SUPPLY CHAIN DELIVERY PERFORMANCE:

POINTS OF VIEW OF A SUPPLIER AND A BUYER

A dissertation submitted to
Kent State University Graduate School of Management

By
Maxim A. Bushuev
Dept. of Management and Information Systems
Kent State University

Dissertation Committee:
Dr. Alfred L. Guiffrida (Chair)
Dr. Butje Eddy Patuwo
Dr. Murali Shanker
Dr. Emmanuel Dechenaux
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>vii</td>
</tr>
<tr>
<td>Abstract</td>
<td>viii</td>
</tr>
<tr>
<td>Chapter 1: Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Modeling supply chain delivery performance</td>
<td>3</td>
</tr>
<tr>
<td>with a delivery window</td>
<td>3</td>
</tr>
<tr>
<td>1.3. Limitations in supply chain delivery performance models</td>
<td>4</td>
</tr>
<tr>
<td>1.4 Research objectives</td>
<td>6</td>
</tr>
<tr>
<td>Chapter 2: Literature review</td>
<td>9</td>
</tr>
<tr>
<td>2.1. Supply chain delivery window</td>
<td>9</td>
</tr>
<tr>
<td>2.1.1. Examples of delivery windows</td>
<td>9</td>
</tr>
<tr>
<td>2.1.2. Empirical studies on the importance of on time delivery</td>
<td>10</td>
</tr>
<tr>
<td>2.1.3. Models for evaluating delivery performance</td>
<td>13</td>
</tr>
<tr>
<td>2.2. Inventory management models</td>
<td>17</td>
</tr>
<tr>
<td>Chapter 3: Concept of optimal position of supply chain delivery window</td>
<td>20</td>
</tr>
<tr>
<td>3.1. Determining the optimal position of delivery window</td>
<td>20</td>
</tr>
<tr>
<td>3.2. Optimal position of delivery window and expected penalty cost</td>
<td>25</td>
</tr>
<tr>
<td>for untimely delivery for several distributions with closed form cdf</td>
<td>25</td>
</tr>
<tr>
<td>3.2.1. Uniform distribution</td>
<td>25</td>
</tr>
<tr>
<td>3.2.2. Exponential distribution</td>
<td>26</td>
</tr>
</tbody>
</table>
3.2.3. Logistic distribution

3.2.4. Asymmetric Laplace distribution

3.3. Optimal position of delivery window and expected penalty cost
for untimely delivery for several distributions without closed form cdf

3.4. Numerical example

Chapter 4: Delivery performance improvement

4.1. Effect of parameters defined in a contract on expected penalty cost

4.1.1. Width of the on-time portion of the delivery window

4.1.2. Penalty costs for early and late deliveries

4.2. Effect of uniform delivery time distribution parameters
on expected penalty cost

4.3. Effect of exponential delivery time distribution parameters
on expected penalty cost

4.4. Effect of logistic delivery time distribution parameters
on expected penalty cost

4.5. Effect of Laplace delivery time distribution parameters
on expected penalty cost

4.6. Effect of normal delivery time distribution parameters
on expected penalty cost

4.7. Effect of gamma delivery time distribution parameters
on expected penalty cost

4.8. Discussion
List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Illustration of a Delivery Window for Gaussian Delivery Times</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Positioning of delivery window</td>
<td>22</td>
</tr>
<tr>
<td>3a</td>
<td>Position of delivery window for asymmetric Laplace distribution (Case 1)</td>
<td>29</td>
</tr>
<tr>
<td>3b</td>
<td>Position of delivery window for asymmetric Laplace distribution (Case 2)</td>
<td>29</td>
</tr>
<tr>
<td>3c</td>
<td>Position of delivery window for asymmetric Laplace distribution (Case 3)</td>
<td>29</td>
</tr>
<tr>
<td>4a</td>
<td>The optimal position of delivery window as function of $QH/K$</td>
<td>31</td>
</tr>
<tr>
<td>4b</td>
<td>The optimal position of delivery window as function of $Ac$</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>Graph of a convex function</td>
<td>33</td>
</tr>
<tr>
<td>6</td>
<td>The expected penalty cost ($Y$) as function of the position of the delivery window ($c_1$)</td>
<td>35</td>
</tr>
<tr>
<td>7</td>
<td>Typical cases</td>
<td>69</td>
</tr>
<tr>
<td>8a</td>
<td>Case 2: On time delivery</td>
<td>71</td>
</tr>
<tr>
<td>8b</td>
<td>Case 3: Early delivery</td>
<td>72</td>
</tr>
<tr>
<td>8c</td>
<td>Case 1: Late delivery</td>
<td>73</td>
</tr>
<tr>
<td>9</td>
<td>Optimal reorder point for different values of $H$</td>
<td>90</td>
</tr>
<tr>
<td>10</td>
<td>Holding cost as a percent of total cost for different values of $H$</td>
<td>90</td>
</tr>
<tr>
<td>11</td>
<td>Difference of total costs for $R_{opt}$ and $R = 17,257.5$</td>
<td>91</td>
</tr>
</tbody>
</table>
List of Tables

Table 1. Delivery Window Definitions 10

Table 2. Summary of Empirical Studies on the Importance of On Time Delivery 11

Table 3. Classification of supply chain delivery performance models 16

Table 4. Expected penalty costs for different values of $QH/K$ 37

Table 5. Connection between parameters of supplier’s and buyer’s cost functions 83
Supply Chain Delivery Performance: 
Points of View of a Supplier and a Buyer

ABSTRACT

The need for performance measurement and evaluation in supply chain management is well recognized in the literature. The timeliness of delivery is a key concern to customers and numerous empirical studies have documented the importance that on time delivery plays in the operation of the supply chain. Supply chain delivery performance models are based on the concept of the delivery window, which is defined as the difference between the earliest acceptable delivery date and the latest acceptable delivery date. In the dissertation supply chain delivery performance is evaluated from a supplier’s and a buyer’s prospective.

The research introduces a concept of the optimal positioning of the delivery window in a serial supply chain. Optimally positioning the delivery window minimizes the expected penalty cost due to early and late delivery. The conditions for the optimal position of the delivery window are derived for the general form of a delivery time distribution.

The research herein addresses strategies of delivery performance improvement using a cost based delivery performance model and evaluates the effect of different parameters on the expected penalty cost. An understanding of these analytical properties provides a strong foundation for identifying and integrating strategies to improve delivery performance.

Furthermore, we investigate how the timeliness of the delivery will affect the inventory cost structure of a buyer in a two stage supply chain. From the perspective of the buyer, untimely delivery can impact inventory holding and stockout costs. We formulate the supply chain delivery window problem as a stochastic model with three possible delivery outcomes (early, on time, and late delivery) and integrate this feature with an inventory model with two levels of storages (owned warehouse and rented warehouse).

This comparison and supporting analysis bridges existing gaps found in the literature and contributes to linking and coordinating the delivery and inventory sub processes within supply chains. Theoretical and managerial implications of the findings are discussed.
CHAPTER 1
INTRODUCTION

1.1. Background

Supply chain management serves as the foundation of an organization’s overall competitive strategy for attaining and maintaining competitive advantage (Kristal et al., 2010; Nollet et al., 2005; Bechtel & Jayaram, 1997). Under the supply chain management philosophy, value-adding activities such as raw materials acquisition, production processing and physical distribution are all coordinated to insure that customer demand is met with a correct order quantity that is delivered in a timely manner (Wisner et al., 2012). Based on empirical findings found in the Supply Chain Management 2010 and Beyond research initiative, Melnyk et al. (2010) identify six outcomes that supply chains should provide: cost, responsiveness, security, sustainability, resilience, and innovation. In this paper we present a model for determining the optimal position of the delivery window to the end customer in a serial supply chain. Our modeling directly addresses improving the timeliness and reliability of delivery thus contributing to the potential improvement of the supply chain in terms of the outcomes of cost, responsiveness and resilience as identified in Melnyk et al. (2010).

The delivery process is one of five supply chain processes (plan, source, make, deliver, and return) found in the Supply Chain Operations Reference-model (SCOR) (Lockamy and McCormack, 2004; Huan et al., 2004). Delivery lead time to the end customer in a supply chain is defined to be the elapsed time from the receipt of an order by the supplier to the receipt of the product ordered by the customer and is composed of a series of internal (manufacturing and processing) lead times and external (distribution and transportation) lead times found at the
various stages of the supply chain. The timeliness of delivery is a key concern to customers and numerous empirical studies have documented the importance that on-time delivery plays in the operation of the supply chain (da Silveira & Arkader, 2007; Iyer et al., 2004; Salvador et al., 2001).

The delivery process within an integrated supply chain is attractive to study for several reasons. First, the importance of time in establishing competitive performance was well established in the literature in the 1980s by Porter (1980) and Stalk (1988) and serves today a fundamental aspect of overall supply chain success. As a time based measure, delivery performance to the final customer in a supply chain is firmly based in this foundation. Second, given the direct impact that the timeliness of delivery has on customer satisfaction, improving the performance of the delivery process is a key concern of supply chain and logistics managers (Forslund et al., 2009). The need for performance measurement and evaluation in supply chain management is well recognized in the literature (see for example, Martin & Patterson, 2009; Gunasekaran & Kobu, 2007; and Lockamy & McCormack, 2004). Lastly, several researchers have examined the relationship between delivery performance and supply chain operations. Anderson et al. (2011) presented empirical evidence on the importance of delivery performance in the selection third party logistics providers. Delivery performance and supplier selection has been investigated by Shin et al. (2009), Morgan and Dewhurst (2008) and Ernst et al. (2007). Golini and Kalchschmidt (2010) addressed the relationship between globalization of sales, investments in supply chains and delivery performance. Lane and Szwejczewski (2000) investigated the link between delivery performance and production planning and control systems.

A taxonomy of supply chain performance metrics that spans the strategic, tactical and operational levels of supply chain operation has been effectively summarized in Bhagwat and
Sharma (2007) and Gunasekaran et al. (2004). Delivery performance is acknowledged as a key metric for supporting operational excellence of supply chains and is classified as a strategic level performance measure by Gunasekaran et al. (2004). Rao et al. (2011) characterize delivery performance as the most important metric in a supply chain since it serves to integrate and involve the measurement of performance throughout the stages of the supply chain. Cost-based models that translate time-based delivery performance measures into financial based delivery performance metrics serve as a precursor for identifying and managing improvements in delivery in the delivery process (Guiffrida and Nagi, 2006b).

1.2. Modeling Supply Chain Delivery Performance with a Delivery Window

An integral component found in many supply chain delivery performance models is the concept of the delivery window. In the context of a delivery performance model, a delivery window is defined as the difference between the earliest acceptable delivery date and the latest acceptable delivery date. When an order is placed, the customer is typically given a fixed due date. Under the concept of a delivery window, the customer supplies benchmarks in time which are used to classify deliveries as being early, on-time, and late (see Figure 1). Delivery lead time, $X$, is a random variable with probability density function $f(x)$. The on-time portion of the delivery window is defined by $\Delta c$. Ideally, $\Delta c = 0$. However, the extent to which $\Delta c > 0$ may be measured in hours, days, or weeks depending on the industrial situation.

Early and late deliveries introduce waste in the form of excess cost into the supply chain; early deliveries contribute to excess inventory holding costs while late deliveries may contribute to production stoppage costs, lost sales and loss of goodwill. These costs have been characterized by Guiffrida and Nagi (2006a) as “penalty costs” that are incurred in addition to the normal
operating costs of the supply chain and are hence considered to be forms of waste. When a delivery is within the on time portion of the delivery window, no penalty cost is incurred.

![Figure 1. Illustration of a Delivery Window for Gaussian Delivery Times]

Legend: $f(x)$ is the probability density function (pdf) of delivery time, $C_t$ is beginning of on time delivery, $\Delta C$ is the width of the delivery window.

1.3. Limitations in Supply Chain Delivery Performance models

The supply chain delivery performance models reported in the literature are elegant in their application of probability theory to the task of financially evaluating supply chain delivery performance; however, these models do have a set of limitations. A critical examination of the literature on cost-based delivery performance models in supply chain management reveals the following three limitations.

The current set of supply chain delivery performance models assume that the position of the delivery window is fixed and make no attempt to find the optimal position of the delivery window. A contractually agreed upon delivery window which is not optimally positioned within
the delivery distribution contributes to a normative level of delivery performance that is an inaccurate financial characterization of the waste due to untimely (early and late) delivery inherent to the true underlying delivery process. This inaccuracy can lead to management’s adoption of a presumed level of performance that is not indicative of the actual level of performance that could be achieved.

A second limitation is that the models define delivery time as a Gaussian random variable. The widespread use of the Gaussian distribution suggests two major limitations in supply chain delivery performance models found in the literature. First, use of the Gaussian implicitly implies that the resulting delivery time distribution is always symmetric which may be an over restrictive assumption. Second, negative delivery times are also possible under the Gaussian model. By definition, a Gaussian random variable is defined over the range of negative infinity to positive infinity. In reality, a random variable for representing delivery time can only be non-negative. As demonstrated in Guiffrida and Jaber (2006), using the Gaussian to model a time-based random variable such as lead time can lead to significant underestimation when the coefficient of variation in the lead time distribution exceeds 0.25.

A third limitation is the lack of evaluating the supply chain delivery performance from a buyer’s point of view. From a supplier’s prospective, the effect of the timeliness of delivery can be described by supply chain delivery performance models; but the delivery performance models do not look at the buyer’s costs associated with delivery performance. On the buyer’s side, early and late deliveries introduce waste in the form of excess cost into the supply chain; early deliveries contribute to excess inventory holding costs while late deliveries may contribute to production stoppage costs, lost sales and loss of goodwill. Thus, delivery will affect buyer’s
inventory level and inventory models represent buyer’s point of view on supply chain delivery. It is in buyer’s best interests to minimize inventory costs and probability of stockouts.

1.4. Research objectives

The objectives of this research are as follows:

• Define the concept and derive the conditions of the optimal position of the delivery window.

• Evaluate delivery performance and develop strategies of delivery performance improvement.

• Define a buyer’s point of view on supply chain delivery performance and develop a strategy for delivery performance optimization.

In satisfying the first research objective, a methodology for determining the optimal position of the delivery window which minimizes the expected cost of untimely (early and late) delivery will be developed. The conditions for the optimal position of the delivery window will be derived for a general form of the delivery time distribution in a two stage supply chain and shown to be a function of the delivery window width and ratio of early to late penalty costs per unit per time. The position can be chosen by the supplier without incurring additional costs due to the positioning decision. Closed form expressions for the optimal position of the delivery window will be derived for several different forms of delivery time distributions that have been used in delivery performance models found in the literature. A search algorithm will be presented for determining the optimal position of the delivery window for delivery distributions that do not have a closed form for their cumulative density function. The research reported herein
contributes a methodology for determining the optimal position of the delivery window which minimizes the expected penalty cost of untimely delivery.

Once the currently level of delivery performance has been evaluated, the next natural step is to find ways to improve delivery performance which is the second research objective. Strategies for improving delivery performance will be addressed using a cost based delivery performance model. A set of supporting propositions will be presented to provide an analytical analysis of the delivery performance model. The propositions examine the robustness of the optimal positioning of the delivery window and the expected penalty cost of untimely delivery in terms of the width of the on-time portion of the delivery window, penalty costs for early and late delivery, and delivery time distribution parameters. The results will be provided for the following distributions: uniform, exponential, logistic, asymmetric Laplace, normal, and gamma. Our model and the supporting propositions derived herein contribute to the literature along the following four dimensions. First, the model overcomes the two aforementioned limitations that are inherent to Gaussian-based supply chain delivery models found in the literature. Second, the optimal positioning of the delivery window is used. Third, the model represents a generalized modeling approach to the evaluation of supply chain delivery performance. Fourth, the analytical analysis of the key attributes of the model that are presented in the set of supporting model propositions provide managerial insight into the dynamics of integrating the model into the long term continuous improvement of delivery performance within the supply chain.

This research contributes to the third research objective as follows. The interaction of delivery performance and inventory models and the effect of delivery performance on a buyer’s costs in a two-stage supply chain will be examined. The models define supplier’s and buyer’s points of view on supply chain delivery performance and illustrate how these different points of
view affect managerial decisions. Buyer’s point of view on supply chain delivery is represented by inventory models with two levels of storage.

Inventory models assume that a reorder point (a time when a buyer places an order) defines when the product will be shipped and, as a result, when it will arrive to the buyer. In reality a supplier ships the product and the time when the product will be shipped is the supplier’s decision. How the supplier’s delivery decision will affect buyer costs? Could the buyer influences on the supplier’s delivery decision? The answer can be found comparing delivery and inventory models as supplier’s and buyer’s points of view. However, the existing literature has a gap between two research streams (supply chain delivery performance and inventory management) and fails to provide a research which considers the problems together. The paper bridges existing gaps found in the literature and contributes to linking and coordinating the delivery and inventory sub processes within supply chains. This supports the third research objective addressed in this dissertation and is directly in agreement with the need for such integrated models (see for example, Jaber et al., 2010; Khouja, 2003; Fawcett & Magnan, 2001; Thomas & Griffin, 1996).

The paper explains how supply chain delivery performance and inventory models define a supplier’s and buyer’s points of view respectively and how different points of view affect managerial decisions.

In addition to the main contribution, the paper extends research in the area of stochastic models with two levels of inventory. The new model proposed herein differs from existing models by cost dimensions. The optimal solution for the model is found.
CHAPTER 2
LITERATURE REVIEW

The chapter is organized as follows. In section 2.1 the supply chain delivery window literature is reviewed. Section 2.1.1 describes examples of using the delivery windows in industry. The importance of delivery windows is analyzed in section 2.1.2. Section 2.1.3 reviews the models for evaluating delivery performance which use delivery window. Section 2.2 focuses on areas in the inventory management literature and describes how the buyer’s point of view on supply chain delivery performance can be defined.

2.1. Supply chain delivery window

2.1.1. Examples of delivery windows

Delivery windows are an effective tool for modeling the expected costs associated with untimely delivery and provide a means for financially quantifying delivery performance thereby serving the need identified in Gunasekaran et al. (2001) for delivery performance to be measured in financial terms. The first application of a delivery window model that financially penalizes a supplier for early and late delivery was introduced for a two stage (supplier-buyer) supply chain by Guiffrida et al. (1990). The attractiveness of delivery windows was further advanced by Corbett (1992) who demonstrated how a furniture manufacturer placed a higher value on delivery reliability (conformance to the on time delivery within a window) than on delivery speed. Examples of delivery windows reported in the literature are found in Table 1.

Discussion of factors as to why untimely delivery occurs is found in Peng et al. (2010), Hadavi (1996) and Hanfield and Pannes (1992).
<table>
<thead>
<tr>
<th>On Time Delivery Window</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 minutes early to 30 minutes late (food distribution)</td>
<td>Shokri et al. (2010)</td>
</tr>
<tr>
<td>Four day delivery window (Wal-Mart)</td>
<td>Anonymous (2010)</td>
</tr>
<tr>
<td>45 minute delivery window (automotive assembly)</td>
<td>Holecek (2002)</td>
</tr>
<tr>
<td>Four hour delivery window (automotive seats supplier)</td>
<td>Anonymous (1998)</td>
</tr>
<tr>
<td>Three days early to zero days late, (Hewlett-Packard)</td>
<td>Schneiderman (1996)</td>
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<tr>
<td>Two weeks early to zero days late (Analog Devices, Inc.)</td>
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<tr>
<td>Zero days late and no more than four days early (telecommunications firm)</td>
<td>Lockamy (1994)</td>
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<tr>
<td>Two hour delivery window (parts delivery in automotive industry)</td>
<td>Bowman (1992)</td>
</tr>
<tr>
<td>5.1 days early to 1.8 days early (survey of purchasing managers)</td>
<td>Raia (1990)</td>
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2.1.2. Empirical studies on the importance of on time delivery

Numerous empirical studies have documented the importance that on time delivery plays in the operation of the supply chain (see Table 2).

Anderson et al. (2011) presented empirical evidence on the importance of delivery performance in the selection third party logistics providers. Boon-itt and Wong (2011) investigated the effect of uncertainties on the relationships between supply chain integration and customer delivery performance. Golini and Kalchschmidt (2010) addressed the relationship between globalization of sales, investments in supply chains and delivery performance. da Silveira and Arkader (2007) showed that the timeliness of delivery is a key concern to customers. Lastly, several researchers have examined the relationship between delivery performance and supply chain operations.
<table>
<thead>
<tr>
<th>Study</th>
<th>Sample</th>
<th>Finding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prajogo and Olhager (2012)</td>
<td>232 responses from managers of Australian manufacturing firms</td>
<td>Long term relationship with suppliers has a positive relationship with performance</td>
</tr>
<tr>
<td>Anderson et al. (2011)</td>
<td>309 Asia Pacific customers of large multinational 3PL providers</td>
<td>Delivery performance plays an important role in the selection of third party logistics providers</td>
</tr>
<tr>
<td>Boon-itt and Wong (2011)</td>
<td>151 first tier suppliers in the Thai automotive industry</td>
<td>Technological and demand uncertainties moderate the relationships between supply chain integration and customer delivery performance</td>
</tr>
<tr>
<td>Golini and Kalchschmidt (2010)</td>
<td>485 companies from 19 countries in metal products, machinery and equipment assembly industries</td>
<td>A complex interrelationship exists between globalization of sales, supply chain investment and delivery performance</td>
</tr>
<tr>
<td>da Silveira and Arkader (2007)</td>
<td>243 manufacturers from 13 countries in metal, machinery, electrical and transportation products and equipment industries</td>
<td>Direct relationship identified between customer coordination investment and delivery speed and reliability</td>
</tr>
<tr>
<td>Tan et al. (2002)</td>
<td>1500 senior purchasing and materials managers of US firms</td>
<td>On time delivery reported as the most important (out of 24) supply chain performance practices</td>
</tr>
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</table>
Salvador et al. (2001) empirically demonstrate the effect of punctuality of delivery on the overall time-based performance in a study of 164 manufacturing plants from 5 different counties who were engaged in the electronic, machinery, and transportation equipment industrial sectors. Fawcett et al. (1997) studied the importance of delivery capability using a cross-national survey of 131 organizations. The analysis revealed that firms who are able to develop high levels of cross functional delivery performance consistently outperformed organizations that either failed to recognize the importance of delivery capability or have been unable to manage their resources to achieve delivery competence. Liukko et al. (1997) explored manufacturers’ choices of subcontractors in the Finnish metal industry. The authors concluded that delivery reliability influenced the shares of available business a manufacturer awarded to subcontractors.

Choi and Hartley (1996) in a study of the supplier-selection practices in the US auto industry found that quality and delivery formed a single construct that was viewed as the most important supplier selection criteria. Lockamy (1993) in a study of six manufacturing firms reported that all firms used on-time delivery not only as a metric in their distribution function but also as a performance measure used for value-evaluating delivery lead time management. Handfield and Pannesi (1992) defined a path analysis model to test the impact of production plan goals, inventory goals, master production schedule performance, forecasting accuracy and the number of end items on delivery reliability. All factors except forecasting accuracy had a significant effect on delivery reliability having explained approximately 55 percent of the variance in a sample 285 manufacturing firms.

2.1.3. Models for evaluating delivery performance
Modeling performance using delivery windows has spread to several related research streams such as: (i) supply chain agility (Swafford et al., 2006), (ii) mass customization (Squire et al., 2006) and (iii) retail strategy and location analysis (Shang et al., 2009). Delivery windows appearing in supply chain delivery models have their genesis in the time window constrained models for production scheduling and vehicle routing. An early survey of time window constrained scheduling and routing problems is found in Solomon and Desrosiers (1988). As evident by a sampling of the recent literature, time windows continue to be integral component in models for project scheduling (Cesta & Addi, 2002; Hurink, 2011), job shop scheduling (Brucker & Kravchenko, 2008; Huang & Yang, 2008), and vehicle routing (Dondo & Cerda, 2007; Benjamin & Beasley, 2010).

The problem of interest in the research herein is the delivery time window within a delivery model of untimely delivery in a supply chain. This specific class of time window models evolved from the general class of time window constrained models found in the operations research and operations management literature and representative examples of delivery performance models with time windows may be found in Garg et al. (2006), Guiffrida & Nagi (2006a) and Shin et al. (2009).

An early application of a delivery window model that financially penalizes a supplier for early and late delivery in a two stage (supplier-buyer) supply chain is found in Guiffrida et al. (1990). Early and late deliveries introduce waste in the form of excess cost into the supply chain; early deliveries contribute to excess inventory holding costs while late deliveries may contribute to production stoppage costs, lost sales and loss of goodwill. It is a common purchasing agreement practice to allow the buyer to charge the supplier for untimely deliveries (Schneiderman, 1996; Freehand, 1991).
These costs have been characterized by Guiffrida and Nagi (2006a) as “penalty costs” that are incurred in addition to the normal operating costs of the supply chain and are hence considered to be forms of waste. When a delivery is within the on-time portion of the delivery window, no penalty cost is incurred. Delivery windows are an effective tool for modeling the expected costs associated with untimely delivery and models containing them contribute to meeting the need identified in Gunasekaran et al. (2001) for delivery performance to be measured in financial terms.

The attractiveness of delivery windows was further advanced by Corbett (1992) who demonstrated how a furniture manufacturer placed a higher value on delivery reliability (conformance to the on-time delivery within a window) than on delivery speed. Guiffrida and Nagi (2006a) categorized the cost of early and late delivery in a supply chain as “penalty costs” that are incurred in addition to the normal operating costs of the supply chain and are hence considered to be forms of waste. When a delivery is within the on-time portion of the delivery window, no penalty cost is incurred. Boyer et al. (2009) identified delivery windows as a key factor in modeling the “last mile challenge” in supply chains.

Models for evaluating delivery performance within supply chains using delivery windows can be categorized into two groups: i) index based models, and ii) penalty cost based models. Table 3 provides an overview of this literature.

Both categories of models are similar in that delivery timeliness to the final customer is analyzed with regard to the customer’s specification of an on-time delivery window. The models differ in how they report delivery performance in terms of an overall metric. Index based models translate the probability of untimely (early and late delivery) into a “delivery capability index” measure which is similar to the family of process capability indexes that have been traditionally
used in statistical process control activities in manufacturing. Penalty cost based models translate the probability of untimely delivery into an expected cost measure.

Table 3. Classification of supply chain delivery performance models

<table>
<thead>
<tr>
<th>Article</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Index based models</strong></td>
<td></td>
</tr>
<tr>
<td>Wang and Du (2007)</td>
<td>Develop a capability index to model continuous improvement when delivery performance is measured subject to a delivery window</td>
</tr>
<tr>
<td>Lu (2007)</td>
<td>Constructs and demonstrates for a case study in the apparel industry a uniformly minimum variance unbiased estimator for a delivery performance capability index</td>
</tr>
<tr>
<td>Choudhary et al. (2006)</td>
<td>Present a non linear optimization model for minimizing the cost of untimely delivery when delivery performance is measured as a six-sigma based capability index measure</td>
</tr>
<tr>
<td>Garg et al. (2006)</td>
<td>Utilize a six-sigma based delivery capability index to optimally distribute the pool of individual stage activity variance in a multi-stage supply chain to satisfy customer delivery expectations</td>
</tr>
<tr>
<td><strong>Penalty cost based models</strong></td>
<td></td>
</tr>
<tr>
<td>Shin et al. (2009)</td>
<td>Develop an analytical model for comparing sourcing alternatives based on supplier quality and delivery performance</td>
</tr>
<tr>
<td>Guiffrida et al. (2008)</td>
<td>Develop an analytical model that financially quantifies the benefit of reducing delivery variance subject to an investment budget</td>
</tr>
<tr>
<td>Guiffrida and Nagi (2006a)</td>
<td>Develop a framework for financial modeling and evaluation of continuous improvement in delivery performance</td>
</tr>
<tr>
<td>Guiffrida and Nagi (2006b)</td>
<td>Quantify the concept of managerial neglect as the opportunity cost of management neglecting to improve delivery performance</td>
</tr>
<tr>
<td>Grout (1998)</td>
<td>Demonstrates delivery windows as a tool for reducing the delivery window</td>
</tr>
</tbody>
</table>
After evaluating delivery performance, the next natural step is to find ways to improve delivery performance. In this case, penalty cost based models have huge advantage over index and six-sigma based models, because improvement is easier to understand in cost values than in index values. Moreover, a decision can be easily made if a monetary advantage of the decision is known. The first attempt to analyze delivery performance improvement was done by Guiffrida and Nagi (2006a). The paper analyzed an effect of changing parameters of the model (delivery time distribution parameter, etc.) on the expected penalty cost and modeled improvement in delivery performance for normally distributed delivery time. The paper assumes the delivery time has Gaussian distribution and does not optimize the position of the delivery window. There are limitations as it was explained in chapter 1.

2.2. Inventory management models

Under the concept of the delivery window used in delivery performance models, the customer sets benchmarks in time which are used to classify deliveries as being early, on-time, or late. In general, when the delivery window is imposed on the underlying delivery time pdf, the delivery performance model becomes a stochastic model with three possible outcomes: early, on-time, and late delivery. In classical inventory models, there are two possible outcomes (stockouts and no stockouts) in each inventory cycle that can occur as a result of the demand during lead time. Late delivery contributes to stockouts; on-time delivery contributes to no stockouts. In practice, usually the storage capacity for inventory is limited and industry has a long history of working to address the effects of limited storage space (Kempf et al., 2011). When a limited storage space is considered three outcomes are now possible: stockouts, overloading, and no
stockouts and no overloading; where overloading means the level of inventory exceeds the storage capacity. Delivery time is a common element that affects the delivery outcomes (early, on-time, and late) and inventory outcomes (stockouts, overloading, and no stockouts and no overloading). Hence it is an attractive factor for further study.

The efficient management of inventory requires managers to make decisions on two fundamental questions: (1) how large should an inventory replenishment order be, and/or (2) when should an inventory replenishment order be placed. Answers to these questions are used to establish policies for inventory planning and control and are typically made with the objective of providing a desired level of customer service at a minimal cost. Beginning in 1913 with Harris’ economic order quantity model (EOQ), numerous mathematical inventory models have been published in the inventory literature to assist managers in determining the size and/or timing of an inventory replenishment order in support of this objective. Delivery performance models investigate penalty costs for untimely delivery for a supplier; hence the models can be used as a supplier’s point of view on supply chain delivery performance.

Two formulation approaches have been reported in the literature for inventory models with limited storage space. The first approach takes the form of a resource constrained optimization problem with storage capacity as the constraining resource (see for example Hausman et al., 1998; Zhao, 2007). A second modeling approach due to Hartley (1976) recognizes that outside storage space can be rented hence the optimization of the inventory decision variables is addressed from a modeling perspective that accommodates two levels of storages: owned warehouse (OW) and rented warehouse (RW). Hence, the models with capacity constraints are just a special case of the models with two levels of inventory with the costs for rented warehouse equal to infinity. Previous researchers mostly focused on deterministic models
with two storage facilities (for example, Sarma, 1983; Lee and Hsu, 2009). The first stochastic model with two warehouses was developed by Hariga (1998). Lately, he extended the model to a stochastic continuous review \((Q, R)\) model with two warehouses and backorder (Hariga, 2009). The lack of the stochastic models with two levels of inventory suggests that there is an opportunity for research in this area. In inventory models that allow shortage there are multiple ways in which the shortage cost is defined. Shortage cost coefficient dimension can be \([$/\text{unit/unit time}]\) (for example, Naddor, 1966), \([$/\text{unit}]\) (for example, Hadley and Whitin, 1963), or \([$/\text{unit time}]\). It is most commonly assumed that the shortage cost is linear, i.e. has dimension \([$/\text{unit/unit time}]\) (Rosling, 2002).

There is a well-known joint economic lot size model (JELS) (Banerjee, 1986) which optimizes inventory associated costs for a supplier and a buyer together. The classical JELS model and its extensions might look similar to the approach proposed herein. The same as JELS models, we optimize the costs within two stage supply chain with one supplier and one buyer, but there are several significant differences between the JELS models and the model herein. Firstly, the JELS models optimize the lot size (or order quantity); and the model herein assume that the lot size is defined by a buyer and optimizes the delivery process (reorder point and shipping time). Therefore, the model herein can be applied when a supplier has several buyers and/or a buyer has several suppliers and the joint lot size cannot be defined. Secondly, the JELS models assume that a supplier and a buyer would share all information needed for the models. The model herein does not require information sharing.
CHAPTER 3
CONCEPT OF OPTIMAL POSITION
OF SUPPLY CHAIN DELIVERY WINDOW

The chapter is organized as follows. In Section 3.1, the concept of the optimal position of the delivery window is described and the conditions for the optimal position of the delivery window are derived. The optimal position of the delivery window is derived for delivery distributions with closed form cdf (uniform, exponential, logistic, and asymmetric Laplace) in Section 3.2. In section 3.3, algorithm of finding optimal position of delivery window and expected penalty cost is developed for untimely delivery for several distributions with closed form cdf. In Section 3.4 we demonstrate the optimal positioning of the delivery window for an industrial case study.

3.1. Determining the optimal position of delivery window

Guiffrida and Nagi (2006a) defined the expected penalty cost per period for untimely delivery for a two-stage serial supply chain as:

\[
Y = QH \int_a^{c_1} (c_1 - x)f(x)dx + K \int_{c_1 + \Delta c}^b (x - (c_1 + \Delta c))f(x)dx,
\]

(3-1)

where

\[
QH \int_a^{c_1} (c_1 - x)f(x)dx = \text{expected penalty cost of early delivery},
\]

\[
K \int_{c_1 + \Delta c}^b (x - (c_1 + \Delta c))f(x)dx = \text{expected penalty cost of late delivery},
\]

\[Q = \text{constant delivery lot size},\]

\[H = \text{supplier inventory holding cost per unit per unit time},\]
\( K = \text{penalty cost per time unit late (levied by the buyer)}, \)
\[ x = \text{the delivery time for } Q, \]
\[ f(x) = \text{probability density function (pdf) of delivery time}, \]
\[ a = \text{earliest acceptable delivery time}, \]
\[ b = \text{latest acceptable delivery time}, \]
\[ c_1 = \text{beginning of on time delivery}, \]
\[ \Delta c = \text{the width of the delivery window}. \]

We make the following modeling assumptions: (i) there is a single product with a fixed delivery lot size \( Q \), (ii) the delivery time for \( Q \) consists of the internal manufacturing lead time(s) of the supplier plus the external lead time associated with transporting the delivery lot size from supplier to buyer, and (iii) a make-to-order replenishment policy is in effect.

Therefore, the expected penalty cost for untimely delivery is a function of the following parameters: the beginning of on time delivery \( (c_1) \), the width of the delivery window \( (\Delta c) \), penalty cost per time unit late \( (K) \), penalty cost per time unit early \( (QH) \), and parameters of \( f(x) \).

From the supplier’s point of view most of the parameters cannot be changed instantly. The width of the delivery window and penalty costs per time unit late and early are specified by the contract between the buyer and the supplier. The parameters cannot be changed without resigning the contract. Also, the form of the delivery time distribution cannot be changed in the short term. Thus, in short term these parameters are fixed.

The last parameter, the beginning of on time delivery (which with \( \Delta c \) defines the position of the delivery window) can be chosen by supplier. Of course, the real position of the delivery window which includes a certain date and time is fixed by contract, but a supplier can select the date and time when the product will be sent (the earliest delivery time). To decrease the earliest
delivery time by certain $\Delta t$ time units implies to start to deliver $\Delta t$ time units earlier. For the fixed real position of the delivery window decreasing the earliest delivery time implies increasing $c_1$ which is a difference between the earliest delivery time and the real position of the delivery window.

Suppose that the on time delivery window for a product is from 12 pm to 2 pm, thus the width of the delivery window is 2 hours (as it shown on Figure 2). Assume that the starting time of the shipment contemporizes with an earliest delivery time of $a = 0$. If the supplier starts to deliver at 11 am, the beginning of on time delivery is $c_1 = 12 - 11 = 1$ hour. But the time of beginning the delivery could be changed. For example, for the new time $a = 10$ am, the beginning of on time delivery is $c_1 = 12 - 10 = 2$ hour. Thus a supplier can choose the position of the delivery window with respect to the distribution curve.

![Figure 2. Positioning of delivery window.](image)

The position of the delivery window affects the expected penalty cost for untimely delivery. Hence for fixed $Ac$, $K$, $QH$, and $f(x)$, the optimal position of the delivery window which will minimize the expected penalty cost can be found. Thus, the optimal position of the delivery window which defined by the beginning of on-time delivery ($c_1^*$) is a function of the width of the
delivery window ($\Delta c$), penalty cost per time unit late ($K$), penalty cost per time unit early ($QH$), and parameters of the delivery time distribution ($f(x)$). To find an optimal position of the delivery window, which will minimize the expected penalty cost for untimely delivery, we should find a derivative of the cost function (equation 3-1) with respect to $c_1$. Firstly, we will find the derivatives for both the expected penalty cost of earliness and the expected penalty cost of lateness functions separately. In this case, we require $f(x)$ to be a continuously differentiable function, which is true for the most continuous pdfs.

For the expected cost of early delivery:

$$\frac{dY_{early}}{dc_1} = QH \frac{d}{dc_1} \int_a^c (c_1 - x) f(x) dx .$$

(3-2)

Because $f(x)$ and $a$ are not functions of $c_1$, $df(x)/dc_1 = 0$ and $da/dc_1 = 0$. Using this information and applying Leibniz rule yields

$$\frac{dY_{early}}{dc_1} = QH \int_a^c f(x) dx .$$

(3-3)

Since $P_{early} = \int_a^{c_1} f(x) dx$ is the probability of early delivery, (3-3) can be written as

$$\frac{dY_{early}}{dc_1} = QH \cdot P_{early} .$$

(3-4)

For the expected cost of late delivery:

$$\frac{dY_{late}}{dc_1} = K \frac{d}{dc_1} \int_{c_1+\Delta c}^b (x - (c_1 + \Delta c)) f(x) dx .$$

(3-5)

Observing that $\Delta c$ and $b$ are not functions of $c_1$ ($d(c_1+\Delta c)/dc_1 = 1$ and $db/dc_1 = 0$), and applying Leibniz rule yields
Recognizing that $P_{\text{late}} = \int_{c_1 + \Delta c}^{b} f(x)dx$ is the probability of late delivery, (3-6) simplifies to

$$\frac{dY_{\text{late}}}{dc_1} = -K \cdot P_{\text{late}}$$  \hspace{1cm} (3-7)$$

Equating $dY/dc_1=0$ gives

$$\frac{dY}{dc_1} = QH \cdot P_{\text{early}} - K \cdot P_{\text{late}} = 0.$$  \hspace{1cm} (3-8)$$

Thus $c_1$ will take a critical value when

$$\frac{P_{\text{late}}}{P_{\text{early}}} = \frac{QH}{K}.$$  \hspace{1cm} (3-9)$$

Examining the second derivative

$$\frac{d^2Y}{(dc_1)^2} = \frac{QH \cdot P_{\text{early}} - K \cdot P_{\text{late}}}{dc_1}$$  \hspace{1cm} (3-10)$$

and considering only

$$\frac{dP_{\text{early}}}{dc_1} = \frac{d}{dc_1} \left( \int_a^c f(x)dx \right) = f(c_1),$$  \hspace{1cm} (3-11)$$

and

$$\frac{dP_{\text{late}}}{dc_1} = \frac{d}{dc_1} \left( \int_{c_1 + \Delta c}^{b} f(x)dx \right) = -f(c_1 + \Delta c)$$  \hspace{1cm} (3-12)$$

we note that

$$\frac{d^2Y}{(dc_1)^2} = QH \cdot f(c_1) + K \cdot f(c_1 + \Delta c).$$  \hspace{1cm} (3-13)$$
Because \( f(x) \) cannot take on negative values, \( \frac{d^2Y}{(dc_1)^2} \geq 0 \) and \( Y \) is a convex function with respect to \( c_1 \). Hence (3-14) determines the optimal position of the delivery window which minimizes the total penalty cost for untimely delivery.

3.2. Optimal position of delivery window and expected penalty cost for untimely delivery for several distributions with closed form cdf

As shown in the previous section the optimal position of the delivery window is a function of the delivery time distribution. In general, we cannot evaluate the optimal position of delivery window from (3-1), but it is possible to find the optimal position of the delivery window as a function of \( \Delta c, K, QH, \) and \( f(x) \), if cumulative distribution function (cdf) \( F(x) \) of the delivery distribution has a closed form. In this section we present expressions for the optimal position of the delivery window for when the delivery distribution follows the uniform, exponential, logistic, and asymmetric Laplace pdf. The four distributions selected are representatives of delivery distributions which exhibit differing characterizations of skewness and kurtosis.

3.2.1. Uniform distribution

The uniform pdf is selected as the base case to illustrate the step-by-step derivation of the optimal position of the delivery window due to its commonly known and mathematically simple pdf. This base case will prove a useful introduction to the more advanced pdf’s which are forthcoming in this section.

Let the delivery time \( X \) be uniformly distributed with density function

\[
f(x) = \frac{1}{b-a} \text{ for } x = [a,b].
\] (3-14)
The cumulative distribution function of $X$ is

$$F(x) = \frac{x-a}{b-a} \text{ for } x = [a,b].$$

(3-15)

When delivery times are uniformly distributed the optimal position of the delivery window is (see Appendix A.1 for derivation)

$$c^*_1 = \frac{K \cdot (b - \Delta c) + QH \cdot a}{QH + K}.$$  

(3-16)

The expected penalty cost for untimely delivery is (see Appendix B.1)

$$Y = \frac{QH \cdot K}{2(b-a)(QH + K)}((b-a) - \Delta c)^2.$$  

(3-17)

3.2.2. Exponential distribution

The most reasonable distribution that can be used for the delivery time random variable should be non symmetric and has left finite (equal to 0) and right infinite bounds. The exponential distribution possesses these attributed as well as a closed form of cdf. These characteristics have lead to the use of the exponential as a popular density for modeling lead time demand (see for example He et al., 2005).

Let the exponential random variable $X$ with density functions $f(x)$ be defined over the supports $x = (0, \infty)$. The probability density function of $X$ is

$$f(x) = \frac{1}{\beta} \exp(-x/\beta).$$

(3-18)

The cumulative distribution function of $X$ is

$$F(x) = 1 - \exp(-x/\beta).$$

(3-19)

The optimal position of the delivery window for exponential random variable will be (Appendix A.2):
\[ c_1^* = \beta \cdot \ln \left[ \frac{K \cdot \exp(-\Delta c/\beta)}{QH} + 1 \right]. \] (3.20)

The expected penalty cost for untimely delivery is (see Appendix B.2)

\[ Y = QH \cdot \beta \cdot \ln \left[ \frac{K \cdot e^{-\Delta c/\beta}}{QH} + 1 \right]. \] (3-21)

### 3.2.3. Logistic distribution

As seen in Table 3, the Gaussian distribution is predominately used to model delivery performance. However, the lack of a closed form for the Gaussian cumulative distribution function greatly complicates the convexity argument for \( c_1^* \). As identified by van Beek (1978), the logistic distribution, whose CDF exists in closed form, is a reasonable surrogate to the Gaussian.

Let the logistic random variable \( X \) with density functions \( f(x) \) be defined over \( x = (-\infty, \infty) \). The probability density function of \( X \) is

\[ f(x) = \frac{\exp(-(x-\alpha)/\beta)}{\beta(1 + \exp(-(x-\alpha)/\beta))^2}. \] (3-22)

The cdf of \( X \) is

\[ F(x) = 1 - \frac{1}{1 + \exp((x-\alpha)/\beta)}. \] (3-23)

The optimal position of the delivery window for logistic random variable will be (Appendix A.3)

\[ c_1^* = \alpha + \beta \cdot \ln \left[ \frac{-\left(QH \cdot K^{-1} - 1\right) + \sqrt{\left(QH \cdot K^{-1} - 1\right)^2 + 4QH \cdot K^{-1} \exp(\Delta c/\beta)}}{2QH \cdot K^{-1} \exp(\Delta c/\beta)} \right]. \] (3-24)
The expected penalty cost for untimely delivery is (see Appendix B.3)

\[
Y = QH\beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\frac{\Delta c}{QH \cdot K^{-1} e^{\frac{\Delta c}{QH}}} + 1}}}{2QH \cdot K^{-1} e^{\frac{\Delta c}{QH}}} \right\} + \\
+ K\beta \ln \left\{ \frac{QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\frac{\Delta c}{QH}} + 1}}{2e^{\frac{\Delta c}{QH}}} \right\}.
\]

(3-25)

3.2.4. Asymmetric Laplace distribution

The asymmetric Laplace distribution is a distribution that can be used to model a wide range of delivery distributions which exhibit symmetry or skewness as well as a wide range of kurtosis values. Kozubowski and Podgorski (2000) define the probability density function and distribution function of the asymmetric Laplace (AL) density as

\[
f(x) = \frac{k}{\beta(1 + k^2)} \begin{cases} 
\exp\left[-\frac{k}{\beta}(x - \alpha)\right], & x \geq \alpha \\
\exp\left((\beta k)^{-1}(x - \alpha)\right), & x < \alpha
\end{cases}
\]

(3-26)

and

\[
F(x) = \begin{cases} 
1 - \frac{1}{1+k^2} \exp\left[-\frac{k}{\beta}(x - \alpha)\right], & x \geq \alpha \\
\frac{k^2}{1+k^2} \exp\left((\beta k)^{-1}(x - \alpha)\right), & x < \alpha
\end{cases}
\]

(3-27)

for \(-\infty < x < \infty\), location parameter \(\alpha \in (-\infty, \infty)\), shape parameter \(\beta > 0\) and skewness parameter \(k > 0\).

As seen in (3-27), the cdf is a piecewise continuous function which leads to an interesting piecewise continuous function for describing the optimal value of \(c_1\). We address the
optimal positioning of the delivery window for the asymmetric Laplace distribution in three cases that are parameter dependent (see Figure 3 for illustrations of densities for the three cases).

Figure 3a. Position of delivery window for asymmetric Laplace distribution (Case 1).

Figure 3b. Position of delivery window for asymmetric Laplace distribution (Case 2).

Figure 3c. Position of delivery window for asymmetric Laplace distribution (Case 3).

Case 1. \( c_1 < \alpha < c_1 + \Delta c \) or \( \alpha - \Delta c < c_1 < \alpha \):

The optimal position of the delivery window for Laplace random variable will be (Appendix A.4):
\[ c_i^* = \alpha - \Delta c - \frac{k^2}{k^2 + 1} + \frac{\beta k}{k^2 + 1} \ln \left( \frac{k^2 QH}{K} \right). \] (3-28)

**Case 2.** \( c_i < c_i + \Delta c < \alpha \) or \( c_i < \alpha - \Delta c < \alpha \):

In this case, the optimal position of the delivery window is (Appendix A.4):

\[ c_i^* = \alpha - \beta k \ln \left( \frac{k^2}{1 + k^2} \left[ \frac{QH}{K} + \exp \left( \frac{\Delta c}{\beta k} \right) \right] \right). \] (3-29)

**Case 3.** \( \alpha - \Delta c < \alpha < c_i < c_i + \Delta c \) or \( \alpha - \Delta c < \alpha < c_i \):

The optimal position of the delivery window is (Appendix A.4):

\[ c_i^* = \alpha - \Delta c - \frac{\beta}{k} \ln \left( (1 + k^2) \frac{QH}{K} \right) + \frac{\beta k}{k^2} \ln \left( 1 + \frac{QH}{K} \exp \left( \frac{k \Delta c}{\beta k} \right) \right). \] (3-30)

Combining all three cases we have a continuous curve. The examples of the curves are shown in the next section.

If \( k = 1 \) the distribution is symmetric and called Laplace distribution. The optimal position of delivery window and the expected penalty cost for untimely delivery for Laplace distribution (see Appendix B.4) are

**Case 1.** If \( c_i < \alpha < c_i + \Delta c \), the expected penalty cost is

\[ c_i^* = \alpha - \frac{\Delta c}{2} + \frac{\beta}{2} \ln \left( \frac{QH}{K} \right). \] (3-31)

\[ Y = \sqrt{QH \cdot K \cdot \beta} \cdot e^{-\frac{\Delta c}{2\beta}}. \] (3-32)

**Case 2.** If \( c_i < c_i + \Delta c < \alpha \), the expected penalty cost is

\[ c_i^* = \alpha - \beta \ln \left( \frac{QH}{2K} + \frac{1}{2} \exp \left( \frac{\Delta c}{\beta} \right) \right). \] (3-33)
\[ Y = K \cdot \beta \left[ 1 + \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) - \frac{\Delta c}{\beta} \right]. \]  \hspace{1cm} (3-34)

Case 3. If \( \alpha < c_1 < c_1 + \Delta c \), the expected penalty cost is

\[ c^*_i = \alpha - \beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right). \]  \hspace{1cm} (3-35)

\[ Y = QH \cdot \beta \left[ 1 - \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right) \right]. \]  \hspace{1cm} (3-36)

Combining all three cases we have a curve. For \( \alpha = 0, \beta = 1, \Delta c = 1 \) the optimal position of delivery window as function of \( QH/K \) is shown on figure 4.

![Figure 4a](image)

Figure 4a. The optimal position of delivery window as function of \( QH/K \)

For \( \alpha = 0, \beta = 1, QH/K = 0.8 \) the optimal position of delivery window as function of \( \Delta c \) is shown on figure 4b.
3.3. Optimal position of delivery window and expected penalty cost

for untimely delivery for several distributions without closed form cdf

There are many distributions that do not have closed form of cdf (for example, normal and gamma). For this type of delivery time distributions, the closed form of the optimal position of the delivery window \( (c_f^*) \) cannot be found. In this case, convex optimization algorithms can be applied to solve the problem.

Convex optimization is a relatively new, yet well-defined approach in mathematical optimization which includes reliable and efficient solution algorithms. Boyd and Vandenberghe (2004, p. 8) note that: “It is reasonable to expect that solving general convex optimization problems will become a technology within a few years.”

If a problem can be formulated as a convex optimization problem, it can be solved effectively. To be treated as a convex problem, the problem must meet the following three requirements:

- the objective function must be convex;
- the inequality constraint functions must be convex;
• the equality constraint functions must be affine.

A set $C$ is convex if the line segment between any two points in $C$ lies in $C$, i.e., if for any $x_1, x_2 \in C$ and any $\theta$ with $0 \leq \theta \leq 1$, we have $\theta x_1 + (1-\theta)x_2 \in C$. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom } f$ is a convex set and if for all $x, y \in \text{dom } f$, and $\theta$ with $0 \leq \theta \leq 1$, we have $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ (Boyd & Vandenberghe, 2004).

Geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$, which is the chord from $x$ to $y$, lies above the graph of $f$ (Figure 5).

![Figure 5. Graph of a convex function](image)

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant, i.e., if it has the form $f(x) = Ax + b$, where $A \in \mathbb{R}^{mxn}$ and $b \in \mathbb{R}^m$.

Thus, convex optimization takes the form

$$
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
$$

where the functions $f_0(x), f_1(x), \ldots, f_m(x)$ are convex.

Convex optimization algorithms are described in textbooks (Boyd & Vandenberghe, 2004, Horst et al. 1995), in papers (Strekalovsky 1993, Dur et al., 1998), and surveys (Benson 1995). There are several methods of solving convex optimization problems. The interior-point
algorithm (see for example Boyd & Vandenberghe, 2004) is one of the most widely used convex optimization solution techniques.

The problem of finding the optimal position of the delivery window has no constraints and the objective function is presented in (3-1). The expected penalty cost function is convex because the second derivative with respect to $c_1$ (3-13) is always positive. Thus convex optimization techniques can be applied to the problem.

3.4. Numerical example

In this section we present an application of optimally positioning the delivery window for the asymmetric Laplace expected cost model using data on supplier delivery performance collected from the disguised manufacturing firm ABC incorporated (Guiffrida, 2008). ABC Incorporated is a manufacturer of computer controlled stamping machinery and is located in the northeast region of the United States. Six months of data on vendor delivery performance was collected from company records.

The data were maintained in the “Purchased Parts Vendor Performance Module” of the firm’s manufacturing and inventory control database. The timeliness of vendor delivery performance was measured in terms of the deviation from the due date of delivery. Deliveries were considered on time for deviations from four days early to zero days late ($c_1 = -4, \Delta c = 4$). Any delivery with a deviation of one or more days beyond the due date was considered late; any delivery with a deviation of five or more days early was considered early. The company policy defining the timeliness of delivery as per the stated delivery window was communicated to all vendors in written form by the director of materials. During the six month period under study, ABC utilized a vendor base of 519 different vendors. Many of the vendors were given multiple
purchase orders during the time period with each purchase order often containing several items. The order quantity of specific items was predominately one unit. As a result of the lack of repeated ordering for specific items during the duration of the study, a vendor delivery time distribution was created by aggregating the delivery deviation data across all vendors and items.

Based on the collected data, maximum likelihood estimates for the parameters of the asymmetric Laplace expected cost model were (Guiffrida, 2008): \( \hat{k} = 0.85, \hat{p} = 9.18, \hat{\alpha} = 0.0 \). Guiffrida (2008) assumed \( \frac{QH}{K} = 0.05 \). To be more specific we are choosing \( QH = $10 \) and \( K = $200 \). Other values of \( K \) have been chosen to show the results for different ratios \( \frac{QH}{K} \): \( K = $50 \) (\( \frac{QH}{K} = 0.2 \)), and \( K = $10 \) (\( \frac{QH}{K} = 1 \)). For these parameters the expected penalty cost as a function of the position of the delivery window is shown on Figure 6.

![Figure 6. The expected penalty cost (Y) as function of the position of the delivery window (c1).](image)

Figure 6 shows that for any proportion of \( QH \) to \( K \) there is a unique position of the delivery window that minimizes the expected penalty cost. The larger the value of \( \frac{QH}{K} \) the
lower the optimal value of $c_1$. Suggesting that the higher the penalty cost (per unit per time) of early deliver relatively to the penalty cost of late delivery the later a supplier should deliver product. By delivering later a supplier decreases the probability of early delivery and increases the probability of late delivery. Hence the expected penalty cost for early delivery is reduced while the expected penalty cost for late delivery is increased. If the penalty cost (per unit per time) of early delivery is high (and the penalty cost of late delivery is low) the supplier should avoid early deliveries. If the penalty cost of early delivery is low (and the penalty cost of late delivery is high) the supplier should avoid late deliveries.

For these parameter inputs (including $c_1 = -4$), the expected penalty cost is $Y = $1,274. The initial position of the delivery window ($c_1 = -4$) implies a large probability of late delivery (0.58) which is much higher than the probability of early delivery (0.25). Knowing that penalty for the late delivery is 20 times higher than penalty for early delivery it is easy to conclude that the time of the beginning of delivery (and position of delivery window) was chosen wrong. In this case, for the supplier it is better to deliver earlier and pay a relatively small penalty than to deliver late with huge penalty.

Applying the optimal position of the delivery window approach we have $c_1^* = 23.24$. The position means that the probability of early delivery is 0.93 and the probability of late delivery is 0.05. The optimal position assumes that the firm should start to deliver much earlier to decrease the probability of late delivery. In this case, the expected penalty cost is $Y = $310. Thus the firm can decrease the expected penalty cost by more than 4 times by applying the optimal positioning concept.

The economical effect of applying the optimal positioning concept for values of the ratio $QH/K$ ranged from 0.05 to 1 is shown in Table 4.
Table 4. Expected penalty costs for different values of $QH/K$

<table>
<thead>
<tr>
<th>$QH$</th>
<th>$K$</th>
<th>$QH/K$</th>
<th>Expected penalty cost</th>
<th>$c_I = -4$</th>
<th>Optimal $c_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10$</td>
<td>$200$</td>
<td>0.05</td>
<td>$1,274$</td>
<td>$310$ ($c_I^* = 23.2$)</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$100$</td>
<td>0.1</td>
<td>$647$</td>
<td>$243$ ($c_I^* = 16.5$)</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$50$</td>
<td>0.2</td>
<td>$333$</td>
<td>$181$ ($c_I^* = 10.3$)</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$20$</td>
<td>0.5</td>
<td>$145$</td>
<td>$113$ ($c_I^* = 3.5$)</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$10$</td>
<td>1</td>
<td>$82$</td>
<td>$76$ ($c_I^* = -0.2$)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 shows that the supplier chose a wrong position of the delivery window that makes the supplier to pay higher penalties for untimely delivery. In long run the supplier will pay four times ($1,274/310$) more than if it would select the optimal position of the delivery window. The position of the delivery window chosen by supplier is close to the optimal position of the delivery window for $QH/K = 1$ (see the last row in Table 4).
CHAPTER 4
DELIVERY PERFORMANCE IMPROVEMENT

The chapter is organized as follows. In Section 4.1, the effect of parameters defined in a contract on the optimal position of the delivery window and the expected penalty cost is evaluated. The parameters include a width of the on-time portion of the delivery window and penalty costs for early and late deliveries. The following sections evaluate the effect of different delivery time distribution parameters on the optimal position of the delivery window and the expected penalty cost. It is done for the delivery distributions with closed form cdf:

- For uniform distribution in Section 4.2,
- For exponential distribution in Section 4.3,
- For logistic distribution in Section 4.4,
- For asymmetric Laplace distribution in Section 4.5;

and distributions without closed form cdf:

- For normal distribution in Section 4.6,
- For gamma distribution in Section 4.7.

Section 4.8 includes discussion of the results.

4.1. Effect of parameters defined in a contract on expected penalty cost

There are several key parameter of the expected penalty cost function that can be used to determine the potential for improvement in delivery performance: the width of the on time portion of the delivery window, the penalty cost for early delivery, the penalty cost for late delivery, and the parameters of delivery time distribution. The width of the on time portion of the delivery window, the penalty cost for early delivery are defined in a contract. Thus, they are
chosen during a negotiation process between a buyer and a supplier. The delivery time
distribution and its parameters can be estimated based on statistics of the past deliveries. These
parameters can be changed by a supplier, but it requires time and resources.

In this section we present propositions which can be used for examining how the optimal
value of $c_1$ which defines the optimal position of the delivery window and the expected penalty
cost behave as a function of the parameters defined in a contract.

The width of the on-time portion of the delivery window and the accompanying penalties
costs for early and late delivery are defined by a contract between a buyer and a supplier.
Understanding the robustness of $Y$ as a function of the width of the on-time portion of the
delivery window and the penalties costs for untimely delivery is of interest since changes to this
width impact the buyer and seller differently. Typically, a buyer would like to decrease the width
of the on-time portion of the delivery window because it will decrease delivery time uncertainty.
Alternatively, a supplier would like to increase the width of on-time portion of the delivery
window because it decreases the expected penalty cost for untimely delivery.

4.1.1. Width of the on-time portion of the delivery window

**Proposition 4-1.** For a fixed delivery time distribution parameter and penalty costs per
unit per time early and late, increasing the width of the on-time portion of the delivery window
will decrease the optimal position of the delivery window and the total expected penalty cost.

**Proof.** The optimal position of the delivery window is

$$P_{late} = \frac{QH}{K} P_{early} \text{ or } K \cdot P_{late} = QH \cdot P_{early}.$$  \hspace{1cm} (4-1)

Using the derivatives
\[
\frac{dP_{\text{late}}}{d\Delta c} = 0 - \left( \frac{dc_i^*}{d(\Delta c)} + 1 \right) f(c_i^* + \Delta c) + 0 = \left( \frac{dc_i^*}{d(\Delta c)} + 1 \right) f(c_i^* + \Delta c), \quad (4-2)
\]

\[
\frac{dP_{\text{early}}}{d\Delta c} = \frac{dc_i^*}{d(\Delta c)} f(c_i^*) - 0 + 0 = \frac{dc_i^*}{d(\Delta c)} f(c_i^*), \quad (4-3)
\]

we have

\[
-\left( \frac{dc_i^*}{d(\Delta c)} + 1 \right) f(c_i^* + \Delta c) = \frac{QH}{K} \frac{dc_i^*}{d(\Delta c)} f(c_i^*); \quad (4-4)
\]

\[
\frac{dc_i^*}{d(\Delta c)} \left( f(c_i^* + \Delta c) + \frac{QH}{K} f(c_i^*) \right) = -f(c_i^* + \Delta c); \quad (4-5)
\]

\[
\frac{dc_i^*}{d(\Delta c)} = -\frac{K f(c_i^* + \Delta c)}{K f(c_i^* + \Delta c) + QH f(c_i^*)} = -\left( 1 + \frac{K f(c_i^* + \Delta c)}{QH f(c_i^*)} \right). \quad (4-6)
\]

Because the derivative is always negative, increasing the width of the on-time portion of the delivery window will require delivering earlier to keep the position of the delivery window optimal.

Using the envelope theorem \( \frac{dy}{da} = \frac{\partial y}{\partial a} \{x = x^*(a)\} \) the expected penalty cost is

\[
\frac{\partial Y}{\partial(\Delta c)} = 0 + K \left[ 0 - 1 \cdot ((c_1 + \Delta c - (c_1 + \Delta c)) f(c_1 + \Delta c)) + \int_{c_1 + \Delta c}^{b} (0 - 1) f(x) dx \right]; \quad (4-7)
\]

\[
\frac{\partial Y}{\partial(\Delta c)} = -K \cdot P_{\text{late}}. \quad (4-8)
\]

Thus, increasing the width of the delivery window will decrease the expected penalty cost. ■
4.1.2. Penalty costs for early and late deliveries

**Proposition 4-2.** For the fixed delivery time distribution parameter, the width of the delivery window, and penalty cost for late delivery, increasing penalty costs per unit time early will decrease the optimal position of the delivery window and increase the total expected penalty cost.

**Proof.** Using the derivatives

\[
\frac{dP_{\text{late}}}{d(QH)} = 0 - \frac{dc_1^*}{d(QH)} f(c_1^* + \Delta c) + 0 = -\frac{dc_1^*}{d(QH)} f(c_1^* + \Delta c),
\]

\[
\frac{dP_{\text{early}}}{d(QH)} = \frac{dc_1^*}{d(QH)} f(c_1^*) - 0 + 0 = \frac{dc_1^*}{d(QH)} f(c_1^*),
\]

we have

\[
K \left( -\frac{dc_1^*}{d(QH)} \right) f(c_1^* + \Delta c) = P_{\text{early}} + QH \frac{dc_1^*}{d(QH)} f(c_1^*); \quad (4-11)
\]

\[
\frac{dc_1^*}{d(QH)} (K f(c_1^* + \Delta c) + QH f(c_1^*)) = -P_{\text{early}}; \quad (4-12)
\]

\[
\frac{dc_1^*}{d(QH)} = -\frac{P_{\text{early}}}{K f(c_1^* + \Delta c) + QH f(c_1^*)}. \quad (4-13)
\]

The derivative is always negative.

Using envelope theorem \( \frac{dy^*}{da} = \frac{\partial y}{\partial a} \{x = x^*(a)\} \) the expected penalty cost is

\[
\frac{\partial Y}{\partial (QH)} = Y_{\text{early}} + 0 = Y_{\text{early}}. \quad (4-14)
\]

The derivative is always positive. ■
**Proposition 4-3.** For the fixed delivery time distribution parameter, the width of the delivery window, and penalty cost for early delivery increasing penalty costs per unit per time late will increase the optimal position of the delivery window and the total expected penalty cost.

**Proof.** Using the derivatives

\[
\frac{dP_{\text{late}}}{dK} = 0 - \frac{dc_1^*}{dK} f(c_1^* + \Delta c) + 0 = - \frac{dc_1^*}{dK} f(c_1^* + \Delta c), \tag{4-15}
\]

\[
\frac{dP_{\text{early}}}{dK} = \frac{dc_1^*}{dK} f(c_1^*) - 0 + 0 = \frac{dc_1^*}{dK} f(c_1^*), \tag{4-16}
\]

we have

\[
P_{\text{late}} + K \left( - \frac{dc_1^*}{dK} \right) f(c_1^* + \Delta c) = QH \frac{dc_1^*}{dK} f(c_1^*); \tag{4-17}
\]

\[
\frac{dc_1^*}{dK} \left( K f(c_1^* + \Delta c) + QH f(c_1^*) \right) = P_{\text{late}}; \tag{4-18}
\]

\[
\frac{dc_1^*}{dK} = \frac{P_{\text{late}}}{K f(c_1^* + \Delta c) + QH f(c_1^*)}. \tag{4-19}
\]

The derivative is always negative.

Using the envelope theorem \( \frac{dy^*}{da} = \frac{\partial y}{\partial a} \{ x = x^* (a) \} \) the expected penalty cost is

\[
\frac{\partial Y}{\partial K} = 0 + Y_{\text{late}} = Y_{\text{late}}. \tag{4-20}
\]

The derivative is always positive. ■

It can be interesting to estimate the effect of penalties in other forms. From (3-9) we can see that the optimal value of \( c_1 \) is a function of \( QH \) and \( K \), but if the proportion of \( QH \) to \( K \) does not change the optimal value of \( c_1 \) does not change too. So we could use a coefficient which will combine both values of \( QH \) and \( K \).
We suppose to use the next coefficient

\[ K = K_0 \cdot K_1; \quad QH = (1 - K_0)K_1; \quad QH + K = K_0 \cdot K_1. \quad (4-21) \]

And we have

\[ \frac{QH}{K} = \frac{(1 - K_0)K_1}{K_0 \cdot K_1} = \frac{1 - K_0}{K_0} = 1 - \frac{1}{K_0}. \quad (4-22) \]

Expected penalty cost is

\[ Y = K_1 \left[(1 - K_0)Y_{\text{early}} + K_0 Y_{\text{late}}\right], \quad (4-23) \]

and the optimal position of the delivery window is

\[ K_0 \cdot \text{P}_{\text{late}} = (1 - K_0) \cdot \text{P}_{\text{early}}. \quad (4-24) \]

The coefficient \( K_1 \) is a common part of early and late penalties per unit per time.

Increasing (decreasing) \( K_1 \) means that both \( QH \) and \( K \) increasing (decreasing) but the proportion \( QH/K \) stays the same.

**Proposition 4-4.** For the fixed delivery time distribution parameter and the width of the delivery window increasing penalty costs per unit per time early and late with the same proportion \( QH/K \) will increase the total expected penalty cost.

**Proof.** The first derivative is

\[ \frac{dY}{dK_1} = (1 - K_0)Y_{\text{early}} + K_0 Y_{\text{late}}. \quad (4-25) \]

The derivative is always positive, because \( 0 < K_0 < 1 \). Thus increasing \( K_1 \) will increase \( Y \). ■

**Proposition 4-5.** For a fixed delivery time distribution parameter and the width of the delivery window

- increasing \( K_0 \) will increase the optimal position of delivery window;
- and the total expected penalty cost is concave by \( K_0 \) with maximum at \( Y_{\text{early}} = Y_{\text{late}} \).
Proof. Using the derivatives

\[
\frac{dP_{\text{late}}}{dK_0} = 0 - \frac{dc_i^*}{dK_0} f(c_i^* + \Delta c) + 0 = -\frac{dc_i^*}{dK_0} f(c_i^* + \Delta c), \quad (4-26)
\]

\[
\frac{dP_{\text{early}}}{dK_0} = \frac{dc_i^*}{dK_0} f(c_i^*) - 0 + 0 = \frac{dc_i^*}{dK_0} f(c_i^*), \quad (4-27)
\]

we have

\[
P_{\text{late}} + K_0 \left( - \frac{dc_i^*}{dK_0} f(c_i^* + \Delta c) \right) = -P_{\text{early}} + (1 - K_0) \frac{dc_i^*}{dK_0} f(c_i^*); \quad (4-28)
\]

\[
\frac{dc_i^*}{dK_0} \left( K_0 f(c_i^* + \Delta c) + (1 - K_0) f(c_i^*) \right) = P_{\text{late}} + P_{\text{early}}; \quad (4-29)
\]

\[
\frac{dc_i^*}{dK_0} = \frac{P_{\text{late}} + P_{\text{early}}}{K_0 f(c_i^* + \Delta c) + (1 - K_0) f(c_i^*)}. \quad (4-30)
\]

The derivative is always positive.

The first derivative of the total cost is

\[
\frac{dY}{dK_0} = K_1 (Y_{\text{late}} - Y_{\text{early}}). \quad (4-31)
\]

Using that

\[
\frac{dY_{\text{late}}}{dK_0} = \frac{dc_i^*}{dK_0} (c_i^* - c_i^*) f(c_i^*) - 0 + \int_{c_i^* - \Delta c}^{c_i^*} \frac{dc_i^*}{dK_0} f(x) dx = \frac{dc_i^*}{dK_0} P_{\text{early}}, \quad (4-32)
\]

\[
\frac{dY_{\text{late}}}{dK_0} = 0 - \frac{dc_i^*}{dK_0} (c_i^* + \Delta c - (c_i^* + \Delta c)) f(c_i^* + \Delta c) + \int_{c_i^* + \Delta c}^{c_i^*} \frac{dc_i^*}{dK_0} f(x) dx = -\frac{dc_i^*}{dK_0} P_{\text{late}}; \quad (4-33)
\]

the second derivative is

\[
\frac{d^2Y}{d(K_0)^2} = -K_1 \frac{dc_i^*}{dK_0} \left( P_{\text{early}} + P_{\text{late}} \right). \quad (4-32)
\]

The second derivative is always negative and the cost function is concave by $K_0$. ■
4.2. Effect of uniform delivery time distribution parameters on expected penalty cost

This and the following sections investigate the effect of the delivery time distribution parameters on the expected penalty cost when the shape parameter and the width of the on-time portion of the delivery window are fixed.

**Proposition 4-6.** Increasing parameter $a$ of uniform distribution will increase the optimal position of the delivery window and reduce the expected penalty cost.

**Proof.** The derivatives are

$$\frac{dc^*_1}{da} = \frac{QH}{QH + K} > 0 \quad (4-33)$$

$$dY = \frac{QH \cdot K}{2(QH + K)} \left[ -2((b - a) - \Delta c)(b - a) - ((b - a) - \Delta c)^2 (-1) \right] \quad (4-34)$$

For the realistic case $\Delta c < (b - a)$, the derivative is negative. ■

**Proposition 4-7.** Increasing parameter $b$ of uniform distribution will increase the optimal position of the delivery window and increase the expected penalty cost.

**Proof.** The derivatives are

$$\frac{dc^*_1}{db} = \frac{K}{QH + K} > 0 \quad ; \quad (4-37)$$

$$dY = \frac{QH \cdot K}{2(QH + K)} \left[ \frac{\Delta c^2}{(b - a)^2} - 1 \right] \quad (4-36)$$

$$dY = \frac{QH \cdot K}{2(QH + K)} \left[ \frac{2((b - a) - \Delta c)(b - a) - ((b - a) - \Delta c)^2}{(b - a)^2} \right] \quad , \quad (4-38)$$
\[
\frac{dY}{db} = \frac{QH \cdot K}{2(QH + K)} \frac{2(b - a)^2 - 2\Delta c(b - a) - (b - a)^2 + 2\Delta c(b - a) - \Delta c^2}{(b - a)^2} ,
\] (4.39)

\[
\frac{dY}{db} = \frac{QH \cdot K}{2(QH + K)} \left(1 - \frac{\Delta c^2}{(b - a)^2}\right) .
\] (4.40)

For the realistic case \(\Delta c < (b - a)\), the derivative is positive. ■

**Proposition 4-8.** For uniform distributed delivery time changing mean will have no effect on the expected penalty cost.

**Proof.** The mean of uniform distribution is

\[
\mu = \frac{a + b}{2} .
\] (4.41)

The derivatives are

\[
1 = \frac{1}{2} \frac{da}{d\mu} , \text{ and } \frac{da}{d\mu} = 2 ;
\] (4.42)

\[
1 = \frac{1}{2} \frac{db}{d\mu} , \text{ and } \frac{db}{d\mu} = 2 .
\] (4.43)

Using that \(Y'_\mu = Y'_a a'_\mu + Y'_b b'_\mu\) we have

\[
\frac{dY}{d\mu} = 2 \left( \frac{QH \cdot K}{(b - a)^2} - 1 \right) + 2 \left( \frac{QH \cdot K}{2(QH + K)} \left(1 - \frac{\Delta c^2}{(b - a)^2}\right) \right) = 0 . ■
\] (4.44)

**Proposition 4-9.** Decreasing the variance uniform distributed delivery time will decrease the expected penalty cost.

**Proof.** The variance of uniform distribution is

\[
v = \frac{(b - a)^2}{12} .
\] (4.45)

The derivatives are

\[
1 = 2(b - a) \frac{da}{dv} , \text{ and } \frac{da}{dv} = \frac{6}{b - a} ;
\] (4.46)

46
1 = \frac{2(b-a)}{12} \left(-\frac{db}{dv}\right), \text{ and } \frac{da}{dv} = -\frac{6}{b-a}. \quad (4-47)

Using that \( Y_v' = Y_v'a_v' + Y_v'b_v' \) we have

\[
\frac{dY}{dv} = \frac{6}{b-a} \frac{QH \cdot K}{2(QH + K)} \left(\frac{\Delta c^2}{(b-a)^2} - 1\right) - \frac{6}{b-a} \frac{QH \cdot K}{2(QH + K)} \left(1 - \frac{\Delta c^2}{(b-a)^2}\right), \quad (4-48)
\]

\[
\frac{dY}{dv} = \frac{6}{b-a} \frac{QH \cdot K}{(QH + K)} \left(\frac{\Delta c^2}{(b-a)^2} - 1\right). \quad (4-49)
\]

The derivative is positive. ■

4.3. Effect of exponential delivery time distribution parameters

on expected penalty cost

Proposition 4-10. Increasing parameter \( \beta \) of exponential distribution will increase the optimal position of the delivery window and increase the expected penalty cost.

Proof.

\[
\frac{dc^v_i}{d\beta} = \ln \left(\frac{K}{QH} \exp\left(-\Delta c/\beta\right) + 1\right) + \beta \frac{QH}{K \exp(-\Delta c/\beta) + QH} \frac{K}{QH} \exp(-\Delta c/\beta) \frac{\Delta c}{\beta^2}. \quad (4-50)
\]

\[
\frac{dc^v_i}{d\beta} = \ln \left(\frac{K}{QH} \exp\left(-\Delta c/\beta\right) + 1\right) + \frac{\Delta c}{\beta} \frac{K}{QH \exp(\Delta c/\beta) + K}. \quad (4-51)
\]

The derivative is positive.

\[
\frac{dY}{d\beta} = QH \ln \left(\frac{K}{QH} \exp\left(-\Delta c/\beta\right) + 1\right) + \frac{\Delta c}{\beta} \frac{QH \cdot K}{QH \exp(\Delta c/\beta) + K}. \quad (4-52)
\]

The derivative is positive. ■
Proposition 4-11. Increasing mean or variance of exponential distribution will increase the expected penalty cost.

Proof. The mean of uniform distribution is

\[ \mu = \beta. \] (4-53)

\[ \frac{dY}{d\mu} = \frac{dY}{d\beta} = QH \ln\left( \frac{K}{QH} \exp\left(-\frac{\Delta c}{\beta}\right) + 1 \right) + \Delta c \frac{QH \cdot K}{\beta \cdot QH \exp\left(\frac{\Delta c}{\beta}\right) + K}. \] (4-54)

The derivative is positive.

The variance of uniform distribution is

\[ v = \beta^2. \] (4-55)

The derivatives is

\[ 1 = 2 \frac{d\beta}{dv}, \quad \text{and} \quad \frac{d\beta}{dv} = \frac{1}{2}. \] (4-56)

Using that \( Y'_v = Y'_\beta \beta'_v \) we have

\[ \frac{dY}{dv} = \frac{QH}{2} \ln\left( \frac{K}{QH} \exp\left(-\frac{\Delta c}{\beta}\right) + 1 \right) + \Delta c \frac{QH \cdot K}{2\beta \cdot QH \exp\left(\frac{\Delta c}{\beta}\right) + K}. \] (4-57)

The derivative is positive. ■

4.4. Effect of logistic delivery time distribution parameters on expected penalty cost

Proposition 4-12. Increasing parameter \( \alpha \) (which is the mean) of logistic distribution will increase the optimal position of the delivery window and have no effect on the expected penalty cost.

Proof. \[ \frac{dc}{d\alpha} = 1; \] (4-58)
\[
\frac{dY}{d\alpha} = 0. \quad \blacksquare
\]  

**Proposition 4-13.** Increasing parameter $\beta$ (and as a result variance) of logistic distribution will increase the optimal position of the delivery window and the expected penalty cost.

**Proof.** The following notation will be used $Z = QH \cdot K^{-1}$:

\[
\frac{dc_i}{d\beta} = \ln \left[ \frac{-(Z-1) + \sqrt{(Z-1)^2 + 4Z \exp(\Delta c/\beta)}}{2Z \exp(\Delta c/\beta)} \right] + \frac{\Delta c}{\beta^2} \left[ 1 - \frac{2Z \cdot e^{\Delta c/\beta}}{4Z \cdot e^{\Delta c/\beta} + (Z-1)(Z-1) - \sqrt{(Z-1)^2 + 4Ze^{\Delta c/\beta}}} \right].
\]  

(4-60)

It is easy to show that the derivative is always positive.

\[
\frac{dY}{d\beta} = Y + QH \left( \frac{\Delta c}{\beta} \right) \left( \frac{-2Ze^{\Delta c/\beta}(Z-1)^2 + 4Ze^{\Delta c/\beta}}{-Z + 1 + \sqrt{(Z-1)^2 + 4Ze^{\Delta c/\beta} + 2Ze^{\Delta c/\beta}}} \right) + \left( \frac{-2Ze^{\Delta c/\beta}(Z-1)^2 + 4Ze^{\Delta c/\beta}}{-Z + 1 + \sqrt{(Z-1)^2 + 4Ze^{\Delta c/\beta} + 2Ze^{\Delta c/\beta}}} \right)
\]  

(4-61)

\[
+ K \left( \frac{\Delta c}{\beta} \right) \left( \frac{-2Ze^{\Delta c/\beta}(Z-1)^2 + 4Ze^{\Delta c/\beta} + Z - 1 + \sqrt{(Z-1)^2 + 4Ze^{\Delta c/\beta} + 2Ze^{\Delta c/\beta}}}{Z - 1 + \sqrt{(Z-1)^2 + 4Ze^{\Delta c/\beta} + 2Ze^{\Delta c/\beta}}} \right);
\]
\[
\frac{dY}{d\beta} = \frac{Y}{\beta} + QH \frac{\Delta c}{\beta} \left\{ 1 - \frac{2Ze^{\frac{\Delta c}{\beta}} + 2Ze^{\frac{\Delta c}{\beta}} (Z - 1)^2 + 4Ze^{\frac{\Delta c}{\beta}}}{(Z - 1)^2 + 4Ze^{\frac{\Delta c}{\beta}}} + \right. \\
\left. + K \frac{\Delta c}{\beta} \left( 1 - \frac{2Ze^{\frac{\Delta c}{\beta}} + Ze^{\frac{\Delta c}{\beta}} (Z - 1)^2 + 4Ze^{\frac{\Delta c}{\beta}}}{(Z - 1)^2 + 4Ze^{\frac{\Delta c}{\beta}}} \right) \right\} \quad (4-62)
\]

It is easy to show that the derivative is always positive. ■

### 4.5. Effect of Laplace delivery time distribution parameters

**Proposition 4-14.** Increasing location parameter \( \alpha \) (and as a result mean) of Laplace distribution will increase the optimal position of the delivery window and has no effect on the expected penalty cost.

**Proof.** For all three cases \( \alpha \):

\[
\frac{dc_i^*}{d\alpha} = 1. \quad (4-63)
\]

For all three cases \( \alpha \) is not includes in the expected penalty cost, therefore

\[
\frac{dY}{d\alpha} = 0. \quad (4-64)
\]

**Proposition 4-15.** Increasing shape parameter \( \beta \) (and as a result variance) of Laplace distribution will increase the optimal position of the delivery window if \( QH > K \) and always increase the expected penalty cost.
Proof. For the case 1:

\[
\frac{dc_i^*}{d\beta} = 0 - 0 + \frac{1}{2} \ln \left( \frac{QH}{K} \right) = \frac{1}{2} \ln \left( \frac{QH}{K} \right). 
\]  
(4-65)

For the case 1, \( \alpha - \Delta c < c_i^* < \alpha \), thus \( -\frac{\Delta c}{\beta} < \ln \left( \frac{QH}{K} \right) < \frac{\Delta c}{\beta} \). Therefore, \( \frac{dc_i^*}{d\beta} \) both positive and negative.

\[
\frac{dY}{d\beta} = \sqrt{QH \cdot K} \cdot e^{\Delta c / 2\beta} + \sqrt{QH \cdot K} \cdot \beta \cdot e^{\Delta c / 2\beta} \left( \frac{\Delta c}{2\beta^2} \right) = 
\]  
(4-66)

For the case 2:

\[
\frac{dc_i^*}{d\beta} = 0 - \ln \left( \frac{QH}{2K} + \frac{1}{2} \exp \left( \frac{\Delta c}{\beta} \right) \right) - \beta \left( \frac{QH}{2K} + \frac{1}{2} \exp \left( \frac{\Delta c}{\beta} \right) \right)^{-1} \left( \frac{1}{2} - \frac{\Delta c}{\beta^2} \right) \exp \left( \frac{\Delta c}{\beta} \right) = 
\]  
(4-67)

For the case 2, \( c_i^* < \alpha - \Delta c \), thus \( \ln \left( \frac{QH}{2K} + \frac{1}{2} \exp \left( \frac{\Delta c}{\beta} \right) \right) > \frac{\Delta c}{\beta} \). Therefore, \( \frac{dc_i^*}{d\beta} < 0 \).

\[
\frac{dY}{d\beta} = K \left[ 1 + \ln \left( \frac{1}{2} e^{\Delta c / \beta} + \frac{QH}{2K} \right) - \frac{\Delta c}{\beta} \right] + K \beta \left[ \frac{1}{2} e^{\Delta c / \beta} + \frac{QH}{2K} \right]^{-1} \frac{1}{2} e^{\Delta c / \beta} \left( \frac{\Delta c}{\beta^2} \right) + \frac{\Delta c}{\beta} = 
\]  
(4-68)
For the case 3:

\[
\frac{dc_i^*}{d\beta} = -\ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right) - \beta \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right)^{-1} 2 \left( 1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}} \right)^{-2} QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}} \left( -\frac{\Delta c}{\beta^2} \right),
\]

(4-69)

\[
\frac{dc_i^*}{d\beta} = -\ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right) + \frac{\Delta c/\beta}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}}. \tag{4-70}
\]

For the case 3, \( c_i^* > \alpha \), thus \( \ln \left( 2 - 2 \left( 1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}} \right) \right) < 0 \). Therefore, \( \frac{dc_i^*}{d\beta} > 0 \).

\[
\frac{dY}{d\beta} = QH \left[ 1 - \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right) \right] + QH \cdot \beta \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right)^{-1} 2 \left( 1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}} \right)^{-2} QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}} \left( -\frac{\Delta c}{\beta^2} \right);
\]

\[
\frac{dY}{d\beta} = QH \left[ 1 - \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} \right) \right] + QH \frac{\Delta c/\beta}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{\beta}}} > 0. \tag{4-71}
\]

4.6. Effect of normal delivery time distribution parameters on expected penalty cost

For normal distribution, the total expected penalty cost per period for normally distributed delivery time is (Guiffrida and Nagi, 2006)
\[ Y = QH \left[ \sigma \phi \left( \frac{c^*_i - \mu}{\sigma} \right) + (c^*_i - \mu) \left( \Phi \left( \frac{c^*_i - \mu}{\sigma} \right) \right) \right] + \\
+ K \left[ \sigma \phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) - (c^*_i + \Delta c - \mu) \left( 1 - \Phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) \right) \right], \]  

(4-72)

where \( \phi(X) = \frac{1}{\sqrt{2\pi}} \exp(X) \), \( \Phi(X) = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} \exp(x)dx \).

The optimal position of delivery window is when

\[ QH \cdot \Phi \left( \frac{c^*_i - \mu}{\sigma} \right) = K \cdot \left[ 1 - \Phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) \right]. \]  

(4-73)

**Proposition 4-16.** Increasing the mean of normal distribution will increase the optimal position of the delivery window and has no effect on the expected penalty cost.

**Proof.** Using the derivative

\[ \frac{\Phi(X)}{dX} = \frac{1}{\sqrt{2\pi}} \exp(X) - 0 + \int_{-\infty}^{X} dx \phi(X). \]  

(4-74)

the derivative for the optimal position of the delivery window is

\[ \frac{QH}{\sigma} \left( \frac{dc^*_i}{d\mu} - 1 \right) \phi \left( \frac{c^*_i - \mu}{\sigma} \right) = K \left[ - \left( \frac{dc^*_i}{d\mu} - 1 \right) \phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) \right]; \]  

(4-75)

\[ \frac{dc^*_i}{d\mu} \left[ QH \phi \left( \frac{c^*_i - \mu}{\sigma} \right) + K \phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) \right] = QH \phi \left( \frac{c^*_i - \mu}{\sigma} \right) + K \phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right); \]  

(4-76)

\[ \frac{dc^*_i}{d\mu} = 1. \]  

(4-77)

From (Guiffrida and Nagi, 2006)

\[ \frac{dY}{d\mu} = QH \left[ - \Phi \left( \frac{c^*_i - \mu}{\sigma} \right) \right] + K \left[ 1 - \Phi \left( \frac{c^*_i + \Delta c - \mu}{\sigma} \right) \right], \]  

(4-78)
Knowing that the optimal position of the delivery window is used, the equation (4-73) is applied and \( dY/d\mu = 0 \). Thus the means of the delivery time has no effect on the expected penalty cost function. ■

**Proposition 4-17.** Increasing the standard deviation (and as a result variance) of normal distribution will increase the optimal position of the delivery window if

\[
c_i^* > \mu + \Delta c \left[ QH \cdot \phi \left( \frac{c_i^*-\mu}{\sigma} \right) + K \cdot \phi \left( \frac{c_i^*+\Delta c - \mu}{\sigma} \right) \right]^{-1}. \tag{4-79}
\]

**Proof.** Using the derivative (4-74), the derivative for the optimal position of the delivery window is

\[
\frac{QH}{\sigma^2} \left( \sigma \frac{dc_i^*}{d\sigma} - (c_i^* - \mu) \phi \left( \frac{c_i^* - \mu}{\sigma} \right) \right) = \frac{K}{\sigma^2} \cdot \left[ - \left( \sigma \frac{dc_i^*}{d\sigma} - (c_i^* + \Delta c - \mu) \phi \left( \frac{c_i^* + \Delta c - \mu}{\sigma} \right) \right) \right]; \tag{4-80}
\]

\[
\frac{dc_i^*}{d\sigma} \left[ QH \sigma \cdot \phi \left( \frac{c_i^* - \mu}{\sigma} \right) + K \sigma \cdot \phi \left( \frac{c_i^* + \Delta c - \mu}{\sigma} \right) \right] = QH(c_i^* - \mu) \phi \left( \frac{c_i^* - \mu}{\sigma} \right) + K(c_i^* + \Delta c - \mu) \phi \left( \frac{c_i^* + \Delta c - \mu}{\sigma} \right); \tag{4-81}
\]

\[
\frac{dc_i^*}{d\sigma} = \frac{c_i^* - \mu}{\sigma} + \frac{\Delta c}{\sigma} \left[ QH \cdot \phi \left( \frac{c_i^* - \mu}{\sigma} \right) + K \cdot \phi \left( \frac{c_i^* + \Delta c - \mu}{\sigma} \right) \right]^{-1}. \tag{4-82}
\]

The proof that \( dY/d\sigma > 0 \) is given in (Guiffrida and Nagi, 2006).

4.7. Effect of gamma delivery time distribution parameters

**on expected penalty cost**

The probability density function for gamma distributed delivery time \( x \) is
where \( \Gamma(k) = \int_{0}^{\infty} e^{-t} t^{k-1} dt \), scale parameter \( \alpha > 0 \), and shape parameter \( k > 0 \).
\[ \frac{d\gamma(z, y)}{dy} = 1 \cdot e^{-y} y^{z-1} - 0 + \int_0^y \alpha dt = e^{-y} y^{z-1}. \] (4-88)

we have:

\[ \frac{dP_{early}}{d\alpha} = \frac{e^{-\alpha \gamma c} (\alpha c)^{k-1}}{\Gamma(k)} \left( c_i^* + \alpha \frac{dc_i^*}{d\alpha} \right) = f(c_i^*) \left( c_i^* + \alpha \frac{dc_i^*}{d\alpha} \right); \] (4-89)

\[ \frac{dP_{late}}{d\alpha} = -\frac{e^{-\alpha (c_i^* + \Delta c)}}{\Gamma(k)} \left( c_i^* + \Delta c + \alpha \frac{dc_i^*}{d\alpha} \right) = -f(c_i^* + \Delta c) \left( \frac{c_i^* + \Delta c}{\alpha} + \frac{dc_i^*}{d\alpha} \right) \] (4-90)

Thus

\[ -K \cdot f(c_i^* + \Delta c) \left( \frac{c_i^* + \Delta c}{\alpha} + \frac{dc_i^*}{d\alpha} \right) = QH \cdot f(c_i^*) \left( \frac{c_i^*}{\alpha} + \frac{dc_i^*}{d\alpha} \right) \] (4-91)

\[ -K \cdot f(c_i^* + \Delta c) \cdot \frac{c_i^* + \Delta c}{\alpha} - QH \cdot f(c_i^*) \cdot \frac{c_i^*}{\alpha} = \frac{dc_i^*}{d\alpha} \left( QH \cdot f(c_i^*) + K \cdot f(c_i^* + \Delta c) \right) \]

\[ \frac{dc_i^*}{d\alpha} = -\frac{QH \cdot c_i^* f(c_i^*) + K \cdot (c_i^* + \Delta c) f(c_i^* + \Delta c)}{\alpha (QH \cdot f(c_i^*) + K \cdot f(c_i^* + \Delta c))}. \] (4-92)

Since \( dc_i^*/d\alpha < 0 \), increasing \( \alpha \) will decrease \( c_i^* \).  ■

**Proposition 4-19.** For a fixed width of the on-time portion of the delivery window, constant shape parameter of gamma distributed delivery time and an optimally positioned delivery window, increasing the scale parameter will decrease the total expected penalty cost.

**Proof.** The first derivatives of the earliness and lateness components of the expected penalty cost with respect to the scale parameter are, respectively

\[ \frac{dY_{early}}{d\alpha} = QH \frac{(dc_i^*/d\alpha) \gamma(k, \alpha c_i^*) + c_i^* e^{-\alpha c_i^*} (\alpha c_i^*)^{k-1} (c_i^* + \alpha dc_i^*/d\alpha)}{\Gamma(k)} - \] (4-93)

\[ -QH \frac{e^{-\alpha c_i^*} (\alpha c_i^*)^{k} (c_i^* + \alpha dc_i^*/d\alpha) \alpha \Gamma(k) - \Gamma(k) \gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma^2(k)} \]
\[
\frac{dY_{\text{early}}}{d\alpha} = QH \left( \frac{dc_i^*}{d\alpha} \right) \gamma(k, \alpha c_i^*) + QH \frac{c_i^* e^{-\alpha c_i^*}}{\Gamma(k)} (\alpha c_i^*)^{k-1} \left( c_i^* + \alpha dc_i^*/d\alpha \right) - \\
- QH \frac{c_i^* e^{-\alpha c_i^*}}{\Gamma(k)} (\alpha c_i^*)^{k-1} \left( c_i^* + \alpha dc_i^*/d\alpha \right) + QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma(k)}
\]
\[
\frac{dY_{\text{early}}}{d\alpha} = QH \frac{dc_i^*}{d\alpha} P_{\text{early}} + QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma(k)}
\]
\[
\frac{dY_{\text{late}}}{d\alpha} = K \frac{-e^{-\alpha c_i^* + \Delta c}}{\alpha^2 \Gamma^2(k)} (\alpha c_i^* + \Delta c + \alpha dc_i^*/d\alpha) \Gamma(k) - \\
- K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha^2 \Gamma^2(k)} - K \frac{dc_i^*/d\alpha}{\Gamma(k)} \Gamma(k) - \\
- K \frac{-(c_i^* + \Delta c) \left[ -e^{-\alpha c_i^* + \Delta c} (\alpha c_i^* + \Delta c)^{k-1} (c_i^* + \Delta c + \alpha dc_i^*/d\alpha) \right]}{\Gamma(k)}
\]
\[
\frac{dY_{\text{late}}}{d\alpha} = -K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha^2 \Gamma(k)} - K \frac{dc_i^*}{d\alpha} P_{\text{late}}
\]

Combining (4-95) and (4-97) yields
\[
\frac{dY}{d\alpha} = QH \frac{dc_i^*}{d\alpha} P_{\text{early}} + QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma(k)} - K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha^2 \Gamma(k)} - K \frac{dc_i^*}{d\alpha} P_{\text{late}}
\]
\[
\frac{dY}{d\alpha} = \frac{dc_i^*}{d\alpha} (QH \cdot P_{\text{early}} - K \cdot P_{\text{late}}) + QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma(k)} - K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha^2 \Gamma(k)}
\]

Using (3-9) we have
\[
\frac{dY}{d\alpha} = QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha^2 \Gamma(k)} - K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha^2 \Gamma(k)}
\]

The derivative is negative if the first element of (4-100) is smaller than the second one.

Dividing the first element by the right part of equation (3-9) and the second element by the left part of (3-9) and multiplying the both elements by \( \alpha \) yields the comparison
\[
QH \frac{\gamma(k + 1, \alpha c_i^*)}{\alpha \Gamma(k)} \frac{\alpha}{QH \cdot P_{\text{early}}} \text{ vs } K \frac{\Gamma(k + 1, \alpha c_i^* + \Delta c)}{\alpha \Gamma(k)} \frac{\alpha}{K \cdot P_{\text{late}}}
\]

\[\text{(4-101)}\]
The left ratio is a mean of right truncated gamma distribution (with mean range from 0 to $c_i^*$). The right ratio is a mean of left truncated gamma distribution (with mean range from $c_i^*$ to $\infty$). Thus the right ratio is bigger than the left one and the derivative (41) is negative. Thus, increasing $\alpha$ will decrease $Y$. ■

This section outlines an analysis of the expected penalty cost with respect to the shape parameter of the gamma delivery distribution when all other parameters ($\alpha$ and $\Delta c$) are held constant.

Using the derivatives of the gamma and incomplete gamma functions by $z$

$$\frac{d\Gamma(z)}{dz} = \int_0^\infty e^{-t}t^{z-1}\ln(t)dt = \lambda(z), \quad (4-103)$$

Knowing that $y$ is a function of $z$:

$$\frac{d\Gamma(z,y)}{dz} = 0 - \frac{dy}{dz} e^{-y} y^{z-1} + \int_y^\infty e^{-t}t^{z-1}\ln(t)dt = \int_y^\infty e^{-t}t^{z-1}\ln(t)dt - \frac{dy}{dz} e^{-y} y^{z-1} =$$

$$= \lambda(z,y) - \frac{dy}{dz} e^{-y} y^{z-1}, \quad (4-104)$$

$$\frac{dy(z,y)}{dz} = \int_0^y e^{-t}t^{z-1}\ln(t)dt + \frac{dy}{dz} e^{-y} y^{z-1} = \tau(z,y) + \frac{dy}{dz} e^{-y} y^{z-1}. \quad (4-105)$$

**Proposition 4-20.** For a fixed width of the on-time portion of the delivery window and constant scale parameter of gamma distributed delivery time, reducing shape parameter will decrease optimal value of $c_l$ when $K \cdot \lambda(k, \alpha(c^*_i + \Delta c)) > QH \cdot \tau(k, \alpha c^*_i)$.

**Proof.** Using (4-103), (4-104) and (4-105) we have:
\[
\frac{dP_{\text{early}}}{dk} = \frac{\Gamma(k)\left[\alpha \left(\frac{dc_1^*}{dk}\right) e^{-\alpha c_1^*} (\alpha c_1^*)^{k-1} + \tau(k, \alpha c_1^*)\right] - \gamma\left(k, \alpha c_1^*\right) \lambda(k)}{\Gamma^2(k)} = f(c_1^*) \frac{dc_1^*}{dk} + \frac{\tau(k, \alpha c_1^*)}{\Gamma(k)} - \frac{\gamma\left(k, \alpha c_1^*\right) \lambda(k)}{\Gamma^2(k)}
\]

\[
\frac{dP_{\text{late}}}{dk} = \frac{\Gamma(k)\left[\lambda(k, \alpha(c_1^* + \Delta c)) - \alpha \left(\frac{dc_1^*}{dk}\right) e^{-\alpha(c_1^* + \Delta c)} (\alpha(c_1^* + \Delta c))^{k-1}\right] - \Gamma(k, \alpha(c_1^* + \Delta c)) \lambda(k)}{\Gamma^2(k)}
\]

\[
= -f(c_1^* + \Delta c) \frac{dc_1^*}{dk} + \frac{\lambda(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)} - \frac{\Gamma(k, \alpha(c_1^* + \Delta c)) \lambda(k)}{\Gamma^2(k)}
\]

Thus

\[
QH \cdot \left[ f(c_1^*) \frac{dc_1^*}{dk} + \frac{\tau(k, \alpha c_1^*)}{\Gamma(k)} - \frac{\gamma\left(k, \alpha c_1^*\right) \lambda(k)}{\Gamma^2(k)} \right] = K \left[ -f(c_1^* + \Delta c) \frac{dc_1^*}{dk} + \frac{\lambda(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)} - \frac{\Gamma(k, \alpha(c_1^* + \Delta c)) \lambda(k)}{\Gamma^2(k)} \right]
\]

\[
(QH \cdot f(c_1^*) + K \cdot f(c_1^* + \Delta c)) \frac{dc_1^*}{dk} = \left[-QH \frac{\tau(k, \alpha c_1^*)}{\Gamma(k)} + K \frac{\lambda(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)}\right] + 
\]

\[
+ \frac{\lambda(k)}{\Gamma(k)} \left[ QH \frac{\gamma\left(k, \alpha c_1^*\right)}{\Gamma(k)} - K \frac{\Gamma(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)} \right]
\]

Using (3-9) we can conclude that the second element on the right hand side of the equation (4-108) is equal to 0.

\[
\frac{dc_1^*}{dk} = \left[ K \frac{\lambda(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)} - QH \frac{\tau(k, \alpha c_1^*)}{\Gamma(k)} \right] (QH \cdot f(c_1^*) + K \cdot f(c_1^* + \Delta c))^{-1}
\]

Because \(QH \cdot f(c_1^*) + K \cdot f(c_1^* + \Delta c) > 0\), the sign of \(\frac{dc_1^*}{dk}\) depends on the sign of

\[
K \frac{\lambda(k, \alpha(c_1^* + \Delta c))}{\Gamma(k)} - QH \frac{\tau(k, \alpha c_1^*)}{\Gamma(k)}, \text{ or}
\]

\[
K \cdot \lambda(k, \alpha(c_1^* + \Delta c)) - QH \cdot \tau(k, \alpha c_1^*).
\]

(4-111)
Since \( \frac{dc_i^*}{dk} / dk > 0 \) when \( K \cdot \lambda(k, \alpha(c_i^* + \Delta c)) > QH \cdot \tau(k, \alpha c_i^*) \), decreasing \( k \) will decrease \( c_1^* \). \( \blacksquare \)

To further investigate the effect of changing the shape parameter on the optimal value of \( c_1 \) we evaluate the following function

\[
A(k, \alpha, \Delta c, QH, K) = K \frac{\lambda(k, \alpha(c_i^* + \Delta c))}{\Gamma(k)} - QH \frac{\tau(k, \alpha c_i^*)}{\Gamma(k)}
\]

(4-112)

**Remark 4-1.** For any values of parameters \( A(k, \alpha, \Delta c, QH, K) \) is always positive (see Appendix C.1 for proof).

From Remark 4-1 we can conclude that reducing shape parameter will always decrease the optimal value of \( c_1 \).

**Proposition 4-21.** For a fixed width of the on-time portion of the delivery window, constant scale parameter of gamma distributed delivery time and an optimally positioned delivery window, reducing shape parameter will decrease the total expected penalty cost for \( y_{early} + y_{late} > Y \cdot \lambda(k)/\Gamma(k) \), where

\[
y_{early} = QH \int_0^{c_1^*} (c_1^* - x)f(x) \ln(\alpha x) dx \quad \text{and} \quad y_{late} = K \int_{c_1^* + \Delta c}^{\infty} (x - (c_1^* + \Delta c))f(x) \ln(\alpha x) dx.
\]

**Proof.** We will use the results from Proposition 4-5 here to simplify calculations.

\[
\frac{dY_{early}}{dk} = QH \left[ \frac{(dc_i^*/dk) \gamma(k, \alpha c_i^*) + c_i^* (e^{-\alpha c_i^*} (\alpha c_i^*)^{-k-1} \alpha (dc_i^*/d\Delta c) + \tau(k, \alpha c_i^*)) \Gamma(k)}{\Gamma^2(k)} - \right]
\]

\[
- QH \frac{c_i^* \gamma(k, \alpha c_i^*) \lambda(k)}{\Gamma^2(k)} - QH \frac{(e^{-\alpha c_i^*} (\alpha c_i^*)^k \alpha (dc_i^*/d\Delta c) + \tau(k + 1, \alpha c_i^*) ) \Gamma(k) - \gamma(k + 1, \alpha c_i^*) \lambda(k)}{\alpha T^2(k)}
\]

(4-113)
\[
\frac{dY_{\text{early}}}{dk} = QH \left( c_i^* \gamma(k, \alpha c_i^*) + QH \left[ \frac{c_i^* \tau(k, \alpha c_i^*)}{\Gamma(k)} - \frac{\tau(k+1, \alpha c_i^*)}{\alpha \Gamma(k)} \right] - QH \left[ \frac{c_i^* \gamma(k, \alpha c_i^*)}{\Gamma^2(k)} - \frac{\gamma(k+1, \alpha c_i^*)}{\alpha \Gamma^2(k)} \right] \right) \]

(4-114)

\[
\frac{dY_{\text{early}}}{dk} = QH \left( - \frac{c_i^* \gamma(k, \alpha c_i^*)}{\Gamma(k)} + \frac{\tau(k+1, \alpha c_i^*)}{\alpha \Gamma(k)} \right) - QH \left[ \frac{c_i^* \gamma(k, \alpha c_i^*)}{\Gamma^2(k)} - \frac{\gamma(k+1, \alpha c_i^*)}{\alpha \Gamma^2(k)} \right]
\]

(4-115)

where \( y_{\text{early}} = QH \left[ \frac{c_i^* \tau(k, \alpha c_i^*)}{\Gamma(k)} - \frac{\tau(k+1, \alpha c_i^*)}{\alpha \Gamma(k)} \right] \)

\[
\int_0^x (c_i^* - x) f(x) \ln(\alpha x) dx.
\]

(4-116)

\[
\frac{dY_{\text{late}}}{dk} = K \left[ \Gamma(k) \left( - e^{-a(c_i^* + \Delta c)} (\alpha(c_i^* + \Delta c)) \frac{\alpha(d c_i^*/d \Delta c) + \lambda(k + 1, \alpha(c_i^* + \Delta c))}{\alpha \Gamma^2(k)} \right) - K \Gamma(k) \left( (d c_i^*/d \Delta c) \Gamma(k, \alpha(c_i^* + \Delta c)) \right) - K \Gamma(k) \left( e^{-a(c_i^* + \Delta c)} \alpha(c_i^* + \Delta c)) \frac{\alpha(d c_i^*/d \Delta c) + \lambda(k, \alpha(c_i^* + \Delta c))}{\alpha \Gamma^2(k)} \right) - K \Gamma(k) \left( e^{-a(c_i^* + \Delta c)} \alpha(c_i^* + \Delta c)) \frac{\alpha(d c_i^*/d \Delta c) + \lambda(k, \alpha(c_i^* + \Delta c))}{\alpha \Gamma^2(k)} \right) \]

(4-117)

where \( y_{\text{late}} = K \left[ \frac{\lambda(k + 1, \alpha(c_i^* + \Delta c))}{\alpha \Gamma(k)} - \frac{(c_i^* + \Delta c) \lambda(k, \alpha(c_i^* + \Delta c))}{\alpha \Gamma(k)} \right] = K \int_{c_i^* + \Delta c}^\infty (x - (c_i^* + \Delta c)) f(x) \ln(\alpha x) dx \)

(4-118)

Since \( dY/dk > 0 \) when \( y_{\text{early}} + y_{\text{late}} > Y \cdot \lambda(k)/\Gamma(k) \), decreasing \( k \) will decrease \( Y \).
Remark 4-2. For any values of parameters \((k, \alpha, \Delta c, QH,\text{ and } K)\), \(dY/dk\) is always positive (see Appendix C.2 for proof).

In managing delivery performance, a logistics manager may find using the mean and variance of delivery time to be more meaningful than using shape and scale parameters. The mean and variance of the gamma pdf in terms of its shape and scale parameters are \(\mu = \sqrt{k/\alpha}\) and \(\nu = k/\alpha^2\).

Proposition 4-22. For a fixed width of the on-time portion of the delivery window and constant variance of gamma distributed delivery time, reducing the mean will decrease the optimal value of \(c_1\) and the total expected penalty cost.

Proof. The derivatives are
\[
\frac{d\mu}{dk} = \frac{1}{\alpha}, \quad \text{or} \quad \frac{dk}{d\mu} = \alpha \tag{4-120}
\]
\[
\frac{d\mu}{d\alpha} = -\frac{k}{\alpha^2}, \quad \text{or} \quad \frac{d\alpha}{d\mu} = -\frac{\alpha^2}{k} \tag{4-121}
\]

Using that \(dc_1^*/d\mu = dc_1^*/d\alpha \cdot \alpha_{\mu} + dc_1^*/dk \cdot k_{\mu}\) and \(Y_{\mu}' = Y_{\alpha}' \alpha_{\mu} + Y_{k}' k_{\mu}\) we have
\[
\frac{dc_1^*}{d\mu} = -\frac{\alpha^2}{k} \frac{dc_1^*}{d\alpha} + \alpha \frac{dc_1^*}{dk}, \quad \tag{4-122}
\]

The derivative (4-122) is positive, because \(dc_1^*/d\alpha < 0\) (Proposition 4-18) and \(dc_1^*/dk > 0\) (Remark 4-1).

\[
Y_{\mu}' = -\frac{\alpha^2}{k} \frac{dY}{d\alpha} + \alpha \frac{dY}{dk}, \quad \tag{4-123}
\]

Knowing from Proposition 4-19 that \(\frac{dY}{d\alpha} < 0\) and from Remark 4-2 that \(\frac{dY}{dk} > 0\), we can conclude that the derivative (4-123) is positive. ■
Proposition 4-23. For a fixed width of the on-time portion of the delivery window and constant mean of gamma distributed delivery time, reducing the variance will decrease the optimal value of $c_1$ and the total expected penalty cost.

Proof. The derivatives are

$$\frac{dv}{dk} = \frac{1}{\alpha^2}, \text{ or } \frac{dk}{dv} = \alpha^2 \quad (4-124)$$

$$\frac{dv}{d\alpha} = -\frac{k}{\alpha^3}, \text{ or } \frac{d\alpha}{dv} = -\frac{\alpha^3}{2k} \quad (4-125)$$

Using that $\frac{dc_1^*}{dv} = \frac{dc_1^*}{d\alpha} \cdot \alpha' + \frac{dc_1^*}{dk} \cdot k'$ and $Y_\alpha = Y_\alpha' \alpha' + Y_\alpha' k'$ we have

$$\frac{dc_1^*}{dv} = -\frac{\alpha^3}{2k} \frac{dc_1^*}{d\alpha} + \alpha^2 \frac{dc_1^*}{dk} \quad (4-126)$$

The derivative (4-126) is positive, because $\frac{dc_1^*}{d\alpha} < 0$ (Proposition 4-18) and $\frac{dc_1^*}{dk} > 0$ (Remark 4-1).

$$Y_\alpha' = -\frac{\alpha^3}{2k} \frac{dY}{d\alpha} + \alpha^2 \frac{dY}{dk} \quad (4-127)$$

Knowing from Proposition 4-19 that $\frac{dY}{d\alpha} < 0$ and from Remark 4-2 that $\frac{dY}{dk} > 0$, we can conclude that the derivative (4-127) is positive.

4.8. Discussion

As shown in Proposition 4-1, a supplier would prefer to increase the width of the on-time portion of the delivery window thereby reducing the expected penalty cost for untimely delivery. However, from the buyer’s perspective, a wider width for the on-time portion of the delivery window increases delivery time uncertainty. Higher levels of delivery uncertainty could contribute to unwillingness by the buyer to enter into a contractually defined delivery window.
with the supplier. Hence understanding the effect of the width of the on-time portion of the delivery window on the expected penalty cost, a supplier and a buyer can better work in partnership to find a value of the width of the on-time portion of the delivery window which will satisfy the desired levels of expected cost and delivery time uncertainty valued by the supplier and the buyer.

Unlike the width of the on-time portion of the delivery window the parameters of delivery time distribution (shape and scale) can be changed by a supplier alone, but the changes require time and investment. Thus the supplier should compare benefits of the delivery performance improvement and expenses needed for that. There are several practices that can improve delivery performance, among them are Just-in-time (Mackelprang & Nair, 2010), communication and information sharing (Carr & Kaynak, 2007), design for manufacturability (Ketokivi & Schroeder, 2004).

Propositions from 4-6 to 4-23 demonstrates how a supplier should change the parameters of the distributed delivery time to decrease the expected penalty cost. At the same time, it is shown how these changes will affect the optimal value of $c_1$ and when a supplier should ship a product to minimize the expected penalty cost. That suggests that the supplier should coordinate the shipment of the product so that the product arrives on the buyer’s requested delivery date as dictated by the optimal value of $c_1$. This action may require the supplier to adjust their production schedule to avoid increases in inventory levels and related costs and if needed, adjust delaying the shipping of the product to avoid incurring higher in house inventory levels and related costs.

Logistics and supply chain managers may find it easier to work with the mean and variance of the delivery time distribution as opposed to the distribution parameters. The mean
and variance are the most well-known parameters of a random variable and might be easily understood and estimated. Therefore, managers might prefer delivery performance improvement guidance in terms of mean and variance. The chapter (for example, propositions 4-22 and 4-23) provide insight on how reducing both the mean and variance of the delivery distribution leads to a decrease the expected penalty cost.

Delivery performance improvement can focus on all distribution parameters together. A value of one parameter does not affect the direction the other parameter should be changed.
CHAPTER 5

INVENTORY MODELS AS A BUYER’S POINT OF VIEW
ON SUPPLY CHAIN DELIVERY PERFORMANCE

To facilitate the comparison of supplier’s and buyer’s points of view on supply chain delivery performance we compare the delivery performance model and the inventory model. The delivery performance model, which captures the supplier’s point of view, is presented in Chapter 3. The inventory model, which captures the buyer’s point of view, is developed in this chapter below. In Section 5.1, new inventory models are developed proposing a buyer’s point of view on the delivery. Optimal solutions for the models are derived in Section 5.2. The comparison of delivery performance and inventory models as supplier’s and buyer’s points of view is presented in Section 5.3. A buyer’s strategy for delivery performance optimization is discussed in Section 5.4. Numerical example is provided in Section 5.5.

5.1. Model Development

In this section, we define the mathematical form for two stochastic continuous review \((Q, R)\) models with linear shortage cost and two levels of storage: owned warehouse (OW) and rented warehouse (RW). The models assume constant demand and variable leadtime.

In the supply chain delivery performance model, penalty per unit time late is defined by \(K\) which has dimension \([$/unit time]\). Penalty per early delivery is based on \(QH\) where \(Q\) is a lot size and \(H\) is a coefficient that has dimension \([$/unit/unit time]\). Thus, penalties for early and late delivery can be defined in any of two dimensions \([$/unit time]\) or \([$/unit/unit time]\). To maintain consistency between buyer and supplier cost functions the same dimensions for shortage and overload costs are assumed. Two models are proposed:
• ut-model where cost of overload and cost of stockout are defined per unit per unit time;

• t-model where costs of overload and stockout are defined per unit time.

Any combination of these two models can be easily created. For example, a model with cost of overload per unit time and cost of stockout per unit per unit time.

To construct the model, we employ the following notations and assumptions.

5.1.1. Notations

\[ C_{o}^{ut} = \text{Cost of overload (cost of holding extra inventory when inventory level is higher than size of warehouse, $/unit/unit time)}; \]

\[ C'_{o} = \text{Cost of overload ($/unit time)}; \]

\[ C_{s}^{ut} = \text{Cost of stockout ($/unit/unit time)}; \]

\[ C'_{s} = \text{Cost of stockout ($/unit time)}; \]

\[ D = \text{Demand (units/unit time)}; \]

\[ H = \text{Holding cost per unit per year ($/unit/unit time)}; \]

\[ L = \text{Leadtime, random variable with probability density function (pdf) } f(L); \]

\[ Q = \text{Order quantity (Lot size)}; \]

\[ Q_{\text{max}} = \text{Size of warehouse (units)}; \]

\[ R = \text{Reorder point}; \]

\[ S = \text{Ordering cost ($/order)}; \]

5.1.2. Assumptions

The following assumptions about the model are made.
(1) Average leadtime and pdf of leadtime are known.

(2) A single product, instantaneously replenishment, and no quantity discounts.

(3) Stockouts are backordered.

(4) Storage is limited by the capacity of the OW however additional storage can be purchased at an RW.

(5) Demand is first met from RW until it is emptied. Items in OW are used only after RW is depleted.

(6) The transportation cost between warehouses is negligible.

(7) Costs of overload and stockout are higher than holding cost \( C_o > H \) and \( C_s > H \).

(8) Reorder point \( R > 0 \).

(9) No interaction between orders \( 0 \leq L \leq Q/D \).

However, the "interval between placing orders is usually large enough so that there is essentially no interaction between orders" (Hadley & Whitin, 1963, p. 203) or the probability of crossover is small enough to be ignored.

5.1.3. Three outcomes

As shown on Figure 7, there are three possible (probabilistic) outcomes:

1. \( L > R/D \). Inventory level reaches zero level and in addition to inventory holding cost we are paying for stockouts, because of late delivery.

2. \( A/D \leq L \leq R/D \), where \( A = R - (Q_{max} - Q) \) (B area, \( B = Q_{max} - Q \)). No stockouts, no overload. Thus, we have inventory holding cost only.
3. \( L < \frac{A}{D} \). When delivery occurred, the inventory level will exceed \( Q_{\text{max}} \), so we are paying for overloading and inventory holding cost.

Here is a numerical example that illustrates the cases. Assume, \( Q_{\text{max}} = 20 \), \( R = 10 \),

\[ Q = 16, \ D = 2. \] Then \( \frac{R}{D} = 5 \), \( B = Q_{\text{max}} - Q = 4 \), \( \frac{B}{D} = \frac{Q_{\text{max}} - Q}{D} = 2 \),

\[ \frac{A}{D} = \frac{R}{D} - \frac{Q_{\text{max}} - Q}{D} = 5 - 2 = 3. \]

The three cases are:

1. If \( L = 7 \) (\( L > \frac{R}{D} \)), we have stockout (the inventory level before new inventory arrived is

\[ I = R - L \cdot D = 10 - 7 \cdot 2 = -4 < 0 \). \]
2. If \( L = 4 \left( \frac{A}{D} \leq L \leq \frac{R}{D} \right) \), no stockouts \( (I = R - L \cdot D = 10 - 4 \cdot 2 = 2 > 0 \) the inventory level, before new inventory arrived), no overloading \( (I = R + Q - L \cdot D = 10 + 16 - 4 \cdot 2 = 9 < Q_{\text{max}} \) the inventory level, when new inventory arrived).

3. If \( L = 2 \left( L < \frac{A}{D} \right) \), no stockouts, but we have overloading

\( (I = R + Q - L \cdot D = 10 + 16 - 2 \cdot 2 = 22 > Q_{\text{max}} \) the inventory level, when new inventory arrived).

So we can find probability of each outcome. The probability of late delivery, when stockout occur (case 1) is

\[
P_{\text{late}} = P(L > R/D) = \int_{R/D}^{\infty} f(L) dL. \quad (5-1)
\]

The probability of on time delivery without stockouts and overloading (case 2) is

\[
P_{\text{ontime}} = P(A/D \leq L \leq R/D) = \int_{A/D}^{R/D} f(L) dL. \quad (5-2)
\]

The probability of early delivery, when overload occur (case 3) is

\[
P_{\text{early}} = P(L < A/D) = \int_{0}^{A/D} f(L) dL. \quad (5-3)
\]

Now we are calculating costs for each of three cases (excluding ordering cost).

5.1.4. On time delivery

In the case of on time delivery we have inventory holding cost only. The cost per cycle is equal to the squired of two trapezoids (see Figure 8a areas A and B) multiplied by holding cost \((H)\). The cost is the same for both ut- and t-models.
\[ C_{\text{on time}} = H \frac{Q}{2D} \left[ (R + Q - LD) + (R - LD) \right] = H \frac{Q}{2D} (Q + 2(R - LD)). \]  

(5-4)

Figure 8a. Case 2: On time delivery

The expected cost for on time delivery is:

\[ E(C_{\text{on time}}) = \frac{H}{2D} \int_{R/D}^{Q/D} Q(Q + 2(R - LD)) f(L) dL. \]  

(5-5)

5.1.5. Early delivery

**ut-model**

For early delivery cost includes holding and backordering costs (Figure 8b). Holding cost is described the areas A and B (excluding C). And the backordering cost is shown as C area.

\[ C_{\text{early}}^{\text{ut}} = H \frac{Q}{2D} (Q + 2(R - LD)) + (C_{\text{o}}^{\text{ut}} - H) \frac{(A - LD)^2}{2D}. \]  

(5-6)
The expected cost for early delivery is:

\[
E(C_{\text{early}}) = \frac{H}{2D} \int_0^{\frac{A}{2}} Q(Q + 2(R - LD))f(L)dL + \frac{C_o}{2} - \frac{H}{2D} \left( \int_0^{\frac{A}{2}} (A - LD)f(L)dL \right)^2 = 
\]

\[
= \frac{H}{2d} \int_0^{\frac{A}{2}} Q(Q + 2(R - LD))f(L)dL + \frac{C_o}{2} - \frac{H}{2D} (Y_{\text{early}})^2 
\]

where \( Y_{\text{early}} = \int_0^{\frac{A}{2}} (A - LD)f(L)dL \).

The cost of early delivery includes holding and backordering costs. The holding cost is described the sum of the areas A and B on Figure 8b. And the backordering cost is defined by the length C

\[
C_{\text{early}} = H \frac{Q}{2D} (Q + 2(R - LD)) + C_o \frac{A - LD}{D}. 
\]

The expected cost for early delivery is
\[
E(C_{\text{early}}) = \frac{H}{2D} \int_0^A Q(Q + 2(R - LD))f(L)\,dL + \frac{C_o}{D} \int_0^A (A - LD) f(L)\,dL = \\
= \frac{H}{2D} \int_0^A Q(Q + 2(R - LD))f(L)\,dL + \frac{C_o}{D} Y_{\text{early}}.
\] (5-9)

5.1.6. Late delivery

For late delivery inventory cost is described by A and B areas (Figure 8c):

\[
C_{\text{Inv late}} = \frac{H}{2} \left( \frac{Q}{D} - \frac{L - R}{D} \right) (Q + R - LD),
\] (5-10)

or

\[
C_{\text{Inv late}} = H \frac{Q}{2D} (Q + 2(R - LD)) + \frac{H}{2D} (R - LD)^2.
\] (5-11)

**ut-model**

Knowing that the stockout is equal C area (Figure 8c), the cost for late delivery is:

\[
C_{\text{late}}^{\text{ut}} = H \frac{Q}{2D} (Q + 2(R - LD)) + (H + C_s^{\text{ut}}) \frac{(LD - R)^2}{2D}.
\] (5-12)

![Figure 8c. Case 1: Late delivery](image-url)
The expected cost for late delivery is:

\[ E(C_{\text{late}}^t) = \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H + C_{S}^{t}}{2D} \left( \int_{R/D}^{\infty} (LD - R)f(L)dL \right)^2 = \]

\[ = \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H + C_{S}^{t}}{2D} (Y_{\text{late}})^2 \]

where \( Y_{\text{late}} = \int_{R/D}^{\infty} (LD - R)f(L)dL \).

\[ t\text{-model} \]

Knowing that the stockout length of time is equal to C (Figure 8c), the cost for late delivery is

\[ C_{\text{late}}^t = H \frac{Q}{2D} (Q + 2(R - LD)) + H \frac{(R - LD)^2}{2D} + C_{S}^{t} \frac{LD - R}{D}. \]  \hfill (5-14)

The expected cost for late delivery is

\[ E(C_{\text{late}}^t) = \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \]

\[ + \frac{H}{2D} \left( \int_{R/D}^{\infty} (R - LD)f(L)dL \right)^2 + \frac{C_{S}^{t}}{D} \int_{R/D}^{\infty} (LD - R)f(L)dL = \]

\[ = \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H}{2D} (Y_{\text{late}})^2 + \frac{C_{S}^{t}}{D} Y_{\text{late}}. \]

5.1.7. Total costs

\[ ut\text{-model} \]

The total expected cost per cycle including ordering cost is:
\begin{align}
E(C''') & = E(C_{\text{continue}}) + E(C_{\text{early}}') + E(C_{\text{late}}') = S + \frac{H}{2D} \int_{A/D}^{R/D} Q(Q + 2(R - LD))f(L)dL + \\
& + \frac{H}{2D} \int_{0}^{A/D} Q(Q + 2(R - LD))f(L)dL + \frac{C_{o}'}{D} Y_{\text{early}}' + \\
& + \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H + C_{s}'}{2D} (Y_{\text{late}}')^2. (5-16)
\end{align}

\begin{align}
E(C') & = S + \frac{H}{2D} \int_{0}^{\infty} Q(Q + 2(R - LD))f(L)dL + \\
& + \frac{C_{o}'}{2D} \left( \int_{0}^{A/D} (A - LD)f(L)dL \right)^2 + \frac{H + C_{s}'}{2D} \left( \int_{R/D}^{\infty} (LD - R)f(L)dL \right)^2. (5-17)
\end{align}

\begin{align}
E(C') & = S + \frac{H}{2D} Q(Q + 2(R - E(L)D)) + \frac{C_{o}'}{2Q} (Y_{\text{early}}')^2 + \frac{H + C_{s}'}{2Q} (Y_{\text{late}}')^2. (5-18)
\end{align}

where \( E(L) \) is expected value of \( L \).

Using that the number of cycles per unit time is \( D/Q \), the total cost per unit time is:

\begin{align}
E(TC''') & = S \frac{D}{Q} + \frac{H}{2} Q + 2(R - E(L)D)) + \frac{C_{o}'}{2Q} (Y_{\text{early}}')^2 + \frac{H + C_{s}'}{2Q} (Y_{\text{late}}')^2. (5-19)
\end{align}

**t-model**

The total expected cost per cycle including ordering cost is

\begin{align}
E(C') & = E(C_{\text{continue}}') + E(C_{\text{early}}') + E(C_{\text{late}}') = S + \frac{H}{2D} \int_{A/D}^{R/D} Q(Q + 2(R - LD))f(L)dL + \\
& + \frac{H}{2D} \int_{0}^{A/D} Q(Q + 2(R - LD))f(L)dL + \frac{C_{o}'}{D} Y_{\text{early}}' + \\
& + \frac{H}{2D} \int_{R/D}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H}{2D} (Y_{\text{late}}')^2 + \frac{C_{s}'}{D} Y_{\text{late}}'. (5-20)
\end{align}

\begin{align}
E(C') & = S + \frac{H}{2D} \int_{0}^{\infty} Q(Q + 2(R - LD))f(L)dL + \frac{H}{2D} (Y_{\text{late}}')^2 + \frac{C_{o}'}{D} Y_{\text{early}}' + \frac{C_{s}'}{D} Y_{\text{late}}'. (5-21)
\end{align}

\begin{align}
E(C') & = S + \frac{H}{2D} Q(Q + 2(R - E(L)D)) + \frac{H}{2D} (Y_{\text{late}}')^2 + \frac{C_{o}'}{D} Y_{\text{early}}' + \frac{C_{s}'}{D} Y_{\text{late}}'. (5-22)
\end{align}
Using that the number of cycles per unit time is \( \frac{D}{Q} \), the total cost per unit time is

\[
E(TC') = S \frac{D}{Q} + H \left( \frac{Q + 2(R - E(L)D)}{2} \right) + \frac{H}{2Q} (Y_{\text{late}})^2 + \frac{C'_o}{Q} Y_{\text{early}} + \frac{C'_s}{Q} Y_{\text{late}}. \tag{5-23}
\]

### 5.2. Optimal solutions

#### 5.2.1. ut-model

**Optimal reorder point**

We are starting with simplified reorder-point system where \( Q \) is fixed and we need to find \( R^* \) only. To find the optimal reorder point we take derivative of the total cost by \( R \). Using that

\[
\frac{dY_{\text{early}}}{dR} = \sqrt{D} \left( A - \frac{A}{D} D \right) f(\frac{A}{D}) - 0 + \int_0^{A/D} f(L) dL = P_{\text{early}}, \tag{5-24}
\]

where \( \frac{dA}{dR} = 1 \), and

\[
\frac{dY_{\text{late}}}{dR} = 0 - \sqrt{D} \left( \frac{R}{D} D - R \right) f(\frac{R}{D}) + \int_{R/D}^\infty (-1) f(L) dL = -P_{\text{late}}, \tag{5-25}
\]

we have

\[
\frac{dE(TC^{ut})}{dR} = 0 + \frac{H}{2} \left( - \frac{H}{2Q} 2Y_{\text{early}} P_{\text{early}} - \frac{H}{2Q} 2Y_{\text{late}} P_{\text{late}} \right) =
\]

\[
= H + \frac{C'_o}{Q} Y_{\text{early}} P_{\text{early}} - \frac{H + C'_s}{Q} Y_{\text{late}} P_{\text{late}}. \tag{5-26}
\]

The optimal reorder point is when

\[
HQ + (C'_o - H) Y_{\text{early}} P_{\text{early}} = (H + C'_s) Y_{\text{late}} P_{\text{late}}. \tag{5-27}
\]

To prove convexity of the total cost by \( R \) we find the second derivative. Using
\[ \frac{dP_{\text{early}}}{dR} = \frac{1}{D} f\left(\frac{A}{D}\right) - 0 + \int_0^{A/D} 0 dL = \frac{1}{D} f\left(\frac{A}{D}\right), \quad (5-28) \]

\[ \frac{dP_{\text{late}}}{dR} = 0 - \frac{1}{D} f\left(\frac{R}{D}\right) + \int_{R/D}^{\infty} 0 dL = -\frac{1}{D} f\left(\frac{R}{D}\right), \quad (5-29) \]

we have

\[ \frac{d^2E(TC_{\text{util}})}{dR^2} = 0 + \frac{C_o^{\text{util}} - H}{Q} \left[ \left( P_{\text{early}} \right)^2 + Y_{\text{early}} \frac{1}{D} f\left(\frac{A}{D}\right) \right] - \frac{H + C_s^{\text{util}}}{Q} \left( -P_{\text{late}} \right) P_{\text{late}} - Y_{\text{late}} \frac{1}{D} f\left(\frac{R}{D}\right) \]

\[ = \frac{C_o^{\text{util}} - H}{DQ} \left[ D \left( P_{\text{early}} \right)^2 + Y_{\text{early}} f\left(\frac{A}{D}\right) \right] + \frac{H + C_s^{\text{util}}}{DQ} \left( D \left( P_{\text{late}} \right)^2 + Y_{\text{late}} f\left(\frac{R}{D}\right) \right) \]

(5-30)

Because we assume \( C_o^{\text{util}} \geq H \), \( \frac{d^2E(TC_{\text{util}})}{dR^2} > 0 \) and \( E(TC_{\text{util}}) \) is convex by \( R \). Therefore, convex optimization methods can be applied to find the optimal reorder point \((R^*)\).

**Optimal order quantity**

Now we extend the model assuming that \( Q \) is not fixed and we can choose the optimal order quantity (reorder-point-lot-size system). Thus, the question is what order quantity and reorder point will minimize total cost. To solve the problem we take derivative of the total cost by \( Q \). We need:

\[ \frac{dY_{\text{early}}}{dQ} = \frac{1}{D} (A - \frac{A}{D} D) f\left(\frac{A}{D}\right) - 0 + \int_0^{A/D} f(L) dL = P_{\text{early}}, \quad (5-31) \]

where \( \frac{dA}{dQ} = 1 \),

\[ \frac{dY_{\text{late}}}{dQ} = 0. \quad (5-32) \]

The derivative is:
The optimal order quantity is when

\[ HQ^2 = 2SD + (C_o - H)[(Y_{early})^2 - 2Y_{early}P_{early}Q] + (H + C_s)(Y_{late})^2. \]  

(5-34)

To prove convexity we take the second derivative. Using that

\[ \frac{dP_{early}}{dQ} = \sqrt{D} f(A/D) - 0 + \int_0^{A/D} dL = \sqrt{D} f(A/D), \]  

(5-35)

we have

\[ \frac{d^2E(TC^{ut})}{dQ^2} = 0 + 2S \frac{D}{Q^3} + 2 \frac{C_o - H}{2Q^2}(Y_{early})^2 - \frac{C_o - H}{2Q^2}2Y_{early}P_{early} - \frac{C_o - H}{Q^2}Y_{early}P_{early} + \frac{C_o - H}{Q}((P_{early})^2 + Y_{early}f (A/D)/D) + \frac{H + C_s}{Q^3}(Y_{late})^2; \]  

(5-36)

\[ \frac{d^2E(TC^{ut})}{dQ^2} = 2S \frac{D}{Q^3} + \frac{C_o - H}{Q^2}(Y_{early})^2 - 2 \frac{C_o - H}{Q^2}Y_{early}P_{early} + \frac{C_o - H}{Q}((P_{early})^2 + Y_{early}f (A/D)/D) + \frac{H + C_s}{Q^3}(Y_{late})^2. \]  

(5-37)

Using (5-34) we have

\[ \frac{d^2E(TC^{ut})}{dQ^2} = \frac{H}{Q} + \frac{C_o - H}{Q}((P_{early})^2 + Y_{early}f (A/D)/D). \]  

(5-38)

The second derivative of the total cost by \( Q \) is positive, thus the total cost is a convex function of \( Q \). To check convexity of the total cost as a function of two parameters \( Q \) and \( R \) we need to calculate Hessian. The derivative is
\[
\frac{d}{dR} \left( \frac{dE(TC^{ut})}{dQ} \right) = H + C_{S}^{ut} \frac{Y_{late} P_{late}}{Q^2} - \frac{C_{o}^{ut} - H}{Q^2} Y_{early} P_{early} + \\
\frac{C_{o}^{ut} - H}{Q} \left( (P_{early})^2 + Y_{early} f(A/D)/D \right)
\]

(5-39)

Using (5-27) we have

\[
\frac{d}{dR} \left( \frac{dE(TC^{ut})}{dQ} \right) = H + \frac{C_{o}^{ut} - H}{Q} \left( (P_{early})^2 + Y_{early} f(A/D)/D \right)
\]

(5-40)

The Hessian is:

\[
H = \frac{d^2E(TC^{ut})}{dR^2} \frac{dE(TC^{ut})}{dQ^2} - \left[ \frac{d}{dR} \left( \frac{dE(TC^{ut})}{dQ} \right) \right]^2
\]

\[
= \left[ \frac{C_{o}^{ut} - H}{Q} \left( (P_{early})^2 + Y_{early} f(A/D)/D \right) + \frac{H + C_{S}^{ut}}{Q} \left( (P_{late})^2 + Y_{late} f(R/D)/D \right) \right] \\
\times \left[ \frac{H}{Q} + \frac{C_{o}^{ut} - H}{Q} \left( (P_{early})^2 + Y_{early} f(A/D)/D \right) \right] - \\
\left[ \frac{H}{Q} + \frac{C_{o}^{ut} - H}{Q} \left( (P_{early})^2 + Y_{early} f(A/D)/D \right) \right]^2
\]

(5-41)

\[
H = \frac{H + C_{S}^{ut}}{Q} \left( (P_{late})^2 + Y_{late} f(R/D)/D \right) - \frac{H}{Q}.
\]

(5-42)

Using (5-27) we have

\[
H = \frac{H + C_{S}^{ut}}{Q} \left( (P_{late})^2 + Y_{late} f(R/D)/D \right) - \frac{H + C_{S}^{ut}}{Q^2} Y_{late} P_{late} + \frac{C_{o}^{ut} - H}{Q^2} Y_{early} P_{early} = \\
= \frac{H + C_{S}^{ut}}{Q} \left( P_{late} \left( P_{late} - Y_{late} / Q \right) + Y_{late} f(R/D)/D \right) + \frac{C_{o}^{ut} - H}{Q^2} Y_{early} P_{early}.
\]

(5-43)

The Hessian is positive if \( P_{late} > Y_{late} / Q \).

\[
P_{late} - Y_{late} / Q = \int_{R/D}^{\infty} \left( 1 - \frac{LD}{Q} + \frac{R}{Q} \right) f(L) dL.
\]

(5-44)
From assumption (9) No interaction between orders \((0 \leq L \leq Q/D)\), therefore the integral is positive and \(P_{\text{late}} > Y_{\text{late}}/Q\). Thus the Hessian is positive, which means that the total cost as a function of \(Q\) and \(R\) is convex.

5.2.2. \(t\)-model

**Optimal reorder point**

To find the optimal reorder point we take derivative of the total cost by \(R\). Using (5-24) and (5-25) we have

\[
\frac{dE(TC')}{dR} = 0 + \frac{H}{2} - \frac{H}{2Q} (2Y_{\text{late}} P_{\text{late}}) + \frac{C_o'}{Q} P_{\text{early}} - \frac{C_s'}{Q} P_{\text{late}} =
\]

\[
= H - \frac{H}{Q} (Y_{\text{late}} P_{\text{late}}) + \frac{C_o'}{Q} P_{\text{early}} - \frac{C_s'}{Q} P_{\text{late}},
\]

The optimal reorder point is when

\[
H(Q - Y_{\text{late}} P_{\text{late}}) + C_o' P_{\text{early}} = C_s' P_{\text{late}}.
\]

(5-46)

To prove convexity of the total cost by \(R\) we take the second derivative. Using (5-28) and (5-29) we have

\[
\frac{d^2E(TC')}{dR^2} = 0 - \frac{H}{Q} \left( -Y_{\text{late}} \frac{1}{D} f(R/D) - (P_{\text{late}})^2 \right) + \frac{C_o'}{Q D} f(A/D) + \frac{C_s'}{Q D} f(R/D),
\]

(5-47)

\[
\frac{d^2E(TC')}{dR^2} = \frac{H}{Q} \left( \frac{1}{D} f(R/D) Y_{\text{late}} + (P_{\text{late}})^2 \right) + \frac{C_o'}{Q D} f(A/D) + \frac{C_s'}{Q D} f(R/D).
\]

(5-48)

Because all elements of (5-48) are positive, \(\frac{d^2E(TC')}{dR^2} > 0\) and \(E(TC')\) is convex by \(R\).

**Optimal order quantity**

Using (5-31) and (5-32)
\[
\frac{dE(TC')}{dQ} = \frac{-SD}{Q} + \frac{H}{2} - \frac{H}{2Q^2} (Y_{late})^2 = \frac{C_o'}{Q^2} Y_{early} + \frac{C_o'}{Q} P_{early} - \frac{C_s'}{Q^2} Y_{late} = \\
= \frac{-SD}{Q^2} + \frac{H}{2} - \frac{H}{2Q^2} (Y_{late})^2 - \frac{C_o'}{Q^2} Y_{early} + \frac{C_o'}{Q} P_{early} - \frac{C_s'}{Q^2} Y_{late}. 
\]

(5-49)

The equation for the optimal lot size is

\[
\frac{H}{2} Q^2 + C_o' Q \cdot P_{early} - C_o' Y_{early} = SD + \frac{H}{2} (Y_{late})^2 + C_s' Y_{late}. 
\]

(5-50)

Using (5-35) we find the second derivative of the cost function by \(Q\)

\[
\frac{d^2E(TC')}{dQ^2} = 2 \frac{SD}{Q^2} + \frac{H}{Q} + 2 \frac{H}{2Q^2} (Y_{late})^2 + 2 \frac{C_o'}{Q^3} Y_{early} - \\
\frac{C_o'}{Q^2} Y_{late} - \frac{C_o'}{Q^2} P_{early} = \frac{C_o'}{Q} f(A/D) = \\
= 2 \frac{SD}{Q^3} + \frac{H}{Q^3} (Y_{late})^2 + 2 \frac{C_o'}{Q^2} Y_{early} - 2 \frac{C_o'}{Q^2} P_{early} + 2 \frac{C_s'}{Q^2} Y_{late} + \frac{C_o'}{QD} f(A/D). 
\]

(5-51)

Using (5-50) the derivative is

\[
\frac{d^2E(TC')}{dQ^2} = \frac{H}{Q} + \frac{C_o'}{QD} f(A/D). 
\]

(5-52)

The derivative is positive and \(E(TC')\) is convex by \(Q\).

We use Hessian to check convexity of the total cost function of two parameters \(R\) and \(Q\).

\[
\frac{d}{dQ} \left( \frac{dE(TC')}{dR} \right) = -\frac{H}{Q^2} (Y_{late} P_{late}) - \frac{C_o'}{Q^2} P_{early} + \frac{C_o'}{QD} f(A/D) + \frac{C_s'}{Q^2} P_{late}. 
\]

(5-53)

Using (5-45) the derivative is

\[
\frac{d^2E(TC')}{dQ^2} = -\frac{H}{Q} + \frac{C_o'}{QD} f(A/D). 
\]

(5-54)

The Hessian is:
Using (5-46) we have

\[ H = \frac{H}{Q^2} \left( C_s' f(R/D) / D - P_{late}/Q \right) + H / D \cdot f(R/D) Y_{late} + H P_{late} \cdot (P_{late} Y_{late}/Q) + \frac{C_o'}{Q} P_{early} \] + 
\[ \left( \frac{C_o'}{Q} + \frac{C_s'}{QD} \right) \left( \frac{H}{Q} \left( V_{D} \cdot f(R/D) Y_{late} + (P_{late})^2 + 3 \right) + \frac{C_s'}{QD} f(R/D) \right) \] (5-57)

As it was shown \( P_{late} > Y_{late}/Q \) and the only negative element is \( P_{late}/Q \). Convexity of the backorder function \( (Y_{late}) \) was discussed by Brooks & Lu (1969).

### 5.3. Comparison of delivery performance and inventory models

Beforehand, the costs associated with delivery process were defined for a supplier and a buyer. Both a supplier and a buyer are interested in minimizing their costs, but that might be contrary goals. They have a certain influence on the delivery process attempting to minimize their costs, but no one of them can change all parameters which affect the cost functions along. A supplier has an effect on the time of delivery only and would apply the concept of the optimal position of the delivery window to find an optimal time of delivery. A buyer can affect the supplier’s optimal time of delivery defining the penalties for untimely delivery in the contract.
which is a common purchasing agreement practice. A supplier would prefer a time of delivery that will minimize its costs and penalties equal to costs of overload and stockout. How a buyer can achieve this goal? Supplier’s and buyer’s points of view on delivery performance are compared to answer the question.

The supply chain delivery performance model shown in Chapter 3 describes a supplier’s point of view on delivery performance. In Section 5.2 an inventory model was introduced as a buyer’s point of view. Both of these models (delivery performance and inventory) demonstrate costs related to a product delivery for a supplier and a buyer (equations (3-1) and (5-19)/(5-23) for ut- \( t \)-model respectively). Table 5 summarizes the key parameters in the cost functions of each model and identifies the model commonalities which support a comparison of the points of view of the supplier and buyer.

Table 5. Connection between parameters of supplier’s and buyer’s cost functions

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Buyer</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( QH )</td>
<td>( C_o )</td>
<td>Cost of early delivery / overload</td>
</tr>
<tr>
<td>( K )</td>
<td>( C_s )</td>
<td>Cost of late delivery / stockout</td>
</tr>
<tr>
<td>( x )</td>
<td>( L )</td>
<td>Random variable: Delivery lead time (unit time)</td>
</tr>
<tr>
<td>( \Delta c )</td>
<td>( Q_{max} - Q )</td>
<td>Width of on-time portion of delivery window (unit time) / (units)</td>
</tr>
<tr>
<td>( c_{l} )</td>
<td>( R )</td>
<td>Decision variable: Beginning of on-time delivery (unit time) / Reorder point (units)</td>
</tr>
</tbody>
</table>

Both models (supplier’s and buyer’s) have three possible outcomes: early delivery (overload), on-time delivery (no overload and no stockouts), and late delivery (stockouts). These outcomes are related to the same random variable in both models: delivery lead time. This random variable is compared with a decision variable to define the outcomes.
For the $t$-model, $C'_O$ and $C'_S$ have the same dimensions as $QH$ and $K$ respectively. For the $ut$-model, $C'^{ut}_O$ and $C'^{ut}_S$ have the same dimensions as $QH/Q$ and $K/Q$ respectively. Thus knowing $Q$ which is not a decision variable in the delivery performance model $C'^{ut}_O$ and $C'^{ut}_S$ can be compared with $QH$ and $K$.

The decision variable in the delivery performance model (beginning of on-time delivery, $c_1$) is connected with the reorder point ($R$) decision variable in the inventory model. They have different dimensions, however the ratio $\frac{R - (Q_{\text{max}} - Q)}{D}$ which used in the inventory model has the same meaning as $c_1$ in the delivery performance model. The width of on-time portion of the delivery window ($\Delta c$) is measured in time units in the delivery performance model and in units of product in the inventory model ($Q_{\text{max}} - Q$). Through the conversion $\frac{Q_{\text{max}} - Q}{D}$ the common unit of time can be introduced into the inventory model thus enabling a direct comparison between the delivery and inventory models. For further discussion we define $R_b^*$ as the optimal reorder point for a buyer and $R_s^* = c_1^* D + Q_{\text{max}} - Q$ as the optimal reorder point for a supplier. A buyer would prefer to have $R_b^* = R_s^*$, because in this case a supplier choosing an optimal delivery will minimize buyer costs too. Differences between $R_b^*$ and $R_s^*$ are determined by differences in the delivery performance and inventory models.

There are several differences in the models. First, the delivery performance model determines the total cost per delivery while the inventory model finds the cost per unit time. For the inventory models the costs per delivery (cycle) is shown in equation (5-18)/(5-22). The optimal reorder points for both equations (5-18)/(5-22) and (5-19)/(5-23) are the same, thus the dimension of the inventory cost function does not affect the optimal solution. Hence, the
difference does not forbid comparing the optimal solutions of the delivery performance and inventory models. Second, the delivery performance model has one decision variable \( c_1 \), while classical inventory models assume two decision variables: \( R \) and \( Q \). The lot size \( Q \) does not affect the supplier’s decision about the time of delivery; therefore \( Q \) is not used as a decision variable in the inventory model herein. Thus, the difference might be eliminated to keep a connection with the delivery performance model. Third, the inventory model has two extra parameters which are not included in the delivery performance model: \( S \) and \( H \) which define ordering and inventory holding costs. \( S \) does not affect the optimal value of \( R \) which is clear from \((5-27)/(5-46)\).

The value of \( H \) defines the difference between the difference between the optimal values for the delivery performance and inventory models and, as a result, the difference between the supplier’s and buyer’s optimal decisions \( (R_s^* \text{ and } R_b^*) \). If \( H = 0 \), there are no differences between \( R_s^* \) and \( R_b^* \) and there is accordance between optimal \( R \) and \( c_1 \). The higher \( H \) the bigger impact of inventory holding cost on optimal value of \( R \) for a buyer and the bigger difference between supplier’s and buyer’s optimal decisions.

5.4. Buyer’s strategy for delivery performance optimization

From a supplier’s prospective, a delivery strategy is pretty simple. Assuming that the width of on-time portion of the delivery window and penalty costs per time unit early and late are defined by the contract, a buyer should choose a shipping time based on OPDW to minimize its costs. From a buyer’s prospective, a delivery strategy is more complicated. In the inventory model that defines a buyer’s point of view on supply chain delivery performance, the decision
variable is a reorder point which defines a shipping time. Unfortunately for a buyer, they cannot
directly force a supplier to ship at the time optimal for the buyer.

Assume the supplier choose a time when the product will be shipped based on the
concept of the optimal position of the delivery window. How it will affect the buyer’s costs?
What the buyer should do to optimize its own costs? The only way a buyer can influence a
supplier’s delivery decision is during the contract negotiation process. Assume the buyer has
enough power to choose any reasonable penalties for untimely delivery (\(QH\) and \(K\)). If a supplier
has more power than a buyer, then the supplier will ask for no penalties for untimely delivery
and will ship the product when it is appropriate from supplier’s production prospective. Thus, the
buyer should choose the values of \(QH\) and \(K\) so that the shipping time which optimal for the
supplier is optimal for the buyer too. How to choose these values? There are two questions that
should be answered.

The first question is how to choose the ratio \(QH/K\) which is optimal for the buyer? From
equation 2, it can be concluded that the ratio \(QH/K\) defines the optimal value of \(c_1\). Thus,
knowing the ration is enough to find the optimal shipping time for a supplier. Buyer’s strategy
should be the following:

1. Find an optimal reorder point for the buyer.
2. Based on the optimal reorder point calculate the ratio of probability of late to probability
   of early deliveries.
3. Use this ratio to define penalties (\(QH/K\) should be equal to the ratio).

The second question is how high the values of \(QH\) and \(K\) should be? If the values will be
too low, the possible penalties will be too low and will not force the supplier to delivery on-time.
If the values will be too high, it will affect the wiliness of the supplier to sign the contract. The
penalties should be reasonable and a buyer should explain why such high penalties should be applied to the buyer. Therefore, a buyer can penalize a supplier based on the buyer’s costs of overload and stockout. When an early delivery occurs the supplier is penalized for early delivery and the buyer incurs an additional cost to rent a warehouse. It is reasonable to assume that a buyer will ask a supplier to pay for the rented warehouse and per the model $QH = C'_o$ (t-model) or $QH = Q C'_o$ (ut-model). Late delivery will lead to stockouts for a buyer on one side and to penalties for late delivery for a supplier on another. In this case, a buyer can charge its supplier by the amount equal to its stockout costs, then the penalty per unit time is $K = C'_s$ (t-model) or $K = Q C'_s$ (ut-model). On-time delivery does not assume any penalties for a supplier and no additional costs (except inventory holding cost) for a buyer.

Assume a buyer wants its supplier to pay for overloads and stockouts choosing $QH = C'_o$ (or $QH = Q C'_o$) and $K = C'_s$ (or $K = Q C'_s$). The problem is that if a buyer choose $QH = C'_o$ (or $QH = Q C'_o$) and $K = C'_s$ (or $K = Q C'_s$), the supplier’s and buyer’s optimal decisions will not be the same ($R^*_s \neq R^*_b$) for any $H \neq 0$ and the buyer’s costs will not be minimized. Thus, the effect of holding cost on the buyer’s optimal reorder point should be counterbalanced by changing the penalty costs for early and/or late delivery. How the values $QH$ and $K$ of should be changed? To answer the question, the effect of holding cost and costs of stockout and overload on the optimal reorder point is investigated in the following propositions. The propositions are proved for the $t$-model, but it is not hard to prove it for $ut$-model too.

**Proposition 5-1.** For fixed parameters of delivery time distribution and fixed cost of stockout, holding cost, order quantity, and size of warehouse, increasing the cost of overload ($C'_o$) will decrease the optimal reorder point ($R^*$) (see Appendix C.3 for proof).
**Proposition 5-2.** For fixed parameters of delivery time distribution and fixed cost of overload, holding cost, order quantity, and size of warehouse, increasing cost of stockout ($C'_S$) will increase the optimal reorder point ($R^*$) (see Appendix C.4 for proof).

**Proposition 5-3.** For fixed parameters of delivery time distribution and fixed costs of overload and stockout, order quantity, and size of warehouse, increasing holding cost ($H$) will decrease the optimal reorder point ($R^*$) (see Appendix C.5 for proof).

From proposition 3, it can be concluded that the optimal reorder point for a buyer is less than the supplier’s optimal reorder point ($R_b^* \leq R_s^* = R_b^*(H = 0)$) assuming that $QH = C'_O$, $K = C'_S$, and $H > 0$. So the supplier’s optimal reorder point ($R_s^*$) should be decreased. It can be done in two ways:

- Setting the penalty for early delivery higher than the cost of overload ($QH > C'_O$);
- Setting the penalty for late delivery lower than the cost of stockout ($K < C'_S$).

If a buyer will choose to increase the penalty for early delivery comparing to the cost of overload ($QH > C'_O$), it means that the supplier will have to pay for overload more than it requires to cover rented warehouse costs. Thus, the penalties paid buy supplier will cover some inventory holding costs too. If a buyer will choose to set the penalty for late delivery lower than the cost of stockout ($K < C'_S$), the buyer will have to pay the difference in a case of stockout. It could be a combination when a buyer increases the penalty for early delivery and decreases the penalty for late delivery at the same time.
5.5. Numerical example

Delivery performance and inventory models were introduced as supplier’s and buyer’s points of view on supply chain delivery. We employ the following example to compare the optimal solutions for the delivery performance and inventory ($t$-model) models. The parameters for the inventory model (buyer’s point of view) are: $S = $7,000, $D = 50$ units/unit time, $C_S = $5,000/unit time, $C_O = $1,000/unit time; $Q_{\text{max}} = 2,000$ units, and $Q = 500$ units.

The related delivery performance model has the following parameters: $Q_H = C_O = $1,000/unit time, $K = C_S = $5,000/unit time, $\Delta c = (Q_{\text{max}} - Q)/D = (2,000 - 500)/50 = 30$ time units. The delivery lead time $L$ (or $x$) is normally distributed with $\mu = 300$, $\sigma = 40$.

The optimal solution for the delivery performance model is $c_1^* = 315.15$ time units, and $Y = $37,625 /cycle. The related reorder point and the total cost per unit time are

\[
R = (c_1 + \Delta c)D = (315.15 + 30)50 = 17,257.5, \quad (5-58)
\]

\[
TC = (Y + S)\frac{D}{Q} = (37,625 + 7,000)\frac{50}{500} = 4,462.5, \quad (5-59)
\]

which is solution for the inventory models with $H = 0$.

If the supplier minimizes its own costs choosing the optimal time for shipment, how much buyer will overpay? As it was stated, it depends on value of $H$. If $H = 0$, the optimal solution for the supplier is $c_1 = 315.15$ time units which leads to optimal solution for the buyer ($R = 17,257.5$). Increasing value of $H$ will decrease the optimal reorder point (Figure 9) and increase the percent of the holding cost in the total cost (Figure 10).
Figure 9. Optimal reorder point for different values of $H$

Figure 10. Holding cost as a percent of total cost for different values of $H$

Increasing $H$ will increase the difference between the optimal solution for the supplier (which is always $c_1 = 315.15$ or $R = 17,257.5$) and the optimal solution for the buyer. Hence, the shipment time chosen by the supplier will not minimize the buyer’s inventory costs. As a result the total cost based on the supplier’s optimal solution will be higher for the buyer than the optimal buyer’s solution (Figure 11).
Figure 11. Difference of total costs for $R_{opt}$ and $R = 17,257.5$

for different values of $H$

The example shows that if $H > 0$ a supplier’s optimal time of shipment differs from a buyer’s optimal time. Because it is a supplier’s decision when a product should be shipped, a buyer will have inventory holding costs higher than the optimal value. Although the highest difference for $H = 1$ (when holding costs are 78% or total) is $285.3$ or 1.33% of the optimal total cost. Thus a supplier’s shipment decision will not affect a buyer’s inventory costs significantly and the costs will be close to the optimal.
CHAPTER 6

SUMMARY AND FUTURE RESEARCH

This chapter presents a review of the dissertation. Section 6.1 provides a summary of the research contributions of the dissertation. Section 6.2 suggests directions for future research.

6.1. Summary of research contribution

A review of the literature on supply chain performance measurement in Chapter 1 identified the following three concerns among researchers: (1) the current set of models assume that the position of the delivery window is fixed and make no attempt to find the optimal position of the delivery window, (2) that the models define delivery time as a Gaussian random variable, and (3) the lack of evaluating the supply chain delivery performance from a buyer’s point of view. This dissertation addressed these concerns with respect to one aspect of supply chain management by modeling delivery performance to the end customer in a serial supply chain using a cost-based modeling approach.

Chapter 3 introduced the concept of the optimal position of the delivery window into a cost-based delivery performance model. It has been shown that the optimal position that minimizes the expected penalty cost exists and can be found from equation (3-9) for any continuously differentiable probability density function. If a delivery time distribution has a closed form cumulative distribution function, the closed form of the optimal position of the delivery window can be derived. This feature was demonstrated for four distributions (uniform, exponential, logistic, and asymmetric Laplace) which are representative of delivery distributions exhibiting symmetry, skewness and varying levels of kurtosis. For distributions which do not have closed form of cdf (for example, normal and gamma) the optimal position of the delivery
window cannot be derived in closed form. In this case, convex optimization algorithms can be
applied to solve the problem.

The position of the delivery window can be chosen by changing the earliest delivery time
with no cost, because a supplier can decide when the product will be shipped to its buyer. The
optimal position of the delivery window allows a supplier to minimize the expected penalty cost
for fixed parameters of the delivery time distribution, the width of the delivery window, and
penalty cost per unit early and late. The example in Section 3.4 demonstrates that optimizing the
delivery window may lead to a significant economical effect. Based on the data used in this
example, the company would decrease the expected penalty cost for untimely delivery by four
times by choosing the optimal position of the delivery window.

Coordination between suppliers and buyers has strategic implications to the operation of
a supply chain. The model for optimally positioning the delivery window presented herein can be
used as a decision tool to contribute to the coordination between suppliers and buyers in better
managing the delivery process of the supply chain. The first step a supplier can take using the
concept is to choose an optimal position of the delivery window which will minimize the
expected penalty cost of untimely (early and late) delivery. With this information the supplier
can assess their ability to meet the buyer’s expectation of delivery performance in terms of both
the timeliness of delivery and the financial impact of untimely delivery as measured by expected
penalty costs of early and late delivery. The model supports conducting this analysis prior to
committing to a replenishment contract and can be used to identify root causes inherent in the
current delivery process that are contributing to costs due to early and late deliveries.

This information generated from optimally positioning the delivery window can also be
used in planning delivery performance improvement. A common feature of the set of delivery
performance models summarized in Table 3 is that each model quantifies the expected cost of untimely delivery. A subset of these models addresses improvement in the delivery process through either the reduction of the variance of the delivery time distribution or in the evaluation process by which suppliers are selected. Optimal positioning the delivery window adds a robust feature to these delivery models since it advances the models from being purely descriptive (measuring the cost of untimely delivery based on an ad hoc delivery window position) to that of measuring cost-based delivery performance based on an optimally determined delivery window position. Optimally positioning the delivery window prior to committing resources to an improvement program for reducing the variance of delivery times helps to insure that these resources are being applied to their maximum effectiveness. Similarly, when using delivery performance as a key input into the supplier selection decision, optimal positioning the delivery window helps to insure that the delivery performance data generated from the delivery window based model reflects the outcomes of the delivery process under it best operating condition.

Chapter 4 investigates an effect of different parameters on the expected penalty cost function. The results allow developing strategies for improving delivery performance from a supplier’s perspective and answer the question what supplier should do to decrease the expected penalty cost. The research relies on the expected penalty cost function developed by Guiffrida and Nagi (2006a) and assumes that a supplier will choose the optimal position of delivery window proposed in Chapter 3.

The results presented in the paper can be used by both researchers and practitioners. The managerial implication of the paper is that it can serve as guidance for practitioners in delivery performance improvement. A supplier can use the knowledge about the effect of the width of on-time portion of the delivery window on the expected penalty cost during the negotiation process.
with a buyer. Understanding the effect of delivery time distribution parameters a supplier can develop a strategy for continuous improvement of delivery performance.

The models presented in the Chapter 5 coordinates the logistics and inventory management sub systems within a supply chain and represents a generalized modeling approach to the evaluation of supply chain delivery performance. An effect of the timeliness of delivery within a two-stage supply chain on both a supplier and a buyer is evaluated. From a supplier’s prospective, the effect of the timeliness of delivery can be described by supply chain delivery performance models. The buyer’s point of view on supply chain delivery is represented by inventory models with two levels of storage. In addition to the main contribution, the dissertation extends research in the area of stochastic models with two levels of inventory. The new models proposed in the Chapter 5 differ from existing models by cost dimensions. The optimal solution for the model is found.

The Chapter 5 explains how supply chain delivery performance and inventory models define a supplier’s and buyer’s points of view respectively and how different points of view affect managerial decisions. A supplier’s optimal decision is based on the optimal position of the delivery window developed in the Chapter 3. The OPDW concept minimizes a supplier’s costs for untimely delivery choosing an optimal shipping time. The shipping time chosen by a supplier might be nonoptimal for the buyer and the buyer should penalize the supplier such that the supplier’s optimal decision is optimal for the buyer too.

Managerial implications of the findings which include a buyer’s strategy for delivery performance optimization are discussed in Section 5.4. It is based on the assumption that the supplier utilizes an optimally positioned delivery window to determine a time when the product will be shipped. The dissertation suggests a buyer to penalize a supplier for untimely delivery
based on the costs of stockout and overload \((QH = C'_o\) or \(QH = Q C'_o\) and \(K = C'_s\) or \(K = Q C'_s\)). In this case, the penalties will be high enough to cover the buyer’s expenses associated with untimely delivery. If the penalties are matched with the costs, the supplier’s optimal shipping time is optimal for the buyer only if there is no inventory holding cost. The buyer can force the supplier to choose a shipping time that will be optimal for the buyer setting the penalty for early delivery higher than the cost of overload \((QH > C'_o\) or \(QH > Q C'_o\)) and/or setting the penalty for late delivery lower than the cost of stockout \((K < C'_s\) or \(K < Q C'_s\)).

6.2. Directions for future research

There are several aspects of this research that could be expanded.

Firstly, several case studies with additional real world examples across a set of differing industries should be conducted to extend the scope of the research. This would require an industry sponsor willing to provide temporal data on delivery performance so as to be able to capture the improvement in delivery performance resulting from applying the optimal position of the delivery window and changing delivery time distribution parameters.

Secondly, the research can be extended in several directions. Two stage supply chain can be extended to 3-stage and n-stage supply chains. More than one supplier and/or one buyer can be included in the system. The model can include more than one product. The model could be advanced by allowing the costs associated with early and late delivery to be represented stochastically. Chapter 4 can serve as a basis for modeling continuous delivery performance improvement. In Chapter 5, a buyer can have other dimensions of stockout and overload costs and/or reorder policies. Research links the delivery performance model and inventory models with different cost dimensions and/or reorder policies should be done.
Lastly, the models presented in the dissertation can be combined or integrated with other OR/OM models. The possible OR/OM areas include production planning and warehouse location problems.
Appendix A. Optimal position of delivery window

A.1. Uniform distribution

The optimal position of the delivery window for uniform random variable will be:

\[
\frac{\int_{c_1}^{b} (b-a)^{-1} \, dx}{\int_{a}^{c_1} (b-a)^{-1} \, dx} = \frac{QH}{K};
\]
(A.1-1)

\[
\frac{(b-(c_1+\Delta c))(b-a)^{-1}}{(c_1-a)(b-a)^{-1}} = \frac{QH}{K};
\]
(A.1-2)

\[
\frac{b-(c_1+\Delta c)}{c_1-a} = \frac{QH}{K};
\]
(A.1-3)

\[
[b-(c_1+\Delta c)]K = QH (c_1-a);
\]
(A.1-4)

\[
K \cdot b - K \cdot c_1 - K \cdot \Delta c = QH \cdot c_1 - QH \cdot a;
\]
(A.1-5)

\[
K \cdot b - K \cdot c_1 - K \cdot \Delta c = QH \cdot c_1 - QH \cdot a;
\]
(A.1-6)

\[
K \cdot b - K \cdot \Delta c + QH \cdot a = QH \cdot c_1 + K \cdot c_1;
\]
(A.1-7)

\[
K \cdot (b-\Delta c) + QH \cdot a = (QH + K) \cdot c_1;
\]
(A.1-8)

\[
c_1^* = \frac{K \cdot (b-\Delta c) + QH \cdot a}{QH + K}.
\]
(A.1-9)

A.2. Exponential distribution

The optimal position of the delivery window for exponential random variable will be:

\[
\frac{\int_{c_1}^{\infty} \beta^{-1} \exp(-x/\beta) \, dx}{\int_{0}^{c_1} \beta^{-1} \exp(-x/\beta) \, dx} = \frac{QH}{K};
\]
(A.2-1)
\[
\frac{1 - (1 - \exp(-(c_1 + \Delta c)/\beta))}{1 - \exp(-c_1/\beta)} = \frac{QH}{K};
\]  
(A.2-2)

\[
\exp(- (c_1 + \Delta c)/\beta) = \frac{QH}{K}(1 - \exp(-c_1/\beta));
\]  
(A.2-3)

\[
\exp(-c_1/\beta - \Delta c/\beta) + \frac{QH}{K}\exp(-c_1/\beta) = \frac{QH}{K};
\]  
(A.2-4)

\[
\exp(-c_1/\beta)\left(\exp(-\Delta c/\beta) + \frac{QH}{K}\right) = \frac{QH}{K};
\]  
(A.2-5)

\[
\exp(-c_1/\beta) = \frac{QH \cdot K^{-1}}{\exp(-\Delta c/\beta) + QH \cdot K^{-1}};
\]  
(A.2-6)

\[
\exp(c_1/\beta) = \frac{\exp(-\Delta c/\beta)}{QH \cdot K^{-1}} + 1;
\]  
(A.2-7)

\[
\frac{c_1}{\beta} = \ln\left[\frac{K \cdot \exp(-\Delta c/\beta)}{QH} + 1\right];
\]  
(A.2-8)

\[
c_1^* = \beta \cdot \ln\left[\frac{K \cdot \exp(-\Delta c/\beta)}{QH} + 1\right].
\]  
(A.2-9)

A.3. Logistic distribution

The optimal position of the delivery window for logistic random variable will be

\[
\int_{c_1 + \Delta c}^{\infty} \exp(-(x - \alpha)/\beta) \beta^{-1}(1 + \exp(-(x - \alpha)/\beta))^2 dx
\]  
\[
\int_{-\infty}^{c_1 + \Delta c} \exp(-(x - \alpha)/\beta) \beta^{-1}(1 + \exp(-(x - \alpha)/\beta))^2 dx
\]  
= \frac{QH}{K};
\]  
(A.3-1)

\[
1 - \left(1 - \left(1 + \exp((c_1 + \Delta c - \alpha)/\betai)^{-1}\right)\right) = \frac{QH}{K};
\]  
(A.3-2)

\[
\frac{1 + \exp((c_1 + \Delta c - \alpha)/\betai)^{-1}}{1 - \left(1 + \exp((c_1 - \alpha)/\betai)^{-1}\right)} = \frac{QH}{K};
\]  
(A.3-3)
\[
\frac{1 + \exp((c_1 - \alpha) / \beta)}{(1 + \exp((c_1 + \Delta c - \alpha) / \beta))(1 + \exp((c_1 - \alpha) / \beta) - 1)} = \frac{QH}{K} ; \quad (A.3-4)
\]

\[
1 + \exp((c_1 - \alpha) / \beta) = \frac{QH}{K} \left( \exp((c_1 - \alpha) / \beta) + \exp((2c_1 + \Delta c - 2\alpha) / \beta) \right) ; \quad (A.3-5)
\]

\[
\frac{QH}{K} e^{\frac{\beta}{\alpha}} \left( \exp((c_1 - \alpha) / \beta) \right)^2 + \left( \frac{QH}{K} \right)^{-1} \exp((c_1 - \alpha) / \beta) - 1 = 0 . \quad (A.3-6)
\]

Replace \( \exp((c_1 - \alpha) / \beta) = y \), then

\[
QH \cdot K^{-1} \exp(\Delta c / \beta) y^2 + \left( QH \cdot K^{-1} - 1 \right)y - 1 = 0 . \quad (A.3-7)
\]

\[
y_1 = -\frac{\left( QH \cdot K^{-1} - 1 \right) + \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 - 4QH \cdot K^{-1} \exp(\Delta c / \beta)(-1)}}{2QH \cdot K^{-1} \exp(\Delta c / \beta)} ; \quad (A.3-8)
\]

\[
y_2 = -\frac{\left( QH \cdot K^{-1} - 1 \right) - \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 - 4QH \cdot K^{-1} \exp(\Delta c / \beta)(-1)}}{2QH \cdot K^{-1} \exp(\Delta c / \beta)} . \quad (A.3-9)
\]

It is important to mention that \( y_2 \leq 0 \). It means that \( \ln(y_2) \) does not exist and we cannot use \( y_2 \) in farther accounts.

Using that \( \frac{c_1 - \alpha}{\beta} = \ln y \), or \( c_1 = \alpha + \beta \cdot \ln y \), we have

\[
c_1^* = \alpha + \beta \cdot \ln \left[ -\frac{\left( QH \cdot K^{-1} - 1 \right) + \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 + 4QH \cdot K^{-1} \exp(\Delta c / \beta)}}{2QH \cdot K^{-1} \exp(\Delta c / \beta)} \right] . \quad (A.3-10)
\]

A.4. Asymmetric Laplace distribution

Case 1. \( c_1 < \alpha < c_1 + \Delta c \) or \( \alpha - \Delta c < c_1 < \alpha \):

Using that \( P_{late} = \int_{c_1 + \Delta c}^{b} f(x)dx = 1 - F(c_1 + \Delta c) \), we have:
\[
\frac{1 - \left(1 - \left(1 + k^2 \right)^{-1} \exp\left(-\left(c_i + \Delta c - \alpha\right)k\right)\right)}{k^2 \left(1 + k^2 \right)^{-1} \exp\left((c_i - \alpha)k\right)} = \frac{QH}{K}; \quad (A.4-1)
\]

\[
\frac{\exp\left(-\left(c_i + \Delta c - \alpha\right)k\right)}{k^2 \exp\left((c_i - \alpha)k\right)} = \frac{QH}{K}; \quad (A.4-2)
\]

\[
\exp\left(-kc_i - k\Delta c + k\alpha - c_i/k + \alpha/k\right) = k^2 \frac{QH}{K}; \quad (A.4-3)
\]

\[
-c_i(k + k^{-1}) - k\Delta c + k\alpha = \beta \ln\left(k^2 \frac{QH}{K}\right); \quad (A.4-4)
\]

\[
-c_i(k + k^{-1}) = k\Delta c - \alpha(k + k^{-1}) + \beta \ln\left(k^2 \frac{QH}{K}\right); \quad (A.4-5)
\]

\[
c_i = \alpha - \Delta c - \frac{k^2}{k^2 + 1} \frac{\beta k}{k^2 + 1} \ln\left(k^2 \frac{QH}{K}\right). \quad (A.4-6)
\]

**Case 2.** \(c_i < c_i + \Delta c < \alpha\) or \(c_i < \alpha - \Delta c < \alpha\):

In this case, the optimal position of the delivery window is:

\[
1 - k^2 \left(1 + k^2 \right)^{-1} \exp\left((c_i + \Delta c - \alpha)k\right) = \frac{QH}{K}; \quad (A.4-7)
\]

\[
\frac{1 + k^2}{k^2} \exp\left(-\frac{c_i - \alpha}{\beta k}\right) - \exp\left(\frac{c_i + \Delta c - \alpha - (c_i - \alpha)}{\beta k}\right) = \frac{QH}{K}; \quad (A.4-8)
\]

\[
\frac{1 + k^2}{k^2} \exp\left(-\frac{c_i - \alpha}{\beta k}\right) - \exp\left(\frac{\Delta c}{\beta k}\right) = \frac{QH}{K}; \quad (A.4-9)
\]

\[
\exp\left(-\frac{c_i - \alpha}{\beta k}\right) = \frac{k^2}{1 + k^2} \left[\frac{QH}{K} + \exp\left(\frac{\Delta c}{\beta k}\right)\right]; \quad (A.4-10)
\]

\[
-\frac{c_i - \alpha}{\beta k} = \ln\left(\frac{k^2}{1 + k^2} \left[\frac{QH}{K} + \exp\left(\frac{\Delta c}{\beta k}\right)\right]\right); \quad (A.4-11)
\]

101
\[ c_i = \alpha - \beta k \ln \left( \frac{k^2}{1 + k^2} \left[ \frac{QH}{K} + \exp \left( \frac{\Delta c}{\beta k} \right) \right] \right). \] (A.4-12)

**Case 3.** \( \alpha < c_i < c_i + \Delta c \) or \( \alpha - \Delta c < \alpha < c_i \):

The optimal position of the delivery window is:

\[ 1 - \left( 1 - (1 + k^2)^{-1} \exp \left( -c_i + \Delta c - \alpha \right) k(\beta)^{-1} \right) = \frac{QH}{K}; \] (A.4-13)

\[ (1 + k^2)^{-1} \exp \left( -k \frac{c_i + \Delta c - \alpha}{\beta} \right) = \frac{QH}{K} \left[ 1 - (1 + k^2)^{-1} \exp \left( -k \frac{c_i - \alpha}{\beta} \right) \right]; \] (A.4-14)

\[ (1 + k^2)^{-1} \exp \left( -k \frac{c_i - \alpha}{\beta} \right) = \exp \left( -k \frac{\Delta c}{\beta} \right) + \frac{QH}{K} = \frac{QH}{K}; \] (A.4-15)

\[ \exp \left( -k \frac{c_i - \alpha}{\beta} \right) = (1 + k^2) \frac{QH}{K} \left[ \exp \left( -k \frac{\Delta c}{\beta} \right) + \frac{QH}{K} \right]^{-1}; \] (A.4-16)

\[ \exp \left( -k \frac{c_i - \alpha}{\beta} \right) = (1 + k^2) \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) \left[ 1 + \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) \right]^{-1}; \] (A.4-17)

\[ -k \frac{c_i - \alpha}{\beta} = \ln \left( 1 + k^2 \right) \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) - \ln \left[ 1 + \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) \right]; \] (A.4-18)

\[ c_i = \alpha - \frac{\beta}{k} \ln \left( 1 + k^2 \right) \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) + \frac{\beta}{k} \ln \left[ 1 + \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) \right]; \] (A.4-19)

\[ c_i = \alpha - \Delta c - \frac{\beta}{k} \ln \left( 1 + k^2 \right) \frac{QH}{K} + \frac{\beta}{k} \ln \left[ 1 + \frac{QH}{K} \exp \left( k \frac{\Delta c}{\beta} \right) \right], \] (A.4-20)
Appendix B. Expected penalty cost for untimely delivery

B.1. Uniform distribution

Expected penalty cost for untimely delivery:

\[
Y = QH \int_a^{c_i^*} (c_i^* - x) \frac{1}{b-a} \, dx + K \int_{c_i^* + \Delta c}^b (x - (c_i^* + \Delta c)) \frac{1}{b-a} \, dx; \quad (B.1-1)
\]

\[
Y = QH \frac{1}{b-a} \left[ c_i^* \left( \frac{x^2}{2} \right) \right]_a^{c_i^*} + K \frac{1}{b-a} \left[ \frac{x^2}{2} - (c_i^* + \Delta c)x \right]_{c_i^* + \Delta c}^b; \quad (B.1-2)
\]

\[
Y = QH \frac{1}{b-a} \left( c_i^2 - \frac{c_i^2}{2} - c_i^* a + \frac{a^2}{2} \right) + K \frac{1}{b-a} \left( \frac{b^2}{2} - (c_i^* + \Delta c)b - \frac{(c_i^* + \Delta c)^2}{2} + (c_i^* + \Delta c)^2 \right); \quad (B.1-3)
\]

\[
Y = QH \frac{1}{b-a} \left( c_i^2 - c_i^* a + \frac{a^2}{2} \right) + K \frac{1}{b-a} \left( \frac{b^2}{2} - (c_i^* + \Delta c)b + \frac{(c_i^* + \Delta c)^3}{2} \right); \quad (B.1-4)
\]

\[
Y = \frac{1}{2(b-a)} \left[ QH (c_i^* - a)^2 + K ((c_i^* + \Delta c) - b)^2 \right]. \quad (B.1-5)
\]

For uniform distributed delivery time random variable the delivery window has the optimal position if

\[
c_i^* = \frac{K \cdot (b - \Delta c) + QH \cdot a}{QH + K}. \quad (B.1-6)
\]

Substituting \( c_i^* \) we have

\[
Y = \frac{1}{2(b-a)} \left[ QH \left( \frac{K \cdot (b - \Delta c) + QH \cdot a}{QH + K} - a \right)^2 + K \left( \frac{K \cdot (b - \Delta c) + QH \cdot a}{QH + K} + \Delta c - b \right)^2 \right]; \quad (B.1-7)
\]

\[
Y = \frac{QH}{2(b-a)} \left( \frac{K \cdot (b - \Delta c) + QH \cdot a - a(QH + K)}{QH + K} \right)^2 +
\]

\[
+ \frac{K}{2(b-a)} \left( \frac{K \cdot (b - \Delta c) + QH \cdot a + \Delta c(QH + K) - b(QH + K)}{QH + K} \right)^2; \quad (B.1-8)
\]
\[ Y = \frac{QH}{2(b-a)} \left( \frac{K \cdot (b - \Delta c) - K \cdot a}{QH + K} \right)^2 + \frac{K}{2(b-a)} \left( \frac{QH \cdot a + QH \cdot \Delta c - QH \cdot b}{QH + K} \right)^2; \quad (B.1-9) \]

\[ Y = \frac{QH \cdot K}{2(b-a)(QH + K)^2} \left[ K(b - a - \Delta c)^2 + QH(a - b + \Delta c)^2 \right]; \quad (B.1-10) \]

Knowing that \((x - y)^2 = x^2 - 2xy + y^2 = (y - x)^2\), we have

\[ Y = \frac{QH \cdot K}{2(b-a)(QH + K)^2} \left[ K(b - a - \Delta c)^2 + QH(b - a - \Delta c)^2 \right]; \quad (B.1-11) \]

\[ Y = \frac{QH \cdot K}{2(b-a)(QH + K)} \left[ (b - a) - \Delta c \right]^2. \quad (B.1-12) \]

**B.2. Exponential distribution**

For exponential distribution expected penalty cost for untimely delivery:

\[ Y = QH \int_0^{c_i^*} (c_i^* - x) \frac{1}{\beta} e^{-\frac{x}{\beta}} dx + K \int_0^{c_i^*} (x - (c_i^* + \Delta c)) \frac{1}{\beta} e^{-\frac{x}{\beta}} dx; \quad (B.2-1) \]

\[ Y = QH \int_0^{c_i^*} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx - QH \int_0^{c_i^*} e^{-\frac{x}{\beta}} dx + K \int_0^{c_i^*} x \frac{1}{\beta} e^{-\frac{x}{\beta}} dx - K \int_0^{c_i^*} (c_i^* + \Delta c) \frac{1}{\beta} e^{-\frac{x}{\beta}} dx; \quad (B.2-2) \]

\[ Y = -c_i^* QH e^{-\frac{x}{\beta}} \bigg|_0^{c_i^*} + QH \cdot e^{-\frac{x}{\beta}} (\beta + x) \bigg|_0^{c_i^*} - Ke^{-\frac{x}{\beta}} (\beta + x) \bigg|_0^{c_i^*} + K(c_i^* + \Delta c) e^{-\frac{x}{\beta}} \bigg|_0^{c_i^*}; \quad (B.2-3) \]

Using that \( \lim_{x \to \infty} x \cdot e^{-\frac{x}{\beta}} = 0 \), we have

\[ Y = -c_i^* QH \cdot e^{-\frac{x}{\beta}} + c_i^* QH + QH \cdot e^{-\frac{x}{\beta}} (\beta + c_i^*) - QH(\beta + 0) - \\
-0 + Ke^{-\frac{x}{\beta}} (\beta + c_i^* + \Delta c) + 0 - K(c_i^* + \Delta c) e^{-\frac{x}{\beta}} \bigg|_0^{c_i^*}; \quad (B.2-4) \]

\[ Y = c_i^* \cdot QH + QH \cdot \beta \cdot e^{-\frac{x}{\beta}} - QH \cdot \beta + K \cdot \beta \cdot e^{-\frac{x}{\beta}} \bigg|_0^{c_i^*}; \quad (B.2-5) \]
\[ Y = QH \cdot (c_1^* - \beta) + \beta \cdot e^{\frac{-c_1^*}{\beta}} \left( QH + K \cdot e^{\frac{-\Delta c}{\beta}} \right) \]  
(B.2-6)

For exponential distributed delivery time random variable the optimal position of the delivery window is

\[ c_1^* = \beta \cdot \ln \left( \frac{K \cdot e^{\frac{-\Delta c}{\beta}}}{QH} + 1 \right) \]  
(B.2-7)

Using that

\[ e^{\frac{-c_1^*}{\beta}} = \exp \left\{ -\frac{1}{\beta} \beta \cdot \ln \left( \frac{K \cdot e^{\frac{-\Delta c}{\beta}}}{QH} + 1 \right) \right\} = \frac{1}{K \cdot (QH)^{-1} \cdot e^{\frac{\Delta c}{\beta}} + 1} = \frac{QH}{K \cdot e^{\frac{\Delta c}{\beta}} + QH} \]  
(B.2-8)

Thus we have

\[ Y = QH \cdot (c_1^* - \beta) + \beta \cdot \frac{QH}{K \cdot e^{\frac{-\Delta c}{\beta}} + QH} \left( QH + K \cdot e^{\frac{-\Delta c}{\beta}} \right) = QH \cdot (c_1^* - \beta) + \beta \cdot QH \]  
(B.2-9)

\[ Y = QH \cdot c_1^* \]  
(B.2-10)

Substituting \( c_1^* \) the expected penalty cost for untimely delivery

\[ Y = QH \cdot \beta \cdot \ln \left( \frac{K \cdot e^{\frac{-\Delta c}{\beta}}}{QH} + 1 \right) \]  
(B.2-11)

B.3. Logistic distribution

For logistic distribution expected penalty cost for untimely delivery:
\[ Y = QH \int_{-\infty}^{c_i^*} (c_i^* - x) e^{-x/\beta} \left(1 + e^{-x/\beta}\right)^{-\alpha} \, dx + K \int_{c_i^* + \Delta c}^{\infty} (x - (c_i^* + \Delta c)) e^{-x/\beta} \left(1 + e^{-x/\beta}\right)^{-\alpha} \, dx; \quad (B.3-1) \]

Using that
\[
\int e^{-x/\beta} \left(1 + e^{-x/\beta}\right)^{-\alpha} \, dx = \int e^{-x/\beta} \left(1 + e^{-x/\beta}\right)^{-\alpha} d\left(-\frac{x - \alpha}{\beta}\right) = -\int \frac{1}{e^{-x/\beta}} \left(1 + e^{-x/\beta}\right)^{-\alpha} d\left(e^{-x/\beta}\right) =
\]
\[
= \frac{1}{1 + e^{-x/\beta}} \tag{B.3-2}
\]

And using the previous result
\[
\int x e^{-x/\beta} \left(1 + e^{-x/\beta}\right)^{-\alpha} \, dx = \int x d \left(\frac{1}{e^{-x/\beta}} \left(1 + e^{-x/\beta}\right)^{-\alpha}\right) = \frac{x}{1 + e^{-x/\beta}} - \int \frac{1}{1 + e^{-x/\beta}} \, dx
\]
\[
= \frac{x}{1 + e^{-x/\beta}} \tag{B.3-3}
\]

The subtrahend is
\[
\int \frac{1}{e^{-x/\beta}} \, dx = \left\{ u = e^{-x/\beta}; du = -\beta e^{-x/\beta} \, dx; \frac{du}{u} = dx \right\} = -\beta \int \frac{1}{u(u+1)} \, du =
\]
\[
= -\beta \int \frac{1}{u+1} \, du = -\beta \int \left(\frac{1}{u} - \frac{1}{u+1}\right) \, du = -\beta \int \frac{1}{u} \, du + \beta \int \frac{1}{u+1} \, du =
\]
\[
= -\beta \ln u + \beta \ln(u+1) = \beta \left(\ln \left(e^{-x/\beta} + 1\right) - \ln \left(e^{-x/\beta}\right)\right) =
\]
\[
= \beta \left(\ln \left[\frac{e^{-x/\beta} + 1}{e^{-x/\beta}}\right]\right) = \beta \ln \left[ e^{-x/\beta} + 1 \right] \tag{B.3-4}
\]

Thus
\[
\int x \frac{e^{-\frac{x-a}{\beta}}}{\beta \left(1 + e^{-\frac{x-a}{\beta}}\right)} \, dx = \frac{x}{1 + e^{-\frac{x-a}{\beta}}} - \beta \ln \left(\frac{e^{-\frac{x-a}{\beta}} + 1}{1 + e^{-\frac{x-a}{\beta}}}\right)
\]

(B.3-5)

Expected penalty cost for untimely delivery:

\[
Y = QH \left[\frac{c_1^*}{x-a} - \frac{x}{x-a} + \beta \ln \left(\frac{x-a}{\beta} + 1\right)\right]_{x=\infty} +
\]

\[
+ K \left[\frac{x}{x-a} - \beta \ln \left(\frac{x-a}{\beta} + 1\right) - \frac{c_1^* + \Delta c}{x-a} \right]_{x=\infty};
\]

(B.3-6)

\[
Y = QH \left[\frac{c_1^*}{x-a} - \frac{x}{x-a} + \beta \ln \left(\frac{x-a}{\beta} + 1\right)\right]_{x=\infty} +
\]

\[
- \lim_{x \to \infty} \left(\frac{x}{x-a} - \frac{x}{x-a} + \beta \ln \left(\frac{x-a}{\beta} + 1\right)\right) +
\]

\[
+ K \left[\frac{x}{x-a} - \beta \ln \left(\frac{x-a}{\beta} + 1\right) - \frac{c_1^* + \Delta c}{x-a} \right]_{x=\infty};
\]

(B.3-7)

Let us take a look at the first limit:

\[
\lim_{x \to \infty} \left(\frac{x}{x-a} - \frac{x}{x-a} + \beta \ln \left(\frac{x-a}{\beta} + 1\right)\right) = 0 - 0 + \beta \ln(0+1) = 0
\]

(B.3-8)

The second limit is:
\[
\lim_{x \to 0^+} \frac{x}{1 + e^{\alpha - x}} = \beta \ln \left( e^{\beta} + 1 \right) - \frac{c_i^*}{1 + e^{\alpha - x}} = \lim_{x \to 0^+} \left( \frac{x}{1 + 0} - \beta \ln \left( e^{\beta} + 1 \right) - \frac{c_i^*}{1 + 0} \right) = \lim_{x \to 0} \left( x - \beta \frac{x - \alpha}{\beta} - (c_i^* + \Delta c) \right) = \lim_{x \to 0} \left( x + x - \alpha - (c_i^* + \Delta c) \right) = \alpha - (c_i^* + \Delta c). \tag{B.3-9}
\]

Thus expected penalty cost for untimely delivery:

\[
Y = QH \left[ \frac{c_i^*}{1 + e^{\alpha - x}} - \frac{c_i^*}{1 + e^{\alpha - x}} + \beta \ln \left( e^{\beta} + 1 \right) \right] + K \left[ \alpha - (c_i^* + \Delta c) = \frac{c_i^* + \Delta c - x}{1 + e^{\alpha - x}} \right] + \beta \ln \left( e^{\beta} + 1 \right) \left[ \frac{c_i^* + \Delta c - x}{1 + e^{\alpha - x}} \right] + \frac{c_i^* + \Delta c}{1 + e^{\alpha - x}} \right]. \tag{B.3-10}
\]

\[
Y = QH \cdot \beta \ln \left( e^{\beta} + 1 \right) + K \left[ \alpha - (c_i^* + \Delta c) + \beta \ln \left( e^{\beta} + 1 \right) \right]. \tag{B.3-11}
\]

The optimal position of the delivery window is

\[
c_i^* = \alpha + \beta \ln \left[ \frac{-\left( QH \cdot K^{-1} - 1 \right) + \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 + 4QH \cdot K^{-1} e^{\Delta c}}}{2QH \cdot K^{-1} e^{\beta}} \right]. \tag{B.3-12}
\]

Using the equation for the optimal position of the delivery window, we have

\[
\frac{c_i^* - \alpha}{e^{\beta}} = \exp \left( \alpha + \beta \ln \left[ \frac{-\left( QH \cdot K^{-1} - 1 \right) + \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 + 4QH \cdot K^{-1} e^{\Delta c}}}{2QH \cdot K^{-1} e^{\beta}} \right] - \alpha \right) \beta^{-1} = \frac{-QH \cdot K^{-1} + 1 + \sqrt{\left( QH \cdot K^{-1} - 1 \right)^2 + 4QH \cdot K^{-1} e^{\Delta c}}}{2QH \cdot K^{-1} e^{\beta}}. \tag{B.3-13}
\]

Cost of early delivery:

\[
\]
\[ Y_{\text{early}} = QH \cdot \beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} + \Delta c \]. \quad (B.3-14)

Cost of late delivery:

\[ Y_{\text{late}} = K \left\{ \alpha - (\alpha + \beta \ln \left\{ \frac{-QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} + \Delta c \right\} + \]

\[ + K \beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} + \Delta c \]; \quad (B.3-15)

\[ Y_{\text{late}} = K \left\{ -\beta \ln \left\{ \frac{-QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1}} \right\} + \beta \ln e^{\beta} - \Delta c \right\} + \]

\[ + K \beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} ; \quad (B.3-16) \]

\[ Y_{\text{late}} = -K \beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} + \]

\[ + K \beta \ln \left\{ \frac{QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1}} \right\} ; \quad (B.3-17) \]

\[ Y_{\text{late}} = K \beta \ln \left\{ \frac{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} + \]

\[ + K \beta \ln \left\{ \frac{QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} e^{\beta}}}{2QH \cdot K^{-1} e^{\beta}} \right\} ; \quad (B.3-18) \]
$$Y_{late} = K\beta \cdot \ln \left( \frac{QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}}{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}} \right),$$  \hspace{1cm} \text{(B.3-19)}$$

$$Y_{late} = K\beta \cdot \ln \left( \frac{QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}}{-QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}} \right).$$  \hspace{1cm} \text{(B.3-20)}$$

Multiplying both numerator and denominator of the logarithm by

$$QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}$$

we have the numerator of the logarithm is

$$\left( QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} \right) \times$$

$$\times \left( QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} \right) =$$

$$\left( QH \cdot K^{-1} + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} \right)^2 - 1 =$$

$$= (QH \cdot K^{-1})^2 + 2QH \cdot K^{-1} \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} +$$

$$+ (QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} - 1 =$$

$$= (QH \cdot K^{-1})^2 + 2QH \cdot K^{-1} \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} +$$

$$+ (QH \cdot K^{-1})^2 - 2QH \cdot K^{-1} + 1 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} - 1 =$$

$$= 2(QH \cdot K^{-1})^2 + 2QH \cdot K^{-1} \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}} -$$

$$- 2QH \cdot K^{-1} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} =$$
\[ 2QH \cdot K^{-1} \left[ QH \cdot K^{-1} + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} + 2e^{\frac{\Delta c}{\beta}} - 1 \right] \]

(B.3-21)

The denominator of the logarithm is

\[ 1 + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} \times \]

\[ QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} = \]

\[ = -\left(QH \cdot K^{-1} - 1\right)^2 + \left(QH \cdot K^{-1} - 1\right)^2 + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} = 4QH \cdot K^{-1} \frac{\Delta c}{\beta}. \]

(B.3-22)

Thus the cost of late delivery is

\[ Y_{late} = K\beta \cdot \ln \left( \frac{2QH \cdot K^{-1} \left[ QH \cdot K^{-1} + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} + 2e^{\frac{\Delta c}{\beta}} - 1 \right]}{4QH \cdot K^{-1} \frac{\Delta c}{\beta}} \right); \]

(B.3-23)

\[ Y_{late} = K\beta \cdot \ln \left( \frac{QH \cdot K^{-1} + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta} + 2e^{\frac{\Delta c}{\beta}} - 1}{2e^{\frac{\Delta c}{\beta}}} \right); \]

(B.3-24)

\[ Y_{late} = K\beta \cdot \ln \left( \frac{QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2} + 4QH \cdot K^{-1} \frac{\Delta c}{\beta}}{2e^{\frac{\Delta c}{\beta}}} + 1 \right). \]

(B.3-25)

The expected penalty cost for untimely delivery
\[ Y = QH\beta \ln \left( -\frac{QH \cdot K^{-1} + 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1}e^{\Delta c}}} {2QH \cdot K^{-1}e^{\Delta c}} \right) + \\
+ K\beta \ln \left( \frac{QH \cdot K^{-1} - 1 + \sqrt{(QH \cdot K^{-1} - 1)^2 + 4QH \cdot K^{-1}e^{\Delta c}}} {2e^{\Delta c}} \right) + 1. \] 

(B.3-26)

**B.4. Laplace distribution**

Cost of early delivery:

\[ Y_{\text{early}} = QH \int_{-\infty}^{c_1^*} (c_1^* - x) \frac{1}{2\beta} e^{\frac{|x-s|}{\beta}} dx \]  

(B.4-1)

Cost of early delivery:

\[ Y_{\text{late}} = K \int_{c_1^* + \Delta c}^{\infty} (x - (c_1^* + \Delta c)) \frac{1}{2\beta} e^{\frac{|x-s|}{\beta}} dx \]  

(B.4-2)

Let’s take a look at cost of early delivery. There are two cases: \( c_1^* < \alpha \) and \( c_1^* \geq \alpha \).

For \( c_1^* < \alpha \) the expected penalty cost for early delivery is

\[ Y_{\text{early}} = QH \int_{-\infty}^{c_1^*} (c_1^* - x) \frac{1}{2\beta} e^{\frac{\alpha-x}{\beta}} dx = QH \cdot c_1^* \int_{-\infty}^{c_1^*} \frac{1}{2\beta} e^{\frac{\alpha-x}{\beta}} dx - QH \int_{-\infty}^{c_1^*} x \frac{1}{2\beta} e^{\frac{\alpha-x}{\beta}} dx = \\
= QH \frac{c_1^*}{2} e^{\frac{\alpha-x}{\beta}} \bigg|_{-\infty}^{c_1^*} + QH \cdot \frac{1}{2} e^{\frac{\alpha-x}{\beta}} (\beta - x) \bigg|_{-\infty}^{c_1^*} \]  

(B.4-3)

Using that \( \lim_{x \to \infty} x \cdot e^{-\frac{x}{\beta}} = 0 \), we have

\[ Y_{\text{early}} = QH \frac{c_1^*}{2} e^{\frac{\alpha-c_1^*}{\beta}} - 0 + QH \cdot \frac{1}{2} e^{\frac{\alpha-c_1^*}{\beta}} (\beta - c_1^*) - 0 = QH \cdot \frac{\beta}{2} e^{\frac{\alpha-c_1^*}{\beta}}. \]  

(B.4-4)
For $c_i^* \geq \alpha$ the curve of the pdf has two parts: $(-\infty, \alpha)$ and $[\alpha, c_i^*]$. In this case, the expected penalty cost for early delivery is

$$Y_{early} = QH \int_{-\infty}^{c_i^*} \frac{1}{2\beta} e^{\frac{x-c_i^*}{\beta}} \, dx + QH \int_{c_i^*}^{\infty} \frac{1}{2\beta} e^{\frac{x-c_i^*}{\beta}} \, dx =$$

$$= QH \left[ \frac{c_i^*}{2} e^{\frac{\alpha-c_i^*}{\beta}} \int_{-\infty}^{\alpha} + \frac{1}{2} e^{\frac{\alpha-x}{\beta}} (\beta-x) \right] + QH \left[ -\frac{c_i^*}{2} e^{\frac{\alpha-x}{\beta}} \int_{\alpha}^{\infty} + \frac{1}{2} e^{\frac{\alpha-x}{\beta}} (\beta+x) \right] =$$

$$= QH \left[ \frac{c_i^*}{2} e^{\frac{\alpha-c_i^*}{\beta}} - 0 + \frac{1}{2} e^{\frac{\alpha-c_i^*}{\beta}} (\beta - \alpha) - 0 \right] +$$

$$+ QH \left[ -\frac{c_i^*}{2} e^{\frac{\alpha-c_i^*}{\beta}} + \frac{c_i^*}{2} e^{\frac{\alpha-c_i^*}{\beta}} + \frac{1}{2} e^{\frac{\alpha-c_i^*}{\beta}} (\beta+c_i^*) - \frac{1}{2} e^{\frac{\alpha-c_i^*}{\beta}} (\beta+\alpha) \right] =$$

$$= QH \left[ \frac{c_i^*}{2} + \frac{\beta-\alpha}{2} + \frac{c_i^*}{2} + \frac{\beta}{2} e^{\frac{\alpha-c_i^*}{\beta}} - \frac{\beta+\alpha}{2} \right] = QH \left[ c_i^* - \alpha + \frac{\beta}{2} e^{\frac{\alpha-c_i^*}{\beta}} \right]. \quad \text{(B.4-5)}$$

Cost of late delivery will differ too. For $c_i^* + \Delta c \geq \alpha$ the expected penalty cost for late delivery is

$$Y_{late} = K \int_{c_i^*+\Delta c}^{\infty} (x-(c_i^*+\Delta c)) \frac{1}{2\beta} e^{\frac{x-c_i^*}{\beta}} \, dx =$$

$$= K \int_{c_i^*+\Delta c}^{\infty} \frac{x}{2\beta} e^{\frac{x-c_i^*}{\beta}} \, dx - K \int_{c_i^*+\Delta c}^{c_i^*+\Delta c} \frac{x-c_i^*}{2\beta} e^{\frac{x-c_i^*}{\beta}} \, dx =$$

$$= -K \frac{1}{2} e^{\frac{x-c_i^*}{\beta}} (\beta + x) \int_{c_i^*+\Delta c}^{\infty} + K \frac{c_i^*+\Delta c}{2} e^{\frac{x-c_i^*}{\beta}} \int_{c_i^*+\Delta c}^{\infty} =$$

$$= -0 + K \beta + c_i^* + \Delta c \int_{c_i^*+\Delta c}^{\infty} e^{\frac{x-c_i^*+\Delta c}{\beta}} \, dx + 0 - K \frac{c_i^*+\Delta c}{2} e^{\frac{x-c_i^*}{\beta}} \int_{c_i^*+\Delta c}^{\infty} = K \beta e^{\frac{c_i^*+\Delta c-a}{\beta}}. \quad \text{(B.4-6)}$$
For \( c_i^* + \Delta c < \alpha \) and two parts of the curve of the pdf \( \left[ c_i^* + \Delta c, \alpha \right] \) and \( (\alpha, \infty) \) the expected penalty cost for late delivery is

\[
Y_{late} = K \int_{c_i^* + \Delta c}^{\alpha} (x - (c_i^* + \Delta c)) \frac{1}{2\beta} e^{-\frac{a-x}{\beta}} \, dx + K \int_{a}^{\infty} (x - (c_i^* + \Delta c)) \frac{1}{2\beta} e^{-\frac{x-\alpha}{\beta}} \, dx =
\]

\[
= K \left[ e^{-\frac{a-x}{\beta}} x - \frac{x^2}{2} \right]_{c_i^* + \Delta c}^{\alpha} - \frac{c_i^* + \Delta c}{\beta} e^{-\frac{a-x}{\beta}} \left. \right|_{c_i^* + \Delta c} + K \left[ -e^{-\frac{x-\alpha}{\beta}} x + \frac{x^2}{\alpha} \right]_{a}^{\infty} + \frac{c_i^* + \Delta c}{\beta} e^{-\frac{x-\alpha}{\beta}} \left. \right|_{a}^{\infty} =
\]

\[
= K \left[ e^{-\frac{a-a}{\beta}} \alpha - \beta - e^{-\frac{a-(c_i^* + \Delta c)}{\beta}} \frac{c_i^* + \Delta c}{2} - \frac{c_i^* + \Delta c}{\beta} e^{-\frac{a-a}{\beta}} + c_i^* + \Delta c e^{-\frac{a-a}{\beta}} \right] +
\]

\[
+ K \left[ -0 + e^{-\frac{a-a}{\beta}} \frac{\alpha + \beta}{2} + 0 - \frac{c_i^* + \Delta c}{\beta} e^{-\frac{a-a}{\beta}} \right] =
\]

\[
= K \left[ \frac{\alpha - \beta}{2} + \beta e^{-\frac{a-(c_i^* + \Delta c)}{\beta}} - \frac{c_i^* + \Delta c}{2} + \frac{\alpha + \beta}{2} - \frac{c_i^* + \Delta c}{2} \right] =
\]

\[
= K \left[ \alpha - (c_i^* + \Delta c) + \frac{\beta}{2} e^{-\frac{a-(c_i^* + \Delta c)}{\beta}} \right]. \quad \text{(B.4-7)}
\]

Using the results we find the expected penalty cost for untimely delivery for each case.

Case 1. If \( c_i < \alpha < c_i + \Delta c \), the expected penalty cost is

\[
Y = QH \cdot \frac{\beta}{2} e^{-\frac{a-c_i}{\beta}} + K \frac{\beta}{2} e^{-\frac{c_i + \Delta c - a}{\beta}}. \quad \text{(B.4-8)}
\]

Using that \( c_i^* = \alpha - \frac{\Delta c}{2} - \frac{\beta}{2} \cdot \ln \left( \frac{QH}{K} \right) \), we have

\[
Y = QH \cdot \frac{\beta}{2} e^{-\frac{a-(\alpha - \frac{\Delta c}{2} - \frac{\beta}{2} \cdot \ln (QH/K^{-1}))}{\beta}} + K \frac{\beta}{2} e^{-\frac{a-0.5\Delta c-0.5\beta \cdot \ln (QH/K^{-1})+\Delta c - a}{\beta}} =
\]

\[
= QH \cdot \frac{\beta}{2} e^{-\frac{\Delta c}{2\beta} \cdot \ln (QH/K^{-1})} + K \frac{\beta}{2} e^{-\frac{\Delta c}{2\beta} \cdot \ln (QH/K^{-1})} =
\]

114
\[
= QH \cdot (QH \cdot K^{-1})^{0.5} \cdot \beta e^{-\frac{\Delta c}{2\beta}} + K \cdot (QH \cdot K^{-1})^{0.5} \cdot \frac{\beta}{2} e^{-\frac{\Delta c}{2\beta}} = \\
= \sqrt{QH \cdot K} \cdot \frac{\beta}{2} e^{-\frac{\Delta c}{2\beta}} + \sqrt{QH \cdot K} \cdot \frac{\beta}{2} e^{-\frac{\Delta c}{2\beta}} = \sqrt{QH \cdot K} \cdot \beta e^{-\frac{\Delta c}{2\beta}}. 
\]

**Case 2.** If \( c_1 < c_i + \Delta c < \alpha \), the expected penalty cost is

\[
Y = QH \cdot \beta e^{-\frac{\alpha - c_i}{\beta}} + K \left[ \alpha - (c_1^* + \Delta c) + \frac{\beta}{2} e^{-\frac{\alpha - (c_i^* + \Delta c)}{\beta}} \right]. 
\]  

(B.4-10)

Knowing that the optimal position of the delivery window is

\[
c^*_i = \alpha - \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right), 
\]

we have

\[
Y = QH \cdot \frac{\beta}{2} e^{-\frac{\alpha - \beta \ln(0.5 e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1})}{\beta}} + \\
+ K \left[ \alpha - (\alpha - \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) + \Delta c) + \frac{\beta}{2} e^{-\frac{\alpha - \beta \ln(0.5 e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1}) + \Delta c}{\beta}} \right] = \\
= QH \cdot \frac{\beta}{2} e^{-\ln(0.5 e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1})} + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) - \Delta c + \frac{\beta}{2} e^{-\ln(0.5 e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1}) + \Delta c}{\beta} \right] = \\
= QH \cdot \frac{\beta}{2} (0.5 \cdot e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1})^{-1} + \\
+ K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) - \Delta c + \frac{\beta}{2} (0.5 \cdot e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1})^{-1} e^{\frac{\Delta c}{\beta}} \right] = \\
= QH \cdot \frac{\beta}{e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1}} + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) - \Delta c + \frac{\beta}{e^{\frac{\Delta c}{\beta}} + 0.5QH \cdot K^{-1}} e^{\frac{\Delta c}{\beta}} \right] = \\
= QH \cdot \frac{\beta}{e^{\frac{\Delta c}{\beta}} + QH \cdot K^{-1}} + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{\beta}} + \frac{QH}{2K} \right) - \Delta c + \frac{\beta}{e^{\frac{\Delta c}{\beta}} + QH \cdot K^{-1}} e^{\frac{\Delta c}{\beta}} \right] = 
\]
\[
\begin{align*}
= & \frac{\beta}{e^{\Delta c/\beta} + QH \cdot K^{-1}} \left[ QH + K \cdot e^{\Delta c/\beta} \right] + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{e^\beta}} + \frac{QH}{2K} \right) - \Delta c \right] = \\
= & \frac{K \cdot \beta}{e^{\Delta c/\beta} + QH \cdot K^{-1}} \left[ QH \cdot K^{-1} + e^{\Delta c/\beta} \right] + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{e^\beta}} + \frac{QH}{2K} \right) - \Delta c \right] = \\
= & K \cdot \beta + K \left[ \beta \cdot \ln \left( \frac{1}{2} e^{\frac{\Delta c}{e^\beta}} + \frac{QH}{2K} \right) - \Delta c \right] = K \cdot \beta \left[ 1 + \ln \left( \frac{1}{2} e^{\frac{\Delta c}{e^\beta}} + \frac{QH}{2K} \right) - \frac{\Delta c}{\beta} \right]. \quad \text{(B.4-12)}
\end{align*}
\]

Case 3. If \( \alpha < c_i < c_i + \Delta c \), the expected penalty cost is

\[
Y = QH \left[ c_i^* - \alpha + \frac{\beta}{2} e^{\frac{c_i - \alpha}{\beta}} \right] + K \frac{\beta}{2} e^{\frac{c_i + \Delta c - \alpha}{\beta}}. \quad \text{(B.4-13)}
\]

Using \( c_i^* = \alpha - \beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) \), the expected penalty cost

\[
Y = QH \left[ \alpha - \beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) - \alpha + \frac{\beta}{2} e^{-\frac{\alpha - \beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) - \alpha}{\beta}} \right] + \\
+ K \frac{\beta}{2} e^{-\frac{\alpha - \beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) + \Delta c - \alpha}{\beta}} = \\
= QH \left[ -\beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) + \frac{\beta}{2} e^{\ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right)} \right] + K \frac{\beta}{2} e^{\ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{\Delta c}{e^\beta}}} \right) - \frac{\Delta c}{\beta}} =
\]
\[\begin{align*}
&= QH \left[ -\beta \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right) + \frac{\beta}{2} \left( 2 - 2 \left[ 1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}} \right] \right) \right] + \\
&+ K \frac{\beta}{2} \left( 2 - 2 \left[ 1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}} \right] \right) e^{\frac{-\Delta c}{\beta}} = \\
&= \beta \cdot \left[ 1 - \frac{1}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right] \left( QH + K \cdot e^{\frac{-\Delta c}{\beta}} \right) - QH \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right) = \\
&= \beta \cdot \left[ \frac{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}} - 1}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right] \left( QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}} + 1 \right) K \cdot e^{\frac{-\Delta c}{\beta}} - QH \cdot \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right) = \\
&= QH \cdot \beta \cdot \left[ 1 - \ln \left( 2 - \frac{2}{1 + QH \cdot K^{-1} e^{\frac{-\Delta c}{\beta}}} \right) \right]. \hspace{1cm} (B.4-14)
\end{align*}\]
Appendix C. Proof of remarks and propositions

C.1. Remark 4-1.

We will use the following derivatives:

\[
\frac{d \tau(z, y)}{d \alpha} = e^{-y} y^{z-1} \ln(y) \quad \text{(C.1-1)}
\]

\[
\frac{d \lambda(z, y)}{d \alpha} = -e^{-y} y^{z-1} \ln(y) \quad \text{(C.1-2)}
\]

First, we evaluate the effect of scale parameter on the function taking the derivative of \( A \) by \( \Delta c \):

\[
\frac{dA}{d\Delta c} = -K \frac{\alpha (dc^*_i/d\Delta c + 1) e^{-\alpha (c^*_i + \Delta c)} (\alpha (c^*_i + \Delta c))^{k-1} \ln(\alpha (c^*_i + \Delta c))}{\Gamma(k)} - QH \frac{\alpha (dc^*_i/d\Delta c) e^{-\alpha (c^*_i)} (\alpha (c^*_i))^{k-1} \ln(\alpha (c^*_i))}{\Gamma(k)} \quad \text{(C.1-3)}
\]

\[
\frac{dA}{d\Delta c} = -K (dc^*_i/d\Delta c + 1) f(c^*_i + \Delta c) \ln(\alpha (c^*_i + \Delta c)) - QH (dc^*_i/d\Delta c) f(c^*_i) \ln(\alpha (c^*_i)) \quad \text{(C.1-4)}
\]

Using (4-4) we have

\[
\frac{dA}{d\Delta c} = QH (dc^*_i/d\Delta c) f(c^*_i) \ln(\alpha (c^*_i + \Delta c)) - QH (dc^*_i/d\Delta c) f(c^*_i) \ln(\alpha (c^*_i)) \quad \text{(C.1-5)}
\]

\[
\frac{dA}{d\Delta c} = QH (dc^*_i/d\Delta c) f(c^*_i) \left[ \ln(\alpha (c^*_i + \Delta c)) - \ln(\alpha (c^*_i)) \right] \quad \text{(C.1-6)}
\]

\[
\frac{dA}{d\Delta c} = QH (dc^*_i/d\Delta c) f(c^*_i) \ln \left( 1 + \frac{\Delta c}{c^*_i} \right) \quad \text{(C.1-7)}
\]

The derivative (C.1-7) is always negative and increasing \( \Delta c \) will decrease \( A \). Thus \( A \) is taken the lowest value if \( \Delta c \to \infty \) and \( \lim_{\Delta c \to \infty} c^*_i = 0 \). To find the minimal possible value of \( A \) we find the following limit with \( \Delta c \to \infty \) and nonzero finite values of \( a, QH \) and \( K \).
\[
\lim_{\Delta \to \infty} A = K \frac{\int_{0}^{\infty} e^{-t} \ln(t) \, dt}{\Gamma(k)} - QH \frac{\int_{0}^{\infty} e^{-t} \ln(t) \, dt}{\Gamma(k)} = 0. \tag{C.1-8}
\]

Hence the lowest value that \( A \) can take is 0 when \( \Delta c = \infty \), for all other cases \( A > 0 \).

**C2. Remark 4-2**

First, we focus on the first two elements of the derivative \( dY/dk \). We find a value \( \alpha \) of that will minimize \( y_{\text{early}} + y_{\text{late}} \) taking a derivative

\[
\begin{align*}
\frac{dy_{\text{early}}}{d\alpha} &= QH \frac{dc^*_i / d\alpha \cdot \tau(k, \alpha c^*_i) + c^*_i (c^*_i + \alpha dc^*_i / d\alpha) e^{-\alpha c^*_i} (\alpha c^*_i)^{-1} \ln(\alpha c^*_i)}{\Gamma(k)} \\
&\quad - QH \frac{\alpha (c^*_i + \alpha dc^*_i / d\alpha) e^{-\alpha c^*_i} (\alpha c^*_i)^{-1} \ln(\alpha c^*_i) - \tau(k + 1, \alpha c^*_i)}{\alpha^2 \Gamma(k)} \\
&\quad = QH \frac{dc^*_i / d\alpha}{\Gamma(k)} \tau(k, \alpha c^*_i) + QH \frac{\tau(k + 1, \alpha c^*_i)}{\alpha^2 \Gamma(k)} \\
\frac{dy_{\text{late}}}{d\alpha} &= K \frac{-\alpha (c^*_i + \Delta c + \alpha dc^*_i / d\alpha) e^{-\alpha (c^*_i + \Delta c)} (\alpha (c^*_i + \Delta c))^k \ln(\alpha (c^*_i + \Delta c))}{\alpha^2 \Gamma(k)} \\
&\quad - K \frac{\lambda (k + 1, \alpha (c^*_i + \Delta c))}{\alpha^2 \Gamma(k)} - K \frac{dc^*_i / d\alpha \lambda(k, \alpha (c^*_i + \Delta c))}{\Gamma(k)} \\
&\quad - K \frac{(c^*_i + \Delta c) (c^*_i + \Delta c + \alpha dc^*_i / d\alpha) e^{-\alpha (c^*_i + \Delta c)} (\alpha (c^*_i + \Delta c))^k \ln(\alpha (c^*_i + \Delta c))}{\Gamma(k)} \\
&\quad = K \frac{-\lambda (k + 1, \alpha (c^*_i + \Delta c))}{\alpha^2 \Gamma(k)} - K \frac{dc^*_i / d\alpha \lambda(k, \alpha (c^*_i + \Delta c))}{\Gamma(k)} \\
&\quad - K \frac{(c^*_i + \Delta c) (c^*_i + \Delta c + \alpha dc^*_i / d\alpha) e^{-\alpha (c^*_i + \Delta c)} (\alpha (c^*_i + \Delta c))^k \ln(\alpha (c^*_i + \Delta c))}{\Gamma(k)} \\
&\quad = K \frac{\lambda (k + 1, \alpha (c^*_i + \Delta c))}{\alpha^2 \Gamma(k)} - K \frac{dc^*_i / d\alpha \lambda(k, \alpha (c^*_i + \Delta c))}{\Gamma(k)} \\
&\quad = K \frac{dc^*_i / d\alpha \lambda(k, \alpha (c^*_i + \Delta c))}{\alpha^2 \Gamma(k)} - K \frac{dc^*_i / d\alpha \lambda(k, \alpha (c^*_i + \Delta c))}{\Gamma(k)}. \tag{C.2-4}
\end{align*}
\]
Knowing from Proposition 4-18 that \( \frac{dc_i^*}{d\alpha} < 0 \) and from Remark 4-1 that \( A(k, \alpha, \Delta c, QH, K) > 0 \) we can conclude \( \frac{d(y_{early} + y_{late})}{d\alpha} > 0 \). Thus \( y_{early} + y_{late} \) takes a minimum when \( \alpha = 0 \).

The last component of \( dY/dk \) includes \( Y \) and \( \frac{\lambda(k)}{\Gamma(k)} \) which is not a function of \( \alpha \).

From proposition 4-19 to maximize \( Y \) (minimize \(-Y\)) \( \alpha \) should be minimized (\( \alpha \to 0 \)). Thus to find the minimum value of \( \frac{dY}{dk} \) the limit for \( \alpha \to 0 \) should be taken.

We define

\[
\frac{dY}{dk} = QH \int \left( c_i^* - x \right) f(x) \left( \ln(\alpha x) - \frac{\lambda(k)}{\Gamma(k)} \right) dx +
\]

\[
+ K \int_{c_i^* + \Delta c}^{\infty} \left( x - (c_i^* + \Delta c) \right) f(x) \left( \ln(\alpha x) - \frac{\lambda(k)}{\Gamma(k)} \right) dx
\]

Knowing that (L'Hôpital's rule) \( \lim_{\alpha \to 0} \alpha^k \ln \alpha = \lim_{\alpha \to 0} \frac{\ln \alpha}{\alpha^{-k}} = \lim_{\alpha \to 0} \frac{\alpha^{-1}}{-k \cdot \alpha^{-k-1}} = \lim_{\alpha \to 0} (-k \cdot \alpha^{-k}) = 0 \), we have \( \lim_{\alpha \to 0} \frac{dY}{dk} = 0 \). Hence, the derivative \( dY/dk \) is always positive.
C3. Proposition 5-1.

The following derivatives are used

\[
\frac{dP_{\text{early}}}{dC_{O}'} = \frac{1}{D} (dR^*/dC_{O}') f(A^*/D) - 0 + \int_0^{A'/\rho_D} 0dL = \frac{1}{D} f(A^*/D)(dR^*/dC_{O}'); \quad (C.3-1)
\]

\[
\frac{dP_{\text{late}}}{dC_{O}'} = 0 - \frac{1}{D} (dR^*/dC_{O}') f(R^*/D) + \int_{R^*/D}^{\infty} 0dL = -\frac{1}{D} f(R^*/D)(dR^*/dC_{O}'); \quad (C.3-2)
\]

\[
\frac{dY_{\text{late}}}{dC_{O}'} = 0 - \frac{1}{D} (R^*/D - R^*) f(R^*/D)(dR^*/dC_{O}') + \\
\int_{R^*/D}^{\infty} (-dR^*/dC_{O}') f(L)dL = -P_{\text{late}}(dR^*/dC_{O}'). \quad (C.3-3)
\]

Hence, we have

\[
H\left( \frac{1}{D} f(R^*/D)(dR^*/dC_{O}') + P_{\text{late}}(dR^*/dC_{O}')P_{\text{late}} \right) + \\
P_{\text{early}} + C_{O}' \frac{1}{D} f(A^*/D)(dR^*/dC_{O}') = -C_{S}' \frac{1}{D} f(R^*/D)(dR^*/dC_{O}'), \quad (C.3-4)
\]

\[
dR^*/dC_{O}' \left( Y_{\text{late}} H/D f(R^*/D) + H(P_{\text{late}})^2 + C_{O}' /D f(A^*/D) + C_{S}' /D f(R^*/D) \right) = \\
= -P_{\text{early}}', \quad (C.3-5)
\]

\[
\frac{dR^*}{dC_{O}'} = -\frac{D \cdot P_{\text{early}}}{H \cdot Y_{\text{late}} f(R^*/D) + H \cdot D(P_{\text{late}})^2 + C_{O}' f(A^*/D) + C_{S}' f(R^*/D)} \quad (C.3-6)
\]

All parameters in the derivative (C.3-6) are always positive. Thus, the derivative is negative, that means that increasing \( C_{O}' \) will decrease \( R^* \). ■


The following derivatives are used

\[
\frac{dP_{\text{early}}}{dC_{S}'} = \frac{1}{D} (dR^*/dC_{S}') f(A^*/D) - 0 + \int_0^{A'/\rho_D} 0dL = \frac{1}{D} f(A^*/D)(dR^*/dC_{S}'); \quad (C.4-1)
\]
\[
\frac{dP_{\text{late}}}{dC_S} = 0 - \frac{1}{D} (dR^*/dC_S) f(R^*/D) + \int_{R^*/D}^{\infty} dL = -\frac{1}{D} f(R^*/D)(dR^*/dC_S); \quad (C.4-2)
\]

\[
\frac{dY_{\text{late}}}{dC_S} = 0 - \frac{1}{D} \left( \frac{R^*/D - R^*}{D} \right) f(R^*/D)(dR^*/dC_S) + \\
+ \int_{R^*/D}^{\infty} \left( -dR^*/dC_S \right) f(L)dL = -P_{\text{late}}(dR^*/dC_S).
\]

Hence, we have

\[
H \left( Y_{\text{late}} \frac{1}{D} f(R^*/D)(dR^*/dC_S) + P_{\text{late}}(dR^*/dC_S) \right) + \\
+ C_O \frac{1}{D} f(A^*/D)(dR^*/dC_S) = P_{\text{late}} - C_S \frac{1}{D} f(R^*/D)(dR^*/dC_S),
\]

\[
\frac{dR^*}{dC_S} = \frac{D \cdot P_{\text{late}}}{H \cdot Y_{\text{late}} f(R^*/D) + H \cdot D \cdot (P_{\text{late}})^2 + C_O f(A^*/D) + C_S f(R^*/D)}, \quad (C.4-5)
\]

All parameters in the derivative (C.4-5) are always positive. Thus, the derivative is negative, that means that increasing \( C_S \) will decrease \( R^* \). ■

**C5. Proposition 5-3.**

The follow derivatives are used

\[
\frac{dP_{\text{early}}}{dH} = \frac{1}{D} (dR^*/dH) f(A^*/D) - 0 + \int_{0}^{A^*/D} \frac{A^*/D}{D} f(A^*/D)(dR^*/dH); \quad (C.5-1)
\]

\[
\frac{dP_{\text{late}}}{dH} = 0 - \frac{1}{D} (dR^*/dH) f(R^*/D) + \int_{R^*/D}^{\infty} dL = -\frac{1}{D} f(R^*/D)(dR^*/dH); \quad (C.5-2)
\]

\[
\frac{dY_{\text{late}}}{dH} = 0 - \frac{1}{D} \left( \frac{R^*/D - R^*}{D} \right) f(R^*/D)(dR^*/dH) + \\
+ \int_{R^*/D}^{\infty} \left( -dR^*/dH \right) f(L)dL = -P_{\text{late}}(dR^*/dH).
\]

Hence, we have
\[
Q - Y_{late}P_{late} + H\left(\frac{Y_{late}}{D}f(R^*/D)(dR^*/dH) + P_{late}(dR^*/dH)P_{late}\right) + \\
C_o^i\frac{1}{D}f(A^*/D)(dR^*/dH) = -C_s^i\frac{1}{D}f(R^*/D)(dR^*/dH),
\]

(C.5-4)

\[
dR^* = \frac{D \cdot (Y_{late}P_{late} - Q)}{H \cdot Y_{late}f(R^*/D) + H \cdot D(P_{late})^2 + C_o^i f(A^*/D) + C_s^i f(R^*/D)}. \quad \text{(C.5-5)}
\]

Because the denominator is always positive, the sign of the derivative (C.5-5) depends on numerator.

\[
Y_{late}P_{late} - Q = (Y_{late} - Q)P_{late} - Q(1 - P_{late}) = \\
\left(\int_{R^*/D}^{\infty} (LD - R^* - Q)f(L)dL - Q \int_{R^*/D}^{\infty} f(L)dL \right)P_{late} - Q(1 - P_{late}) = \\
\left(\int_{R^*/D}^{\infty} (LD - R^* - Q)f(L)dL - Q(1 - P_{late}) \right)P_{late} - Q(1 - P_{late})
\]

(C.5-6)

Because \(0 \leq P_{late} \leq 1\), the only element that can be positive in equation (C.5-6) is the first integral. Using the assumption (9) No interaction between orders \((0 \leq L \leq Q/D)\), we can write

\[
\int_{R^*/D}^{\infty} (LD - R^* - Q)f(L)dL = \int_{R^*/D}^{Q/D} (LD - R^* - Q)f(L)dL = \int_{R^*/D}^{Q/D} (LD - R^* - Q)dL.
\]

(C.5-7)

The value of \((LD - R^* - Q)\) is always negative for any \(R^*/D \leq L \leq Q/D\), hence the integral (C.5-7) is negative. Thus, it can be concluded that the derivative (C.5-5) is negative, that means that increasing \(H\) will decrease \(R^*\). ■
REFERENCES


Hariga, M., 1998. Single period inventory models with two levels of storage. Production Planning and Control, 9, 553-560.


