LOW-DIMENSIONAL MODELING AND ANALYSIS OF
HUMAN GAIT WITH APPLICATION TO THE GAIT OF
TRANSTIBIAL PROSTHESIS USERS

DISSER TATION

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By

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This dissertation uses a robotics-inspired approach to develop a low-dimensional forward dynamic model of normal human walking. The analytical model captures the dynamics of walking over a complete gait cycle in the sagittal plane. The model for normal walking is extended to model asymmetric gait. The asymmetric model is applied to study the gait dynamics of a transtibial prosthesis user.

Modeling human walking is complex because walking involves (i) the body’s many degrees of freedom (DOF), (ii) constraints that change, and (iii) intermittent contact with the environment that may be impulsive. Complex forward dynamic models that attempt to capture details such as joints with multiple DOF, musculature, etc., are analytically intractable; it is impossible to describe the model’s behavior in mathematically manageable terms because of the enormous number of variables and redundancies involved. Observation of human walking from a systems point of view reveals that the human body coordinates its many DOF in a parsimonious manner to accomplish the task of moving the body’s center of mass from one point to another. This dissertation’s approach exploits this parsimony to derive an analytically tractable model that has the minimum DOF necessary to describe the task of walking in the sagittal plane.

The low-dimensional hybrid model is derived as an exact sub-dynamic of a higher-dimensional anthropomorphic hybrid model. The hybrid nature is the result of continuous sub-models of single support (SS) and double support (DS), and discrete maps that model
the transitions from SS to DS and DS to SS. The modeling is validated using existing gait data.

To extend the clinical usefulness of the modeling approach, the model for normal walking is extended to model asymmetric gait. The asymmetric model can accommodate asymmetries in the parameters and joint motions of the left and right legs. The asymmetric model is applied to analyze the gait dynamics of a transtibial prosthesis user. Cost functions are used to evaluate the effect of varying prosthetic alignment, prosthesis mass distribution, and prosthetic foot stiffness. The results agree well with clinical observations and the results of related gait studies reported in the literature.
To my family
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<table>
<thead>
<tr>
<th>Field</th>
<th>Professors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamical systems</td>
<td>Professors E. R. Westervelt and R. G. Parker</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>Prosthetics</td>
<td>Professor B. L. Bowyer</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>viii</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xiv</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xv</td>
</tr>
</tbody>
</table>

Chapters:

1. Introduction .................................................. 1
   1.1 Background on human walking ................................ 2
   1.2 Modeling human walking .................................... 7
       1.2.1 Analytical models and analyses ....................... 9
       1.2.2 Statistical methods .................................. 11
   1.3 Lower limb prosthetics ................................. 12
   1.4 Research approach .................................... 15
   1.5 Organization of dissertation .......................... 18

2. Modeling ......................................................... 19
   2.1 Methodology ............................................... 19
   2.2 Anthropomorphic model development ..................... 21
       2.2.1 Single support ....................................... 21
       2.2.2 Double support ...................................... 26
       2.2.3 Transition mappings ............................... 32
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Anthropometric parameters for the gait analysis subject in [1]</td>
<td>53</td>
</tr>
<tr>
<td>5.1 Normalized anthropometric parameters for a sound leg [1] and a prosthetic leg [2]</td>
<td>95</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>The three reference planes and six fundamental directions of the human body, with reference to the anatomical position.</td>
</tr>
<tr>
<td>1.2</td>
<td>Movements about the hip, knee and ankle joints in the sagittal plane.</td>
</tr>
<tr>
<td>1.3</td>
<td>Movements about the hip and knee joints in the frontal and transverse planes.</td>
</tr>
<tr>
<td>1.4</td>
<td>One gait cycle of human walking is comprised of two consecutive steps.</td>
</tr>
<tr>
<td>1.5</td>
<td>Illustration of the parsimony of human gait over a normal gait cycle.</td>
</tr>
<tr>
<td>2.1</td>
<td>Coordinates for a minimal anthropomorphic model in double support (DS) and single support.</td>
</tr>
<tr>
<td>2.2</td>
<td>The roll-over shape [3] is obtained by representing the center of pressure in a shank-based coordinate system with the ankle as origin.</td>
</tr>
<tr>
<td>2.3</td>
<td>Discrete-event system corresponding to the dynamics of one step (half a normal gait cycle).</td>
</tr>
<tr>
<td>3.1</td>
<td>The scalar quantity chosen to represent forward progression in single support, $\theta_s$ and double support, $\theta_d$ corresponds to the angles shown.</td>
</tr>
<tr>
<td>3.2</td>
<td>Joint motions from the gait data in Winter [1] versus the model, (2.51), on the limit cycle (i.e., at steady-state) for one step.</td>
</tr>
<tr>
<td>3.3</td>
<td>The joint velocities from the simulation of the dynamics.</td>
</tr>
<tr>
<td>3.4</td>
<td>Comparison of the angle between the line connecting the center of mass and the ground for the gait data in [1] versus the model, (2.51), on the limit cycle (i.e., at steady-state) for one step.</td>
</tr>
</tbody>
</table>
3.5 Comparison of ground reaction forces computed from the model with the values from force plate measurements reported in [1]. 66

3.6 Stick figure animation of one step of the example model. 67

3.7 Simulation results for $z_{d_1,1}$ in DS, $z_{s,1}$ in SS over one gait cycle using the full model and using the low-dimensional model. 67

3.8 Simulation results for $z_{d_2,1}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full model and using the low-dimensional model. 67

3.9 Error in $z_{d_1,1}$ in DS, $z_{s,1}$ in SS over one gait cycle using the full model versus using the low-dimensional model. 68

3.10 Error in $z_{d_2,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full model versus using the low-dimensional model. 69

3.11 The Poincaré map of the full hybrid model (2.113) versus its linear approximation (3.47). 71

3.12 Simulation of the full hybrid model (2.51) for 35 steps illustrating convergence to a limit cycle as predicted by the fixed point in the Poincaré map. 71

3.13 Numerical evaluation of $V_s$ indicates that it may be approximated by a linear relation. 72

3.14 Numerical evaluation of $V_d$ indicates that it may be approximated by a linear relation. 72

4.1 The phases and transitions that are required to describe one gait cycle with asymmetric steps. 75

4.2 Discrete-event system corresponding to the dynamics of a gait cycle consisting of one left and one right step. 80

4.3 Simulation results for $z_{d_1,1}$ in DS, $z_{s,1}$ in SS for one gait cycle using the full asymmetric model and using the low-dimensional asymmetric model. 88

4.4 Simulation results for $z_{d_2,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full asymmetric model and using the low-dimensional asymmetric model. 88
4.5 Error in \( z_{d,1} \) in DS, \( z_{s,1} \) in SS over one gait cycle using the full asymmetric model versus using the low-dimensional asymmetric model. 89

4.6 Error in \( z_{d,2} \) in DS, \( z_{s,2} \) in SS over one gait cycle using the full asymmetric model versus using the low-dimensional asymmetric model. 89

4.7 Simulation of the low-dimensional asymmetric hybrid model (4.29) for 40 steps (20 gait cycles) illustrates convergence to a period-two limit cycle. 90

5.1 Six degrees of freedom associated with transtibial prosthesis alignment. 94

5.2 The parameters that define a roll-over shape (ROS) modeled as a circular arc. 97

5.3 The total joint torque and joint power costs for one gait cycle with varying alignments. 98

5.4 The joint torque costs for the hip and knee of the prosthetic leg in swing. 100

5.5 The total joint torque costs when the mass distribution and A-P alignment are varied. 104

5.6 The total joint power costs when the mass distribution and A-P alignment are varied. 104

5.7 The total joint torque costs for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. 105

5.8 The total joint torque costs for different A-P alignments (+30 mm to +60 mm) when the mass distribution is varied. 106

5.9 The total joint power costs for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. 107

5.10 The total joint power costs for different A-P alignments (+30 mm to +60 mm) when the mass distribution is varied. 108

5.11 Steady-state walking speeds for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. 109
5.12 Steady-state walking speeds for different A-P alignments (+30 mm to +60 mm) when the mass distribution is varied. ................................................. 109

5.13 The total joint torque cost for one gait cycle for different radii of the prosthetic side roll-over shape (ROS) and its variation with A-P alignment. . . . . 111

5.14 The total joint power cost for one gait cycle for different radii of the prosthetic side roll-over shape (ROS) and its variation with A-P alignment. . . . . 112

5.15 Steady-state walking speed for different radii of the prosthetic side roll-over shape (ROS) and different alignments. ........................................... 113

5.16 Comparison of the step length of the prosthetic or the sound leg for different radii of the prosthetic side roll-over shape (ROS) and different alignments. . 114

5.17 Comparison of the step times of the prosthetic and the sound leg for different radii of the prosthetic side roll-over shape (ROS) and different alignments. 114

B.1 The ankle-foot roll-over shape (ROS) is obtained by representing the center of pressure (COP) over a step in a shank-based coordinate system with the ankle as the origin [3]. ......................................................... 131

B.2 Ankle-foot ROS calculation. ............................................................. 132

B.3 The knee-ankle-foot roll-over shape is obtained by representing the center of pressure over a step in a coordinate system based on the hip and the ankle [4]. ................................................................. 133
CHAPTER 1

INTRODUCTION

A defining trait that separates humans from other primates is habitual bipedal walking. Darwin recognized that walking on two legs freed up the arms and hands for tasks like carrying, tool making, and tool use, which paved the way for further human development. Bipedalism is therefore considered a fundamental first step in human evolution.

Human walking is a complex phenomenon that involves the coordination of various joints, muscles and neurons to maintain stability while moving the center of mass (COM) forward. Walking is a learned process [5], and the characteristic patterns\(^1\) of adult walking take several years to develop. In the process, an individual imprints distinctive mannerisms on his style of walking. Gait recognition tools, for instance, are based on these distinctive aspects of an individual’s gait. However, the subtle variations among individuals during walking appear to be superimposed on a common pattern of coordination of the limbs. This fundamental pattern appears to dominate the manner in which a person walks and appears to be responsible for the efficiency of walking.

\(^1\)For example, children may take up to 7 years of age to achieve a three-dimensional COM motion similar to that of an adult [6].
Understanding the pattern of limb coordination to accomplish human walking is essential to help a person affected by a disability, such as an amputation, walk again. Restoring the ability to walk is vital for the physical, psychological, and social rehabilitation of the individual.

1.1 Background on human walking

Any discussion of the mechanics of human walking requires the introduction of some basic anatomical terminology. The motion of body segments is described as occurring in three planes that are referenced to the anatomical position (see Figure 1.1). The anatomical position is the position in which a person is standing upright with the feet together and the arms by the side of the body, with the palms facing forward [7]. A sagittal plane divides the body into right and left halves. A frontal plane divides the body into front (anterior) and back (posterior) portions. A frontal plane is also known as a coronal plane. A transverse plane divides the body into upper (superior) and lower (inferior) portions. Within a single part of the body, relative anatomical locations are described using specific terms: Medial indicates that the location is toward the midline of the body. Lateral indicates a location away from the midline of the body. For example, the big toe is on the medial side of the foot while the little toe is on the lateral side. Proximal refers to an anatomical location that is close to a point of reference, usually the rest of the body. Distal refers to a location far from a point of reference. For example, the hip joint is the proximal part of the thigh while the knee joint is the distal part.

The joint motions during walking can be described using the definitions of motion in the three reference planes. Motions of the hip, knee, and ankle in the reference planes are shown in Figures 1.2 and 1.3. For the hip and knee, the motions in the sagittal plane are
Figure 1.1: The three reference planes and six fundamental directions of the human body, with reference to the anatomical position. Figure courtesy of [7].
referred to as flexion and extension. In the ankle, the sagittal plane motions are referred to as dorsiflexion and plantarflexion. The terms used to describe motions in the frontal plane are abduction and adduction. Possible motions in the transverse plane are described as internal and external rotations.

The primary task of human walking is to translate the body’s center of mass (COM) in the direction of progression. The plane of progression is parallel to the sagittal plane. Observation reveals that walking is accomplished by a pattern of repeatable movements that occur every step. Figure 1.4 depicts one gait cycle, which is composed of a right step and a left step. Each step is composed of two different phases. The swing phase or single support phase is when one foot is on the ground while the other leg swings. This phase makes up the majority (80–90%) of the duration of the walking step. Single support begins with the moment of toe-off and ends with the impact of the swing foot with the ground. In the double support phase, both feet are on the ground while the body is moving forward.
Figure 1.3: Movements about the hip and knee joints in the frontal and transverse planes. Abduction and adduction take place in the frontal plane while the internal and external rotations take place in the transverse plane. Figure courtesy of [7].
Figure 1.4: One gait cycle of human walking is comprised of two consecutive steps. The left and right legs are denoted by “L” and “R”. A step is the period from heel contact (HC) to opposite heel contact (OHC). A normal gait cycle is made up of two symmetric steps with each step comprised of single support (SS) and double support (DS) phases. Toe-off (TO) signals the transition from DS to SS while HC marks the transition from SS to DS. In the figure, the right leg is the stance leg and the left leg is the swing leg for step 1. The left and right legs switch roles for step 2.

During this period, the support of the body is transferred from the leading leg to the trailing leg. The double support phase usually makes up only a small part (10–20\%) of the human walking step. Thus, in walking, one or two feet are always on the ground. As the walking speed increases, the period of double support diminishes. The absence of a period of double support distinguishes running from walking. The cyclic alternation of the support function and the existence of the transfer period of double support are essential characteristics of walking.

The leg that performs the supporting function is termed the stance leg. In double support, the leading leg is the stance leg since it is assuming the support function. The distance the right foot moves forward in front of the left foot is referred to as the right step length. The left step length may be defined similarly. Each stride or gait cycle is composed of
one right and one left step. In pathological gait, it is common for the two step lengths in one stride to be different. **Cadence** is defined as the number of steps per unit time (e.g., steps/min). **Walking speed** is the distance travelled per unit time (e.g., m/s).

Whether the walking is normal or pathological, two basic requisites of walking are the presence of ground reaction forces to support the body and periodic movement of each foot from one position of support to the next in the direction of progression. With each step, the body rises and falls, slows down and speeds up, and weaves from side to side in a systematic manner as the COM moves forward.

The translational movement of the body’s COM in the direction of progression is achieved by angular displacements of the various segments about many joints. In the frontal and transverse planes, the joints undergo small displacements making the variations among individuals significant and contributing to the differences observed in the walking styles of people. However, observation reveals that the displacements parallel to the plane of progression (the sagittal plane) are large with small variations among individuals. Data from multiple subjects and various trials indicate that during normal walking, movements of the hip, knee and ankle lie within narrow bands (see Figure 1.5). This observation implies that the human body makes parsimonious use of the various DOF to achieve a common periodic pattern that enables the forward progression of the COM. This observed parsimony of human gait forms the basis for the modeling approach in this dissertation.

### 1.2 Modeling human walking

The approaches to the modeling and analysis of human gait can be divided into two classes [9]: analytical models and analyses, and statistical methods.
Figure 1.5: Illustration of the parsimony of human gait over a normal gait cycle. Average joint angles as a percentage of gait cycle for five adult subjects with normal gait using five trials each (bold). Dashed lines are the point-wise 1 standard deviation minimum and maximum joint angles. A normal gait cycle is made up of two symmetric steps with each step comprised of single support (SS) and double support (DS) phases. The superscripts “+” and “−” indicate the beginning and end of each phase, respectively. (Data courtesy of J. Linskell, Limb Fitting Centre, Dundee, Scotland [8].)
1.2.1 Analytical models and analyses

Zajac et al. [10] provide a rather complete summary of gait modeling that divides the analytical models and analyses into three subclasses: simplified mechanical models, inverse dynamic models, and forward dynamic models. Simplified mechanical models are low-dimensional models that are idealizations of human motion. They include the idealization of human walking as an inverted pendulum [11, 12] and passive models [13, 14, 15]. Though the simplified models are well-suited for analysis and are able to predict some of the important characteristics of walking, such as step length, speed [16], and metabolic cost [17], these models are approximations and do not provide insight into the control mechanisms used by the human to generate gait. Inverse dynamic models [18, 19] proceed by using measured kinematics and external forces and moments to calculate the resultant inter-segmental forces and moments. These models, while useful for diagnosing some pathologies, lack the ability to predict the effect of anthropometric changes, for instance. Using these models, the only way of determining the efficacy of a treatment, whether it is a surgical alteration, or gait training by physical therapy, is by collecting gait data before and after the treatment. This approach can be expensive, time-consuming and hinges largely on the experience of the clinical team that the right change was effected by the treatment performed.

Forward dynamic models [20, 21, 22] proceed by integration of rigid-body equations of motion over a given time period to determine kinematics as a consequence of the applied inter-segmental forces and moments, given initial conditions. Since the measurement of these forces and moments in vivo is difficult and there is a lack of sufficient control algorithms to specify these quantities, most forward dynamic models use feedforward strategies [20, 23, 24]. Forward dynamic models can be powerful tools for prediction, but when open
loop strategies are taken, small errors in the joint moments result in accumulating kinematic errors that quickly become large, resulting in a fall [1, p. 159]. This phenomenon is one reason forward dynamic models can incorrectly predict stability of the system.

The trend in forward dynamics approaches appears to be the development of detailed musculoskeletal models to simulate and analyze movement [25]. The goal of the traditional approaches is the determination of the contribution of individual muscles to the movement. Static or dynamic optimization procedures are used in these models to compute the muscle excitation patterns necessary to achieve a desired movement [26]. In static optimization, the joint torques are first computed from gait data using inverse dynamics. The joint torques are then input into the forward dynamic model, and optimization is used to estimate the individual muscle forces with, for example, a minimization objective of metabolic energy expenditure per unit distance traveled. These muscle forces are then used in the forward dynamic model, and the resulting kinematics and ground reaction forces are compared with the actual gait data to verify the accuracy of the model [22, 25]. In another forward dynamics approach, termed dynamic optimization, initial and terminal states of the model that correspond to, for example, the beginning and end of a step are specified and the intermediate states are determined by optimization of an appropriate objective function [27, 28]. A common application of static and dynamic optimization approaches is in the analysis of irregular muscular contribution patterns in pathological gait [29, 30, 31]. However, because of the large numbers of degrees of freedom and actuator and neuronal redundancies involved, the computational requirements of these analyses are intensive, and it is not easy to discern the patterns of coordination or interrelationships that act to reduce walking to a
low-dimensional, periodic pattern in humans. The complexity of forward dynamic models makes intractable analysis of the underlying gait mechanics to explore, for example, perturbation recovery patterns.

1.2.2 Statistical methods

A review of existing literature indicates that pathological gait is most often studied by comparison with data from normal gait [32, 33]. Kinematic gait data is collected using a motion analysis system, such as a camera-based system, in addition to data from force plates or EMG, or both. Normative data is collected using healthy subjects who exhibit no gait pathologies, walking at self-selected speeds on level ground. Additional data is gathered using subjects exhibiting the pathologies of interest. The motion analysis system performs the inverse dynamics and outputs measures such as joint angles, velocities, net joint reaction forces, net joint torques, and net joint powers. Common comparisons are performed using various quantities output by the inverse dynamics algorithm [34, 35, 36], while other studies compare characteristics such as cadence and VO2 (volume of oxygen) consumption [37]. Most comparison studies between normal and pathological gait involve only certain extracted quantities such as peak values of joint powers, for instance, or are subjective analyses of the patterns exhibited by the various quantities.

In a sense, by using the idea that a pattern of normal gait exists for the purpose of comparison, statistical analyses exploit the idea that human gait is parsimonious. Examples of methods that attempt to make statistical inferences from gait data include the Determinants of Gait [5], Principal Component Analysis [38, 39] and neural network classifiers [40, 41]. Neural network and pattern recognition techniques offer ways of classification according to the waveform patterns. Principal component modeling (PCM) is complementary to these.
techniques and emphasizes comparison to a reference or normal gait pattern [38]. Factor analysis is a technique similar to PCM in which complex waveforms are represented using just a few eigenvectors [42].

Statistical methods attempt to make inferences from the enormous amounts of data generated by gait analysis but do not model the dynamics of walking.

### 1.3 Lower limb prosthetics

Regaining the ability to walk with a prosthesis is a major challenge for a person who undergoes an amputation of the lower leg. An understanding of the mechanics of human walking is essential for the engineer who designs prostheses and for the clinical team involved in the person’s rehabilitation.

There are various causes of amputation. Common causes are vascular disease, cancer, infection, trauma, and birth defects. Lower limb amputations may be performed at different levels\(^2\) along the lower limb. A transtibial or below-knee (B/K) amputation is an amputation performed through the shank (tibia), while a transfemoral or above-knee (A/K) amputation is an amputation performed through the thigh (femur). Other levels of amputation include partial foot amputations and amputations through the ankle, knee or hip joints.

The challenge of rehabilitation increases with a higher level of the amputation because more of the anatomical structures, joints, musculature, ligaments, and nerves, are lost as a result of the amputation. In general, the shorter the residual limb and the fewer joints that are preserved, the more difficult it is to fit and enable use of a prosthesis for rehabilitation. The most common form of lower limb amputation is transtibial amputation.

\(^2\)The level indicates a point on the lower limb. The portions distal to that point are removed surgically in the amputation.
A lower limb prosthesis is an artificial leg used to restore the ability to walk for someone with a lower limb amputation. A lower limb prosthesis must provide stability (that is, it must not collapse) during standing and walking. In addition, a prosthesis may need to provide shock absorption to cushion the impacts during walking, energy storage and return to improve the efficiency of walking, and a pleasing cosmetic appearance. Some prostheses need to accommodate higher functional demands such as running, jumping, swimming, or other athletic activities.

Various factors affect the function of a lower limb prosthesis and its ability to restore the user’s mobility. The prosthesis user controls the entire prosthesis via the socket, which is the interface between the prosthesis and the residual limb. For this reason, a comfortable fit of the socket is of primary importance.

The mass and mass distribution of a prosthesis are also important factors affecting the prosthesis’s use. These factors are important not only because the musculature and control of the residual limb are impaired as a result of the amputation, but also because the mass and its distribution affect the pendular dynamics when the prosthetic leg is swung forward [43, 44]. Choosing the right components for a prosthesis can be a challenge because of the variety available [45]. For example, the prosthetic feet on the market vary widely in their cost, weight and functional benefits [46].

Prosthetic alignment is another major factor influencing the successful use of a prosthesis. The alignment of a prosthesis is the relative positioning of the various components of the prosthesis to each other and to the anatomy. Prosthetic alignment plays a major role in the successful fitting and ultimate acceptance of a prosthesis by an amputee [47]. Alignment affects the stability\(^3\) of the prosthesis, the fluidity and symmetry of the gait, the

\(^3\)In this context, stability is the ability of the prosthesis to provide adequate support without buckling during standing or walking.
compensatory moments on the contralateral sound side, and the effort required to walk normally. In static (also known as bench) alignment, the components are positioned relative to each other based on principles from statics [48]. This alignment is used as the starting point for the dynamic alignment. In dynamic alignment, the amputee dons the prosthesis and a prosthetist (a professional who chooses, fabricates, and fits prostheses) performs the alignment by watching the prosthesis user walk. As techniques for prosthetic alignment stand currently, the dynamic alignment is largely based on heuristics—primarily observation of the amputee’s gait and subjective feedback from the amputee on how the prosthesis feels. As a result, the final alignment varies from prosthetist to prosthetist [49], and is rarely repeatable, even by the same prosthetist [50].

The effects of poor alignment can be dramatic: changes of a few millimeters or a few degrees from a functional alignment can render the prosthesis unstable (it collapses under normal use) or so uncomfortable to walk with that the patient is unwilling to use the device [51]. The quality of the alignment, and therefore performance of a prosthesis, is highly dependent upon the prosthetist’s skill. Attempts to determine scientific bases for alignment have been mostly with respect to static considerations [48]. Other studies deal with effects of alignment on the geometry of motion, such as the roll-over shape approach [3].

The asymmetric gait model developed in this dissertation allows examination of various aspects of the gait of transtibial prosthesis users. The gait of transtibial prosthesis users and the influence of prosthetic alignment on the gait dynamics are the application areas of focus in this dissertation.
1.4 Research approach

This dissertation attempts to address the many challenges of modeling human walking that arise from (i) the body’s many degrees of freedom (DOF), (ii) changing constraints due to the single and double support phases, and (iii) intermittent contact that may be impulsive, when the swinging leg strikes the ground. Due to the complexity of modeling both normal and pathological human gait, many approaches model only specific parts of the gait cycle. For example, the single support phase of walking has been modeled using an inverted pendulum [11, 12] and passive models [13, 14, 15]. Several ballistic swing models based on a double pendulum have been developed to model the swing leg [52]. This dissertation uses a robotics-inspired approach to model a complete gait cycle taking into account the cycle’s hybrid nature.

The choice of anthropomorphic features used to represent the human is an important modeling decision, especially when the goal of modeling is to gain clinical insight into gait control. Forward dynamic modeling approaches tend to use models that are rich in anthropomorphic details. Complex models that attempt to capture details such as joints with multiple DOF, musculature, etc. are analytically intractable—it is not possible to describe their behavior in mathematically manageable terms because of the large number of variables and redundancies involved. The approach taken in this dissertation uses an anthropomorphic model with minimal features to strike a balance between analytical tractability and clinical usefulness.

The modeling approach taken in this dissertation is based on the observed parsimony of human gait (see Figure 1.5). Based on the parsimony, the modeling approach hypothesizes that the human applies joint torques to control the posture in a specific manner that enables forward progression of the COM. This parsimony hypothesis is used to derive a
low-dimensional hybrid model that is an exact sub-dynamic of a higher-dimensional anthropomorphic hybrid model. The result is an analytically tractable, forward dynamic model for normal human walking that captures the essence of walking dynamics in the sagittal plane over a complete gait cycle.

It is hoped that the model developed in this dissertation will serve as a template dynamical system for the analysis of more complex, anthropomorphic models of walking [53]. The control mechanisms necessary to reduce a higher-dimensional anthropomorphic model (known as an “anchor”) to the low-dimensional template will provide insight that could be clinically useful for the treatment of pathologies. What differentiates such a low-dimensional model from a simplified model, such as the inverted pendulum, is that this model is not an approximation, but a true sub-dynamic of a more complex, anthropomorphic model.

The idea of a dynamical systems approach to studying human gait and its benefits has been explored by Clark [54] to study the evolution of walking in infants and compare it to adult gait. Clark advocates looking at gait data to identify the patterns of human gait and to identify control parameters that affect this pattern. Barela et al. [55] also use gait data to perform analysis of hemiparetic gait based on a dynamical systems approach.

A low-dimensional modeling approach has been used to control biped walking in robots [56, 57] and is the inspiration for the modeling of human walking described here. The robotics work is applicable to planar biped robots with point feet and to walking such that the double support phase is instantaneous. The presence of feet and a non-instantaneous double support phase are essential characteristics of human walking and are taken into account in the modeling of human gait undertaken here.

Hemiparesis is paralysis affecting one side of the body. Hemiparetic gait is the gait of a person with hemiparesis.
This dissertation’s approach addresses the inherent underactuation present in human walking. In walking, if too much torque is applied at the ankle of the supporting foot, the foot rolls over. The robotics work that is the inspiration for this work models the underactuation using point feet. In this dissertation, the stance foot-ankle complex is represented using the roll-over shape (ROS) [3]. The resultant rolling motion of the foot captures the effective underactuation in human walking. In addition, use of the ROS enables the modeling of prosthetic feet of varying stiffnesses and the parameters of the ROS can be varied to represent different prosthetic alignments.

To make the modeling approach clinically useful, the low-dimensional model developed for normal human walking is extended to model asymmetric or pathological gait. In one gait cycle of pathological gait, the two steps are generally asymmetric. The asymmetry arises from unequal parameters of the right and left legs, or differing joint motions when the left and right legs alternate in the role of the stance leg, or both. The asymmetric gait hybrid model enables the modeling of pathologies such as leg length discrepancies or walking with a prosthesis. The application of focus in this research is asymmetric gait that results from walking with a transtibial prosthesis. Cost functions are proposed and used to quantify the change in gait dynamics when parameters associated with a prosthesis are changed. The perturbations studied were changes in sagittal-plane prosthetic alignment, variation in the mass distribution of the prosthesis, changes in the stiffness of the prosthetic foot being used, and combinations of these variations. Simulation of the model with these perturbations suggests uses for the asymmetric model in a clinical setting. For example, subject-specific parameters could be used with the asymmetric gait model to prescribe an optimal prosthetic alignment, or to evaluate the costs and potential benefits of using different prosthetic components that may vary in weight and expense.
1.5 Organization of dissertation

Chapter 2 describes the modeling methodology based on the parsimony of human gait using a general anthropomorphic model. The details of the full hybrid model for one complete gait cycle and the derivation of the low-dimensional model for human walking are presented. Chapter 3 provides details of the model’s application to normal human walking based on anthropometric data and joint trajectories obtained from [1]. Various simulation results are presented. Chapter 4 extends the modeling approach to model asymmetric gait. Scenarios related to gait with a transtibial prosthesis are investigated in Chapter 5 using the asymmetric gait model developed. Chapter 6 presents a summary of the dissertation, ideas for future work and conclusions. Appendix A provides a nomenclature table. Appendix B presents details of the roll-over shape approach used for modeling the stance ankle-foot complex. Appendix C presents some techniques for analyzing the dynamics of constrained systems.
CHAPTER 2

MODELING

This chapter details the modeling approach to human walking taken in this dissertation. The general methodology used to develop the gait model is presented, followed by details of the development of the full anthropomorphic hybrid model of a complete gait cycle. A hypothesis based on the observed parsimony of human gait is used to derive a low-dimensional hybrid model that is an exact sub-dynamic of the full hybrid model. The method of Poincaré is used to study the stability of the gait cycle.

2.1 Methodology

The primary motions of the body segments during walking that contribute to the forward progression of the center of mass (COM) are flexion and extension, which occur in the sagittal plane (the plane that divides the body into left and right halves, see Figure 1.1). As a result, the most significant dynamics occur in the sagittal plane, and, consequently, planar models are common in gait modeling [58, 59, 60, 61]. A planar model is also assumed in this dissertation.

One gait cycle of normal walking is assumed to be made up of two symmetric steps. Four different parts are identified in each step: the single support (SS) phase, impact at heel contact that marks the transition from SS to double support (DS), the DS phase and a
continuous transition from DS to SS (see Figure 1.4). Correspondingly, four sub-models are developed. The sub-models for SS and DS are described by continuous differential equations while the transition sub-models are algebraic maps. Thus, the overall model for a step is necessarily hybrid—a combination of continuous and discrete sub-models. The switching between the different sub-models is assumed to be kinematically driven. Taken together, the four sub-models form a hybrid model for one half of the gait cycle.

Based on the observed parsimony of human gait (Figure 1.5), the modeling approach hypothesizes that the effect of the joint torques applied by the human is to impose holonomic constraints on the posture as a function of forward progression. Using this hypothesis, a high-dimensional hybrid anthropomorphic model is reduced to a low-dimensional hybrid model to describe the task of walking in the sagittal plane.

A planar, anthropomorphic rigid-body model with \( n \) links is used to represent the human. The \( n \) generalized coordinates needed to describe this model are chosen such that one coordinate is absolute and determines the orientation of the entire body in the sagittal plane while the other coordinates determine the posture of the body. Assuming there is no actuation between the foot and the ground, the posture assumed by the human at any instant can be enforced by constraints on the at most\(^5\) \((n - 1)\) coordinates that determine the posture.

The choice of coordinates facilitates application of the parsimony hypothesis. Holonomic constraints are used to regulate the shape (posture) coordinates as a function of forward progression. With the posture constrained, the dynamics associated with the unactuated coordinate, which directly relates to forward progression, constitutes a low-dimensional description of the walking dynamics in the sagittal plane.

\(^5\)In DS, the legs form a closed chain. For this constrained system, fewer coordinates are needed to uniquely specify the posture.
The constraints to enforce the posture of the human are obtained from gait analysis data. The joint angles are parameterized in terms of the forward progression. The model parameters, masses, link lengths, and moments of inertia, correspond to the subject’s anthropometric data.

There are differences between this work and the work of Westervelt et al. [62] in developing a low-dimensional model used in the control of robotic biped walking. Rather than assuming point contact with the ground, in this work, the foot is modeled using Hansen’s [3] roll-over shape (ROS) approach. In addition, rather than assuming the DS phase is instantaneous, the DS phase is assumed to have finite duration. These extensions are necessary because the presence of feet and existence of a non-instantaneous DS phase are essential features of human walking.

2.2 Anthropomorphic model development

The model of a step consists of sub-models for single support, double support, the impact at heel contact and the transition from double to single support. The model for normal walking is assumed to satisfy the following hypotheses.

Walking hypotheses: Walking is assumed to be

WH1) steady-state walking on level ground;

WH2) comprised of non-instantaneous single and double support phases; and

WH3) such that a gait cycle consists of two symmetric steps.

2.2.1 Single support

Consider an anthropomorphic model comprised of \( n_s \) rigid links to represent the human. A minimal anthropomorphic rigid-body model for SS consists of two shanks, two
Figure 2.1: Coordinates for a minimal anthropomorphic model in double support (DS) and single support. Note the choice of coordinates as one absolute ($q_a$) and the others relative. The relative coordinates specify the shape or posture of the model, and the circles indicate the internal degrees of freedom (DOF). Note also the additional DOF at the trailing ankle in the DS model. The stance foot is modeled using the roll-over shape (ROS) approach [3]. The ROS does not apply to the swing foot, and, hence, that foot is depicted differently. The Cartesian coordinates of the rolling point, the swing heel, and the trailing toe are $(x_R, y_R)$, $(x_{h_2}, y_{h_2})$, and $(x_{t_2}, y_{t_2})$, respectively.
thighs and a HAT segment that represents collectively, the head, arms, and trunk; see Figure 2.1. A minimal anthropomorphic model for DS has an additional joint at the trailing ankle to account for the increased plantarflexion that occurs during DS. The model development given here is applicable to more complex anthropomorphic models comprising, for example, additional links to represent the arms, etc. For this reason, the model derivation presented here is for a general anthropomorphic model. The anthropomorphic model for normal gait is assumed to satisfy the hypotheses given below.

_Anthropomorphic model hypotheses:_ The anthropomorphic model is assumed to

HM1) be comprised of rigid links with mass, connected by revolute joints;

HM2) model motions only in the sagittal plane;

HM3) be expressed in angular coordinates such that one coordinate is absolute and the rest are relative;

HM4) have a stance foot that rolls on the ground based on the roll-over shape approach [3];

HM5) have an additional DOF at the trailing ankle joint in double support; and

HM6) have symmetric legs.

In SS, only one leg is in contact with the ground as the body moves forward. In this phase, human walking is inherently underactuated because if too much torque is applied about the ankle joint of the supporting leg, the foot rolls over [63, 64]. To capture this underactuation, the ankle-foot complex is modeled using the roll-over shape approach [3]. The ROS captures the kinematics of the motion of the shank, ankle and foot of the supporting limb in SS. With the one foot rolling on the ground, the model has \( n_s \) DOF: \( (n_s - 1) \)
Figure 2.2: The roll-over shape [3] is obtained by representing the center of pressure in a shank-based coordinate system with the ankle as origin. It represents the effective rocker the ankle-foot complex conforms to in the period between heel contact and opposite heel contact. The shape can be well-approximated by a circular arc and allows the stance foot motion to be modeled as a rolling contact with the ground.

internal angles that determine the posture of the body and an absolute angle that specifies the orientation of the human in the sagittal plane.

Using the ROS approach [3], the shape of the ankle-foot complex is modeled as a circular arc (see Figure 2.2). The ROS approach models the location of the center of pressure\(^6\) (COP) in a shank-based coordinate system between heel contact and opposite heel contact (OHC) and is thus an effective rocker\(^7\) to which the foot conforms during this phase of gait. With the ROS approach, the rolling motion of the foot in SS happens without actuation between the foot and the ground (there is no ankle joint present), and hence it captures the effective underactuation. Between OHC and toe-off, the trailing limb is being rapidly unloaded and ceases to act as a rocker.

\(^6\)The COP is the location of the net ground reaction force (GRF).

\(^7\)Perry [65] describes the action over a step of the ankle-foot complex using three rockers based on the center of rotation of the shank with respect to the foot. The ROS is a model that uses the center of pressure (COP) to generate a single rocker to describe the action of the ankle-foot complex.
The forward kinematics of the rigid multi-body system representing the human is computed using standard robotics techniques. Knowing the positions and velocities of the various links and the model parameters, the kinetic energy $K$ and potential energy $V$ of the system are computed. The equations of motion (EOM) are computed using the method of Lagrange [66].

The EOM for the single support phase can be expressed as

$$D_s(q_s)\ddot{q}_s + C_s(q_s, \dot{q}_s)\dot{q}_s + G_s(q_s) = B_su_s,$$

where $q_s := (q_1, \ldots, q_{n_s}) \in Q_s, Q_s \subset T^{n_s}$, and $q_s$ is a set of angular coordinates specifying the configuration of the human, $\dot{q}_s$ represent the associated velocities and $u_s$ are the $(n_s - 1)$ input joint torques. $D_s$ is the generalized inertia matrix, $C_s$ is the matrix representing the Coriolis and centripetal terms, $G_s$ represents the contributions due to gravity, and $B_s$ is the Jacobian mapping the applied torques to the respective joints. Let $q_a \in S^1$ denote the absolute coordinate and $q_{rel,s} \in T^{n_s-1}$ denote the coordinates required to define the shape. Then, $q_s = (q_{rel,s}, q_a)$.

The SS sub-model may be written in state space form as

$$\dot{x}_s = \begin{bmatrix} \dot{q}_s \\ D_s^{-1}(-C_s\dot{q}_s - G_s) \end{bmatrix} + \begin{bmatrix} 0 \\ D_s^{-1}B_s \end{bmatrix} u_s,$$

$$=: f_s(x_s) + g_s(x_s) u_s,$$

where $x_s := (q_s, \dot{q}_s)$. The state space of the sub-model is $TQ_s := Q_s \times \mathbb{R}^{n_s}$.

The curves for the joint angles over a step are parameterized by forward progression. The scalar quantity used to represent forward progression, $\theta_s : Q_s \rightarrow \mathbb{R}$ ($\theta_d$ in DS), is chosen to be a function on the generalized coordinates that monotonically increases over a

8$T^p$ indicates the $p$-dimensional torus.
step, i.e., over half the gait cycle. As a result, the time derivative of the forward progression is related to the walking speed.

The choice of the generalized coordinates as one absolute and \((n_s - 1)\) relative enables the application of the hypothesis based on the parsimony of human gait: the human moves forward by coordinating the body’s posture in a specific manner. It is assumed that the torques are applied to joints \(q_{rel,s}\) to enforce the shape of the body as a function of \(\theta_s\) and not time. Therefore, the closed-loop system is autonomous (time-invariant). This fact will facilitate the stability analysis of Section 2.4.

2.2.2 Double support

During double support, the trailing foot is assumed to pivot about the end point of the ROS. The end point is the location where rolling ends at the instant of OHC. This point will hereafter be referred to as the toe. An additional DOF is introduced at the trailing ankle since the ankle undergoes considerable plantarflexion in DS. Let \(n_d = n_s + 1\) denote the total number of links (joints) in the DS sub-model. Both feet are in contact with the ground, and the trailing foot pivots about the toe as the leading foot continues to roll forward. The pivoting of the trailing foot does not conflict with the ROS approach since the ROS applies only to the stance foot. The two additional constraints introduced lead to a \((n_d - 2)\)-DOF double support sub-model.

The EOM for double support are of the form

\[
D_d(q_d)\ddot{q}_d + C_d(q_d, \dot{q}_d)\dot{q}_d + G_d(q_d) = B_d u_d + A^T \lambda, \tag{2.4}
\]

where \(q_d := (q_1, \ldots, q_{n_d}) \in Q_d, Q_d \subset \mathbb{T}^{n_d}\), and \(\lambda = (\lambda_1, \lambda_2)\) are the Lagrange multipliers corresponding to the constraints on the trailing toe. The state space of the sub-model is \(TQ_d := Q_d \times \mathbb{R}^{n_d}\).
The angular coordinates in DS can be partitioned as $(q_d, q_{d_d})$ where $q_d$ are the $(n_d - 2)$ independent generalized coordinates and $q_{d_d}$ are the 2 dependent coordinates. The relationship between the independent and dependent coordinates is non-linear, and so the dependent coordinates cannot be eliminated easily from the EOM. The constraints on the trailing toe are introduced into the EOM using the Lagrange multipliers. Partitioning the coordinates into independent and dependent coordinates enables the use of the embedding method [67] to eliminate the Lagrange multipliers and develop the EOM in terms of only the independent coordinates. The independent coordinates are chosen to include the absolute coordinate. Let $q_a$ denote the absolute coordinate while the other $(n_d - 3)$ independent coordinates $q_{rel,d}$ are actuated to establish the posture in DS as a function of forward progression. Thus, $q_{d_i} = (q_{rel,d}, q_a)$.

Let $(x_{t_2}, y_{t_2})$ denote the Cartesian coordinates of the position of the trailing toe, with the constant position $(k, 0)$ in DS. The constraints can be written in the form

$$g(q_d) := \begin{bmatrix} x_{t_2}(q_d) - k \\ y_{t_2}(q_d) \end{bmatrix} = 0,$$  \hspace{1cm} (2.5)

which implies

$$A\dot{q}_d = 0,$$  \hspace{1cm} (2.6)

where

$$A = \partial g / \partial q_d.$$  \hspace{1cm} (2.7)

Differentiating (2.6) with respect to time results in

$$A\ddot{q}_d + \dot{A}\dot{q}_d = 0,$$  \hspace{1cm} (2.8)

which implies

$$A\ddot{q}_d = -\dot{A}\dot{q}_d = g_c.$$  \hspace{1cm} (2.9)
With $q_d$ partitioned into independent and dependent coordinates, and $A$ partitioned accordingly into $A_i$ and $A_d$, (2.6) implies

$$A_i \dot{q}_{di} + A_d \dot{q}_{dd} = 0,$$

(2.10)

which implies

$$\dot{q}_{dd} = - A_d^{-1} A_i \dot{q}_{di},$$

(2.11)

and (2.9) becomes

$$A \ddot{q}_d = A_i \dot{q}_{di} + A_d \ddot{q}_{dd} = g_c.$$

(2.12)

Let

$$A_{di} := \begin{bmatrix} I & \varepsilon(q_{di}) \\ -A_d^{-1} A_i & \end{bmatrix} \quad \text{and} \quad g_{dc} := \begin{bmatrix} 0 \\ A_d^{-1} g_c \end{bmatrix}. \quad (2.13)$$

**Remark 1** Since there exists a solution to $g = 0$ at HC, and since $|A_d| = |\partial g / \partial q_{dd}|$ is non-zero at that point, by the Implicit Function Theorem, there exists a nonlinear mapping, $q_{dd} = \varepsilon(q_{di})$, from the independent coordinates in DS to the dependent coordinates.

Therefore, the configuration in DS is given by

$$q_d = \begin{bmatrix} q_{di} \\ \varepsilon(q_{di}) \end{bmatrix} =: \nu(q_{di}),$$

(2.14)

and the velocities are given by

$$\dot{q}_d = A_{di}(q_d) \dot{q}_{di} = A_{di}(\nu(q_{di})) \dot{q}_{di}.$$

(2.15)

**Proposition 1** Using the embedding method [67], the Lagrange multipliers are eliminated to yield the DS EOM in the form

$$D_d(q_d) \ddot{q}_d = N(q_d, \dot{q}_d) + A_{di}^T(q_d) B_d u_d,$$

(2.16)
where
\[ D_{d_i} := A_{d_i}^T D_d A_{d_i}, \quad (2.17a) \]
\[ N := -A_{d_i}^T (D_d q_{d_{de}} + C_d \dot{q}_d + G_d). \quad (2.17b) \]

Proof. The EOM for the constrained system are
\[ D_d \ddot{q}_d + C_d \dot{q}_d + G_d = B_d u_d + A^T \lambda, \quad (2.18) \]

where \( \lambda = (\lambda_1, \lambda_2) \) are the Lagrange multipliers corresponding to the constraints. Use of the embedding method [67] to eliminate the Lagrange multipliers proceeds as follows. Project (2.18) onto the unconstrained directions of motion by pre-multiplication by \( A_{d_i}^T \). This calculation results in
\[ A_{d_i}^T D_d \ddot{q}_d + A_{d_i}^T C_d \dot{q}_d + A_{d_i}^T G_d = A_{d_i}^T B_d u_d + A_{d_i}^T A^T \lambda. \quad (2.19) \]

Note that
\[ A_{d_i}^T A^T = \begin{bmatrix} I \\ -A_d^{-1} A_i \end{bmatrix}^T A^T, \quad (2.20a) \]
\[ = \begin{bmatrix} I^T & -(A_d^{-1} A_i)^T \end{bmatrix} \begin{bmatrix} A_i^T \\ A_d^T \end{bmatrix}, \quad (2.20b) \]
\[ = A_i - A_i^T A_d^{-T} A_d^T = 0. \quad (2.20c) \]

Therefore, the resulting differential equation (2.19) does not involve \( \lambda \). The EOM for the constrained system may be expressed as
\[ D_d \ddot{q}_d = N + A_{d_i}^T B_d u_d, \quad (2.21) \]
where

\[ D_{d_i} := A_{d_i}^T D_d A_{d_i} \]  

(2.22a)

\[ N := -A_{d_i}^T (D_d g_{dc} + C_d \dot{q}_d + G_d) \]  

(2.22b)

\[ g_{dc} = \begin{bmatrix} 0 \\ -A_d^{-1} g_c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -A_d^{-1} \dot{A}_d \dot{q}_d \end{bmatrix} \]  

(2.22c)

\[ g_{dc} = \begin{bmatrix} 0 \\ -A_d^{-1} \dot{A}_d A_{d_i} \end{bmatrix} \dot{q}_{d_i}. \]  

(2.22d)

This proves Proposition 1.

**Proposition 2** Taking into consideration Remark 1 and the form of the EOM given in Proposition 1, the EOM (2.16) can be written as

\[ D_{d_i}(q_{d_i}) \ddot{q}_{d_i} + \Gamma(q_{d_i}, \dot{q}_{d_i}) \dot{q}_{d_i} + E(q_{d_i}) = M(q_{d_i}) u_d, \]  

(2.23)

where

\[ \Gamma(q_{d_i}, \dot{q}_{d_i}) = A_{d_i}^T D_d \zeta + A_{d_i}^T C_d A_{d_i}, \]  

(2.24a)

\[ E(q_{d_i}) = A_{d_i}^T G_d, \]  

(2.24b)

\[ M(q_{d_i}) = A_{d_i}^T B_d, \]  

(2.24c)

and where

\[ \zeta(q_{d_i}, \dot{q}_{d_i}) = \begin{bmatrix} 0 \\ -A_d^{-1} \dot{A}_d A_{d_i} \end{bmatrix} \]  

(2.25)

*Proof.* From (2.22d),

\[ g_{dc} = \zeta(q_{d_i}, \dot{q}_{d_i}) \dot{q}_{d_i}, \]  

(2.26)

where

\[ \zeta(q_{d_i}, \dot{q}_{d_i}) = \begin{bmatrix} 0 \\ -A_d^{-1} \dot{A}_d A_{d_i} \end{bmatrix} \]  

(2.27)
Then,

\[
D_{d_i} \ddot{q}_{d_i} = -A_{d_i}^T (D dg_{dc} + C_{d_i} \dot{q}_{d_i} + G_d) + A_{d_i}^T B_{d_i} u_d
\]  
\tag{2.28}

\[
= - (A_{d_i}^T D_d \zeta + A_{d_i}^T C_d A_{d_i}) \dot{q}_{d_i} - A_{d_i}^T G_d + A_{d_i}^T B_{d_i} u_d. 
\]  
\tag{2.29}

Let

\[
\Gamma(q_{d_i}, \dot{q}_{d_i}) := A_{d_i}^T D_d \zeta + A_{d_i}^T C_d A_{d_i},
\]  
\tag{2.30a}

\[
E(q_{d_i}) := A_{d_i}^T G_d (q_{d_i}),
\]  
\tag{2.30b}

\[
M(q_{d_i}) := A_{d_i}^T B_{d_i}.
\]  
\tag{2.30c}

The DS EOM in terms of only the independent coordinates is

\[
D_{d_i} (q_{d_i}) \ddot{q}_{d_i} + \Gamma(q_{d_i}, \dot{q}_{d_i}) \dot{q}_{d_i} + E(q_{d_i}) = M(q_{d_i}) u_d. 
\]  
\tag{2.31}

This proves\(^9\) Proposition 2.

The DS sub-model (2.23) may be written in state space form as

\[
\dot{x}_{d_i} = \begin{bmatrix} \dot{q}_{d_i} \\ D_{d_i}^{-1} (-\Gamma \dot{q}_{d_i} - E) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{d_i}^{-1} M \end{bmatrix} u_d 
\]  
\tag{2.32}

\[
=: f_{d_i}(x_{d_i}) + g_{d_i}(x_{d_i}) u_d, 
\]  
\tag{2.33}

where \(x_{d_i} := (q_{d_i}, \dot{q}_{d_i})\). The state space of the sub-model is taken as \(T Q_{d_i} = Q_{d_i} \times \mathbb{R}^{n_d-2}\), where \(Q_{d_i}\) is a simply connected, open subset of \(\mathbb{T}^{n_d-2}\) corresponding to the independent configuration variables.

In DS, the system is, in fact, overactuated since there are more inputs than the number of DOF. Despite this, it is assumed, as in SS, that the system has one degree of underactuation. This assumption is made to model the ‘passive’ nature of human walking at steady-state in

\(^9\)This proof is a result of collaboration with Ioannis Raptis [68].
DS. Namely, it is hypothesized that the posture imposed is driven by forward progression and not time.

Hence, in double support, as in single support, torques are applied to all the joints corresponding to independent coordinates, except for the joint corresponding to \( q_a \). Note that \( q_a \) represents the absolute orientation of the entire human.

Double support is assumed to end when the body attains a certain posture. The body then transitions into single support with the trailing leg lifting off the ground.

### 2.2.3 Transition mappings

**Transition from single support to double support**

The transition from single support to double support is modeled as a rigid impact that occurs when the heel of the swinging limb contacts the ground. This transition mapping is referred to as the impact map. At heel contact, the impact is assumed to be instantaneous and inelastic, i.e., the heel sticks on contact instead of slipping or rebounding. It is also assumed that there is no change in the configuration variables during the impact. The assumptions are similar to those for the instantaneous DS map in [62] and can be summarized as follows.

*Impact map hypotheses:* The impact map is assumed to be such that

IH1) the impact is instantaneous;

IH2) the configuration remains unchanged after impact although there is an instantaneous change in the velocities;

IH3) the impact is inelastic: the swing leg heel sticks on contact; and

IH4) there are no impulsive joint torques generated during the impact.
The impact map uses an extended dynamic model with \((n_s + 2)\) DOF. \(n_s\) DOF correspond to joints \(q_1\) to \(q_{n_s}\), and 2 DOF correspond to the Cartesian coordinates of the point of rolling contact. The impulse-momentum balance equations at impact [69] are

\[
D_e(q^+_e)\dot{q}^+_e = D_e(q^-_e)\dot{q}^-_e + J^T(q^-_e)\delta F
\]

(2.34)

where the subscript \(e\) denotes the extended model, \(J\) is the Jacobian of the functions representing the position of the impacting heel and \(\delta F\) are the impulses at the impact point. The superscripts “+” and “−” indicate the state immediately after and before impact, respectively.

The post-impact velocities are computed using the impact map, which equates the change in the generalized momenta to the generalized impulses generated as the heel of the swing leg contacts the ground. The impact map also relabels the coordinates such that the roles of the swing and stance legs are swapped.

If the stance foot is rolling without slipping, the extended coordinates \(q_e\) and their velocities \(\dot{q}_e\) are related to \(q_s\) and \(\dot{q}_s\) by

\[
q_e = \Upsilon(q_s)
\]

(2.35a)

\[
\dot{q}_e = \frac{\partial \Upsilon(q_s)}{\partial q_s} \dot{q}_s,
\]

(2.35b)

where \(\Upsilon(q_s) := (q_s, x_R, y_R)^T\) and \(x_R\) and \(y_R\) are the horizontal and vertical positions of the point of rolling contact on the stance foot; see Figure 2.1.

Since the configuration remains unchanged after impact,

\[
q^+_e = q^-_e.
\]

(2.36)

The swing leg heel sticks on contact, which implies

\[
J\dot{q}^+_e = 0,
\]

(2.37)
where \( J = \partial E(q_e) / \partial q_e \) and \( E(q_e) = (x_{h_2}, y_{h_2}) \) are the Cartesian coordinates of the swing leg heel. With the impulse-momentum balance equations given by (2.34), the post-impact velocities and the impulses can be calculated using

\[
\begin{bmatrix}
D_e(q_e^-) & -J^T \\
J & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_e^+ \\
\delta F
\end{bmatrix} =
\begin{bmatrix}
D_e(q_e^-) \dot{q}_e^- \\
0
\end{bmatrix},
\]

Let

\[
\Pi := \begin{bmatrix}
D_e(q_e^-) & -J^T \\
J & 0
\end{bmatrix}^{-1}.
\]

The map from velocities just prior to impact to just after impact (without relabeling) is obtained by partitioning \( \Pi(q_e^-) \) and is

\[
\dot{q}_e^+ = \Pi_{11} (q_e^-) D_e (q_e^-) \dot{q}_e^-.
\]

Performing the leg swapping and extracting the independent variables of double support, the positions at the beginning of DS are given by

\[
q_{di}^+ = \Omega_{di} R_0 q_{di}^+ = \Omega_{di} R_0 \Upsilon (q_s^-),
\]

where the constant matrix \( \Omega_{di} \in \mathbb{R}^{(n_d-2) \times (n_s+2)} \) extracts the independent variables in double support and the constant matrix \( R_0 \in \mathbb{R}^{(n_s+2) \times (n_s+2)} \) relabels the coordinates of the swing and stance legs. The velocities at the beginning of DS are mapped according to

\[
\dot{q}_{di}^+ = \Omega_{di} R_0 \dot{q}_e^+ = \Omega_{di} R_0 \Pi_{11} (q_e^-) D_e (q_e^-) \dot{q}_e^- = \Omega_{di} R_0 \Pi_{11} (\Upsilon(q_s^-)) D_e (\Upsilon(q_s^-)) \frac{\partial \Upsilon (q)}{\partial q} \bigg|_{q=q_s^-}.
\]

Let \( x_{di}^+ := (q_{di}^+, \dot{q}_{di}^+) \) and \( x_s^- := (q_s^-, \dot{q}_s^-) \). The mapping that relates the state at the beginning of DS to the state at the end of SS just before impact is denoted by

\[
x_{di}^+ = \Delta_{s}^{di} (x_s^-),
\]
where
\[
\Delta_{d_i}^{d_s}(x_s^-) := \begin{bmatrix}
\Delta_{q_s}^{-}(q_s^-) \\
\Delta_{\dot{q}_s}^{-}(q_s^-) \dot{q}_s^-
\end{bmatrix},
\] (2.44)
and
\[
\Delta_{q_s} := \Omega_{d_i} R_0 \Upsilon(q_s^-),
\] (2.45a)
\[
\Delta_{\dot{q}_s} := \Omega_{d_i} R_0 \Pi_{11} \left( \Upsilon(q_s^-) \right) D_e \left( \Upsilon(q_s^-) \right) \frac{\partial \Upsilon(q)}{\partial q} \bigg|_{q=q_s^-}.
\] (2.45b)

The superscripts “+” and “−” denote the beginning and end of the corresponding phase, respectively.

The transition from SS to DS takes place when
\[
H_{s_i}^{d_s}(x_s) := y_{h_2}(x_s) = 0
\]
where \( y_{h_2} : TQ_s \rightarrow \mathbb{R} \) gives the vertical height of the swing heel. The contact of the swing heel takes place for a specific posture which corresponds to a particular value of \( \theta_s \). Therefore, the condition for transition from SS to DS can also be expressed as
\[
H_{s_i}^{d_s}(x_s) := \theta_s(q_s) - \theta_s^{-},
\]
where \( \theta_s^- \) denotes the value of \( \theta_s \) that corresponds to contact of the swing heel with the ground.

**Transition from double support to single support**

Let \( H_{d_i}^{s}(x_{d_i}) = 0 \) denote the condition for the transition from DS to SS to take place. The transition from double to single support takes place when \( \theta_d \), the variable corresponding to forward progression, achieves a specific value \( \theta_d^- \). The value of \( \theta_d^- \) is determined from gait data using the joint angles at the instant of toe-off. Accordingly,
\[
H_{d_i}^{s}(x_{d_i}) := \theta_d(q_{d_i}) - \theta_d^-.
\]

The configuration and velocities at the end of DS are obtained using (2.14) and (2.15), respectively. Then, the configuration at the beginning of SS is given by
\[
q_s^+ = \Omega_s q_d^-,
\] (2.46)
which implies
\[ q^+ = \Omega_s \nu (q_{d_i}^-) =: \Delta_{q_{d_i}}(q_{d_i}^-); \]  
(2.47)

and the velocities are given by
\[ \dot{q}^+_s = \Omega_s A_{d_i} (\nu (q_{d_i}^-)) \dot{q}_{d_i}^- =: \Delta_{\dot{q}_{d_i}}(q_{d_i}^-) \dot{q}_{d_i}^-, \]  
(2.48)

where \( \Omega_s \) is a constant matrix that performs the reordering of the coordinates from DS to those required by the SS sub-model.

If \( x^+_s := (q^+_s, \dot{q}^+_s) \) represents the state at the beginning of SS and \( x^-_{d_i} := (q^-_{d_i}, \dot{q}^-_{d_i}) \) represents the state at the end of DS, the mapping from DS to SS can be expressed as
\[ x^+_s = \Delta^s_{d_i}(x^-_{d_i}), \]  
(2.49)

where
\[ \Delta^s_{d_i}(x^-_{d_i}) = \begin{bmatrix} \Delta_{q_{d_i}}(q^-_{d_i}) \\ \Delta_{\dot{q}_{d_i}}(q^-_{d_i}) \dot{q}_{d_i}^- \end{bmatrix}. \]  
(2.50)

### 2.2.4 Overall full hybrid model

The overall model can be expressed as a non-linear hybrid system containing two charts (see Figure 2.3),

\[
\begin{align*}
\Sigma_s : & \quad \mathcal{X}_s = TQ_s \\
& \quad \mathcal{F}_s : \dot{x}_s = f_s(x_s) + g_s(x_s)u_s \\
& \quad S^d_{d_i} = \{ x_s \in TQ_s \mid H^s_{d_i}(x_s) = 0 \} \\
& \quad T^d_{d_i} : x^+_s = \Delta^d_{d_i}(x^-_{d_i}) \\
\Sigma_{d_i} : & \quad \mathcal{X}_{d_i} = TQ_{d_i} \\
& \quad \mathcal{F}_{d_i} : \dot{x}_{d_i} = f_{d_i}(x_{d_i}) + g_{d_i}(x_{d_i})u_{d} \\
& \quad S^s_{d_i} = \{ x_{d_i} \in TQ_{d_i} \mid H^s_{d_i}(x_{d_i}) = 0 \} \\
& \quad T^s_{d_i} : x^+_s = \Delta^s_{d_i}(x^-_{d_i})
\end{align*}
\]  
(2.51)
Figure 2.3: Discrete-event system corresponding to the dynamics of one step (half a normal gait cycle).

where $\mathcal{F}$ denotes the vector field on manifold $\mathcal{X}$, $\mathcal{S}$ is the switching hyper surface and $T: \mathcal{S} \rightarrow \mathcal{X}$ is the transition function applied when $x \in \mathcal{S}$.

2.3 Derivation of low-dimensional model

To make more intuitive the derivation of the low-dimensional model, consider a simple example: a rod moving in a plane. When unconstrained, the dynamics of the rod can be described in terms of 3 independent quantities, the Cartesian coordinates $p_x$ and $p_y$ of one of the rod’s ends and the angular orientation $\phi$. Thus, the system has 3 DOF. Assuming the uniform rod has length $2l$, mass $m$ and moment of inertia $I_z$ about its COM, the EOM of the rod are given by

$$ m\ddot{p}_x + ml\sin(\phi)\ddot{\phi} + m\dot{\phi}^2l\cos(\phi) = F_x, \quad (2.52a) $$

$$ m\ddot{p}_y - ml\cos(\phi)\ddot{\phi} + m\dot{\phi}^2l\sin(\phi) + mg = F_y, \quad (2.52b) $$

$$ ml\sin(\phi)\ddot{p}_x - ml\cos(\phi)\ddot{p}_y + (ml^2 + I_z)\ddot{\phi} + mgl\cos(\phi) = u. \quad (2.52c) $$
Let \( q := (p_x, p_y, \phi) \in Q, Q = \mathbb{R}^2 \times S^1 \), and \( F := (F_x, F_y, u) \in \mathbb{R}^3 \) represents the non-conservative applied forces on the system. The state space of the system is \( TQ := Q \times \mathbb{R}^3 \).

When the end, \((p_x, p_y)\), is constrained by a smooth holonomic constraint \( h(p_x, p_y) = 0 \), the motion of the rod will be restricted to a surface, a sub-manifold of the configuration space, \( Q \).

Consider the simple constraint on the end, \( p_x \equiv c_x, p_y \equiv c_y \), i.e., the end is fixed. The state of the rod now evolves in the manifold \( Z := \{(q, \dot{q}) \in TQ \mid p_x = c_x, p_y = c_y, \dot{p}_x = 0, \dot{p}_y = 0\} \). Differentiating the constraint gives \( \dot{p}_x = 0 \) and \( \dot{p}_y = 0 \), and differentiating again gives \( \ddot{p}_x = 0 \) and \( \ddot{p}_y = 0 \). Then, the dynamics of the rod on the constrained manifold reduce to the dynamics of the inverted pendulum and can be completely described by the dynamics of the coordinate \( \phi \). Substituting \( \ddot{p}_x = 0, \ddot{p}_y = 0 \) into (2.52c), the reduced EOM of this constrained system is given by

\[
(ml^2 + I_z) \ddot{\phi} + mgl \cos(\phi) = u. \tag{2.53}
\]

The forces \( F_x \) and \( F_y \) represent the (unique) constraint forces that enforce the constraint and can be computed using the full model and the constraint equations. The constraint forces are obtained by substituting \( \ddot{p}_x = 0, \ddot{p}_y = 0, \dot{p}_x = 0, \) and \( \dot{p}_y = 0 \) into (2.52a) and (2.52b) and are given by

\[
F_x \mid_Z = ml \sin(\phi) \ddot{\phi} + m\dot{\phi}^2 l \cos(\phi), \tag{2.54a}
\]
\[
F_y \mid_Z = -ml \cos(\phi) \ddot{\phi} + m\dot{\phi}^2 l \sin(\phi) + mg. \tag{2.54b}
\]

This example illustrates the concept of deriving a low-dimensional model as a sub-dynamic of a higher dimensional model by applying suitable holonomic constraints. Along similar lines, for human walking, a model that is a sub-dynamic of a more complex, higher DOF model may be obtained by applying holonomic constraints on the posture as a function
of forward progression. This model constitutes a low-dimensional description of sagittal plane walking. The low-dimensional model for walking is derived based on the parsimony hypothesis, the assumption that the human uses joint torques to coordinate body shape in a specific manner to achieve forward progression of the center of mass.

The derivation of the low-dimensional model for normal walking involves using inverse dynamics to compute the joint torques necessary to enforce the constraints on the shape (i.e., all of the actuated coordinates) as a function of the forward progression. The sub-dynamic obtained with the constraints exactly imposed and including the discrete transition maps then constitutes a low-dimensional description of the dynamics of walking in the sagittal plane.

Consider the output function (taken from [62])

$$y = h(q) := h_0(q) - h_d \circ \theta(q), \quad (2.55)$$

where the smooth function $h_0 : Q \rightarrow \mathbb{R}^{n-1}$ specifies the independent quantities to be controlled, and the smooth function $h_d : \mathbb{R} \rightarrow \mathbb{R}^{n-1}$ specifies the desired evolution of the independent quantities as a function of the monotonic quantity $\theta(q) : Q \rightarrow \mathbb{R}$. The output $y = h(q)$ is a holonomic constraint that depends only on the configuration variables. Driving $y$ to zero forces $h_0(q)$ to track $h_d \circ \theta(q)$. In this manner, the joint angles that determine the posture or shape of the anthropomorphic model are forced to follow the quantities $h_d$, which are parameterized by $\theta(q)$, and are therefore independent of time. The technical conditions that restrict the choice of $h(q)$ and $\theta(q)$ are outlined in [62].
The EOM for single and double support can be expressed as \( D\ddot{q} + C\dot{q} + G = Bu \), and in state-space form as \( \dot{x} = f(x) + g(x)u \). Differentiating \( y \) with respect to time results in

\[
\frac{dy}{dt} = \frac{\partial h}{\partial x} \dot{x}
\]

\[
= \left[ \frac{\partial h}{\partial q} \ 0 \right] \begin{bmatrix} \dot{q} \\ D^{-1}(-C\dot{q} - G) \end{bmatrix} =: L_f h
\]

\[
+ \left[ \frac{\partial h}{\partial q} \ 0 \right] \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u
\]

\[
= \frac{\partial h}{\partial q} \dot{q},
\]

(2.56a)

(2.56b)

(2.56c)

where the Lie derivative notation \( L_a b = (\partial b/\partial x)a \) has been used. Differentiating the output again computes the accelerations resulting in

\[
\frac{d^2y}{dt^2} = \left[ \frac{\partial}{\partial q} \left( \frac{\partial h}{\partial q} \dot{q} \right) \ \frac{\partial h}{\partial q} \right] \begin{bmatrix} \dot{q} \\ D^{-1}(-C\dot{q} - G) \end{bmatrix} =: L_f^2 h
\]

\[
+ \left[ \frac{\partial}{\partial q} \left( \frac{\partial h}{\partial q} \dot{q} \right) \ \frac{\partial h}{\partial q} \right] \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u
\]

\[
= L_f^2 h(q, \dot{q}) + L_f L_f h(q) u.
\]

(2.57a)

(2.57b)

Under the technical assumptions [62, HH1–HH4],

\[
u^*(x) = - (L_f L_f h(x))^{-1} L_f^2 h(x)
\]

(2.58)

is the unique control that is associated with \( y \equiv 0 \).

### 2.3.1 Single support

The EOM for single support are given by (2.1). Let \( h_{d_s} \) denote the smooth functions that represent the desired trajectories of the actuated joints in SS. Using the smooth functions
\( h_{ds}(\theta_s) \) to constrain the joint trajectories that specify the posture in SS, the dynamics of single support reduce to the dynamics of the unactuated coordinate, \( q_a \), given by the second order differential equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0,
\]

where \( L = K - V \) is the Lagrangian of the system.

By a suitable choice of coordinates, the low-dimensional model can be expressed in state-space form. The analysis for single support parallels the analysis for a biped robot described in [62] but with one important difference: the presence of a rolling foot contact instead of the point foot, which changes the form of the low-dimensional model. With a rolling foot, the kinetic energy is no longer independent of the absolute orientation of the entire human. The analysis leads to the following result specifying the form of the low-dimensional model for SS.

**Proposition 3 (Single support low-dimensional model)** Suppose the coordinates of the anthropomorphic model are chosen as one absolute and the others relative, and holonomic constraints \( h_{ds} \), parameterized by a monotonic function representing forward progression \( \theta_s \), are imposed on the posture. Let \( h_0(q_s) = H_s q_s \) represent the independent quantities to be controlled, where \( H_s \) is a constant matrix, and \( \theta_s = \zeta_s q_s \). Let \((z_{s,1}, z_{s,2})\) be chosen as coordinates for the low-dimensional model. Then, \((z_{s,1}, z_{s,2})\) parameterize the constraint surface

\[
Z_s := \left\{ x_s \in TQ_s \mid q_{\text{rel},s} = h_{ds}(\theta_s), \dot{q}_{\text{rel},s} = \frac{\partial h_{ds}}{\partial \theta_s} \dot{\theta}_s \right\},
\]

and, for

\[
z_{s,1} = \theta_s(q_s)|_{Z_s},
\]

\[
z_{s,2} = (D_{\theta_s}(q_s)\dot{\theta}_s)|_{Z_s},
\]
where $D_{sa}$ is the row of matrix $D_s$ corresponding to the unactuated coordinate $q_a$, the model takes the form

$$
\dot{z}_{s,1} = \kappa_1(z_{s,1})z_{s,2},
$$

(2.62a)

$$
\dot{z}_{s,2} = \kappa_2(z_{s,1}, z_{s,2}),
$$

(2.62b)

where

$$
\kappa_1 := \frac{\partial \theta_s}{\partial q_s} \left[ H_s - \frac{\partial h_{ds}}{\partial \theta_s} c_s \right]^{-1} \left[ \begin{array}{cc} 0 \\ 1 \end{array} \right] \Bigg|_{Z_s},
$$

(2.63a)

$$
\kappa_2 := \left( \frac{\partial K}{\partial q_a} - \frac{\partial V}{\partial \theta_a} \right) \Bigg|_{Z_s}.
$$

(2.63b)

**Proof.** The constraints on the actuated joints in single support can be expressed as $y = h(q_s) = h_0(q_s) - h_{ds}(\theta_s)$. On the constraint surface $Z_s$, $y \equiv 0$ implies

$$
h_0(q_s) = H_s q_s = h_{ds}(\theta_s).
$$

(2.64)

Then,

$$
\begin{bmatrix}
H_s \\
\cdot_c s
\end{bmatrix} q_s = \begin{bmatrix} h_{ds}(\theta_s) \\ \theta_s \end{bmatrix},
$$

(2.65)

or

$$
q_s = \begin{bmatrix} H_s \\
\cdot_c s
\end{bmatrix}^{-1} \begin{bmatrix} h_{ds}(\theta_s) \\ \theta_s \end{bmatrix}.
$$

(2.66)

Again, $\dot{y} = 0$ implies

$$
H_s \dot{q}_s - \frac{\partial h_{ds}}{\partial \theta_s} \dot{\theta}_s = 0,
$$

(2.67)

which leads to

$$
\begin{bmatrix} H_s - \frac{\partial h_{ds}}{\partial \theta_s} c_s \\
D_{sa}(q_s)
\end{bmatrix} \dot{q}_s = 0.
$$

(2.68)

Then,

$$
\begin{bmatrix} H_s - \frac{\partial h_{ds}}{\partial \theta_s} c_s \\
D_{sa}(q_s)
\end{bmatrix} \dot{q}_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix} z_{s,2}
$$

(2.69)
\[
\dot{q}_s = \begin{bmatrix}
H_s - \frac{\partial h_{ds}}{\partial q_s} c_s \\
D_{sa}(q_s)
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
1
\end{bmatrix} z_{s,2}.
\] (2.70)

For the chosen coordinates,
\[
\dot{z}_{s,1} = L_f \dot{z}_{s,1} = \frac{\partial z_{s,1}}{\partial q_s} \dot{q}_s,
\] (2.71)

and
\[
\dot{z}_{s,2} = L_f \dot{z}_{s,2} = \left[ q_s^T \frac{\partial D^T_{sa}}{\partial q_s} D_{sa} \right] \begin{bmatrix}
\dot{q}_s \\
-D_s^{-1}(C_s \dot{q}_s + G_s)
\end{bmatrix} (2.72a)
= q_s^T \frac{\partial D^T_{sa}}{\partial q_s} \dot{q}_s - C_{sa} \dot{q}_s - G_{sa}. (2.72b)
\]

For the kinetic energy in the single support phase,
\[
\frac{\partial K}{\partial q_a} = D_{sa}(q_s) \dot{q}_s, \tag{2.73}
\]
\[
\frac{\partial K}{\partial \dot{q}_a} = \frac{1}{2} q_s^T \frac{\partial D_s(q_s)}{\partial q_a} \dot{q}_s. \tag{2.74}
\]

Note that [66]
\[
C_{sa} = q_s^T \frac{\partial D^T_{sa}}{\partial q_s} - \frac{1}{2} q_s^T \frac{\partial D_s(q_s)}{\partial q_a} \dot{q}_s. \tag{2.75}
\]

Then, from (2.72c),
\[
\dot{z}_{s,2} = \frac{1}{2} q_s^T \frac{\partial D_s(q_s)}{\partial q_a} \dot{q}_s - G_{sa} = \frac{\partial K}{\partial q_a} - \frac{\partial V}{\partial q_a}. \tag{2.76}
\]

Therefore, the low-dimensional model for SS takes the form
\[
\dot{z}_{s,1} = \kappa_1(z_{s,1}) \dot{z}_{s,2}, \tag{2.77a}
\]
\[
\dot{z}_{s,2} = \kappa_2(z_{s,1}, \dot{z}_{s,2}). \tag{2.77b}
\]

This proves\textsuperscript{10} Proposition 3.

\textsuperscript{10}This proof is a result of collaboration with Ioannis Raptis [68].
Note that (2.77b) can also be derived directly from the Lagrangian. Since $q_a$ is unactuated,

$$\frac{d}{dt} \frac{\partial L}{\partial ˙q_a} - \frac{\partial L}{\partial q_a} = 0,$$

(2.78)

Since $(\partial V/\partial ˙q_a) = 0$, (2.78) reduces to

$$\frac{d}{dt} \frac{\partial K}{\partial ˙q_a} = \frac{\partial K}{\partial q_a} - \frac{\partial V}{\partial q_a}.$$

(2.79)

**Remark 2** For single support, the choice of the low-dimensional model’s coordinates has special significance. The coordinate $z_{s,1}$ represents the forward progression, and $z_{s,2} = \partial K/\partial ˙q_a$ represents the angular momentum of the entire human about the contact point of the stance leg with the ground.

### 2.3.2 Double support

The DS EOM in the space of the independent coordinates are given by Proposition 2. The independent coordinates are chosen to include $q_a$, which is assumed to be unactuated. Similar to SS, the other $(n_d - 3)$ independent coordinates are regulated to establish the posture in DS as a function of forward progression. Let $h_{dd,i}$ denote the smooth functions that represent the desired trajectories, $h_d$, of the actuated joints in DS.

**Proposition 4 (Double support low-dimensional model)** Suppose the coordinates of the anthropomorphic model are chosen as one absolute and the others relative, and holonomic constraints $h_{dd}$ are imposed on the posture that are parameterized by a monotonic function representing forward progression $θ_d$. Let $h_0(q_d) = H_{di}q_d$ represent the independent quantities to be controlled, where $H_{di}$ is a constant matrix and $θ_d = c_dq_d$. Let $(z_{d,i,1}, z_{d,i,2})$ be chosen as coordinates for the low-dimensional model. Then, $(z_{d,i,1}, z_{d,i,2})$ parameterize the
constraint surface

\[ Z_d := \left\{ q_{rel,d} \in T Q_d \mid q_{rel,d} = h_{dd_i}(\theta_d), \dot{q}_{rel,d} = \frac{\partial h_{dd_i}}{\partial \theta_d} \dot{\theta}_d \right\}, \quad (2.80) \]

and, for

\[ z_{d,1} = \theta_d(q_d)|_{Z_d}, \quad (2.81a) \]
\[ z_{d,2} = (D_{d,a}(q_d)\dot{q}_d)|_{Z_d}, \quad (2.81b) \]

where \( D_{d,a} \) is the row of matrix \( D_d \) corresponding to the unactuated coordinate, \( q_a \). The model takes the form

\[ \dot{z}_{d,1} = \psi_1(z_{d,1}) z_{d,2}, \quad (2.82a) \]
\[ \dot{z}_{d,2} = \psi_2(z_{d,1}, z_{d,2}), \quad (2.82b) \]

where

\[ \psi_1 = \frac{\partial \theta_d}{\partial q_{d_i}} \left[ H_{d_i} - \frac{\partial h_{dd_i}}{\partial q_{d_i}} c_{d_i} \right]^{-1} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] |_{Z_d}, \quad (2.83a) \]
\[ \psi_2 = \left( \frac{\partial D_{d,a}}{\partial q_{d_i}} \dot{q}_d - \Gamma_a \dot{q}_d - E_a \right) |_{Z_d}, \quad (2.83b) \]

and where the subscript \( a \) indicates the rows corresponding to the unactuated coordinate \( q_a \).

**Proof.** The constraints on the actuated joints in double support can be expressed as \( y = h(q_d) = h_0(q_d) - h_{dd_i}(\theta_d) \). As in the case of SS,

\[ q_d = \left[ \begin{array}{c} H_{d_i} \\ c_{d_i} \end{array} \right]^{-1} \left[ \begin{array}{c} h_{dd_i}(\theta_d) \\ \theta_d \end{array} \right]. \quad (2.84) \]

and

\[ \left[ H_{d_i} - \frac{\partial h_{dd_i}}{\partial \theta_d} c_{d_i} \right] \dot{q}_{d_i} = 0. \quad (2.85) \]
The kinetic energy is expressed as

\[
K = \frac{1}{2} \dot{q}^T_d D_d(q_d) \dot{q}_d
\]

(2.86)

\[
= \frac{1}{2} \dot{q}^T_{d_i} A^T_{d_i} D_d(q_d) A_{d_i} \dot{q}_{d_i}
\]

(2.87)

\[
= \frac{1}{2} \dot{q}^T_{d_i} D_{d_i}(q_{d_i}) \dot{q}_{d_i}.
\]

(2.88)

So

\[
\frac{\partial K(q_{d_i}, \dot{q}_{d_i})}{\partial \dot{q}_a} = D_{d_i,a} \dot{q}_{d_i}.
\]

(2.89)

where \(D_{d_i,a}\) is the row of matrix \(D_{d_i}\) corresponding to the unactuated coordinate, \(q_a\). Then,

\[
\dot{z}_{d_i,1} = L_f z_{d_i,1} = \frac{\partial z_{d_i,1}}{\partial q_{d_i}} \dot{q}_{d_i},
\]

(2.90)

and

\[
\dot{z}_{d_i,2} = L_f z_{d_i,2}
\]

(2.91a)

\[
= \left[ \dot{q}_d^T \frac{\partial D_{d_i,a}}{\partial q_{d_i}} D_{d_i,a} \right] \left[ \begin{array}{c} \dot{q}_{d_i} \\ D_{d_i}^{-1}(-\Gamma \dot{q}_{d_i} - E) \end{array} \right]
\]

(2.91b)

\[
= \dot{q}_d^T \frac{\partial D_{d_i,a}}{\partial q_{d_i}} \dot{q}_{d_i} - \Gamma_a \dot{q}_{d_i} - E_a.
\]

(2.91c)

Therefore, the low-dimensional model for DS takes the form

\[
\dot{z}_{d_i,1} = \psi_1(z_{d_i,1}) z_{d_i,2},
\]

(2.92a)

\[
\dot{z}_{d_i,2} = \psi_2(z_{d_i,1}, z_{d_i,2}).
\]

(2.92b)

This proves\(^{11}\) Proposition 4.

\(^{11}\)This proof is a result of collaboration with Ioannis Raptis [68].
procedure similar to SS, with the choice of coordinates,
\[ z_{d_{i,1}} = \theta_d(q_d)|_{Z_d} \quad \text{and} \quad z_{d_{i,2}} = D_{d_{i,a}}(q_d)\dot{q_d}|_{Z_d}, \]  
(2.93)
the low-dimensional model for DS takes the form,
\[ \dot{z}_{d_{i,1}} = \left. \frac{\partial \theta_d}{\partial q_d} z_{d_{i,2}} \right|_{Z_d}, \]  
(2.94)
\[ \dot{z}_{d_{i,2}} = \left( \frac{1}{2} \dot{q_d}^T \frac{\partial D_d(q_d)}{\partial q_a} \dot{q_d} - G_{s_a} - \Lambda_a \right) \bigg|_{Z_d}, \]  
(2.95)
where \( \Lambda_a \) is the row of the matrix \( A^T \lambda \) corresponding to the unactuated coordinate, \( q_a \).

### 2.3.3 Transition from single support to double support

The transition from SS to DS occurs when the value of \( \theta_s \) attains the value corresponding to contact of the swing heel, \( \theta_s^- \). This can be expressed as
\[ S_{s}^{d_{i}} \cap Z_{s} = \left\{ (z_{s,-}^{-}, z_{s,-}) \left| z_{s,-} = \theta_s^{-} , z_{s,-} \in \mathbb{R} \right. \right\}. \]  
(2.96)
Let the superscripts “+” and “−” indicate the start and end of the corresponding phase, respectively. Then,
\[ z_{d_{i,1}}^{+} = \theta_d(q_{d_{i}}^{+}) = \theta_d(\Delta q_s(q_{s}^{-})), \]  
(2.97)
and
\[ z_{d_{i,2}}^{+} = D_{d_{i,a}}(q_{d_{i}}^{+})\dot{q}_{d_{i}}^{+} \]  
(2.98a)
\[ = D_{d_{i,a}}(\Delta q_s(q_{s}^{-})) \cdot \Delta \dot{q}_s(q_{s}^{-})\dot{q}_{s}^{-} \]  
(2.98b)
or
\[ z_{d_{i,2}}^{+} = \delta_{s}^{d_{i}} z_{s,-}, \]  
(2.99)
where, using (2.70),
\[ \delta_{s}^{d_{i}} = D_{d_{i,a}}(\Delta q_s(q_{s}^{-})) \cdot \Delta \dot{q}_s(q_{s}^{-}) \left[ H_s - \frac{\partial h_{d_{i,s}}}{\partial q_s} \right]^{-1} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \]  
(2.100)
and \( \delta_{s}^{d_{i}} \) is a constant since it is only configuration dependent.
2.3.4 Transition from double support to single support

The transition from DS to SS occurs when the value of $\theta$ attains a specific value, $\theta_{d}^{-}$. This can be expressed as

$$S_{d}^{s} \cap Z_{d} = \left\{ (z_{d_{i},1}^{-}, z_{d_{i},2}^{-}) \mid z_{d_{i},1}^{-} = \theta_{d}^{-}, z_{d_{i},2}^{-} \in \mathbb{R} \right\}. \quad (2.101)$$

Then,

$$z_{s,1}^{+} = \theta_{s}(q_{s}^{+}) = \theta_{s}(\Delta_{q_{d_{i}}}(q_{d_{i}}^{-})), \quad (2.102)$$

and

$$z_{s,2}^{+} = D_{s_{a}}(q_{s}^{+})\dot{q}_{s}^{+} \quad (2.103a)$$

$$= D_{s_{a}}(\Delta_{q_{d_{i}}}(q_{d_{i}}^{-})) \cdot \Delta_{q_{d_{i}}}(q_{d_{i}}^{-})\dot{q}_{d_{i}}^{-} \quad (2.103b)$$

or

$$z_{s,2}^{+} = \delta_{d_{i}} s_{d_{i}} z_{s,2}^{-}, \quad (2.104)$$

where

$$\delta_{d_{i}} s_{d_{i}} = D_{s_{a}}(\Delta_{q_{d_{i}}}(q_{d_{i}}^{-})) \cdot \Delta_{q_{d_{i}}}(q_{d_{i}}^{-}) \left[ H_{d_{i}} - \frac{\partial h_{d_{i}}}{\partial q_{d_{i}}} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (2.105)$$

where $\delta_{d_{i}} s_{d_{i}}$ is a constant since it is only configuration-dependent.

2.3.5 Overall low-dimensional hybrid model

The low-dimensional model for a complete step can be expressed as

$$\Sigma_{s} : \begin{cases} \bar{X}_{s} = Z_{s} \\ \bar{F}_{s} : \dot{z}_{s} = \bar{f}_{s}(z_{s}) \\ \bar{S}_{d_{i}}^{s} = S_{d_{i}}^{s} \cap Z_{s} \\ \bar{T}_{d_{i}}^{s} : z_{d_{i}}^{+} = \Delta_{d_{i}}^{s}(z_{s}^{-}) \end{cases} \quad (2.106a)$$
\[\Sigma_{d_i} : \begin{cases} 
\bar{X}_{d_i} = Z_{d_i} \\
\bar{F}_{d_i} : \dot{z}_{d_i} = \bar{f}_{d_i}(z_{d_i}) \\
\bar{S}_{d_i}^s = S_{d_i}^s \cap Z_{d_i} \\
\bar{T}_{d_i}^s : z_{d_i}^+ = \bar{\Delta}_{d_i}^s(z_{d_i}^-) 
\end{cases} \tag{2.106b}\]

where \(z\) denotes the state of the low-dimensional model and the subscripts denote the corresponding phase. Note that the dimensions of \(\bar{X}_s\) and \(\bar{X}_{d_i}\) are each two whereas the dimensions of their counterparts, \(X_s\) and \(X_{d_i}\), are \(2n_s\) and \(2n_d\), respectively.

### 2.4 Stability properties of the model

Deriving a low-dimensional model for one complete step allows investigation of the stability properties of the limit cycle. The method of Poincaré is used to examine the stability of the periodic gait. Based on the methodology of [62, 70], the complete Poincaré map is composed of the mappings associated with single support and double support\(^{12}\).

#### 2.4.1 Mapping for the single support phase

The low-dimensional model for single support is given in Proposition 3. Since \(z_{s,1}\) is monotonic over SS, i.e., \(dz_{s,1}/dt \neq 0\), it follows that

\[\frac{dz_{s,2}}{dz_{s,1}} = \frac{\kappa_2(z_{s,1}, z_{s,2})}{\kappa_1(z_{s,1}) z_{s,2}}.\tag{2.107}\]

Unlike the model in [62], \(\kappa_2\) depends on both \(z_{s,1}\) and \(z_{s,2}\) because of the presence of the rolling foot. Hence, it is not possible to derive a closed-form expression for (2.107) by integration along \(Z_s\). Integration from \(\theta_s^-\) to \(\theta_s^+\) can be expressed as

\[z_{s,2}^- - z_{s,2}^+ = V_s(z_{s,2}^+).\tag{2.108}\]

\(^{12}\)The mathematical development is a result of collaboration with Ioannis Raptis [68].
Note that the integral $V_s$ is a function of both $z_{s,1}^+$ and $z_{s,2}^+$ but can be expressed as $V_s(z_{s,2}^+)$ since $z_{s,1}^+$ is a constant. With $z_{s,2}^+$ given by (2.104), the map for the single support phase $\rho_s : S_d^s \cap Z_d \rightarrow S_d^s \cap Z_s$ is defined as

$$\rho_s(z_{d,2}^-) = \delta_{d}^s z_{d,2}^- + V_s(\delta_{d}^s z_{d,2}^-).$$

(2.109)

### 2.4.2 Mapping for the double support phase

The low-dimensional model for DS is given in Proposition 4. The development of the mapping for DS follows that for SS. Since $z_{d,1}^+$ is monotonic,

$$\frac{dz_{d,2}}{dz_{d,1}} = \frac{\psi_2(z_{d,1}^+, z_{d,2}^+)}{\psi_1(z_{d,1}^+)z_{d,2}^+}. \quad (2.110)$$

As before, the second state, $\psi_2$, depends on both $z_{d,1}^+$ and $z_{d,2}^+$. Hence, it is not possible to derive a closed-form expression for (2.110) by integration along $Z_d$. Integration from $\theta_d^-$ to $\theta_d^+$ can be expressed as

$$z_{d,2}^- - z_{d,2}^+ = V_d(z_{d,2}^+). \quad (2.111)$$

Note that the integral $V_d$ is a function of both $z_{d,1}^+$ and $z_{d,2}^+$, but can be expressed as $V_d(z_{d,2}^+)$ since $z_{d,1}^+$ is a constant.

Substituting for $z_{d,2}^+$ using (2.99), which gives the transition map from SS to DS, the map for the double support phase $\rho_{d} : S_d^d \cap Z_s \rightarrow S_d^d \cap Z_d$ is defined as

$$\rho_{d}(z_{s,2}^-) = \delta_{d}^s z_{s,2}^- + V_d(\delta_{d}^s z_{s,2}^-).$$

(2.112)

### 2.4.3 Overall Poincaré mapping for the low-dimensional model

The Poincaré map for the step, $\rho : S_d^d \cap Z_s \rightarrow S_d^d \cap Z_s$ is given by

$$\rho = \rho_s \circ \rho_{d}.$$

(2.113)
Using (2.109) and (2.112), the overall Poincaré map is given by

$$
\rho(z_{n,2}) = \delta_d^s \left( \delta_d^s z_{n,2} + V_d(\delta_d^s z_{n,2}) \right) + V_s \left( \delta_d^s \left( \delta_d^s z_{n,2} + V_d(\delta_d^s z_{n,2}) \right) \right).
$$

(2.114)

The next chapter presents the validation of the hybrid models using actual gait data of normal human walking.
CHAPTER 3

ANALYSIS OF NORMAL HUMAN GAIT

Chapter 2 presented the derivation for the full hybrid model and the low-dimensional hybrid model of walking using an anthropomorphic model to represent the human. This chapter describes the use of the hybrid models to analyze normal human walking using gait analysis data reported by Winter [1].

The analysis presented here is based on a minimal anthropomorphic model (see Figure 2.1). In single support, this model consists of links representing two shanks, two thighs and a head-arms-trunk (HAT) segment and has 5 DOF. In double support, the model has an additional link representing the trailing foot pivoting about the ground. The anthropomorphic model for DS has 6 links but only 4 DOF because the pivot introduces two additional constraints on the trailing toe.

The joint angles, masses, and link lengths are obtained from anthropometric data in Winter [1]. The parameters provided in [1] for the gait analysis subject are shown in Table 3.1. Other parameters such as the moments of inertia and locations of the centers of mass of the link segments are derived using the anthropometric formulae provided in [1]. The EOM of the system are derived using the method of Lagrange [66]. All the symbolic computations are performed using Maple®. MATLAB® is used for numerical computations.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body mass</td>
<td>56.7</td>
<td>kg</td>
</tr>
<tr>
<td>Shank length (ankle to knee)</td>
<td>0.425</td>
<td>m</td>
</tr>
<tr>
<td>Thigh length (knee to hip)</td>
<td>0.314</td>
<td>m</td>
</tr>
<tr>
<td>HAT length (hip to base of rib cage)</td>
<td>0.25</td>
<td>m</td>
</tr>
</tbody>
</table>

Table 3.1: Anthropometric parameters for the gait analysis subject in [1]

3.1 Single support

In the single support model, \( q_s = (q_1, q_2, q_3, q_4, q_a)^T = (q_{rel,s}, q_a)^T \) represents the independent generalized coordinates specifying the configuration, \( \dot{q}_s \) represents the velocities and \( u_s = (u_1, u_2, u_3, u_4)^T \) is the vector of joint torques corresponding to \( q_1, q_2, q_3, \) and \( q_4 \). The functions \( b_{s,i}, i = 1, ..., 4 \) that give the desired shape (posture), where \( h_{ds} = (b_{s,1}, b_{s,2}, b_{s,3}, b_{s,4})^T \), are chosen to be Bézier polynomials parameterized by the scalar quantity representing forward progression, \( \theta_s \). The polynomial coefficients are chosen by a least squares fit to the joint motions obtained from the gait data. Choosing \( \theta_s(q_s) := c_s q_s \) with \( c_s = (-2, 0, -1, 0, -2) \) satisfies the condition of \( \theta_s \) being monotonically increasing over a step. This choice of \( \theta_s \) corresponds to the angle shown in Figure 3.1.

With this choice for \( \theta_s \), the output function (2.55) for SS is given by

\[
y = h(q_s) := h_0(q_s) - h_{ds} \circ \theta_s(q_s),
\]

(3.1)

where

\[
h_0(q_s) := H_0 q_s
\]

(3.2)
Figure 3.1: The scalar quantity chosen to represent forward progression in single support, \( \theta_s \) and double support, \( \theta_d \) corresponds to the angles shown. This absolute angle, which is between the vertical axis and the line joining the hip to the point of rolling contact, is monotonic over a step.

and

\[
H_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \
\end{bmatrix}.
\]  \hspace{1cm} (3.3)

The choice results in

\[
q_s = H_s^{-1} \begin{bmatrix} h_{ds} \\ \theta_s \end{bmatrix},
\]  \hspace{1cm} (3.4)

because

\[
H_s := \begin{bmatrix} H_0 \\ c_s \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-2 & 0 & -1 & 0 & -2 
\end{bmatrix}.
\]  \hspace{1cm} (3.5)

is invertible. The posture is constrained by applying torques at joints 1 to 4 by feedback using the method of computed torque, see (2.58). As a result of constraining the posture,
the dynamics of single support reduce to the dynamics associated with the unactuated coordinate, $q_a$. This dynamics is described by the second order differential equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0,$$

(3.6)

where $L = K - V$ is the Lagrangian of the system. By choosing coordinates $(z_{s,1}, z_{s,2})$, Proposition 3 is used to describe the state of the low-dimensional model in single support where $z_{s,1} = \theta_s$ represents the forward progression, and $z_{s,2} = \partial K/\partial \dot{q}_a$ represents the angular momentum of the entire human about the contact point of the stance leg with the ground.

### 3.2 Double support

In the double support model, $q_{d_i} = (q_1, q_2, q_3, q_a)^T = (q_{rel,d}, q_a)^T$ represents the independent generalized coordinates. Because the system is constrained, two coordinates, here chosen to be $q_4$ and $q_5$, are dependent on the independent coordinates. Thus, in DS, only three joints are actuated to establish the desired shape of the body. It is assumed that coordinates $q_1$ to $q_3$ are regulated as a function of the scalar quantity representing forward progression in DS, $\theta_d$ (see Figure 3.1). The function that gives the desired shape, $h_{dd_i} = (b_{d,1}, b_{d,2}, b_{d,3})^T$, is chosen as in SS. For DS, choosing the output function (2.55) as

$$h_0(q_{d_i}) := H_0 q_{d_i},$$

(3.7)

$$\theta_d(q_{d_i}) := c_d q_{d_i},$$

(3.8)

where

$$H_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

(3.9)
and \( c_d = (-2, 0, -1, -2) \) results in

\[
q_{di} = H_{di}^{-1} \begin{bmatrix} h_{dd_i} \\ \theta_d \end{bmatrix},
\]

(3.10)

because

\[
H_{di} := \begin{bmatrix} H_0 \\ c_d \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2 & 0 & -1 & -2
\end{bmatrix}
\]

is invertible. The body is constrained to maintain its shape by applying torques at joints 1 to 3 by feedback using the method of computed torque, see (2.58). Proposition 4 is used to obtain the evolution of the dynamics using the low-dimensional model for double support.

### 3.3 Transitions

Since normal human gait is assumed here to be periodic, the kinematics of the constrained motions need to obey the transitions in the sense that if the constraints are satisfied at the end of single support, the constraints will be satisfied at the beginning of double support, despite the jump in the velocities. In other words, the transitions from SS to DS and DS to SS need to occur such that the state of the low-dimensional walking model always evolves on either \( Z_s \) or \( Z_d \)—that is, these manifolds are invariant under the impact map and the map from DS to SS. Since the transition maps and output derivatives are linear in the joint velocities, invariance at one velocity guarantees invariance at all velocities. The next section describes the details involved in ensuring periodicity at the transitions.

### 3.4 Enforcing periodicity

The errors inherent in data acquisition by gait analysis necessitate slight modifications to achieve periodicity for the analytical model. When fitting the functions that give the
desired posture, \( h_{ds} \) and \( h_{dd} \), to the gait data in [1], properties of the Bézier polynomials and the properties of the impact map are used to make the desired posture periodic.

A Bézier polynomial has the form

\[
b(s) := \sum_{k=0}^{N} \alpha_k \frac{N!}{k!(N-k)!} s^k (1 - s)^{(N-k)}, \quad s \in [0, 1] \tag{3.12}
\]

where \( N \) is the degree of the Bézier polynomial and the \( \alpha_k \)'s are the \( N + 1 \) coefficients defining the polynomial. The derivative of the Bézier polynomial at \( s = 0 \) is given by

\[
\left. \frac{\partial b(s)}{\partial s} \right|_{s=0} = N (\alpha_1 - \alpha_0), \tag{3.13}
\]

while the derivative at \( s = 1 \) is given by

\[
\left. \frac{\partial b(s)}{\partial s} \right|_{s=1} = N (\alpha_N - \alpha_{N-1}). \tag{3.14}
\]

Let \( \alpha_d \) and \( \alpha_s \) denote the vectors of coefficients of the Bézier polynomial fits for the desired joint trajectories in DS and SS, respectively. The parameter corresponding to \( s \) in SS, denoted by \( s_s \), normalizes \( \theta_s \) as

\[
s_s = \frac{\theta_s - \theta_s^+}{\theta_s^- - \theta_s^+}, \tag{3.15}
\]

and the parameter corresponding to \( s \) in DS, denoted by \( s_d \), normalizes \( \theta_d \) as

\[
s_d = \frac{\theta_d - \theta_d^+}{\theta_d^- - \theta_d^+}. \tag{3.16}
\]

Note that \( \theta_s^+ = \theta_d^- \), since the beginning of single support corresponds to the end of double support.

### 3.4.1 Periodicity in the transition from single support to double support

Enforcing periodicity implies that the desired configuration and velocities at the beginning of DS match the configuration and velocities obtained by applying the transition map...
from SS to DS. The configuration in SS and DS in terms of the virtual constraints and the forward progression are given by (3.4) and (3.10), respectively. Relabeling the coordinates at the end of SS and extracting the independent coordinates of DS leads to

\[ q_{d_i}^+ = \Omega_{d_i} R_0 q_s^-, \]  

which implies

\[ H_{d_i}^{-1} \begin{bmatrix} h_{dd_i}(\theta_d^+) \\ \theta_d^+ \end{bmatrix} = \Omega_{d_i} R_0 H^{-1}_s \begin{bmatrix} h_{ds}(\theta_s^-) \\ \theta_s^- \end{bmatrix}. \]  

(3.18)

In terms of the Bézier coefficients,

\[ \begin{bmatrix} \alpha_{d0} \\ \theta_d^+ \end{bmatrix} = H_{d_i}^{-1} \Omega_{d_i} R_0 H_s^{-1} \begin{bmatrix} \alpha_{sN_s} \\ \theta_s^- \end{bmatrix}, \]  

(3.19)

where \( \alpha_d \) and \( \alpha_s \) represent the vectors of Bézier coefficients of the Bézier polynomial fits for the DS and SS joint trajectories, respectively. The degree of the polynomial fit in SS is given by \( N_s \). The impact map \( \dot{q}_s^+ = \Delta \dot{q}_s \dot{q}_s \) is linear in the velocities since \( \Delta \dot{q}_s \) is a function of the configuration alone (2.44), which remains unchanged during the impact. The velocities of the controlled variables in DS and SS are given by

\[ \dot{q}_{d_i} = H_{d_i}^{-1} \begin{bmatrix} \frac{\partial h_{dd_i}(\theta_d)}{\partial \theta_d} \\ 1 \end{bmatrix} \dot{\theta}_d, \]  

(3.20)

and

\[ \dot{q}_s = H_s^{-1} \begin{bmatrix} \frac{\partial h_{ds}(\theta_s)}{\partial \theta_s} \\ 1 \end{bmatrix} \dot{\theta}_s. \]  

(3.21)

In terms of the derivatives of the Bézier polynomials, the relation between the velocities can be expressed as

\[ \begin{bmatrix} N_d(\alpha_{d1} - \alpha_{d2}) \\ \theta_s^+ - \theta_d^+ \end{bmatrix} \dot{\theta}_s^+ \]  

\[ = H_{d_i} \Omega_{d_i} R_0 \Delta \dot{q}_s H_s^{-1} \begin{bmatrix} N_s(\alpha_sN_s - \alpha_sN_s-1) \\ \theta_s^+ - \theta_s^- \end{bmatrix} \dot{\theta}_s^- \]  

(3.22)

where \( N_d \) is the order of the Bézier polynomial fits in DS. Equations (3.20) and (3.22) are used along with (3.13) to calculate \( \alpha_{d0} \), \( \alpha_{d1} \), \( \theta_d^+ \), and \( \dot{\theta}_d^+ \) to ensure periodicity in the transition from SS to DS.
3.4.2 Periodicity in the transition from double support to single support

The transition from DS to SS is continuous and occurs at a particular value of the forward progression, $\theta_d^-$. Periodicity in the configuration requires that

$$\alpha_{s_0} = \alpha_{d_{N_d}},$$  \hspace{1cm} (3.23)

where $\alpha_{d_{N_d}}$ for the dependent coordinate, $q_4$ in DS satisfies the DS constraint (2.14). Periodicity in the velocities in the transition from DS to SS requires that

$$\dot{q}_s^+ = \dot{q}_d^-.$$

The dependent velocities, $\dot{q}_4$ and $\dot{q}_5$ in DS are given by

$$\dot{q}_{d_d} = \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} = -A_d^{-1} A_i \dot{q}_{d_i} =: \begin{bmatrix} A_a \\ A_b \end{bmatrix} \dot{q}_{d_i}.$$

Therefore,

$$\dot{q}_4 = A_a \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_a \end{bmatrix},$$ \hspace{1cm} (3.26)

which implies

$$\frac{N_d(\alpha_{d_{N_d,4}} - \alpha_{d_{N_d-1,4}})}{\theta_d^- - \theta_d^+} \dot{\theta}_d^- = A_a H_d^{-1} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\theta}_d^- \end{bmatrix} = A_a H_d^{-1} \begin{bmatrix} N_d(\alpha_{d_{N_d,1}} - \alpha_{d_{N_d-1,1}}) \\ \theta_d^- - \theta_d^+ \\ N_d(\alpha_{d_{N_d,2}} - \alpha_{d_{N_d-1,2}}) \\ \theta_d^- - \theta_d^+ \\ N_d(\alpha_{d_{N_d,3}} - \alpha_{d_{N_d-1,3}}) \\ \theta_d^- - \theta_d^+ \\ 1 \end{bmatrix} \dot{\theta}_d^-.$$

\hspace{1cm} (3.27)
Then, to ensure a continuous transition in the velocities,

\[
\begin{bmatrix}
N_s(\alpha_{s1,1} - \alpha_{s0,1}) \\
\theta_d^+ - \theta_d^- \\
N_s(\alpha_{s1,2} - \alpha_{s0,2}) \\
\theta_d^+ - \theta_d^- \\
N_s(\alpha_{s1,3} - \alpha_{s0,3}) \\
\theta_d^+ - \theta_d^- \\
N_s(\alpha_{s1,4} - \alpha_{s0,4}) \\
\theta_d^+ - \theta_d^- \nend{bmatrix}
\begin{bmatrix}
\dot{\theta}_d^- 
\end{bmatrix}
= \begin{bmatrix}
N_d(\alpha_{dN_d,1} - \alpha_{dN_d-1,1}) \\
\theta_d^+ - \theta_d^- \\
N_d(\alpha_{dN_d,2} - \alpha_{dN_d-1,2}) \\
\theta_d^+ - \theta_d^- \\
N_d(\alpha_{dN_d,3} - \alpha_{dN_d-1,3}) \\
\theta_d^+ - \theta_d^- \\
N_d(\alpha_{dN_d,4} - \alpha_{dN_d-1,4}) \\
\theta_d^+ - \theta_d^- 
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_d^- 
\end{bmatrix}. \tag{3.28}
\]

Equations (3.23) and (3.28) are used to calculate the Bézier coefficients \(\alpha_{s0}\) and \(\alpha_{s1}\) necessary to enforce periodicity in the transition from DS to SS.

### 3.5 Computing the ground reaction forces

The ground reaction forces (GRF) can be computed using the extended model and the method of Lagrange multipliers.

#### 3.5.1 Ground reaction forces during single support

If the stance foot is rolling without slipping, the extended coordinates \(q_e\) are related to \(q_s\) by

\[
q_e := (q_s, x_R, y_R)^T,
\]

where \(x_R\) and \(y_R\) are the horizontal and vertical positions of the point of rolling contact on the stance foot; see Figure 2.1. Then, the EOM in SS can be expressed as

\[
D_e \ddot{q}_e + C_e \dot{q}_e + G_e = B_e u_s + J_s^T \begin{bmatrix}
F_{1x} \\
F_{1y}
\end{bmatrix} \tag{3.29}
\]

\[
= \begin{bmatrix}
B_s u_s \\
F_{1x} \\
F_{1y}
\end{bmatrix}, \tag{3.30}
\]

where the subscript \(e\) denotes the extended model, \(J_s\) is the Jacobian of the functions representing the position of the rolling contact point and \(F_{1x}\) and \(F_{1y}\) are the GRF at the rolling contact point. In single support, \(J_s = [0_{2 \times 5}; I_{2 \times 2}]\). Since the foot is rolling without slipping
at the contact point, the positions, velocities and accelerations of the contact point are

\[ x_R = Rq_v, \quad y_R = 0, \quad (3.31) \]
\[ \dot{x}_R = 0, \quad \dot{y}_R = 0, \quad (3.32) \]
\[ \ddot{x}_R = 0, \quad \ddot{y}_R = Rq_v^2, \quad (3.33) \]

where \( q_v = (-q_1 - q_3 - q_a + \pi) \) is the absolute angle of the shank to the vertical and \( R \) is the radius of the roll-over shape. Let \( C = C_e \dot{q}_e + G_e \). Partitioning \( D_e \) and \( C \),

\[ D_e = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{bmatrix}, \quad (3.34) \]

and

\[ C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad (3.35) \]

results in

\[ \ddot{q}_s = D_{11}^{-1} \left( B_s u_s - C_1 - D_{12} \begin{bmatrix} \ddot{x}_R \\ \ddot{y}_R \end{bmatrix} \right), \quad (3.36) \]

and

\[ \begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} = D_{12}^T \ddot{q}_s + D_{22} \begin{bmatrix} \ddot{x}_R \\ \ddot{y}_R \end{bmatrix} + C_2. \quad (3.37) \]

### 3.5.2 Ground reaction forces during double support

During double support, GRF act on the stance leg at the point of rolling contact and on the trailing toe. The extended coordinates for the double support model are given by

\[ q_e := (q_d, x_R, y_R)^T. \]

The EOM in DS can be expressed as

\[ D_e \ddot{q}_e + C_e \dot{q}_e + G_e = B_e u_d + J_d^T \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{bmatrix}, \quad (3.38) \]
where \( J_d \) is the Jacobian of the functions representing the positions of the rolling contact point and of the trailing toe, \( F_{1x} \) and \( F_{1y} \) are the GRF at the rolling contact point, and \( F_{2x} \) and \( F_{2y} \) are the GRF at the trailing toe. The velocity constraints on the trailing toe and rolling contact point can be expressed as

\[
J_d \ddot{q}_e = 0, \tag{3.39}
\]

which implies

\[
J_d \ddot{q}_e + \dot{J}_d \dot{q}_e = 0 \tag{3.40}
\]

or

\[
J_d \ddot{q}_e = -\dot{J}_d \dot{q}_e. \tag{3.41}
\]

The constraints for rolling without slipping are

\[
x_R = R q_v \quad \text{and} \quad y_R = 0 \tag{3.42}
\]

where \( q_v = (-q_1 - q_3 - q_a + \pi) \) is the absolute angle of the shank with respect to the vertical and \( R \) is the radius of the roll-over shape. Thus,

\[
\begin{bmatrix}
D_e & -J_d^T \\
J_d & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_e \\
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y}
\end{bmatrix}
= \begin{bmatrix}
B_e u_d - C_e \dot{q}_e - G_e \\
-\dot{J}_d \dot{q}_e
\end{bmatrix}, \tag{3.43}
\]

which can be solved for the GRF in double support.

### 3.6 Simulation results of the full and low-dimensional hybrid models

A simulation of (2.51) reveals that the evolution of the unactuated coordinate in both SS and DS closely matches the data. Figure 3.2 gives the joint trajectories from the gait data.
Figure 3.2: Joint motions from the gait data in Winter [1] (dashed) versus the model, (2.51), (solid) on the limit cycle (i.e., at steady-state) for one step. Note that the HAT angle, $q_a$, is unactuated, and, therefore, its evolution is not directly controlled.

Figure 3.3: The joint velocities from the simulation of the dynamics.
and the corresponding trajectory of the model, (2.51), on the limit cycle (i.e., at steady-state). Figure 3.3 gives the model’s joint velocity trajectories. Since the posture of the human is constrained, the center of mass of the human is only a function of the unactuated coordinate. The low-dimensional hybrid model’s prediction of the motion of the center of mass closely follows the gait data (the data used to derive the model); see Figure 3.4.

Figure 3.5 compares the vertical GRF on the stance leg calculated from the model with the values from force plate measurements reported in [1]. The model values are computed using the joint torques that are required to control the posture in DS and SS as a function of the forward progression. Aside from the minimal data provided in Table 3.1, the model requires anthropometric data regarding individual link masses, mass centers, moments of inertia, etc. These anthropometric parameter calculations are based on tables that estimate the values from cadavers. It is highly likely that there are errors between these estimates and the subject’s actual measures, which cannot be directly determined. In addition, errors are inherent in video gait analysis, which uses high resolution cameras to track markers attached to a subject’s skin or clothing to recover the subject’s motions. The many sources of error are likely to be compounded when using the model to calculate the GRF. Despite this limitation, the GRF values from the model are reasonably close to the direct measurement using force plates. A typical GRF curve from measurements shows two peaks and a valley in between. The absence of the valley in the model’s prediction is likely because the model lacks compliance, unlike the human body. The first peak of the GRF force in the data is the result of the push-off action of the plantarflexors of the trailing foot towards the end of double support. The peak is not evident in the model’s prediction since push-off is not modeled. The shapes of the joint motion curves necessitate the use of high order polynomials to obtain a least-squares fit. The quality of the high-order polynomial fit to
data can be poor especially near the boundaries. In addition, the slopes of the joint motion curves at the end of SS are modified in the model to ensure invariance of the constraints with respect to the impact map. The rigid impact map used does not take into account the shock absorbing mechanisms of the human body. This fact coupled with the sometimes poor quality of the high-order polynomial fit results in torque spikes that cause the unnatural spike in the GRF towards the end of SS. The discontinuity in the transition from DS to SS occurs because of the numerical error in matching the first and second derivatives of the joint motion curves for SS and DS at this junction. Moreover, since the model is planar, the influence of motions in the frontal plane that contribute to the pattern of the GRF seen in normal human walking is absent. Despite these limitations and the likely errors due to the anthropometric parameter estimation, the model’s prediction is reasonable for about 80% of the step, from 0 to 0.4 s in Figure 3.5.

The results of the simulation support the hypothesis that the human uses joint torques to control posture in a specific manner to enable forward progression. Figure 3.6 gives a stick animation of a simulation of one step.

The existence of a low-dimensional model for human gait is thus validated. When torques are applied to the internal joints to maintain the shape of the anthropomorphic model as a function of the forward progression, the dynamics evolves such that the unactuated coordinate undergoes motion that is the result of the body moving forward. This fact enables the sagittal plane dynamics of walking to be described by a single DOF model.

Figures 3.7 and 3.8 compare the evolution of full- and low-dimensional models’s coordinates over one gait cycle. Figures 3.9 and 3.10 plot the error in the coordinates. The close match validates the derivation of the low-dimensional model.
Figure 3.4: Comparison of the angle between the line connecting the center of mass (COM) and the ground for the gait data in [1] (dashed) versus the model, (2.51), (solid) on the limit cycle (i.e., at steady-state) for one step. The plot indicates a close match.

Figure 3.5: Comparison of ground reaction forces computed from the model with the values from force plate measurements reported in [1].
Figure 3.6: Stick figure animation of one step of the example model. Note the rolling motion of the stance foot. The swing leg is shown with dashed lines.

Figure 3.7: Simulation results for $z_{d,1}$ in DS, $z_{s,1}$ in SS over one gait cycle using the full model and using the low-dimensional model.
Figure 3.8: Simulation results for $z_{d,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full model and using the low-dimensional model.

Figure 3.9: Error in $z_{d,1}$ in DS, $z_{s,1}$ in SS over one gait cycle using the full model versus using the low-dimensional model. The absolute and relative integration tolerances in MATLAB® were set at $10^{-9}$ and $10^{-7}$, respectively.
Figure 3.10: Error in $z_{d,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full model versus using the low-dimensional model. The absolute and relative integration tolerances in MATLAB® were set at $10^{-9}$ and $10^{-7}$, respectively.

### 3.7 Stability analysis

Analysis using the method of Poincaré on the low-dimensional model indicates the existence of a fixed point (see Figure 3.11). Simulation of the dynamics for multiple steps confirms the existence of a limit cycle (see Figure 3.12).

The simulation reveals that the integrals $V_d$ and $V_s$ can be expressed by linear relations,

$$V_d \approx \vartheta_d z_{d,2}^+ + \varphi_d$$

and

$$V_s \approx \vartheta_s z_{s,2}^+ + \varphi_s,$$

where $\vartheta_d$ and $\vartheta_s$ represent the respective slopes, and $\varphi_d$ and $\varphi_s$ represent the respective $y$-intercepts (see Figures 3.13 and 3.14). As a result, the Poincaré map for the overall low-dimensional model may be approximated as follows.
First, identify that

\[ z_{s,2}^- = z_{s,2}^+ + V_s(z_{s,2}^+) \]  

(3.44a)

\[ = z_{s,2}^+ + \vartheta_s z_{s,2}^+ + \varphi_s \]  

(3.44b)

and

\[ z_{d,2}^- = z_{d,2}^+ + V_d(z_{d,2}^+) \]  

(3.45a)

\[ = \delta_d^s z_{s,2}^- + \vartheta_d \delta_d^s z_{s,2}^- + \varphi_d \]  

(3.45b)

so that

\[ z_{s,2}^+ = \delta_d^s z_{d,2}^- \]  

(3.46a)

\[ = \delta_d^s (\delta_d^s z_{s,2}^- + \vartheta_d \delta_d^s z_{s,2}^- + \varphi_d). \]  

(3.46b)

Therefore, the Poincaré map for the step simplifies to the expression

\[ \rho(z_{s,2}^-) = \vartheta z_{s,2}^- + \varphi, \]  

(3.47)

where

\[ \vartheta := (1 + \vartheta_d + \vartheta_s + \vartheta_d \vartheta_s)\delta_d^s \delta_d^s \]  

(3.48a)

\[ \varphi := \varphi_d \vartheta_s \delta_d^s + \varphi_d \delta_d^s + \varphi_s. \]  

(3.48b)

The fixed point predicted by the empirically linearized Poincaré map, (3.47), is within 0.02% of the value achieved in the limit cycle.

The step length of 0.64 m predicted by the model is less than the 0.7 m calculated from the data [1]. The discrepancy can be attributed to the swiveling movement of the hips that happens in the transverse plane during human walking. This movement is not taken into
Using (3.47)
Using (2.113)

Figure 3.11: The Poincaré map of the full hybrid model (2.113) versus its linear approximation (3.47).

Figure 3.12: Simulation of the full hybrid model (2.51) for 35 steps illustrating convergence to a limit cycle as predicted by the fixed point in the Poincaré map. The triangles indicate HC, the beginning of DS and the transition to SS.
Figure 3.13: Numerical evaluation of $V_s$ indicates that it may be approximated by a linear relation.

Figure 3.14: Numerical evaluation of $V_d$ indicates that it may be approximated by a linear relation.
account in the planar model. Winter reports a walking rate of 1.4 m/s while the model predicts a steady-state walking rate of 1.3 m/s. Since the data in [1] is reported for only one complete gait cycle, it is difficult to ascertain whether the walking rate reported is the steady-state walking rate. A walking rate of 1.2 m/s, which is the rate typically used to establish crossing times at traffic intersections, is generally considered normal for adults [71].

The next chapter presents the extension of the hybrid models of Chapter 2 to model asymmetric gait.
Chapter 2 presented the model for normal human walking in which one gait cycle was assumed to consist of two symmetric steps, and the right and left legs were assumed to be identical. The role of the stance leg during walking is assumed alternately by the right and left legs. In one gait cycle of asymmetric gait, the two steps are generally asymmetric. The asymmetry can arise from different kinematic and inertial properties of the two legs. Asymmetry can also arise from differing joint motions of each leg.

The hybrid model for one gait cycle of asymmetric gait, therefore, consists of two phases of double support, two phases of single support, and four transition maps. Figure 4.1 shows the ordering of these phases and maps for one gait cycle. If the gait starts from DS with the left leg as the stance leg, the gait transitions to and continues in SS with the same stance leg. When SS ends, the transition to DS results in the right leg assuming the role of the stance leg. The next step is completed with the right leg as the stance leg. The completion of the second step signals the end of one gait cycle.

4.1 Full asymmetric model for one gait cycle

The four sub-models for each step of asymmetric gait take the same form as the four sub-models for a step described in Chapter 2 for symmetric gait. However, the sub-models
Figure 4.1: The phases and transitions that are required to describe one gait cycle with asymmetric steps. The “left” and “right” denote the stance leg for the particular phase. At each of the transitions from single support (SS) to double support (DS), the stance leg is switched. The value of the quantity representing forward progression, $\theta$, is indicated at the beginning and end of each phase. The subscripts “$L$” and “$R$” indicate the left and right leg as the stance leg. The superscripts “$+$” and “$-$” indicate the beginning and end of each phase.
for the model of asymmetric gait differ depending on which leg is the stance leg. Hence, eight sub-models are needed to describe a gait cycle. In the process, the constraints on the posture need to be made invariant across the four transitions.

Assume that the left leg is the stance leg for step 1 starting from double support. The phases and transitions that comprise one gait cycle follow the order depicted in Figure 4.1. The notation used is similar to that in Chapter 2 with the addition of a subscript “L” or “R” to indicate whether the left or the right leg is the stance leg for the phase.

4.1.1 Model for one step with the left leg as the stance leg

**Double support** By Proposition 2, the EOM for DS with the left leg as stance leg is given by

\[
D_{d_L} (q_{d_L}) \ddot{q}_{d_L} + \Gamma_{L}(q_{d_L}, \dot{q}_{d_L}) \dot{q}_{d_L} + E_L(q_{d_L}) = M_L(q_{d_L}) u_{d_L}.
\]

(4.1)

**Transition from double support to single support** The transition from DS (left) to SS (left) takes place when \( \theta_{d_L} \), the variable corresponding to forward progression, achieves a specific value, \( \theta_{d_L}^- \) (see Figure 4.1). The configuration and velocities at the end of DS are obtained using (2.14) and (2.15), respectively.

The mapping from DS (left) to SS (left) can be expressed as

\[
x_{s_L}^+ = \Delta_{d_L}^s (x_{d_L}^-),
\]

(4.2)

where

\[
\Delta_{d_L}^s (x_{d_L}^-) = \begin{bmatrix} \Delta q_{d_L} (q_{d_L}) \\ \Delta \dot{q}_{d_L} \end{bmatrix}.
\]

(4.3)

The left leg continues to be the stance leg after this transition.
**Single support**  Using (2.1), the EOM for SS with the left leg as the stance leg are given by

\[
D_{s_L}(q_{s_L})\ddot{q}_{s_L} + C_{s_L}(q_{s_L}, \dot{q}_{s_L})\dot{q}_{s_L} + G_{s_L}(q_{s_L}) = B_{s_L}u_{s_L},
\]  

(4.4)

**Transition from single support to double support**  The transition from SS (left) to DS (right) is modeled using the rigid impact map developed in Section 2.2.3. The transition occurs when the heel of the right leg contacts the ground, which corresponds to \(\theta_{s_L} = \theta_{s_L}^-\) (see Figure 4.1). The mapping that relates the state at the beginning of DS (right) to the state at the end of SS (left) just before impact is given by

\[
x_{d_R}^+ = \Delta_{s_L}^{d_R}(x_{s_L}^-),
\]  

(4.5)

where

\[
\Delta_{s_L}^{d_R}(x_{s_L}^-) = \begin{bmatrix}
\Delta_{q_{s_L}}(q_{s_L}^-) \\
\Delta_{\dot{q}_{s_L}}(q_{s_L}^-) \dot{q}_{s_L}^-
\end{bmatrix}.
\]  

(4.6)

This transition marks the end of step 1. The right leg becomes the stance leg for step 2.

**4.1.2 Model for one step with the right leg as the stance leg**

**Double support**  Using Proposition 2, the EOM for DS with the right leg as the stance leg are given by

\[
D_{d_R}(q_{d_R})\ddot{q}_{d_R} + \Gamma_{R}(q_{d_R}, \dot{q}_{d_R})\dot{q}_{d_R} + E_{R}(q_{d_R}) = M_{R}(q_{d_R})u_{d_R},
\]  

(4.7)

**Transition from double support to single support**  The transition from DS (right) to SS (right) takes place when \(\theta_{d_R}\) achieves a specific value, \(\theta_{d_R}^-\) (see Figure 4.1). The configuration and velocities at the end of DS (right) are obtained using (2.14) and (2.15), respectively.

The mapping from DS (right) to SS (right) can be expressed as

\[
x_{s_R}^+ = \Delta_{d_R}^{s_R}(x_{d_R}^-),
\]  

(4.8)
where
\[ \Delta_{d_{i_R}}^s (x_{d_R}) = \begin{bmatrix} \Delta_{\dot{q}_{d_{i_R}}} (\dot{q}_{d_{i_R}}^-) \\ \Delta_{\dot{\dot{q}}_{d_{i_R}}} (\dot{\dot{q}}_{d_{i_R}}^-) \end{bmatrix}. \tag{4.9} \]

The right leg continues to be the stance leg after the transition.

**Single support** Using (2.1), the EOM for SS with the right leg as the stance leg are given by
\[ D_{s_{s_R}} (q_{s_{s_R}}) \ddot{q}_{s_{s_R}} + C_{s_{s_R}} (q_{s_{s_R}}, \dot{q}_{s_{s_R}}) \dot{q}_{s_{s_R}} + G_{s_{s_R}} (q_{s_{s_R}}) = B_{s_{s_R}} u_{s_{s_R}}. \tag{4.10} \]

**Transition from single support to double support** The transition from SS (right) to DS (left) is modeled using the rigid impact map developed in Section 2.2.3. The transition occurs when the heel of the left leg contacts the ground, which corresponds to \( \theta_{s_{s_R}} = \theta_{s_{s_R}}^- \) (see Figure 4.1). With the right leg as stance leg, the mapping that relates the state at the beginning of DS (left) to the state at the end of SS (right) just before impact is given by
\[ x_{d_{i_L}}^+ = \Delta_{d_{i_R}}^{s_{s_R}} (x_{s_{s_R}}^-), \tag{4.11} \]

where
\[ \Delta_{s_{s_R}}^{d_{i_L}} (x_{s_{s_R}}^-) = \begin{bmatrix} \Delta_{q_{s_{s_R}}} (q_{s_{s_R}}^-) \\ \Delta_{\dot{q}_{s_{s_R}}} (\dot{q}_{s_{s_R}}^-) \end{bmatrix}. \tag{4.12} \]

This transition marks the end of step 2 and the end of one gait cycle.
4.1.3 Overall full hybrid model

The overall model can be expressed as a non-linear hybrid system containing four charts (see Figure 4.2),

\[
\begin{align*}
\Sigma_{d_L} : & \quad \mathcal{X}_{d_L} = TQ_{d_L} \\
& \quad \mathcal{F}_{d_L} : \dot{x}_{d_L} = f_{d_L}(x_{d_L}) + g_{d_L}(x_{d_L})u_{d_L} \\
& \quad S^d_{d_L} = \{x_{d_L} \in TQ_{d_L} \mid H^d_{d_L}(x_{d_L}) = 0\} \\
& \quad T^d_{d_L} : x^+_{d_L} = \Delta^d_{d_L}(x^-_{d_L}) \\
\end{align*}
\]

(4.13a)

\[
\begin{align*}
\Sigma_{s_L} : & \quad \mathcal{X}_{s_L} = TQ_{s_L} \\
& \quad \mathcal{F}_{s_L} : \dot{x}_{s_L} = f_{s_L}(x_{s_L}) + g_{s_L}(x_{s_L})u_{s_L} \\
& \quad S^d_{s_L} = \{x_{s_L} \in TQ_{s_L} \mid H^d_{s_L}(x_{s_L}) = 0\} \\
& \quad T^d_{s_L} : x^+_{s_L} = \Delta^d_{s_L}(x^-_{s_L}) \\
\end{align*}
\]

(4.13b)

\[
\begin{align*}
\Sigma_{d_R} : & \quad \mathcal{X}_{d_R} = TQ_{d_R} \\
& \quad \mathcal{F}_{d_R} : \dot{x}_{d_R} = f_{d_R}(x_{d_R}) + g_{d_R}(x_{d_R})u_{d_R} \\
& \quad S^d_{d_R} = \{x_{d_R} \in TQ_{d_R} \mid H^d_{d_R}(x_{d_R}) = 0\} \\
& \quad T^d_{d_R} : x^+_{d_R} = \Delta^d_{d_R}(x^-_{d_R}) \\
\end{align*}
\]

(4.13c)

\[
\begin{align*}
\Sigma_{s_R} : & \quad \mathcal{X}_{s_R} = TQ_{s_R} \\
& \quad \mathcal{F}_{s_R} : \dot{x}_{s_R} = f_{s_R}(x_{s_R}) + g_{s_R}(x_{s_R})u_{s_R} \\
& \quad S^d_{s_R} = \{x_{s_R} \in TQ_{s_R} \mid H^d_{s_R}(x_{s_R}) = 0\} \\
& \quad T^d_{s_R} : x^+_{s_R} = \Delta^d_{s_R}(x^-_{s_R}) \\
\end{align*}
\]

(4.13d)

where \(\mathcal{F}\) denotes the vector field on manifold \(\mathcal{X}\), \(S\) is the switching hyper-surface and \(T : S \rightarrow \mathcal{X}\) is the transition function applied when \(x \in S\).
Figure 4.2: Discrete-event system corresponding to the dynamics of a gait cycle made up of one left and one right step. The subscripts “L” and “R” indicate the left and right legs, respectively.

4.2 Low-dimensional asymmetric model for one gait cycle

The low-dimensional model for a step has the same form for the single and double support phases and the transition maps as the low-dimensional model in Chapter 2. The sub-models differ in the implementation depending on which leg is the stance leg. This is indicated by the use of subscripts “L” and “R” to denote the left and right leg accordingly.

4.2.1 Model for one step with the left leg as the stance leg

**Double support** Using Proposition 4, the coordinates for DS (left) are chosen as

\[
\begin{align*}
\dot{z}_{d,L}^{1} &= \theta_{d,L}(q_{d,L})\bigg|_{Z_{d,L}}, \\
\dot{z}_{d,L}^{2} &= \left(D_{d,L} \left(q_{d,L}\right) \dot{q}_{d,L}\right)\bigg|_{Z_{d,L}},
\end{align*}
\]

(4.14a) (4.14b)
where $D_{d_{L,a}}$ is the row of matrix $D_{d_{L}}$ corresponding to the unactuated coordinate. With this choice, the low-dimensional model takes the form

$$
\dot{z}_{d_{L},1} = \psi_{1_{L}}(z_{d_{L,1}}, z_{d_{L},2}), \quad (4.15a)
$$

$$
\dot{z}_{d_{L},2} = \psi_{2_{L}}(z_{d_{L,1}}, z_{d_{L},2}). \quad (4.15b)
$$

**Transition from double support to single support** The transition from DS (left) to SS (left) occurs when the value of $\theta_{d_{L}}$ attains a specific value, $\theta_{d_{L}}$. This condition can be expressed as

$$
S_{d_{L}} \cap Z_{d_{L}} = \left\{ (z_{d_{L,1}}, z_{d_{L},2}) \mid z_{d_{L,1}} = \theta_{d_{L}}, z_{d_{L},2} \in \mathbb{R} \right\}. \quad (4.16)
$$

From Section 2.3.4, the transition can be expressed as

$$
\dot{z}_{s_{L},2}^{+} = \delta_{s_{L}}^{d_{L}} \dot{z}_{d_{L},2}, \quad (4.17)
$$

where $\delta_{d_{L}}^{s_{L}}$ is a constant since it is only configuration dependent.

**Single support** Using Proposition 3, the coordinates for SS (left) are chosen as

$$
z_{s_{L},1} = \theta_{s_{L}}(q_{s_{L}})|_{Z_{s_{L}}}, \quad (4.18a)
$$

$$
z_{s_{L},2} = \left( D_{s_{L,a}}(q_{s_{L}}, \dot{q}_{s_{L}}) \right)|_{Z_{s_{L}}}, \quad (4.18b)
$$

where $D_{s_{L,a}}$ is the row of matrix $D_{s_{L}}$ corresponding to the unactuated coordinate. With this choice, the low-dimensional model takes the form

$$
\dot{z}_{s_{L},1} = \kappa_{1_{L}}(z_{s_{L},1}) z_{s_{L},2}, \quad (4.19a)
$$

$$
\dot{z}_{s_{L},2} = \kappa_{2_{L}}(z_{s_{L},1}, z_{s_{L},2}). \quad (4.19b)
$$
**Transition from single support to double support**  The transition from SS to DS occurs when the value of $\theta_{s_L}$ attains the value $\theta_{s_L}^-$ corresponding to contact of the swing heel. This condition can be expressed as

$$
S_{s_L}^{d_R} \cap Z_{s_L} = \left\{ (z_{s_L}^-, z_{s_L}^-) \mid z_{s_L}^-, z_{s_L}^- \in \mathbb{R} \right\}.
$$

(4.20)

From Section 2.3.3, the transition can be expressed as

$$
z_{d_R}^+ \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \delta_{s_L}^{d_R} z_{s_L}^- \begin{bmatrix} 2 \\ 2 \end{bmatrix},
$$

(4.21)

where $\delta_{s_L}^{d_R}$ is a constant since it is only configuration dependent.

### 4.2.2 Model for one step with the right leg as the stance leg

**Double support**  Using Proposition 4, the coordinates for DS (right) are chosen as

$$
z_{d_R,1} = \theta_{d_R} \left( q_{d_R} \right) |_{z_{d_R}^+},
$$

(4.22a)

$$
z_{d_R,2} = \left( D_{d_{R,\alpha}} \left( q_{d_R} \right) \dot{q}_{d_R} \right) |_{z_{d_R}^+},
$$

(4.22b)

where $D_{d_{R,\alpha}}$ is the row of matrix $D_{d_R}$ corresponding to the unactuated coordinate. With this choice, the low-dimensional model takes the form

$$
\dot{z}_{d_R,1} = \psi_{1_R} \left( z_{d_R,1}, z_{d_R,2} \right),
$$

(4.23a)

$$
\dot{z}_{d_R,2} = \psi_{2_R} \left( z_{d_R,1}, z_{d_R,2} \right).
$$

(4.23b)

**Transition from double support to single support**  The transition from DS (right) to SS (right) occurs when the value of $\theta_{d_R}$ attains a specific value, $\theta_{d_R}^-$. From Section 2.3.4, the transition can be expressed as

$$
z_{s_R,2}^+ = \delta_{s_R}^{s_R} z_{s_R,2}^-,
$$

(4.24)

where $\delta_{s_R}^{s_R}$ is a constant since it is only configuration dependent.
Single support  Using Proposition 3, the coordinates for SS (right) are chosen as

\[
\begin{align*}
    z_{sR,1} &= \theta_{sR}(q_{sR})|_{Z_{sR}}, \\
    z_{sR,2} &= \left(D_{sRa}(q_{sR})\dot{q}_{sR}\right)|_{Z_{sR}},
\end{align*}
\]  

(4.25a)  

(4.25b)

where \(D_{sRa}\) is the row of matrix \(D_{sR}\) corresponding to the unactuated coordinate, the low-dimensional model takes the form

\[
\begin{align*}
    \dot{z}_{sR,1} &= \kappa_{1R}(z_{sR,1})z_{sR,2}, \\
    \dot{z}_{sR,2} &= \kappa_{2R}(z_{sR,1}, z_{sR,2}).
\end{align*}
\]  

(4.26a)  

(4.26b)

Transition from single support to double support  The transition from SS (right) to DS (right) occurs when the value of \(\theta_{sR}\) attains the value \(\theta_{sR}^-\) corresponding to contact of the swing heel. This can be expressed as

\[
S_{sR}^{d_1L} \cap Z_{sR} = \left\{ (z_{sR,1}^-, z_{sR,2}^-) \left| \begin{array}{c}
    z_{sR,1}^- = \theta_{sR}^-, \\
    z_{sR,2}^- \in \mathbb{R}
\end{array} \right. \right\}.
\]  

(4.27)

From Section 2.3.3, the transition can be expressed as

\[
\begin{align*}
    z_{sR,2}^+ &= \delta_{sR}^d z_{sR,2}^-,
\end{align*}
\]  

(4.28)

where \(\delta_{sR}^d\) is a constant since it is only configuration dependent.

The completion of the second step marks the end of one gait cycle with a left and a right step.
4.2.3 Overall low-dimensional hybrid model

The low-dimensional model for a complete gait cycle can be expressed as

\[
\begin{align*}
\bar{\Sigma}_{d_l} : & \quad \bar{X}_{d_l} = Z_{d_l} \\
& \quad \bar{F}_{d_l} : \ z_{d_l} = \bar{f}_{d_l}(z_{d_l}) \\
& \quad \bar{S}_{d_l} = S_{d_l}^d \cap Z_{d_l} \\
& \quad \bar{T}_{d_l} : \ z_{d_l}^+ = \bar{\Delta}_{d_l}(z_{d_l}^-) \\
\end{align*}
\]

(4.29a)

\[
\begin{align*}
\bar{\Sigma}_s : & \quad \bar{X}_s = Z_s \\
& \quad \bar{F}_s : \ z_s = \bar{f}_s(z_s) \\
& \quad \bar{S}_s = S_s^s \cap Z_s \\
& \quad \bar{T}_s : \ z_s^+ = \bar{\Delta}_s(z_s^-) \\
\end{align*}
\]

(4.29b)

\[
\begin{align*}
\bar{\Sigma}_{d_r} : & \quad \bar{X}_{d_r} = Z_{d_r} \\
& \quad \bar{F}_{d_r} : \ z_{d_r} = \bar{f}_{d_r}(z_{d_r}) \\
& \quad \bar{S}_{d_r} = S_{d_r}^d \cap Z_{d_r} \\
& \quad \bar{T}_{d_r} : \ z_{d_r}^+ = \bar{\Delta}_{d_r}(z_{d_r}^-) \\
\end{align*}
\]

(4.29c)

\[
\begin{align*}
\bar{\Sigma}_s : & \quad \bar{X}_s = Z_s \\
& \quad \bar{F}_s : \ z_s = \bar{f}_s(z_s) \\
& \quad \bar{S}_s = S_s^s \cap Z_s \\
& \quad \bar{T}_s : \ z_s^+ = \bar{\Delta}_s(z_s^-) \\
\end{align*}
\]

(4.29d)

where \( z \) denotes the state of the low-dimensional model and the subscripts denote the corresponding phase and corresponding leg. Note that the dimensions of \( \bar{X}_{d_l} \), \( \bar{X}_{d_r} \), \( \bar{X}_s \), and \( \bar{X}_s \) are each two whereas the dimensions of \( X_{d_l} \) and \( X_{d_r} \) are \( 2n_d \), and the dimensions of \( X_s \) and \( X_r \) are \( 2n_s \).
4.3 Stability properties of the asymmetric gait model

Asymmetric gait is a period-two gait. Unlike the case of symmetric gait, the low-dimensional model for one complete gait cycle is needed to investigate the stability properties of the motion in the limit cycle. The method of Poincaré is used to examine the stability of the periodic gait as in Section 2.4. Based on the methodology of [62, 70], the complete Poincaré map is composed of the mappings associated with SS and DS for each of the two steps making up the gait cycle. Assume that the left leg is the stance leg for step 1.

Mapping for double support (left) From (2.112), the map for DS (left) \( \rho_{d_L} : S_{s_L}^d \cap Z_{s_L} \rightarrow S_{d_L}^s \cap Z_{d_L} \) is given by

\[
\rho_{d_L}(z_{s_L}^-) = \delta_{s_L}^d z_{s_L}^- + V_{d_L}(\delta_{s_L}^d z_{s_L}^-).
\] (4.30)

Mapping for single support (left) From (2.109), the map for SS (left) \( \rho_{s_L} : S_{d_L}^s \cap Z_{d_L} \rightarrow S_{s_L}^d \cap Z_{s_L} \) is given by

\[
\rho_{s_L}(z_{d_L}^-) = \delta_{s_L}^s z_{d_L}^- + V_{s_L}(\delta_{s_L}^s z_{d_L}^-).
\] (4.31)

Mapping for double support (right) Similar to DS (left), the map for DS (right) \( \rho_{d_R} : S_{s_R}^d \cap Z_{s_R} \rightarrow S_{d_R}^s \cap Z_{d_R} \) is given by

\[
\rho_{d_R}(z_{s_R}^-) = \delta_{s_R}^d z_{s_R}^- + V_{d_R}(\delta_{s_R}^d z_{s_R}^-).
\] (4.32)

Mapping for single support (right) Similar to SS (left), the map for SS (right) \( \rho_{s_R} : S_{d_R}^s \cap Z_{d_R} \rightarrow S_{s_R}^d \cap Z_{s_R} \) is given by

\[
\rho_{s_R}(z_{d_R}^-) = \delta_{s_R}^s z_{d_R}^- + V_{s_R}(\delta_{s_R}^s z_{d_R}^-).
\] (4.33)
Overall Poincaré mapping for the low-dimensional model

The Poincaré map for the gait cycle, \( \rho : S_{s_R}^{d_R} \cap Z_{s_L} \rightarrow S_{s_R}^{d_R} \cap Z_{s_L} \) is given by

\[
\rho = \rho_{s_R} \circ \rho_{d_R} \circ \rho_{s_L} \circ \rho_{d_L}(z_{s_L}^2). \tag{4.34}
\]

The Poincaré map is computed numerically by evaluating \( z_{s,2} \) at the end of every other step.

### 4.4 Asymmetric gait model simulation

The previous chapter presented simulation results of the normal gait models using the minimal anthropomorphic model and gait analysis data reported by Winter [1]. The next chapter details the application of the asymmetric model to study the gait of a transtibial prosthesis user. In this section, the simulation of a leg length discrepancy is used to illustrate another application of the asymmetric model to study pathological gait.

To validate the derivation of the asymmetric gait models, asymmetry was introduced in the parameters of [1] (see Table 3.1) by increasing the length of the right shank by 0.01 m. This 1 cm increase represents a small (2.4%) increase in the shank length. Leg length discrepancies of up to 3 cm can be corrected using lifts in the shoe. The simulation compared the results of the full and low-dimensional asymmetric models.

Assuming that the posture is exactly enforced using feedback (2.58), the hybrid model starting from DS with the left leg as the stance leg may be simulated by specifying an initial value for the scalar quantity representing forward progression \( \theta = \theta_{d_L}^+ \) (see Figure 4.1) and its rate \( \dot{\theta} = \dot{\theta}_{d_L}^+ \). The desired values of the controlled coordinates \( h_{id} \) and the desired values of the joint velocities \( (\partial h_{id}/\partial \theta) \dot{\theta} \) are enforced by feedback. The change from DS to SS occurs when \( \theta = \theta_{d_L}^- \). When \( \theta = \theta_{d_L}^- \), integration of the DS model stops, the necessary transition map is computed and the model is switched to the SS model. Integration of the SS model proceeds until \( \theta = \theta_{s_L}^- \). This condition signifies heel contact of the right leg.
Integration of the SS model stops and the transition map for SS to DS is computed. This event marks the end of one step. The model for integration is then switched to the DS model of the new stance leg. Completion of the step with the new stance leg denotes the end of one gait cycle. Several gait cycles may be simulated to determine the existence of a stable gait (stable limit cycle).

Figures 4.3 and 4.4 compare the evolution of the low-dimensional model’s coordinates over one gait cycle using both the full- and low-dimensional models. Figures 4.5 and 4.6 are plots of the associated errors. The close match validates the derivation of the low-dimensional model for asymmetric gait. Figure 4.7 shows the convergence of the gait to a period-two limit cycle.

The right and left step lengths as predicted by the model are 0.58 m and 0.68 m, respectively. As expected, the right step length is shorter since the right leg is longer and as a result, heel contact occurs sooner than with the left leg. This example demonstrates the pronounced asymmetry that can occur in the gait because of a small asymmetry in the parameters of the right and left legs: an asymmetry of 0.01 m in the leg lengths produces an asymmetry of 0.1 m in the step lengths. The asymmetry in the two steps is also evident in Figures 4.3 and 4.4.

The next chapter describes the application of the hybrid models for asymmetric gait to study the gait of a transtibial prosthesis user.
Figure 4.3: Simulation results for $z_{d,1}$ in DS, $z_{s,1}$ in SS for one gait cycle using the full asymmetric model and using the low-dimensional asymmetric model. Note the asymmetry in the two steps because of the leg length discrepancy.

Figure 4.4: Simulation results for $z_{d,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full asymmetric model and using the low-dimensional asymmetric model. Note the asymmetry in the two steps because of the leg length discrepancy.
Figure 4.5: Error in $z_{d,1}$ in DS, $z_{s,1}$ in SS over one gait cycle using the full asymmetric model versus using the low-dimensional asymmetric model. The absolute and relative integration tolerances were set to $10^{-9}$ and $10^{-7}$, respectively.

Figure 4.6: Error in $z_{d,2}$ in DS, $z_{s,2}$ in SS over one gait cycle using the full asymmetric model versus using the low-dimensional asymmetric model. The absolute and relative integration tolerances were set to $10^{-9}$ and $10^{-7}$, respectively.
Figure 4.7: Simulation of the low-dimensional asymmetric hybrid model (4.29) for 40 steps (20 gait cycles) illustrates convergence to a period-two limit cycle. The dashed lines and the solid lines represent the steps with the left and the right leg as the stance leg, respectively.
CHAPTER 5

ANALYSIS OF THE GAIT OF TRANSTIBIAL PROSTHESIS USERS

The hybrid model for normal gait assumes that consecutive steps are symmetric. In the previous chapter, the hybrid model is extended to account for asymmetry in the anthropometric parameters and, possibly, joint motions, when the left and right legs alternate in the role of the stance leg. This chapter details the application of the asymmetric model to study the gait of a transtibial prosthesis user.

Several research questions are of interest in the study of gait of prosthesis users. Of prominent interest is the effect of varying inertial parameters, namely, the mass and mass distribution of prosthetic components [43, 44]. Other research has focused on studying the changes in user gait due to the use of different prosthetic components [45] or variations in prosthetic alignment [46, 48]. To study these effects, gait studies are usually conducted with a group of prosthesis users and inferences are drawn using statistical methods.

Prosthetic alignment is a critical factor in the successful use of a prosthesis. Current techniques for prosthetic alignment are primarily based on principles from statics. The dynamic alignment performed as a patient walks is largely based on heuristics from observation of a user’s gait and subjective user feedback on how the prosthesis feels. As a result, final alignment varies with the prosthetist [49] and is rarely repeatable, even by the
same prosthetist [50]. Moreover, classifying an alignment as good or bad is subjective.
The development of tools to quantify and analyze gait changes resulting from changing the
alignment of a prosthesis would be a significant step toward making alignment systematic
and specifying a desirable alignment based on lowered torque and energy costs.

The asymmetric gait model developed in the previous chapter is used to examine the
effect of varying a set of parameters associated with a transtibial prosthesis. The parameters
varied separately and in combination were (i) prosthetic alignment (ii) the mass and mass
distribution of a prosthesis, and (iii) stiffness of the prosthetic foot.

5.1 Cost functions

To compare the effect of different perturbations to the model, two cost functions are
used: total joint power and total joint torque. The costs are evaluated over one stride and
normalized by the steady-state walking speed. To compare costs for a given stance leg, the
cost functions may be evaluated over a step and normalized by walking speed. The cost
functions may also be evaluated for a given joint. Quantities of interest for the analysis,
such as walking speed and joint torques are computed for steady-state walking.

Total joint power as a measure of mechanical energy is considered a reliable indicator
of corresponding metabolic energy consumption [1]. The function used to calculate total
joint power is the integral of the absolute values of the joint powers over a stride (or step)
normalized by walking speed:

\[ C_p = \frac{1}{L_{st}} \int_0^T \sum_{i=1}^4 |u_i \dot{q}_i| \, dt, \]  

(5.1)

where \( L_{st} \) is the stride (or step) length, and \( T \) is the stride (or step) time. The torques \( u \) are
computed using (2.58). The joint power cost has units of Joules per meter.
The total joint torque cost is evaluated by the integral of the absolute values of the joint torques, normalized by walking speed:

$$C_u = \frac{1}{L_{st}} \int_0^T \sum_{i=1}^4 |u_i| dt.$$  \hspace{1cm} (5.2)

The torque cost has units of Newton-seconds and relates to the loading experienced by the joints. This in turn relates to the pain a prosthesis user may experience at the interface between the socket and the residual limb.

5.2 Application to the analysis of gait with a transtibial prosthesis

The gait of persons with unilateral amputations is asymmetric primarily because the inertial properties of the prosthetic leg are different from the inertial properties of the contralateral sound leg. In addition, because of the missing musculature, asymmetry in the joint motions over a stride is possible when the prosthetic and sound legs alternate their roles as the stance and swing legs. However, in the case studies that follow, an assumption of kinematic invariance is made. The assumption implies that a prosthesis user maintains a gait whose kinematics are symmetric with respect to the sound and prosthetic sides. As a result, the gait kinetics, namely the joint torques and joint powers, will differ. For transtibial prosthesis users, this assumption has been shown to be true by gait studies [72, 44]. Prosthesis users may exhibit kinematic invariance because the usual goal of rehabilitation is a symmetric gait [72].

5.2.1 Effect of varying prosthetic alignment in the sagittal plane

Figure 5.1 shows the six degrees of freedom associated with transtibial prosthesis alignment. To maintain symmetry while standing, the height of the prosthesis is adjusted so that the two hip joints are level. Hence, shortening and lengthening the leg (Figure 5.1(c)) was
Figure 5.1: Six degrees of freedom associated with transtibial prosthesis alignment: (a) dorsiflexion and plantarflexion; (b) anterior and posterior shift; (c) shortening and lengthening of the leg; (d) inversion and eversion of the foot; (e) medial and lateral shift; (f) toe-in and toe-out of the foot (rotation). Figure courtesy of [4].

not considered a variation in alignment in the studies that follow. Possible variations of alignment in the sagittal plane include anterior-posterior (A-P) shift of the prosthetic foot with respect to the socket and plantarflexion or dorsiflexion of the foot. Varying prosthetic alignment changes the position or orientation, or both, of the ROS of the ankle-foot complex with respect to the residual limb socket [73]. Alignment related to dorsiflexion and plantarflexion of the foot is usually performed to ensure comfortable standing. Then, further changes in dynamic alignment correspond to A-P shifts of the foot with respect to the socket. In this work, changes in alignment were incorporated in the model as changes in the A-P positioning of the ROS under the knee on the prosthetic side.
Table 5.1: Normalized anthropometric parameters for a sound leg [1] and a prosthetic leg [2]. The shank length is the distance from the ankle to the knee of the sound leg.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sound leg</th>
<th>Prosthetic leg</th>
<th>Normalized by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower leg mass</td>
<td>0.061</td>
<td>0.046</td>
<td>Body mass</td>
</tr>
<tr>
<td>Center of mass distance from knee</td>
<td>0.606</td>
<td>0.560</td>
<td>Shank length</td>
</tr>
<tr>
<td>Radius of gyration around knee</td>
<td>0.735</td>
<td>0.730</td>
<td>Shank length</td>
</tr>
</tbody>
</table>

**Methods**

For the sound leg, the joint angles, masses, and link lengths were obtained from anthropometric data [1]. The other parameters, such as locations of the link centers of mass and moments of inertia, were derived using standard anthropometric formulae [1]. The anthropometric parameters of the prosthetic leg were obtained from Selles et al. [2]. Table 5.1 lists the parameters that differ for the sound leg and the prosthetic leg. The assumption of kinematic invariance implies that the joint angle trajectories for the prosthetic leg are the same as those of the sound leg\(^\text{13}\). The EOM of the system were derived using the method of Lagrange [66], symbolic computations were performed using Maple\(^\text{®}\), and MATLAB\(^\text{®}\) was used for numerical computations.

The parameters of the ROS are shown in Figure 5.2. Investigation of the roll-over shapes of four different prosthetic feet in [74] indicated little change in their arc lengths. Gait analysis subjects used in the studies reported in [74] indicated a preference for a foot whose ROS matched the sound side ROS. For the alignment study with the model, the nominal alignment was taken to be that which mimics the ROS of the sound leg, and the

\(^{13}\)The asymmetric gait model allows the use of different joint trajectories for the left and right legs. In the case of users who make modifications to their gait, it is possible to study the dynamics and evaluate the costs of such a gait.
prosthetic foot was assumed to have a ROS of the same radius, arc length, and orientation as the sound side ROS. Then, with the ROS modeled as a circular arc, different alignments were represented by an A-P shift of the center of the circular arc.

**Results and discussion**

The simulation results indicate that the total joint torque and joint power costs are lowest for a specific alignment, and increase when the foot is moved to either side of this position; see Figure 5.3. In fact, there appears to be a zone between an A-P shift of 40 and 45 mm where the costs remain low, indicating some leeway for finding a desirable alignment. This study suggests that the desirable alignment may be the one where the joint torque or joint power costs are lowest. This may vary with each user and is likely the “sweet spot” prosthetists try to find when aligning feet in the sagittal plane.

The results appear to be in good agreement with the findings of Knox [75] and somewhat in agreement with Blumentritt’s [76] recommendations for static alignment. Knox developed a rigid wooden rocker prosthetic foot called the Shape Foot. He used a radius similar to that used by the physiologic ankle-foot system and placed the foot 40 mm in front of the knee axis on the prosthesis for his research subjects as a static (bench) alignment. Knox reports that almost no dynamic alignment was necessary to achieve a smooth gait in his subjects [75]. Blumentritt [76] indicates that positioning the load line 10 to 30 mm anterior to the knee center constitutes an optimal static prosthetic alignment.

Positioning the foot posterior to the desirable alignment tends to drive the knee into flexion at heel contact. To maintain weight-bearing stability, the user exerts greater effort to extend the knee. On the other hand, if the foot is set too far forward, the user expends greater effort to prevent the knee from hyperextending. A greater torque cost is directly related to the pain experienced by the user at the interface of the socket and residual limb,
Figure 5.2: The parameters that define a roll-over shape (ROS) modeled as a circular arc. The coordinates of the center of the circular arc are expressed in terms of a reference frame located at the ankle.
Figure 5.3: The total joint torque and joint power costs for one gait cycle with varying alignments. A positive value of the A-P shift indicates that the foot is moved anterior to the nominal.

while a greater power cost implies greater metabolic energy consumption. It may be concluded from this numerical study that proper A-P alignment is important to reduce total joint torque and total joint power costs.

In this study, the joint torque and joint power costs for the different alignments were normalized by their corresponding steady-state walking speeds. The costs can be evaluated at speeds other than the steady-state walking speed. If a prosthesis user is likely to walk in a range of speeds, the model provides the ability to evaluate the costs at these various speeds for different alignments. It would then be possible to find an alignment that represents a good compromise for the different speeds that the prosthesis user is likely to use on a regular basis.
5.2.2 Prosthesis mass perturbation

Inspired by the work of Selles et al. [43], the asymmetric gait model was used to study the variation in the gait of prosthesis users when the inertial properties of the prosthetic leg were changed by adding mass to different locations along the prosthesis. Gait studies conducted by Selles et al. [43] found that prosthesis users adapt to mass changes in the prosthesis by varying the kinetics in order to maintain the same kinematics.

Methods

The anthropometric parameters of the prosthetic leg were obtained from Selles et al. [2] and the parameters for the sound leg were obtained from Winter [1] (see Table 5.1). The five mass conditions studied by Selles et al. [72] were: (i) no additional mass, (ii) 1 kg added at the COM of the original prosthesis, (iii) 2 kg added at the COM of the original prosthesis, (iv) 1 kg added proximally, just below the knee, and (v) 1 kg added to the prosthesis at the ankle. The asymmetric gait model was simulated for masses added at different places along the shank length. This simulation study included the cases studied in [72]. In addition, to test the general belief that a lighter prosthesis is always preferable over a heavier one, the model was simulated for the case where 1 kg was removed from the original prosthesis at different locations. The added (or removed) masses were treated as point masses to calculate the new COM locations and radii of gyration of the prosthesis. The joint torque and joint power costs were evaluated at the steady-state walking speed.

In addition to mass perturbation, the model provides the ability to evaluate other scenarios, such as the effect of combined variations in mass distribution and alignment. The A-P shift of the ROS and the mass and the mass distribution of the prosthetic leg were varied in the study.
Figure 5.4: The joint torque costs for the hip (dashed) and knee (solid) of the prosthetic leg in swing. The costs increase with increased mass and more distal location of the additional mass.

**Results and discussion**

Figure 5.4 is a plot of the joint torque cost at the optimal alignment (A-P shift of +40 mm) for the prosthetic leg in swing for various mass distributions. The results support the findings of Selles et al. [72]: the knee and hip torques increase with increased distal mass. Figures 5.5 and 5.6 are plots of the total joint torque and total joint power costs computed over a gait cycle for different alignments. For each alignment (see Figures 5.7, 5.8, 5.9, and 5.10), the results support the findings of Mattes et al. [44] and Lehmann et al. [77] that addition of mass at a more distal location generally increases the energetic cost of walking. Figures 5.11 and 5.12 are plots of the steady-state walking speeds. The joint torque and joint power costs are normalized by the corresponding steady-state walking speeds.
Figures 5.7 and 5.9 indicate that for each mass condition, and away from the optimal alignment zone, the costs decrease with a more anterior alignment. Thus, the model indicates that a heavier prosthesis should be aligned more anterior than a lighter prosthesis to offset the costs of the additional mass. Addition of mass at locations in the proximal part of the shank (up to 40% of the shank length from the knee) cause the costs to increase gradually while the costs increase more sharply when the mass is added at more distal locations. At a more anterior alignment, the costs increase more gradually with added mass. An instance where mass is added to a prosthesis is when a component such as a shock absorber is incorporated in the prosthesis to improve the comfort of walking. In such a case, the model suggests that the additional component should be located more proximally and the alignment made more anterior to minimize the effects of the additional mass.

Figures 5.8 and 5.10 indicate that after a certain point, the joint torque costs increase if the alignment is made anterior to this point. This point of optimal alignment where the cost is minimum varies slightly for the different mass conditions. The optimal alignment point for the heavier prostheses tends to be more anterior than for the cases where the prosthesis is lighter than the sound side. Near the optimal alignment, the costs are less sensitive to added mass.

The results of varying mass distribution and A-P alignment indicate that the cost of added mass can be minimized by appropriately varying the A-P alignment. Specifically the alignment needs to be made more anterior if the original prosthesis is not at optimal alignment. There may be instances where the anterior positioning necessary for an optimal alignment may not be physically realizable. For example, the componentry available for performing the alignment may limit the amount of A-P shift, or the shift required may be
cosmetically unacceptable. In such a case, the model indicates that the anterior shift should be maximized to reduce the costs.

The plots also indicate that a lighter prosthesis may not always reduce the joint power costs (see Figure 5.10) although the joint torque costs decrease with reduced mass (see Figure 5.8). While other considerations such as strength or fatigue life may determine how light a prosthesis can be, the study shows that it is possible that making a prosthesis lighter may not always be beneficial. In some cases, the joint torque cost may be the deciding factor for a prosthesis user with a short residual limb and compromised musculature. In such a case, a user may be able to apply only limited torque at the knee on the prosthetic side to control the prosthesis because of limited musculature or because of pain at the socket-residual limb interface.

Figures 5.8 and 5.10 show that near the optimal alignment (A-P shift of 40 to 50 mm) the costs are less sensitive to the addition of mass to the prosthesis. Overall, this study underscores the importance of appropriate alignment to reduce the costs of walking since it appears that alignment has a greater influence on the costs than the mass of the prosthesis. Thus, the model can be used to quantify the costs and potential benefits of using different prosthetic components. For example, an implication of this study is that the addition of a component that provides A-P shift adjustability where none existed is potentially beneficial despite the added mass because the component may be used to reduce the cost of walking.

In another example, such a study may help quantify a choice between a more affordable but heavier prosthesis and a more expensive but lighter prosthesis. Stainless steel is the conventional choice for lower-limb prosthetic components because it possesses good structural properties and is corrosion-resistant. Titanium is the newer choice because of its
higher strength-to-weight ratio in comparison to stainless steel. A prosthesis made of tita-
nium components, however, is considerably more expensive than a stainless steel version. The model can be used to compare the costs and optimal alignments for the two prostheses of different weights. The comparison would indicate whether a significant reduction in the walking costs is evident to justify the use of the more expensive prosthesis. If the anterior shift necessary for optimal alignment is not realizable, the greater expense for the lighter prosthesis may be justifiable since the model indicates that the costs go up significantly with added mass away from the optimal alignment zone.

From a designer’s standpoint, the model provides the ability to evaluate various mass distributions. Computer-aided design can be used to optimize the mass distribution in the design stage. Designs can also be conceived such that the foot placement occurs in the optimal alignment zone. Incorporating some of the anterior shift in the design stage can be used to optimize the weight, improve the structural integrity of the prosthesis and address cosmesis issues.

### 5.2.3 Effect of varying the type of prosthetic foot

Different prosthetic feet conform to ROS of different radii, reflecting different stiffness properties. The relationship between the stiffness of a prosthetic foot and its ROS has not been quantified. Future experiments using the Prosthetic Foot Loading Apparatus described in [74] may provide empirical relationships connecting the prosthetic foot stiffness and the radius of its ROS. In general, a softer foot is one that conforms to a smaller radius. In addition, different feet will likely be aligned differently. The asymmetric gait model developed was used to study the effect of variations in the radius and A-P shift of the ROS.
Figure 5.5: The total joint torque costs when the mass distribution and A-P alignment are varied.

Figure 5.6: The total joint power costs when the mass distribution and A-P alignment are varied.
Figure 5.7: The total joint torque costs for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. The different A-P shifts are represented by dashdot (−10 mm), solid (zero), dotted (+10 mm), and dashed (+20 mm) lines. The costs decrease with a more anterior alignment.
Figure 5.8: The total joint torque costs for different A-P alignments when the mass distribution is varied. The different A-P shifts are represented by dashdot (+30 mm), solid (+40 mm), dotted (+50 mm), and dashed (+60 mm) lines. Depending on the mass and mass distribution, the costs are lowest for the +40 mm or +50 mm A-P shifts. For anterior alignments close to the optimal (+40 to +50 mm), the costs are less sensitive to added mass. Appropriate choice of A-P alignment can offset the addition of mass to the prosthesis and decrease the cost of walking.
Figure 5.9: The total joint power costs for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. The different A-P shifts are represented by dashdot (−10 mm), solid (zero), dotted (+10 mm), and dashed (+20 mm) lines. For alignments posterior to, and far from, the optimal, joint power costs generally increase with increased distal mass.
Figure 5.10: The total joint power costs for different A-P alignments (+30 mm to +60 mm) when the mass distribution is varied. The different alignments are represented by dashdot (+30 mm), solid (+40 mm), dotted (+50 mm), and dashed (+60 mm) lines. For anterior alignments close to the optimal (+40 to +50 mm), the costs are less sensitive to added mass and may decrease with added mass.
Figure 5.11: Steady-state walking speeds for different A-P alignments (−10 mm to +20 mm) when the mass distribution is varied. The different alignments are represented by dashdot (−10 mm), solid (zero), dotted (+10 mm), and dashed (+20 mm) lines.

Figure 5.12: Steady-state walking speeds for different A-P alignments (+30 mm to +60 mm) when the mass distribution is varied. The different alignments are represented by dashdot (+30 mm), solid (+40 mm), dotted (+50 mm), and dashed (+60 mm) lines.
Methods

The radius of the ankle-foot ROS of the sound leg equals approximately 36% of the leg length, where the leg length is the distance from the hip to the ground when standing [4]. This radius is hereafter referred to as the biomimetic radius. The total joint torque and joint power costs (see Figures 5.13 and 5.14) were evaluated for radii both smaller and larger than the biomimetic radius, with varying A-P alignments. The simulation results of varying the ROS radii were compared with recent studies that explored the effect of varying foot curvature on the energetics of walking [78].

Results and Discussion

The total joint power cost results given in Figure 5.14 support the findings of Adamczyk et al. [78]: metabolic energy consumption is lowered as the radius of the ROS is increased. The minima of the total joint torque and total joint power curves (see Figures 5.13 and 5.14) reduce dramatically as the radius is increased up to the biomimetic radius (36% LL). Above this radius, there is not much reduction in cost. Figure 5.15 is a plot of the steady-state walking speeds. An increased radius of the ROS results in lower steady-state walking speeds. As a result, because of insufficient angular momentum of the COM about the point of contact of the foot with the ground, a step cannot be completed for the more anterior alignments. Figure 5.16 compares the step lengths of the prosthetic leg and the normal leg. For a ROS radius of 36% LL and a ROS radius of 42% LL, i.e., at the legs’ respective optimal alignments (based on minimum total joint torque and total joint power costs), the results indicate that the step lengths of the sound and prosthetic legs are nearly symmetric. In feet that have ROS of smaller radii than the biomimetic radius, this asymmetry in step lengths may be more pronounced according to the study. Figure 5.17 compares the step
times of the prosthetic and sound legs. The results indicate that the stance times are similar between the sound and prosthetic limbs regardless of the radius of the ROS.

For prosthesis users, the results indicate that (i) a radius of the ROS on the prosthetic side that matches the sound side or is slightly larger results in gait that is closer to symmetric in terms of the step lengths and step times, and (ii) with an appropriate alignment, these radii also result in lower total joint torque and joint power costs while walking.

5.3 Clinical usefulness of the asymmetric model

The asymmetric gait model allows the assignment of unequal values for the kinematic properties, inertial properties, and the joint motions of the right and left legs. Using the model to perform systematic selection and alignment of prosthetic components would
Figure 5.14: The total joint power cost for one gait cycle for different radii of the prosthetic side roll-over shape (ROS) and its variation with A-P alignment. The different radii are expressed as a percentage of leg length (LL).

greatly reduce the time and expense involved in the process for the prosthetist and the prosthesis user. A few examples of how the model could be used in a clinical setting to perform an informed selection and alignment of transtibial prostheses are outlined below.

With an appropriately designed user interface, the asymmetric gait model could be used in a clinical setting such as a prosthetist’s office. In order to use the model, a prosthetist would input subject-specific parameters corresponding to Table 3.1 into the model. In addition, gait analysis could be performed to obtain the subject-specific and leg-specific joint motions. The COP data from the gait analysis could be used to obtain the ROS of the sound ankle-foot complex. The prosthetist would input the joint motion data and the ROS data into the model. If subject-specific gait data is not available, the model could still be used with a default set of joint motions derived based on average values because of the
Figure 5.15: Steady-state walking speed for different radii of the prosthetic side roll-over shape (ROS) and different alignments. The different radii are expressed as a percentage of leg length (LL). For the larger radii, a step cannot be completed for the more anterior alignments.
Figure 5.16: Comparison of the step length of the prosthetic (dotted lines) or the sound leg (solid lines) for different radii of the prosthetic side roll-over shape (ROS) and different alignments. The different radii are expressed as a percentage of leg length (LL).

Figure 5.17: Comparison of the step times of the prosthetic (dotted lines) and the sound leg (solid lines) for different radii of the prosthetic side roll-over shape (ROS) and different alignments. The different radii are expressed as a percentage of leg length (LL).
observed parsimony of human gait. The ROS parameters could also be estimated based on average values parameterized by leg length [74].

Using a database of feet classified by their ROS, and a database of prosthetic components, a prosthesis could be virtually assembled in software. Computer-aided design programs can provide good estimates of the inertial properties for assemblies of components. These tools could be integrated with the database and used with geometric models of the residual limb [43] to estimate the inertial properties on the prosthetic leg for the specific combination of components selected.

With this subject-specific data, the asymmetric model can be simulated to determine the joint torque and joint power costs, and temporal measures of the associated gait. The model could be simulated for varying alignments. The model can then be used to predict an optimal alignment based on an associated minimum joint torque or joint power cost. This starting point for alignment is likely to be close to the final alignment since the model predicts that there is a zone where the costs remain low.

The asymmetric gait model could be used to help with the selection of prosthetic components to include in a prosthesis. Using the model, various combinations of prosthetic components can be evaluated to determine the corresponding recommended alignment, and to determine if the recommended optimal alignment is physically feasible. In some cases, the recommended alignment may not be realizable because the components used may limit the amount of A-P shift, or the shift may be too extreme to be able to apply a cosmetic cover to the prosthesis. Typical prostheses use a system of connectors, known as the pyramid system, at the bottom of the socket and at the top of the foot. In these cases, A-P shifts of the foot with respect to the socket are achieved by angular changes at the two ends. The amount of A-P shift that can be achieved in this manner is limited by the distance
between the bottom of the socket and the top of the foot. For instance, with this method, only a limited A-P shift can be achieved for a short prosthesis user with a long residual limb. In such a case, the asymmetric gait model could be used to determine whether a slide adapter is necessary for further alignment changes. Since the results of the model indicate that alignment has a greater influence on the costs than the mass of the prosthesis, the additional mass of the adapter would likely impact the costs less and provide a much greater benefit in terms of the alignment capability. On the other hand, the model may determine that an additional slide adapter is unnecessary for a specific prosthesis user since the recommended alignment may be achieved by existing means. Thus, the model can be used to make informed choices of components to be included or excluded from the prosthesis.

The costs of including components such as a shock absorber or a torsional absorber can be evaluated using the model. These components are typically used to improve the comfort of the prosthesis user, but they add more weight and expense to a prosthesis. The asymmetric model can be used to select components that present an appropriate mass distribution. The model can also determine an optimal alignment that could offset the additional mass and enhance the benefits to the user.

The asymmetric gait model could also be used to justify the use of a lighter but more expensive prosthesis or the selection of a foot with a more appropriate ROS if the results indicate that these choices reduce the cost of walking. Reducing the torque costs for a prosthesis user is important from the standpoint of reducing damage to the joints. Reducing the energetic costs is important so that a prosthesis user stays active. Staying active can help prevent additional health problems. Preventing further damage is especially important if the prosthesis user underwent the amputation because of vascular disease.
The asymmetric gait model provides the ability to evaluate multiple options in prosthesis selection and alignment in a short amount of time. This would enable clinicians to devote more time to patient care.

Other clinical applications that were not explored in this dissertation include predicting the outcomes of orthotic\textsuperscript{14} treatments or physical therapy. For example, if a subject has restricted motion in one knee, the model can be used to determine the joint torque and energetic costs of this gait. If a particular physical therapy is prescribed for this patient, the model can be used to simulate the condition to see if the desired outcome is realized.

5.4 Gait studies

This chapter illustrates the application of the asymmetric gait model when various parameters related to a transtibial prosthesis are varied. The asymmetric gait model’s predictions agree well with clinical observations and with results of gait studies reported in the literature. The initial results demonstrate the versatility of an analytical forward dynamic model in studying the variation of parameters separately or in combination. In a clinical setting, the model can be used as a basis for analytical tools that systematize the process of prosthesis selection and alignment.

More extensive gait studies are required to validate the results of this model. The model’s prediction that optimal alignment is the result of a minimum joint torque or joint power cost needs to be validated. To validate the alignment study results of the model, gait studies need to be conducted using several transtibial prosthesis users. The gait studies would gather kinematic and ground reaction force data using a motion analysis system, compute the joint torques using inverse dynamics and evaluate the torque cost. The torque

\textsuperscript{14}An orthosis is an assistive device to treat pathologies while a prosthesis is a replacement device.
cost obtained using gait analysis can be compared with the torque cost predicted by the model. The trials need to be repeated for different A-P shifts of the foot with respect to the prosthetic socket. For each trial, the VO2 (volume of oxygen) consumption could be measured to estimate the metabolic cost of walking. The metabolic cost measures the biochemical energy required to perform a certain task. It is not possible to directly estimate the metabolic cost using a mechanical model but the joint power cost is considered a reasonable estimate of the metabolic cost [1]. The joint torque costs and energy consumption data should be analyzed to determine the existence of minima as predicted by the model. The model would be validated if such minima exist, and if the minima occur at the A-P shifts predicted by the model. For each subject, 19 trials (to cover A-P shifts from $-30$ mm to $+60$ mm in increments of 5 mm) need to be performed to cover the range of alignments studied using the model.

To study the effects of mass perturbation and alignment, the gait studies need to gather data when the alignment is varied in conjunction with adding masses at various locations along the prosthesis. To study the conditions studied by the model, 104 gait studies (to cover 6 mass distributions for the 2 mass conditions, 1 kg added and 2 kg added, 12 trials are needed, plus 1 trial for the original prosthesis with no mass added makes 13 trials times 8 trials to cover A-P shifts from $-10$ mm to $+60$ mm in 10 mm increments) would be required for each subject. Trials where 1 kg mass is removed from the prosthesis at different levels are not physically feasible. However, the conditions where mass is added could provide sufficient validation.

To study the effect of using feet that conform to different ROS, the feet can be classified according to the radii they conform to using the experimental set up described in [74]. Each subject would undergo gait analysis using the different feet with the alignment varied
during the trials with each foot. 114 trials (to cover the 19 A-P shifts from −30 mm to +60 mm in increments of 5 mm for each of the 6 different stiffnesses of the prosthetic foot) with each subject would be required to produce data corresponding to the case studies of the model. Thus, an experimental study of the variations studied using the model would require numerous gait studies involving considerable time and effort.
CHAPTER 6

SUMMARY, FUTURE WORK, AND CONCLUSION

6.1 Summary

This dissertation presents a model for human walking that addresses the many challenges in gait modeling such as multiple degrees of freedom (DOF), changing constraints, and occurrence of intermittent contact with the environment that may be impulsive. The modeling is a robotics-inspired approach and results in an analytical model for a complete gait cycle that takes into account the cycle’s hybrid nature. The modeling method uses measured gait data to derive a predictive, forward dynamic model that incorporates the advantages of low-dimensional analysis and accommodates the need to include minimal anthropomorphic features for clinical usefulness.

Inspiration for this approach comes from the fact that the primary task of walking is to translate the center of mass (COM) of the body from one position to another in the direction of progression. When gait is examined in the plane of progression (also known as the sagittal plane), the trajectories of the hip, knee, and ankle lie within narrow bands, across various subjects and multiple trials. This observation implies that walking is an activity that makes parsimonious and predictable use of the body’s various DOF to execute a repeatable pattern that accomplishes the task of forward progression of the COM. The observed parsimony forms the basis for a true low-dimensional forward dynamic model.
that captures the essence of the dynamics of walking in the sagittal plane over a complete gait cycle.

To derive the low-dimensional model, a planar anthropomorphic model that has five links in single support and six links in double support is used to represent the human. The foot-ground interface is modeled as a rolling contact using the roll-over shape (ROS) approach [3]. The coordinates of the anthropomorphic model are chosen such that the relative coordinates determine the posture and an absolute coordinate is left unconstrained. A specific control hypothesis is used in this research to reduce the anthropomorphic model to a low-dimensional model: it is hypothesized that the effect of the joint torques is to impose holonomic constraints on the posture as a function of the percentage of the gait cycle. This hypothesis enables the derivation of a low-dimensional model that is an exact sub-dynamic of a higher-dimensional anthropomorphic model.

Constraining the posture as a function of the forward progression in the gait cycle results in the evolution of the COM of the anthropomorphic model. The evolution as predicted by the model matches closely with the gait data. This result implies that the dynamics of normal walking can be described by means of the unconstrained coordinate alone in the continuous phases. In addition to the continuous phase sub-models, transition maps and constraint invariance are incorporated by design to lead to a low-dimensional hybrid model of human walking for a complete gait cycle. The stability properties of the low-dimensional model are evaluated using the method of Poincaré. The validation using measured human gait data supports the parsimony hypothesis that underlies the model’s derivation and demonstrates the observed stability of the measured gait.
To extend the clinical usefulness of the model, the low-dimensional hybrid model developed for normal gait is extended to derive an asymmetric hybrid model that can accommodate asymmetry in the anthropometric parameters and joint motions. The gait of a transtibial prosthesis user was studied with this asymmetric model. Two cost functions, total joint torque and total joint power, were used to quantify the effect of changing a set of parameters associated with the use of a prosthesis. The different scenarios studied were (i) varying the prosthesis’s alignment, (ii) varying the prosthesis’s inertial parameters, namely the mass and mass distribution, (iii) varying stiffness of the prosthetic foot, and (iv) combinations of these variations. The results from the study agree with clinical observations and the results of gait studies reported in the literature.

A list of the specific contributions of this dissertation follows.

- The low-dimensional modeling approach of [62] was extended to develop a hybrid model for a complete gait cycle of human walking. The extensions included the incorporation of feet and modeling of a non-instantaneous double support phase [79, 68].

- The choice of the ROS approach of [3] to model the foot-ground interface allows the incorporation of actual human gait data in the analytical models since the ROS approach is based on center of pressure data from gait studies. The advantage of using the effective rocker shape is that it elegantly captures the motions of the foot and ankle, and the deformations of the shoe. In addition, for normal human walking, the roll-over shape has been found to be invariant with changes in walking speed, with using shoes of different heel heights (within a reasonable range), and with carrying loads. In this work, the choice of the ROS approach enabled the representation of
the normal ankle-foot complex, prosthetic feet of varying stiffnesses, and different prosthetic alignments in the gait models.

- The modeling approach for normal gait was extended to model asymmetric gait. The asymmetric gait model allows for differences in anthropometric parameters and joint motions between the right and left legs.

- The asymmetric gait model was used to examine the influence of varying the alignment, inertial properties, and foot stiffness of the prosthesis on the gait dynamics [80]. Use of the model suggests that lowered joint torque or joint power cost may be a good goal for optimal alignment. The model was used to provide guidelines for alignment when the mass or mass distribution of a prosthesis is changed, and to suggest appropriate choices for prosthetic feet based on the ROS radius. The simulations show good agreement with published gait studies and demonstrate the versatility of the analytical model.

6.2 Future work

The hybrid model of walking developed in this thesis and its application to the analysis of gait using a transtibial prosthesis suggest areas for further research. Some of these areas are detailed next.

Gait studies

The simulation studies for transtibial prosthesis users assume that alignment is optimal when the total joint torque or joint power cost is at a minimum. Gait studies are necessary to validate this assumption and to validate the results of the model regarding the effects of alignment changes. Gait studies can also help establish the relationship between the static
alignment recommendations generally used by clinicians and the results of the asymmetric gait model.

The use of the ROS to model the foot-ground interface introduces some limitations; for example, in analyzing the gait of a prosthesis user, different combinations of alignment may give rise to the same ROS. Clinically, one of these alignments may be preferable to the others. The model, however, cannot distinguish between these alignments. Gait studies are necessary to establish a one-to-one relationship between the ROS and the preferred alignment to which it corresponds.

**Extension to model general foot motion**

The models in this work, by virtue of using the ROS, assume that there is always rolling motion from heel contact to opposite heel contact. Prosthetic feet may have shorter arc lengths leading to rocking motions at the heel, the toe, or both with rolling in between. The models developed in this dissertation can be extended to account for these phenomena. This extension would enable, for example, study of the effect of prosthetic foot drop-off on the dynamics of gait [81].

**Extension to three dimensions**

The models for normal gait and asymmetric gait only model motions in the sagittal plane. Hence, stability in the frontal plane cannot be studied. Extension of the model to three dimensions is necessary to investigate mediolateral stability. With this extension, it would be possible, for example, to predict the effect of variations in alignment in the frontal and transverse planes, and any associated coupling with the sagittal plane alignment.
Introducing compliance in the model

The anthropomorphic model used in this work is comprised of rigid links. In addition, heel contact is modeled with a rigid impact map. The compliance that is present in the human body is not included in the model. Incorporation of compliance in the model will add to modeling and computational complexity but will likely lead to more realistic results for the ground reaction forces.

Study of gait adaptations

The models developed in this dissertation can be used to study kinematic gait adaptations. Optimization can be used with the analytical models to determine a different set of joint motions that minimize the energetic cost of walking. The optimization results could influence training methods for new prosthesis users adapting to a prosthesis.

Extension of the model to study a transfemoral prosthesis user’s gait

Modeling a transfemoral prosthesis user’s gait involves additional complexity due to the lack of actuation at the prosthetic knee. When the prosthetic leg acts as the stance leg, the additional underactuation in the artificial knee can be handled by using the effective knee-ankle-foot\(^{15}\) ROS. When the prosthetic leg is the swing leg, modeling is more challenging. The modeling may need to incorporate spring-damper elements in the knee and incorporate the kinematics of polycentric knees.

\(^{15}\)The knee-ankle-foot (KAF) ROS is found by transforming the COP in laboratory coordinates to a coordinate system based on the ankle and hip markers. More details are presented in Appendix B.
6.3 Conclusion

The modeling approach taken in this dissertation presents an alternative to traditional forward dynamic approaches, which are computationally intensive and are unable to make predictions about the overall stability of the gait. The approach described attempts to bridge the gap between simplified models that are approximations, and complex anthropomorphic models that are rich in details but analytically intractable.

This approach aims to balance the clinical necessity of incorporating anthropomorphic details with analytical tractability. The approach used to derive the low-dimensional models for walking uses a minimal anthropomorphic model to represent the human. The models’s low dimension makes the analysis of walking tractable. The methodology used can be extended to cases where more complex models are used to represent the human. Such complex models may be clinically necessary to diagnose pathologies. The low-dimensional models developed in this work could form the template for these other, more elaborate “anchors”, models that capture more of the anthropomorphic details [53]. The template is necessary to provide the goal behavior for the anchor. Clinical insight can be achieved based on how the higher-dimensional anchor coordinates its elements to achieve the template behavior.

Extension of the template dynamical model for normal walking to model asymmetric gait aims to extend the usefulness of the approach. Using such an analytical model to better quantify the energetic costs of walking can lead to more informed design of prosthetic components with regard to their mass distribution and enable better design and selection of prosthetic feet. It is also hoped that the asymmetric model used in this work, once validated by gait studies, will form the basis for the development of analytical tools that will lead to objective prosthetic alignment techniques.
APPENDIX A

NOMENCLATURE

The subscript “s” corresponds to the single support phase and the subscript “d” corresponds to the double support phase. The subscript “di” corresponds to the double support model expressed in terms of the independent coordinates alone. The superscripts “+” and “−” define the beginning and end of a phase, respectively.

Applying the subscript “L” or “R” to any quantity associated with the single support phase or the double support phase indicates that the left or right leg is the stance leg for that phase.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Defined</th>
</tr>
</thead>
<tbody>
<tr>
<td>θs, θd</td>
<td>scalar, monotonic quantity corresponding to forward progression</td>
<td>Sec. 2.2.1</td>
</tr>
<tr>
<td>ns, nd</td>
<td>number of links in the anthropomorphic model</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>K</td>
<td>kinetic energy of the system</td>
<td>Sec. 2.2.1</td>
</tr>
<tr>
<td>V</td>
<td>potential energy of the system</td>
<td>Sec. 2.2.1</td>
</tr>
<tr>
<td>Ds, Dd, Ddi</td>
<td>inertia matrices</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>Cs, Cd</td>
<td>coriolis and centrifugal terms</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>Gs, Gd</td>
<td>gravity terms</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>Bs, Bd</td>
<td>Jacobian mapping applied torques to the respective joints</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>us, ud</td>
<td>applied torques</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>qs, qd</td>
<td>generalized coordinates</td>
<td>Sec. 2.2</td>
</tr>
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<td>Qs, Qd, Qdi</td>
<td>configuration spaces</td>
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</tr>
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<td>TQs, TQd, TQi</td>
<td>state spaces</td>
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<td>$q_{rel,s}$, $q_{rel,d}$</td>
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<td>absolute coordinate</td>
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<tr>
<td>$A$</td>
<td>Jacobian of the functions representing the position of the trailing toe</td>
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<td>$\lambda$</td>
<td>Lagrange multipliers</td>
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<td>constraints on the trailing toe in DS</td>
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<td>independent and dependent coordinates in DS</td>
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<td>generalized coordinates</td>
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<td>mapping from the independent to the dependent coordinates in DS</td>
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<td>$\upsilon$</td>
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<td>Cartesian coordinates of the trailing toe</td>
<td>Sec. 2.2.2</td>
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<td>$x_s, x_d, x_{d_i}$</td>
<td>state vector</td>
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<td>$D_e, q_e$, etc.</td>
<td>$e$ denotes objects related to the extended model</td>
<td>Eqn. (2.34)</td>
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<td>$J$</td>
<td>Jacobian of the functions representing the position of the impacting heel</td>
<td>Eqn. (2.34)</td>
</tr>
<tr>
<td>$\delta F$</td>
<td>impulses at the impact point</td>
<td>Eqn. (2.34)</td>
</tr>
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<td>transition map from “subscript” to “superscript”</td>
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<td>$\Upsilon$</td>
<td>mapping from the coordinates in SS to the extended model</td>
<td>Sec. 2.2.3</td>
</tr>
<tr>
<td>$\Omega_s$</td>
<td>mapping that extracts the coordinates required by the SS model</td>
<td>Sec. 2.2.3</td>
</tr>
<tr>
<td>$\Omega_{d_i}$</td>
<td>mapping that extracts the coordinates required by the DS model after relabeling the coordinates for the leg role-swapping</td>
<td>Sec. 2.2.3</td>
</tr>
<tr>
<td>$R_0$</td>
<td>circular matrix representing coordinate relabeling at impact</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>$\mathcal{X}<em>s, \mathcal{X}</em>{d_i}$</td>
<td>state spaces of the hybrid model’s charts</td>
<td>Sec. 2.2.4</td>
</tr>
<tr>
<td>$\mathcal{F}<em>s, \mathcal{F}</em>{d_i}$</td>
<td>vector field on $\mathcal{X}<em>s$ and $\mathcal{X}</em>{d_i}$</td>
<td>Sec. 2.2.4</td>
</tr>
<tr>
<td>$S_{d_i}^i, S_{d_i}^s$</td>
<td>switching function from “subscript” to “superscript”</td>
<td>Sec. 2.2.4</td>
</tr>
<tr>
<td>$T_{s_i}^d, T_{d_i}^s$</td>
<td>transition function from “subscript” to “superscript”</td>
<td>Sec. 2.2.4</td>
</tr>
<tr>
<td>$y, h(q)$</td>
<td>output, output function</td>
<td>Eqn. (2.55)</td>
</tr>
<tr>
<td>$h_0(q)$</td>
<td>independent quantities to be controlled</td>
<td>Eqn. (2.55)</td>
</tr>
<tr>
<td>$h_d$</td>
<td>functions representing desired evolution of independent quantities</td>
<td>Eqn. (2.55)</td>
</tr>
<tr>
<td>$h_{d_s}, h_{d_d_i}$</td>
<td>desired joint trajectories for actuated joints</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>$u^*$</td>
<td>unique control that enforces $y \equiv 0$</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
<td>Defined</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>Lagrangian</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>$z_{s,1}, z_{s,2}$</td>
<td>variables representing the state of the low-dimensional model in SS</td>
<td>Sec. 2.3.1</td>
</tr>
<tr>
<td>$Z_s, Z_d$</td>
<td>constrained surface</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>$D_{s,a}, D_{d,a}, \text{etc.}$</td>
<td>subscript $a$ corresponds to rows of respective matrices corresponding to $q_a$</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>$z_{d,1}, z_{d,2}$</td>
<td>variables representing the state of the low-dimensional model in DS</td>
<td>Sec. 2.3.2</td>
</tr>
<tr>
<td>$\delta_s^d, \delta_d^s$</td>
<td>constant that represents the transition from “subscript” to “superscript”</td>
<td>Sec. 2.3</td>
</tr>
<tr>
<td>$\rho_s, \rho_d$</td>
<td>Poincaré map</td>
<td>Sec. 2.4</td>
</tr>
<tr>
<td>$V_s, V_d$</td>
<td>integral in the Poincaré map</td>
<td>Sec. 2.4</td>
</tr>
<tr>
<td>$\rho$</td>
<td>overall Poincaré map</td>
<td>Sec. 2.4.3</td>
</tr>
<tr>
<td>$\vartheta_s, \vartheta_d, \vartheta$</td>
<td>slope of the linearized expression for the Poincaré map</td>
<td>Sec. 2.4</td>
</tr>
<tr>
<td>$\varphi_s, \varphi_d, \varphi$</td>
<td>intercept of the linearized expression for the Poincaré map</td>
<td>Sec. 2.4</td>
</tr>
<tr>
<td>$b_s, b_d$</td>
<td>Bézier polynomial representation of a desired joint trajectory</td>
<td>Sec. 3.1</td>
</tr>
<tr>
<td>$\alpha_s, \alpha_d$</td>
<td>coefficients of the Bézier polynomial</td>
<td>Sec. 3.4.1</td>
</tr>
<tr>
<td>$N_s, N_d, N_d$</td>
<td>order of the Bézier polynomial</td>
<td>Sec. 3.4.1</td>
</tr>
<tr>
<td>$s, s_s, s_d$</td>
<td>parameter of the Bézier polynomial</td>
<td>Sec. 3.4.1</td>
</tr>
<tr>
<td>$x_R, y_R$</td>
<td>coordinates of the point of rolling contact on the stance foot</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$F_{1x}, F_{1y}$</td>
<td>ground reaction forces at the rolling contact point</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$J_s$</td>
<td>Jacobian of the functions representing the positions of the rolling contact point</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$F_{2x}, F_{2y}$</td>
<td>ground reaction forces at the trailing toe</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$J_d$</td>
<td>Jacobian of the functions representing the positions of the rolling contact point and of the trailing toe</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of the roll-over shape</td>
<td>Sec. 3.5</td>
</tr>
<tr>
<td>$C_p$</td>
<td>joint power cost</td>
<td>Sec. 5.1</td>
</tr>
<tr>
<td>$C_u$</td>
<td>joint torque cost</td>
<td>Sec. 5.1</td>
</tr>
<tr>
<td>$T$</td>
<td>stride or step time</td>
<td>Sec. 5.1</td>
</tr>
<tr>
<td>$L_{st}$</td>
<td>stride or step length</td>
<td>Sec. 5.1</td>
</tr>
</tbody>
</table>
APPENDIX B

THE ROLL-OVER SHAPE APPROACH

This appendix presents details of the roll-over shape (ROS) derivation. The ROS approach [3] is used in this dissertation to model the normal and prosthetic ankle-foot complex.

Gait studies show that an able-bodied human appears to adapt to various conditions of level-ground walking such that the ROS remains unchanged. The ROS is found to be invariant with walking speed, changes in foot heel height, and carrying added weight [4].

B.1 The ankle-foot roll-over shape

The ankle-foot ROS models the effect of the ankle and foot motion as a rocker. The model is valid for the stance leg between heel contact (HC) and opposite heel contact (OHC). The ankle-foot ROS is obtained by transforming the center of pressure (COP) coordinates from a laboratory-based Cartesian coordinate system \([X_0, Y_0, Z_0]\) to a shank-based Cartesian coordinate system \([X_1, Y_1, Z_1]\) (see Figure B.1). The laboratory-based coordinate system is usually chosen to be an absolute coordinate system in gait studies.

Let \([\hat{x}_0, \hat{y}_0, \hat{z}_0]\), and \([\hat{x}_1, \hat{y}_1, \hat{z}_1]\) denote a Cartesian set of unit vectors in the absolute and shank-based coordinate systems, respectively. Then \(\hat{x}_0 = [1, 0, 0]\), and \(\hat{y}_0 = [0, 1, 0]\). The unit vectors normal to the sagittal plane are \(\hat{z}_0 = \hat{z}_1 = [0, 0, 1]\). Let \((x_A, y_A, 0)\) and
Figure B.1: The ankle-foot roll-over shape (ROS) is obtained by representing the center of pressure (COP) over a step in a shank-based coordinate system with the ankle as the origin [3]. The rigid ROS replaces the action of the ankle and foot in the period between heel contact and opposite heel contact. Figure adapted from [4].

\((x_K, y_K, 0)\) denote the coordinates of the ankle and knee, respectively, in the absolute coordinate system. The transformation of the COP to obtain the ROS proceeds as follows.

The unit vectors in the shank-based coordinate system are given by

\[
\hat{y}_1 = \frac{(x_K - x_A)\hat{x}_0 + (y_K - y_A)\hat{y}_0}{\sqrt{(x_K - x_A)^2 + (y_K - y_A)^2}}, \quad (B.1)
\]

\[
\hat{x}_1 = \hat{y}_1 \times \hat{z}_1, \quad (B.2)
\]

Then, a point on the ROS, \((x_{11}, y_{11}, 0)\), in the shank-based coordinate system is given by

\[
\begin{bmatrix}
  x_{11} \\
y_{11} \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
  \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 & x_A \\
  \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 & y_A \\
  \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
p_x \\
0 \\
0 \\
1
\end{bmatrix}, \quad (B.3)
\]
Figure B.2: Ankle-foot ROS calculation. (a) Ankle and knee trajectories in the sagittal plane, and COP locations on the floor. The data is plotted in a laboratory-based coordinate system. (b) The data plotted in a shank-based coordinate system. The COP in the shank-based coordinates draws out the effective rocker, or roll-over shape. Data in both plots are shown in black from heel contact (HC) to opposite heel contact (OHC) and in white after OHC. Note that the rocker shape is between HC and OHC. Data taken from [1]. Figure courtesy of [4].

where \((p_x, 0, 0)\) are the coordinates of the corresponding COP in the laboratory-based coordinate system. The locus of \((x_{11}, y_{11}, 0)\) is the ankle-foot ROS with respect to the shank-based coordinate system.

Figure B.2 is a plot of the data from [1]. This data is used to derive the ROS for the normal foot in the simulation studies presented in this dissertation.

### B.2 The knee-ankle-foot roll-over shape

The knee-ankle-foot (KAF) ROS captures the effect of the knee, ankle, and foot motions. The KAF ROS is found by transforming the COP in laboratory coordinates into a coordinate system based on the hip and ankle markers (see Figure B.3). It is thus a model for the effective rocker that the knee-ankle-foot system conforms to between HC and OHC.
Figure B.3: The knee-ankle-foot roll-over shape is obtained by representing the center of pressure over a step in a coordinate system based on the hip and the ankle [4]. Figure adapted from [4].
This appendix presents two techniques commonly used in Analytical Dynamics to analyze the dynamics of constrained systems. The double support phase of walking is an example of a constrained dynamical system.

Two approaches to the derivation of the equations of motion (EOM) for a system subject to holonomic constraints are presented. These approaches are the Newton-Euler or Vectorial approach and the Lagrangian or Analytical approach [82].

C.1 Vectorial approach

This approach involves the direct application of Newton’s second law. The method concentrates on the forces and motions associated with the individual parts of the system, and on the interactions among these parts. In this approach, the force and moment balances are written for each body separately. Kinematical relations and constraint forces are then used to reduce the number of equations. The vectorial approach is iterative, and is not used in this dissertation.
C.2 Analytical approach

In this approach, the system of interest is analyzed as a whole. The number of degrees of freedom (DOF) of the system is determined, and a set of independent generalized coordinates is chosen to represent the DOF. The method uses scalar functions such as the kinetic and the potential energies. The method can be used to obtain a complete set of the EOM without solving for the constraint forces. This approach has some advantages. The computation of accelerations is not necessary, which simplifies the calculations. The method also yields the EOM in a closed form that is suitable for control.

The system is described using a minimum set of generalized coordinates. The constraints on the system may be holonomic or non-holonomic. Constraints are termed holonomic if they depend only on the configuration variables, that is, only on the generalized coordinates. A non-holonomic constraint is usually a non-integrable constraint on the velocities.

In some dynamical systems, it is not possible to find a set of generalized coordinates for a system. An example is a system with non-holonomic constraints that cannot be integrated to displacement expressions. Another instance is when a system has holonomic constraints, but the surplus coordinates cannot be easily eliminated. In other cases, a system may have holonomic constraints but the amplitudes of the reaction forces may be needed. Two techniques commonly used in the analysis of such systems are the Lagrange Multiplier Approach and the Embedding Method (or Kane’s Method).

C.2.1 Lagrange multiplier approach

The Lagrange Multiplier approach is used in the cases where the reaction forces are required for the analysis. To apply the method, the constraints are introduced in the EOM as
generalized constraint forces. The EOM in terms of the full set of \( n \) generalized coordinates are of the form,

\[
D \ddot{q} + C \dot{q} + G = Bu + A^T \lambda
\]  

(C.1)

where \( \lambda \) is a \( m \times 1 \) vector of Lagrange multipliers corresponding to the \( m \) constraints and \( A \) is the \( n \times m \) constraint Jacobian matrix. The \( m \) constraint equations can be written as

\[
g(q) = 0, \quad i = 1, 2, \ldots, m,  
\]

(C.2)

or, after differentiation with respect to time, in matrix form as

\[
A \dot{q} = 0,  
\]

(C.3)

where \( A = \partial g / \partial q \). Differentiating (C.3) with respect to time results in

\[
A \ddot{q} + \dot{A} \dot{q} = 0.  
\]

(C.4)

From (C.1) and (C.4), the \( m + n \) differential-algebraic equations can now be solved for the accelerations and the constraint forces as follows using [83].

\[
\begin{bmatrix}
D & -A^T \\
A & 0 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\dot{\lambda} \\
\end{bmatrix} =
\begin{bmatrix}
-C \dot{q} - G + Bu \\
-\dot{A} \dot{q} \\
\end{bmatrix}.  
\]

(C.5)

The advantage of this method is that the Lagrange multipliers and the accelerations can be determined simultaneously by solving a system of linear equations. The disadvantages are that the method requires a consistent set of initial conditions, and the method is known to have stability problems because of numerical drift.

**C.2.2 Embedding method or Kane’s method**

When there are \( n \) EOM and \( m \) holonomic constraints, \( m \) coordinates are dependent. The embedding method can be used to eliminate the Lagrange multipliers and express the
EOM in terms of the \((n - m)\) independent generalized coordinates. The \(m\) constraints can be expressed as

\[
g(q) = 0, \tag{C.6}
\]

or,

\[
Aq = 0, \tag{C.7}
\]

where \(A = \partial g/\partial q\). Differentiating with respect to time results in

\[
A\ddot{q} + \dot{A}\dot{q} = 0 \quad \Rightarrow \quad A\ddot{q} = -\dot{A}\dot{q} =: gc. \tag{C.8}
\]

Partitioning \(q\) into independent and dependent coordinates,

\[
q = \begin{bmatrix} q_i \\ q_d \end{bmatrix}, \tag{C.9}
\]

yields

\[
A\dot{q} = 0 \quad \Rightarrow \quad A_i\dot{q}_i + A_d\dot{q}_d = 0 \quad \Rightarrow \quad \dot{q}_d = -A_d^{-1}A_i\dot{q}_i, \tag{C.10}
\]

and

\[
A\ddot{q} = A_i\ddot{q}_i + A_d\ddot{q}_d = g_c. \tag{C.11}
\]

If the \(m\) constraints are independent, the independent coordinates can be chosen such that \(A_d^{-1}\) exists. Premultiplying (C.11) by \(A_d^{-1}\) yields

\[
A_d^{-1}A_i\ddot{q}_i + \ddot{q}_d = A_d^{-1}g_c \tag{C.12}
\]

\[
\Rightarrow \ddot{q}_d = -A_d^{-1}A_i\ddot{q}_i + A_d^{-1}g_c \tag{C.13}
\]

and

\[
\ddot{q} = \begin{bmatrix} \ddot{q}_i \\ \ddot{q}_d \end{bmatrix} = \begin{bmatrix} I \\ -A_d^{-1}A_i \end{bmatrix} \dot{q}_i + \begin{bmatrix} 0 \\ A_d^{-1}g_c \end{bmatrix}. \tag{C.14}
\]

Let

\[
\begin{bmatrix} I \\ -A_d^{-1}A_i \end{bmatrix} =: A_{di} \quad \text{and} \quad \begin{bmatrix} 0 \\ A_d^{-1}g_c \end{bmatrix} =: gc. \tag{C.15}
\]
\[ \ddot{q} = A_{di} \dot{q}_i + g_{dc}. \]  \hspace{1cm} (C.16)

The EOM for the constrained system are
\[ D \ddot{q} + C \dot{q} + G = Bu + A^T \lambda, \]  \hspace{1cm} (C.17)

where \( \lambda \) is a \( m \times 1 \) vector of Lagrange multipliers corresponding to the \( m \) constraints.

Premultiplying (C.17) by \( A_{di}^T \) yields
\[ A_{di}^T D \ddot{q} + A_{di}^T C \dot{q} + A_{di}^T G = A_{di}^T Bu + A_{di}^T A^T \lambda. \]  \hspace{1cm} (C.18)

Notice that
\[ A_{di}^T A^T = \begin{bmatrix} I \\ -A_d^{-1} A_i \end{bmatrix}^T A^T = \begin{bmatrix} I^T \\ -(A_d^{-1} A_i)^T \end{bmatrix} \begin{bmatrix} A_{di}^T \\ A_d^T \end{bmatrix} = A_i - A_{di}^T A_d^{-T} A_d^T = 0. \]  \hspace{1cm} (C.19)

As a result, \( \lambda \) is eliminated from (C.18). The EOM for the constrained system are then
\[ D_i \ddot{q}_i = Q + A_{di}^T Bu, \]  \hspace{1cm} (C.20)

where
\[ D_i := A_{di}^T D A_{di}, \]  \hspace{1cm} (C.21)
\[ Q := -A_{di}^T (Dg_{dc} + C \dot{q} + G). \]  \hspace{1cm} (C.22)

This method yields the EOM in a minimum set of coordinates for the system and is suitable for the use of control in constrained systems.
BIBLIOGRAPHY


