf{d}, \quad \mathbf{f} = \mathbf{L} \mathbf{F} \quad (2.10)
\]

By defining \( \mathbf{\tilde{m}} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \alpha \mathbf{I} \end{bmatrix}, \quad \mathbf{\tilde{c}} = \begin{bmatrix} \overline{C}_{11} \mathbf{I} & \overline{C}_{12} \mathbf{I} \\ \overline{C}_{21} \mathbf{I} & \overline{C}_{22} \mathbf{I} \end{bmatrix} \) and \( \mathbf{\tilde{K}} = \begin{bmatrix} \overline{K}_b \mathbf{I} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \), where \( \mathbf{I} \) is a \((2R+1) \times (2R+1)\) identity matrix, substitution of (2.8), (2.9) and (2.10) into (2.3) yields

\[
\mathbf{L}[(\mathbf{\tilde{m}} \mathbf{A} + \mathbf{\tilde{c}} \mathbf{B} + \mathbf{\tilde{K}}) \mathbf{u} - \mathbf{F} + \mathbf{d}] = \mathbf{L} \mathbf{E} = 0
\]

\[
\mathbf{E} = \mathbf{\tilde{K}} \mathbf{u} - \mathbf{F} + \mathbf{d} \quad \mathbf{\tilde{K}} = \mathbf{\tilde{m}} \mathbf{A} + \mathbf{\tilde{c}} \mathbf{B} + \mathbf{\tilde{K}}
\]

The vector \( \mathbf{u} \) that determines \( \mathbf{x} \) is found from

\[
\mathbf{E} = 0 \quad (2.12)
\]
We seek to follow periodic solution branches as a parameter $p$ of the system changes (such as frequency, where $p = \Omega$). In particular, it is desirable to trace stable and unstable branches in the space of $u_i$ and $p$, and these branches generally involve curves that reverse direction. Baker and Overman [54] describe a continuation method approach. In this spirit, we introduce the solution branch arclength parameter $s$ as an independent variable and consider the system parameter $p$ as an unknown as well. The residue $E$ in (2.11) then has $2(2R+1)+1$ unknowns $\bar{u}(s) = \{u(s)^T, p(s)^T\}$ and the infinitesimal arclength $ds$ satisfies

$$ds^2 = \sum_{i=1}^{2R+1} du_i^2 + dp^2 \quad \Rightarrow \quad 1 = \sum_{i=1}^{2R+1} \left(\frac{du_i}{ds}\right)^2 + \left(\frac{dp}{ds}\right)^2$$  \hspace{1cm} (2.13)

According to Newton-Raphson iteration,

$$\bar{u}_{\text{new}} = \bar{u}_{\text{old}} - \left.J\right|_{\text{old}}^{-1} E(\bar{u}_{\text{old}})$$  \hspace{1cm} (2.14)

where the Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial E}{\partial u} & \frac{\partial E}{\partial p} \\ \frac{\partial E}{\partial u} & \frac{\partial E}{\partial p} \end{bmatrix}, \quad \frac{\partial E}{\partial u} = \bar{K} + \frac{\partial d}{\partial x} = \bar{K} + \bar{\Gamma} \frac{\partial h}{\partial x} L$$  \hspace{1cm} (2.15)

and $J^{-1}$ denotes the pseudo-inverse. We have used

$$d = \Gamma h = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} h$$  \hspace{1cm} (2.16)
where $\Gamma_0$ represents the discrete Fourier transformation operator with $\Gamma_{0,n} = \frac{1}{N}$,

$$
\Gamma_{0,2r,n} = \frac{2}{N} \cos \frac{2\pi r(n-1)}{N} \quad \text{and} \quad \Gamma_{0,2r+1,n} = \frac{2}{N} \sin \frac{2\pi r(n-1)}{N}
$$

for $r = 1, 2, \cdots, R$ and $n = 1, \cdots, N$. Iteration continues until $\epsilon = \| \overline{u}_{\text{new}} - \overline{u}_{\text{old}} \|$ is within a specified tolerance.

To improve convergence, it is important to provide an appropriate initial guess for iteration of (2.14). For the two-dimensional case, the tangent line of the solution branch at the current solution $\overline{u}_k(s)$ defines the axis along which the initial guess for the next solution $\overline{u}_{k+1}(s)$ lies [54]. Generalizing to higher dimensions, (2.12) describes $2(2R+1)$ hypersurfaces and the solutions $\overline{u}(s)$ lie on the intersection of these hypersurfaces. The desired initial guess lies along the axis that is the intersection of the tangent planes of these hypersurfaces at the current solution. Differentiating (2.12) at $\overline{u}_k(s)$,

$$
\frac{\partial \mathbf{E}}{\partial \mathbf{u}_{k s}} \frac{\partial \mathbf{u}}{\partial s} + \frac{\partial \mathbf{E}}{\partial p_{k s}} \frac{\partial p}{\partial s} = \mathbf{J}_{k s} \mathbf{\tau} = 0
$$

(2.17)

where $\mathbf{\tau} = \left[ \frac{\partial \mathbf{u}}{\partial s}, \frac{\partial p}{\partial s} \right]^T$. $\mathbf{J}_{k s}$ represents the gradients of the surfaces at $\overline{u}_k(s)$, and $\mathbf{\tau}$ is on the intersection of the tangent planes. $\mathbf{\tau}$ is a basis of the nullspace of $\mathbf{J}_{k s}$ where $\mathbf{J}_{k s}$ has rank $2(2R+1)$ except at bifurcation points. Alternatively, $\mathbf{\tau}$ is a basis of the left nullspace of $\mathbf{J}_{k s}^T$ from $\mathbf{\tau}^T \mathbf{J}_{k s}^T = 0^T$. From QR decomposition of $\mathbf{J}_{k s}^T$, $\mathbf{Q}^T \mathbf{J}_{k s}^T = \mathbf{R}$, where $\mathbf{Q}$ is an orthonormal square matrix and $\mathbf{R}$ is an upper triangular matrix with zero elements in the last row.
Consequently, $q_{2(2R+1)+1}^T J_{l_{in}}^T = 0^T$, and $\tau = q_{2(2R+1)+1}$. Orthonormality of $Q$ implies $\|\tau\| = 1$ in accordance with (2.13). The sign of $\tau$ must be chosen to ensure the solution path is traced “forward” in the arclength direction. If the inner product $\tau_k^T \tau < 0$, let $\tau_k = -\tau$, otherwise $\tau_k = \tau$ (subscript $k$ denotes the current solution). $\Delta s_k$ is determined by step size control [54] and then $\bar{u}^0_{k+1} = \bar{u}_k + \Delta s_k \tau_k$ establishes the initial guess.

To establish the stability of periodic solutions from harmonic balance, much of the literature employs Floquet multipliers [44-48]. Von Groll and Ewins [49] apply Hill’s method. Both methods were used here, and we found Floquet multipliers to produce more reliable results for known example problems. When applying Floquet multipliers, Friedmann et al. [55] compared two numerical schemes to obtain the monodromy matrix. One is developed by Hsu [56,57], which has been widely used for stability analysis [44-48]. That idea is to discretize a period into a number of intervals and consider the periodic coefficient matrices to be constant over each interval. The second approach is a numerical integration scheme improved in [55]. It is based on the fourth order Runge-Kutta method and requires less computer time to converge than Hsu’s method. Here, numerical integration is employed to find the monodromy matrix, whose eigenvalues $\lambda_i, i = 1, \cdots, 4$ are the Floquet multipliers. The solution stability is classified as follows: if
there exists any \( \| \lambda_i \| > 1 \), that eigenvalue (and the solution) is unstable; if all the eigenvalues satisfy \( \| \lambda_i \| < 1 \), the solution is stable.

The values \( R = 18 \) and \( N = 512 \) in (2.7) and (2.8) are used in results to follow. While the shape of the solutions branches is not highly sensitive to \( R \), this number of terms was found to give better stability conclusions than lower values.

In addition to harmonic balance, subsequent results compare findings from numerical integration and the bifurcation software AUTO. AUTO performs bifurcation analysis of systems of the form

\[
y'(t) = f(y(t), p), \quad f(\cdot, \cdot), y(\cdot) \in \mathbb{R}^n
\]

where, \( p \) denotes one or more free parameters. AUTO is employed to compute branches of stable and unstable periodic solutions as well as locate bifurcations along these branches.

### 2.4 Results

#### 2.4.1 Excitation frequency sweep

In this section, the nominal case solutions are examined over a range of excitation frequency. In automotive belt drives, the excitation frequency is the engine firing frequency, which varies across a wide range. Figure 2.5 shows the steady state (RMS) dynamic amplitude of the relative pulley-accessory rotation \( (\theta_p - \theta_a) - (\theta_p - \theta_a)_{\text{mean}} \) as the excitation frequency varies. Results for increasing and decreasing frequency are shown. The parameters considered are those in Table 2.1. For the results from numerical
integration, the increasing and decreasing frequency branches coincide except for the resonant regimes near the natural frequencies \( \Omega_{1,2} = 0.982, 6.81 \) of the linear system corresponding to a clutch of stiffness \( K_d = 27.66 \). Hysteretic behavior corresponding to a softening nonlinearity is evident for \( \Omega \approx \Omega_2 \). The softening behavior is a result of clutch disengagement where the spring between the pulley and accessory shaft is inactive. A slight kink occurs at the transition from clutch engagement at all times (linear system) to frequencies where the clutch disengages during some portion of the periodic solution. Nonlinear disengagement occurs outside the range where multiple steady state periodic solutions are possible. The waterfall plots of Figure 2.6 show the numerical integration spectra as excitation frequency increases and decreases. The higher harmonics of excitation frequency for \( \Omega \approx \Omega_2 \) indicate periodic, but not sinusoidal, response due to the presence of nonlinearity. The presence of these higher harmonics demonstrates the nonlinear clutch disengagement outside the region of multiple solutions. Figure 2.7a shows the time history and clutch torque for \( \Omega = 4.455 \), which lies at the point of local maximum response for decreasing frequency. Clutch disengagement (once per cycle) is apparent from the zero clutch torque.

Results in the range \( \Omega = 0.89 \sim 1.07 \) are more complicated as shown in the exploded figure in Figure 2.5b. In this range, only a few small sections are periodic. In the hysteretic range for \( \Omega \) just above 0.89 (point M), however, solutions are periodic for both increasing and decreasing frequency, though clutch disengagement occurs only along the upper branch achieved for decreasing frequency. Figure 2.7b shows time
response for $\Omega = 0.915$ (point F) on the periodic upper branch at the right boundary of the region where multiple steady state solutions exist. Notice multiple clutch disengagements occur in a single period. Throughout the range from E to A ($0.94 < \Omega < 1.07$), solutions are primarily quasiperiodic or chaotic with a few periodic exceptions indicated by the solid lines (discussed later). Chaotic response is evident in the rich response spectrum in this range for both increasing and decreasing frequency (Figure 2.6). Figure 2.8 shows time histories and clutch torque for the excitation frequencies $\Omega = 0.945$ (quasiperiodic) and $\Omega = 1$ (chaotic). For $\Omega = 0.945$, the spectrum consists of discrete components, the phase portrait is a banded attractor, and the Poincaré map is a closed curve; for $\Omega = 1$, the distributed spectrum, phase portrait, and Poincaré map indicate chaos.
Figure 2.5. RMS of \((\delta \theta - \delta \theta_{\text{mean}})\) versus excitation frequency for the nominal case of Table 2.1. a) Over a range of excitation frequencies including two resonances. b) Zooming a) at the first resonant region. HBM stable (---), unstable (--.--); numerical integration (NI) (---); two-DOF linear system (-----).
Figure 2.6. Waterfall spectra of $\delta \theta$ using numerical integration and the parameter values from Table 2.1. a) Decreasing excitation frequency; b) Increasing excitation frequency.
Figure 2.7. a) Time history of $\delta \theta$ and clutch spring torque $\bar{g}_s(\delta \theta)$ at $\Omega = 4.455$; b) Time history of $\delta \theta$, clutch spring torque $\bar{g}_s(\delta \theta)$ and spectrum of $\delta \theta$ at $\Omega = 0.915$. Results from numerical integration and decreasing frequency with parameter values from Table 2.1.
Figure 2.8. (i) Time history of $\delta \theta$, (ii) Clutch spring torque $g_s(\delta \theta)$, (iii) Spectrum of $\delta \theta$, (iv) Phase diagram, (v) Poincaré map from numerical integration and decreasing frequency with parameter values from Table 2.1. a) $\Omega = 0.945$; b) $\Omega = 1$. 

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Application of HBM with arclength continuation and AUTO provides a more complete picture of the dynamics for varying frequency. The model is equation (2.3) with the smoothing function (2.6). The unknown vector \( \mathbf{u} \) and elements of J in (2.15) are given by

\[
\mathbf{u} = \begin{bmatrix} u^T \ 
\Omega \end{bmatrix}^T
\]

(2.20)

\[
\mathbf{K} = \begin{bmatrix}
A + \overline{C}_{11} \mathbf{B} + \overline{K}_{b} \mathbf{I} & \overline{C}_{12} \mathbf{B} \\
\overline{C}_{21} \mathbf{B} & \alpha \mathbf{A} + \overline{C}_{22} \mathbf{B}
\end{bmatrix},
\frac{\partial \mathbf{d}}{\partial \mathbf{u}} = \Gamma \begin{bmatrix}
\frac{\partial g_s}{\partial \theta_p} & -\frac{\partial g_s}{\partial \theta_p} \\
\frac{\partial g_s}{\partial \theta_p} & \frac{\partial g_s}{\partial \theta_p}
\end{bmatrix} \mathbf{L}
\]

(2.21)

\[
\frac{\partial g_s}{\partial \theta_p} = \text{diag} \left( \sqrt{2} \overline{K}_d \left[ \epsilon \text{sech}^2(\epsilon \delta \theta) \cdot \delta \theta + 1 + \tanh(\epsilon \delta \theta) \right] \right)
\]

(2.22)

\[
\frac{\partial \mathbf{E}}{\partial \Omega} = \begin{bmatrix}
-2\Omega \mathbf{A}_0 + \overline{C}_{11} \mathbf{B}_0 \\
\overline{C}_{21} \mathbf{B}_0 & -\alpha 2\Omega \mathbf{A}_0 + \overline{C}_{22} \mathbf{B}_0
\end{bmatrix} \mathbf{u}
\]

(2.23)

Excellent agreement between HBM and numerical integration is apparent from Figure 2.5. No differences appear in the second resonant region. Harmonic balance gives a more complete picture of the periodic solutions for \( \Omega \approx \Omega_1 \), however. Three lobes of periodic solutions occur on the left of the exploded view in Figure 2.5b. Each of these has a small region of stability at their top, but are otherwise unstable. One can trace the jump-up and jump-down sequence observed in numerical integration as indicated by the arrows. On frequency increase, a dramatic jump up occurs from L to F near \( \Omega \approx 0.91 \). For decreasing frequency, solutions jump from the maximum lobe to the intermediate one and finally to the single solution that exists for \( \Omega \leq 0.89 \). Across the range E-A
(0.94 < \Omega < 1.07), a single unstable periodic solution exists, with the exception of a small interval C-B of stability. That interval can be construed as a fourth lobe. These findings are consistent with numerical integration, where quasiperiodic and chaotic solutions occur along interval E-A (with the exception of the small stable periodic solution interval C-B). While the shape of the first mode resonance in Figure 2.5b suggests near linear response, that is not the case in shape (discussed above) nor amplitude as shown by the linear system response curve in Figure 2.5b. Similar comments apply to the second mode resonance.

To compare results with AUTO, it is most convenient to use the maximum values of \( \theta_p \) and \( \theta_a \). We choose the maximum of \( \theta_p \) within a period to make the comparison in Figure 2.9 between HBM, AUTO, and numerical integration. Figure 2.9a shows excellent agreement of the three methods. More detail is apparent in the zoomed regions of Figure 2.9b and Figure 2.9c. (The letters A-M highlighting points in Figure 2.5 correspond to the same points on Figure 2.9b.) Comparison between HBM and AUTO solutions is indistinguishable even in the details of Figure 2.9b and c with the exception that HBM incorrectly predicts part of the branch F-G to be unstable. Increasing the number of harmonic terms and the temporal discretization resolution improves the results. In the region A-E, no stable periodic solutions exist, except for the small region B-C noted previously in Figure 2.5b. The numerical integration results confirm this conclusion. The pattern of jumps in the numerical integration results is consistent with the rich pattern of stable and unstable solution branches shown in Figure 2.9c (with slight differences resulting from limited frequency resolution).
Figure 2.9. Maximum of $\theta_p$ versus excitation frequency for parameters in Table 2.1. a) Range of excitation frequencies including two resonances; b) First resonant region; c) Zoom of the area highlighted in b). HBM and AUTO stable (---), unstable (----); NI (--.--).
The impact of nonlinearity is much different in the two resonant regimes \( \Omega = \Omega_1, \Omega_2 \). This stems from the difference in linear system vibration modes (Figure 2.2). The second mode involves out-of-phase motion between the pulley and accessory shaft and is more prone to clutch disengagement. This nonlinear softening spreads the resonance curve (and associated clutch disengagement) across a broad frequency range. The first mode generates less relative motion between pulley and accessory because these components move in-phase, so there is less softening nonlinearity from clutch disengagement. The frequency range over which disengagement occurs is markedly more limited. The amplitude is higher in the first mode resonant regime because the excitation from crankshaft fluctuations more directly excites this mode. At these high amplitudes, the system exhibits a rich range of bifurcations of periodic solutions not evident at the lower amplitudes near the second mode resonance.

### 2.4.2 Dependence on the clutch spring stiffness

In practice, the stiffness of the clutch spring can be designed across a wide range, and selected values are illustrated in Figure 2.10. For finite stiffness, the term one-way decoupler is sometimes used. The change in linear system natural frequencies with \( \bar{K}_d \) is shown in Figure 2.11. Only the second mode natural frequency changes meaningfully because this mode is the only one with significant relative pulley/accessory motion (Figure 2.2). The frequency response for various \( \bar{K}_d \) is shown in Figure 2.12 as computed by numerical integration. From the exploded inset, the amplitude of nonlinear response for \( \Omega = \Omega_1 \) is relatively insensitive to \( \bar{K}_d \) except for very low values, as one
would expect from Figure 2.11. Most solutions in this range are aperiodic, as discussed previously. An additional curve is shown in Figure 2.12 for the SDOF system with a locked clutch (typical belt drive pulley). For $\Omega \approx \Omega_1$, the resonant response decreases significantly because of the one-way clutch nonlinearity. The penalty balancing this benefit is the additional resonance region for $\Omega \approx \Omega_2$, much like a vibration absorber. The amplitude of response for $\Omega \approx \Omega_2$ is generally much lower than that for $\Omega \approx \Omega_1$, however. This amplitude decreases monotonically with increasing $\bar{K}_d$, and is negligible for $\bar{K}_d = 276.6$. In contrast, the amplitude for $\Omega \approx \Omega_1$ increases with increasing $\bar{K}_d$, though the changes are minimal compared to amplitude reductions for $\Omega \approx \Omega_2$. For the infinite stiffness limit as approximated by $\bar{K}_d = 2766$, the amplitude for $\Omega \approx \Omega_1$ is higher and noticeably smoother than for lower $\bar{K}_d$; solutions in this range are periodic. These results suggest use of large but not rigid clutch spring stiffness, although the optimal solution depends on the anticipated excitation frequency range. Both options are superior to the SDOF (no clutch) case.

To apply HBM, the unknown in (2.14) is $\bar{u} = \{u^T \quad \bar{K}_d\}^T$. Under the assumption that the damping matrix $\bar{C}$ is the same as for the nominal case, determination of the Jacobian matrix in (2.15) requires

$$
\begin{align*}
\frac{\partial E}{\partial \bar{K}_d} &= \begin{bmatrix}
\frac{\partial g_x}{\partial \bar{K}_d} \\
-\frac{\partial g_x}{\partial \bar{K}_d}
\end{bmatrix} \\
&= \Gamma
\end{align*}
$$

(2.24)
where \( \frac{\partial \tilde{g}}{\partial K_d} = \frac{1}{2} [1 + \tanh(\varepsilon \delta \theta)] \cdot \delta \theta \). This procedure yields results that agree with Figure 2.12 to the same degree as the comparison in Figure 2.9.

![Figure 2.10. Clutch spring torque for various spring stiffnesses \( K_d \).](image-url)
Figure 2.11. Sensitivity of the linear system natural frequencies to clutch stiffness $\bar{K}_d$ for parameters in Table 2.1.
2.4.3 Dependence on excitation amplitude

Now consider the influence of excitation amplitude $A_m$, which in a belt drive represents the magnitude of the crankshaft pulley fluctuations due to engine firing. The above results are for $A_m = 0.001$. When varying amplitude $A_m$ for given excitation frequency $\Omega$, the unknown in (2.14) is $\tilde{u} = \{u^T A_m\}^T$. Equations (2.21) and (2.22) are again used to determine the Jacobian matrix (2.15) with the modification that equation (2.23) is replaced by
\[
\frac{\partial E}{\partial A_m} = -\frac{\partial F}{\partial A_m} = \left\{0 \quad \beta \bar{K}_b \quad 0 \quad \cdots \quad 0\right\}^T_{2(2R+1)}
\] (2.25)

Figure 2.13a,b show the maximum of \( \theta_p \) for varying \( A_m \) and other parameters as in Table 2.1 for three excitation frequencies \( \Omega = 0.92, 1, 2 \) determined by HBM and AUTO (the two results are indistinguishable). \( \Omega = 0.92 \) and \( \Omega = 1 \) are near resonant excitation frequencies for the first mode; \( \Omega = 2 \) is off-resonant. For \( A_m > 0.007 \), all periodic solutions are unstable except for a branch from \( 0.01 < A_m < 0.022 \) for \( \Omega = 1 \). Note that for large excitation amplitudes the off-resonant periodic solution, while unstable, is higher amplitude than those for \( \Omega = 0.92, 1 \). Figure 2.13b zooms the low excitation amplitude region of Figure 2.13a. Here the stable and unstable resonant amplitudes exceed the off-resonant one at \( \Omega = 2 \). A linear region is apparent for small \( A_m \). With \( A_m = 0.001 \), the single periodic solution for \( \Omega = 1 \) is unstable, but there is a stable periodic solution for \( \Omega = 0.92 \), which is in agreement with Figure 2.9b. The cases for \( \Omega = 6 \) and \( \Omega = 6.8 \) (Figure 2.13c) exhibit the classical nonlinear behavior. For \( A_m = 0.001 \), there are three solutions for \( \Omega = 6 \). Two are stable and the intermediate one is unstable. For \( \Omega = 6.8 \) only one solution is generated at all amplitudes. For large amplitudes, the single periodic solution branches for each of \( \Omega = 6, 6.8 \) are unstable and no stable periodic solutions exist.
Figure 2.13. Maximum of $\theta_p$ versus excitation amplitude for parameters in Table 2.1. a) $\Omega = 0.92, 1, 2$; b) Zoomed plot of highlighted area in a); c) $\Omega = 6, 6.8$. HBM and AUTO stable ( ), unstable ( ).
2.4.4 Dependence on the ratio of pulley and accessory inertias

The parameter $\alpha = \frac{J_a}{J_p}$ governs the ratio of the inertia of the pulley to that of the accessory. Figure 2.14 shows the sensitivity of the linear system natural frequencies to $\alpha$. Figure 2.15 illustrates two excitation frequency sweep examples for $\alpha = 5$ and $\alpha = 0.3$; $\alpha = 1.62$ is in Figure 2.9. The results from HBM and AUTO are identical. For $\alpha = 5$, the inertia of the pulley is much smaller than that of the accessory, and the natural frequencies of the linear system are $\Omega_1 = 0.640$ and $\Omega_2 = 5.950$. At the second resonant peak, there is only one stable periodic solution branch at the bottom, and the two upper branches all consist of unstable periodic solutions except a small section close to the tip of the peak. This contrasts with $\alpha = 1.62$ in Figure 2.9, where the upper branch at the second resonant region is always stable. Complex behavior near $\Omega_1 = 0.640$ is shown in the inset to Figure 2.15. For $\alpha = 0.3$, the natural frequencies are $\Omega_1 = 1.42$ and $\Omega_2 = 11$. The peak in the first mode is much more pronounced than the barely evident peak at $\Omega = 11$. All periodic solutions are stable except in a narrow region at the tip of the first peak.
Figure 2.14. Sensitivity of natural frequencies of linear system to $\alpha$ for parameters in Table 2.1.
Physically, imagine the case of a big pulley driving a small accessory, i.e., $\alpha$ is small. In this case, the nonlinear clutch and accessory attached to the pulley have little impact on the belt-pulley system, and the behavior is largely that of a linear SDOF system. For larger $\alpha$ ( $\alpha=1.62$ in Figure 2.9, $\alpha=5$ ), considerably more pulley/accessory coupling occurs. Significant clutch disengagement occurs in the second, out-of-phase mode. The large amplitude response in the first mode creates a complicated periodic solution picture for $\Omega = \Omega_i$ even though this in-phase mode is less prone to disengagement.

Figure 2.15. Maximum of $\theta_p$ versus frequency for $\alpha = 5$ and $\alpha = 0.3$ with other parameters in Table 2.1. HBM and AUTO stable (___), unstable (____).
CHAPTER 3

PERTURBATION ANALYSIS OF A CLEARANCE-TYPE NONLINEAR SYSTEM

In this chapter, the method of multiple scales is applied to obtain periodic solutions of a two-pulley belt system with clearance-type nonlinearity. The purpose is to explain the obtained numerical results in Chapter 2 and clarify how design parameters affect the system dynamics. The validity of the perturbation method for such strong nonlinearity is evaluated. The closed-form frequency response relation is determined at the first order, and an implicit expression is obtained for the second order approximation. The preload applied to the accessory determines the softening level of the nonlinearity. Larger preload leads to less disengagement and less softening. For a considerable range of practical parameter values, the analytical solutions well approximate the numerical results from harmonic balance.
3.1 Introduction

A two-pulley system with clearance-type nonlinearity is illustrated in Figure 3.1a. An application of this system has been presented in [58], where a discontinuous separation function models the alternate engagement and disengagement of the pulley and accessory that functions through a one-way clutch. In [58], the harmonic balance method with arc-length continuation is employed. Results show a rich picture of stable and unstable periodic solutions when the system operates across a range of excitation frequencies. The present work pursues analytical periodic solutions of this system in order to explain the numerical results and clarify how the design parameters affect the dynamics.

Perturbation analysis to obtain periodic solutions of dynamic systems is most commonly applied to weak nonlinearities. A few studies (e.g., [59-62]) deal with the validity of this technique for strongly nonlinear systems, where the Lindstedt-Poincaré perturbation scheme is extended. For discontinuous nonlinearities with piecewise linear or weakly nonlinear characteristics, piecewise analytical methods are generally employed [63-65] where the perturbation technique is used for each linear or nonlinear piece.

The current study applies the method of multiple scales to the system with clearance-type nonlinearity shown in Figure 3.1a and evaluates the validity of the method for such strong nonlinearity. The discontinuous function is expanded as a Fourier series, where the Fourier coefficients are evaluated for given excitation amplitude. The closed-form frequency-response relation is determined at the first order. Periodic approximations
are determined up to the second order. The system is analyzed for a range of excitation frequency with different preload values to characterize where the power transmission is most efficient.

Figure 3.1. a) A two-pulley belt system; b) Wrap spring torque $g(\delta\theta)$ in (3.1) and the smoothed function $g_s(\delta\theta)$ according to (3.3) for different smoothing parameters $e$. 
<table>
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</tr>
</tbody>
</table>

Table 3.1. Nomenclature and dimensionless parameter values for nominal case.

### 3.2 System Model

The driving pulley in Figure 3.1a is subject to periodic motion excitation specified as $\theta_{c/s} = A_m \cos \Omega t$, and the power is transmitted to the driven pulley through a belt modeled as springs with stiffness $\tilde{K}_b/2$. The driven pulley and the accessory shaft are connected with a wrap spring of stiffness $K_d$ that is disconnected when the pulley and the shaft are disengaged. An accessory that acts as a load is rigidly connected to the shaft. The system shows clearance-type nonlinearity when the pulley and shaft are alternately engaged and disengaged. When the rotations of the wrap-spring ends are such that the pulley rotation $\theta_p$ exceeds the accessory shaft rotation $\theta_a$, the clutch is engaged. Alternately, when pulley rotation is less than accessory rotation, the wrap-spring diameter decreases and the clutch disengages. Only engagement of the pulley and shaft allows...
power transmission to the accessory. The wrap-spring torque is mathematically expressed in the dimensionless form (Figure 3.1b)

\[ g(\delta \theta) = \begin{cases} K_d \delta \theta & \delta \theta > 0 \\ 0 & \delta \theta \leq 0 \end{cases} \]  
(3.1)

where \( \delta \theta = \theta_p - \theta_a \). The equations of motion for the pulley and accessory are

\[
\begin{bmatrix} J_p & 0 \\ 0 & J_a \end{bmatrix} \ddot{\theta} + C \dot{\theta} + \begin{bmatrix} K_b & 0 \\ 0 & 0 \end{bmatrix} \theta = \begin{bmatrix} g(\delta \theta) \\ -g(\delta \theta) \end{bmatrix} - \begin{bmatrix} M + K_c \beta A_m \cos \Omega t \\ -M \end{bmatrix}
\]  
(3.2)

where \( \theta = [\theta_p \ \theta_a]^T \), \( K_b = \tilde{K}_b r_p^2 \), \( \beta = r_c / r_p \), and the dot denotes the time derivative \( d/dt \). \( C \) is computed from the modal damping matrix of the two-DOF linear system with the pulley and shaft engaged. See Table 3.1 for the nomenclature and dimensionless parameter values.

A hyperbolic tangent function

\[ g_s(\delta \theta) = K_d \delta \theta f_s, \quad f_s = \frac{1}{2} [1 + \tanh(e \delta \theta)] \]  
(3.3)

is employed in [58] to approximate \( g(\delta \theta) \) for multiple methods. According to [58], use of \( e = 10,000 \) in (3.3) ensures accuracy (Figure 3.1b). In the following, the original discontinuous function (3.1) is considered and the results are compared with those from harmonic balance using (3.3).
3.3 Perturbation Analysis

The method of multiple scales is frequently used to obtain periodic approximations for systems with continuous, weak nonlinearity [66]. This yields closed-form approximations for the frequency-response curve. Furthermore, it can provide theoretical explanation for phenomena observed by numerical methods. Here the method of multiple scales is employed for the system in (3.2) but rewritten as

\[ \begin{bmatrix} J_p & 0 \\ 0 & J_a \end{bmatrix} \theta + C\theta + \begin{bmatrix} K_b + K_d & -K_d \\ -K_d & K_d \end{bmatrix} \theta + \begin{pmatrix} h(\delta \theta) \\ -h(\delta \theta) \end{pmatrix} = \begin{pmatrix} M + K_b \beta A_m \cos \Omega t \\ -M \end{pmatrix} \]  

(3.4)

\[ h = -K_d \delta \theta f(\delta \theta), \quad f(\delta \theta) = \begin{cases} 0 & \delta \theta \geq 0 \\ 1 & \delta \theta < 0 \end{cases} \]  

(3.5)

Let \( U \) be the orthonormalized modal matrix of the linear undamped system. Letting \( \theta = Uq \), one obtains the decoupled form with modal coordinates \( q \) as

\[ \ddot{q}_i + 2\zeta_i \Omega_i \dot{q}_i + \Omega_i^2 q_i + H_i(\delta \theta) = M_i + \tilde{B}_i \cos \Omega t \quad i = 1, 2 \]  

(3.6)

where \( \Omega_i \) are the linear undamped natural frequencies of (3.4), \( H_i(\delta \theta) = \gamma_i h(\delta \theta) \) with \( \gamma_i = u_i - u_2 \) and \( \delta \theta = \gamma_1 q_1 + \gamma_2 q_2 \), \( M_i = \gamma_i M \) and \( \tilde{B}_i = u_i K_b \beta A_m \).

The separation function \( f(\delta \theta) \) in (3.5) is expanded as a Fourier series. To be consistent with numerical results in [58] where the response is periodic at the excitation frequency, \( \Omega \) is the fundamental frequency of \( f(\delta \theta) \), that is,

\[ f = \hat{f}_0 + \left( \sum_{k=1}^{\infty} \hat{f}_k e^{jk\Omega t} + c.c. \right) \]  

(3.7)
where \( c.c. \) denotes complex conjugate. To introduce a perturbation parameter, let \( \varepsilon \) be the fraction of the response period \( T \) where the pulley and shaft are disengaged. This assumption and another that only a single disengagement occurs per cycle lead to Fourier coefficients \( \tilde{f}_k, k \geq 0 \), that are all order \( \varepsilon \) (demonstrated later). Therefore, one can write (3.7) as

\[
f = \varepsilon \left[ \tilde{f}_0 + \sum_{k=1}^{\infty} \tilde{f}_k e^{i\varepsilon k\Omega} + c.c. \right] = \varepsilon \tilde{f} \quad \text{where} \quad \tilde{f}_k = \tilde{f}_k / \varepsilon = O(1).
\]

The quantities \( t_0 = t \) and \( t_n = \varepsilon^n t, n \geq 1 \) are fast and slow times, respectively. The differential operators and the response are expanded as

\[
\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + O(\varepsilon^3), \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + O(\varepsilon^3) \quad (3.8)
\]

\[
q_i = q_{i0}(t_0, t_1, t_2) + \varepsilon q_{i1}(t_0, t_1, t_2) + \varepsilon^2 q_{i2}(t_0, t_1, t_2) + O(\varepsilon^3) \quad (3.9)
\]

Considering the second primary resonance as an example, the excitation frequency is \( \Omega = \Omega_2 + \varepsilon \sigma \) where \( \sigma \) is a detuning parameter. Internal resonance is not considered. The dynamic excitation and damping are specified as \( O(\varepsilon) \) according to \( \tilde{B}_i = \varepsilon B_i \) and \( \zeta_i \omega_i = \varepsilon \mu_i \). Substitution of (3.8) and (3.9) into (3.6) gives

\[
D_0^2 q_{i0} + \Omega_i^2 q_{i0} = M_i \quad (3.10)
\]

\[
D_0^2 q_{i1} + \Omega_i^2 q_{i1} = -2\mu_i D_0 q_{i0} - 2D_0 D_1 q_{i0} - H_{i1} + B_i \cos \Omega t_0 \quad (3.11)
\]

\[
D_0^2 q_{i2} + \Omega_i^2 q_{i2} = -2\mu_i D_0 q_{i0} - 2\mu_i D_0 q_{i1} - 2D_0 D_1 q_{i1} - D_1^2 q_{i0} - 2D_0 D_2 q_{i0} - H_{i2} \quad (3.12)
\]

for \( i = 1, 2 \), and where \( H_{i1} \) and \( H_{i2} \) include the \( \varepsilon^1 \), \( \varepsilon^2 \) order terms in \( H_i \), respectively.
3.3.1 First order approximation

The leading order approximation from (3.10) is

\[ q_{i0}(t_0, t_1, t_2) = m_i + \left[ A_i(t_1, t_2) e^{j\omega_0 t_0} + c.c. \right] \]  

(3.13)

where \( A_i(t_1, t_2) \) is the unknown amplitude and \( m_i = M_i / \Omega_i^2 \). The solvability conditions generated from (3.11) are

\[
2 j \Omega D_i A_i = \left( -2 \mu_i j \Omega_1 + \gamma_i^2 K_d \tilde{f}_0 \right) A_i \\
2 j \Omega_2 D_i A_2 = -2 \mu_i j \Omega_2 A_2 + \gamma_i^2 K_d \left[ \gamma_i A_2 \tilde{f}_0 + (\gamma_i m_1 + \gamma_2 m_2) \tilde{f}_1 e^{j\sigma_i} + \gamma_2 A_2 \tilde{f}_2 e^{j2\sigma_i} \right] + b_2 e^{j\sigma_i} 
\]

(3.14)

with \( b_i = B_i / 2 \). The overbar denotes complex conjugate. At the steady state, \( A_i \to 0 \) as \( t_1 \to \infty \) from the first equation of (3.14). This trivial solution guarantees the response has fundamental frequency \( \Omega \approx \Omega_2 \) according to \( \delta \theta = \gamma_i q_{10} + \gamma_2 q_{20} + O(\varepsilon) \). Now one can take

\[ A_i(t_1, t_2) = a_i(t_1, t_2) e^{j\beta_i(t_1, t_2)} / 2, \quad i = 1, 2 \]  

(3.15)

and write \( q_{20} \) as the real form \( q_{20} = a_2 \cos(\Omega_2 t_0 + \beta_2) + m_2 \). Because

\[ \delta \theta \approx \gamma_2 [a_2 \cos(\Omega_2 t_0 + \beta_2) + m_2] + \gamma_1 m_i \]  

(3.16)

the Fourier coefficients of \( f(\delta \theta) \) are

\[
\tilde{f}_0 = \frac{\Omega}{2\pi} \int_0^{2\pi} f(\delta \theta) e^{j(\Omega_2 t_0 + \beta_2)} d\phi = \frac{1}{2\pi} \int_0^{2\pi} f(\delta \theta) d\phi \\
\tilde{f}_k = \frac{1}{2\pi} e^{j\beta_2 - \sigma_0} \int_0^{2\pi} f(\delta \theta) e^{-jk\phi} d\phi 
\]

(3.17)

In (3.17), \( t_i \) varies slowly compared to \( t_0 \) and is considered as a constant in the integrals.

To evaluate the integrals in (3.17), identification of the region where \( \delta \theta < 0 \) (giving
\( f(\delta \theta) = 1 \) from (3.5)) is needed (see Figure 3.2a). The critical phases \( \phi = \kappa_1, \kappa_2 \) are obtained from \( \delta \theta = 0 \) as

\[
\cos \phi \bigg|_{\phi = \kappa_1, \kappa_2} = -\frac{\left( \gamma_1 m_1 + \gamma_2 m_2 \right)}{\gamma_2 a_2} = -\frac{\left( \gamma_1^2 / \Omega_1^2 + \gamma_2^2 / \Omega_2^2 \right) M}{\gamma_2 a_2} = \lambda \tag{3.18}
\]

Here the case of \( \gamma_2 < 0 \) is discussed (\( \gamma_2 > 0 \) generates the same results). The integration interval is \([0, \kappa_1] \cup [\kappa_2, 2\pi] \), where \( \kappa_1 = \cos^{-1} x \) and \( \kappa_2 = 2\pi - \cos^{-1} x \). For \( x > 1 \) there is no separation and the pulley and shaft are always engaged. In this case, \( f(\delta \theta) = 0 \) and the system operates linearly. On the other hand, \( x < -1 \) implies the pulley and shaft are disengaged for the entire cycle and \( f(\delta \theta) = 1 \). Further expansion of (3.17) gives

\[
\hat{f}_0 = \frac{\kappa_i}{\pi}, \quad \hat{f}_k = \chi_k e^{ik(\beta_1 - \sigma_i)}, \quad \chi_k = \sin k\kappa_i / k\pi, \quad \text{for} \ k \geq 1 \tag{3.19}
\]

For given belt stiffness \( K_b \) and wrap spring stiffness \( K_d \), the preload \( M \) affects the separation for a given amplitude \( a_2 \). Positive preload acts against separation and promotes power transmission, while negative preload promotes separation. For the parameters in Table 3.1, \( \gamma_2 < 0 \) and, according to (3.18), \( \lambda > 0 \) for \( M \geq 0 \). Therefore, for any vibration amplitude where disengagement occurs, \( 0 < \kappa_i \leq \pi / 2 \). This is consistent with the prior assumption that \( \kappa_i / \pi \) is small. Also, \( \hat{f}_0 \in [0, 0.5] \). The mean value \( |\hat{f}_0| \) of the separation function \( f \) is greater than any harmonic amplitude \( |\chi_k| \) because

\[
|\chi_k| = |\sin k\kappa_i / k\pi| \leq |\kappa_i / \pi| = |\hat{f}_0| \tag{Figure 3.2b}
\]

All these validate the earlier stipulation that \( \hat{f}_0, \hat{f}_k = O(\varepsilon) \). In the following, let \( \bar{\chi}_k = \chi_k / \varepsilon \).
Figure 3.2. a) The integration intervals for (3.17)-(3.19) and separation function $f(\delta \theta)$ (— — ) as $\gamma_2 < 0$. b) Magnitudes of Fourier coefficients of the separation function $f$ vary with the response amplitude $a_2$ for $M = 0.0127$. $|f_0|$ (— •), $|x_1|$ (— — — ), $|x_2|$ (— —— ), $|x_3|$ (••••).
Separation of the real and imaginary parts of (3.14) yields

\[
\Omega_1 D_1 a_i = -\Omega_i \mu_i a_i, \quad -\Omega_i a_i D_1 \beta_i = \frac{1}{2} \gamma_1^2 K_d \tilde{f}_0 a_i
\]  
(3.20)

\[
\Omega_2 D_1 a_i = -\Omega_2 \mu_2 a_i + b_2 \sin \lambda
\]
\[
\Omega_2 a_i D_1 \lambda = \Omega_2 a_i \sigma + \gamma_2 K_d \left[ \frac{1}{2} \gamma_2 a_2 \tilde{f}_0 + (\gamma_2 m_1 + \gamma_2 m_2) \tilde{X}_1 + \frac{1}{2} \gamma_2 a_2 \tilde{X}_2 \right] + b_2 \cos \lambda
\]  
(3.21)

where \( \lambda = \sigma t - \beta_2 \). Considering steady state motion where \( D_1(\cdot) = 0 \), the frequency-response relation from (3.20) and (3.21) is \( a_i = 0 \) and

\[
\sigma = -R \pm \sqrt{\frac{b_2^2}{\Omega_2^2 a_i^2} - \mu_i^2}, \quad R = \frac{\gamma_2 K_d}{\Omega_2} \left[ \frac{1}{2} \gamma_2 \tilde{f}_0 + (\gamma_2 m_1 + \gamma_2 m_2) \frac{\tilde{X}_1}{a_2} + \frac{1}{2} \gamma_2 \tilde{X}_2 \right]
\]  
(3.22)

3.3.2 Second order approximation

To seek second order approximations, the reconstitution procedure [39-41, 67] is adopted. After eliminating secular terms, the particular solutions \( q_i(t) \) of (3.11) are obtained. Substitution of \( q_i(t) \) into (3.12) and collection of the secular terms of (3.12) lead to the solvability conditions

\[
-2j\Omega_2 D_2 A_i - D_i^2 A_i - 2\mu_i D_i A_i + I_1 A_i = 0
\]  
(3.23)

\[
-2j\Omega_2 D_2 A_2 - D_2^2 A_2 - 2\mu_2 D_2 A_2 + I_2 = 0
\]  
(3.24)

where \( I_1 = I_1(\tilde{f}_0, \tilde{X}_k) \) and \( I_2 = I_2(\tilde{f}_0, \tilde{X}_k, A_2, \sigma) \) are collected from \( H_{12} \) in (3.12). \( D_i^2 A_i \) in (3.23) and (3.24) are determined by differentiating (3.14) with respect to \( t_i \). Substitution of (3.15) into (3.23) and (3.24) and separation of their real and imaginary parts yield

\[
\Omega_1 D_2 a_i = -\frac{K_d \gamma_i^2}{2 \Omega_i} \mu \eta \xi a_i^2, \quad -\Omega_i a_i D_2 \beta_i = \frac{1}{2} \frac{a_i}{a_i} \left( \mu_i^2 + \frac{K_d^2 \chi_i^2 \tilde{f}_0^2}{\Omega_i^2} - I_1 \right)
\]  
(3.25)
\[ \Omega_2 D_2 a_2 = r_1 + r_1 \sin \lambda + r_2 \cos \lambda, \quad \Omega_2 a_2 D_2 = r_2 + r_1 \sin \lambda + r_2 \cos \lambda \]  \hspace{1cm} (3.26)

\( \eta_k, k \geq 0 \) are determined by differentiating the Fourier coefficients \( \tilde{f}_0, \tilde{\chi}_k \) with respect to \( t_1 \) as
\[ D_1 \tilde{f}_0 = \eta_0 D_1 a_2 \quad \text{and} \quad D_1 \tilde{\chi}_k = \eta_k D_1 a_2 \quad \text{for} \quad k \geq 1, \]
where
\[ \eta_0 = \frac{x}{\varepsilon 2\pi a_2 \sqrt{1-x^2}}, \quad \eta_k = \frac{\varepsilon \cos(k \cos^{-1} x)}{\varepsilon \pi a_2 \sqrt{1-x^2}}, \quad k = 1, 2, \ldots \]  \hspace{1cm} (3.27)

for \(|x| < 1\). When \(|x| \geq 1\), \( \eta_k = 0 \). \( r_i \) and \( r_2 \) consist of the coefficients of \( \sin \lambda \) and \( \cos \lambda \), respectively. \( r_i \) is the collection of all terms except those with \( \sin \lambda \) and \( \cos \lambda \) as coefficients. For steady state motion,
\[ \frac{da_i(t_1, t_2)}{dt} = \varepsilon D_1 a_i + \varepsilon^2 D_2 a_i + O(\varepsilon^3) = 0 \]
\[ \frac{d\beta_i(t_1, t_2)}{dt} = \varepsilon D_1 \beta_i + \varepsilon^2 D_2 \beta_i + O(\varepsilon^3) = 0 \]  \hspace{1cm} (3.28)

Therefore, by combining (3.20) with (3.25) and (3.21) with (3.26), the steady state equations are
\[ -\varepsilon \mu a_1 - \varepsilon^2 \frac{K_d R^2 \mu \eta_0 a_i^2}{2\Omega_i^2} + O(\varepsilon^3) = 0 \]
\[ -\frac{1}{2\Omega_2} \left[ \varepsilon K_d \gamma_i \tilde{f}_0 + \varepsilon^2 \left( \mu_i^2 + \frac{K_d^2 \gamma_i \tilde{f}_0^2}{\Omega_i^2} - I_1 \right) \right] + O(\varepsilon^3) = 0 \]  \hspace{1cm} (3.29)

\[ \frac{1}{\Omega_2^2} \left[ \varepsilon(-\Omega_d \mu a_2 + b_3 \sin \lambda) + \varepsilon^2 (r_1 + r_2 \sin \lambda + r_2 \cos \lambda) \right] + O(\varepsilon^3) = 0 \]
\[ \frac{\varepsilon}{\Omega_d a_2} \left[ \Omega_d \sigma + \gamma K_d \left[ \gamma_2 a_2 \tilde{f}_0 + 2(\gamma m_t + \gamma_2 m_b) \tilde{\chi}_1 \right] + b_3 \cos \lambda \right] + \frac{\varepsilon^2}{\Omega_d a_2} (r_1 + r_2 \sin \lambda + r_2 \cos \lambda) + O(\varepsilon^3) = 0 \]  \hspace{1cm} (3.30)

Clearly, \( a_1 = 0 \) is a solution of (3.29), which implies the first mode is not excited at all.

By ignoring \( O(\varepsilon^3) \) terms, (3.30) is rewritten as
The elimination of \( \sin \lambda \) and \( \cos \lambda \) by using common algebraic manipulations and trigonometric identities yields a polynomial in \( \sigma \). For chosen amplitude \( a_2 \), one finds the roots of the polynomial. Consequently the frequency-response relation analogous to (3.22) but at a higher order is generated.

### 3.4 Results and Discussions

The first order approximation from (3.22) is shown in Figure 3.3 for parameters given in Table 3.1. To implement the stability analysis of the steady state motions at \((a_{20}, \lambda_0)\) from (3.21), a small variation \( \mathbf{v} = \{\alpha, \vartheta\}^T \) near a steady state is introduced as

\[
a_2 = a_{20} + \alpha, \quad \lambda = \lambda_0 + \vartheta
\]

For the first order approximation, the stability analysis can be conducted similarly as [66]. Substitution of (3.32) into (3.21) yields \( D_1 \mathbf{v} = \mathbf{J} \mathbf{v} + O(\mathbf{v}) \), where \( \mathbf{J} \) is the Jacobian matrix consisting of the first order derivatives with respect to \( (\alpha, \vartheta) \) evaluated at \((a_{20}, \lambda_0)\), i.e.,

\[
\mathbf{J} = \begin{bmatrix}
-\mu_2 & -\alpha(\sigma + R) \\
\frac{1}{\alpha}(\sigma + R + \frac{\partial R}{\partial \alpha}) & -\mu_2
\end{bmatrix}_{(\pi_2, \pi_3)}
\]
The real parts of the eigenvalues of $J$ determine the stability at steady state $(a_{20}, \lambda_0)$. One can prove that

$$\Gamma = \mu_2^2 + (\sigma + R) \left( \sigma + R + \frac{\partial R}{\partial \alpha} \right)_{k_{a_0}, \lambda_0} < 0$$

(3.35)

guarantees unstable steady state motions and the roots of $\Gamma = 0$ correspond to the two frequencies $\sigma_1$ and $\sigma_2$ on the turning points $P_1$ and $P_2$ (see Figure 3.3), respectively, where $\frac{\partial \sigma}{\partial a_2} = 0$. On the arc between $P_1$ and $P_2$, $\Gamma < 0$. Accordingly, this branch is unstable [66]. Other than this arc, the solutions are stable.

In comparison with the results yielded by numerical integration and multi-term harmonic balance with arc-length continuation, the perturbation approximation generates lower maximum amplitude with less softening nonlinearity. According to (3.22), the maximum amplitude is determined by

$$a_2 = \left| \frac{b_2}{\Omega_2 \mu_2} \right|$$

(3.36)

where $b_2 = u_{12}K_p\beta A_\omega/2\epsilon$. This is identical to the maximum amplitude of the linear system. The terms outside the square root in (3.22), $-R$, give the backbone curve. The backbone is affected only by the mean value and the first two harmonics of the separation function. These two issues are limitations of the first order solution.
Figure 3.3. RMS of $\delta \theta$ at the 2$^{nd}$ primary resonance yielded by the method of multiple scales, multi-term harmonic balance (HB) and numerical integration (NI) for the parameters in Table 3.1. Stable (---); unstable (—).
Figure 3.4. Separation function $f$ varies with time for points $P_3$ and $P_5$ in Figure 3.3, where RMS of $\delta \theta$ is $9.64 \times 10^{-4}$ ($a_2 = 4.3 \times 10^{-5}$). (---) $f_g$ in (3.3) at $P_5$, from multi-term harmonic balance; (—) analytical solution at $P_3$, from 4-term truncated Fourier series.

The transition frequencies where the disengagement begins and ends can be determined through the backbone terms $-R$ in (3.22). For the linear case, $\tilde{f}_0$, $\tilde{\chi}_k$ in $R$ are all zero. The transition frequencies occur when $\tilde{f}_0$, $\tilde{\chi}_k$ start to be non-zero, which corresponds to a maximum amplitude $a_{cr}$ above which the system behaves nonlinearly (shown in Figure 3.2b). For the nominal case, $\Omega_{cr} \approx 6.24, 7.38$ where points $P_1$ and $P_7$ are located correspondingly in Figure 3.3. These values are consistent with the numerical results and those from analytical single-term harmonic balance.
The second order approximation in the original physical coordinates is also shown in Figure 3.3. One substitutes (3.32) into the derivatives of (3.28) for $i = 2$ and considers $\dot{\lambda}$ instead of $\dot{\beta}_2$ for the stability analysis. $d\mathbf{v}/dt = \mathbf{J}\mathbf{v} + O(\mathbf{v})$ results, where

$$
\mathbf{J} = \begin{bmatrix}
\frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \theta} \\
\frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \theta}
\end{bmatrix}
_{(a_0, \dot{\alpha}_0)} \tag{3.37}
$$

That all eigenvalues of $\mathbf{J}$ have negative real part launches a stable periodic picture. In contrast, any eigenvalue with positive real part indicates the solution is unstable. The solid and dashed curves distinguish the stability properties in Figure 3.3.

The second order approximation somewhat improves the amplitude and softening nonlinearity of the resonant peak in Figure 3.3. Determination of the Fourier coefficients (3.17) and (3.19) is the same as for the first order case, which limits the improvement from a second order approximation. Corresponding to the points $P_3$ and $P_5$ in Figure 3.3, where the maximum amplitude of the first order approximation is $a_2 = 4.3 \times 10^{-5}$ (RMS of $\delta\theta$ is $9.64 \times 10^{-4}$), the time histories of the separation function $f$ in (3.5) are developed in Figure 3.4. The smoothing function $f_g$ in (3.3) is used for multi-term harmonic balance, while a four-term Fourier series of $f$ is used for the analytical approximation. Note that the point $P_4$ with $a_2 = 4.3 \times 10^{-5}$ on the second order branch in Figure 3.3 has a similar shape of time history for $f$ as $P_3$, although a phase difference exists. Apparently, the prediction of the separation (when $f \approx 1$) from the method of multiple scales is poorer than that from multi-term harmonic balance where half period of
the disengagement is shown. It is hard for the procedures (3.16)–(3.19) to determine half time separation ($\hat{f}_0 = 0.5$ only when $M = 0$). When the amplitude is large, the large separation resulting from the nominal parameters according to the numerical solutions violates the assumption of small time fraction of a period where the pulley and shaft are disengaged.

3.4.1 First primary resonance

By considering the first primary resonance $\Omega = \Omega_1 + \epsilon \sigma$ and following the aforementioned perturbation procedures and assumptions, one can obtain an analytical approximation for the first primary resonance. On the frequency-response curve, the maximum amplitude of approximation is the same as the linear solution but the peak bends slightly to the left and all solutions are stable. In contrast, the numerical results in [58] form a complicated bifurcation diagram with alternate stable and unstable branches and the amplitude is distinguished markedly from the linear one. There are several reasons that the first primary resonance approximations are poor. First, the small separation assumption is not appropriate based on numerical results. Second, around the first primary resonance, not only the first mode but the second mode is excited. According to the frequency spectrum in [58], the highest spike occurs at the excitation frequency $\Omega = \Omega_1$ and comparably high spikes occur around $\Omega_2$ as well. Through the perturbation procedures, the first primary resonance does not show second mode participation ($a_2 \rightarrow 0$ as $t \rightarrow \infty$). Third, the harmonic resonance assumption cannot capture multiple disengagements or higher harmonic components. The numerical results
indicate that near the first primary resonance, for a given excitation frequency $\Omega$, the response not only includes the fundamental frequency $\Omega$ but frequency components at $4\Omega$, $5\Omega$ or $6\Omega$ with high amplitudes, where the multiple of $\Omega$ depends on the lobe location [58].

3.4.2 Impact of preload on the system dynamics

Preload $M$ is an important design parameter. Positive preload is a practical requirement that guarantees the desired power transmission from the pulley to the accessory. The higher the preload is, the less disengagement occurs during a cycle.

We investigate the impact of the preload on the second primary resonance. If all the Fourier coefficients are zero, i.e., $f(\delta \theta) = 0$, the system response is linear. Given the linear amplitude according to (3.36), $a_2 = 4.3 \times 10^{-3}$, the Fourier coefficient curves from (3.19) with varying preload are obtained, which shows that $M \geq 0.0377$ prevents disengagement. The critical preload can be analytically determined by setting $\hat{f}_k = 0$ in (3.19), which causes $\cos \kappa = 1$, and from (3.18)

$$M_{cr} = -\frac{\gamma_2 a_2}{\left(\frac{\gamma_1}{\Omega_1^2} + \frac{\gamma_2}{\Omega_2^2}\right)}$$

Equations (3.36) and (3.38) imply that, for the given system, any set of excitation amplitude and damping ratio admits a critical preload beyond which the system behaves linearly at the second primary resonance. If $M \leq 0$, the numerical methods cannot yield steady state periodic solutions, while the perturbation procedure is not applicable because
disengagement would be the leading motion between the pulley and shaft and this violates the basic assumptions.

Figure 3.5 illustrates that large positive preload weakens the softening nonlinearity, and the perturbation method predicts good approximations. In contrast, low preload, for example, $M = 0.0127$ in Figure 3.3, causes strong softening nonlinearity and the perturbation approximations are poorer. For $M \geq 0.0377$, the response overlaps the linear solution curve. Figure 3.6 compares the time histories and the associated spectra of the relative displacements at points $P_6$ (in Figure 3.3) and $P_8$ (in Figure 3.5) from harmonic balance. The fraction of the period where the pulley and shaft are disengaged is not small for $P_6$ ($M = 0.0127$). In addition, the second harmonic of the response in Figure 3.6d is not small, which implies the assumption of harmonic response is not practical. On the other hand, the time history and spectrum of the response when $M = 0.0277$ at $P_8$ validates those assumptions, and perturbation is effective.
Figure 3.5. RMS of δθ at the 2nd primary resonance, yielded by the method of multiple scales and harmonic balance for ζ = 5% and other parameters in Table 3.1. Stable (——); unstable (— —).
3.4.3 Impact of damping on the system dynamics

According to (3.22), the damping ratio does not influence the backbone curve. From (3.36), however, it does impact the response amplitude. When increasing the damping ratio $\zeta$ (or $\mu$), the two branches from (3.22) or from (3.31) close at a lower amplitude $a_2$. For low amplitude, the backbones of the numerical results and the approximations deviate only slightly, and the disengagement fraction of a period from harmonic balance is comparably small. In this case, perturbation generates good approximations. Figure 3.7 verifies this claim by setting $\zeta = 5\%$. 

Figure 3.6. Time history of $\delta \theta$ (---), spring torque $g_s(\delta \theta)$ (----) and spectrum of $\delta \theta$. a) and c) for point $P_s$ as $\Omega = 6.42$ on Figure 3.5 with $M = 0.0277$; b) and d) for point $P_6$ as $\Omega = 4.455$ on Figure 3.3 with $M = 0.0127$. 

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Figure 3.7. RMS of $\delta \theta$ at the 2$^{nd}$ primary resonance, yielded by the method of multiple scales and harmonic balance (HB) for $\zeta = 5\%$ and other parameters in Table 3.1. Stable (---); unstable (---).
CHAPTER 4

PIECE-WISE LINEAR DYNAMIC ANALYSIS OF

SERPENTINE DRIVES WITH A ONE-WAY CLUTCH

In this chapter, the prototypical three-pulley system with belt bending stiffness is extended with a one-way clutch in order to fully understand the nonlinear dynamics of the system with the one-way clutch performance. The clutch is modeled based on the relative velocity of the driven pulley and its accessory. The clutch locks the pulley and accessory for zero relative velocity and produces a positive inner clutch torque. Zero clutch torque starts clutch disengagement with unequal velocities of the two components. This model leads to a piece-wise linear system. The transition matrix is used to evaluate the system response in discrete time series at each linear status that tremendously saves computation time. The system dynamics including dynamic response and dynamic
tension drop are examined through varying excitation frequency, inertia ratio of the pulley and accessory, and the external load. The investigation of the effectiveness of the vibration reduction due to the single-direction transmission of the clutch provides optimal design guideline in practice.

4.1 Introduction

Serpentine belts power automotive front end accessory drives that use a single belt to transmit power from the crankshaft to a variety of accessories. The linear/nonlinear mechanics of this system are of interest in a group of studies [3,4,14,28,30-32,68,69]. A spring-like belt model is deserved in [4,14,28] to focus on the pulley rotational vibration. In [3,30-32], the theory of string-like moving media is adopted and developed for the serpentine belt to examine the belt transverse vibration as well as the pulley rotational vibration. Beikmann et al. [31] develop a prototypical three-pulley system involving a driving pulley, a driven pulley and a dynamic tensioner (Figure 4.1). The tensioner consists of a rigid arm rotating around its pivot and a pulley pinned at the other end of the arm. Its purpose is to adjust the belt tension during operation.

Adopting the prototypical three pulley system, Kong and Parker [68] establish a hybrid continuum/discrete drive model considering belt bending stiffness. The steady state equilibrium solutions are obtained. Unlike the string-like belt model, non-trivial span equilibrium curvature is observed, which induces the coupling of belt transverse vibration and pulley rotational vibration. In their subsequent work [69], the span-pulley
coupling, which does not occur with string models, is investigated through modal analysis of the system. This coupling is observed in experiments.

One-way clutches are used elements in mechanical systems such as the air turbine starters of a jet engine [12], helicopter main rotors, and certain pulleys in FEAD [1,2,13,58,70]. Generally integrated in a driving-driven combination device, the clutch disengages when the driven member of the device overruns the driving member; on the other hand, the two members are either locked together or connected through a spring-like element when their velocities are the same. Engagement ensures power transmission, while disengagement is for decoupling the two members to reduce or eliminate the influence from the heavy driven member during its overrunning. Such influence includes system vibration, belt flapping or wear, and even chirp noise when a belt passes over the driven member. In this study, the application of a one-way clutch in serpentine drives is considered.

Vernay et al. [12] experimentally study the transient behavior of sprag-type overrunning clutches to demonstrate the sliding effects during clutch engagement. King and Monahan [2] introduce the structure of a wrap-spring type overrunning pulley and address its design and installation consideration. Solfrank and Kelm [1] make a comprehensive simulation of the accessory drive operation. The model of the one-way clutch element consists of a velocity-dependent damping and a parallel spring with nonlinear stiffness for torque transmission, which intervenes only when the speed of the driven pulley is lower than that of the driving pulley. Leamy and Wasfy [13] develop a dynamic finite element model to determine the transient and steady-state response of a
pulley belt-drive system. To demonstrate the utility of the method, a one-way clutch that is modeled using a proportional torque law is incorporated to the driven pulley and its transient response is simulated.

The aforementioned literature, however, does not address the detailed dynamics of the one-way clutch. To examine its nonlinear dynamics, Zhu and Parker [58,71] establish a two-pulley belt system with a one-way clutch integrated between the driven pulley and its accessory. The clutch is modeled as a piece-wise linear spring with discontinuous stiffness that engages only for positive relative displacement of its connected components. Through the simulation of its alternate engagement and disengagement behavior, a classical softening nonlinearity is identified. The clutch separates the pulley and accessory into two degrees of freedom (DOF) and functions like a vibration absorber to reduce vibration. Mockensturm and Balaji [70] present a piece-wise linear analytical method to investigate the dynamic behavior of one-way clutches that are modeled based on relative velocity of the pulley and accessory shaft. The clutch operation leads to increase of power transmission and belt tension drop.

In the above studies, the pulley rotational vibration is considered only. To fully understand the dynamic behavior of serpentine belt drive with a one-way clutch, the work in this chapter adopts the typical three-pulley belt system (see Figure 4.1) and employs the continuum-discrete hybrid model in [68,69] that incorporates belt bending stiffness and exhibits belt-pulley coupling. Integrated between the driven pulley and its accessory, the one-way clutch separates and locks the motions of the two components during disengagement and engagement, respectively. For this piece-wise linear system, in each
linear configuration the analytical solutions are evaluated in discrete time series through
the transition matrix [72]. The transition time instants are sought using two criteria, i.e.,
zero relative velocity and zero inner torque. The solutions are confirmed by numerical
integration. To give practical design advice, the impact of the one-way clutch on the
system dynamics is investigated across ranges of excitation frequency, inertia ratio of the
pulley and accessory, and external load. The dynamic tension drop of the belt is
examined.

Figure 4.1 Three-pulley system with a one-way clutch integrated in the driven pulley and
its associated accessory. The tildes on the physical quantities have been dropped for
simplicity.

4.2 System Model

Figure 4.1 shows a typical three-pulley belt system. The belt travels at a constant
speed $s$ to transmit power from the driving pulley to the driven (accessory) pulley. The
tensioner arm is coil-spring loaded with spring stiffness $k_t$ and the accessory is subjected
to a constant external torque $\bar{M}_t$. 
A relative-velocity dependent one-way clutch is installed between the driven pulley and the accessory drawing power from the belt drive. This clutch model applies to sprag-type, roller-type or wrap-spring type one-way clutches. Consider first the disengaged clutch, where the velocity of the pulley $\dot{\theta}_p$ is less than that of the accessory $\dot{\theta}_a$. When the relative velocity approaches zero, the pulley and accessory lock together, i.e., the clutch engages. During engagement, the pulley and accessory are constrained to move together by an inner clutch torque $\tilde{M}_c$. In this case, the pulley with inertia $J_p$ and the accessory with inertia $J_a$ are combined as one body with inertia $J_0 = J_a + J_p$. When the clutch torque reduces from positive to zero, the pulley and accessory disengage. The mathematical expressions for the engagement and disengagement statuses of the clutch and their switching criteria are listed in Table 4.1. This piece-wise linear system behaves linearly both in engagement and disengagement configurations.

<table>
<thead>
<tr>
<th>Status</th>
<th>Switching Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engagement</td>
<td>$\dot{\theta}_a = \dot{\theta}_p$, $M_c &gt; 0$</td>
</tr>
<tr>
<td>Disengagement</td>
<td>$\dot{\theta}_a &gt; \dot{\theta}_p$, $M_c = 0$</td>
</tr>
</tbody>
</table>

Table 4.1 The engagement and disengagement statuses of the clutch and their switching criteria.

By employing Hamilton’s principle, Kong and Parker [68] derive nonlinear equations of motion of the three-pulley belt system, where the belt is modeled as a
moving flexural beam and the departure and end points of the three belt spans are the same as those for the straight belt. The inclusion of finite belt bending stiffness causes non-trivial steady state span curvature which determines the degree of coupling between the pulley and belt behavior, in contrast to decoupled belt-pulley motion based on straight steady state curvature in a string-like belt model. They then linearize the equations about the nontrivial equilibria and investigate the belt-pulley coupling in the free vibration analysis [69].

In the present system, the linearized dimensionless governing equations in the engaged configuration are similar to those in [69] and are written in the extended operator form

$$M\ddot{Y} + G\dot{Y} + KY = F$$

(4.1)

where $$Y = \{y_1, y_2, y_3, \theta_1, \theta_2, \theta_3, \theta_3\}^T$$ and $$\theta_i, i = 1, 2, 3$$, and $$\theta_i$$ denote rotations of the pulleys and tensioner arm about their equilibria. $$\theta_3$$ refers to the rotation of the accessory pulley when engaged, i.e., $$\theta_3 = \theta_a$$. Notice that the accessory rotation $$\theta_a$$ involves rigid translation if disengagement occurs during a cycle. $$y_i$$ is related to the transverse displacement of each span $$w_i$$ as

$$y_1 = w_1 - \frac{r_1}{l_1} x_1 \theta_1 \cos \beta_1, \quad y_2 = w_2 + \frac{r_1}{l_2} (x_2 - 1) \theta_1 \cos \beta_2, \quad y_3 = w_3$$

(4.2)

where $$x_{1,2} \in [0,1]$$ is the spatial variable along each span, $$r_i$$ is the length of the tensioner arm, and $$l_i$$ are span lengths. Note that $$y_i$$ satisfies trivial boundary conditions as those of a pinned beam. The differential operators $$M$$, $$K$$ are self-adjoint and $$G$$ is skew-self-adjoint.
adjoint with an appropriate inner product defined in [69]. $\mathbf{F}$ is a vector composed of the accessory load as

$$
\mathbf{F} = \begin{bmatrix} 0, 0, 0, 0, -\frac{r_a}{r_f}M_l, 0 \end{bmatrix}^T
$$

(4.3)

By adopting the non-dimensionization spirit in [69], the dimensionless quantities here are

$$
x_i = \frac{\ddot{x}_i}{l_i}, \quad w_i = \frac{\ddot{w}_i}{l_i}, \quad l = \frac{l_1 + l_2 + l_3}{3}, \quad t = \frac{P_0}{P_0 l^3}, \quad \eta = \frac{EI}{P_0 l^3}, \quad s = \sqrt{\frac{P}{P_0}}, \quad k_i = \frac{k_r}{P_0 r_i}
$$

(4.4)

where $j = 0, a, p$ and the tilde denotes physical quantities. For the clutch engagement configuration, $m_3 = m_0$. Considering the accessory as a free body, its equation of motion is

$$
m_3 \ddot{\theta}_3 + C_g \dot{\theta}_3 = M_c - M_l
$$

(4.5)

where $C_g$ is the damping to the ground.

Equation (4.1) also applies to the clutch disengagement configuration, but only the pulley inertia is active, i.e., $m_3 = m_p$. In this case, $\theta_3$ in (4.1) represents the accessory pulley rotation only, i.e., $\theta_3 = \theta_p$, and $\mathbf{F} = \mathbf{0}$. The accessory rotates separately from the pulley and carries the load alone. It satisfies (4.5) but with zero clutch torque $M_c = 0$.

Similar to [69], we employ a Galerkin discretization to the hybrid discrete-continuous system (4.1). The extended variable $\mathbf{Y}$ is expanded in a series of basis function as
\[ Y = \sum_{k=1}^{p} a_k(t) \psi_k(x) + \sum_{k=1}^{3} \theta_k(t) \psi_{p+k-1}(x) + \theta_i(t) \psi_i(x) \quad (4.6) \]

where \( p = N_1 + N_2 + N_3 \) and \( N_i \) is the number of basis function for the \( i^{th} \) span. \( \psi_i \) are global comparison functions satisfying all boundary conditions and each of them describes a deflection of the entire system. For the \( i^{th} \) span, the shape of the span deflection is superposed by sinusoidal waves \( \sin(k\pi x) \) for \( k = 1, \cdots, N_i \). For instance, \( \psi_k = \{ \sin k\pi x, 0, 0, 0, 0, 0 \}^T \) for the first span with \( k = 1, \cdots, N_1 \). For the discrete pulleys and tensioner arm, the global comparison function involves only the relevant discrete element, such as \( \psi_{p+2} = \{ 0, 0, 0, 1, 0, 0 \}^T \) for pulley 2. Using the inner product defined in [69], the discretized formulation is

\[ [M]\ddot{A} + [G] \dot{A} + [K] A = F \quad (4.7) \]

where \( A(t) = \{ a_1(t), \cdots, a_p(t), \theta_1, \theta_2, \theta_3, \theta_i \}^T \) are generalized coordinates.

In vehicles, the alternating engine cylinder ignition and compression strokes cause cyclic fluctuations of the crankshaft speed. One specifies the periodic speed irregularity \( \dot{\theta}_i \) based on engine properties. For such specified crankshaft speed, all terms with \( \dot{\theta}_i \) in (4.7) are moved to the right-hand side as excitation, which yields

\[ M^0 \ddot{A}^0 + (G^0 + C^0) \dot{A}^0 + K^0 A^0 = f \]

\[ f = F^0 - \left( [M]_{i,p+1} \dot{\theta}_i + [G]_{i,p+1} \ddot{\theta}_i + [K]_{i,p+1} \theta_i \right), \quad i = 1, \cdots, p+4, \quad i \neq p+1 \quad (4.8) \]
where the superscript 0 denotes the new matrices and vectors induced by the elimination of the \((p+1)\)th row and column from their original ones. \(C^0\) is a damping matrix obtained from modal damping to capture the energy dissipation in spans and bearings. The combination of (4.8), (4.5) and the switching criteria in Table 4.1 fully describe the dynamic motion of the system.

### 4.3 Methods

The system response for either engaged or disengaged clutch status can be found from linear system theory. To decouple the equations in (4.8), a state space variable \(z = \{A^0, \dot{A}^0\}^T\) is introduced and (4.8) is transformed into

\[
\frac{d}{dt}z = Rz + f_F
\]

where \(I\) is a \((p+3)\times(p+3)\) identity matrix. Furthermore, one finds the modal matrix \(V\) of \(R\) and introduces the modal coordinate vector \(q\) as

\[
z = Vq
\]

Substituting (4.10) into (4.9) and left-multiplying \(V^{-1}\) yields the decoupled equation

\[
\dot{q} = \Lambda q + h, \quad \Lambda = V^{-1}RV, \quad h = V^{-1}f_F
\]

where \(\Lambda\) is diagonal. The solution of (4.11) is

\[
q(t) = e^{\Lambda t}q(t_0) + \int_{t_0}^{t} e^{\Lambda(t-\tau)}h(\tau)d\tau
\]

where \(t_0\) is the initial time and \(e^{\Lambda(t-\tau)}\) is the transition matrix.
One can numerically evaluate the integral in (4.12). At any time instant $t$, however, the transition matrix has to be evaluated from $t_0$ to $t$ for integration. For numerical efficiency, one can save the transition matrix at each time instant as $t \in [t_0, t_f]$ to avoid repetitive matrix evaluation, where $t_f$ refers to the final time. This works well for systems with few DOF and few integration time points. For the current system and practical ones having more spans and pulleys, the discretized system has many DOF and many oscillation cycles are needed to ensure the complete decay of the transient response to capture the steady state. Therefore, the analytical solution from (4.12) is inefficient.

Meirovitch [72] presents an analytical approximation in discrete time series that evaluates the solution at $t_0, \cdots t_k, \cdots t_N$ for $t_k = t_0 + k\Delta t$, where $N = (t_f - t_0)/\Delta t$. This approximation is adopted herein as

$$
\{q(t_{k+1}) = \Phi q(t_k) + \Gamma f_F(t_k), \ k = 0, 1, \cdots, N \}
$$

where $\Delta t$ is the sampling period chosen sufficiently small such that the input vector $f_F(t)$ can be regarded as constant over the time interval $t_k < t < t_{k+1}$. Use of (4.10) generates the physical response. The solutions are confirmed by numerical integration. This method costs about 1/3~1/2 of the computation time compared to numerical integration.

The transition time instants for switching between the different clutch configurations must be determined. During engagement, the clutch torque is monitored. At each time instant, by obtaining the response from (4.13) and (4.10), the response of
the accessory $\dot{\theta}_a$ and $\ddot{\theta}_a$ is known because the accessory is moving together with the pulley. Hence, the clutch torque from (4.5) can be determined. If $M_c = 0$ (or less than a specified tolerance in the numerical implementation), disengagement starts from the next time instant. On the other hand, while the clutch is disengaged, the relative velocity is monitored. $\dot{\theta}_p$ is obtained from (4.13) and (4.10), and $\dot{\theta}_a$ is yielded from (4.5) with $M_c = 0$. If $\dot{\theta}_p - \dot{\theta}_a = 0$ (within a specified tolerance), the clutch returns to engagement.

The above method allows for unlimited switches between engagement and disengagement in a single cycle, unlike the analytical solution in [70], where only a single transition is assumed.

4.4 Results and discussions

We consider the steady state periodic response of the system subjected to excitation from the crankshaft speed fluctuation specified as $\dot{\theta}_1 = A \cos \Omega t$. The excitation frequency $\Omega$ is $\sigma$ times the engine (crankshaft) speed, where $\sigma$ is determined by the number of the engine cylinders and $\sigma = 3$ is used in the present work. The belt translation speed varies with the engine speed as $s = r \Omega / \sigma l$. The range $\Omega = 0 \sim 12$ corresponds to the engine speed varying over $0 \sim 5850 \text{ rpm}$ for the parameters in Table 4.2, which is a frequency range of practical importance. The belt speed $s = 0.921$ corresponding to $\Omega = 12$ is lower than any critical speed that results in unstable operation of this gyroscopic system. The amplitude of the crankshaft speed fluctuation $A$ is
typically an estimated percentage $\mu$ of the engine speed, so $A = \mu \Omega / \sigma$. The value $\mu = 10\%$ is chosen, although this depends on engine speed in practice.

<table>
<thead>
<tr>
<th>Pulley radius $r_i$</th>
<th>$0.08125 , m$</th>
<th>Pulley center $(x_i, y_i)$</th>
<th>$(0.5514, 0.0556) , m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulley radius $r_2$</td>
<td>$0.03115 , m$</td>
<td>Pulley center $(x_2, y_2)$</td>
<td>$(0.3601, 0.05715) , m$</td>
</tr>
<tr>
<td>Pulley radius $r_3$</td>
<td>$0.03 , m$</td>
<td>Pulley center $(x_3, y_3)$</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td>Tensioner arm $r_t$</td>
<td>$0.04 , m$</td>
<td>Pulley center $(x_t, y_t)$</td>
<td>$(0.3082, 0.0635) , m$</td>
</tr>
<tr>
<td>Rotational inertia $J_1$</td>
<td>$0.07248 , kg \cdot m^2$</td>
<td>Belt Modulus $EA$</td>
<td>$120000 , N$</td>
</tr>
<tr>
<td>Rotational inertia $J_2$</td>
<td>$0.000293 , kg \cdot m^2$</td>
<td>Initial tension $P_0$</td>
<td>$300 , N$</td>
</tr>
<tr>
<td>Rotational inertia $J_0$</td>
<td>$0.01 , kg \cdot m^2$</td>
<td>Belt mass density $m$</td>
<td>$0.1029 , kg/m$</td>
</tr>
<tr>
<td>Rotational inertia $J_t$</td>
<td>$0.001165 , kg \cdot m^2$</td>
<td>Tensioner stiffness $k_r$</td>
<td>$28.25 , N-m/rad$</td>
</tr>
<tr>
<td>Span length $l_1$</td>
<td>$0.1548 , m$</td>
<td>Alignment angle $\beta_1$</td>
<td>$135.79^\circ$</td>
</tr>
<tr>
<td>Span length $l_2$</td>
<td>$0.3477 , m$</td>
<td>Alignment angle $\beta_2$</td>
<td>$178.74^\circ$</td>
</tr>
<tr>
<td>Span length $l_3$</td>
<td>$0.5518 , m$</td>
<td>Tensioner rotation $\theta_r$</td>
<td>$0.1688 , rad$</td>
</tr>
<tr>
<td>Damping ratio $\zeta$</td>
<td>$5%$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2 Physical properties of the example system.
During calculation, some of the parameters are considered as follows. The modal damping coefficient to obtain $C^0$ in (4.8) is $\zeta = 5\%$. The parameter $\alpha$ is the inertia ratio of the accessory and the driven pulley, \textit{i.e.}, $m_a = \alpha m_p$. Throughout the paper, the combined inertia of the pulley and accessory is unchanged as $m_0 = 26.03$ ($J_0 = 0.01 \text{ kg} \cdot \text{m}^2$) for any $\alpha$. For the discretization, $N_i = 2$ is chosen for $i = 1, 2, 3$, \textit{i.e.}, the half-sine and sine waves are considered.

As the excitation frequency $\Omega$ varies, the belt speed and the eigensolutions of the linear system change [69]. For small bending stiffness, the natural frequencies of transversely dominant vibration modes decrease quickly with increasing speed, while those of rotationally dominant vibration modes change slightly. For large bending stiffness, all natural frequencies decrease comparably slowly and non-monotonically with speed because of the strong belt-pulley coupling. A moderate bending stiffness $\varepsilon = 0.02$ is used in the present work.

Two linear systems are considered for comparison to the nonlinear system with the one-way clutch. One such system corresponds to the permanently engaged clutch case. This is called the \textit{locked clutch} system. It represents the standard arrangement in practical systems where there is no clutch and the pulley and accessory are rigidly connected. In this case, $m_3 = m_0$. The other system refers to the case when the clutch fully disengages and the belt only drives the pulley with inertia $m_3 = m_p = m_0/(1 + \alpha)$. This is called the \textit{disengaged clutch} system.
The active driven inertia is changed from the combined inertia (when the clutch is locked) to the pulley inertia (when the clutch is disengaged), and the natural frequencies vary accordingly. Figure 4.2 illustrates this variation, where $\alpha = 0$ corresponds to the natural frequencies of the locked clutch system. Natural frequency veering occurs with increasing $\alpha$. $\alpha$ affects the modal amplitude of the accessory pulley in each mode, and the modal energies exchange near veering. Away from veering, the modal energy distributions and their associated natural frequencies approach steady configurations for large $\alpha$. For the parameters in Table 4.2, the rotationally dominant modes have the lowest natural frequencies for both linear systems. According to Figure 4.2, $\Omega_i$ initially increases quickly with $\alpha$ because the eigenvalue veering affects $\Omega_i$ at low $\alpha$. The increasement of $\Omega_i$ is indistinguishable for large $\alpha$ and the tensioner arm eventually dominates this mode. In the following, the disengaged clutch system with $\alpha = 10$ and 50 will be the examples in comparison with the locked clutch case $\alpha = 0$. Table 4.3 lists the natural frequencies for $s = 0.069$ within the frequency range of practical importance. Those for $\alpha = 500$ show limiting values, and the last row indicates which system component dominates the mode for $\alpha = 500$. Notice that these natural frequencies vary slightly when the belt speed (excitation frequency) is changed.
Figure 4.2 Natural frequencies of the disengaged clutch system varying with inertia ratio $\alpha$ for belt speed $s = 0.069 \ (\Omega = 0.9)$.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>0.89</td>
<td>2.02</td>
<td>2.65</td>
<td>3.25</td>
<td>4.04</td>
<td>6.50</td>
<td>7.14</td>
</tr>
<tr>
<td>$\alpha = 10$</td>
<td>1.66</td>
<td>2.03</td>
<td>3.18</td>
<td>4.04</td>
<td>4.72</td>
<td>6.53</td>
<td>7.14</td>
</tr>
<tr>
<td>$\alpha = 50$</td>
<td>1.78</td>
<td>2.03</td>
<td>3.20</td>
<td>4.04</td>
<td>6.41</td>
<td>7.13</td>
<td>9.21</td>
</tr>
<tr>
<td>$\alpha = 500$</td>
<td>1.80</td>
<td>2.04</td>
<td>3.20</td>
<td>4.04</td>
<td>6.45</td>
<td>7.14</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 4.3 Natural frequencies for different inertia ratio $\alpha$ when $s = 0.069 \ (\Omega = 0.9)$. 

| Mode dominated by $\theta_t$ | Span 3 | Span 2 | Span 3 | Span 2 | Span 1 | N/A |

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4.4.1 Excitation frequency sweep

Figure 4.3 gives RMS dynamic response of the deflection amplitude of the half-sine wave of three spans and that of the sine wave of the second span ($a_i$). The two linear systems are compared with the nonlinear system. The common resonances occur in transversely dominant modes, like those at $\Omega = 6.4$ of the first span (Figure 4.3a), at $\Omega = 2.0$ of the third span (Figure 4.3c), and at $\Omega = 3.2, 7.1$ of the second span (Figure 4.3b and d). For a given frequency near these modes, the responses of the two linear systems determine the effectiveness of the vibration suppression by the one-way clutch, provided super- or sub-harmonics do not occur. When the response of the locked clutch system is higher than that of the disengaged clutch case, the nonlinear clutch suppresses vibration, such as the modes at $\Omega = 2.0$ in span 3 and $\Omega = 3.2$ in span 2. On the opposite, the vibration increases where the response of the disengaged clutch system is higher, such as the mode at $\Omega = 6.4$ in span 1. At $\Omega = 7.1$ in span 2, the nonlinear vibration is reduced for $\alpha = 50$ while it increases for $\alpha = 10$ because varying inertia ratio $\alpha$ introduces non-monotonic variation of the difference between the linear responses at this mode. In the meanwhile, the nonlinear behavior that the pulley is engaged to the accessory in partial of a cycle, strongly affects the rotationally dominant modes, such as those at $\Omega = 0.9$ and 2.7, where the resonances are markedly reduced in comparison with the locked clutch system. Large nonlinear response occurs at the veering natural frequency for a given $\alpha$, such as the resonances at $\Omega = 4.7$ for $\alpha = 10$ and at $\Omega = 9.2$ for $\alpha = 50$. 

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Figure 4.3 RMS dynamic response of the deflection amplitudes for $M_f = 0.556$ and other parameters in Table 4.2. a) $a_1 - a_{1\text{mean}}$; b) $a_3 - a_{3\text{mean}}$; c) $a_5 - a_{5\text{mean}}$; d) $a_4 - a_{4\text{mean}}$.  

Locked linear;  

- disengaged clutch $\alpha = 10$;  

- disengaged clutch $\alpha = 50$;  

- nonlinear $\alpha = 10$;  

- nonlinear $\alpha = 50$.

Super-harmonics are identified in the span transverse vibrations in Figure 4.3, although they are not harmful to the nonlinear system because of the small amplitudes. If the nonlinear vibration is pronounced at certain resonance, its super-harmonics possibly occur. A strong vibration is shown in span 3 near $\Omega_2 = 2.0$ (Figure 4.3c), and one can see a one-third and a one-half super-harmonic of $\Omega_2$ occurs at $\Omega = 0.7$ and $\Omega = 1$, respectively. Similarly, a one-half super-harmonic of $\Omega_6 = 7.15$ occurs near $\Omega = 3.5$ in
span 1 (Figure 4.3a). There are many other super-harmonics with small amplitudes that are associated with prominent resonances in each span.

![Figure 4.4 RMS dynamic response of a) tensioner pulley \( \theta_2 - \theta_{2\,\text{mean}} \), b) tensioner arm \( \theta_t - \theta_{\text{mean}} \), c) accessory pulley \( \theta_p - \theta_{p\,\text{mean}} \) and d) accessory velocity \( \dot{\theta}_a - \dot{\theta}_{a\,\text{mean}} \) for \( M_i = 0.556 \) and other parameters in Table 4.2. — locked clutch; — disengaged clutch \( \alpha = 10 \);  disengaged linear \( \alpha = 50 \); ... nonlinear \( \alpha = 10 \); — nonlinear \( \alpha = 50 \).

Figure 4.4 RMS dynamic response of a) tensioner pulley \( \theta_2 - \theta_{2\,\text{mean}} \), b) tensioner arm \( \theta_t - \theta_{\text{mean}} \), c) accessory pulley \( \theta_p - \theta_{p\,\text{mean}} \) and d) accessory velocity \( \dot{\theta}_a - \dot{\theta}_{a\,\text{mean}} \) for \( M_i = 0.556 \) and other parameters in Table 4.2. — locked clutch; — disengaged clutch \( \alpha = 10 \);  disengaged linear \( \alpha = 50 \); ... nonlinear \( \alpha = 10 \); — nonlinear \( \alpha = 50 \).

The pulley and tensioner arm vibrations are elaborated in Figure 4.4. The accessory experiences rigid body translation during disengagement, so its rotational velocity is shown in Figure 4.4d for vibration evaluation. When the one-way clutch performs, the locked clutch resonance at \( \Omega = 0.9 \) is removed while disengaged clutch resonances appear near \( \Omega = 1.8 \) for different \( \alpha \) where the response does not increase compared with
that of the locked clutch system though. The nonlinear vibration increases at the
disengaged clutch resonances near $\omega = 4.7$ for $\alpha = 10$ and near $\omega = 9.2$ for $\alpha = 50$ but
diminishes at the locked clutch resonance $\omega = 2.7$. By selecting an appropriate $\alpha$ to
avoid the disengaged clutch resonances in the practically important frequency range, such
as $\alpha > 117.6$, no high rotational vibration occurs. One can also predict that if the
combined inertia of the pulley and accessory is not big enough to avoid the impact of $\alpha$
on the first mode dominated by the tensioner arm, the nonlinear vibration near this mode
will not be mitigated much. The clutch performance suppresses the vibration of the
accessory, especially removing the modal resonances dominated by rotational vibration
in the locked clutch system (Figure 4.4d).

Figure 4.5 shows the time histories of the velocities of the driven pulley and
accessory and the associated clutch torque at different frequencies. The disengagements
occur in the deceleration phase of the pulley and accessory, which exactly match the one-
way clutch function that decouples the pulley and accessory during deceleration to avoid
the impact from the big accessory inertia which decelerates more slowly than the pulley.
Both velocities with more sinusoidal-like profile are associated with less disengagement.
For $\omega = 0.5$, the clutch disengages in a small portion of the cycle. For $\omega = 0.6$, the
disengagement expands and higher harmonics of the excitation frequency $2\omega$, $3\omega$, etc.,
are clearly explored in the velocity spectrum of the accessory pulley. Figure 4.5c, for
$\omega = 2.0$, shows a general case where the clutch torque jumps from zero for
disengagement to non-zero as engagement begins. For $\omega = 5.0$, a sharp positive clutch
torque yields when the disengagement is the majority in a cycle.
Figure 4.5 Time history of the rotational velocities of the accessory pulley and accessory, and the associated clutch torque for $\alpha=10$, $M_f=0.556$ and other parameters in Table 4.2. a) $\Omega = 0.5$; b) $\Omega = 0.6$; c) $\Omega = 2$; d) $\Omega = 5$. $\dot{\theta}_a$, $\dot{\theta}_p$, $M_c$.

To track the engagement and disengagement behavior of the clutch, an engaged time ratio $\gamma$ is introduced which is evaluated by division of the engaged time of a cycle by the cycle period. Figure 4.6 records $\gamma$ varying across the practically important frequency range for the parameters in Table 4.2. At low frequencies, disengagement seldom happens since the deceleration is mild, like $\Omega < 0.5$ and $\Omega = 1.8$ where no disengagement occurs. The deceleration turns stronger at higher frequencies, which leads to more disengagement. For each $\alpha$ branch, the disengagement expands near the modal
resonance of the disengaged clutch system dominated by the accessory pulley, such as at
\[ \Omega = 4.7 \] for \( \alpha = 10 \) and at \( \Omega = 9.2 \) for \( \alpha = 50 \).

![Engaged time ratio varies with excitation frequency for \( M_i = 0.556 \) and other parameters in Table 4.2. \( \cdots \alpha = 10 \); \( \alpha = 50 \).](chart)

**4.4.2 Impact of inertia ratio**

The inertia ratio \( \alpha \) exerts strong impact on the system dynamics. Figure 4.7 depicts how the engaged time ratio \( \gamma \) varies with \( \alpha \). With increasing \( \alpha \), if a given frequency \( \Omega \) induces a resonance of either linear system, the disengagement is intensified near the resonance inertia ratio. For example, \( \Omega = 0.9 \) is near the first mode resonance of the locked clutch system dominated by the accessory pulley, and the quick deceleration of the driven pulley and accessory during engagement at the resonance easily introduces
clutch disengagement. For very low non-zero inertia ratio, the disengaged response is resonant at the first mode of the disengaged clutch system, and the high response potential of the pulley further enlarges the disengagement. With increasing $\alpha$, $\Omega = 0.9$ gradually deviates away from the disengaged clutch resonance and the disengagement is reduced. $\Omega = 4.2$ is resonant during disengagement at $\alpha = 7$ where considerable disengagement occurs. $\Omega = 7.1$ is near one of the natural frequencies of both linear systems across the entire inertia ratio range, and the response is large in both clutch configuration phases. Hence, the curve variation is comparably slight. Under the external load $M_i = 0.556$, $\Omega = 1.8$ does not introduce disengagement for any $\alpha$. With lower load value, the disengagement could occur at $\Omega = 1.8$.

![Engaged time ratio varies with inertia ratio](image-url)

Figure 4.7 Engaged time ratio varies with inertia ratio for $M_i = 0.556$ and other parameters in Table 4.2. $\cdots \Omega = 0.9$; -- $\Omega = 1.8$; --- $\Omega = 2$; ---- $\Omega = 4.2$; $\cdots \cdots \Omega = 7.1$.  

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Figure 4.8 RMS dynamic responses of a) the accessory pulley and b) the accessory velocity vary with inertia ratio for $M_i = 0.556$ and other parameters in Table 4.2.  

- $\Omega = 0.9$;  
- $\Omega = 2$;  
- $\Omega = 4.2$;  
- $\Omega = 7.1$.

Away from the resonance, a large inertia ratio leads to a steady pattern of engagement and disengagement with an almost constant engaged time ratio (Figure 4.7) and so a constant response of each system component for given frequency, although its response variation is not necessarily monotonic at low range of $\alpha$ (Figure 4.8). For instance, at the low range of $\alpha$, the accessory pulley response decreases for frequency branches $\Omega = 0.9$ and increases for $\Omega = 2.0, 4.2, 7.1$ with $\alpha$, and eventually goes to their individual limiting values as $\alpha$ goes to infinity (Figure 4.8a). The constant dynamic response of the accessory velocity with large inertia ratio is always lower than its
dynamic response in the locked clutch case (Figure 4.8b). Therefore, the accessory gains appreciable benefit from the one-way clutch performance if its inertia is large.

### 4.4.3 Impact of external load

During disengagement, the velocity of the accessory pulley easily captures that of the accessory when the accessory undergoes a rapid deceleration resulting from a large external load. Therefore, a higher load prevents more disengagement and a sufficient load can make the disengagement impossible. Furthermore, a lower load preserves the velocity gap between the accessory and pulley, which results in more disengagement.

Figure 4.9 shows the engaged time ratio varying with the load for different frequencies as $\alpha = 50$. The engagement extends with increasing the load until the disengagement completely varnishes. The frequency branches $\Omega = 0.9, 1.8, 2.0, 2.7$ and $7.1$ have critical loads for a locked clutch ($\gamma = 1$) as $M_{i,cr} \approx 15.1, 0.5, 1.0, 19.5$ and $5.0$, respectively. $\Omega = 0.9$ and $2.7$ are near the modal resonance of the locked clutch system dominated by pulley rotations. With increasing the load, the dynamic velocity amplitudes of the pulley and accessory increase during engagement, which promotes disengagement because of their large deceleration potential. This requires a large critical load for full engagement. $\Omega = 2.0$ and $7.1$ are near the resonances of dominantly transverse vibration modes of both linear systems, where the dynamic velocity amplitudes of the pulley and accessory are not affected substantially by the alternate engagement and disengagement. Therefore, a locked clutch is achieved quickly as increasing the load compared with the case for $\Omega = 0.9$ and $2.7$. 
The critical external load, beyond which the system behaves with the locked clutch only, can be obtained analytically. According to (4.5), the maximum clutch torque is proportional to the load when only engagement occurs. In this case, $\dot{\theta}_a$ is a sinusoidal response assumed as $B \sin(\Omega t + \vartheta)$, where $B$ and $\vartheta$ denote the amplitude and phase of the accessory, respectively. The expression of the clutch torque from (4.5) can be expanded as

$$M_c = m_a \ddot{\theta}_a + C_s \dot{\theta}_a + M_l = B \left( \frac{\alpha m_b}{1 + \alpha} \right)^2 + C_s^2 \sin(\Omega t + \vartheta + \varphi) + M_l$$  \hspace{1cm} (4.14)
where $\varphi = \tan^{-1}(C_g (1 + \alpha)/\alpha m_0 \Omega)$. For critical case, $M_c = 0$ and the lowest external load $M_{l,cr}$ that leads to a locked clutch can be determined for given frequencies. To accomplish partial torque transmission from the pulley to accessory within a cycle, a load satisfying $M_l < M_{l,cr}$ is desirable; otherwise, the two components would be locked for the entire cycle. In (4.14), $B$ is determined by the analytical RMS solution $\hat{\theta}_a$ of the locked clutch system.

Figure 4.10 shows two curves generated from (4.14) as $M_c = 0$ for varying excitation frequency. The critical load is large at the locked clutch resonances, such as at

Figure 4.10 Critical external load varies across excitation frequency range for parameters in Table 4.2. ___ $\alpha = 50$; ___ $\alpha = 10$. 
\( \Omega = 0.9 \) and 2.7 where the modal amplitude of the accessory is large. At these frequencies, disengagements are easily introduced when \( M_i < M_{l,cr} \). (4.14) indicates that the critical load is a function of \( \alpha \) and \( \Omega \). As increasing \( \Omega \), the distinction between different \( \alpha \) is increased. Higher \( \alpha \) predicts higher critical load for a given frequency, with a slow increasing rate though.

4.4.4 Impact on dynamic tension drop

The current system model assumes the equilibrium curvatures of the spans and the equilibrium belt tensions do not change when a one-way clutch is involved. The dynamic tension can be evaluated as the one-way clutch performs. According to [69], the linearized dynamic tension of each span is

\[
\begin{align*}
P_{d1} &= \eta \left( -\frac{r_1}{l_1} \theta_2 + \frac{r_1}{l_1} \theta_1 - \frac{r_1}{l_1} \theta_1 \sin \beta_1 + \int_0^1 w_{1,x}^* w_{1,x}^* dx \right) \\
P_{d2} &= \eta \left( -\frac{r_2}{l_2} \theta_3 + \frac{r_2}{l_2} \theta_2 + \frac{r_2}{l_2} \theta_1 \sin \beta_2 + \int_0^1 w_{2,x}^* w_{2,x}^* dx \right) \\
P_{d3} &= \eta \left( -\frac{r_3}{l_3} \theta_1 + \frac{r_3}{l_3} \theta_3 + \int_0^1 w_{3,x}^* w_{3,x}^* dx \right)
\end{align*}
\]

(4.15)

The rotations of the pulleys and tensioner arm have strong impact on the dynamic tension compared with the span curvature from (4.15) where the integrals involving the span slopes are small.
Figure 4.11 Time history of dynamic tensions of three spans for $M_i = 0.167$, $\Omega = 2$, $\alpha = 50$ and other parameters in Table 4.2. — nonlinear; — locked clutch.
Figure 4.12 Maximum and minimum dynamic tensions of three spans vary with external load for $\Omega = 2$, $\alpha = 50$ and other parameters in Table 4.2. --- nonlinear; — locked clutch.

With a big accessory inertia, the belt is relaxed during disengagement and more disengagement induces more tension drop if the excitation frequency is away from a disengaged clutch resonance whose mode is dominated by the accessory pulley. Figure 4.11 exhibits the time history of the dynamic tension from (4.15) in a single cycle at $\Omega = 2$ in comparison with that of the locked clutch system. Clearly, a tension drop is achieved in every span. The non-sinusoidal shapes of the dynamic tensions, especially shown in the second and third spans, reflect the alternate engagement and disengagement of the clutch performance. Figure 4.12 illustrates the maximum and minimum dynamic tension varying with the external load. The tension drop decreases with increasing the
load until the tension curves merge with the linear ones since large load prevents disengagement. According to Figure 4.1, the first span is tight which produces the highest dynamic tension and the third one is slack with the lowest dynamic tension. Large external torque strengthens the loading to the driving pulley, and therefore the linear dynamic tension increases with $M$ in the tight side—the first span, but decreases in the slack side—the third span. The linear dynamic tension of the second span is insensitive to the load. Notice that the maximum and minimum values of the nonlinear system are not symmetric about their mean values of the linear dynamic tensions because of disengagement in partial of a cycle.

The maximum dynamic tensions varying with the excitation frequency are shown in Figure 4.13 and are compared with the locked clutch system. The dynamic tension of each span is mainly determined by the rotations of its adjacent system components. The nonlinear dynamic tension drops near the modal resonances of the locked clutch system dominated by rotational vibration, such as $\Omega = 0.9$ and 2.7, while it increases at those of the disengaged clutch system dominated by the accessory pulley, such as near $\Omega = 4.7$ for $\alpha = 10$ and near $\Omega = 9.2$ for $\alpha = 50$. For the first span, the rotations of the driving pulley, tensioner pulley and arm affect the tension and none of their nonlinear responses is high at the disengaged clutch resonance, and accordingly, the tension increases slightly. On the contrary, the dynamic response of the accessory pulley affects the dynamic tensions of the second and third spans, and hence, high tensions occur at the disengaged clutch resonances dominated by the accessory pulley. The minimum dynamic tensions have similar shapes as the maximum ones, not exactly symmetric though.
Figure 4.13 Maximum dynamic tensions of three spans vary across the frequency range for $M_i = 0.556$ and other parameters in Table 4.2. - - locked clutch; --- nonlinear $\alpha = 50$; ----- nonlinear $\alpha = 10$.

Figure 4.14 takes the second span as an example to illustrate the impact of the inertia ratio on the dynamic tension for a variety of frequencies. Similar as its impact on the dynamic response, non-monotonic variation of the dynamic tension results for a given frequency $\Omega$. High tension occurs when $\Omega$ is a resonant frequency of either linear system as varying $\alpha$ because resonant response produces high dynamic tension from (4.15). Away from the resonant region, the dynamic response and so the belt tension vary slightly with $\alpha$ and eventually approach limiting values which are formed by steady engagement and disengagement patterns. Additionally, a high inertia ratio means a small pulley is associated with a big accessory. In this case, the pulley receives little impact.
from the accessory and disengagement occurs dominantly during a cycle, which releases the entire system from the accessory inertia and its load. As a result, lower tension is induced at very large $\alpha$ than the locked clutch case ($\alpha = 0$).

Figure 4.14 Maximum dynamic tension in span 2 varies with inertia ratio for $M_i = 0.556$ and other parameters in Table 4.2. $\Omega = 0.9$; $\Omega = 2$; $\Omega = 3.2$; $\Omega = 4.2$; $\Omega = 7.1$. 
In this chapter, a model of dry friction tensioner in a belt-pulley system considering transverse belt vibration is developed, and the influence of the dry friction on the system dynamics is examined. The discretized formulation is divided into a linear subsystem including linear coordinates and a nonlinear subsystem addressing tensioner arm vibration, which reduces the dimension of the iteration matrices when employing the harmonic balance method. The Coulomb damping at the tensioner arm pivot mitigates the
tensioner arm vibration but not necessarily the vibrations of other system components. The extent of the mitigation varies for different excitation frequency ranges. The critical amplitude of the dry friction torque beyond which the system operates with a locked arm is determined analytically. Superharmonic resonances are observed in the responses of the generalized span coordinates but their amplitudes are small. The energy dissipation at the tensioner arm hub is discussed, and the stick-slip phenomena of the arm are reflected in the velocity reversals near the arm extreme location. Dependence of the span tension fluctuations on Coulomb torque is explored.

5.1 Introduction

Since the late 1970s, serpentine belts have been widely used in the automotive industry to drive vehicle accessories. A tensioning system, known as a tensioner, plays an important role in serpentine drives by automatically adjusting the belt tension during operation. It consists of an idler pulley (tensioner pulley) at the end of a rigid arm (tensioner arm) (Figure 5.1). The arm pivots around a fixed point and the pulley is pinned at the free end of the arm [3,4,6,7,28,31]. This assembly leads to geometric nonlinearity in the system model [3,4,6,28,31], and the arm motion causes belt-pulley coupling [3,34,69] by moving the end points of the two adjacent spans.

Barker et al. [28] establish a mathematical model for the entire serpentine belt drive with a dynamic tensioner. Determination of the tensioner arm geometric configuration and span tension arising from belt stretching are addressed in detail.
Transient responses to the engine firing pulsations are discussed. Hwang et al. [4] determine the equilibria of the system with a tensioner arm including geometric nonlinearity. The system dynamic characteristics are captured from the eigenvalue problem for the equations linearized about equilibrium. The onset of belt slip at static and dynamic states is predicted.

![Diagram of a prototypical three-pulley serpentine belt system.](image)

Figure 5.1 A prototypical three-pulley serpentine belt system. The tildes on the physical quantities have been dropped for simplicity.

The tensioner arm exhibits stick-slip motion during operation because of dry (Coulomb) friction at its pivot, which is neglected in the above literature. Dry friction is a controlled design input that manufacturers use to distinguish their products. It is one of the few practical ways to introduce system damping to dissipate vibration caused by crankshaft excitations. Few works focus on this subject. Kraver et al. [35] develop a complex modal analysis procedure for a front end accessory drive system. They use
equivalent linear viscous damping to replace the dry friction at the tensioner arm. Leamy et al. [7] adopt the model in [4,28] but add a Coulomb damper at the tensioner arm pivot. Runge-Kutta integration is employed with modifications to surmount numerical difficulties induced by tensioner arm stick-slip motion. Superharmonic responses are found that are absent from models without the Coulomb damper. Leamy and Perkins [14] utilize the incremental harmonic balance method to efficiently predict the nonlinear periodic response. Again secondary resonances are observed. Cheng and Zu [73] analyze a belt drive subject to multi-frequency excitations from both driving and driven pulleys. They assume single-term harmonic response when the arm is slipping and predict the periodic response of the system when the arm either is purely slipping or sticks once during slipping.

The aforementioned works [4,7,14,28,35,73] only consider the pulley rotational vibration, which can not capture the coupling between pulley rotations and belt transverse motions observed in practice and in experimental measurements [3]. Beikmann et al. [3,31] investigate a prototypical three pulley system. The belt is modeled as a continuum string, which shows coupled vibrations only between the tensioner pulley and its adjacent spans; other spans remain straight. Adopting this model, Parker [34] presents a method for calculating the eigensolutions and dynamic response of the system. Kong and Parker [68,69] consider the belt bending stiffness in a system model. This beam-like belt model exhibits new dynamic characteristics of the system and extends the belt drive model to broader operating conditions. In particular, all span motions are coupled to pulley rotations.
In this study, dry friction damping at the arm pivot is incorporated into the three-pulley system with the beam-like belt model established in [69]. The objective is to investigate the effects of dry friction on the coupled belt-pulley system dynamics. Multi-term harmonic balance method is used to find the steady state periodic response of accessory drives subject to periodic excitation from the driving pulley. Energy dissipation occurs when the two mating surfaces at the arm pivot rotate relative to each other, which is examined by varying the dry friction torque. The dynamic span tension fluctuations, an important practical consideration, are investigated for various dry friction torques and other major parameters.

5.2 System Model

Kong and Parker [68,69] establish a hybrid continuum-discrete model incorporating belt bending stiffness. Figure 5.1 depicts this prototypical serpentine belt drive consisting of a driving pulley, a driven pulley, a tensioner and a serpentine belt. The driving pulley 1 and the driven pulley 3 are subject to accessory moments \( \tilde{M}_1 \) and \( \tilde{M}_3 \). The tensioner arm is spring-loaded with rotational stiffness \( k_r \) at its pivot and is assembled with alignment angles \( \beta_i, i = 1,2 \) relative to the two adjacent spans in the reference state, which corresponds to a stationary system without bending stiffness [68]. The belt spans are modeled as moving Euler-Bernoulli beams with constant speed \( c \). The incorporation of finite belt bending stiffness induces non-trivial steady state span deflection, in contrast to the string belt models which result in straight span equilibria. The non-trivial equilibria lead to the span-pulley coupling, and the steady state equilibrium curvature determines
the extent of the coupling. In [69], the governing equations are linearized about the steady state configuration. Eigensolutions from the Galerkin discretized equations show characteristics that differ from the string belt model [3,31]. Unlike string belt models, the spans between fixed-center pulleys are coupled with pulley rotation by the non-trivial steady span deflection.

In the present work, the hybrid continuum-discrete system model in [68,69] is adopted, and a dry friction damper is introduced at the tensioner arm pivot. The dry friction torque is

\[
\tilde{h}_c(t) = \tilde{Q}_m \text{sgn}(\dot{\theta}_i) = \begin{cases} 
\tilde{Q}_m, & \dot{\theta}_i > 0 \\
0, & \dot{\theta}_i = 0 \\
-\tilde{Q}_m, & \dot{\theta}_i < 0 
\end{cases}
\]  

(5.1)

The magnitude \(\tilde{Q}_m\) of the dry friction torque is controlled through preload adjustment between rubbing parts on the arm and fixed hub. The velocity of the arm \(\dot{\theta}_i\) remains zero for a finite time when the arm is locked up.

Similar to [69], the dimensionless governing equations of the system considering dry friction tensioner is written into the extended operator form

\[
M\ddot{Y} + G\dot{Y} + KY + H = F
\]  

(5.2)

where \(Y = \{y_1, y_2, y_3, \theta_1, \theta_2, \theta_3, \theta_i\}^T\), \(\theta_i, i = 1, \cdots, 3\) and \(\theta_i\) denote rotational displacement of the pulley and the tensioner arm about the equilibria, respectively. \(y_i\) is related to the transverse displacement about the equilibrium of each span \(w_i\) as
\[
y_1 = w_1 - \frac{r_i x_i \theta_i \cos \beta_i}{l_i}, \quad y_2 = w_2 + \frac{r_i (x_2 - 1) \theta_i \cos \beta_2}{l_2}, \quad y_3 = w_3
\]

(5.3)

where \( x_{i,2} \in [0,1] \) are the spatial variables along each span, \( r_i \) is the length of tensioner arm, and \( l_i \) are span lengths. Note that \( y_1 \) satisfies trivial boundary conditions, and \( y_1 \) and \( y_2 \) refer to the transverse displacements about the span curvatures which are formed by the rigid translation of the span equilibrium curvatures from moving their end points adjacent to the tensioner pulley from the arm equilibrium position \( \theta_i^* \) to the instantaneous position \( \theta_i \). The differential operators \( M, K \) are self-adjoint and \( G \) is skew-self-adjoint with an appropriate inner product. See [68] for the details of these operators and the definition of the inner product. \( F \) is a vector composed of dynamic accessory torques. In this work, zero dynamic accessory torque is assumed, that is, \( F = 0 \). The term

\[
H = \{0, \ldots, 0, h_c \}^T
\]

(5.4)

in (5.2) represents the nonlinear dry friction torque applied at the arm pivot, where \( h_c = Q_m \text{sgn}(\dot{\theta}_i) \) is the dimensionless torque from (5.1). The dimensionless quantities are

\[
x_i = \frac{\tilde{x}_i}{l_i}, \quad w_i = \frac{\tilde{w}_i}{l_i}, \quad l = \frac{l_1 + l_2 + l_3}{3}, \quad t = \tilde{t} \sqrt{\frac{P_0}{\rho l^2}}, \quad \epsilon^2 = \frac{EI}{P_0 l^2}, \quad \gamma = \frac{EA}{P_0},
\]

(5.5)

where \( \tilde{t} \) is time, \( P_0 \) is the uniform belt tension for the stationary belt system with zero accessory torques, \( EI \) is the bending stiffness, \( EA \) is the belt modulus, \( J_i \) and \( J_i \) denote
the rotational inertias of the pulleys and tensioner arm, and \( \rho \) is the belt mass per unit length.

For Galerkin discretization, the extended variable \( Y \) is expanded in a series of basis function as

\[
Y = \sum_{k=1}^{p} a_k(t) \psi_k(x) + \sum_{k=1}^{3} \theta_k(t) \psi_{p+k}(x) + \theta(t) \psi_i(x)
\]

(5.6)

where \( p = N_1 + N_2 + N_3 \) and \( N_i \) is the number of basis function for the \( i \)th span. Each \( \psi_i \) is a global comparison function that describes deflections of the entire system and satisfies all boundary conditions. For the \( i \)th span, the span deflection is the superposition of sinusoidal waves. For example, \( \psi_{N+k} = \{0, \sin k\pi x, 0, 0, 0, 0\}^T \) for the second span for \( k = 1, \ldots, N_2 \). For the discrete pulleys and tensioner arm, the basis functions are, for example, \( \psi_{p+3} = \{0, 0, 0, 0, 0, 1\} \) for pulley 3. Using the inner product in [69], the discretized formulation from (5.2) is

\[
[M] \ddot{Z} + [G] \dot{Z} + [K] Z + H = 0
\]

(5.7)

where \( Z(t) = \{a_1(t), \ldots, a_p(t), \theta_1, \theta_2, \theta_3, \theta\}^T \) are generalized coordinates. \( H \) is the same as (5.4).

The excitation of the system is from the engine firing pulsations, which induces periodic fluctuations in the crankshaft speed at the firing frequency \( \Omega \). In this paper, pulley 1 represents the crankshaft and its vibration is specified based on engine
properties. For specified $\dot{\theta}_i(t)$, all terms involving $\theta_1$ in (5.7) are moved to the right-hand side as excitation, which results in

$$M^0\ddot{Z}^0 + (G^0 + C^0)\dot{Z}^0 + K^0Z^0 + H^0 = f$$

$$f = -\left[ [M]_{i,p+1} \dot{\theta}_i + [G]_{i,p+1} \dot{\theta}_1 + [K]_{i,p+1} \theta_1 \right], \quad i = 1, \cdots, p + 4, \quad i \neq p + 1$$

(5.8)

Superscript 0 denotes the new matrices and vectors induced by the elimination of the $(p+1)^{th}$ row and column from the original ones. $C^0$ is a damping matrix obtained from modal damping to capture the energy dissipation in the spans and bearings.

5.3 Methods

We seek steady state periodic solutions of (5.8) for periodic driving pulley excitation using the harmonic balance method [58]. A special feature exists in (5.8) that the last equation for $\theta_1$ is the only equation with a nonlinear term, which is similar to the system analyzed by Leamy and Perkins [14]. Adopting the method of incremental harmonic balance [74], they separate the nonlinear equation from the linear ones to obtain periodic solutions. This approach reduces the dimensions of the iteration matrices. According to this spirit, $q = \{a_i, \cdots, a_p, \theta_2, \theta_3\}^T$ are collected as linear unknown variables, and sub-matrices $m^0$, $g^0$, $c^0$ and $k^0$ are introduced by eliminating the last row and column of $M^0$, $G^0$, $C^0$ and $K^0$. The formulation of this subsystem is

$$m^0q + (g^0 + c^0)\dot{q} + k^0q = f^0 - f_a$$

(5.9)
\[ f_a = \begin{pmatrix} \vdots \\ M_{i,p+3}^0 \ddot{\theta}_i + \left( G_{i,p+3}^0 + C_{i,p+3}^0 \right) \dot{\theta}_i + K_{i,p+3}^0 \theta_i \end{pmatrix} \]  

(5.10)

for \( i = 1, \ldots, p + 2 \). Elimination of the last element of \( f \) yields \( f^0 \). The last equation of (5.8) is

\[
M_{p+3,p+3}^0 \ddot{\theta}_i + \left( G_{p+3,p+3}^0 + C_{p+3,p+3}^0 \right) \dot{\theta}_i + K_{p+3,p+3}^0 \theta_i + h_i = \sum_{j=1}^{p+2} \left[ M_{p+3,j}^0 \ddot{\theta}_i + \left( G_{p+3,j}^0 + C_{p+3,j}^0 \right) \dot{\theta}_i + K_{p+3,j}^0 \theta_i \right] = f_{p+3} 
\]

(5.11)

where \( a_{p+1} = \theta_2 \) and \( a_{p+2} = \theta_3 \) are used subsequently for convenience.

To apply harmonic balance, the response is assumed to be periodic and the unknowns are expanded as Fourier series truncated to \( R \) harmonics. Because the nonlinear friction torque (5.1) is an odd function, only odd order harmonics are considered

\[
a_i(t) = \sum_{r=1,3,\cdots}^{R} (u_{r,i} \cos r\Omega t + u_{r+1,i} \sin r\Omega t), \quad i = 1, \ldots, p + 2 
\]

(5.12)

\[
\theta_i(t) = \sum_{r=1,3,\cdots}^{R} (v_{r,i} \cos r\Omega t + v_{r+1,i} \sin r\Omega t) 
\]

(5.13)

The time domain is then discretized into \( N \) intervals as \( t_0, \ldots, t_N \) to form an operator \( L_0 \) such that

\[ a_j = L_0 u_j, \quad \theta_j = L_0 v \]

(5.14)
where \( \mathbf{u}_i = \{u_{i,1}, u_{i,3}, \ldots, u_{i,R}\}^T \), \( \mathbf{v} = \{v_1, v_3, \ldots, v_R\}^T \), and \( \mathbf{L}_0 \) contains \( \cos r\Omega t \) and \( \sin r\Omega t \) evaluated at these discrete times. The time-discretized response vector of the linear subsystem is

\[
\mathbf{q} = \mathbf{L}\mathbf{u}, \quad \mathbf{L} = \begin{bmatrix}
\mathbf{L}_0 & & \\
& \ddots & \\
& & \mathbf{L}_0
\end{bmatrix}
\]  

(5.15)

where \( \mathbf{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_{p+2}\}^T \). Introducing the \((R+1)\times(R+1)\) operator

\[
\mathbf{A}_0 = -\Omega^2 \text{diag}(1^2, 1^2, 3^2, 3^2, \ldots, R^2, R^2)
\]

(5.16)

and a similar operator \( \mathbf{B}_0 \), one has the relations \( \dot{\mathbf{q}} = \mathbf{L}\mathbf{B}\mathbf{u} \) and \( \ddot{\mathbf{q}} = \mathbf{L}\mathbf{A}\mathbf{u} \), where \( \mathbf{A} \) and \( \mathbf{B} \) are block-diagonal matrices constructed from \( \mathbf{A}_0 \) and \( \mathbf{B}_0 \), similar to forming \( \mathbf{L} \) from \( \mathbf{L}_0 \) in (5.15). Furthermore, \( \mathbf{f}^0 = \mathbf{L}\chi \), \( i.e., \chi \) consists of Fourier coefficients of \( \mathbf{f}^0 \).

Substitution of (5.14) into (5.10) generates \( \mathbf{f}_a = \mathbf{L}\mathbf{K}_a\mathbf{v}^0 \), where \( \mathbf{K}_a \) is a block-diagonal matrix with \( M^{0}_{i,p+3}\mathbf{A}_0 + (G^0_{i,p+3} + C^0_{i,p+3})\mathbf{B}_0 + K^0_{i,p+3}\mathbf{I}_0 \) as the diagonal element for \( i = 1, \ldots, p+2 \). \( \mathbf{I}_0 \) is an \((R+1)\times(R+1)\) identity matrix and \( \mathbf{v}^0 = \{\mathbf{v}, \ldots, \mathbf{v}\}^T \). Fourier expansion of (5.9) yields

\[
\mathbf{L}\mathbf{K}_a\mathbf{u} = \mathbf{L}(\chi - \mathbf{K}_a\mathbf{v}^0), \quad \mathbf{K}_a = \tilde{\mathbf{m}}\mathbf{A} + (\tilde{\mathbf{g}} + \tilde{\mathbf{c}})\mathbf{B} + \tilde{\mathbf{k}}
\]

(5.17)

where

\[
\tilde{\mathbf{m}} = \begin{bmatrix}
m^0_{1,1}\mathbf{I}_0 & \cdots & m^0_{1,p+2}\mathbf{I}_0 \\
& \ddots & \\
m^0_{p+2,1}\mathbf{I}_0 & \cdots & m^0_{p+2,p+2}\mathbf{I}_0
\end{bmatrix}
\]

(5.18)
and \( \tilde{g} \), \( \tilde{c} \) and \( \tilde{k} \) are formed similarly. (5.17) produces a relationship between the linear subsystem unknown \( u \) and the nonlinear subsystem unknown \( v \)

\[
u = K_0^{-1}(\chi - K_a v^0)
\]

(5.19)

Fourier expansion also leads to

\[
\dot{\theta}_r = L_0 A_0 v \quad \dot{\theta}_r = L_0 B_0 v \quad h_c = L_0 d \quad f_{p+3} = L_0 \eta
\]

(5.20)

where \( d \) (unknown) and \( \eta \) (known) consist of the Fourier coefficients of the periodic nonlinear function \( h_c \) and the excitation \( f_{p+3} \). Substitution of (5.20) into (5.11) leads to

\[
E = K_b v + d + Pu - \eta
\]

\[
K_b = M^0_{p+3,p+3} A_0 + \left( G^0_{p+3,p+3} + C^0_{p+3,p+3} \right) B_0 + K^0_{p+3,p+3} I_0
\]

(5.21)

where \( E \) represents the numerical residue and \( P \) is derived from the summation term in (5.11).

For a given initial guess \( v \), \( u \) can be determined from (5.19). \( \dot{\theta}_r \) from (5.20) and \( h_c \) from (5.1) are known as well. The discrete Fourier transformation matrix \( \Gamma \) evaluated for \( R \) harmonics yields the Fourier coefficients of the nonlinear function \( h_c \) as \( d = \Gamma h_c \).

The residue \( E \) of (5.21) is then evaluated for the initial guess of \( v \). Newton-Raphson iteration is employed to determine the unknown \( v \) such that \( E \approx 0 \). Similar to the procedure in [58], the Jacobian for Newton-Raphson iteration is

\[
J = \frac{\partial E}{\partial v} = K_b + \frac{\partial d}{\partial v} + P \frac{\partial u}{\partial v}, \quad \frac{\partial d}{\partial v} = \Gamma \text{diag} \left( \frac{\partial h_c}{\partial \theta_r} \right) L_0 B_0
\]

(5.22)
where \( \frac{\partial \mathbf{u}}{\partial \mathbf{v}} \) is determined from (5.19). The iteration \( \mathbf{v}^{\text{new}} = \mathbf{v}^{\text{old}} - \left( \mathbf{J}^{-1} \mathbf{E} \right)^{\text{old}} \) continues until
\[
\| \mathbf{v}^{\text{new}} - \mathbf{v}^{\text{old}} \| / \| \mathbf{v}^{\text{new}} \| \text{ and } \| \mathbf{E} \| \text{ are less than specified tolerances.}
\]

The nonlinear function (5.1) is discontinuous at \( \dot{\theta}_i = 0 \). The tensioner arm exhibits stick-slip motions. When the arm is in the sticking phase, the velocity of the arm is seldom identically zero even though it is small. When sticking is prevalent, the sign of the velocity, and so that of the dry friction torque, changes rapidly, which introduces numerical difficulties. When applying numerical integration, it takes significant time to achieve accurate results and maintain numerical stability because small time steps are required. When employing the harmonic balance technique, the iteration hardly converges across the sticking phase. Many methods have been developed to avoid or solve these difficulties. As recommended in [51], a saturation approximation that removes the discontinuity can yield reliable and numerically efficient results. A saturation function \( h_{cs}(t) \) that approximates \( h_c(t) \) is introduced as (Figure 5.2)

\[
h_{cs} = h_{cs} = \begin{cases} 
Q_m \left| \frac{\dot{\theta}_i}{\sigma} \right| & \dot{\theta}_i > \sigma \\
Q_m & \left| \dot{\theta}_i \right| \leq \sigma \\
-Q_m & \dot{\theta}_i < -\sigma 
\end{cases} \tag{5.23}
\]

where \( \sigma \) is a small positive number. The bound of numerical error from (5.23) during sticking is \( O(\sigma) \) [51]. Use of (5.23) improves the convergence of Newton-Raphson iteration. If divergence still occurs, the Broyden (or secant) method [54,74] is employed in place of Newton-Raphson iteration. In addition, (5.23) alleviates numerical ‘chattering’ and saves substantial computer time when numerically integrating. To
maintain the accuracy and numerical efficiency in this work, $\sigma$ varies between $10^{-5} \sim 10^{-4}$ according to the convergence of the harmonic balance procedures and is fixed at $10^{-6}$ for numerical integration. These values of $\sigma$ ensure satisfaction of the sticking conditions presented in [51]. The variation of $\sigma$ does not affect when the arm sticks or slips, and its impact on the dynamic response is negligible.

Figure 5.2 The dry friction torque $h_c(t)$ (---) and its approximation $h_{cs}(t)$ (---).

Through the above approach, steady state period-$T$ responses are calculated across a wide range of excitation frequency $\Omega$, where $T = 2\pi/\Omega$. Superharmonic resonances are observed if they exist. There is no problem to compute the response to multi-frequency excitation even if the system is extended to a general $n$-pulley system. Most results have
been confirmed by numerical integration, which is much more time-consuming than the harmonic balance method. Because the distinction between the results yielded by the two methods is negligible, only those from harmonic balance are shown in the figures for clarity.

5.4 Results and Discussions

5.4.1 Periodic response

In this section, the response of the three-pulley belt system to periodic driving pulley motion is presented. The nominal physical properties are given in Table 5.1, from which the dimensionless parameters can be calculated.

The crankshaft speed fluctuation is \( \dot{\theta}_i = A \cos \Omega t \). In vehicle applications, the excitation frequency \( \Omega \) is \( \alpha \) times the engine operating speed, where \( \alpha \) is determined by the number of engine cylinders; \( \alpha = 3 \) is used in the present work. \( \Omega = 0 \sim 12 \) corresponds to the engine speed varying over 0~5880 rpm, which is a range of practical importance. The amplitude of the crankshaft speed fluctuation, \( A \), is typically an estimated percentage \( \mu \) of the engine speed, so \( A = \mu \Omega / \alpha \). The value \( \mu = 10\% \) is chosen, although this depends on engine speed in practice. The belt translation speed is \( c = r_1 \Omega \sqrt{P_0 / \rho l^2} / \alpha \), and its dimensionless quantity is obtained by (5.5) as \( s = r_1 \Omega / \alpha l \). As excitation frequency \( \Omega \) varies, the belt speed, and consequently the eigensolutions of the linear system, change. For small bending stiffness, natural frequencies of the dominantly span transverse vibration modes decrease quickly with increasing speed,
while those of the dominantly pulley rotational vibration modes change slightly. For large bending stiffness, all natural frequencies decrease comparably slowly and non-monotonically with the speed because of the strong belt-pulley coupling [69]. Unless indicated otherwise, \( \varepsilon = 0.01 \) is used for belt bending stiffness. The modal damping coefficient to obtain \( C^0 \) in (5.8) is \( \zeta = 5\% \) and the accessory moments \( M_1 \) and \( M_3 \) are assumed zero.

Tensioner dry friction damping is desirable to reduce vibration. This may not be achievable, however, if the friction torque is so large that the tensioner arm is essentially locked. The dynamic belt tension is compensated by the arm’s rotation around its steady state equilibrium. A nearly locked tensioner arm prevents this and may intensify dynamic response and belt tension fluctuations compared to the case of no tensioner dry friction. Two limiting cases of the system are introduced for comparison. The free arm system is when there is no dry friction applied on the arm pivot, \( i.e., Q_m = 0 \). When \( Q_m \) is so large that the tensioner arm does not move, a locked arm system results. Both limiting systems are linear.

The root mean square (RMS) responses of the tensioner arm and pulleys 2 and 3 across the excitation frequency range of practical importance are shown in Figure 5.3 for various dry friction torques. To quantify the belt vibrations, the belt transverse displacements averaged along each span are evaluated as

\[
\overline{y}_i(t) = \sqrt{\int_0^1 y^2_i(x_i,t)dx_i}
\]  
(5.24)

The RMS responses of \( \overline{y}_i(t) \) are shown in Figure 5.4.
Table 5.1 Physical properties of the example system.

| Pulley radius \( r_1 \) & 0.0889 m | Pulley center \((x_1, y_1)\) & (0.5525,0.0556) m |
|----------------------|-----------------|----------------------|------------------|
| Pulley radius \( r_2 \) & 0.0452 m | Pulley center \((x_2, y_2)\) & (0.3477,0.05715)m |
| Pulley radius \( r_3 \) & 0.02697 m | Pulley center \((x_3, y_3)\) & (0,0) |
| Tensioner arm \( r_t \) & 0.0970 m | Pulley center \((x_t, y_t)\) & (0.2508,0.0635) m |
| Rotational inertia \( J_1 \) & 0.07248 kg \( \cdot \) m\(^2\) | Belt Modulus \( EA \) & 120000 N |
| Rotational inertia \( J_2 \) & 0.000292 kg \( \cdot \) m\(^2\) | Initial tension \( P_0 \) & 300 N |
| Rotational inertia \( J_3 \) & 0.000292 kg \( \cdot \) m\(^2\) | Belt mass density \( m \) & 0.1029 kg/m |
| Rotational inertia \( J_r \) & 0.001165 kg \( \cdot \) m\(^2\) | Tensioner stiffness \( k_r \) & 28.25 N-m/rad |
| Span length \( l_1 \) & 0.1548 m | Alignment angle \( \beta_1 \) & 135.79° |
| Span length \( l_2 \) & 0.3449 m | Alignment angle \( \beta_2 \) & 178.74° |
| Span length \( l_3 \) & 0.5518 m | Tensioner rotation \( \theta_r \) & 0.1688 rad |

If the modal amplitude of the tensioner arm in the free arm system is considerable in a specific mode, the nonlinear response at that modal resonance will be mitigated significantly. For example, the tensioner arm motion participates strongly in the pulley dominated vibration modes at \( \Omega = 2.8, 3.3 \) and 8.7 of the free arm system, and therefore the rotational vibrations of the tensioner arm and pulleys 2 and 3 at these resonances are
markedly suppressed by Coulomb damping (Figure 5.3). The suppression is even reflected in the span transverse vibration in Figure 5.4. In contrast, the modal amplitudes of the tensioner arm are small in the dominantly span transverse modes at $\Omega = 6.3$ (span 1), and are moderate in those at $\Omega = 5.7$ and 8.0 (span 2) and at $\Omega = 2.0, 3.8, 5.5$ and 9.4 (span 3). Accordingly, the dry friction has less impact on these resonances.

Figure 5.3 RMS response of a) the tensioner arm, b) pulley 2 and c) pulley 3 for various Coulomb torques for the parameters in Table 5.1. ---- $Q_m = 0$; $Q_m = 0.1$; $Q_m = 1.5$; $Q_m = 5$; (slim) locked arm.
Figure 5.4 RMS response of transverse displacement averaged along each span varies with excitation frequency for the parameters in Table 5.1. --- $Q_m = 0$; --- $Q_m = 0.1$; $Q_m = 1.5$; $Q_m = 5$; (slim) locked arm.

Continued increase of dry friction torque monotonically decreases vibration of the tensioner arm until the arm is fully locked. This is not always true for other components. With an effective loss of one degree of freedom, the eigensolutions of the original system tend to those of the locked arm case for sufficiently large dry friction torque. The pulley dominant modes shift to new frequencies while the span transverse dominant modes change only slightly. The nonlinear response of components other than the tensioner arm approach the new natural frequency resonances for large Coulomb torque. For the system in Table 5.1, one of the pulley 3 dominant modes of the free arm system shifts from $\Omega = 8.7$ to $\Omega = 6.8$ in the locked arm system. The responses of pulley 2, pulley 3
(Figure 5.3) and span 3 (Figure 5.4c) suddenly increase near the locked arm resonance at $\Omega = 6.8$. For span 1 (Figure 5.4a), although the vibration is suppressed for given dry friction torque up to $Q_m = 5$ near $\Omega = 6.8$, for even larger Coulomb torque the vibration increases because the response tends to the linear locked arm case, which is higher than that of the free arm case. Furthermore, a dominantly span 2 transverse vibration mode at $\Omega = 3.0$ replaces the pulley rotational vibration modes at $\Omega = 2.8$ and 3.3 of the free arm system (Figure 5.4b). This mode is barely evident even in the linear locked arm response of span 2, and therefore the vibration is small for the nonlinear system.

The system is subject to an odd excitation function (5.1) at the tensioner arm hub, which leads to odd order secondary resonances. For the current model, superharmonics are observed in the periodic response of the span generalized coordinates. Consider the RMS response and waterfall spectra of the displacement amplitude $a_3$ of the $\sin 3\pi x$ component of span 1 in Figure 5.5. For a given dry friction torque $Q_m = 0.5$, $\Omega = 4.2$ ($s = 0.35$) is one fifth of $\Omega_{15} = 21.0$ and $\Omega = 4.5$ ($s = 0.38$) is one third of $\Omega_{13} = 13.5$, where $\Omega_i$ are the $i^{th}$ natural frequencies of the free arm system with belt speed $s$. The secondary resonances are not large, and when using (5.3) to yield belt deflections, they only modestly influence the physical response. No secondary resonance is possible if the arm is locked with linear response. Higher dry friction torque extends the frequency range with locked-arm response. For instance, observing where the curves in Figure 5.5a separate from the locked-arm case, the arm is locked for $\Omega < 2.8$ for $Q_m = 0.5$ while the range extends to $\Omega < 4.3$ for $Q_m = 1.5$. As a result, an increase of dry friction torque
extends the range without secondary resonance to higher frequency. This property holds for systems with higher bending stiffness. Accordingly, for the current damping ratio $\zeta = 5\%$ and even higher values, the dry friction torque does not introduce harmful secondary resonances to offset its dissipative benefits. Even when the system is reduced to a purely discrete model that considers only pulley rotations, there are no pronounced superharmonics in the response.
Figure 5.5 Superharmonics are shown in a) RMS response and b) spectra waterfall of the displacement amplitude $a_3$ of the $\sin 3\pi x$ component of span 1 for the parameters in Table 5.1. In a) $\ldots\quad Q_m = 0$; $\ldots\quad Q_m = 0.1$; $\ldots\quad Q_m = 0.5$; $\ldots\quad Q_m = 1.5$; $\ldots$ (slim) locked arm. b) $Q_m = 0.5$. 
5.4.2 Impact of dry friction torque

Starting from the linear system \((Q_m = 0)\), the tensioner arm vibration decreases monotonically with increasing \(Q_m\) across the entire frequency range until eventually the arm does not move at all (Figure 5.6a). The rate of decrease depends on which mode is dominantly excited. Dry friction most impacts those modes having large tensioner arm motion, and response in such modes decreases more steeply. For instance, the arm modal amplitudes in the pulley dominant modes at \(\Omega = 2.8\) and 3.3 are more appreciable than in the mode at \(\Omega = 8.7\), so the decrease rate is steeper. For the dominantly span transverse vibration modes near \(\Omega = 5.7, 6.3\) and 8.0, the arm modal displacement is least for the mode at \(\Omega = 6.3\), so this mode requires the highest Coulomb torque to achieve locked arm status.

The response of other system components does not vary monotonically with the friction torque. Moreover, the vibration increases at the frequency range near the resonance of the mode of the locked arm system involving large pulley amplitudes such as the modes at \(\Omega = 5.7, 6.3\) and 6.8; it is diminished outside these frequency ranges. Figure 5.6b and 6c show the response of pulleys 2 and 3 for varying Coulomb torque. The free arm response is the starting point of each frequency branch. The flat portion of each branch begins at a friction torque beyond which the arm is fully locked. The vibration is suppressed for the frequency branches \(\Omega = 2.8, 3.3, 8.0\) and 8.7, while it increases for the branches \(\Omega = 5.7\) and 6.3. Generally the nonlinear response falls between the locked and free arm linear responses, but in some cases there is an extremum.
of the curves in Figure 5.6b and 6c that is outside of this range (e.g., $\theta_2$ response for $\Omega = 8.0$).

Figure 5.6 RMS response of a) the tensioner arm, b) pulley 2 and c) pulley 3 with varying Coulomb torque for the parameters in Table 5.1. $\Omega = 2.8$; (bold) $\Omega = 3.3$; (bold) $\Omega = 5.7$; $\Omega = 6.3$; (slim) $\Omega = 8.0$; (slim) $\Omega = 8.7$.

5.4.3 Energy dissipation

The dissipated energy $\Delta E$ within a cycle, which is a direct measure of dry friction effectiveness, is given by

$$\Delta E = \int_0^T Q_m \text{sgn}(\dot{\theta}) \dot{\theta} dt = \sum_{k=1}^N Q_m \text{sgn}(\dot{\theta}(t_k)) \dot{\theta}(t_k) \Delta t$$

(5.25)
Figure 5.7 shows the dissipated energy for various friction torques across the practically important frequency range. Values near zero in Figure 5.7 imply a nearly stationary arm. At resonances of modes with large tensioner arm rotation, large arm velocity causes considerable dissipation. Larger torque undesirably extends the frequency range where the arm gets stuck and no dissipation occurs. Arm sticking also prevents the tensioner from achieving its main purpose of dynamic tension compensation as operating conditions fluctuate. The energy dissipation is greatest at frequencies where high velocities occur.

![Energy dissipation graph](image)

**Figure 5.7** Energy dissipation varies with excitation frequency for the parameters in Table 5.1. 
- $Q_m = 0.1$; 
- $Q_m = 0.5$; 
- $Q_m = 3$; 
- $Q_m = 5$. 

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The energy dissipation does not vary monotonically with the torque for specific frequencies because the dissipation depends upon the product of arm velocity and Coulomb torque according to (5.25). Larger torque may induce lower velocity, which may lead to lower dissipation. For instance, $Q_m = 3$ and $5$ dissipate more energy than $Q_m = 0.1$ and $0.5$ for $6.0 < \Omega < 7.7$ where lower velocities result for the former dry friction cases, but the opposite holds outside this range. Figure 5.8 addresses how the Coulomb torque affects energy dissipation for given excitation frequencies. With increasing friction torque for a given frequency, the energy dissipation first increases to an extreme value, and then decreases gradually until there is no dissipation for a stationary arm.

![Energy dissipation vs. Coulomb torque](image)

Figure 5.8 Energy dissipation varies with Coulomb torque for the parameters in Table 5.1. $v$ indicates the number of the velocity reversals near the arm extreme location in a half cycle for $\Omega = 6.3$. $\cdots$ $\Omega = 2.8$; (bold) $\Omega = 3.3$; $\cdots$ $\Omega = 5.7$; $\cdots$ $\Omega = 6.3$; (slim) $\Omega = 8.0$. 

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The energy dissipation curves in Figure 5.8 for each frequency branch exhibit lobes. The transitions between lobes reflect a change in the number of arm velocity reversals, as indicated by $\nu$, near the arm extreme location for each half cycle. Within a half cycle, for low friction torque, the arm normally moves back and forth only once because the torques exerted by other system components easily overcome the dry friction torque. See $\nu = 0$ portion of $\omega = 6.3$ branch. An example time history of the tensioner arm and the corresponding dry friction torque for $Q_m = 1.0$ is shown in Figure 5.9a, where the arm vibrates around the equilibrium regularly. With increasing Coulomb torque, the vibratory torques are insufficient to surmount the dry friction and the arm tends to stick near the extreme rotation angle. With increasing $Q_m$, the arm velocity first reverses once while crossing the sticking phase, which corresponds to the second lobe on the dissipation curve where $\nu = 1$. Figure 5.9b illustrates one velocity reversal with $Q_m = 3.5$. After the sign of the arm velocity changes, a short time later, the arm fluctuates back to the extreme angle and then continues its normal trajectory toward the equilibrium. Because the fluctuation is not large, the angular velocity falls into $[-\sigma, \sigma]$ and, according to (5.23), linear dry friction torque is used during calculation.
When further increasing the torque magnitude, the arm velocity may reverse two or more times at the extreme arm location. Eventually, the arm gets fully stuck for large torque. In Figure 5.9c for $Q_m = 6.1$, on the return toward the equilibrium, the arm fluctuates back to the extreme location twice ($\nu = 2$) crossing the sticking phase. In Figure 5.9d for $Q_m = 11$, the arm rotational angle varies only slightly around the equilibrium for the entire cycle. The arm is effectively stuck. The arm velocity is within the interval $[-\sigma, \sigma]$ and the linear friction torque in (5.23) applies.
5.4.4 Critical dry friction torque

There exists a critical dry friction torque that induces locked arm status according to previous sections. With the arm locked, \( \theta_i = 0 \), all terms involving the tensioner arm in (5.11) vanish, and the pivot torque is

\[
Q_m = f_{p+3} - \sum_{i=1}^{p+2} \left[ M_{p+3,i}^0 \ddot{a}_i + \left( G_{p+3,i}^0 + C_{p+3,i}^0 \right) \dot{a}_i + K_{p+3,i}^0 a_i \right] \quad (5.26)
\]

where \( a_i \) refers to the component response of the locked arm system from (5.9) with \( f_a = 0 \). For the sinusoidally driven locked arm linear system, \( Q_m \) varies sinusoidally, \( i.e., \)

\[
Q_m = Q_{mcr} \sin(\Omega t + \phi)
\]

where \( Q_{mcr} > 0 \) is the critical torque. Figure 5.10 shows how the critical Coulomb torque \( Q_{mcr} \) varies with frequency for various belt bending stiffness. These are upper bounds that enforce \( \theta_i(t) = 0 \). The system can be effectively locked, \( \theta_i(t) \approx 0 \), for lower \( Q_m \) as shown in Figure 5.8 for specific frequency. For example, \( Q_{mcr} = 18.7 \) for \( \Omega = 6.3 \) and \( \epsilon = 0.01 \), which is larger than the estimated value \( Q_{mn} = 10.4 \) in Figure 5.8. In the curves near \( \Delta E \approx 0 \) in Figure 5.8, the energy dissipations are nearly, but not exactly, zero and decreasing slowly.

Other than the resonant regions, the curves in Figure 5.10 show that the critical Coulomb torque tends to decrease with increasing bending stiffness. Away from resonance, the torque generated by the summation term of the right-hand side of (5.26) is small compared with that from the first term, \( i.e., \) the torque \( f_{p+3} \) derived from the driving pulley rotation dominates the pivot torque. With this stipulation,
\[ Q_m \approx f_{p+3} = \frac{\gamma r_1}{l_1} \left( \sin \beta_1 - \cos \beta_1 w_1^*(1) \right) \theta_1 \]  

For the specified system parameters and the driving pulley rotation \( \theta_1 \), \( Q_m \approx f_{p+3} \) varies with the equilibrium deflection at the first span boundary \( w_1^*(1) \). From [68], the boundary layers resulting from small bending stiffness \( \varepsilon \) become less pronounced in the equilibrium deflections as \( \varepsilon \) grows and the equilibrium deflections become larger and smoother. For the parameters in Table 5.1, \( \cos \beta_1 \) and \( w_1^*(1) \) are negative. Therefore, \( f_{p+3} \) and so \( Q_m \) decrease with increasing bending stiffness.

At the resonant regions, the system components exert more pronounced torque evaluated by the summation term in (5.26), and the pulley vibrations dominate the peak values of each bending stiffness branch. For example, with \( \varepsilon = 0.01 \), the resonant amplitude of pulley 3 induces a peak value at \( \Omega = 6.8 \) in the critical dry friction torque curve.
Figure 5.10 Critical Coulomb torque varies with excitation frequency for different belt bending stiffness beyond which the tensioner arm is entirely locked for the parameters in Table 5.1. .... $\epsilon = 0.01$; .... $\epsilon = 0.05$; .... $\epsilon = 0.1$; .... $\epsilon = 0.2$.

5.4.5 Dynamic span tension

Span tension fluctuations are important criteria in evaluating the dynamic behavior of the whole system. The tensioning system aims to compensate for operating condition changes and dynamic excitations to reduce the dynamic belt tension amplitudes. The dynamic span tensions $P_{di}$ are [69]

$$
P_{d1} = \gamma \left( -\frac{r_2}{l_1} \theta_2 + \frac{r_1}{l_1} \theta_1 - \frac{r}{l_1} \theta_1 \sin \beta_1 + \int_0^l w_1 w_1^* dx \right)
$$

$$
P_{d2} = \gamma \left( -\frac{r_2}{l_2} \theta_2 + \frac{r_2}{l_2} \theta_2 + \frac{r}{l_2} \theta_2 \sin \beta_2 + \int_0^l w_2 w_2^* dx \right)
$$

$$
P_{d3} = \gamma \left( -\frac{r_3}{l_3} \theta_3 + \frac{r_3}{l_3} \theta_3 + \int_0^l w_3 w_3^* dx \right)
$$

(5.28)
where the quantities with asterisks are from the steady equilibrium solution. The integrals in (5.28) are typically small compared to the dynamic span deflection with the pulley rotations, so the bounding pulley rotational vibrations dominated the tension fluctuation of each span.

The maximum dynamic tension in each span is evaluated as shown in Figure 5.11. On one hand, increasing dry friction torque monotonically diminishes the tensioner arm rotation until the tensioner is effectively locked. On the other hand, pulleys 2 and 3 vibrate more near the resonances of the natural modes of the locked arm system that involve large pulley amplitudes, such as the modes at $\Omega = 5.5, 5.7, 6.3$ (dominated by span vibration) and 6.8 (dominated by pulley 3 vibration). These competing effects combine to impact the dynamic tensions for the first and second spans when varying the dry friction torque. Near these resonances, the balance of the response of the tensioner arm and pulley 2 leads to slight reduction on the dynamic tension of the first span with increasing dry friction torque, while the involvement of pulley 3 vibration increases the dynamic tension of the second span. Pulley 3 vibration independently affects the third span, so large dynamic tension occurs at those resonances where large Coulomb torque drives the arm toward the locked arm configuration. Other than the above resonant regions, higher Coulomb torque produces better tension reduction.

To avoid large belt tension fluctuation, one may tune the natural frequencies of the locked arm system whose modes involve large pulley vibrations away from the operating frequency range if a large dry friction torque is used at the tensioner arm pivot.
Figure 5.11 Maximum dynamic tension in each span varies with excitation frequency for the parameters in Table 5.1. ... $Q_m = 0$; \_\_\_\_ $Q_m = 0.1$; \_\_ \_\_ $Q_m = 1.5$; \_\_ \_\_ \_ $Q_m = 5$. 


CHAPTER 6

SUMMARY AND FUTURE WORK

6.1 Summary

This work investigates, via analytical and numerical methods, nonlinear issues occurring in belt-pulley systems with special application in automotive front end accessory drives. The main results are summarized for each specific topic.

1. Nonlinear dynamics of a one-way clutch in belt-pulley systems

The nonlinear dynamics of a two-pulley belt system with a wrap-spring type one-way clutch are examined using a two degree of freedom model. The nonlinear one-way clutch is modeled as a spring with discontinuous stiffness, where clutch torque acts only for positive relative motion between the pulley and the accessory shaft. The problem is studied with three methods: multi-term harmonic balance, numerical integration, and the bifurcation software AUTO. All yield results with excellent agreement. Harmonic
balance and AUTO yield the stable and unstable periodic solutions. Numerical integration gives details of the quasiperiodic and chaotic solutions that occur in the absence of stable periodic solutions. The main conclusions are as follows.

a) The nonlinear spring changes the SDOF system (no clutch) into a two-DOF system, much like a vibration absorber. The frequency response is concentrated at two resonant regions near the natural frequencies of the two-DOF linear model (clutch engaged). Significant softening nonlinearity occurs due to clutch disengagement at the second natural frequency where the mode involves out-of-phase pulley-accessory motion. The clutch markedly decreases resonant amplitude near the first mode natural frequency, as it is designed to do. Near this region, a complicated pattern of periodic solution bifurcations occurs. Across a range of speeds, aperiodic, quasiperiodic and chaotic responses occur. Multiple disengagements per cycle occur in some periodic solutions.

b) The stiffness of the clutch spring impacts the response considerably. Smaller values tend to minimize the response in the first (in-phase) mode. Larger values tend to minimize response in the second (out-of-phase) mode and simultaneously push this resonance to frequencies that may lie outside the range of practical importance. Large values reduce or eliminate the range in which chaotic or aperiodic solutions occur.

c) Steady state solutions for large and small excitation amplitudes $A_m$ at frequencies near the first mode resonance show a complex assortment of multiple stable/unstable periodic and aperiodic solutions. Calculation of closed-form approximations to these solution curves appears very difficult. Behavior near the second (out-of-phase) mode
follows the classical softening nonlinearity pattern similar to Duffing’s equation, for example.

d) The dynamics are sensitive to the inertia ratio $\alpha = J_a / J_p$ largely because this changes the nature of the vibration modes. For small $\alpha$, or a large driven pulley attached to a small accessory, linear behavior dominates the system dynamics at the first resonance and the second mode resonance is hardly excited. With increasing accessory inertia, the modal force applied to the second mode increases, causing more disengagement in this out-of-phase mode and therefore softening nonlinearity; the dynamics of the first mode also show much more nonlinear behavior.

2. Perturbation analysis of a clearance-type nonlinear system

A classical perturbation technique, the method of multiple scales, is employed to approximate the steady state periodic solutions of a two-pulley belt system with clearance-type nonlinearity. The discontinuous separation function is expanded as a Fourier series in the perturbation analysis. For given amplitude, the Fourier coefficients of the separation function can be evaluated, and the closed-form frequency-response relation is determined at the first order. For the second order approximation, the frequency-response relation is an implicit expression of a fourth order polynomial.

The preload determines the softening level of the nonlinearity. Larger preload induces less disengagement, hence, less softening. In this case, the perturbation method generates good approximations. In contrast, lower preload results in a large disengagement fraction of a cycle, and therefore, greater softening deformation of the
resonant peak. This degrades the small disengagement assumption and the perturbation approximations, including the amplitude and the backbone, deviate from the numerical solutions.

3. *Piece-wise linear dynamic analysis of serpentine drives with a one-way clutch*

The prototypical three-pulley system with belt bending stiffness is extended with a one-way clutch. The clutch is modeled based on the relative velocity of the driven pulley and its accessory. When the relative velocity is zero, the clutch engages that produces a positive inner clutch torque. For unequal velocities, the clutch disengages with zero clutch torque. This model leads to a piece-wise linear system and the switching conditions are zero clutch torque and zero relative velocity for engagement and disengagement configurations, respectively. The transition matrix is used to evaluate the system response in discrete time series, which saves computation time.

For the periodic excitation from the driving pulley, the steady state periodic response is obtained across the practically important frequency range. The natural frequency of the accessory pulley dominant mode of the disengaged clutch system associated with an inertia ratio $\alpha$ veers away from that of the locked clutch system, which strongly impacts the effectiveness of the vibration suppression on the nonlinear system. Compared with the locked clutch system, the rotational vibrations are noticeably suppressed if the frequency is away from the disengaged clutch resonances. For the span transverse vibration, superharmonics of the modes with high nonlinear response occur,
which slightly increase the vibrations at their vicinities but the harmfulness is not crucial. The accessory gains appreciable benefit from the clutch performance with suppressed vibration across the frequency range of practical importance.

The inertia ratio $\alpha$ reflects the distribution of the inertia on the driven pulley and accessory. The natural frequencies of the disengaged clutch system veer as varying $\alpha$. As crossing the range of $\alpha$, a resonance of either linear system occurs for a given frequency and more disengagement is generated at the resonance. For large enough inertia ratio, the engagement and disengagement of the clutch form a steady pattern and the engaged time ratio approaches a limiting value. Larger external load prevents more disengagement. A critical load beyond which the system always behaves with a locked clutch can be determined analytically across the frequency range.

The one-way clutch performance induces dynamic tension drop. The dynamic tension of the spans that are adjacent to the pulley integrated with the one-way clutch receives more impact from the clutch performance than other spans. Smaller external load induces more disengagement for a given frequency, and so more tension drop. The dynamic tension is mainly determined by the pulley rotational vibrations. For a given frequency, across the inertia ratio range, higher dynamic tension is induced at resonances of either linear system because of higher responses of the system components.

4. Influence of dry friction tensioner on belt-pulley systems with belt bending stiffness
A mathematical model for belt-pulley systems with a dry friction tensioner considering belt bending stiffness and transverse span vibration is established, and the influence of dry friction at the tensioner arm hub is examined. The dry friction torque is modeled as a signum function with a constant magnitude, the sign of which is determined by the arm velocity. The governing equations are divided into linear and nonlinear subsystems. The nonlinear one addresses the vibration of the tensioner arm subject to the dry friction torque; the linear one includes the pulley rotations and spatially discretized span motions. The engine firing pulsation is the excitation source driving the entire system, which is reflected in the periodic fluctuation of the crankshaft speed.

The response to periodic excitation is obtained by the harmonic balance method. Suppression of the tensioner arm vibration increases with the dry friction torque until the arm is essentially locked. For other system components, this dissipative benefit is not monotonic and the extent of the vibration mitigation varies for different excitation frequency range. Odd order superharmonics are identified in the response of the generalized span coordinates, but they have minor effect on the physical system because of their small amplitudes.

With increasing dry friction torque, energy dissipation from the dry friction first increases until a maximum value is achieved and then gradually decreases to zero where the arm is locked. Lobes occur in the dissipation curve because the arm velocity reverses near the extreme arm location that reveals the arm sticking feature. More velocity reversals indicate the arm spends more percentage of time in the sticking phase of each
cycle. The critical magnitude of the Coulomb torque that makes the arm essentially locked is determined analytically.

6.2 Future Work

More linear/nonlinear dynamic phenomena observed in belt-pulley systems are of interest to be analyzed. To gain better modeling and understanding of the general belt-pulley systems, the author recommends the following future work.

1. Investigation of transient responses of serpentine drives with one-way dry friction tensioner

In Chapter 5, the dynamics of a dry friction tensioner in a serpentine belt drive is investigated. The tensioner arm shows stick-slip behavior when it vibrates. In slipping phase, the arm pivot is subjected to a constant dry friction torque opposite to the velocity direction whose magnitude is determined by the normal force $F_n$ and the kinetic friction coefficient $\mu_k$. If the arm is stuck (stationary), static dry friction is active and varies within its maximum limit. The maximum static dry friction is determined by $F_n$ as well as the static friction coefficient $\mu_s$. The normal force can be controlled by the installation preload. Figure 6.1 addresses the relation between the dry friction and arm velocity. To simplify the problem and avoid numerical difficulty, one typically assumes $\mu_s = \mu_k$ and a line with slope $\mu_k F_n / \sigma$ replaces the vertical line across the origin at $\dot{\theta} \in [-\sigma, \sigma]$. See
Figure 5.2 for details, where $h_{cs}(t) = -F_i$ and $Q_{ms} = \mu_k F_n$. The sticking phase occurs in $\dot{\theta}_t \in [-\sigma, \sigma]$.

![Diagram of Coulomb friction law](image)

Figure 6.1. A Coulomb friction law.

For certain sets of parameters, the dry friction tensioner effectively dissipates the system energy and reduces the vibration. This type of design that allows the tensioner arm movement for both directions, however, encounters new operation difficulty when the crankshaft drives one of the accessories with large inertia. Without loss of generality, in Figure 6.2, the crankshaft rotates clockwise and a fan with big inertia $J_s$ is the accessory adjacent to it. The tensioner arm adjusts the belt tension with initial alignment angle $\theta_{ir}$ for static belt and zero load (defined as reference state). When the engine starts up, the crankshaft drives the system components to accelerate. The belt span between the crankshaft pulley and the fan pulley is initially slack, and at this moment is tightened (pressed down) by the tensioner arm (clockwise) to adjust the belt tension for a new
equilibrium with the arm rotational angle $\theta_{1}$. When the engine shuts down, the crankshaft decelerates. All the system components are supposed to decrease the speed simultaneously. The fan, however, still has large kinetic energy because of its big inertia. This results in the fan working as a driving pulley. To adjust the belt tension, the tensioner arm is pushed away from the fan pulley, i.e., moves counterclockwise with backward angle $\theta_{2}$ (compared to the reference state). If the crankshaft decelerates sharply and the accessory inertia is sufficiently large, $\theta_{2}$ can be large. If another pulley is located near the tensioner, the tensioner pulley may hit that pulley, such as pulley 3 indicated in Figure 6.2, causing damage to one or both elements.

![Figure 6.2. Schematic diagram for a potential that the tensioner arm hits another pulley during backward movement.](image)

To avoid ineffective operation and potential arm damage, engineers design a one-way damper at the arm pivot. The damper allows the arm to rotate and adjust the belt
tension only when the crankshaft accelerates and/or the deceleration is not large. This rotating direction of the arm is defined as ‘forward’. When the crankshaft decelerates sharply, the damper produces big dry friction which does not allow for backward arm movement. Figure 6.3 gives a typical three-pulley belt system including a one-way damper at the arm pivot. Here clockwise rotational angle \( \theta_i \) of the arm is admitted. The one-way damper is mathematically modeled as

\[
f(\dot{\theta}_i) = \begin{cases} 
-\mu_{k_1} f_n, & \dot{\theta}_i > 0 \\
\mu_{k_1} f_n, & -\sigma < \dot{\theta}_i < 0 \\
\mu_{k_2} f_n, & \dot{\theta}_i < -\sigma 
\end{cases}
\]

(6.1)

where \( \mu_{ni} \) and \( \mu_{ki} \) for \( i = 1, 2 \) are static and dynamic friction coefficients, respectively, and \( \mu_n \geq \mu_{ki} \), \( \mu_{s2} \geq \mu_{s1} \), \( \mu_{k2} \geq \mu_{k1} \). \( f_n \) denotes the normal force. One assumes \( \mu_{s2} f_n \) is adequate to avoid backward slipping. \( \sigma \geq 0 \) is a tolerance that allows for small vibration near the equilibria. Figure 6.4 describes this dry friction law.

Equation (6.1) is an asymmetric expression that can introduce numerical difficulties. One approximation is that \( \mu_{k2} = \mu_{s2} \) and a line crossing the origin with finite slope \( \mu_{k2} F_n / \sigma \) replaces the lines between \( [-\sigma, \varepsilon] \), where \( \varepsilon = \sigma \mu_{k1} / \mu_{k2} \). This approximation is similar to the function in Chapter 5 with an origin translation of \( (\sigma (\mu_{k1} + \mu_{k2}) / 2 \mu_{k2}, F_n (\mu_{k1} + \mu_{k2}) / 2) \). The harmonic balance method can yield steady state periodic solutions, while numerical integration provides transient responses.
Can the approximated dry friction model reveal the device operation function? One can demonstrate the model through investigation of the solutions when the system is subjected to excitation from crankshaft acceleration and deceleration. One way damper similarly has impact on the steady state response especially when the dynamic $\dot{\theta}_i$ is lower.
than $\sigma$. The dynamic excitation sources could come from the crankshaft speed fluctuation, periodic external load, etc. How does this type of damper affect the belt tension? Can the tensioner arm effectively adjust the belt tension? How to control the parameters to avoid belt wear because of one-way tension adjustment? Are there bifurcation phenomena? All above will be of interest and examination of them will provide design guidelines.

2. *Dynamic analysis on belt creep on pulleys with belt bending stiffness*

In the previous chapters, the belt creep on pulleys is not modeled, *i.e.*, the belt and pulley have no relative velocity in the belt-pulley contact zones when the belt transmits power between the pulley elements and the impact of friction at the belt-pulley interface is ignored.

Belt-pulley mechanics has received research attention for decades. Two main theories in this research field are developed as elastic creep theory and shear theory. In the creep theory, the belt behavior is assumed to be governed by the elastic extension or contraction of the belt caused by the tension variation in the belt. Because of the relative sliding motion of the belt and pulley, the friction is developed on the contact interface and is modeled by a Coulomb law [9-11,75-77]. The shear theory assumes the shear deformation dominates the belt behavior [78], which works for inextensible belts. Alciatore and Traver [79] compare these two theories and apply them to different belt systems. Gerbert makes strong contribution to the development of the belt mechanics. In
[8,80], he combines the theory of creep, shear, seating/unseating and rubber compliance to the belt drives with different belt types and makes the experimental measurement and theory fit well.

A lot of literature focuses on steady state analysis of the belt creep on a pulley using classical creep theory. Johnson [75] reviews this theory where the derivation of a belt-pulley system ignores the belt inertia term and centrifugal term. Bechtel, et al. [76] include all effects of inertia in the momentum balance to derive the equations of motion for an extensible belt on a pulley. Based on the spirit in [76], Leamy and Wasfy [10] develop an exact belt-drive solution by employing creep-rate-dependent friction law on the belt-pulley friction and discuss distribution of the sliding and adhesion zones on the contact region. These works model the belt as a translating string where the contact zones start and end at the tangent points of straight free spans between pulleys. Kong and Parker [11] consider the belt bending stiffness, which introduces unknown belt departure and end points on the belt-pulley contact zones because of non-straight free span curvature. The unknown boundary value problem is transformed into a fixed-boundary form, and iteration is implemented to determine numerical solutions.

The literature about dynamic analysis of axially moving materials [25,81-83] is large. Continuous models are developed for axially moving materials. Perturbation methods are employed to obtain analytical approximations for the associated nonlinear differential equations [25,81-83], or the Galerkin method is used to discretize the equations [21,84].
As reviewed in Chapter 1, transmission belts are applications of axially moving materials, and the recent studies about belt drives are extensive [3,4,28,30,31,35,68,69,85,86]. Although they study the interactions between the belt and pulleys, the dynamics in the belt-pulley contact zones are ignored.

Leamy bridges this gap in [77]. He develops the equations of motion for an unsteady belt-drive system. In the analytical approximations from perturbation method, the zero order solution refers to the steady state equilibria for given rotational speed of the driving pulley. In this study, the belt is modeled as a translating string and the two pulleys have an identical radius, which induces straight equilibrium curvature of the belt and the belt-pulley contact arc covers half of the outer circumference of each pulley. To simplify the analysis of the tension distributions, the spatial variation of the mass flow rate is neglected, which results in discontinuous distribution of the mass flow rate along the belt and unchanged belt tension in the free spans for any time instant.

To gain better understanding of the dynamic behavior of the belt drive including belt creep on the pulleys, the future work could use the creep theory to develop equations of motion for an unsteady belt-drive system including the belt bending stiffness and considering all effects of belt inertia. The dynamic analysis is based on the steady state mechanics discussed in [11]. Figure 6.5 is a schematic diagram of a two-pulley belt system considering belt creep on the belt-pulley contact zones. The driver pulley with radius $R_1$ and the driven pulley with radius $R_2$ are subjected to external torques $M_1(t)$ and $M_2(t)$, respectively, with a two-pulley center distance $L$. The power transmission is implemented counter-clockwise. The frictional forces occur between the belt and pulley.
contact arcs. Accordingly, the belt tension varies and the belt will extend or contract elastically and move relative to the rigid pulley surface.

Figure 6.5. Two-pulley belt system considering belt creep on contact zones.

Here is the mechanism of power transmission by the driving pulley to the driven pulley. The belt translates in the tight-side span with speed $V(s_1, t)$ and tension $T(s_1, t)$. The belt runs onto the pulley at $s_1 = 0$ and starts contact with speed $V(0, t)$ and tension $T(0, t)$ in the adhesion zone bounded by $\alpha_i(t)$, where no relative motion of the belt and pulley occurs. At the end of the adhesion zone, the static dry friction approaches its maximum limit and the two contact surfaces start to slide relative to each other and generate dynamic dry friction. This part of the contact arc delineated by $\beta_i(t)$ is called the sliding zone, where the surface speed of the driver pulley $R\omega_{DR}(t)$ will be greater than that of the belt. After the sliding zone, the belt travels as slack span with speed
$V_2(s_2,t)$ and tension $T_2(s_2,t)$. At $s_2 = \tilde{L}_2$, it runs onto the adhesion zone $\alpha_z(t)$ of the driven pulley with constant speed $V_2(\tilde{L}_2,t)$ and tension $T_2(\tilde{L}_2,t)$ for any time instant $t$ until, again, the dry friction reaches its limits and the sliding zone with arc $\beta_z(t)$ starts. Faster speed of the belt surface than that of the pulley surface induces growth of the belt tension and speed in this region until $s_i = \tilde{L}_i$. In the above description, $\tilde{L}_i, i = 1, 2$ refer to the total arc-lengths of the free belt spans.

To develop the governing equations of the system, a control volume with an infinitesimal arc length $ds$ on a moving belt is considered. Its force diagram is shown in Figure 6.6. By considering the balance of angular momentum to the control volume, one obtains

$$Q(s,t) = \frac{\partial M(s,t)}{\partial s}$$

(6.2)

where $M(s,t)$ and $Q(s,t)$ are the moment and shear force on the cross section of the belt segment. $EI$ is the belt bending stiffness and $\kappa(s,t)$ is the belt curvature with

$$\kappa(s,t) = \frac{\partial \theta}{\partial s}, \quad M(s,t) = EI\kappa(s,t)$$

(6.3)

Consequently, the linear momentum balance gives

$$\frac{\partial T(s,t)}{\partial s} + EI\kappa(s,t) \frac{\partial \kappa(s,t)}{\partial s} - f(s,t) = \frac{\partial G(s,t)}{\partial t} + V(s,t) \frac{\partial G(s,t)}{\partial s} + G(s,t) \frac{\partial V(s,t)}{\partial s}$$

(6.4)

$$[T(s,t) - G(s,t)V(s,t)]\kappa(s,t) - EI \frac{\partial^2 \kappa(s,t)}{\partial s^2} = n(s,t)$$

(6.5)
where $T(s,t)$ and $V(s,t)$ denote the belt tension and speed at the arc location $s$ at time instant $t$, respectively. $f(s,t)$ and $n(s,t)$ refer to the frictional force and normal force on the belt surface. $G(s,t)$ is the mass flow rate defined as

$$G(s,t) = \rho(s,t)A(s,t)V(s,t)$$  

(6.6)

where $\rho(s,t)$ and $A(s,t)$ denote the mass density per unit volume and cross-sectional area of the belt, respectively. Conservation of mass requires

$$\frac{\partial \rho(s,t)A(s,t)}{\partial t} + \frac{\partial G(s,t)}{\partial s} = 0$$  

(6.7)

To fully describe the system behavior, a constitutive law is needed. According to [11] and [77], a differential belt element with un-deformed length $ds_0$ has deformed length

$$ds = (1 + \lambda T(s,t)) ds_0$$  

(6.8)

and has deformed area

$$A(s,t) = \left(1 - \nu \lambda T(s,t)\right)^2 A_{ref}$$  

(6.9)

due to Poisson effects, where $\lambda = 1/EA$ is a measure of belt compliance. The subscript 0 means the quantity in the un-deformed state. The expression for the mass density per unit length is

$$\rho(s,t)A(s,t) = \frac{\rho_0 A_0}{1 + \lambda T(s,t)}$$  

(6.10)

and for the mass flow rate is

$$G(s,t) = \frac{\rho_0 A_0 V(s,t)}{1 + \lambda T(s,t)}$$  

(6.11)
The free spans satisfy
\[
\frac{\partial x(s,t)}{\partial s} = \cos \theta(s,t), \quad \frac{\partial y(s,t)}{\partial s} = \sin \theta(s,t)
\] (6.12)

The unknown variable $\tilde{L}(s,t)$ still satisfies
\[
\frac{\partial \tilde{L}(s,t)}{\partial s} = 0
\] (6.13)

One assumes the tension at the start point is known
\[
T(0,t) = T_0(t)
\] (6.14)

The curvatures at the two boundaries are determined by
\[
\kappa(0,t) = -1/R_1, \quad \kappa(\tilde{L},t) = -1/R_2
\] (6.15)

Other geometric boundary conditions are
\[ x(0, t)^2 + y(0, t)^2 = R_1^2, \quad x(0, t) = -R_1 \sin \theta(0, t) \]
\[ (x(L, t) - L)^2 + y(L, t)^2 = R_2^2 \quad -R_2 \sin \theta(L, t) = x(L, t) - L \]  
(6.16)

The combination of (6.4) and (6.5) with \( f(s, t) = 0 \) and \( n(s, t) = 0 \), and (6.7), (6.10)-(6.13) can fully describe the free span belt motion with the boundary conditions (6.14)-(6.16) and initial conditions.

For the belt-pulley contact zones, the curvature is a constant \( \kappa(s, t) = 1/R_i \) for \( i = 1, 2 \) in the governing equations (6.4) and (6.5). In the sliding zones, a Coulomb friction law can be employed as

\[ f(s, t) = \begin{cases} -\mu n(s, t), & \text{sliding zone (driver)} \\ \mu n(s, t), & \text{sliding zone (driven)} \end{cases} \]  
(6.17)

To investigate the driven pulley rotation, one applies the conservation of angular momentum to the control volume shown in Figure 6.7, which includes the driven pulley and the contact zone of the belt on the driven pulley, i.e.,

\[ \mathbf{r} \times \mathbf{F} + \mathbf{M} = \frac{\partial}{\partial t} \int_{C.V.} (\mathbf{r} \times \mathbf{v}) \rho d\mathbf{V} + \int_{C.S.} (\mathbf{r} \times \mathbf{v}) \rho \mathbf{v} \cdot d\mathbf{A} \]  
(6.18)

gives

\[ R_2 \left( T_1(L_1, t) - T_2(L_2, t) \right) - M_2(t) \]
\[ = \frac{\partial}{\partial t} \int_{Bel} R_2 G^{DN}(s, t) ds + I_2 \dot{\omega}_2 + R_2 \left[ G_1(L_1, t)V_1(L_1, t) - G_2(L_2, t)V_2(L_2, t) \right] \]  
(6.19)

where \( C.V. \) and \( C.S. \) in (6.18) are on the control volume and the control surface, respectively, and the integral in (6.19) is over the belt length on the contact zone.
Due to the inclusion of belt bending stiffness, the steady state curvature of the free belt spans is not a straight line, and the departure and end points of the free spans vibrate near the equilibria developed in [11]. Seeking an effective way to solve the equations is challenging. For the dynamic analysis, one can first obtain the dynamic equations of small vibrations near the equilibria based on the steady state solutions. Second, to solve the dynamic equations, application of the method of finite difference to discretize the spatial variable $s$ is desirable, and the solution for time variation can be obtained by using Newmark method. On the other hand, it is worth to check if a PDE solver with function PDEPE in MATLAB works for this problem, where systems with certain type of parabolic and elliptic PDEs in one spatial variable $x$ and time $t$ are required. Perturbation analysis, with an example in [77], might be another way to solve the unsteady problem, but full consideration is needed to deal with the influence of the belt bending stiffness and the distribution of the mass flow rate, belt tension and speed.
The dynamic behavior can result the unsteady driving torque $M_1(t)$, driving pulley speed fluctuation $\omega(t)$, or the unsteady load $M_2(t)$. If the excitation source is given as $M_1(t)$, how do the speed fluctuation or the external load impact the system dynamics, and vice versa? How do these factors influence the variation of the belt tension and speed? How is the mass flow rate distributed along the span and how does it vary with time? Are there any groups of parameters that introduce instability? Can this investigation reveal the belt chirp on the pulley during dynamic or steady state operation? The chirp noise reflects a high-frequency response. Are there any other phenomena? All of these investigations will advance the dynamic analysis on the belt-pulley system with belt creep to a broader view.
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