REDUCED ORDER MODELING, NONLINEAR ANALYSIS
AND CONTROL METHODS FOR FLOW CONTROL
PROBLEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

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2007

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Flow control refers to the ability to manipulate fluid flow so as to achieve a desired change in its behavior, which offers many potential technological benefits, such as reducing fuel costs for vehicles and improving effectiveness of industrial processes. An interesting case of flow control is cavity flow control, which has been the motivation of this study: When air flow passes over a shallow cavity a strong resonance is produced by a natural feedback mechanism, scattering acoustic waves that propagate upstream and reach the shear layer, and developing flow structures. These cause many practical problems including damage and fatigue in landing gears and weapons bays in aircrafts.

Presently there is a lack of sufficient mathematical analysis and control design tools for flow control problems. This includes mathematical models that are amenable to control design. Recently reduced-order modeling techniques, such as those based on proper orthogonal decomposition (POD) and Galerkin projection (GP), have come to interest. However, a main issue with these models is that the effect of boundary conditions, which is where the control input is, gets embedded into system coefficients. This results in a form quite different from what one deals with in standard control systems framework, which is a set of ordinary differential equations (ODE) where the input appears as an explicit term. Another issue with the standard POD/GP models is that they do not yield to systems that have any apparent structure in their
coefficients. This leaves one with little choice other than to neglect the nonlinearities of the models and employ standard linear control theory based designs.

The research presented in this thesis makes an effort at closing the gaps mentioned above by 1) presenting a reduced-order modeling method utilizing a novel technique for input separation on POD/GP models, 2) introducing a technique based on averaging theory and center manifold theory so as to reveal certain structures embedded in the model, and 3) developing nonlinear analysis and control design approaches for the resulting model. The theory is complemented by examples and case studies as appropriate, including the case of cavity flow control.
Sevgili aileme…

To my family…
ACKNOWLEDGMENTS

First, I would like to greatly acknowledge my adviser Prof. Serrani for accepting me as his student and having faith in me throughout my graduate studies, and for all the help, support, guidance, motivation and encouragement that he gave me during our work on this new, interesting and difficult topic. It has been a great challenge and at the same time a great pleasure and privilege working with him, as a result of which I believe I have grown both as a scientist and as a person.

I am also grateful to Prof. Mo Samimy for sharing his extensive and valuable knowledge on flow control, for letting us use the cavity flow experimental facility at the OSU Gas Dynamics and Turbulence Laboratory (GDTL), as well as being a member of my committee.

I would like to thank Prof.s Vadim Utkin, Hooshang Hemami, Eylem Ekici and Stuart Raby for agreeing to serve on my dissertation committee, and for their valuable comments and suggestions.

I am grateful to Dr. Chris Camphouse for our collaborative works, for providing us with computer codes for the convective flow over the obstacle problem, and for taking the time to come to Columbus many times, including the day of my defense.

I would like to thank Prof. Önder Efe for his great help regarding input separation concepts including providing computer codes for the sub-domain method, for giving me the opportunity to be part of Türkiye Odalar ve Borsalar Birliği Ekonomi ve
Teknoloji Üniversitesi (TOBB ETÜ) upon completion of my studies, and for sharing lots of insider information on how things work in this business.

I would like to express my thanks to the members of the flow control group at the OSU GDTL including Edgar Carballo, Kihwan Kim, Jesse Little, Marco DeBiasi and Xin Yuan for their help regarding cavity flow experiments and for useful discussions on flow concepts.

I would like to express gratitude to all my friends from the controls group at the OSU Electrical and Computer Engineering (ECE) Department with whom I shared many memories, including Xingping Chen, Zhen Zhang, David Sigthorsson, Lisa Fiorentini, Kutay İçöz, Zengshi Chen, Suat Gümüşsoy, Tankut Acarman, Veysel Gazi, Nicanor Quijano, Alvaro Gil and many others whose names I could not list here.

Special thanks go to my non-departmental friends who have been there for me when I was in need to come up for air from my work, including Mustafa Zeki, Ömer Acar, Koray Tap, Yusuf Danışman, Yuri Dimitrov, Claudio Caraffi and everyone else whose name I could not mention here.

I would also feel that this section would be incomplete if I did not express my gratitude to Prof.s S. Yurkovich, Y. Zheng, C. Klein, S. Ahalt, K. Boyer, Ü. Özguner, D. Orin, H. Özbay, C. Onorato, P. Nevai, Y. Flicker, J. McNeal, M. Linckelmann, Y. Kodama and K. Nakata, who are among the many fine instructors from various departments, from whom I had the pleasure and opportunity of taking courses and learning, during my studies at OSU.

Last but not least, my dearest appreciation and thanks goes to my family, and especially to my parents Zehra and Haluk Kasnakoglu, for their unconditional and
continuous love and support, and for being great role models both personally and academically. I would never have been where I am now had it not been for them and nor could I have ever become a person capable of achieving what I have achieved in life.
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CHAPTER 1

INTRODUCTION

The goal of this study is to establish theoretically sound analysis and design methods for key aspects of flow control problems, namely: reduced-order modeling, nonlinear analysis, and control design.

Fluid flow can be defined as the motion of liquids or gases, which is a phenomenon that one encounters continuously in everyday life. The flow of air around the body of a car or the wing of an aircraft, the motion of petroleum through pipelines, flow of water in oceans and the motion of air in the atmosphere carrying the clouds are only a few examples of fluid flows. Flow control refers to the ability to manipulate fluid flow so as to achieve a desired change in its behavior. Flow control is very important from a technological point of view and offers many potential benefits, such as reducing fuel costs for land, air and sea vehicles, and improving effectiveness of industrial processes.

A particular interesting case of flow control is the so called cavity flow control problem, which has been the motivation for this study within the activities of the flow control group at the OSU-AFRL Collaborative Center of Control Sciences (CCCS). The flow control group of the OSU Gas Dynamics and Turbulence Laboratory (GDTL), directed by Prof. Mo Samimy, is part of a larger multi-disciplinary effort at the
AFRL/AFOSR CCCS to develop tools and methodologies that can enable the use of closed-loop aerodynamic flow control for manipulating the flow over maneuvering air vehicles. In the past few years, the group has made ground-breaking contributions to the advancement of flow control techniques. In particular, the flow control group at OSU-GDTL was the first to successfully implement feedback controllers based on POD/Galerkin models of subsonic cavity flows in actual experiments. Some of the groups contributions including but not limited to those described above can be found in [1–19].

A schematic of cavity flow is shown in Figure 1.1. The physics of the process be described as follows: When air flow passes over a shallow cavity, this flow is characterized by a strong self-sustained resonance produced by a natural feedback mechanism. Shear layer structures impacting the cavity trailing edge scatter acoustic waves that propagate upstream and reach the shear layer receptivity region, where they tune and enhance the development and growth of shear layer structures. There are many situations and real life applications in which this sort of flow happens, such
as the landing-gears and weapon bays in aircrafts, flow in gas transport systems, flow over sunroofs and windows in automobiles, etc. The resulting acoustic fluctuations in cavities can be very intense and are known to cause, among other problems, store damage and airframe structural fatigue in weapons bay applications. It is therefore desirable to develop methods to analyze and understand as well as eliminate these effects.

Initial efforts on flow control, including cavity flow control in particular, were mostly experimentally oriented. As the desire to achieve more sophisticated and systematic designs increased, the need to establish an adequate theory for the task has become more apparent. Presently there is still a lack of sufficient mathematical analysis and control design tools developed for flow control problems in general. This includes mathematical models that are amenable to control design. Significant portions of modeling efforts in the past has been on obtaining very accurate and high order models to be used for simulations of the flow dynamics. These models are impractical if not impossible to use in control design. Recently reduced-order modeling techniques, such as those based on proper orthogonal decomposition (POD)\textsuperscript{1} and Galerkin projection (GP), have come to interest, which do indeed produce low order models. However, a main issue with these models is that the effect of boundary conditions, which is where the control input is, gets embedded into system coefficients. This results in a form quite different from that one is used to dealing with in standard control systems framework, which is a set of ordinary differential equations (ODE), with the property that the input appears as an explicit term on the right hand side.

\textsuperscript{1}POD is also known as Karhunen-Loeve decomposition.
Another issue with the standard POD/GP models is that they do not yield to systems that have any apparent structure in their coefficients. This leaves one with little choice other than to neglect the nonlinearities of the models and employ standard linear control theory based designs, as more sophisticated designs such as those based on nonlinear control theory require the exploitation of such structures in the models.

The research presented here in this thesis makes an effort at closing the gaps mentioned above by 1) presenting a reduced-order modeling method utilizing a novel technique for input separation on POD/GP models, 2) introducing a technique based on averaging theory and center manifold theory so as to reveal certain structures embedded in the model, and 3) developing nonlinear analysis and control design approaches for the resulting model. The theory will be complemented by examples and case studies as appropriate, including the case of cavity flow control. At this point, we would like reemphasize the interdisciplinary nature of the work, which required the expertise of many people involved in the GTDL research group. In particular, this research has profited by the work of Edgar Caraballo, Marco Debiasi and Jesse Little on reduced-order modeling, data acquisition and analysis, and physical interpretation of the results.

1.1 Review of the Literature

Since it is a characteristic of flow control problems that the control input is applied from the physical boundaries of the system, a literature search on boundary control is in order. When flow systems are modeled as partial differential equations, boundary control corresponds to the control input appearing as a boundary condition. There
are countless examples of boundary control applied to many different types of dynamical equations, among which one can find studies that are of particular importance to the field of flow control. Smaoui [20] analyzed the dynamics of the forced Burgers equation subject to both Neumann boundary conditions and periodic boundary conditions using boundary and distributed control. Hinze and Kunisch [21] devised second-order methods for open loop optimal boundary control problems governed by the non-stationary Navier-Stokes system. Nonlinear boundary control of coupled Burgers’ equations was studied by Kobayashi and Oya [22]. Park and Lee [23] studied boundary control of the Navier-Stokes equation by empirical reduction of modes. Formulations and analysis of a sequential quadratic programming method for the optimal Dirichlet boundary control of Navier-Stokes flow were established by Heinkenschloss [24]. Baramov et al. [25] achieved $H_{\infty}$ control of Navier-Stokes equations governing nonperiodic 2D channel flow, significantly reducing unwanted disturbances. Global stabilization of Burgers’ equation by boundary control was studied by Krstic [26], who derived nonlinear boundary control laws that achieve global asymptotic stability of both the viscous and the inviscid Burgers’ equation, using both Neumann and Dirichlet boundary control. Aamo et al. [27] achieved enhancement of mixing by implementing boundary control for Navier-Stokes equations describing 2D channel flow.

Traditionally the approach to oscillation suppression in cavities has been to utilize passive techniques, implemented through geometric modifications, such as using rigid fixed fences, spoilers and ramps [28, 29]. The area of active control is relatively new, and the more common approach to date is using open-loop control, as in [30–37]. Both passive and open-loop techniques are not well suited for environments requiring high
responsiveness, such as dynamic flight. Closed-loop techniques are gaining interest since the technological advances in sensors, actuation and computing has made them more practical than before, and because they have high potential to adapt to variable conditions and consume less power. Earlier approaches to closed-loop control included quasi-static closed loop control, where the control acts on a slower time scale than the flow itself [38, 39]. First examples of dynamic controllers that act on a time scale comparable to that of the flow phenomena to be controlled are manually-tuned PI controllers, e.g. [40–43]. Better models enable systematic designs using modern tools, e.g. [30, 44, 45].

As for the literature on mathematical modeling, various techniques for model development for flow control have been used, some based on flow physics and others based on empirical identifications from experiments. Some examples of the latter include: Mongeau et al. [40], Kook et al. [41] who determined open-loop transfer functions for low-speed flows at several different flow velocities by observing the response to forcing. Rowley and Williams [46] followed a similar approach but with the difference that the frequency-responses from experiments were studied on the stabilized system with a known controller tuned to stabilize the oscillations. A similar approach using a least-squares method was performed by Cattafesta et al. [30] to identify the parameters in a discrete-time transfer function, and similar techniques were used by Cabell et al. [44] to obtain very high order models. Cattafesta et al. [45] used an adaptive algorithm which could be used either off line, or online as part of an adaptive feedback controller. Various adaptive algorithms to tune coefficients in discrete-time were also used by Kegerise et al. [42], Pillarisetti et al. [47]. The work of Rowley et al. [48, 49], Smith et al. [50], Glauser et al. [51, 52], Rempfer
Lehmann et al. [54] could be listed as examples of physically based methods for reduced-order modeling sharing a methodology: One first starts from the infinite dimensional Navier-Stokes (NS) equations, which are derived based on first-principles fluid physics. Then, a finite dimensional subspace spanned by a set of basis functions is determined using proper orthogonal decomposition methods [55]. Finally the NS stokes equations are projected onto the space spanned by the POD basis by using a procedure called Galerkin Projection (GP), which yields a finite dimensional model of differential equations that approximate the original system dynamics in an energy optimal sense. A reduced order system model obtained in this fashion is called a Galerkin model (GM) and the methodologies used to obtain such model from the original NS equations is termed model reduction. A detailed description of the POD-GP method can be found in Holmes et al. [55]. Reduced order models that are derived using balanced truncation techniques are also of interest [56]. In addition to the work listed above, standard references in the fields of boundary control, control of infinite dimensional systems and model reduction control include Antoulas [57], Aamo and Krstic [58], King et al. [59], Bensoussan et al. [60], Curtain and Zwart [61], Burns and King [62].

The main feature of the reduced-order modeling approach presented in this thesis consists in a methodology for input separation, which is a method that renders the presence of the control input explicit in the reduced-order model. The input separation process takes place during model reduction, and aims at separating the control from the boundary conditions and bringing it into the actual differential equations to appear as a separate term. This is the form that one usually sees in dynamical system models in control problems, and to which the tools of control theory can be applied.
This approach is relatively new; initial research efforts on this topic performed by the OSU GDTL Flow Control Group include the work of Efe and Ozbay [63, 64, 65], in which POD/GP Models are obtained for Burgers’ and Heat Equations, where the control input appears as a free boundary condition. In these works, the input is made to appear as an additive term in the differential equation, by means of a technique termed ”sub-domain” separation method. In [1–5, 8, 10, 12, 13] the same methodology has been applied for ROM modeling of cavity flows. Also worth noting is the work by Camphouse [66], where the reduced order models are obtained with input separation performed through weak formulations of proper orthogonal decomposition based system models.

The mathematical models resulting from the reduced order modeling approach considered in this thesis belong to the class of nonlinear dynamical systems called Galerkin models. Galerkin systems frequently arise in reduced-order modeling of infinite-dimensional systems [67–76]. In the area of aerodynamic flow control, significant effort has been put into both developing Galerkin, including feedback control of cylinder wakes [77–81], control of cavity flow [82–85], and optimal control of vortex shedding [86]. The OSU GDTL Flow Control Group has successfully built and utilized reduced order Galerkin models for cavity flow control [1–9]. In these work the Galerkin models have been used for designing controllers based only on linearized models. Therefore, there has been no attempt to analyze the behavior of the Galerkin systems from a nonlinear perspective, and to design controllers that go beyond a linear approach and hence, the goals of this thesis.

The approach for nonlinear analysis and control of Galerkin systems developed in this thesis is based on center manifold theory, which is widely recognized as an
important tool for studying the local behavior of nonlinear systems [87–89], and has been used extensively in nonlinear control theory in the last two decades. In this respect, application of center manifold theory include, among others, nonlinear output regulation [90], stabilization of systems possessing uncontrollable linearization [91, 92], control of bifurcations [93], and the characterization of the long-term behavior of nonlinear systems that are practically stabilized by high-gain feedback [94]. Reduction methods based on center manifold arguments play a fundamental role in the analysis of bifurcations of fixed points and periodic solutions of nonlinear systems [95–97], as well as in the characterization of finite-dimensional invariant manifolds for infinite-dimensional systems [55]. In this thesis center manifold theory is used to further reduce Galerkin models obtained through reduced order modeling and to reveal more structure that can be exploited in nonlinear analysis and control design.

1.2 Organization of this Thesis

The thesis is organized as follows: Chapter 2 describes a reduced order mathematical modeling approach which relies on POD/GP extended by an $\mathcal{L}_2$ optimization based input separation approach. Chapter 3 presents example case studies for the modeling method proposed, including an application to the cavity flow control problem. Chapter 4 describes the nonlinear analysis and control approach developed based on averaging theory and center manifold theory, and illustrates in simulations its application to cavity flow as well. Chapter 5 applies the modeling, analysis and control design techniques presented in earlier chapters to a case study problem on convective flow over an obstacle, governed by Burgers equations. Chapter 6 summarizes the results, and gives future research directions.
CHAPTER 2

REDUCED-ORDER MODELING

2.1 Introduction

In this chapter a mathematical modeling approach for obtaining low order models for the cavity flow problem is presented. The method relies on POD/GP augmented with an input separation technique based on an $\mathcal{L}_2$ optimization problem. The motivation for this chapter and the points of difference from the studies noted in the literature review (Section 1.1) is summarized below.

Although the works on reduced-order modeling and input separation given in Section 1.1 provide means of obtaining low order models for flow dynamics with explicit input terms, there are still issues associated, and room for improvement, mainly regarding the input separation part. First, usually the separation is performed \textit{a posteriori} with respect to the generation of a POD basis. This results in the issue that the model is not reduced to the unforced baseline case when the input is set to zero. Another problem with the available methods is that the POD basis does not capture the effect of the external forcing, but rather the dynamics of the input under which they were built. This limits the ability of the dynamics to represent inputs that are significantly different from the modeling conditions. A final issue
is the underestimation of the control vector field in the reduced-order model. This creates some mismatch in scale between values obtained in simulation for the control and its actual value to be used in a real-time experiment.

The reduced-order modeling approach described in this chapter is based on a control input separation method that addresses these issues by determining an expansion of the flow field in terms of baseline POD modes and actuation modes, where the latter is obtained from actuated snapshots a posteriori to the baseline POD modes. The method has a number of benefits including the fact that it is optimal in the sense that energy neglected by the expansion is minimal, it yields a finite-dimensional system, and it uses the input explicitly as temporal coefficients of the actuation modes. The chapter is organized as follows: Section 2.2 introduces the problem. Section 2.3 provides a brief review of the classical POD/GP. Section 2.4 explains the derivation of the actuation mode. In Section 2.5 the reduced-order model based on the actuated expansion is derived. Section 2.6 reviews alternative separation methods that were developed by the GDTL research group, and compares them to the one proposed.

The material presented in this chapter also appears in Kasnakoglu et al. [98].

2.2 Problem Description

Consider a flow over a planar spatial domain $\Omega$, where $\Omega \subset \mathbb{R}^2$. Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$. Let $u(x, t)$ be the flow velocity, where $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^2$, $u(\cdot, t) \in \mathbb{H}$, $u(x, \cdot) \in C^k$ and $k \in \mathbb{R}_+$. Here $t \in \mathbb{R}_+$ is the temporal variable and $x \in \Omega$ is the spatial variable. Consider a partial differential equation (PDE) of the form

$$\dot{u} = X(u) \quad (2.1)$$
where the operator $X : \mathbb{H} \to \mathbb{H}$ includes spatial derivatives. Equation (2.1) is subject to the initial condition

$$u(x, 0) = u_{\text{init}}$$

(2.2)

where $u_{\text{init}} \in \mathbb{H}$, and subject to the boundary conditions

$$(B_i(u, \gamma))(x, t) = 0, \quad i = 1 \ldots N_b$$

(2.3)

where in the above $x \in \partial \Omega$, $t \in \mathbb{R}_+$ and $B_i : \mathbb{H} \times C^k \to \mathbb{H}$, $N_b \in \mathbb{N}$. The control input $\gamma$ acts through the boundary conditions. The operator $B$ will include spatial derivatives in general. The goal from this point is to find a reduced-order model suitable for feedback control design through $\gamma$. Unfortunately (2.1)-(2.3) is not in a standard form to which the tools of control systems theory can directly be applied.

Our goal in this chapter is to present an approach to obtain a system to approximate (2.1)-(2.3) which is in a from that one is used to dealing with in control system design. This means an ordinary differential equation (ODE), with the property that the input $\gamma$ enters the dynamics directly. More specifically, we will obtain a finite dimensional system of the form

$$\dot{a} = f(a, \gamma)$$

(2.4)

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, $a \in \mathbb{R}^N$, with an initial condition

$$a(0) = a_0$$

(2.5)

where $a_0 \in \mathbb{R}^N$ such that (2.4)-(2.5) approximate the original PDE (2.1)-(2.3) in some optimal sense.
2.3 Classical POD/GP

The approach we propose builds on classical POD/GP [55]. We will provide only a brief review here.

The solution of the PDE in (2.1)-(2.3) will be denoted by \( u(x, t) \), or by bold \( \mathbf{u}(x, t) \) when we feel the need to emphasize that this is a vector. Let \( u_k(x) = u(x, t_k) \) be a snapshot taken at time \( t_k \) and let \( \{u_k\}_{k=1}^M \subset \mathbb{H} \) be an ensemble of \( M \in \mathbb{N} \) snapshots collected at times \( \{t_k\}_{k=1}^M \). Let \( u_0 = E[u_j] \) where \( E \) is a linear averaging operation over the index \( j \). Among all subspaces \( S \subset \mathbb{H} \) of a given dimension \( N < M \), the one that minimizes the averaged error

\[
J(S) = E[\|u_j - P_Su_j\|^2]
\]

where \( \|\cdot\| \) is the norm in \( \mathbb{H} \) and \( P_S \) denotes projection onto \( S \), is given by the subspace spanned by the orthonormal eigenfunctions \( \phi_i \) corresponding to the \( N \) largest nonzero eigenvalues of the linear operator \( R : \mathbb{H} \rightarrow S \) given by the correlation tensor

\[
R = E[u_j \otimes u_j^*]
\]

where \( u_j^* \) is the dual vector of \( u_j \) in \( \mathbb{H} \) (see Holmes et al. [55], Rowley et al. [82]). The vectors \( \phi_i, i = 1, \ldots, N \), are called the POD modes of the ensemble. The problem posed as such is infinite dimensional and could be difficult to handle, however, it can be shown that it suffices to solve the finite, \( M \)-dimensional eigenvalue problem

\[
C\alpha_i = \lambda_i\alpha_i
\]

where \( \alpha_i = (\alpha_{i1}, \ldots, \alpha_{iM}) \), and \( C \in \mathbb{R}^{M \times M} \) is the correlation matrix with entries \( C_{ij} = \langle u_j, u_i \rangle \). This method is known as the method of snapshots [99]. Once the
POD modes $\phi_i$ are obtained as described here, the flow can be represented as a finite dimensional approximation as

$$u(x, t) \approx \sum_{i=1}^{N} a_i(t) \phi_i(x) \, .$$  \hspace{1cm} (2.6)

It is also customary to remove the mean value $u_0$ from the snapshots prior to the calculation on the POD modes. This is also the route that we will follow: instead of $\{u_k\}_{k=1}^M$, one carries out the above procedure for $\{u_k - u_0\}_{k=1}^M$, obtaining an expansion

$$u(x, t) \approx u_0 + \sum_{i=1}^{N} a_i(t) \phi_i(x) \, .$$ \hspace{1cm} (2.7)

From now on we will also use the Einstein notation when convenient, and omit summation signs. It will be understood that the terms with repeating indices are summed over their possible values. Using Einstein notation (2.7) becomes

$$u(x, t) \approx u_0 + a_i(t) \phi_i(x) \, .$$ \hspace{1cm} (2.8)

Now consider the dynamical system (2.1). Let $S = u_0 + \text{span}\{\phi_1, \ldots, \phi_n\}$. A dynamical system

$$\dot{r} = X_{S_i}(r)$$ \hspace{1cm} (2.9)

on $S$ that approximates (2.1) can be obtained by Galerkin projection as

$$X_S(r) = P_S X(r) \, .$$

The Galerkin projection is optimal in the sense that it minimizes $\|X_S(r) - X(r)\|$. Since $X_S(r) - X(r) \perp S$ one can write

$$\langle X_S(r) - X(r), \phi_i \rangle = 0 \, , \quad i = 1 \ldots N. $$ \hspace{1cm} (2.10)
Substituting \( r = u_0 + a_j \phi_j \) into (2.10) above, using (2.9), using orthogonality and then simplifying one gets the set of nonlinear ODEs

\[
\dot{a}_k = \langle X(r), \phi_k \rangle, \quad k = 1 \ldots N. \tag{2.11}
\]

At this stage, the effect of actuation is still buried in the boundary conditions in (2.3), and does not appear explicitly in (2.11). In the next section we will investigate the method proposed to remedy this situation.

### 2.4 Actuation Mode Expansion

#### 2.4.1 Description

Let \( S = \text{span}\{\phi_i\}_{i=1}^N \) be a set of \( N \) POD modes obtained using the POD procedure described in Section 2.3, but from unforced snapshots, i.e. for \( \gamma = 0 \). These modes will be called *baseline POD modes*. Then let \( \{ (u_k, \gamma_k) \}_{k=1}^M \subset \mathbb{R}^2 \times \mathbb{R} \) be an ensemble of actuated flow snapshots and corresponding control signals, \( u_k = u(x, t_k), \gamma_k = \gamma(t_k) \).

As noted above, \( u_0 \), the mean value of the baseline case, is assumed to already have been removed from the \( u_k \) values. Define the *innovation* as

\[
\tilde{u}_k = u_k - P_S u_k.
\]

An optimization problem on the Hilbert space \( \mathbb{H} \) can be defined as finding

\[
\psi^* = \arg \min_{\psi \in \mathbb{H}} E\{\|\tilde{u}_k - \gamma_k \psi\|^2\}. \tag{2.12}
\]

The element \( \psi^* \in \mathbb{H} \) will be called the *actuation mode*. The squared norm of the velocity represents the energy contained in the flow. Therefore, among all augmented POD expansions in the form

\[
u = u_0 + \sum_{i=1}^N a_i \phi_i + \gamma \psi \tag{2.13}\]

15
where the input $\gamma$ directly appears as the coefficient of $\psi$; the choice $\psi = \psi^*$ is optimal, in the sense that the energy not captured by this expansion achieves its minimum for $\psi = \psi^*$. The theorem below summarizes the main result of this section.

**Theorem 2.4.1.** Let $J(\psi) := E \left[ \| \tilde{u}_k - \gamma_k \psi \|^2 \right]$. Then:

1. The minimum value of the function $J$ is achieved at
   \[ \psi^* = \frac{E [\gamma_k \tilde{u}_k]}{E[(\gamma_k)^2]} . \]

2. $\psi^* \in \mathbb{H}$.

3. $\psi^* \perp \phi_i$ for $i = 1, \ldots, N$.

**Proof.** 1. Note that
   \[
   J(\psi) = E \left[ \| \tilde{u}_k - \gamma_k \psi \|^2 \right] \\
   = E \left[ \| \tilde{u}_k \|^2 - 2 \gamma_k \langle \tilde{u}_k, \psi \rangle + (\gamma_k)^2 \| \psi \|^2 \right].
   \]

Since $J$ is quadratic in $\psi$ with positive leading coefficient $E [\gamma_k^2]$, it has a unique minimum. Computing the first variation of $J$ with respect to $\xi \in \mathbb{H}$

\[
\frac{d}{d\delta} \bigg|_{\delta=0} J(\psi + \delta \xi) = \frac{d}{d\delta} \bigg|_{\delta=0} E \left[ \| \tilde{u}_k \|^2 - 2 \gamma_k \langle \tilde{u}_k, \psi + \delta \xi \rangle + (\gamma_k)^2 \| \psi + \delta \xi \|^2 \right] \\
= E \left[ -2 \gamma_k \langle \tilde{u}_k, \xi \rangle + \gamma_k^2 \langle \psi + \delta \xi, \xi \rangle + \gamma_k^2 \langle \xi, \psi + \delta \xi \rangle \right] \bigg|_{\delta=0} \\
= E \left[ -2 \gamma_k \langle \tilde{u}_k, \xi \rangle + (\gamma_k)^2 \langle \psi, \xi \rangle + \gamma_k^2 \langle \xi, \psi \rangle \right] \\
= E \left[ -2 \gamma_k \langle \tilde{u}_k, \xi \rangle + 2(\gamma_k)^2 \langle \psi, \xi \rangle \right] \\
= E \left( -2 \gamma_k \tilde{u}_k + 2(\gamma_k)^2 \psi, \xi \right).
\]
For \( \psi \) to be an extremum of \( J \) its first variation must vanish \( \forall \xi \in \mathbb{H} \). Therefore

\[
E \left[ -2\gamma_k \tilde{u}_k + 2\gamma_k^2 \psi^* \right] = 0
\]

and thus, by linearity of \( E \)

\[
\psi^* = \frac{E [\gamma_k \tilde{u}_k]}{E [\gamma_k^2]} .
\]  

(2.14)

2. The fact that \( \psi^* \in \mathbb{H} \) follows from the fact that \( E \) is linear, \( \gamma_k \tilde{u}_k \in \mathbb{H} \) and \( E [(\gamma_k)^2] \in \mathbb{R} \).

3. To show that \( \psi^* \perp \phi_i \) for \( i = 1 \ldots N \), first note that \( \tilde{u}_k \perp S \) for all \( k = 1 \ldots N \).

But for any choice of \( i \) and \( k \)

\[
\langle \tilde{u}_k, \phi_i \rangle = \langle u - P_S u, \phi_i \rangle = \langle u - \sum_{j=1}^n \langle u, \phi_j \rangle \phi_j, \phi_i \rangle = \langle u, \phi_i \rangle - \sum_{j=1}^n \langle u, \phi_j \rangle \langle \phi_j, \phi_i \rangle = \langle u, \phi_i \rangle - \langle u, \phi_i \rangle = 0 .
\]

Then, for any \( i \), using the above result, the linearity of \( E \) and the linearity of the inner product

\[
\langle \psi^*, \phi_i \rangle = \langle E [\gamma_k \tilde{u}_k], \phi_i \rangle = \frac{E [\gamma_k \langle \tilde{u}_k, \phi_i \rangle]}{E [\gamma_k]} = \frac{E [0]}{E [\gamma_k]} = 0 .
\]

Therefore it follows from above that \( \psi^* \perp S \), i.e. \( \psi^* \perp \phi_i \) for \( i = 1 \ldots N \). 

\[\square\]
2.4.2 Extension to Multiple Modes

It is possible to extend the above procedure to obtain additional input modes for additional inputs, for derivatives of the input, etc. For instance, assume the derivative of the input $\dot{\gamma}(t)$ is available. Let us be given an ensemble of actuated flows $\{u_k\}_1^m \subset \mathbb{R}^2$ and an ensemble of actuation values $\{\gamma_k\}_1^m \subset \mathbb{R}$ and their derivatives $\{\dot{\gamma}^k\}_1^m \subset \mathbb{R}$. Again, as in the previous section we assume that the baseline mean value $u_0$ is already removed from $\{u_k\}_1^m$. Then one obtains the actuation mode $\psi_1$ for the input itself $\gamma$ using the same procedure in the previous section, i.e.

$$\psi_1 = \arg \min_{\psi \in H} E(\|\tilde{u}_k - \gamma_k \psi\|) .$$

Now let us define

$$\tilde{u} := u_k - P_{S_1} u_k$$

where $S_1 := \text{span}\{\phi_1, \ldots, \phi_n, \psi_1\}$. Note that in general $\|\psi_1\| \neq 1$ hence one needs to use the projection operator of the form

$$P_{\psi_1} u_k = \|\psi_1\|^{-1} \langle u_k, \psi_1 \rangle \psi_1 .$$

One can now use the same procedure as described in Section 2.4 with $\tilde{u}$ replacing $u$ to obtain an actuation mode for $\dot{\gamma}$ as

$$\psi_2 = \arg \min_{\psi \in H} E(\|\tilde{u}_k - \dot{\gamma}^k \psi\|)$$

with $\psi_2 \perp S_1$. This yields an actuated POD expansion of the form

$$u = u_0 + \sum_{i=1}^N a_i \phi_i + \gamma \psi_1 + \dot{\gamma} \psi_2 .$$

It possible to extend the procedure as many times as one has the derivatives of the input available. If one can obtain, say $N_{m\text{ in}}$ derivatives of the input (e.g. through
dynamic extension), then it is possible to obtain an expansion of the form

\[ u = u_0 + \sum_{i=1}^{N} a_i \phi_i + \sum_{i=1}^{N_{in}} \gamma^{(i-1)} \psi_i . \]

A derivation almost identical to the above can be made if one does not have extra derivatives but instead additional control inputs. If one has, say \( N_{in} \) inputs, then it is possible to obtain an expansion of the form

\[ u = u_0 + \sum_{i=1}^{N} a_i \phi_i + \sum_{i=1}^{N_{in}} \gamma_i \psi_i . \]

### 2.5 Obtaining the Reduced-order Model

Consider the dynamical system (2.1). Let \( V_1 = u_0 + \text{span}\{\phi_1, \ldots, \phi_n, \psi_1\} \). A dynamical system

\[ \dot{r} = X_{V_1}(r) \quad (2.15) \]
on \( V_1 \) that approximates (2.1) can be obtained by Galerkin projection as

\[ X_{V_1}(r) = P_{V_1}X(r) . \]

The Galerkin projection is optimal in the sense that it minimizes \( \|X_{V_1}(r) - X(r)\| \).

Since \( X_{V_1}(r) - X(r) \perp V_1 \) one can write

\[ \langle X_{V_1}(r) - X(r), \phi_i \rangle = 0, \quad i = 1 \ldots N \quad (2.16) \]

Substituting \( r = u_0 + a_j \phi_j + \gamma \psi \) into (2.16)

\[ \langle \dot{a}_j \phi_j + \dot{\gamma} \psi - X(r), \phi_k \rangle = 0 \]
\[ \dot{a}_j \langle \phi_j, \phi_k \rangle + \dot{\gamma} \langle \psi, \phi_k \rangle - \langle X(r), \phi_k \rangle = 0 \]
\[ \dot{a}_k \langle \phi_k, \phi_k \rangle - \langle X(r), \phi_k \rangle = 0 \]

and hence

\[ \dot{a}_k = \langle X(r), \phi_k \rangle . \quad (2.17) \]
2.5.1 Special Case: Quadratic Dynamics

Quadratic dynamics

\[ X(u) := L(u) + Q(u, u) \]  \hspace{1cm} (2.18)

where \( L(u) \) is a linear term and \( Q(u, u) \) is a quadratic term in \( u \), is of interest since many flow control problems, ranging from simple ones, such as heat equation, to more complicated ones, such as Burger’s and Navier-Stokes equations, can be represented in this form. Following the procedure outlined in the previous section, let us derive the reduced-order system for (2.18). This derivation will also be useful for the examples in the following section. Substituting \( X \) from (2.18) into (2.17) gives

\[
\dot{a}_k = \langle X(r), \phi_k \rangle \\
= \langle L(u_0 + a_i \phi_i + \gamma \psi), Q(u_0 + a_i \phi_i + \gamma \psi, u_0 + a_j \phi_j + \gamma \psi) \rangle \\
= \langle L(u_0), \phi_k \rangle + \langle L(\phi_i), \phi_k \rangle a_i + \langle L(\psi), \phi_k \rangle \gamma + \langle Q(u_0, u_0), \phi_k \rangle \\
+ \langle Q(u_0, \phi_j), \phi_k \rangle a_j + \langle Q(u_0, \psi), \phi_k \rangle \gamma + \langle Q(\phi_i, u_0), \phi_k \rangle a_i + \langle Q(\phi_i, \phi_j), \phi_k \rangle a_i a_j \\
+ \langle Q(\phi_i, \psi), \phi_k \rangle a_i \gamma + \langle Q(\psi, u_0), \phi_k \rangle \gamma + \langle Q(\psi, \phi_i), \phi_k \rangle a_i \gamma + \langle Q(\psi, \psi), \phi_k \rangle \gamma^2 \\
= F_k + G_{1ik} a_i + g_{2k} \gamma + H_{1ijk} a_i a_j + H_{2ik} a_i \gamma + h_{3k} \gamma^2 \]  \hspace{1cm} (2.19)

where

\[
F_k = \langle L(u_0), \phi_k \rangle + \langle Q(u_0, u_0), \phi_k \rangle \\
G_{1ik} = \langle L(\phi_i), \phi_k \rangle + \langle Q(u_0, \phi_j), \phi_k \rangle + \langle Q(\phi_i, u_0), \phi_k \rangle \\
g_{2k} = \langle L(\psi), \phi_k \rangle + \langle Q(u_0, \psi), \phi_k \rangle + \langle Q(\psi, u_0), \phi_k \rangle \\
H_{1ijk} = \langle Q(\phi_i, \phi_j), \phi_k \rangle \\
H_{3ik} = \langle Q(\phi_i, \psi), \phi_k \rangle + \langle Q(\psi, \phi_i), \phi_k \rangle \\
h_{3k} = \langle Q(\psi, \psi), \phi_k \rangle .
\]
System (2.19) can be represented in compact form as

\[
\dot{a} = F + G_1a + g_2\gamma + H_1(a, a) + H_2(a, \gamma) + h_3\gamma^2
\]  

Extension to multiple modes: The same procedure above can be repeated for the case of more than one modes. For instance if there is a second input mode $\psi_2$ corresponding to the derivative of the input $\dot{\gamma}$, then following the derivation steps above, one obtains a reduced system of the form

\[
\dot{a} = F + G_1a + g_2\gamma + g_3\dot{\gamma} + H_1(a, a) + H_2(a, \gamma) + h_3\gamma^2 + H_4(a, \dot{\gamma}) + h_5\dot{\gamma}^2 + h_6\dot{\gamma} \cdot
\]  

Similarly a Galerkin system corresponding to an expansion with multiple control inputs will be of the form

\[
\dot{a}_k = F_k + G_{1ik}a_i + G_{2ik}\gamma_i + H_{1ijk}a_i a_j + H_{2ijk}a_i \gamma_i + H_{3ijk}\gamma_i \gamma_j .
\]

2.6 Alternative Methods for Input Separation

In this section a brief summary of two other important input separation techniques that were developed by the OSU GDTL flow control group will be given. One is the sub-domain separation method [2, 63, 100] and the other is the actuated POD expansion/stochasctic estimation method [101]. From this point on, we will also use the OSU GDTL notation established by Edgar Caraballo and refer to the sub-domain method as M0, the actuated POD/stochastic estimation method as M1, and the $L_2$ optimization as M2.

2.6.1 Sub-domain Separation Method

The idea behind the sub-domain method is to divide the entire flow domain $\Omega$ into two sub-regions, so that $\Omega = \Omega_1 \cup \Omega_2$. The smaller domain $\Omega_1$ comprises the
physical region where the actuation enters the flow field. Separation is performed at
the level of the Galerkin projection by splitting the inner product as $< \cdot, \cdot >_\Omega =< \cdot, \cdot >_{\Omega_1} + < \cdot, \cdot >_{\Omega_2}$. The boundary conditions are imposed on $\Omega_1$. This procedure
yields a non-autonomous set of ODEs in the following form

$$\dot{a}_k = F_k + G_{1ik}a_i + g_{2k}\gamma + H_{1ijk}a_ia_j + H_{2ik}a_i\gamma$$

where $\gamma$ is the input signal (for instance, the voltage applied to a synthetic jet-like
actuator.) Further details on this method and its application to the cavity flow control
problem can be found in [2, 63, 100]. This was the method used by the OSU GDTL
flow control group for the development of reduced-order model-based controllers in the
past. It was later observed that this method suffers from issues such as mismatch with
the baseline model, a need for an identifiable control region and the underestimation
of the effect of control. These undesirable effects lead to development and application
of other separation methods, one being the $L_2$ separation method already described,
and another one being the actuated POD/stochastic estimation method of Caraballo
et al. [101] to be briefly summarized next.

### 2.6.2 Actuated POD Expansion/Stochastic Estimation

This method developed by Caraballo et al. [101] is closely related to the $L_2$ method;
as a matter of fact, the baseline POD modes $\phi_i$ and innovations $\tilde{u}_i$ are built in exactly
the same way. One then seeks to find an expansion of the flow field in the form

$$u(x, t) \approx u_0(x) + \sum_{i=1}^{N} \phi_i(x)a_i(t) + \sum_{i=1}^{N_{\text{ac}}} \psi_i(x)a_i^{\text{ac}}(t) \quad (2.22)$$

where $\psi_i$ are actuation modes which are determined using the POD procedure de-
scribed in Section 2.3, but this time applied to snapshots of the innovation $\tilde{u}_i$. The
next step is the correlation of the forcing input $\gamma$ with the actuation mode coefficients $a_{i}^{ac}$ in (2.22) using a second order stochastic estimation

$$a_{i}^{ac} = M_{i}\gamma + O_{i}\gamma^{2}$$  \hfill (2.23)

where $M$ and $O$ are the estimation vectors for the linear and quadratic portion correspondingly. Substituting (2.23) into (2.22) and then carrying out GP yields Galerkin model of the form

$$\dot{a}_{k} = F_{k} + G_{1ik}a_{i} + g_{2k}\gamma + H_{1ijk}a_{i}a_{j} + H_{2ik}a_{i}\gamma$$

$$+ (h_{3k} + g_{3k})\gamma^{2} + H_{3ijk}a_{i}\gamma^{2} + h_{4k}\gamma^{4} + h_{5k}\gamma^{3}.$$ \hfill (2.24)

As stated initially, this method has similarities in design to $\mathcal{L}_{2}$ method and the results obtained from the two in open loop validation and closed loop control experiments are usually close to one another. $\mathcal{L}_{2}$ method can said to have an advantage in the sense that it is more compact and simpler to implement, but the actuated POD/stochastic method has the advantage of having more flexibility through additional actuation modes and the stochastic estimation step. However, both methods have clear advantages over the sub-domain method, as summarized in Table 2.1. Table entries in red indicate undesirable properties.

### 2.7 Summary and Comments

In this chapter, an input separation method for the boundary control of flow problems was described. First a POD expansion of the unforced baseline case was given. This was then augmented with an optimally chosen actuation mode, which
Table 2.1: Comparison of Methods: Sub-domain separation (M0), actuated POD with stochastic estimation (M1), and $L_2$ optimization (M2)

<table>
<thead>
<tr>
<th>Criteria</th>
<th>M0</th>
<th>M1</th>
<th>M2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Provides the input $\gamma$ as a separate term</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Requires identification of a control region</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Consistent with baseline flow for $\gamma = 0$</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Correctly estimates magnitudes of control terms</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

minimizes the energy not captured by the augmented set of modes. Then a reduced-order Galerkin model was derived for this expansion, in which the control input enters the dynamics directly.

Recall that the main goal is to obtain a reduced-order model in which the input enters the dynamics explicitly. The first step is to obtain a POD expansion for the baseline, i.e., no input case. It is known that the POD modes are optimal in the sense that the energy left out of the subspace spanned by the modes is minimized on average. The sense of optimality described in the previous sentence however, does not enforce the boundary conditions per se, i.e., the approximation at the boundaries will be as good as it is elsewhere. One might be surprised by this since the input enters through the boundary. However, recall that the baseline POD modes are computed under no forcing. The effect of the input is added to the baseline modes, via an actuation mode. The optimality condition for the actuation mode, requires that the input appear as a coefficient for the mode. This is different from the baseline POD modes, since in that case the coefficients are computed through projections. In the case of the actuation mode, the input itself is directly the coefficient. This assures that the contribution of the actuation term is zero under no forcing, thus the augmented POD expansion reduces exactly to the baseline case. Also note that any point on the
boundary directly dictated by the input will force all the baseline modes to vanish at that point and the actuation mode to be unitary. Hence the augmented expansion at that point will exactly equal the value of the input.
CHAPTER 3

CASE STUDIES ON REDUCED-ORDER MODELING

3.1 Introduction

In this chapter a number of illustrative case studies are presented in which the modeling approach of Chapter 2 is applied to flow problems. The cases considered include a 2D heat equation in Section 3.2 where the goal is to start with a simple example to illustrate the ideas, and also to present a comparison with another input separation method, namely the subdomain separation method, that had been used in earlier CCCS work, as mentioned in Section 2.6. Next case considered in the 2D incompressible Navier Stokes equations (Section 3.3), studied on an abstract boundary control problem, with different types of boundary conditions so as to demonstrate the capability of the models built in capturing the behavior of complicated dynamics and boundary conditions, as well as their usefulness for feedback control design. The focus of this chapter however, is still on modeling and therefore the control designs are kept simple and minimal, which are in fact linear quadratic regulator (LQR) designs. A more sophisticated and nonlinear approach to analysis and control will be considered in Chapter 4. As a final example, a study on cavity flow control using the GDTL cavity flow experimental facility is presented in Section 3.4. This study has been
made possible thanks to the generous work of all the members of the flow control group at GDTL, in particular of Edgar Caraballo, who provided the tools for the construction of the reduced-order models from experimental data, and contributed to the analysis of the results. In this example, the reduced-order model is tested in its ability to reconstruct flow fields and also closed loop performance in peak reduction. A comparison with the alternative actuated POD expansion/stochastic estimation method is discussed as well.

### 3.2 Example 1: 2D Heat Equation and Comparison with Subdomain Method

Let us first consider a simple illustrative example to present a comparison between the proposed modeling method and a previously used subdomain separation method. The comparison will be done in open loop. The main reason for this is the fact that the differences are more apparent. The concept of feedback by itself introduces some level of robustness (even if one does not do robust control explicitly). Feedback can take care of a certain degree of model mismatch, whereas the open loop has no such forgiveness. Hence it provides a stricter test for comparison.

The method that we will use for comparison is the sub-domain separation method described in Efe and Ozbay [63, 64, 65]. We will compare the two approaches on a simple problem, the 2D heat equation in $u(x, y, t)$, i.e.

$$\frac{\partial u}{\partial t} = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$  \hspace{1cm} (3.1)

on a square domain $\Omega = [0, 1] \times [0, 1]$, with $\mu = 4$, starting from the initial condition

$$u(x, y, 0) = 16x(x - 1)y(y - 1)$$  \hspace{1cm} (3.2)
and subject to boundary conditions

\[
\begin{align*}
    u(0,0,t) &= \gamma(t), \quad u(1,y,t) = 0, \quad u(x,1,t) = 0, \\
    \frac{\partial u}{\partial t}(0,y,t) &= \mu \frac{\partial^2 u}{\partial y^2}(0,y,t), \quad \frac{\partial u}{\partial t}(x,0,t) = \mu \frac{\partial^2 u}{\partial x^2}(x,0,t).
\end{align*}
\] (3.3)

The reason for the choice for these boundary conditions is that they are what were used in the sub-domain separation work [63–65].

### 3.2.1 Reduced-order Model for Proposed Method

As described in Section 2.4, we first obtained an actuated POD expansion of the form (2.13). We took \( N = 5 \) in (2.13), which can be shown to preserve more than 99\% of the energy. Snapshots were taken from a MATLAB simulation with the following parameters: spatial grid size is \( 25 \times 25 \), time between adjacent snapshots is 1 millisecond, total number of snapshots is 1001, total simulation time is 1 second. A chirp signal actuation of the form \( \gamma(t) = \sin(2\pi 10t^3) \) was used to collect actuation snapshots from which the input mode \( \psi \) was built. The reduced-order model was then obtained as described in Section 2.5. Note that (3.1) is in the same form \( \dot{u} = X(u) \) used in Section 2.5, with the operator \( X \) being

\[
X = \mu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]

Hence, following the steps described there yields

\[
\frac{d}{dt} a_i(t) = C + L a_i(t) + L_m \gamma(t)
\] (3.4)
where

\[
C_i = \left\langle \mu \left( \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial^2}{\partial y^2} u_0 \right), \phi_i \right\rangle
\]

\[
L_{ij} = \left\langle \mu \left( \frac{\partial^2}{\partial x^2} \phi_i + \frac{\partial^2}{\partial y^2} \phi_i \right), \phi_j \right\rangle
\]

\[
L_{in,i} = \left\langle \mu \left( \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial y^2} \psi \right), \phi_i \right\rangle.
\]

### 3.2.2 Reduced-order Model for Sub-domain Method

As mentioned in Section 2.6 sub-domain method is another approach adopted for model reduction that was used in earlier GDTL works [63–65]. Below is a brief summary of how the reduced-order model is obtained for the 2D heat equation, provided courtesy of Dr. Onder Efe, who is the developer of the sub-domain method.

Consider the 2D heat equation given by (3.1) and assume the sole excitation is of Dirichlet type and at \( x = 0 \) and \( y = 0 \) corner. Let

\[
u(x, y, t) = \sum_{i=1}^{N} \phi_i(x, y) a_i(t)
\]

be the solution to the given PDE. Naturally, the prescribed solution given above is also valid at the boundaries and this leads to the following observation

\[
\sum_{i=1}^{N} \dot{a}_i(t) \phi_i(x, y) = \mu \sum_{i=1}^{N} a_i(t) \zeta_i(x, y)
\]

where \( \zeta_i(x, y) = \frac{\partial^2 \phi_i}{\partial x^2}(x, y) + \frac{\partial^2 \phi_i}{\partial y^2}(x, y) \). Clearly taking the inner product of both sides with \( \phi_k(x, y) \) yields

\[
\dot{a}_k = \mu \sum_{i=1}^{N} a_i(t) \left\langle \phi_k(x, y), \zeta_i(x, y) \right\rangle,
\]

where \( k = 1, 2, \ldots, N \). The representation above contains the effect of the boundary excitations implicitly. To overcome this problem, a subdomain \( \Omega_2 \) in which the
boundary excitation directly affects the flow is identified. Then the domain is partitioned as $\Omega = \Omega_2 \cup \Omega \setminus \Omega_2$. and the inner product is computed separately over the two domains as

$$\langle \phi_k(x,y), \zeta_i(x,y) \rangle_{\Omega} = \langle \phi_k(x,y), \zeta_i(x,y) \rangle_{\Omega_2} + \langle \phi_k(x,y), \zeta_i(x,y) \rangle_{\Omega \setminus \Omega_2}.$$  \tag{3.8}

This basically corresponds to the repartitioning of the domain by changing the limits of a Riemann integral computed over a non-overlapping subdomains embodying the domain of the original integral when they are united. For the 2D heat equation, this approach yields a model that can be computed as follows;

$$\frac{d}{dt} a_k(t) = A a(t) + B \gamma(t),$$  \tag{3.9}

where $a(t) = (a_1(t), a_2(t), \ldots, a_N(t))^T$ and the matrices $A$ and $B$ are computed as below

$$A_{ki} = \frac{\mu}{M} \left( \langle \phi_k(x,y), \zeta_i(x,y) \rangle_{\Omega} - \langle \phi_i(x,y), \zeta_k(x,y) \rangle_{\Omega_2} \right)$$  \tag{3.10}

$$B_k = \frac{\mu}{M} \zeta_k(x,y)|_{(x,y)\in\Omega_2}$$  \tag{3.11}

We have given only a brief overview here. The reader interested in the technical details and derivations is referred to Efe and Ozbay [63, 64, 65].

3.2.3 Simulation Results

In the previous subsections 3.2.1 and 3.2.2, derivations of reduced-order models using the proposed method and the sub-domain method were obtained. In this section we will show MATLAB simulation results and present a comparison.

The snapshots obtained from the unforced flow is shown in Figure 3.1. Figure 3.3 shows the baseline POD modes. Figure 3.4 shows the actuation mode. Figure 3.2 shows the system under a ramp actuation from $-2$ to $2$. We will test and compare

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the two reduced-order models in their capability of reproducing this flow. Let us first present a comparison of POD coefficients obtained directly from the snapshots, i.e. $a_i(t_k) = \langle u_k, \phi_i \rangle$ to those obtained from the Galerkin system, denoted $a_G$. For the proposed method Figure 3.5(a) shows the norm of error $\|a - a_G\|$ for the baseline case and Figure 3.5(b) shows the error for ramp excitation. The value of the error is reasonably small, which suggests good agreement between the $a$ and $a_G$. For the sub-domain method Figure 3.7(a) shows the norm of error $\|a - a_G\|$ for the baseline case and Figure 3.7(b) shows the error for ramp excitation. It is clear that the error in both cases is much larger compared to the proposed method, but the difference for the baseline case is much more pronounced. Let us now compare flows reconstructed using these two methods. This is done by using (2.13) for the proposed method and (3.5) for the sub-domain method. The coefficients $a_i$ in the expansions are computed from the corresponding reduced-order systems, i.e. from (3.4) for the proposed method and from (3.9) for the sub-domain method. The reconstruction error using the sub-domain method is shown in Figure 3.8. The reconstruction error using the proposed method is shown in Figure 3.9. It can be seen that the proposed method does a good job in reconstructing the flow during both initial and later time values. It can also be seen that the reconstruction becomes almost identical to the original flow after about 0.091 seconds. The reconstruction before this time, while not identical to the original, is still satisfactory in most of the domain. It is notably superior to the results from the sub-domain method, shown in Figure 3.8. In Figure 3.8, one can see that the reconstruction error for the sub-domain method is quite high during initial times, and takes about 0.364 seconds for the error to reach a small value.
Figure 3.1: Unforced Flow

Figure 3.2: Actuated Flow
Figure 3.3: Baseline Modes

Figure 3.4: Actuation Mode
Figure 3.5: Error between actual time coefficients and those from the Galerkin model for proposed method

Figure 3.6: POD modes for Sub-domain method
Figure 3.7: Error between actual time coefficients and those from the Galerkin model for sub-domain method

Figure 3.8: Error in reconstructed flow for sub-domain method
Figure 3.9: Error in reconstructed flow for proposed method
3.3 Example 2: Boundary Control of 2D Incompressible Navier-Stokes Equation

To illustrate the usefulness and effectiveness of the input separation technique proposed for control design, let us consider the boundary control of the incompressible Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p + \nu \Delta \mathbf{u}. \quad (3.12)$$

subject to the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. The Navier-Stokes equations are a key equations for fluid flows. We will consider the 2D version of (3.12). Let $\mathbf{u}(x, y, t) = (u(x, y, t), v(u, x, t)) \in \mathbb{R}^2$ be the flow variable where $u$ and $v$ are the components in the longitudinal and latitudinal directions. The system we will consider is

$$\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v &= \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v &= \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (3.13)
\end{align*}$$

subject to the initial condition

$$u(x, y, t) = v(x, y, t) = 0$$

and boundary conditions

$$\begin{align*}
u(x, 0, t) &= 1, \quad v(x, 0, t) = 0 \\
u(x, 1, t) &= 1, \quad v(x, 1, t) = 0 \\
u(0, y, t) &= 0, \quad \frac{\partial v}{\partial x}(0, y, t) = 0 \\
u(1, y, t) &= \begin{cases} 0, & y \in [0, 0.42); \\
\gamma(t), & y \in [0.42, 0.58]; \\
0, & y \in (0.58, 1]. \end{cases} \\
v(1, y, t) &= 0
\end{align*}$$

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where $\gamma$ is the control input. Under no forcing the system acts as shown in Figures 3.10 and 3.11 and goes to some steady state. As described in Section 2.4, first the baseline POD expansion is obtained using these snapshots. We will take the number of baseline modes to be $N = 3$, which can be shown to preserve about 99% of the energy. Figures 3.12 and 3.13 show these baseline POD modes. In addition, an actuation mode $\psi$ needs to be obtained using actuated snapshots of the system. Figures 3.14 and 3.15 show the system under an oscillatory excitation of increasing frequency. Figure 3.16 shows the actuation mode computed from these snapshots. The input mode considered here was obtained from a chirp like excitation, however, we also tried obtaining it from different kinds of excitations such as a square, sinusoidal and ramp input, which gave similar results to that in Figure 3.16. Following the procedure outline in Section 2.5, one obtains the Galerking system for the 2D Navier-Stokes equation (3.13) as

$$\dot{a} = C + La + L_{in}\gamma + Q(a,a) + Q_{in}\gamma^2 + Q_{ain}a\gamma$$

where

$$C_i = \left\langle \begin{bmatrix} -u_0 \frac{\partial}{\partial x} u_0 + \nu \left( \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial^2}{\partial y^2} u_0 \right) - v_0 \frac{\partial}{\partial y} u_0 \\ -u_0 \frac{\partial}{\partial x} v_0 + \nu \left( \frac{\partial^2}{\partial x^2} v_0 + \frac{\partial^2}{\partial y^2} v_0 \right) - v_0 \frac{\partial}{\partial y} v_0 \end{bmatrix}, \phi_i \right\rangle$$

$$L_{ij} = \left\langle \begin{bmatrix} -(\phi_{u,i}) \frac{\partial}{\partial x} u_0 + \nu \left( \frac{\partial^2}{\partial x^2} (\phi_{u,i}) + \frac{\partial^2}{\partial y^2} (\phi_{u,i}) \right) -(\phi_{v,i}) \frac{\partial}{\partial y} u_0 \\ -(\phi_{u,i}) \frac{\partial}{\partial x} v_0 + \nu \left( \frac{\partial^2}{\partial x^2} (\phi_{v,i}) + \frac{\partial^2}{\partial y^2} (\phi_{v,i}) \right) -(\phi_{v,i}) \frac{\partial}{\partial y} v_0 \end{bmatrix}, \phi_i \right\rangle$$

$$L_{in,i} = \left\langle \begin{bmatrix} -u_0 \frac{\partial}{\partial x} \psi_u - \psi_v \frac{\partial}{\partial y} u_0 + \nu \left( \frac{\partial^2}{\partial x^2} \psi_u + \frac{\partial^2}{\partial y^2} \psi_u \right) - \psi_u \frac{\partial}{\partial x} u_0 - v_0 \frac{\partial}{\partial y} \psi_u \\ -u_0 \frac{\partial}{\partial x} \psi_v - \psi_v \frac{\partial}{\partial y} v_0 + \nu \left( \frac{\partial^2}{\partial x^2} \psi_v + \frac{\partial^2}{\partial y^2} \psi_v \right) - \psi_u \frac{\partial}{\partial x} v_0 - v_0 \frac{\partial}{\partial y} \psi_v \end{bmatrix}, \phi_i \right\rangle$$
\[ Q_{ijk} = \left\langle \left[ - (\phi_{u,i}) \frac{\partial}{\partial x} (\phi_{u,j}) - (\phi_{v,i}) \frac{\partial}{\partial y} (\phi_{u,j}) \right], \phi_k \right\rangle \]

\[ Q_{in,j} = \left\langle \left[ - \psi_{v,i} \frac{\partial \psi_u}{\partial y} - \psi_{u,i} \frac{\partial \psi_u}{\partial x} \right], \phi_i \right\rangle \]

\[ Q_{ain,ij} = \left\langle \left[ - (\phi_{v,i}) \frac{\partial}{\partial y} \psi_v - (\phi_{u,i}) \frac{\partial}{\partial x} \psi_u - (\phi_{u,i}) \frac{\partial}{\partial x} \psi_v - (\phi_{v,i}) \frac{\partial}{\partial y} \psi_v \right], \phi_i \right\rangle \]  (3.14)

**Remark 3.3.1.** We have neglected the pressure term, which is standard practice when obtaining Galerkin models for Navier-Stokes equations. This is either justified theoretically from the boundary conditions or numerically from computational fluid dynamics (CFD) simulations. We have done the latter by confirming that the effect of the projected pressure terms are small compared to others. If it is found that the pressure term is not negligible, there are ways to include it without changing the model structure [77].

The unforced and actuated responses of the Galerkin system as compared to the actual values of the time coefficients obtained from projecting the snapshots onto the basis can be seen in Figures 3.17(a) and 3.17(b). For this problem, the control objective selected will be to drive the trajectories of the systems to a desired 2D profile given by \( \mathbf{u}_d(x, y) = (u_d(x, y), v_d(x, y)) \). Because the trajectories generated by the Galerkin system always lie in \( S = \text{span}\{\phi_i, \psi\} \), the objective is only feasible if \( \mathbf{u}_d \in S \). Keeping this in mind we select a desired profile as shown in Figure 3.18. Note that this is a non-equilibrium state of the system, in absence of an appropriate control action. From the desired profile \( \mathbf{u}_d \), one obtains the corresponding time coefficients \( a_d = (a_d)_i^{N} \) as

\[ a_{d_i} = \langle \mathbf{u}_d, \phi_i \rangle . \]

\(^2\)If \( \mathbf{u}_d \notin S \), one can attempt to drive the system to \( P_S \mathbf{u}_d \in S \).
Assuming it is possible to find $\gamma_d$ such that

$$C + La_d + L_{in}\gamma_d + Q(a_d, a_d) + Q_{in}\gamma_d^2 + Q_{ain}a_d \gamma_d = 0$$

one can define a shift of coordinates $\tilde{a} = a - a_d$, $\tilde{\gamma} = \gamma - \gamma_d$ which yields

$$\dot{\tilde{a}} = \tilde{L}\tilde{a} + Q(\tilde{a}, \tilde{a}) + \tilde{L}_{in}\tilde{\gamma} + Q_{in}\gamma^2 + Q_{ain}\tilde{a}\tilde{\gamma} \quad (3.15)$$

where

$$\tilde{L} = L + Q(\cdot, a_d) + Q(a_d, \cdot) + Q_{ain}\gamma_d$$

$$\tilde{L}_{in} = L_{in} + 2Q_{in}\gamma_d + Q_{ain}a_d .$$

For $\tilde{\gamma} = 0$, i.e., $\gamma = \gamma_d$, (3.15) now has an equilibrium at $\tilde{a} = 0$, i.e., $a = a_d$, which is the desired state. Therefore all that needs to be done is to stabilize its equilibrium. Here we designed an LQR control on its linearization

$$\dot{\tilde{a}} = \tilde{L}\tilde{a} + \tilde{L}_{in}\tilde{\gamma} .$$

minimizing the objective function

$$J(\tilde{\gamma}) = \int_0^\infty \left( a^TQa + \tilde{\gamma}^TR\tilde{\gamma} \right) dt$$

with $Q = I_N$, and $R = 0.1$. This yields a controller $\tilde{\gamma} = K\tilde{a}$ with $K = [-0.7972 \ -0.0133 \ 0.0914]$ which places the system poles at $[-174.93 \ -8.85 \ -33.46]$, achieving the desired regulation. The gains were computed using the lqr command in MATLAB. The controller expressed in the original coordinates becomes

$$\gamma = \gamma_d + \tilde{\gamma}$$

$$= \gamma_d + K\tilde{a}$$

$$= \gamma_d + K(a - a_d)$$

$$= \gamma_d + Ka - Ka_d$$

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Remark 3.3.2. Note that control design is based on the linearization of the system. The controller also uses state feedback, which for the original system are computed as $a = \langle U, \phi_i \rangle$. It is possible to consider more complicated control designs, observer based controls, and so on; however, we have chosen to keep it simple here, for the fact that the designs in this section are just tools to illustrate the proposed separation method.

Figures 3.19 and 3.20 show the result of the implementation of this control in the original Navier-Stokes equations numerical simulation. Figure 3.21 compares the final state reached, to the desired profiles. The majority of the error is in the region from where the input is applied. This is to be expected since the desired profile selected was an intermediate non-equilibrium state under no forcing. However, in order to render this profile an equilibrium and stabilize it, control effort must be applied. Aside from the regions where the external excitation enters, it can be seen that the final state is almost identical to the desired profile with a small residual error.
Figure 3.10: Unforced Flow - $U$ Component

Figure 3.11: Unforced Flow - $V$ Component
Figure 3.12: Baseline Modes - $U$ Component

Figure 3.13: Baseline Modes - $V$ Component
Figure 3.14: Actuated Flow - $U$ Component

Figure 3.15: Actuated Flow - $V$ Component
Figure 3.16: Actuation Mode

(a) No forcing  
(b) -2 to 2 ramp forcing

Figure 3.17: Error between actual time coefficients and those from the Galerkin model
Figure 3.18: Reference Profile for Flow Velocity

Figure 3.19: Controlled Flow - $U$ Component
Figure 3.20: Controlled Flow - V Component

Figure 3.21: Final State of Controlled Flow vs. Reference Profile
3.3.1 Control Design Using a Two Input Mode POD Based Model

We will now investigate a control design based on a reduced-order model obtained from a POD expansion with two actuation modes. For this purpose we first need to compute a second actuation mode \( \psi_2 \) for the derivative of the input, \( \dot{\gamma} \). This is again done from the actuated snapshots in Figures 3.14 and 3.15, based on the innovations computed as differences from span \( \{ \phi_i, \psi \} \), as was described in Section 2.4.2. Figure 3.22 shows the actuation mode \( \psi_2 \) computed from these snapshots. The augmented POD expansion with the additional mode is then

\[
\mathbf{u}(x, t) = \mathbf{u}_0(x) + \sum_{i=1}^{N} \phi_i(x) a_i(t) + \psi(x) \gamma(t) + \psi_2(x) \dot{\gamma}(t). \tag{3.16}
\]

As described in Section 2.5, substituting (3.16) into the 2D Navier-Stokes equation (3.13) one obtains

\[
\dot{\mathbf{a}} = \mathbf{C} + L \mathbf{a} + L_{\text{in}} \gamma + L_{2\text{in}} \gamma + Q(a, a) + Q_{\text{in}} \gamma^2 + Q_{2\text{in}} \dot{\gamma}^2 + Q_{\text{ain}} a \gamma + Q_{2\text{ain}} a \dot{\gamma} + Q_{\text{min}} \gamma \dot{\gamma}
\]

where \( \mathbf{C}, L, Q, Q_{\text{in}}, Q_{\text{ain}} \) are as given in (3.14) and

\[
L_{2\text{in}, i} = \left\langle \begin{bmatrix} -u_0 \frac{\partial}{\partial y} \psi_{2u} - \psi_{2u} \frac{\partial}{\partial y} u_0 + \nu \left( \frac{\partial^2}{\partial x^2} \psi_{2u} + \frac{\partial^2}{\partial y^2} \psi_{2u} \right) - \psi_{2u} \frac{\partial}{\partial x} u_0 - v_0 \frac{\partial}{\partial y} \psi_{2u} \\ -u_0 \frac{\partial}{\partial x} \psi_{2v} - \psi_{2v} \frac{\partial}{\partial y} v_0 + \nu \left( \frac{\partial^2}{\partial x^2} \psi_{2v} + \frac{\partial^2}{\partial y^2} \psi_{2v} \right) - \psi_{2v} \frac{\partial}{\partial x} v_0 - v_0 \frac{\partial}{\partial y} \psi_{2v} \end{bmatrix}, \phi_i \right\rangle
\]

\[
Q_{2\text{in}, i} = \left\langle \begin{bmatrix} -\psi_{2u} \frac{\partial}{\partial y} \psi_{2u} - \psi_{2u} \frac{\partial}{\partial x} \psi_{2u} \\ -\psi_{2v} \frac{\partial}{\partial y} \psi_{2v} - \psi_{2v} \frac{\partial}{\partial x} \psi_{2v} \end{bmatrix}, \phi_i \right\rangle
\]

\[
Q_{\text{ain}, ij} = \left\langle \begin{bmatrix} -\left( \phi_{u,i} \right) \frac{\partial}{\partial y} \psi_{2u} - \left( \phi_{u,i} \right) \frac{\partial}{\partial x} \psi_{2u} - \psi_{2u} \frac{\partial}{\partial x} \left( \phi_{u,i} \right) - \psi_{2u} \frac{\partial}{\partial y} \left( \phi_{u,i} \right) \\ -\left( \phi_{v,i} \right) \frac{\partial}{\partial y} \psi_{2v} - \left( \phi_{v,i} \right) \frac{\partial}{\partial x} \psi_{2v} - \psi_{2v} \frac{\partial}{\partial x} \left( \phi_{v,i} \right) - \psi_{2v} \frac{\partial}{\partial y} \left( \phi_{v,i} \right) \end{bmatrix}, \phi_i \right\rangle
\]

\[
Q_{\text{min}, 2i} = \left\langle \begin{bmatrix} -\psi_{2u} \frac{\partial}{\partial x} \psi_{1,u} - \psi_{1,u} \frac{\partial}{\partial x} \psi_{2,u} - \psi_{2,u} \frac{\partial}{\partial y} \psi_{1,u} - \psi_{1,u} \frac{\partial}{\partial y} \psi_{2,u} \\ -\psi_{2u} \frac{\partial}{\partial x} \psi_{1,v} - \psi_{1,v} \frac{\partial}{\partial x} \psi_{2,v} - \psi_{2,v} \frac{\partial}{\partial y} \psi_{1,v} - \psi_{1,v} \frac{\partial}{\partial y} \psi_{2,v} \end{bmatrix}, \phi_i \right\rangle
\]

The error between the actuated response of the Galerkin system and the actual values of the time coefficients obtained from projecting the snapshots onto the basis can be
seen in Figure 3.23. As described in Section 2.5, with a shift of coordinates the above can be transformed into

\[ \dot{\tilde{a}} = \tilde{L}\tilde{a} + Q(\tilde{a}, \tilde{a}) + \tilde{L}_{\text{in}}\dot{\gamma} + \tilde{L}_{2\text{in}}\dot{\gamma} + Q_{\text{in}}\dot{\gamma}^2 + Q_{2\text{in}}\dot{\gamma}^2 + Q_{\text{ain}}\tilde{a}\dot{\gamma} + + Q_{\text{ain2}}\tilde{a}\dot{\gamma} + Q_{\text{inin2}}\dot{\gamma}\dot{\gamma} \]

and then linearized into

\[ \dot{\tilde{a}} = \tilde{L}\tilde{a} + \tilde{L}_{\text{in}}\dot{\gamma} + \tilde{L}_{2\text{in}}\dot{\gamma}. \]

The above can be written in the following standard state space form

\[
\begin{bmatrix}
\dot{\tilde{a}} \\
\dot{\gamma}
\end{bmatrix} =
\begin{bmatrix}
\tilde{L} & \tilde{L}_{\text{in}} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{a} \\
\dot{\gamma}
\end{bmatrix} +
\begin{bmatrix}
\tilde{L}_{2\text{in}} \\
1
\end{bmatrix} \gamma_2
\]

where \( \gamma_2 := \dot{\gamma} \). An LQR design on this system yields a control of the form \( \gamma_2 = K\col{\tilde{a}, \dot{\gamma}} \) where \( K = \begin{bmatrix} -0.1237 & 0.0303 & 0.0061 & 1.1650 \end{bmatrix} \). This control places the system poles at \([-174.7179, -31.2961, -6.7433, -1.1953]\) achieving the regulation desired. Recalling that \( \gamma_2 := \dot{\gamma} \), this control is actually a dynamic one since

\[
\begin{align*}
\gamma_2 &= K \begin{bmatrix}
\tilde{a} \\
\dot{\gamma}
\end{bmatrix} \\
\dot{\gamma} &= [K_1 \ k_2] \begin{bmatrix}
\tilde{a} \\
\dot{\gamma}
\end{bmatrix} \\
\dot{\gamma} &= k_2\dot{\gamma} + K_1\tilde{a}.
\end{align*}
\]

Figures 3.24 and 3.25 show the result of the implementation of this control in the original Navier-Stokes equations numerical simulation. Figure 3.26 compares the final state reached, to the desired profiles. As it was the case for the single actuation mode based control of the previous section, the majority of the error is in the region from where the input is applied and aside from the regions where the external excitation enters, it can be seen that the the final state is almost identical to the desired profile with a small residual error.
If one is to compare the results obtained from the two actuation mode based implementation to the one actuation mode based implementation, it is difficult to say that one approach provides a significant improvement over the other. This could be expected since the actuation mode $\psi_2$ for $\dot{\gamma}$, as shown in Figure 3.22, is of quite small magnitude, compared to $\psi$, the actuation mode for $\gamma$ (Figure 3.16). The reason for this could be that the trend of the input is not as important as its value, for the dynamics of the system given in this problem. It could also be interpreted as the input mode $\psi$ being able to capture and reconstruct enough of the actuation effect, so that when the new innovations are built as differences from $\text{span}\{\phi_i, \psi\}$, not much is left for $\psi_2$ to work with. In any case, since the value of $\psi_2$ is very small, its contribution to the augmented POD expansion, as well as the final reduced-order model is expected to be minimal. Comparing the final tracking errors in Figures 3.21 and 3.26, the one mode approach actually yields somewhat less error in the actuation region; however for all practical purposes it can be said that the results are almost identical. As to the complexity in implementation of the one-mode vs. two-mode approach, the two-mode approach obviously requires extra steps, such as: the computation of a second set of innovations, a second optimization for finding another mode, computing additional terms for the Galerkin projection, converting the linearized system into standard form, and implementing a dynamic controller instead of a static one. While for the given example, the augmentation with an additional mode does not yield an improvement, the situation could be different for more complicated dynamics with more complicated interaction with the actuation. Thus, it is could be beneficial to have a systematic methodology in hand for an extension beyond the basic approach when needed.
Figure 3.22: Second Actuation Mode

Figure 3.23: Error between actual time coefficients and those from the Galerkin model under forcing
Figure 3.24: Controlled Flow with Controller from Two Mode Based Model - \textit{U} Component

Figure 3.25: Controlled Flow Controller from Two Mode Based Model - \textit{V} Component
Figure 3.26: Final State of Controlled Flow vs. Reference Profile for Controller from Two Mode Based Model
3.4 Example 3: Cavity Flow Control

In this section, the input separation technique developed in the preceding sections for reduced order modeling is applied to the cavity flow control problem, through experiments carried out at the OSU GTDL cavity flow facility (see Appendix A). The reduced order modeling method proposed is first evaluated in its ability to achieve reconstruction of the flow. Then the performance of closed loop controllers built from a reduced order model based on the new input separation techniques is analyzed. This study has been performed in closed collaboration with the other members of the flow control group. We would like to again express our thanks to the OSU GTDL director Prof. Mo Samimy for making these experiments possible, as well as to Jesse Little, Kihwan Kim and especially Edgar Caraballo, for providing the data and figures from the experiments, and helping with the construction of the reduced-order models.

3.4.1 Modeling

To describe the dynamics of the flow process in the experimental, we use the isentropic compressible Navier-Stokes equations [16, 100, 102]

\[
\frac{Dw}{Dt} + \frac{2}{Ma^2 \alpha - 1} \nabla c = \frac{1}{Re} \nabla^2 w \\
\frac{Dc}{Dt} + \frac{\alpha - 1}{2} c \text{div} w = 0
\]  

where \( w(x,t) = (u(x,t), v(x,t)) \) is the flow velocity in the stream-wise and vertical direction, \( c(x,t) \) is the local speed of sound, the operator \( D/Dt = \partial/\partial t + w \cdot \nabla \) stands for the material derivative, \( t \in \mathbb{R}_+ \) is the temporal variable, \( x \in \Omega \) is the spatial variable, and \( \Omega \subset \mathbb{R}^2 \) is the spatial domain of flow. The constants \( \alpha, Re, \) and \( Ma \) denote respectively ratio of specific heats, Reynolds number, and Mach number.
These equations can be expressed in compact form as

\[
\dot{\mathbf{u}} = X(\mathbf{u}) := L(\mathbf{u}) + Q(\mathbf{u}, \mathbf{u})
\]

(3.18)

where \( \mathbf{u} := (u, v, c) \) is the augmented flow velocity, \( L(\mathbf{u}) \) is linear in \( \mathbf{u} \), and \( Q(\mathbf{u}, \mathbf{u}) \) is quadratic in \( \mathbf{u} \). The derivation of the compact form from (3.17) is given in Appendix B. The first task of reduced order modeling is the building of the baseline modes \( \phi_i(x) \), which are extracted from the unactuated flow (i.e. the \( \gamma = 0 \) case, termed ‘B’ for ‘baseline’ or ‘nc’ for ‘no control’ – see Table 3.1) using standard POD as explained in Section 2.3. As a compromise between accuracy and simplicity, we select the number of modes as four, which are shown in Figure 3.27. The next step is the obtaining the actuation model \( \psi \), through solving the optimization problem as described in Section 2.4. Figure 3.28 shows the actuation mode \( \psi \) computed using three different actuation conditions: F4, F4 & F1 combined, and white noise. The

<table>
<thead>
<tr>
<th>Name</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>B or nc</td>
<td>Baseline flow (no excitation)</td>
</tr>
<tr>
<td>F1</td>
<td>Flow under 1610 Hz open loop forcing</td>
</tr>
<tr>
<td>F2</td>
<td>Flow under 1830 Hz open loop forcing</td>
</tr>
<tr>
<td>F3</td>
<td>Flow under 3250 Hz open loop forcing</td>
</tr>
<tr>
<td>F4</td>
<td>Flow under 3920 Hz open loop forcing</td>
</tr>
<tr>
<td>Wn</td>
<td>Flow under bandlimited white noise forcing</td>
</tr>
<tr>
<td>Bi</td>
<td>Combination of snapshots of B and i</td>
</tr>
<tr>
<td>Bi j</td>
<td>Combination of snapshots of B, i and j</td>
</tr>
<tr>
<td>MkBi</td>
<td>Model built using method Mk from Bi snapshots.</td>
</tr>
<tr>
<td>MkBi j</td>
<td>Model built using method Mk from Bi j snapshots.</td>
</tr>
<tr>
<td>MkBi c</td>
<td>Model MkBi under forcing c</td>
</tr>
<tr>
<td>MkBi j c</td>
<td>Model MkBi j under forcing c</td>
</tr>
</tbody>
</table>

\( i, j \in \{F1, F2, F3, F4, Wn\}, \ k \in \{0, 1, 2\}, \ c \in \{\text{nc}, F1, F2, F3, F4, Wn\} \)

Table 3.1: Nomenclature
Figure 3.27: Baseline POD modes.

Figure 3.28: Control modes for separation method M2, based on different forcing conditions; a) F4 forcing, b) combination of F1 and F4 forcing, and c) White noise forcing.
Galerkin model obtained for these baseline and actuation modes has the form

\[
\dot{a}_k = F_k + G_{1ik}a_i + g_{2k}\gamma + H_{1ijk}a_ia_j + H_{2ik}a_i\gamma + h_{3k}\gamma^2
\]

Figure 3.29 shows the numerical solution of the Galerkin system above for three different flow combinations. It can be observed that when the control is not present (nc), the solution of the different systems match the baseline case solution (M0B). However, when the forcing is introduced (i.e. F4), there is a change in the amplitude and frequency of the modal coefficient. This is important the previous input separation method, i.e. the subdomain separation method had a sensitivity issue in this regards and needed ad hoc modifications such as multiplication by 1000, to show any effect [100].

**3.4.2 Flow Field Reconstruction**

As a first test of the proposed method, we examine how well the POD expansion augmented with the actuation mode is able to reconstruct flow snapshots from experiments. We choose to do so as the necessary preliminary step for two reasons: First, it allows us to confirm that the newly developed technique works well at the POD stage before performing Galerkin projection. This is important because the proposed method provides modifications and augmentations to the POD expansion. The second reason for this test is that it could shed some light on what flow snapshots should be used to derive reduced-order models that can encompass a dynamic flow field. To ascertain the ability of each model to recover the forced flows, we compare the original velocity field with the reconstructed one using the control modes from the new separation method. For the reconstruction of the velocity field, the control modes from two of the control cases, namely F1-F4 and Wn (see Table 3.1) were used
Figure 3.29: Time series and power spectrum of the first modal amplitude for the Galerkin system M2; nc (no control) and F4 (f = 3920 Hz).
in an attempt to establish the ideal forcing to use in the model reduction process.

The velocity fields for the forced cases F1 and F4 are reconstructed using white noise or F1-F4 basis. To this end, we took the following steps: First, the modal amplitude of the baseline portion are obtained by projecting the velocity field (PIV images) of the forced flow onto the baseline basis

\[ a_i(t) = \langle u(x,t) - u_0(x), \phi_i(x) \rangle \]  

(3.19)

where \( u(x,t) \) is the forced flow velocity field, \( u_0(x) \) is the mean velocity of the baseline flow and \( \phi_i(x) \) are the POD bases of the baseline flow. To add the control effect, the control modes are multiplied by the corresponding voltage \( \gamma \) measured at the time the PIV images were taken, which corresponds to the operation shown in (2.13). Then, we define the averaged error as the mean value of the squared difference between the actual velocity and the reconstructed value

\[ e(x) = \sqrt{\frac{1}{M} \sum_{k=1}^{M} (u_r(x,t_k) - u(x,t_k))^2} \]  

(3.20)

This procedure is applied using the F1-F4 or the Wn control modes to reconstruct the forced cases F1 and F4. Figures 3.30 and 3.31 show the mean square error of the reconstruction for the F4 and F1 cases. Figures 3.30(a) and 3.31(a) shows the mean square error when the flow is reconstructed using the white noise control basis. Similarly, Figures 3.30(b) and 3.31(b) shows the error for the case based on the combination of the forced cases F1-F4. The figure also contains the mean square error and velocity components. It is clear from both figures that the error levels are lower for the Wn case (a). In this case the error is concentrated in the shear layer region and close to the leading edge. We suspect that this is due to the difference in mean flow, as we are using the baseline mean flow as the overall mean. For the cases based
Figure 3.30: Mean error in the velocity reconstruction for the forced case (F4) for the new model, with the control modes based on: a) white noise forcing (Wn), and b) the combination of F1F4 forcing.

Figure 3.31: Mean error in the velocity reconstruction for the forced case (F1) for the new model, with the control modes based on: a) white noise forcing (Wn), and b) the combination of F1F4 forcing.
on F1-F4, it can be observed that the error spreads in a larger region midway point
the cavity and towards the trailing edge. We believe that the main reason for the
difference in the mean error is due to the nature of the structures of the control mode
basis: for the F1-F4 case, the basis contains large and well organized structures, while
the Wn case has more scattered structures. This in turn generates larger fluctuations
in the velocity reconstruction when the control portion is introduced.

3.4.3 Feedback Control Design

After obtaining satisfactory results in velocity reconstruction with the newly devel-
oped method, the next step is to compare the performance of a closed loop controller
built from the model based on the new technique. As noted before, satisfactory exper-
imental results in closed loop with the old subdomain method was already obtained
in our previous works [2, 100]. Still, the need for the development of alternative
modeling techniques was felt, as the previous approach was deemed to be intrinsi-
cally empirical, required numerous trial-and-error iterations, and ultimately lacked
adequate theoretical support. The new model was used in a feedback control design
shown in Figure 3.32, which was developed in earlier CCCS works, which has the
following steps: 1) state estimation to estimate time coefficients from pressure mea-
surers, 2) LQR control with scaling to achieve reduce oscillations while avoid actuation
saturation and 3) actuator compensation to compensate for the unwanted dynamics
that the actuator introduces to the system [2, 8]. Controllers are obtained for the
four different models: 1) Our previous model (M0B), 2) Model built using method M2
from B and F4 (M2BF4) snapshots combined, 3) Model built using method M2 from
B, F1 and F4 snapshots combined (M2BF1F4), and 4) Model built using method
M2 from B and Wn snapshots combined (M2BWn). Figure 3.33 shows the sound pressure level (SPL) reduction obtained by the LQ state feedback control for the different models tested in Mach 0.3 cavity flow for which the models were derived. The thin red line yields the SPL of the unforced baseline flow, whereas the thick line corresponds to the SPL of the flow at the same location under state feedback control. All the models show improvement with respect to the uncontrolled flow by reducing the resonant peak by more than 18 dB, without the addition of any addition peak. The performance of the control law was also tested in the closed-loop experiments for different flows conditions in the neighborhood of Mach 0.30. Figure 3.34 shows the behavior of the flow under off design conditions for all the models. It can be noticed that in each case the controller is capable of maintaining the same general characteristics and benefits as in the Mach 0.30 design condition. This is consistent with previous CCCS results[2], which showed the robustness of the feedback control loop under off design conditions.
Figure 3.33: Sound pressure level (SPL) obtained under LQ control
Figure 3.34: SPL under LQ control for models built with M2 at off design conditions Ma = 0.28 and Ma = 0.32
3.5 Summary and Comments

In this chapter a number of illustrative case studies were given in which the modeling approach of Chapter 2 was applied to flow problems. The reduced order model was first compared with sub-domain separation method, which is a previously developed technique for input separation. The usefulness and power of the proposed method was illustrated next by a flow control example, which is the boundary control of 2D incompressible Navier-Stokes equations. A controller was designed so as to achieve a desired spatial profile, using a single actuation mode and also using two actuation modes. It was seen that both controls achieve the desired profile with little error for the most part. The next example was application to the OSU GDTL cavity flow experiment. The new method was evaluated first in its ability to reconstruct actuated flows at the POD level. It was observed that the model improved over the past results in its ability to reconstruct a wide range of flows that are different from the modeling conditions, especially when built from white noise excitation. Next, we tested LQR controllers derived on the basis of the models obtained from the new separation method. Experimental results showed that these controllers significantly reduce the resonant peak of the single-mode Mach 0.3 flow, for which they were designed, and also performed satisfactorily for off-design conditions at Mach 0.28 and Mach 0.32.
CHAPTER 4

NONLINEAR ANALYSIS AND CONTROL DESIGN

4.1 Introduction

In this chapter, nonlinear analysis and control design tools are developed for the reduced order Galerkin models for flow control problems, where the Galerkin system is characterized by the presence of a stable limit cycle, generating a self-sustained oscillation. The control objective is to suppress or attenuate the oscillation by means of feedback, which is generally pursued by modifying the linearization of the dynamics about an equilibrium point by means of linear feedback. However, analyzing the effect of linear control on the amplitude of the limit cycle using standard arguments involving Poincaré normal forms and center manifold theory, it is found that the oscillation amplitude depends both on terms linear in the control and nonlinear terms that depend on the center manifold. To exploit these latter, in this chapter a control law is proposed that aims at reducing the oscillation by a method that relies solely on nonlinear arguments whose main idea is to shape by nonlinear feedback a suitably-defined center manifold for the Galerkin system, so as to attenuate the amplitude of the steady-state oscillation without modifying the linear terms directly. The approach
aims at exploiting the effect of the control-dependent nonlinear terms in the expansion of the center manifold, rather than neglecting them in the analysis. This feature makes the method appealing in case the Galerkin system at issue has limited or null linear controllability. Towards this goal, averaging theory is used to define a class of nonlinear feedback laws that enforce a particular structure for the averaged closed-loop system that is instrumental for the analysis. Then, a center manifold reduction is employed to obtain an expression of the closed-loop system amenable to analyzing the number and type of the steady state solutions as a function of the control parameters. Furthermore, all analysis and design are performed under the constraint that the oscillation is preserved, so as to remain within the bounds of physical meaningfulness and practical implementability when applications to flow control problems is considered. Finally the developed results are applied to attenuation of oscillations in cavity flow problems, and the outcome of the analysis is compared with simulation results.

The material presented in this chapter also appears in Kasnakoglu and Serrani [103, 104].

4.2 Problem Formulation and Background

In this chapter, we consider a class of \(n\)-dimensional nonlinear control systems described by equations of the form

\[
\dot{a}_i = \sum_{j=1}^{n} l_{ij} a_j + \sum_{j,k=1}^{n} q_{ijk} a_j a_k + r_i \gamma + \sum_{j=0}^{n} s_{ij} a_j \gamma, \quad i = 1 \ldots n \quad (4.1)
\]

which can be expressed in a more compact notation as

\[
\dot{a} = La + Q(a) + (R + Sa) \gamma,
\]
where $a \in \mathbb{R}^n$, is the state vector, $\gamma \in \mathbb{R}$ is the control input, and the plant parameter matrices are respectively given by $L = \{l_{ij}\}_{i,j=1}^n \in \mathbb{R}^{n \times n}$, $Q(a) = \{a^T Q_i a\}_{i=1}^n \in \mathbb{R}^n$, $Q_i = \{q_{ijk}\}_{j,k=1}^n \in \mathbb{R}^{n \times n}$, $R = \{r_i\}_{i=1}^n \in \mathbb{R}^n$, and $S = \{s_{ij}\}_{i,j=1}^n \in \mathbb{R}^{n \times n}$.

**Remark 4.2.1.** This form is slightly simpler than the forms obtained from reduced order modeling in the previous chapter, in the sense that the quadratic term in the input has been removed. We found this term to have significantly small values in the examples considered, and therefore it has been decided to neglect it in the derivations here so as to simplify the notations.

In absence of external control, the plant model (4.1) is assumed to possess a stable limit cycle, compatible with the following assumption:

**Assumption 4.2.2.** The spectrum of $L$ is of the form $\text{spec}(L) = \{\sigma + j\omega, \sigma - j\omega, -\lambda_1, \cdots, -\lambda_{n-2}\}$ where $\sigma > 0$, $\omega > 0$, $\lambda_i > 0$, and $\lambda_i \neq \lambda_j$ for all $i \neq j$.

Using a non-singular linear transformation, system (4.1) is represented in modal form as

\[
\begin{align*}
\dot{\eta} &= F_1 \eta + \varphi_1(\eta, \zeta) + (G_1 + \chi_1(\eta, \zeta)) \gamma \\
\dot{\zeta} &= F_2 \zeta + \varphi_2(\eta, \zeta) + (G_2 + \chi_2(\eta, \zeta)) \gamma,
\end{align*}
\]

(4.2)

where $\eta = \text{col}(\eta_1, \eta_2)$, $\zeta = \text{col}(\zeta_1, \ldots, \zeta_{n-2})$,

\[
F_1 = \begin{bmatrix}
\sigma & -\omega \\
\omega & \sigma
\end{bmatrix}, \quad G_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix}, \quad F_2 = \begin{bmatrix}
-\lambda_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\lambda_{n-2}
\end{bmatrix}, \quad G_2 = \begin{bmatrix} g_{21} \\ \vdots \\ g_{2n-2} \end{bmatrix}
\]

and $\varphi_1, \chi_1 : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$ and $\varphi_2, \chi_2 : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ are continuously differentiable functions, vanishing at the origin with their first derivatives. As mentioned

\[^3\text{The } \lambda_i \neq \lambda_j \text{ part of the assumption is only to simplify the presentation, and could be relaxed easily.}\]
before, the in model (4.2), commonly referred to as a *Galerkin system*, is representative of a class of systems that arises frequently in flow control problems, when one considers finite-dimensional reduced-order models obtained through proper orthogonal decomposition (POD) and Galerkin projection [55], such as done in Chapter 2. The control objective is that of reducing the amplitude of the steady-state oscillation that arises from the stable limit cycle, or to dissipate it completely whenever it is possible to do so. This can be accomplished through the introduction of adequate dissipation by means of feedback. For reasons that will become clear later, only the ‘oscillating components’ \( \eta \) are used for feedback control.

To introduce the motivation for the nonlinear analysis developed in this chapter, in what follows we discuss and review the basic mechanism behind the attenuation of the oscillation proposed in the literature (as found, for instance, in [1, 8, 78, 80, 81, 84, 105].) In order to maintain physical meaningfulness for applications to flow control problems, the control input is constrained to preserve the structure of the linear approximation of (4.2). Specifically, we make use of the following definition:

**Definition 4.2.3.** Consider a continuously differentiable control law of the form \( \gamma = \gamma(\eta, K) \) satisfying \( \gamma(0, K) = 0 \) and \( \gamma(\eta, 0) = 0 \), where \( K \in \mathbb{R}^p \) is a vector of control gains. Let

\[
\gamma(\eta, K) = K_1 \eta + \bar{\gamma}(\eta, K_2), \quad \bar{\gamma}(\eta, K_2) = O(\|\eta\|^2)
\]

where \( K_1 = [k_{11}, k_{12}] \in \mathbb{R}^{1 \times 2} \) and \( K_2 \in \mathbb{R}^{p-2} \) denote the gain parameters of the linear and nonlinear terms of the control, respectively. The control \( \gamma(\eta, K) \) is said to be oscillation-preserving if the closed loop system (4.2)-(4.3) preserves the eigenvalue structure in Assumption 4.2.2.
A few comments on the motivation behind introducing the concept of oscillation preserving control are in order. For physical systems which are oscillatory in nature, it may be impractical (if not impossible) to completely quench the oscillation by feedback. When simple (i.e., reduced-order) mathematical models for these systems are derived, the structure of the model may allow in principle the oscillation to be suppressed via feedback. However, this has little physical meaning, as the capability of the model to describe the behavior of the system usually breaks apart when aggressive control actions are performed. Moreover, in many cases the nature of the actuation mechanism itself imposes an oscillatory input. A good example for a system possessing all of the aforementioned issues is the very problem in hand, i.e. cavity flow control. This system has a naturally oscillatory dynamics which can be only approximately described by the aid of a Galerkin model. While, in principle one could render the Galerkin model non-oscillatory by supplying enough linear dissipation if \((F_1, G_1)\) is controllable, this would correspond to quenching the natural oscillation of the cavity, which is physically unrealistic, also in consideration of the fact that synthetic jets and pressure transducers commonly implemented as actuating and sensing devices, respectively, operate under oscillatory regimes. Therefore, for any analysis and derivation of control algorithms on the flow control problem to have physical significance and possibility of implementation, the oscillation-preserving constraint should be respected.

**Assumption 4.2.4.** Throughout this chapter, the control input \(\gamma\) is assumed to be oscillation-preserving.
Under the feedback law (4.3), the closed loop system reads as

\[ \dot{\eta} = (F_1 + G_1 K_1)\eta + \varphi(\eta, \zeta) + \chi_1(\eta, \zeta) K_1 \eta + (G_1 + \chi_1(\eta, \zeta))\bar{\gamma}(\eta, K_2) \]

\[ \dot{\zeta} = F_2 \zeta + G_2 K_1 \eta + \varphi(\eta, \zeta) + \chi_2(\eta, \zeta) K_1 \eta + (G_2 + \chi_2(\eta, \zeta))\bar{\gamma}(\eta, K_2), \] (4.4)

where

\[ F_1 + G_1 K_1 = \begin{bmatrix} \sigma + g_{11} k_{11} & -\omega + g_{11} k_{12} \\ \omega + g_{12} k_{11} & \sigma + g_{12} k_{12} \end{bmatrix}. \]

Under the oscillation preserving constraint, the eigenvalues of \( F_1 + G_1 K_1 \) remain as a complex conjugate pair \( \lambda(K_1) := \bar{\sigma}(K_1) \pm j\bar{\omega}(K_1), \) where

\[ \bar{\sigma}(K_1) = \sigma + \frac{1}{2} g_{11} k_{11} + \frac{1}{2} g_{12} k_{12} \]

\[ \bar{\omega}(K_1) = \frac{1}{2} \left( 4 \omega^2 - 4 \omega g_{11} k_{12} + 4 \omega g_{12} k_{11} - g_{12}^2 k_{12}^2 - 2 g_{11} g_{12} k_{11} k_{12} - g_{11}^2 k_{11}^2 \right)^{1/2}. \]

The oscillation-preserving constraint also dictates that the spectra of \( F_1 + G_1 K_1 \) and \( F_2 \) are disjoint. Transforming the above system into modal form (and keeping the same notation for the sake of simplicity) yields

\[ \dot{\eta} = \bar{F}_1(K_1) \eta + \bar{\Phi}_1(\eta, \zeta, K) \]

\[ \dot{\zeta} = F_2 \zeta + \bar{\Phi}_2(\eta, \zeta, K), \] (4.5)

where

\[ \bar{F}_1(K_1) := F_1 + G_1 K_1 = \begin{bmatrix} \bar{\sigma}(K_1) & -\bar{\omega}(K_1) \\ \bar{\omega}(K_1) & \bar{\sigma}(K_1) \end{bmatrix} \]

and \( \bar{\Phi}_1, \bar{\Phi}_2 : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) collect nonlinear terms. An analysis of the local behavior of the trajectories of system (4.5) can be carried out by using center manifold theory, regarding the parameter \( \bar{\sigma} \) as an additional state obeying the trivial dynamics \( \dot{\bar{\sigma}} = 0 \) (see [87, 95]). Consider a second-order polynomial approximation of the center
manifold of the form $\zeta = h(\bar{\sigma}, \eta, K)$, where

$$\zeta_i = h_i(\bar{\sigma}, \eta, K) = \sum_{j=1}^{2} h^0_{ij}(K)\bar{\sigma}(K_1)\eta_j + \sum_{j,k=1}^{2} h^1_{ijk}(K)\eta_j\eta_k, \quad i = 1, \ldots, n - 2$$

and $h^0_{ij}(K), h^1_{ijk}(K)$ are coefficients to be determined. Substituting the approximation into the center manifold equation

$$\frac{\partial h}{\partial \eta} \left[ \bar{F}_1(K_1)\eta + \bar{\Phi}_1(\eta, h(\bar{\sigma}, \eta, K), K) \right] = F_2 h(\bar{\sigma}, \eta, K) + \bar{\Phi}_2(\eta, h(\bar{\sigma}, \eta, K), K)$$

yields $h^0_{ij} = 0$, for all $i = 1, \ldots, n - 2$ and all $j = 1, 2$. Therefore, an approximation of the least nontrivial order for the center manifold is of the form $\zeta_i = \sum_{j,k=1}^{2} h^1_{ijk}(K)\eta_j\eta_k$. The dynamics of system (4.5) reduced to the center manifold, that is, the system

$$\dot{\eta} = \bar{F}_1(K_1)\eta + \bar{\Phi}_1(\eta, h(\bar{\sigma}, \eta, K), K) + \text{h.o.t.}$$

can be equivalently expressed in a neighborhood of the origin $(\bar{\sigma}, \eta) = (0, 0)$ by using the Poincarè normal form. Following [88, 89], the normal form of system (4.5) reads as

$$\dot{\eta} = \left[ \bar{F}_1(K_1) + \bar{H}(\eta, K) \right] \eta + \mathcal{O}(\|\eta\|^5),$$

where

$$\bar{H}(\eta, K) = \begin{bmatrix} \bar{\alpha}(K)(\eta_1^2 + \eta_2^2) & -\bar{\beta}(K)(\eta_1^2 + \eta_2^2) \\ \bar{\beta}(K)(\eta_1^2 + \eta_2^2) & \bar{\alpha}(K)(\eta_1^2 + \eta_2^2) \end{bmatrix}$$

It should be clear that the presence of a term of the form $h_i\bar{\sigma}^2$ is inessential.

It should be noted that the center manifold is not unique, and the form given here for $h$ is not the only possibility.

The notation h.o.t. stands for ‘higher order terms’.

---

4It should be clear that the presence of a term of the form $h_i\bar{\sigma}^2$ is inessential.

5It should be noted that the center manifold is not unique, and the form given here for $h$ is not the only possibility.

6The notation h.o.t. stands for ‘higher order terms’.
and $\bar{\alpha}(K)$, $\bar{\beta}(K)$ are functions of the control parameters resulting from the given approximation of the center manifold. Converting the reduced system to polar coordinates $\rho = (\eta_1^2 + \eta_2^2)^{1/2}$, $\theta = \arctan(\eta_2/\eta_1)$, yields a system of the form

$$
\dot{\rho} = \bar{\sigma}(K_1)\rho - \bar{\alpha}(K)\rho^3 + O(\rho^5)
$$
$$
\dot{\theta} = \omega(K_1) + \bar{\beta}(K)\rho^2 + O(\rho^4).
$$

(4.6)

Note that Assumption 4.2.2 implies that $\bar{\sigma}(0) > 0$ and $\bar{\alpha}(0) > 0$. From (4.6), one sees that if $K$ is such that $\bar{\sigma}(K_1) < 0$, then $\rho = 0$ is an asymptotically stable equilibrium for the first equation in (4.6). If, in addition, $\bar{\alpha}(K) < 0$ then the system has an unstable limit cycle. On the other hand, if $\bar{\sigma}(K_1) > 0$, the equilibrium $\rho = 0$ is unstable. The most interesting case occurs when $\bar{\sigma}(K_1) > 0$ and $\bar{\alpha}(K) > 0$; for this case, the system has a stable limit cycle, with amplitude and frequency of the steady-state oscillation given respectively by

$$
\rho^* = \sqrt{\frac{\bar{\sigma}(K_1)}{\bar{\alpha}(K)}}, \quad \omega^* = \omega(K_1) + \bar{\beta}(K)\frac{\bar{\sigma}(K_1)}{\bar{\alpha}(K)}.
$$

(4.7)

Clearly, the oscillation is completely quenched if $K_1$ can be chosen such that $\bar{\sigma}(K_1) < 0$. If this is not the case, from the first identity in (4.7) it is readily seen that the amplitude $\rho^*$ can be still decreased if the control parameter $K$ can be chosen such that $\bar{\sigma}(K_1)/\sigma < \bar{\alpha}(K)/\bar{\alpha}(0)$. One sees immediately that there are two ways to achieving this goal: to decrease the value of $\bar{\sigma}(K_1)$ and/or to increase $\bar{\alpha}(K)$. Recalling that $\bar{\sigma}(K_1)$ depends only on the linear term of the control, the easiest strategy to accomplishing the goal of reducing the amplitude of the steady-state oscillation is to set $K_2$ to zero in (4.3), and disregard in the analysis the terms related to the center manifold, implicitly assuming that the variation of $\bar{\sigma}(K_1)$ with respect to $K_1$ dominate that of $\bar{\alpha}(K)$. In [81, 105] and [84], an approach of this sort is applied to
cylinder wake and cavity flow oscillations, respectively. In their works, the authors transform the system into polar coordinates \((r, \theta)\), average with respect to the angle \(\theta\), and choose \(K_1\) in such a way that the control vector is oriented in the radial direction. Specifically, the control gains \(k_{11}\) and \(k_{12}\) are parameterized as

\[
k_{11} = -k \cos(\phi), \quad k_{12} = -k \sin(\phi), \quad (k, \phi) \in \mathbb{R}_{>0} \times \mathbb{S}
\]

and then, once a value for \(k\) has been fixed, obtain the phase shift solving

\[
\phi = \arg \max_{\phi \in \mathbb{S}} \left( g_{11} \cos(\phi) + g_{12} \sin(\phi) \right)
\]

which yields\(^7\) \(\phi = \text{atan2}(g_{12}, g_{11})\), and thus \(\bar{\sigma}(K_1) = \sigma - k(g_{11}^2 + g_{12}^2)^{1/2}\). In [1, 8], the matrix \(K_1\) is chosen solving a linear-quadratic control problem based on the Jacobian linearization of the system around the origin. The design exploits the fact that, when the quadratic penalty on the control is sufficiently large, the open-loop right-half-plane eigenvalues are mirrored in closed-loop, and therefore the influence of the linear control on the imaginary part on the complex conjugate eigenvalues of \(F_1\) is negligible. It should be kept in mind that in all the cited works, the control gains depend explicitly on the plant parameters, which opens the door for some concerns about robustness of the design.

With this in mind, the goal of the present chapter is to provide an analysis of the variation of the coefficient \(\bar{\alpha}(K')\) with the control parameters on the magnitude of the steady-state oscillation, by exploiting, rather than neglecting, the higher order terms in the expansion of the center manifold mapping. The analysis, besides being of its own interest, becomes especially important in case the system has limited linear controllability (that is, when the values of \(g_{11}\) and \(g_{12}\) are small,) which prevents one

\(^7\)Here, the function \(\text{atan2}(\cdot, \cdot)\) denotes the four-quadrant inverse tangent.
from attaining an adequately low value for $\bar{\sigma}$ by using linear control. In this case, increasing $\bar{\alpha}(K)$ by means of *nonlinear feedback* be the only practical way to lower the amplitude of the steady-state oscillation. Furthermore, the analysis will be performed assuming only limited a priori knowledge of the plant parameters for the selection of the controller gains.

### 4.3 Analysis of Nonlinear Feedback

In this section, in interest of clarity and without any loss of generality, the analysis will be restricted to the case $n = 4$; it should be clear how to extend the results in case there are additional stable modes. Recall that we are only interested in analyzing the effect of nonlinear feedback; consequently, we consider a parameterized family of control laws of the form (4.3), where $K_1$ is set to zero to single out the linear part of the control. Accordingly, the closed loop system is written as

$$
\dot{\eta} = F_1\eta + \varphi_1(\eta, \zeta) + (G_1 + \chi_1(\eta, \zeta))\bar{\gamma}(\eta, K_2) =: F_1\eta + f_\eta(\eta, \zeta, K_2)
$$

$$
\zeta = F_2\zeta + \varphi_2(\eta, \zeta) + (G_2 + \chi_2(\eta, \zeta))\bar{\gamma}(\eta, K_2) =: F_2\zeta + f_\zeta(\eta, \zeta, K_2). \quad (4.8)
$$

Since our goal is to control the amplitude of the limit cycle only by shaping the center manifold, we need to identify a subset of the family $\bar{\gamma}(\eta, K_2)$ that does not affect directly the coefficient of $\rho$ in the normal form (4.6). The method of averaging [88, 89] offers at the same time a simple procedure for obtaining explicitly the coefficients of the Poincarè normal form and a convenient way of defining a class of control laws that accomplish the stated objectives.
4.3.1 Averaged Model

We begin with applying an appropriate version of Krylov-Bogoliubov averaging to our system. Let $\omega_c$ be the frequency of oscillation of the limit cycle in steady state, and define a time-varying periodic change of coordinates as

$$\eta^\vartheta = R(\vartheta) \eta, \quad R(\vartheta) = \begin{bmatrix} \cos(\omega_c t) & \sin(\omega_c t) \\ -\sin(\omega_c t) & \cos(\omega_c t) \end{bmatrix}. \quad (4.9)$$

Using the above transformation, and recalling the structure of $F_1$, one obtains the following expression for (4.8) in the new coordinates

$$\dot{\eta}^\vartheta = (F_1 + \Omega_c) \eta^\vartheta + f_\vartheta^\eta(\vartheta, \eta^\vartheta, \zeta, K_2)$$

$$\dot{\zeta} = F_2 \zeta + f_\vartheta^\zeta(\vartheta, \eta^\vartheta, \zeta, K_2)$$

where $\vartheta = \omega_c t$, $\Omega_c = \dot{R}(\vartheta) R^T(\vartheta)$ is a skew-symmetric matrix, and

$$f_\vartheta^\eta(\vartheta, \eta^\vartheta, \zeta, K_2) := R(\vartheta) f_\eta(R^T(\vartheta) \eta^\vartheta, \zeta, K_2)$$

$$f_\vartheta^\zeta(\vartheta, \eta^\vartheta, \zeta, K_2) := f_\zeta(R^T(\vartheta) \eta^\vartheta, \zeta, K_2).$$

Letting $\epsilon = \omega_c^{-1}$ and rescaling the time as $t \to \vartheta$, one obtains

$$\frac{d\eta^\vartheta}{d\vartheta} = \epsilon(F_1 + \Omega_c) \eta^\vartheta + \epsilon f_\vartheta^\eta(\vartheta, \eta^\vartheta, \zeta, K_2)$$

$$\frac{d\zeta}{d\vartheta} = \epsilon F_2 \zeta + \epsilon f_\vartheta^\zeta(\vartheta, \eta^\vartheta, \zeta, K_2),$$

which is in a form for which classic averaging over $\vartheta \in [0, 2\pi]$ can be applied, as both $f_\vartheta^\eta$ and $f_\vartheta^\zeta$ are bounded functions with respect to the argument $\vartheta$ (see [88, 89]). As a result, one obtains the averaged system with states $(\bar{\eta}^\vartheta, \bar{\zeta})$, which in the original time scale $t$ reads as

$$\dot{\bar{\eta}}^\vartheta = (F_1 + \Omega_c) \bar{\eta}^\vartheta + f^\vartheta_{\eta, \text{avg}}(\bar{\eta}^\vartheta, \bar{\zeta}, K_2)$$

$$\dot{\bar{\zeta}} = F_2 \bar{\zeta} + f^\vartheta_{\zeta, \text{avg}}(\bar{\eta}^\vartheta, \bar{\zeta}, K_2). \quad (4.10)$$
where
\[
\begin{align*}
 f_{\eta, \text{avg}}^{\vartheta}(\bar{\eta}^{\vartheta}, \bar{\zeta}, K_2) & := \frac{1}{2\pi} \int_0^{2\pi} f_{\vartheta}^{\vartheta}(\vartheta, \bar{\eta}^{\vartheta}, \bar{\zeta}, K_2) \, d\vartheta \\
 f_{\zeta, \text{avg}}^{\vartheta}(\bar{\eta}^{\vartheta}, \bar{\zeta}, K_2) & := \frac{1}{2\pi} \int_0^{2\pi} f_{\zeta}^{\vartheta}(\vartheta, \bar{\eta}^{\vartheta}, \bar{\zeta}, K_2) \, d\vartheta.
\end{align*}
\]

According to the results in [95], the averaged system (4.10) yields the same information on the local behavior of the trajectory of the closed loop system (4.8) as its Poincarè normal form. While averaged models of open-loop Galerkin systems have been widely used in flow control application (see [80, 81]), the focus here is on analyzing the dependence of averaged closed-loop Galerkin systems on the control. As a matter of fact, notice that the structure of (4.10) depends implicitly on the choice of the input \( \bar{\gamma}(\eta, K_2) \). As mentioned earlier in the section, the goal is to single out the effect of those terms in the control that affect directly the coefficient of \( \rho \) in the normal form of the closed-loop system reduced to its center manifold. It is easy to verify that any phase-independent control, i.e. a control law of the form \( \bar{\gamma} = \bar{\gamma}(\rho, K_2) \) which depends only on the amplitude \( \rho = (\bar{\eta}_1^2 + \bar{\eta}_2^2)^{1/2} \) of the oscillating components of the original system, results in the following expression for the averaged dynamics (4.10)
\[
\begin{align*}
 \dot{\bar{\eta}}^{\vartheta} &= (F_1 + \Omega_c) \bar{\eta}^{\vartheta} + \Phi_1(\bar{\eta}^{\vartheta}, \bar{\zeta}) + g_1(\bar{\eta}^{\vartheta}) \bar{\gamma}(\bar{\rho}^{\vartheta}, K_2) \\
 \dot{\bar{\zeta}} &= F_2 \bar{\zeta} + \Phi_2(\bar{\eta}^{\vartheta}, \bar{\zeta}) + [B_2 + g_2(\bar{\zeta})] \bar{\gamma}(\bar{\rho}^{\vartheta}, K_2),
\end{align*}
\]

(4.11)

where \( \bar{\rho}^{\vartheta} := (\bar{\eta}_1^{\vartheta^2} + \bar{\eta}_2^{\vartheta^2})^{1/2} \) and

\[
\begin{align*}
 \Phi_1(\bar{\eta}^{\vartheta}, \bar{\zeta}) &= \begin{bmatrix} \phi_{11} \bar{\zeta}_1 + \phi_{12} \bar{\zeta}_2 & \phi_{21} \bar{\zeta}_1 + \phi_{22} \bar{\zeta}_2 \\ -\phi_{21} \bar{\zeta}_1 - \phi_{22} \bar{\zeta}_2 & \phi_{11} \bar{\zeta}_1 + \phi_{12} \bar{\zeta}_2 \end{bmatrix} \begin{bmatrix} \bar{\eta}_1^{\vartheta} \\ \bar{\eta}_2^{\vartheta} \end{bmatrix}, \\
 \Phi_2(\bar{\eta}^{\vartheta}, \bar{\zeta}) &= \begin{bmatrix} \phi_{31} (\bar{\eta}_1^{\vartheta^2} + \bar{\eta}_2^{\vartheta^2}) + \phi_{32} \bar{\zeta}_1^2 + \phi_{33} \bar{\zeta}_2^2 + \phi_{34} \bar{\zeta}_1 \bar{\zeta}_2 \\ \phi_{41} (\bar{\eta}_1^{\vartheta^2} + \bar{\eta}_2^{\vartheta^2}) + \phi_{42} \bar{\zeta}_1^2 + \phi_{43} \bar{\zeta}_2^2 + \phi_{44} \bar{\zeta}_1 \bar{\zeta}_2 \end{bmatrix}, \\
 B_2 &= \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}, \quad g_1(\bar{\eta}^{\vartheta}) = \begin{bmatrix} g_{11} & g_{12} \\ -g_{12} & g_{11} \end{bmatrix} \begin{bmatrix} \bar{\eta}_1^{\vartheta} \\ \bar{\eta}_2^{\vartheta} \end{bmatrix}, \quad g_2(\bar{\zeta}) = \begin{bmatrix} g_{31} & g_{32} \\ g_{41} & g_{42} \end{bmatrix} \begin{bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{bmatrix}.
\end{align*}
\]
The values of the parameters of (4.11) in terms of the parameters of the original Galerkin system are given in Appendix C. Notice that knowledge of $\omega_c$ is not required to compute $f_{\eta,\text{avg}}^\theta$ and $f_{\zeta,\text{avg}}^\theta$. This is desirable, as the determination of the actual frequency of oscillation of the limit cycle is often a difficult and tedious task. However, the averaged system under phase-independent control, i.e. system (4.11), still depends on $\omega_c$ by way of the matrix $\Omega_c$. To remove the dependence on $\omega_c$, we revert the transformation (4.9) on the state of the averaged system and define the change of coordinates

$$\bar{\eta} = R^T(\omega_c t) \hat{\eta}^\theta.$$ 

This yields the averaged model in the new coordinates

$$\dot{\bar{\eta}} = F_1 \bar{\eta} + R^T(\omega_c t) f_{\eta,\text{avg}}^\theta (R(\omega_c t) \bar{\eta}, \bar{\zeta}, K_2)$$

$$\dot{\bar{\zeta}} = F_2 \bar{\zeta} + f_{\zeta,\text{avg}}^\theta (R(\omega_c t) \bar{\eta}, \bar{\zeta}, K_2).$$

Finally, it is easy to see that the skew-symmetric structure of $\Phi_1(\bar{\eta}^\theta, \bar{\zeta})$ and $g_1(\bar{\eta}^\theta)$, and the fact that $\Phi_2(\bar{\eta}^\theta, \bar{\zeta})$ depends only on $\bar{\rho}^\theta$ imply that

$$R^T(\omega_c t) f_{\eta,\text{avg}}^\theta (R(\omega_c t) \bar{\eta}, \bar{\zeta}, K_2) = f_{\eta,\text{avg}}^\theta (\bar{\eta}, \bar{\zeta}, K_2)$$

$$f_{\zeta,\text{avg}}^\theta (R(\omega_c t) \bar{\eta}, \bar{\zeta}, K_2) = f_{\zeta,\text{avg}}^\theta (\bar{\eta}, \bar{\zeta}, K_2),$$

and notice that $\bar{\rho} := (\bar{\eta}_1^2 + \bar{\eta}_2^2)^{1/2} = \bar{\rho}^\theta$. As a result, the final expression of the averaged model obtained from application of a phase-independent control is given as

$$\dot{\bar{\eta}} = F_1 \bar{\eta} + \Phi_1(\bar{\eta}, \bar{\zeta}) + g_1(\bar{\eta}) \bar{u}(\bar{\rho}, K_2)$$

$$\dot{\bar{\zeta}} = F_2 \bar{\zeta} + \Phi_2(\bar{\eta}, \bar{\zeta}) + [B_2 + g_2(\bar{\zeta})] \bar{u}(\bar{\rho}, K_2),$$

(4.12)

which does not depend explicitly on $\omega_c$, as desired.
4.3.2 Phase-Invariant Control

It is important to notice that while any nonlinear feedback law that depends only on \( \rho \) results in an averaged system of the form (4.12), the same expression of the averaged closed-loop system might as well be obtained by a different control law which depends explicitly on the phase \( \theta = \tan^{-1}(\eta_2/\eta_1) \). For example, the phase-independent control \( \bar{\gamma} = K_2 \rho^2 \) and the phase-dependent control \( \bar{\gamma} = \frac{\pi}{2} K_2 \rho^2 \sin(\frac{1}{2} \theta) \) yield the same averaged closed-loop dynamics. This motivates the following definition:

**Definition 4.3.1.** The parameterized families of control laws \( \gamma = \bar{\gamma}_1(\eta, K_2) \) and \( u = \bar{\gamma}_2(\eta, K_2) \) are said to be equivalent with respect to averaging if they yield the same expression for the averaged closed-loop system (4.12).

This allows us to introduce formally the concept of phase-invariant control as follows:

**Definition 4.3.2.** A control law \( \gamma = \bar{\gamma}(\eta, K_2) \) is said to be phase-invariant if it is equivalent with respect to averaging to a phase-independent control.

It is obvious that, as long as the analysis is performed on the averaged system (or, equivalently, using the Poincarè normal form), a control \( u = \bar{\gamma}(\eta, K_2) \) is a representative of the entire equivalence class characterized by Definition 4.3.1. The importance of the concept of equivalence with respect to averaging lies in the fact that only the simplest representative of the equivalence class need to be used for the computation of the normal form. The possibility of realizing a phase-independent control by means of a phase-dependent one by way of phase-invariance is instrumental in almost all applications encountered in flow control, where the control policy must be implemented...
using a zero net-mass or zero net-moment actuator (such as a synthetic jet-like actuator) which necessitates an oscillating component. For example, the control \( \gamma = K_2 \rho^2 \), which can not be physically realized with a synthetic jet-like actuator, can be replaced in the actual implementation by the equivalent control \( \gamma = \frac{\pi}{2} K_2 \rho^2 \sin(\frac{1}{2} \theta) \), while the former is used in the analysis. It must be kept in mind however, that the two control laws yield the same results only in a qualitative sense as far as the original system (4.8) is concerned.

### 4.3.3 Center Manifold Reduction

In what follows, we restrict our attention to the class of phase-invariant nonlinear feedback laws. From the previous discussion, we only need to consider phase-independent feedback laws in the analysis. In order to obtain a reduced-order dynamic model for (4.12), we first change coordinates polar coordinates representation, obtaining the system

\[
\begin{align*}
\dot{\rho} &= [\sigma + \phi_{11} \zeta_1 + \phi_{12} \zeta_2 + g_{11} \bar{\gamma}(\rho, K_2)] \rho \\
\dot{\theta} &= \omega - \phi_{21} \zeta_1 - \phi_{22} \zeta_2 - g_{12} \bar{\gamma}(\rho, K_2) \\
\dot{\zeta}_1 &= -\lambda_1 \zeta_1 + \phi_{31} \rho^2 + \phi_{32} \zeta_1^2 + \phi_{33} \zeta_2^2 + \phi_{34} \zeta_1 \zeta_2 + [b_{21} + g_{31} \zeta_1 + g_{32} \zeta_2] \bar{\gamma}(\rho, K_2) \\
\dot{\zeta}_2 &= -\lambda_2 \zeta_2 + \phi_{41} \rho^2 + \phi_{42} \zeta_1^2 + \phi_{43} \zeta_2^2 + \phi_{44} \zeta_1 \zeta_2 + [b_{22} + g_{41} \zeta_1 + g_{42} \zeta_2] \bar{\gamma}(\rho, K_2)
\end{align*}
\]

where the original notation \((\rho, \theta, \zeta)\) has been used for the sake of simplicity. Note that the \(\theta\)-subsystem is in cascade with the rest of the dynamics. Since by assumption the control laws employed are at least continuously differentiable functions of \(\eta\) which possess a vanishing Jacobian, we are bound to consider phase-independent control
laws with Taylor expansion about $\rho = 0$ as
\[
\bar{\gamma}(\rho, K_2) = K_{22}\rho^2 + K_{23}\rho^3 + \text{h.o.t.}
\] (4.14)

Since $[\partial \bar{u}/\partial \rho]_{\rho=0} = 0$, the coefficient of $\rho$ in the right hand side of the first equation in (4.13) does not depend on the control. By treating $\sigma$ as a state with trivial dynamics, system (4.13) is written as
\[
\begin{bmatrix}
\dot{\sigma} \\
\dot{\rho}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\sigma \\
\rho
\end{bmatrix} + \begin{bmatrix}
0 \\
\varphi_{11}(\sigma, \rho, \zeta, K_2)
\end{bmatrix}
\]
\[
\dot{\theta} = \omega + \varphi_{12}(\rho, \zeta, K_2)
\]
\[
\begin{bmatrix}
\dot{\zeta}_1 \\
\dot{\zeta}_2
\end{bmatrix} = \begin{bmatrix}
-\lambda_1 & 0 \\
0 & -\lambda_2
\end{bmatrix} \begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix} + \begin{bmatrix}
\varphi_{21}(\rho, \zeta, K_2) \\
\varphi_{22}(\rho, \zeta, K_2)
\end{bmatrix},
\] (4.15)

where $\varphi_{11}()$ and $\varphi_2() = [\varphi_{21}(\cdot) \varphi_{22}(\cdot)]^T$ collect the nonlinear terms in the state variables $(\sigma, \rho, \theta, \zeta) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{S} \times \mathbb{R}^2$. For system (4.15), there exists a parameterized family of mappings $\zeta = \bar{\zeta}(\sigma, \rho, K_2)$, satisfying
\[
\bar{\zeta}_i(0, 0, K_2) = 0, \quad \frac{\partial \bar{\zeta}_i}{\partial \rho}(0, 0, K_2) = 0, \quad i = 1, 2
\] (4.16)
for all $K_2 \in \mathbb{R}^{p_2}$, and
\[
\frac{\partial \bar{\zeta}_i}{\partial \rho} \varphi_{11}(\sigma, \rho, \bar{\zeta}(\sigma, \rho, K_2), K_2) = -\lambda_i \bar{\zeta}_i(\sigma, \rho, K_2) + \varphi_{2i}(\rho, \bar{\zeta}(\sigma, \rho, K_2), K_2), \quad i = 1, 2
\] (4.17)
which locally describe a center manifold to which the solutions of the system (4.15) are locally exponentially attracted [89]. Since the partial differential equations in (4.17) can not be solved analytically, a suitable approximation must be employed. It was found that a more accurate approximation of the solution of (4.17) than the usual polynomial one in the variables $({\sigma, \rho})$ (with coefficients dependent on the control parameter vector $K_2$) is given by an expansion of the form
\[
\bar{\zeta}_i(\sigma, \rho, K_2) = \sum_{j=1}^\ell c_{ij}(\sigma, K_2)\rho^j + \mathcal{O}(\rho^{\ell+1}), \quad i = 1, 2
\] (4.18)

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where \( c_{ij}(\cdot), i = 1, 2, j = 1, \ldots, \ell, \) are functions which depend on the choice of the control. The expansion automatically fulfills (4.16), while solving (4.17) term by term using (4.14) yields the coefficients \( c_{ij}(\cdot) \). When restricted to the center manifold, the dynamics of (4.15) are described by equations similar to (4.6), that is,

\[
\dot{\rho} = \sigma \rho + \varphi_{11}(\rho, K_2) + \text{h.o.t.} \\
\dot{\theta} = \omega + \varphi_{12}(\rho, K_2) + \text{h.o.t.}
\]

with \( \left[ \frac{\partial \varphi_{11}}{\partial \rho} \right]_{\rho=0} = 0 \), and \( \left[ \frac{\partial \varphi_{12}}{\partial \rho} \right]_{\rho=0} = 0 \). Note that, as expected, the term linear in \( \rho \) does not depend on the control.

### 4.3.4 Analysis of the Reduced System

A meaningful case which is simple enough to be presented here, yet informative, is given by the class of phase-invariant controls corresponding to \( \bar{\gamma} = K \rho^2 \), with control gain \( K \in \mathbb{R} \). For this selection, an expansion of order \( \ell = 4 \) yields rational functions
\( c_{ij}(\cdot) \) whose expression is given by

\[
\begin{align*}
    c_{11} &= 0, \quad c_{21} = 0, \quad c_{13} = 0, \quad c_{23} = 0 \\
    c_{12} &= \frac{2 \sigma K b_{21} + 2 \sigma \phi_{31} + \lambda_2 K b_{21} + \lambda_2 \phi_{31}}{4 \sigma^2 + 2 \sigma \lambda_2 + 2 \lambda_1 \sigma + \lambda_1 \lambda_2}, \quad c_{22} = \frac{2 \sigma K b_{22} + 2 \sigma \phi_{41} + \lambda_1 K b_{22} + \lambda_1 \phi_{41}}{4 \sigma^2 + 2 \sigma \lambda_2 + 2 \lambda_1 \sigma + \lambda_1 \lambda_2} \\
    c_{14} &= \left[ \lambda_2 c_{22} g_{32} K + \lambda_2 c_{12} c_{12} \phi_{34} - 2 \lambda_2 c_{12} \phi_{12} c_{22} - 2 \lambda_2 c_{12} g_{11} K \right. \\
    &\quad + 4 \sigma c_{22} c_{12} \phi_{34} + 4 \sigma c_{22} g_{32} K + 4 \sigma c_{12} g_{31} K - 8 \sigma c_{12} \phi_{12} c_{22} - 8 \sigma c_{12} g_{11} K \\
    &\quad - 8 \sigma \phi_{11} c_{12}^2 + 4 \sigma c_{12}^2 \phi_{32} + 4 \sigma c_{22}^2 \phi_{33} - 2 \lambda_2 \phi_{11} c_{12}^2 + \lambda_2 c_{12}^2 \phi_{32} + \lambda_2 c_{22}^2 \phi_{33} \big] \times \\
    &\quad \left[ 4 \sigma \lambda_2 + 4 \lambda_1 \sigma + \lambda_1 \lambda_2 + 16 \sigma^2 \right]^{-1} \\
    c_{24} &= \left[ \lambda_1 c_{22} c_{12} \phi_{44} - 2 \lambda_1 c_{22} g_{11} K - 2 \lambda_1 c_{22} \phi_{11} c_{12} + 4 \sigma c_{12} g_{41} K + 4 \sigma c_{22} g_{42} K \right. \\
    &\quad + 4 \sigma c_{22} c_{12} \phi_{44} - 8 \sigma c_{22} g_{11} K - 8 \sigma c_{22} \phi_{11} c_{12} + \lambda_1 c_{12} g_{41} K + \lambda_1 c_{22} g_{42} K \\
    &\quad + 4 \sigma c_{22}^2 \phi_{43} + 4 \sigma c_{12}^2 \phi_{42} + \lambda_1 c_{12}^2 \phi_{43} + \lambda_1 c_{22}^2 \phi_{42} - 8 \sigma \phi_{12} c_{22}^2 - 2 \lambda_1 \phi_{12} c_{22}^2 \big] \times \\
    &\quad \left[ 4 \sigma \lambda_2 + 4 \lambda_1 \sigma + \lambda_1 \lambda_2 + 16 \sigma^2 \right]^{-1}.
\end{align*}
\]

Collecting terms, one obtains

\[
\begin{align*}
    c_{12} &= \mu_{121} + \mu_{122} K, \quad c_{14} = \mu_{141} + \mu_{142} K + \mu_{143} K^2 \\
    c_{22} &= \mu_{221} + \mu_{222} K, \quad c_{24} = \mu_{241} + \mu_{242} K + \mu_{243} K^2
\end{align*}
\]

(4.20)

where the coefficients \( \mu_{ijk} \) depend only on the parameters of the plant model. Substituting (4.20) into (4.18), and then into the first two equations in (4.13) yield the reduced system

\[
\begin{align*}
    \dot{\rho} &= \sigma \rho + d_1(K) \rho^3 + d_2(K) \rho^5 + \text{h.o.t} \\
    \dot{\theta} &= \omega - d_3(K) \rho^2 - d_4(K) \rho^4 + \text{h.o.t},
\end{align*}
\]

(4.21)
where
\[
\begin{align*}
d_1(K) &= \phi_{11}(\mu_{121} + \mu_{122}K) + \phi_{12}(\mu_{221} + \mu_{222}K) + g_{11}K \\
d_2(K) &= \phi_{11}(\mu_{141} + \mu_{142}K + \mu_{143}K^2) + \phi_{12}(\mu_{241} + \mu_{242}K + \mu_{243}K^2) \\
d_3(K) &= \phi_{21}(\mu_{121} + \mu_{122}K) + \phi_{22}(\mu_{221} + \mu_{222}K) + g_{11}K \\
d_4(K) &= \phi_{21}(\mu_{141} + \mu_{142}K + \mu_{143}K^2) + \phi_{22}(\mu_{241} + \mu_{242}K + \mu_{243}K^2).
\end{align*}
\] (4.22)

The existence and the stability type of limit cycles for the averaged closed-loop system (4.10) are related to the existence of positive roots of the polynomial \( p(\rho) = \sigma + d_1\rho^2 + d_2\rho^4 \). The roots of the polynomial in question are easily computed as
\[
\rho_{1,2}^* = \sqrt{-\frac{-d_1(K) \pm \sqrt{d_1^2(K) - 4d_2(K)\sigma}}{2d_2(K)}}.
\] (4.23)

The solutions in (4.23) will now be analyzed based on the signs of \( d_1 \) and \( d_2 \): First recall that \( \sigma > 0 \). Next, note that it is required to have \( d_1^2 - 4\sigma d_2 > 0 \) to have any real roots. Assuming that this is the case, the following cases may occur:

(a) If it is the case that \( d_2 > 0 \), in order to have real roots for (4.23) it is required that \( -d_1 \pm \sqrt{d_1^2 - 4\sigma d_2} > 0 \). If \( d_1 > 0 \), then \( -d_1 - \sqrt{d_1^2 - 4\sigma d_2} > 0 \) is impossible, and to have \( -d_1 + \sqrt{d_1^2 - 4\sigma d_2} > 0 \) it is required that \( d_2 < 0 \), which is also not true for this particular case; hence the case \( d_2 < 0 \) and \( d_1 > 0 \) gives no real solutions. If \( d_1 < 0 \), then \( -d_1 + \sqrt{d_1^2 - 4\sigma d_2} > 0 \) holds true, and \( -d_1 - \sqrt{d_1^2 - 4\sigma d_2} > 0 \) is also true, since \( d_2 > 0 \) for this case. Hence, for the case \( d_2 < 0 \) and \( d_1 < 0 \) there are two positive solutions.

(b) If it is the case that \( d_2 < 0 \), in order to have real roots for (4.23) it is required that \( -d_1 \pm \sqrt{d_1^2 - 4\sigma d_2} < 0 \). If \( d_1 > 0 \), then \( -d_1 + \sqrt{d_1^2 - 4\sigma d_2} < 0 \) cannot hold, as \( d_2 > 0 \). However, \( -d_1 - \sqrt{d_1^2 - 4\sigma d_2} < 0 \) holds true, so when \( d_2 < 0 \)
and $d_1 > 0$ there is a single real solution. If $d_1 < 0$, then $-d_1 + \sqrt{d_1^2 - 4\sigma d_2} < 0$ is false, but $-d_1 - \sqrt{d_1^2 - 4\sigma d_2} < 0$ holds true for this case, so there is also one real solution. Combining the two cases, it is concluded that there exists a single solution for $d_2 < 0$.

Keeping in mind that the equilibrium solution $\rho = 0$ is unstable, case (a) is readily seen to correspond to the existence of a pair of limit cycles, of which the one with lower amplitude is stable, and the one with higher amplitude is unstable. Case (b), on the other hand, corresponds to the existence of one stable limit cycle. Note that, by design, the origin $\rho = 0$ is always unstable, hence the stable oscillation corresponding to either one of the two cases can not be suppressed by feedback. In both cases, the amplitude of the limit cycle depends nonlinearly on the controller gain, whereas the variation with respect to $K$ can be analyzed graphically.

### 4.4 Application to Cavity Flow Control

As mentioned in Chapter 1, cavity flow is characterized by a strong self-sustained resonance produced by a natural feedback mechanism similar to that occurring in impinging or screeching jet. Shear layer structures impacting the cavity trailing edge scatter acoustic waves that propagate upstream and reach the shear layer receptivity region, where they tune and enhance the development and growth of shear layer structures. The resulting acoustic fluctuations can be very intense and are known to cause, among other problems, store damage and airframe structural fatigue in weapons bay applications. To suppress or reduce the pressure fluctuations inside the cavity, feedback control is applied to the flow by using a synthetic jet-like actuator, which is typically an acoustic actuator located at the cavity trailing edge.
Figure 1.1. The modeling process for the cavity flow was described in Section 3.4. To summarize, in deriving a control-oriented model for the actuated flow dynamics, one starts with the governing Navier-Stokes equations, obtains a set of POD modes from snapshots of the flow, applies an input separation method to get the control as an explicit term and finally applies Galerkin projection to obtain a ODEs of time coefficients \( \{a_i(t)\}_{i=0}^n \) of the form

\[
\dot{a}_i(t) = \frac{1}{Re} \sum_{j=0}^{n} l_{ij} a_j(t) + \sum_{j,k=0}^{n} q_{ijk} a_j a_k(t) + r_i \gamma(t) + \sum_{j=0}^{n} s_{ij} a_j(t) \gamma(t), \quad i = 1, \ldots, n,
\]

where \( \mathbf{u}(x,t) \) is the velocity field, \( p(x,t) \) is pressure, \( Re \) is the Reynolds number, \((x,t)\) is the spatial and temporal variables, \( a_0(t) \equiv 1 \), and \( \gamma(t) \) is the control. Shifting the origin of the coordinate system of (4.24) to the equilibrium point \( a^* \) satisfying

\[
a_0^* = 1, \quad \frac{1}{Re} \sum_{j=0}^{n} l_{ij} a_j^* + \sum_{j,k=0}^{n} q_{ijk} a_j^* a_k^* = 0, \quad i = 1, \ldots, n
\]

the constant terms are removed from the equations, and one obtains a system of the same form (4.1) that was analyzed in this chapter. As a case study, we consider the model of the cavity flow dynamics for the OSU GDTL experimental setup (Appendix A), which was made available to us courtesy the Prof. Mo Samimy, and all numerical data was provided by Edgar Caraballo. Specifically, the Galerkin system considered in the case study is a 4-dimensional model of the cavity flow with free-stream flow velocity \( U_\infty = 100 \ [\text{m/s}] \), corresponding to a Mach number \( M = 0.3 \). A non-dimensional time unit \( \tau = t U_\infty / D \) is employed in the simulation, where \( D = 12.7 \times 10^{-3} \ [\text{m}] \) is the depth of the cavity. Here, we investigate the effect of nonlinear feedback on the Galerkin model of the cavity flow, restricting ourselves to class of phase-invariant control laws corresponding to \( \gamma = K \rho^2 \), and applying the
analysis developed in Section 4.3. For the considered model, the values of \( d_1(K) \) and \( d_2(K) \) in (4.22) have been computed as

\[
d_1(K) = -5.0579 \times 10^{-4} K - 0.0876
\]
\[
d_2(K) = -7.4279 \times 10^{-8} K^2 - 1.0121 \times 10^{-4} K - 0.0149,
\]

from which the discriminant reads as

\[
d_1^2 - 4\sigma d_2 = 3.1053 \times 10^{-7} K^2 + 1.6316 \times 10^{-4} K + 0.0187.
\]

Analyzing the identities above, it is readily seen that \( d_1 \) is positive if \( K < -173.2063 \) and negative for \( K > -173.2063 \). On the other hand, \( d_2 \) is positive for \( K < -1194.3 \) or \( K > -168.2720 \), and it is negative for \( -1194.3 < K < -168.2720 \). Finally,
\[d_1^2 - 4\sigma d_2\] is positive for \(K < -357.0527\) or \(K > -168.3783\), and it is negative for \(-357.0527 < K < -168.3783\). Based on analysis developed in Section 4.3, the conclusion is the following:

1. No positive real solutions exist for \(K \in (-1194.3, -168.3783)\);

2. One positive real solution exists for \(K \in (-\infty, -1194.3) \cup (-168.2720, \infty)\);

3. Two positive real solutions exists for \(K \in (-168.3783, -168.2720)\).

The variation of the oscillation amplitude in steady-state \(\rho^*\) versus the controller gain \(K\) in the range \(K \in [-1000, 1000]\) is shown in Figure 4.1. The graph is computed on the basis of the center manifold analysis. In this range, one can observe that \(\rho^*\) decreases with \(K > 0\), while it increases with respect to the baseline (uncontrolled) oscillation when \(K < 0\). In the latter case, the amplitude of the oscillation diverge to infinity as \(K\) approaches the lower bound \(-168\), after which the analysis fails to predict the existence of a stable oscillation.

To illustrate the validity of the analysis, and to check how well the averaged reduced system approximates the original Galerkin system, a comparative simulation study were performed for systems (4.1) and (4.19). Figure 4.2 shows the results for the original and averaged reduced system in open loop. Both systems eventually sustain an oscillation of amplitude \(\rho^* \approx 1\), which is consistent with what is predicted by the center manifold analysis in Figure 4.1. For the system in closed loop with the feedback control \(\gamma = K \rho^2\), Figure 4.3 shows the results when the gain is selected as \(K = 500\). Again, for both cases a stable oscillation with amplitude reduced to \(\rho^* \approx 0.65\), consistent with the graph in Figure 4.1, is observed. Note that the reduced averaged system overestimates the rate of convergence to the steady state. When the
gain is chosen as $K = -100$, the original and averaged systems behave in agreement with the analysis, as an oscillation with amplitude increased to about $\rho^* \approx 1.5$ is observed, as seen in Figure 4.4. Finally, Figure 4.5 shows the simulation results for the original and averaged system when $K = -200$. For this value of the controller gain, the trajectories of both systems do not converge to a stable oscillation, and instability occurs. This confirms the validity of the center manifold analysis, as $K = -200$ is in the range $(-168.3783, -168.2720)$ for which it was concluded that no positive real roots $\rho_{1,2}^*$ in (4.23) exist. Note that the onset of instability occurs much earlier for the averaged system with respect to the original one, consistently with the remark made previously on the rate of convergence. In addition, simulations were also performed using the control law $\gamma = \frac{\pi}{2} K \rho^2 \sin\left(\frac{1}{2}\theta\right)$. Recall from the discussion in Section 4.3.2 that this control can be implemented by a zero-net mass actuator, and yields the same averaged system as $\gamma = K \rho^2$; therefore, simulation results for the averaged system are identical to those given by the bottom plot of Figure 4.3 through Figure 4.5. On the other hand, it is interesting to compare the effect of the phase-invariant control on the original Galerkin system. Since the control in this case depends on the phase explicitly, a somewhat noticeable difference from the previous results is to be expected, as the impact of averaging is more substantial. However, the results of the simulations indicate that a qualitatively similar behavior to the phase-independent control is achieved. The case $K = 500$ is shown in Figure 4.6. Comparing this with the results in Figure 4.3, it is readily seen that a stable oscillation occurs, with a reduction of the steady-state amplitude that is larger than the one predicted by averaging analysis. This may indicate that the contribution of the oscillatory term is beneficial, but this—as expected—can not be quantified in an averaged-based
analysis. As far as negative values for $K$ are concerned, simulations show that still a stable limit cycle with larger amplitude than the uncontrolled case is observed; as the magnitude of $K$ is increased to $K = -200$, the stability of the periodic orbit is lost and the trajectories of the system eventually blow up. This behavior is quite similar to that observed in the top plots of Figure 4.5, and it is therefore omitted.

4.5 Summary and Comments

In this chapter, nonlinear analysis and control aspects for the cavity flow problem were investigated, within the general context of controlled Galerkin models which exhibit an oscillation caused by a stable limit cycle for zero input. All analysis and design were subjected to the constraint that system oscillations need to be preserved by the control, which assures that the physical meaningfulness and validity of the model are kept, and that the resulting controllers are practically implementable. A basic analysis on the effect of linear control on this model, performed by using Poincaré normal form, was first recalled. It was observed that the amplitude of oscillation of the closed loop system depends both on terms that are linear in control as well as those that depend on a center manifold. It was argued that, while the latter is often disregarded for linear control designs, one can still control the oscillation by exploiting the effect of the center manifold related terms. Towards this goal, a simplification of the system model was performed using a time varying periodic change of coordinates and averaging. It was shown that for certain types of control, this procedure yields a much simpler system with much more apparent structure than the original. The averaged system was further simplified using center manifold theory, after which it was possible to obtain conditions governing the number and stability type of the limit
cycles of the closed loop system, and to derive analytical expressions for the amplitude of oscillation. The results obtained were tested and verified on the cavity flow control problem.
Figure 4.2: Trajectories of the open loop Galerkin system. Top: original model. Bottom: Averaged model.
Figure 4.3: Trajectories of the closed loop Galerkin system with $\gamma = K\rho^2$, $K = 500$. Top: original model. Bottom: Averaged model.
Figure 4.4: Trajectories of the closed loop Galerkin system with $\gamma = K \rho^2$, $K = -100$. Top: original model. Bottom: Averaged model.
Figure 4.5: Trajectories of the closed loop Galerkin system with $\gamma = K\rho^2$, $K = -200$. Top: original model. Bottom: Averaged model.
Figure 4.6: Trajectories of the closed loop Galerkin system with $\gamma = \frac{\pi}{2} K \rho^2 \sin\left(\frac{1}{2} \theta\right)$, $K = 500$. Original model.
CHAPTER 5

CASE STUDY: MODELING AND CONTROL OF CONVECTIVE FLOW OVER AN OBSTACLE

5.1 Introduction

In this chapter we consider a case study of a boundary flow control problem of convective flow over an obstacle governed by the Burgers equations. Flows over obstacles are important as they arise in many practical configurations. Burgers equations are also significant as they represent a simpler form of the more general Navier-Stokes equations. The goal of this chapter is to apply the reduced order modeling methods developed in Chapter 2 as well as the control design methods developed in Chapter 4 to this sample problem, and guide the reader systematically through the steps involved from start to the end. The results are then verified using full order CFD simulations of the governing Burgers equations, and are then compared to a standard LQR design based on linearization.

We are grateful to Dr. Chris Camphouse for making this chapter possible by proving us with the program code for the CFD simulations for the case study problem considered.
5.2 Problem Description

The case study considered in this chapter is a distributed parameter system which models convective flow over an obstacle [106, 107]. A reduced-order model of this system will be developed for feedback control design. Let \( \Omega_1 \subseteq \mathbb{R}^2 \) be the rectangle given by \((a,b] \times (c,d)\). Let \( \Omega_2 \subseteq \Omega_1 \) be the rectangular domain given by \([a_1,a_2] \times [b_1,b_2]\) where \(a < a_1 < a_2 < b\) and \(c < b_1 < b_2 < d\). The problem domain, \( \Omega \), is given by \( \Omega = \Omega_1 \setminus \Omega_2 \) where \( \Omega_2 \) is the obstacle. Dirichlet boundary controls are located on the obstacle bottom and top, denoted by \( \Gamma_{\text{bottom}} \) and \( \Gamma_{\text{top}} \), respectively. A schematic representation of the problem geometry is given in Figure 5.1. The dynamics of the system are described by the two-dimensional Burgers equation

\[
\frac{\partial}{\partial t} u(x, y, t) + \nabla \cdot F(u) = \frac{1}{Re} \Delta u(x, y, t) \tag{5.1}
\]

where \( t > 0 \) is the temporal variable and \((x, y)\) are the coordinates of a point in \( \Omega \). In (5.1), \( F(u) \) has the form

\[
F(u) = \begin{bmatrix}
\frac{k_1 u^2(x, y, t)}{2} \\
\frac{k_2 u(x, y, t)^2}{2}
\end{bmatrix}^T, \tag{5.2}
\]
where $k_1, k_2$ are nonnegative constants. This equation has a convective nonlinearity like the one found in the Navier-Stokes momentum equation modeling fluid flow [108]. The quantity $Re$, a nonnegative constant, is analogous to the Reynolds number in the Navier-Stokes momentum equation. We will also write (5.1) in compact form as

$$\dot{u} = X(u)$$

(5.3)

where $X(u) = -\nabla \cdot F(u) + Re^{-1}\Delta u$. For simplicity, boundary controls are assumed to be separable. With this assumption, we specify conditions on the obstacle bottom and top of the form

$$u(x, y, t) = \gamma_{\text{bottom}}(t)\Psi_B(x) \text{ for } (x, y) \in \Gamma_{\text{bottom}},$$

(5.4)

$$u(x, y, t) = \gamma_{\text{top}}(t)\Psi_T(x) \text{ for } (x, y) \in \Gamma_{\text{top}}.$$  

(5.5)

where $\gamma_{\text{bottom}}(t)$ and $\gamma_{\text{top}}(t)$ are the control inputs on the bottom and top of the obstacle, respectively. The profile functions $\Psi_B(x)$ and $\Psi_T(x)$ describe the spatial influence of the controls on the boundary. A parabolic inflow condition is specified as follows

$$u(x, y, t) = f(y) \text{ for } (x, y) \in \Gamma_{\text{in}}.$$ 

(5.6)

At the outflow, a Neumann condition is specified according to the relationship

$$\frac{\partial}{\partial x} u(x, y, t) = 0 \text{ for } (x, y) \in \Gamma_{\text{out}}.$$ 

(5.7)

For notational convenience, the remaining boundary is denoted by $\Gamma_U$. We require that values of $u$ be fixed at zero along $\Gamma_U$ as time evolves. The resulting boundary condition is of the form

$$u(x, y, t) = 0 \text{ for } (x, y) \in \Gamma_U.$$ 

(5.8)
The initial condition of the system is given as

\[ u(0, x, y) = u_{\text{init}}(x, y) \in L^2(\Omega). \]  

(5.9)

Throughout the chapter, for numerical simulations a positive parabolic profile with unit maximum amplitude is specified for the inlet condition in boundary condition (5.6), as shown in Figure 5.2. In (5.2), we set \( k_1 = 1 \) and \( k_2 = 0 \) in order to obtain solutions that convect from left to right for the positive inlet. In addition, we set \( Re = 300 \). Parameters for the problem domain \( \Omega \) are specified as \( a = 0, \ b = 0.99, \ c = 0, \ d = 0.48, \ a_1 = 0.15, \ a_2 = 0.24, \ b_1 = 0.15, \ b_2 = 0.33 \) and the problem domain is discretized, resulting in a uniform grid with spatial step-size \( h = 0.015 \).
We utilize the finite difference scheme in [106, 107] to numerically solve the model problem in MATLAB.

5.3 Reduced Order Modeling

The theory for reduced order modeling derived Chapter 2 is used to obtain a reduced order model for the problem at hand. To summarize, one finds an augmented POD expansion of the form

\[ u = u_0 + \sum_{i=1}^{N} a_i \phi_i + \sum_{i=1}^{N_{\text{in}}} \gamma_i \psi_i. \]  

(5.10)

where the baseline modes \( \phi_i \) are determined from a standard POD expansion on the baseline flow (Section 2.3), and actuation modes \( \phi_i \) are to determined from the innovation flow field computed from forced snapshots (Section 2.4). Then one applies Galerkin projection, which involves substituting (5.10) into (5.3) and simplifying, to obtain a dynamical system that approximates the flow dynamics as

\[ \dot{a}_k = \langle X(r), \phi_k \rangle, \quad k = 1 \ldots N \]  

(5.11)

where \( r = u_0 + a_i \phi_i + \gamma_i \psi_i \) (Section 2.5). For the problem at hand, we will define the baseline conditions as the those where the control inputs as well as the inlet flow vanish. Note that he inlet flow is not an actual control input, however, to improve accuracy and of the model resulting, we treat it as a pseudo-input \( \gamma_{\text{inlet}} \) in such a way that the inlet flow is present for \( \gamma_{\text{inlet}} = 1 \) and vanishes for \( \gamma_{\text{inlet}} = 0 \). Figure 5.3 shows the snapshots of the solution of the system (5.1) under no forcing and no inlet. The the initial condition at \( t = 0 \) shown in the upper left plot and the lower right plot shows the steady state. Figure 5.4 shows the baseline modes built from these snapshots. The number of baseline modes are taken to be \( N = 4 \), which is a good compromise
Figure 5.3: Flow with no inlet and no excitation

Figure 5.4: Baseline POD modes
between the amount of energy captured and the complexity of the reduced-order model. Higher values of $N$ would result in higher-order models that capture more energy, but these would be less tractable from control design perspective. In addition to the baseline modes, actuation modes are obtained using actuated snapshots of the system. Figures 5.5, 5.7 and 5.8 show the system under actuation from the inlet, inlet and from the top of the obstacle, and inlet and from the bottom of the obstacle respectively. When present, the actuation signal applied from the top or bottom of the obstacle is shown in Figure 5.6. Snapshots from these flow conditions are used to build the actuation modes $\psi_{\text{inlet}}$, $\psi_{\text{top}}$ and $\psi_{\text{bottom}}$, which are shown in Figure 5.9. Following the procedure outlined in Section 5.3, one obtains the Galerkin system for
Figure 5.6: Excitation signal for modeling

Figure 5.7: Flow with inlet and top excitations
Figure 5.8: Flow with inlet and bottom excitations

Figure 5.9: Actuation modes
(5.1) having the form

\[ \dot{a}_i = C_i + L_{ik}a_k + L_{in,ik}\gamma_k + Q_{ijk}a_k a_j + Q_{ain,ijk}a_k \gamma_j + Q_{in,ijk}\gamma_k \gamma_j \]  

(5.12)

where \( \gamma = [\gamma_{\text{top}} \gamma_{\text{bottom}} \gamma_{\text{inlet}}]^T \) and

\[ C_i := \left\langle -k_1 u_0 \frac{\partial}{\partial x} u_0 - k_2 u_0 \frac{\partial}{\partial y} u_0 + \mu \left( \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial^2}{\partial y^2} u_0 \right), \phi_i \right\rangle \]

\[ Q_{ain,ijk} := \left\langle -k_1 \phi_j \frac{\partial}{\partial x} \psi_k - k_2 \phi_j \frac{\partial}{\partial y} \psi_k - k_1 \psi_j \frac{\partial}{\partial x} \phi_k - k_2 \psi_j \frac{\partial}{\partial y} \phi_k, \phi_i \right\rangle \]

\[ Q_{in,ijk} := \left\langle -k_1 \psi_j \frac{\partial}{\partial x} \psi_k - k_2 \psi_j \frac{\partial}{\partial y} \psi_k, \phi_i \right\rangle \]

\[ L_{in,ik} := \left\langle -k_1 u_0 \frac{\partial}{\partial x} \psi_k - k_2 u_0 \frac{\partial}{\partial y} \psi_k - k_1 \psi_k \frac{\partial}{\partial x} u_0 - k_2 \psi_k \frac{\partial}{\partial y} u_0 \right. \]

\[ \left. + \mu \left( \frac{\partial^2}{\partial y^2} \psi_k + \frac{\partial^2}{\partial x^2} \psi_k \right), \phi_i \right\rangle \]

\[ Q_{ijk} := \left\langle -k_1 \phi_j \frac{\partial}{\partial x} \phi_k - k_2 \phi_j \frac{\partial}{\partial y} \phi_k, \phi_i \right\rangle \]

\[ L_{ik} := \left\langle -k_1 \phi_k \frac{\partial}{\partial x} u_0 - k_2 \phi_k \frac{\partial}{\partial y} u_0 \right. \]

\[ \left. + \mu \left( \frac{\partial^2}{\partial y^2} \phi_k + \frac{\partial^2}{\partial x^2} \phi_k \right) - k_1 u_0 \frac{\partial}{\partial x} \phi_k - k_2 u_0 \frac{\partial}{\partial y} \phi_k, \phi_i \right\rangle \]

As a test for the Galerkin model’s ability to represent the flow, consider the excitation shown in Figure 5.10 applied from the top and bottom of the obstacle, which results in the flow whose snapshots are shown in Figure 5.11. The response of the Galerkin system as compared to the actual values of the modal coefficients obtained from projecting these snapshots onto the basis can be seen in Figure 5.12. One sees that the reduced order Galerkin model response captures the qualitative behavior of the actual values, as well as being close quantitatively. As discussed above, increasing \( N \) could result in an high order model with better quantitative performance but this would make the control design more involved and is therefore undesirable.
Figure 5.10: Excitation signal for test

Figure 5.11: Flow with test excitation signal applied to top and bottom
Figure 5.12: Response of Galerkin system (blue) vs. modal coefficients from direct projection (green)

5.4 Control Design

In this section a control law will be designed on the basis of the reduced order model derived in Section 5.3. We will employ two different approaches: 1) a design based on averaging and center manifold techniques (Section 4 and Kasnakoglu and Serrani [103, 104]) and 2) an LQR design based on the linearization of the model (Camphouse [106]). For the control objective, we specify a fixed 2D reference profile $u_d$ and design a controller to drive the trajectories of the systems to this profile. The profile reference is chosen to be a non-equilibrium state of the system in absence of an appropriate control action. Projecting $u_d$ onto the baseline POD basis yields the reference modal coefficients $a_d = (a_{d,i})_{i=1}^N$ for the reduced order model, that
is
\[ a_d = P_s u_d . \]

As the reference is selected so that there exists a \( \gamma_d \) such that
\[ C + La_d + L_{in} \gamma_d + Q(a_d, a_d) + Q_{in}(\gamma_d, \gamma_d) + Q_{ain}(a_d, \gamma_d) = 0 \]

one can define a shift of coordinates \( \tilde{a} = a - a_d, \tilde{\gamma} = \gamma - \gamma_d \) which yields
\[ \dot{\tilde{a}} = \tilde{L}\tilde{a} + Q(\tilde{a}, \tilde{a}) + \tilde{L}_{in} \tilde{\gamma} + Q_{in}(\tilde{\gamma}, \tilde{\gamma}) + Q_{ain}(\tilde{a}, \tilde{\gamma}) \quad (5.13) \]

where
\[ \tilde{L} = L + Q(\cdot, a_d) + Q(a_d, \cdot) + Q_{ain}(\cdot, \gamma_d) \]
\[ \tilde{L}_{in} = L_{in} + Q_{in}(\gamma_d, \cdot) + Q_{in}(\cdot, \gamma_d) + Q_{ain}(a_d, \cdot) . \]

For \( \tilde{\gamma} = 0 \), i.e., \( \gamma = \gamma_d \), system (5.13) has an equilibrium at \( \tilde{a} = 0 \), which corresponds the desired state. We also eliminate \( \gamma_{inlet} \) at this stage as it was a pseudo-input introduced for modeling purposes only. Under normal operation we have \( \gamma_{inlet} = 1 \) (i.e. \( \tilde{\gamma}_{inlet} = 0 \)) hence substitution into (5.13) yields a system of the same structure, with the corresponding dimensions of the matrices reduced by one. To keep the notation simple and avoid clutter, we will not introduce new names for these but it will be understood from this point on that when referring to (5.13) we mean \( \gamma = [\gamma_{top} \ gamma_{bottom}]^T \) and \( \tilde{\gamma} = [\tilde{\gamma}_{top} \tilde{\gamma}_{bottom}]^T \).

For the numerical simulation, one sees that the system (5.13) has an unstable equilibrium, as the eigenvalues of \( \tilde{L} \) turn out to be \( \{-1.4476 - 1.2619i, \ -1.4476 + 1.2619i, \ 0.3360, \ -0.1690\} \). The task is to stabilize the equilibrium with a domain of attraction large enough to include initial conditions of interest.
5.4.1 Control Design Based on Averaging/Center Manifold Reduction

The first approach to control design is based on phase-averaging and center manifold theory described in Chapter 4. The reason for this choice is that it is a novel technique that provides a means of reducing the Galerkin model (5.13) even further by looking at the behavior of an averaged system corresponding to (5.13) which exhibits certain symmetries and structures that allows one to eliminate dimension of the system (which will be \( \theta \) in the below derivations). Further simplification is achieved by looking at the behavior of the dynamics on the center manifold, which includes only one state (which will be \( \rho \) in the below derivations). The derivations in Chapter 4 were given from an oscillation suppression perspective and for a different control law than that considered here, however, the concepts presented still apply, as will be seen in what follows below.

As noted in Section 5.4, the eigenvalues of \( \tilde{L} \) are of the form \( \text{spec}(\tilde{L}) = \{-\sigma + j\omega, -\sigma - j\omega, \lambda_1, -\lambda_2\} \) where \( \sigma > 0, \omega > 0, \lambda_1, \lambda_2 > 0 \). Using a non-singular transformation, the system in (5.13) can be expressed in modal form as

\[
\begin{align*}
\dot{\eta} &= F_1\eta + \varphi_1(\eta, \zeta) + G_1\gamma + \xi_1(\eta, \zeta, \gamma) \\
\dot{\zeta} &= F_2\zeta + \varphi_2(\eta, \zeta) + G_2\gamma + \xi_2(\eta, \zeta, \gamma)
\end{align*}
\]

(5.14)

where

\[
\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \quad \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -\sigma & -\omega \\ \omega & -\sigma \end{bmatrix}, \quad F_2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}
\]

and \( G_1, G_2, \varphi_1, \xi_1, \varphi_2, \xi_2 \) are continuously differentiable functions which vanish at the origin together with their first derivatives. We will look for a control of the form
\[ \gamma = K \text{col}(\eta, \zeta), \text{ where } \text{col}(\eta, \zeta) = [\eta^T \zeta^T]^T. \] Substituting this into (5.14) yields

\[ \dot{\eta} = F_1 \eta + \varphi_1(\eta, \zeta) + G_1 K \text{col}(\eta, \zeta) + \xi_1(\eta, \zeta, K \text{col}(\eta, \zeta)) := f_\eta(\eta, \zeta, K) \]

\[ \dot{\zeta} = F_2 \zeta + \varphi_2(\eta, \zeta) + G_2 K \text{col}(\eta, \zeta) + \xi_2(\eta, \zeta, K \text{col}(\eta, \zeta)) := f_\zeta(\eta, \zeta, K). \] (5.15)

The approach that will be employed here involves the determination of a set of conditions on the gains \( K \) so as to achieve the desired stabilization, and then set up a numerical search to find a set of gains that satisfy these criteria.

First, it will be required that the control does not alter the structure of the eigenvalue spectrum, i.e. the closed loop system will be assumed to have a spectrum of the form \( \{-\sigma_{rmm} + j\omega_{m}, -\sigma_{m} - j\omega_{m}, -\lambda_{m1}, -\lambda_{m2}\} \) where \( \sigma_{m}, \omega_{m}, \lambda_{m1}, \lambda_{m2} > 0. \) Respecting the original eigenvalue structure of the system, even in closed loop, is important for preserving the domain of validity of reduced order flow models, as discussed in depth in Chapter 4. Transforming (5.15) into modal form gives

\[ \dot{\eta}_m = F_{m1}(K)\eta_m + \varphi_{m1}(\eta_m, \zeta_m, K) \]

\[ \dot{\zeta}_m = F_{m2}(K)\zeta_m + \varphi_{m2}(\eta_m, \zeta_m, K) \] (5.16)

where

\[ \eta_m = \begin{bmatrix} \eta_{m1} \\ \eta_{m2} \end{bmatrix}, \zeta_m = \begin{bmatrix} \zeta_{m1} \\ \zeta_{m2} \end{bmatrix}, F_{m1}(K) = \begin{bmatrix} -\sigma_m & -\omega_m \\ \omega_m & -\sigma_m \end{bmatrix}, F_{m2}(K) = \begin{bmatrix} -\lambda_{m1} & 0 \\ 0 & -\lambda_{m2} \end{bmatrix}. \]

We next define a change of coordinates parameterized by a dummy variable \( \vartheta \in [0, 2\pi] \) described as \( \eta^\vartheta = R(\vartheta)\eta_m \) where

\[ R(\vartheta) = \begin{bmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{bmatrix}. \]

Using the above transformation and differentiating one gets

\[ \dot{\eta}^\vartheta = R(\vartheta) f_\eta(R^T(\vartheta)\eta^\vartheta, \zeta_m, K) =: f_\eta^\vartheta(\vartheta, \eta^\vartheta, \zeta_m, K) \]

\[ \dot{\zeta}_m = f_\zeta(R^T(\vartheta)\eta^\vartheta, \zeta_m, K) =: f_\zeta^\vartheta(\vartheta, \eta^\vartheta, \zeta_m, K). \] (5.17)
One then averages over $\vartheta \in [0, 2\pi]$ to get the averaged system with states $\text{col}(\bar{\eta}^\vartheta, \bar{\zeta}^\vartheta)$, to obtain [78]

$$
\dot{\bar{\eta}} = \frac{1}{2\pi} \int_0^{2\pi} f_\eta^\vartheta(\vartheta, \bar{\eta}^\vartheta, \bar{\zeta}, K) \, d\vartheta =: f_{\eta,\text{avg}}^\vartheta(\bar{\eta}^\vartheta, \bar{\zeta}, K)
$$

$$
\dot{\bar{\zeta}} = \frac{1}{2\pi} \int_0^{2\pi} f_\zeta^\vartheta(\vartheta, \bar{\eta}^\vartheta, \bar{\zeta}, K) \, d\vartheta =: f_{\zeta,\text{avg}}^\vartheta(\bar{\eta}^\vartheta, \bar{\zeta}, K).
$$

From this point on, for ease of notation we will drop the bars, superscripts $\vartheta$ and subscripts $m$ from variables $\eta, \zeta, \sigma, \omega$ and $\lambda$. The averaged system is then written as

$$
\begin{align*}
\dot{\eta} &= f_{\eta,\text{avg}}(\eta, \zeta, K) \\
\dot{\zeta} &= f_{\zeta,\text{avg}}(\eta, \zeta, K).
\end{align*}
$$

(5.18)

Converting the averaged system above into polar coordinates, i.e. $\rho = \sqrt{\eta_1^2 + \eta_2^2}$ and $\theta = \arctan(\eta_2/\eta_1)$, and viewing the constant $\sigma$ as a state with trivial dynamics, one can write

$$
\begin{bmatrix}
\dot{\sigma} \\
\dot{\rho}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} 
\begin{bmatrix}
\sigma \\
\rho
\end{bmatrix} +
\begin{bmatrix}
0 \\
\varphi_{12}(\sigma, \rho, \zeta)
\end{bmatrix},
$$

$$
\begin{bmatrix}
\dot{\zeta}_1 \\
\dot{\zeta}_2
\end{bmatrix} = F_{m2}(K) 
\begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix} +
\begin{bmatrix}
\varphi_{21}(\sigma, \rho, \zeta) \\
\varphi_{22}(\sigma, \rho, \zeta)
\end{bmatrix}
$$

(5.19)

where $\varphi_{12} = [\varphi_{21} \varphi_{22}]^T$ encapsulate the nonlinear terms in (5.19) and $\varphi_{12}(0, 0, 0) = 0$ and $\varphi_{2}(0, 0, 0) = 0$. As explained in depth in Chapter 4, looking at $\sigma$ as a state makes it possible to view $\rho$ as center state. Also as analyzed in Chapter 4, the system (5.19) resulting from averaging is such that none of the state dynamics in depend on $\theta$. Therefore the state $\theta$ is immaterial in the stabilization of the system and hence will be dropped from now on. Recall that the gains $K$ are selected such that $F_{m2}$ is Hurwitz, so this partitions the system in (5.19) into center states $\sigma$ and $\rho$, and stable states $\zeta_1$ and $\zeta_2$. This allows one to perform an analysis of the system (5.19)
on a center manifold, which is guaranteed to exists per center manifold theory [89].

Recall from Chapter 4 that the center manifold is an invariant manifold of the form

\[ \zeta = \bar{\zeta}(\sigma, \rho), \]

where \( \bar{\zeta} = \begin{bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{bmatrix}^T \), satisfying

\[ \frac{\partial \bar{\zeta}_i}{\partial \rho} \varphi_{12}(\sigma, \rho, \bar{\zeta}(\sigma, \rho)) = -\lambda_i \bar{\zeta}_i(\sigma, \rho) + \varphi_{2i}(\sigma, \rho, \bar{\zeta}(\sigma, \rho)) \tag{5.20} \]

for \( i = 1, 2 \), to which the dynamics of the system (5.19) will be locally attracted.

The differential equations in (5.20) are cannot be solved directly, so one looks for an approximation of the form

\[ \bar{\zeta}_i(\rho, \sigma) = c_{i,1}(K)\sigma^2 + c_{i,2}(K)\sigma \rho + c_{i,3}(K)\rho^2 + O(\rho^3) \tag{5.21} \]

for \( i = 1, 2 \). One then substitutes (5.21) into (5.20) and solves for the coefficients. Then substituting (5.21) into (5.19) gives the system dynamics on the approximate center manifold, which is of the form

\[ \dot{\rho} = (d_1(K) + d_2(K)\rho + d_3(K)\rho^2)\rho . \tag{5.22} \]

where parameters \( d_i \) depend on parameters \( c_{i,j} \) in (5.21). Explicit expressions for the parameters in the above derivations are complicated and difficult to analyze analytically, so a numerical search problem over the range \( K_{ij} \in [-150, 150] \) for \( i = 1, 2 \) \( j = 1, \ldots, 4 \) is implemented to select proper values for \( K \) to satisfy the following criteria:

1. The original eigenvalue structure of the system is respected in closed loop, i.e., the gains \( K \) must be chosen such that the closed loop spectrum is of the form

\[ \{-\sigma_m + j\omega_m, -\sigma_m - j\omega_m, -\lambda_{m1}, -\lambda_{m2}\} \]

where \( \sigma_m, \omega_m, \lambda_{m1}, \lambda_{m2} > 0 \).

2. The center manifold needs to be attractive for \( \zeta \in Z \) and \( \rho \in D \) for some given domains \( Z \) and \( D \) that include the initial conditions of interest. More precisely,
define the off-manifold distance as \( \tilde{\zeta} := \zeta - \bar{\zeta}(\sigma, \rho) \). Differentiating yields the off-manifold dynamics which has the form

\[
\dot{\tilde{\zeta}} = F_{m2} \tilde{\zeta} + \varphi_3(\sigma, \rho, \tilde{\zeta})
\]  (5.23)

for which \( \tilde{\zeta} = 0 \) is an equilibrium as the center manifold is invariant, and \( \varphi_3 \) encapsulates nonlinear terms. Define a candidate Lyapunov function as

\[
V := \tilde{\zeta}^T P \tilde{\zeta}
\]  (5.24)

where \( P = P^T > 0 \) is the solution to the Lyapunov equation \( PF_{m2}^T + F_{m2} P = -I \). This choice for \( P \) is the one that maximizes the estimate for the domain of attraction [109]. Differentiating (5.24) yields the time derivative \( \dot{V} \) of the form

\[
\dot{V} = 2\tilde{\zeta}^T P F_{m2} \tilde{\zeta} + 2\tilde{\zeta}^T P \varphi_3(\sigma, \rho, \tilde{\zeta})
\]  (5.25)

To achieve convergence of trajectories to the manifold one then needs to have \( \dot{V} < 0 \) for \( \rho \in D \) and \( \tilde{\zeta} \in Z \), where \( Z \) needs to be big enough to include a level set of \( V \) containing the value of \( \tilde{\zeta}(0) \) corresponding to the initial condition of interest.

3. The center manifold dynamics need to be such that \( \rho \) converges to zero on the manifold, for \( \rho \in D \). Consider the on manifold dynamics in (5.22). Define a candidate Lyapunov function as

\[
V := \rho^2
\]  (5.26)

and differentiate to get

\[
\dot{V} = 2\rho^2(d_1(K) + d_2(K)\rho + d_3(K)\rho^2)
\]  (5.27)
To achieve the converging of $\rho$ to 0 once the trajectories of the system is close to the center manifold, one needs to have $\dot{V} < 0$ for all $\rho \in D$, where $D$ needs to be big enough to contain the value of $\rho(0)$ corresponding to the initial condition of interest. Since in (5.27) we have $2\rho^2 > 0$, having $\dot{V} < 0$ is equivalent to having $(d_1(K) + d_2(K)\rho + d_3(K)\rho^2) < 0$ for $\rho \in D$.

In the implementation, starting the search from a randomly picked initial condition $K_0$ such that $K_{0ij} \in [-150, 150]$ for $i = 1, 2 \ j = 1, \ldots, 4$, the search terminated successfully at

$$K = \begin{bmatrix} 0.0297 & 37.0349 & -39.4230 & 37.0349 \\ 0.0297 & 37.0349 & -39.4230 & 37.0349 \end{bmatrix}$$

which satisfies all the required criteria. Figure 5.13 shows a plot of the on-manifold dynamics $\dot{\rho}$ vs. $\rho$ for the gains above. The value of $\rho$ corresponding to our initial point of interest is also marked on the plot. It can be seen that the gains render $\dot{\rho} < 0$ for $\rho$ in the domain $D = [0, 0.5]$, which includes the value of $\rho$ of interest. Figure 5.14 shows the Lyapunov function (5.24) for the off-manifold behavior, where $\tilde{\zeta}$ in the domain $Z = [-0.4, 0.4] \times [-0.4, 0.4]$, which includes the value of $\tilde{\zeta}$ corresponding to the initial condition and some level sets containing it. Figure 5.15 shows the derivative of $V$ for some values of $\rho$ in $D$. It can be seen that $\dot{V}$ is negative for $\tilde{\zeta} \in Z$ and $\rho \in D$.

Figure 5.16 shows the application of the controller resulting from these gains to the nonlinear Galerkin system in 5.13. It can be seen that the desired stabilization is achieved for this controller. The control is then tested in the full order Burgers CFD simulation, which is shown Figure 5.17. The control input signals applied are shown in Figure 5.18. Also, Figure 5.19 shows a comparison of the desired state, final state and the absolute error between the two. It can be seen that the averaging/center
Figure 5.13: $\dot{\rho}$ vs. $\rho$ for the on-manifold dynamics. The star is the value of $\rho(0)$ corresponding to the initial condition.
Figure 5.14: Lyapunov function $V$ for off manifold dynamics. The star is the value of $\zeta(0)$ corresponding to initial condition.
Figure 5.15: Derivative of Lyapunov function $V$ for the off-manifold dynamics for different values of $\rho$. The star is the value of $\tilde{\zeta}(0)$ corresponding to initial condition.
Figure 5.16: Modal coefficients $a$ for control derived from averaging/center-manifold manifold based design achieves the desired regulation with very little error for the most part of the domain.

5.4.2 Linear Quadratic Regulator (LQR) Control Design Based on Linearization

In this section a linearization based LQR controller will be developed for comparison with the results in the previous section. The linearization of the system (5.13) is

$$\dot{\tilde{a}} = \tilde{L}\tilde{a} + \tilde{L}_{\text{in}}\tilde{\gamma}.$$
Figure 5.17: Averaging/center manifold based control applied to full order Burgers CFD

Figure 5.18: Control signals for averaging/center manifold based control
Classical LQR control attempts to design a control law of the form $\tilde{\gamma} = -K\tilde{a}$, where $K$ is selected to minimize the objective function

$$J(\tilde{\gamma}) = \int_0^\infty (a^TQa + \tilde{\gamma}^TR\tilde{\gamma}) \, dt .$$

Despite numerous trials with many different values for the weighing parameters $Q$ and $R$, it was not possible to obtain a controller that achieves stabilization starting from the desired initial condition $a_0$, when the control was applied to the nonlinear Galerkin system in (5.13). For this reason a modification of the standard LQR approach was used [106, 110]. This approach tries to minimize a modified objective function of the form

$$J(\tilde{\gamma}) = \int_0^\infty (a^TQa + \tilde{\gamma}^TR\tilde{\gamma}) \, e^{2\alpha t} \, dt .$$

Figure 5.19: Final state, reference profile and absolute error for averaging/center manifold based control
where $\alpha > 0$ is an additional parameter to be selected. After a high number of trials with different values for $\alpha$, $Q$ and $R$, the best results that could be obtained were for $\alpha = 0.1$, $Q = 2500I_4$ and $R = I_2$, which yields a controller $\tilde{\gamma} = -K\tilde{a}$ with

$$K = \begin{bmatrix} -27.9149 & 117.1223 & 50.2110 & -52.4688 \\ -27.9159 & 117.1197 & 50.2104 & -52.4673 \end{bmatrix}.$$  

Figures 5.20 shows the result of the implementation of this control law on the nonlinear Galerkin system (5.13). It can be seen that control achieves the desired stabilization, but the response is not as smooth and nice compared to the the response obtained using the averaging/center manifold controller in Figure 5.16. The control is also implemented in the full order Burgers CFD simulation, which is shown Figure 5.21. The control input signals applied are shown in Figure 5.22. Figure 5.23 shows a comparison of the desired state, final state and the absolute error between the two.
It can be seen that the control achieves the regulation with small error for the most part of the domain. However, the control effort is higher compared to that of the averaging/center manifold controller (see Figure 5.18) and for the parts of the domain where the error is visible, its value is notably higher than the averaging/center manifold case (see Figure 5.19).

5.5 Summary and Comments

In this chapter, we studied a sample boundary flow control problem of convective flow over an obstacle governed by the Burgers equations. This case study is regarded as an important problem since flows over obstacles arise in many situations in practice and in addition, Burgers equations are of significance as they are a simpler form of the more general Navier-Stokes equations. A reduced order model was built based
Figure 5.22: Control signals for LQR control

Figure 5.23: Final state, reference profile and absolute error for LQR control
on a POD expansion consisting of baseline modes and actuation modes, where the former were built using unactuated snapshots and the latter were built in an energy optimal fashion using innovation vectors which represent the flow not captured by the baseline space, as described in detail in Chapter 2. This was followed by a controller design with the control objective of driving the system state to a desired 2D profile. The controller design was based on an analysis using averaging theory and center manifold theory to further simplify the system and reveal more structure, as described in Chapter 4. From this analysis, certain conditions for the controller gains to satisfy were derived, and a numerical search problem was set up and solved using MATLAB. Then for comparison, a standard LQR design based on the linearization of the reduced order model was implemented and tested. It was seen through full order CFD simulations that while both controllers achieve the desired objective, the averaging/center manifold based controller design results in a smoother response, lesser control effort and smaller tracking error over the flow domain.

We believe that the importance of this chapter is that it provides a complete and detailed case study on a sample problem regarding the control of flow over an obstacle using the modeling and control techniques developed in the earlier chapters. The case study is on a specific nonlinear convection problem governed by Burgers equations, however, the methods will extend straightforwardly to different flow configurations that may be governed by possibly different set of PDEs. It is our opinion that by systematically guiding the reader through a complete case study on the sample problem, this chapter will aid in introducing interested researchers to the novel modeling and control tools built in this thesis and also help them apply the results to their own problems at hand.
CHAPTER 6

CONTRIBUTIONS AND FUTURE WORK

In this thesis, tools for reduced order mathematical modeling, nonlinear analysis and control of flow control problems were developed, with specific application to the cavity flow control problem in mind. Attempt was made to develop the methods in a general context, so as to open the door for potential applications to different kinds of flow control problems.

To summarize the contributions believed to have been made by this thesis: Presently there is a lack of sufficient mathematical analysis and control design tools developed for flow control problems in general, and cavity flow in particular. As a result, not all but a substantial portion of the work in the field is of empirical or heuristic nature. One key aspect of flow control that has not received deserved attention is the concept of input separation, for which this research proposes solutions based on optimization on Hilbert spaces. As to model reduction, current standard techniques do not offer models that reveal enough structure to be exploited for advanced control design tools. This research offers systematic methods based on averaging and center manifold theory to simplify the system beyond what is achieved by the standard approaches, while simultaneously revealing more structure on key behavior. The tools for modeling and control proposed in this work are
In summary, the work presented aims to establish a complete mathematical framework in terms of modeling analysis and control under which flow control problems can handled, with specific application to cavity flow control problems as well.

Future directions include extending the input separation approaches to include different, perhaps constrained optimization problems, developing nonlinear analysis methods for more general flow system structures, designing more advanced controllers including adaptive and robust designs, establishing formal connections between the original Navier-Stokes equations and the reduced models, and applying the results to different real life flow control problems.
APPENDIX A

CAVITY FLOW EXPERIMENTAL FACILITY AT OSU GDTL

In this appendix we provide a brief description of the experimental setup (Figure A.1(a)) at The Ohio State University (OSU) Gas Dynamics and Turbulence Laboratory (GDTL) directed by Prof. Mo Samimy, on which we had the chance and opportunity to carry out experiments and test and verify our results. We are also grateful to all the other members of the group, Kihwan Kim, Jesse Little, Marco DeBiasi and especially Edgar Caraballo, who has kindly provided most of the information on the experimental facility reported below. Full details about the facility can be found in [2, 15, 39].

The experimental facility is an instrumented, optically accessible wind-tunnel that operates in a blow-down fashion with atmospheric exhaust. Filtered, dried air is conditioned in a stagnation chamber before entering a smoothly contoured converging nozzle to the 2 inch by 2 inch test section. The facility can run continuously in the subsonic range between Mach 0.25 and 0.70. A shallow cavity is recessed in the test section with a depth $D = 12.7$ mm and length $L = 50.8$ mm for a length to depth aspect ratio $L/D = 4$. For control the cavity is forced in the shear-layer receptivity region by a 2-D synthetic-jet type actuator issuing at 30 degrees relative
to the main flow from a 1 mm slot embedded in the cavity leading edge and spanning the width of the cavity (see Figure A.1(b)). A Selenium D3300Ti compression driver provides the mechanical oscillations necessary to create the zero net mass, non-zero net momentum flow for actuation. The actuator signals are produced by either a BK Precision 3011A function generator for open-loop forcing or by a dSPACE 1103 DSP control board in closed-loop studies and are amplified by a Crown D-150A amplifier.

The “snapshots” of the flow field required for the development of the low dimensional model are acquired and processed using a LaVision Inc. PIV system[111]. The main flow is seeded with Di-Ethyl-Hexyl-Sebacat (DEHS) particles by using a 4-jet atomizer upstream of the stagnation chamber. A dual-head Spectra Physics PIV-400 Nd:YAG laser operating at the 2nd harmonic (532 nm) is used in conjunction with spherical and cylindrical lenses to form a thin (1mm), vertical sheet spanning the cavity in the streamwise direction at the middle of test section width. The time separation between the lasers pulses for the flow at Mach 0.30 in our experimental setup is 1.8 microseconds. The acquired images are divided into 32 by 32 pixel interrogation windows that contain 6-10 seed particles each. For each image sub regions are cross correlated by using multi-pass processing with 50% overlap. The resulting vector fields are post-processed to remove any remaining spurious vectors. This setup gives a velocity vector grid with vector separation of approximately 0.4 mm.

Flush-mounted Kulite transducers are placed at various locations on the walls of the test section for dynamic surface pressure measurements. Figure A.1(c) shows the locations of the transducers used in this study. All these transducers have an almost flat frequency response up to about 50 kHz and are powered by a dedicated signal conditioner that amplifies and low-pass filters at 10 kHz their signals.
Figure A.1: The OSU GDTL cavity flow experimental facility
For state estimation, dynamic pressure measurements are recorded simultaneously with the PIV measurements using a National Instruments (NI) PCI-6143 S-Series data acquisition board mounted on a Dell Precision Workstation 650. Each pressure recording is band-pass filtered between 100 Hz and 10 kHz to remove spurious frequency components. In the current study 1000 PIV snapshots are recorded for each flow/actuation condition explored. For each PIV snapshot 128 pressure samples from the laser Q-switch signal and from each of the transducers of Figure A.1(c) are acquired at 50 kHz. The NI board was triggered by a programmable timing unit (PTU) housed in the PIV system that activated the beginning of the acquisition to allow the Q-switch TTL to fall approximately in the middle of the 256 data points. The simultaneous sampling of the laser Q-switch signal with the pressure signals allows for each snapshot the identification of the section of pressure time traces corresponding to the instantaneous PIV velocity field. Additional, longer recordings of 262,144 samples per channel acquired at 200 kHz were also used to derive sound pressure level (SPL) spectra.

For closed-loop control of the flow, a dSPACE 1103 DSP board connected to the Dell Precision Workstation 650 is used. This system utilizes four independent, 16-bit A/D converters each with 4 multiplexed input channels and allows simultaneous acquisition and control processing of 4 signals and almost simultaneous, due to multiplexing, acquisition and processing of additional signals at a rate up to 50 kHz per channel to produce at the same rate a control signal from a 14-bit output channel. Similar to state estimation pressure data, the pressure signals are band-pass filtered between 100 Hz and 10 kHz to remove spurious frequency components.
APPENDIX B

NAVIER STOKES EQUATIONS

To describe the dynamics of the cavity flow process one uses Navier-Stokes equations. Consider the non-dimensional\(^8\) isentropic compressible Navier-Stokes equations

\[
\frac{D\mathbf{w}}{Dt} + \frac{1}{M^2} \frac{2}{\gamma - 1} \nabla c = \frac{1}{Re} \nabla^2 \mathbf{w}
\]

\[
\frac{Dc}{Dt} + \frac{\gamma - 1}{2} c \text{div} \mathbf{w} = 0 \tag{B.1}
\]

where \(\mathbf{w}(\mathbf{x}, t) = (u(\mathbf{x}, t), v(\mathbf{x}, t))\) is the flow velocity in the stream-wise and vertical direction, \(c(\mathbf{x}, t)\) is the local speed of sound, the operator \(D/Dt = \partial/\partial t + \mathbf{w} \cdot \nabla\) stands for the material derivative, and \(\mathbf{x} = (x, y)\) denotes Cartesian coordinates over the spatial domain \(\Omega \subset \mathbb{R}^2\). The constants \(\gamma, \text{Re},\) and \(\text{M}\) denote respectively ratio of specific heats, Reynolds number, and Mach number. It is shown in Rowley [102] that these equations can be expressed in compact form as

\[
\dot{\mathbf{u}} = X(\mathbf{u}) = \text{Re}^{-1} L_1(\mathbf{u}) + M^{-2} Q_1(\mathbf{u}, \mathbf{u}) + Q_2(\mathbf{u}, \mathbf{u}) \tag{B.2}
\]

\(^8\)The governing equations have been non-dimensionalized by scaling \(\mathbf{w}\) by the freestream velocity \(U_\infty\), the local speed of sound by the ambient sound speed \(c_\infty = (\gamma RT_\infty)^{1/2}\), where \(T_\infty\) is the ambient temperature, the cartesian coordinates \(\mathbf{x}\) by the cavity depth \(D\), time by \(D/U_\infty\), and pressure by \(\bar{\rho}U_\infty^2\), where \(\bar{\rho}\) denotes mean density.
where \( \mathbf{u} = (u, v, c) \) represents the nondimensionalized state and

\[
L_1(\mathbf{u}) = \begin{pmatrix}
u_{xx} + v_{yy} \\
v_{xx} + v_{yy} \\
0
\end{pmatrix}
\]

\[
Q_1(\mathbf{u}_1, \mathbf{u}_2) = \frac{2}{\gamma - 1} \begin{pmatrix} c_1 c_{2x} \\
c_1 c_{2y} \\
0
\end{pmatrix}
\]

\[
Q_2(\mathbf{u}_1, \mathbf{u}_2) = \begin{pmatrix} u_1 u_{2x} + v_1 u_{2y} \\
u_1 v_{2x} + v_1 v_{2y} \\
u_1 c_{2x} + v_1 c_{2y} + \frac{\gamma - 1}{2} c_1 (u_{2x} + v_{2y})
\end{pmatrix}.
\]

Defining \( L := Re^{-1} L_1 \) and \( Q := M^{-2} Q_1 + Q_2 \) in (B.2) yields

\[
\dot{\mathbf{u}} = X(\mathbf{u}) = L(\mathbf{u}) + Q(\mathbf{u}, \mathbf{u})
\]

(B.3)

which is the compact form used in this thesis.
APPENDIX C

PARAMETERS OF THE AVERAGED SYSTEM

The values of the parameters of (4.12) in terms of the original Galerkin system (4.1) are given by:

\[ b_{21} = r_{31}, \quad b_{22} = r_{41} \]

\[ \phi_{11} = \frac{1}{2} q_{131} + \frac{1}{2} q_{223} + \frac{1}{2} q_{113} + \frac{1}{2} q_{232}, \quad \phi_{12} = \frac{1}{2} q_{114} + \frac{1}{2} q_{224} + \frac{1}{2} q_{141} + \frac{1}{2} q_{242} \]

\[ \phi_{21} = \frac{1}{2} q_{123} - \frac{1}{2} q_{213} - \frac{1}{2} q_{231} + \frac{1}{2} q_{132}, \quad \phi_{22} = -\frac{1}{2} q_{214} + \frac{1}{2} q_{124} + \frac{1}{2} q_{142} - \frac{1}{2} q_{241} \]

\[ \phi_{31} = \frac{1}{2} q_{322} + \frac{1}{2} q_{311}, \quad \phi_{32} = q_{333}, \quad \phi_{33} = q_{344}, \quad \phi_{34} = q_{334} + q_{343} \]

\[ \phi_{41} = \frac{1}{2} q_{422} + \frac{1}{2} q_{411}, \quad \phi_{42} = q_{433}, \quad \phi_{43} = q_{444}, \quad \phi_{44} = q_{434} + q_{443} \]

\[ g_{11} = \frac{1}{2} s_{221} + \frac{1}{2} s_{111}, \quad g_{12} = -\frac{1}{2} s_{211} + \frac{1}{2} s_{121}, \quad g_{31} = s_{331}, \quad g_{32} = s_{341} \]

\[ g_{41} = s_{431}, \quad g_{42} = s_{441} \]
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