ANALYZING THE RELATIONSHIP BETWEEN PLAYER PERSONNEL AND
OPTIMAL MIXED STRATEGIES IN AMERICAN FOOTBALL

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Erin McGough

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ANALYZING THE RELATIONSHIP BETWEEN PLAYER PERSONNEL AND OPTIMAL MIXED STRATEGIES IN AMERICAN FOOTBALL

Erin McGough

Thesis

Approved: 
Advisor 
Dr. Gerald Young 

Co-Advisor 
Dr. Curtis Clemons

Co-Advisor 
Dr. Michael Ferrara

Accepted: 
Dean of the College 
Dr. Chand Midha 

Dean of the Graduate School 
Dr. George Newkome

Date 
Department Chair 
Dr. Joseph Wilder
ABSTRACT

The purpose of this paper is to explore how the optimal mix of run and pass is affected by a change in player personnel in American football. To investigate this notion, we construct a model under the hypothesis that the offense has recently acquired a proven quarterback that will increase the production of the passing game. We then solve the game, and in the process construct Nash equilibrium functions that depend on the influence of the new quarterback. Lastly, as an example, we use empirical data and model results to examine how the addition of Jay Cutler impacts the mix of run and pass for the 2009 Chicago Bears.
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Thank you Dad for showing me the competitive game of football.

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CHAPTER I
INTRODUCTION

There is a growing interest in the application of game theory to American football. This is natural since the strategy sets for each opposing team are vast, and oftentimes play selection determines the odds of success of a given play. Coaches who employ the principles of game theory experience benefits ranging from optimal game planning to more efficient in-game play calling [1]. This aids in their pursuit of maximizing expected returns, which is any coach’s foremost objective. Although the use of game theory is a powerful tool for coaches, its benefits are not limited to this tier of the professional organization. Since optimal mix depends heavily on the ability of the players, a change in player personnel affects optimal balance. Therefore, a game-theoretic mindset is crucial when considering trades, signing free agents, and releasing current players.

This paper implements the use of game theory to find the optimal mix of run and pass while also exploring the relationship between player personnel and optimal mix. In short, we aim to answer how the acquisition of a proven quarterback changes the optimal mix for a team with a known payoff matrix. This paper discusses how improvement in the passing game alters optimal balance. We begin by defining the payoffs and introducing the model. We then solve the game and in the process, con-
struct Nash equilibrium functions that depend on the ability of the new quarterback. After the theoretical framework has been established, we explore how the acquisition of Jay Cutler impacts the play-calling strategies of the Chicago Bears. Lastly, we discuss some results and a few possible routes of extension.

There have been several relevant contributions in this area, many focus on the risk-averse behavior exhibited by coaches in critical game situations. Carter and Machol gather data from the 1971 NFL season that suggests coaches opt for field goals when the expected point values of “going for it” are much higher \[2\]. A similar analysis is captured in the work of Romer, who argues that coaches ought to go for it on fourth down in some situations instead of kicking frequently \[3\]. Alamar continues, declaring that although risks associated with the passing game have steadily decreased since the 1960s and its expected returns have increased, the passing game is under-utilized \[4\]. Conversely, in response to Alamar’s paper, Rockerbie assumes a risk-averse utility function to model coaching behavior, and shows that teams pass too often and that there exists a running premium. Rockerbie also shows that there is a correlation between win percentage and teams who employ their optimal mix strategies \[5\].

All of these analyses have proven to be consequential, since each has shown that coaches do not maximize expected returns \[6\]. We do not construct a risk-averse model, instead, we maintain the philosophy of finding optimal solutions, while disregarding the human tendency to be risk-averse in pressure situations. This produces a truly optimal result, since it is well known that being risk-averse lowers the expected
value of games. The model we discuss is not one that reflects human behavior, however it produces optimal results under the hypothesis that a coach is risk-indifferent when it comes to calling run plays and pass plays.

Other related papers focus on stochastic methods that determine expected point values for different field positions. These findings are often used as evidence to support the claims of Carter and Machol, Romer, etc. In a 1971 paper, Carter and Machol find expected point values for several field positions using empirical data [7]. Likewise, Boronico and Newbert use a dynamic programming model to analyze Monmouth University football games in order to determine the odds of scoring a touchdown in any situation inside the ten yard-line [8]. Their approach is highly game-theoretic, as they examine four different types of offensive plays pitted against two types of defenses. The paper by Boronico and Newbert is particularly important here since, like them, we analyze a numerical example consisting of empirical data, desiring to find nontrivial results that will aid in the development of coaching strategy.

We proceed by defining the payoffs and setting up the model. Then in Chapter 3, we solve the game and discuss some theoretical results. In Chapter 4, we apply the results, and examine how the acquisition of Jay Cutler might impact the run/pass balance of the 2009 Chicago Bears.
CHAPTER II

THE OFFENSE VS. DEFENSE MODEL

2.1 Setting Up the Game

The games we discuss are two-by-two zero-sum games where the offense either runs or passes, and the defense either defends against the run or defends against the pass. Each payoff represents an expected gain for the offense under the given strategic situation. For any serious football team, it is reasonable to suggest that the payoffs can be found, since every coaching staff spends much of their time breaking down film. To find the payoffs, we take an expected value, and like Alamar and Rockerbie, we enforce a 45-yard penalty for fumbles and interceptions. Next, we formulate this idea.

Table 2.1: Initial Payoffs

<table>
<thead>
<tr>
<th>Offense</th>
<th>Defense</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Defend Run</td>
<td>Defend Pass</td>
<td></td>
</tr>
<tr>
<td>Run</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Pass</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
To find the payoffs for cells \( j = 1, 2, 3, 4 \) in Table 2.1, let \( n_j \) be the number of observations in cell \( j \). So, if \( O_i \) is the outcome of play \( i \) in yardage, then \( Y_j = \sum_{i=1}^{n_j} O_i \) is the total yardage in cell \( j \). Now, let \( T_j \) be the number of turnovers in cell \( j \). Then the payoff is

\[
P_j = \frac{Y_j - 45T_j}{n_j}, \quad \text{for} \quad j = 1, 2, 3, 4. \tag{2.1}
\]

We note that to find the payoffs for Table 2.1 using (2.1) the defensive strategies Defend Run and Defend Pass must be clearly defined. We suggest an algorithm that partitions the defensive plays by taking into account the alignment of the defense, the defensive coverage, and situational factors such as third and long. In this paper, we do not create such a partition, instead we find the payoffs using basic linear algebra and a few simple assumptions. Our goal is not to exactly quantify the initial payoffs, instead it is to find the exact initial offensive mixed strategy. Once this is found, we desire to understand how it changes as passing improves.
CHAPTER III
THE IMPROVEMENT IN PASSING MODEL

3.1 The Model

We assume that acquiring a proven quarterback will improve the output of the passing game. Therefore, we anticipate that yards per game and yards per attempt will both increase from a season ago. To determine how an improvement in the passing game affects the running game we argue as follows. Since it is common belief that the success of the passing game is intertwined with the success of the running game, we can suppose that the quarterback’s production will also improve the running payoffs. Although the quarterback will positively influence the running game, he will not influence the running game as much as he influences the passing game. This is because production in the passing game squarely rests upon his ability and performance, whereas in most cases, his contributions are limited in the running game. We now introduce the model.

Observe that each entry in Table 3.1 is a function of $x$. Here $x$ measures the influence of the quarterback; it can be thought of as a quarterback rating. In this model, we see that if $x = 0$ we retrieve the initial payoff matrix which reflects the production of the team prior to the acquisition of the quarterback. For convenience,
let \( A = \begin{pmatrix} a_{r,r} & a_{r,p} \\ a_{p,r} & a_{p,p} \end{pmatrix} \) be this matrix. Also, suppose that throughout this analysis \( A \) contains no dominant rows and no dominant columns, since we seek to solve games that have only mixed strategy Nash equilibria [9]. This is because no professional football team would ever play a pure strategy. Now, we assume that each payoff will increase as a function of the initial payoff, that is why \((1 + \delta x)\) multiplies the initial payoffs in row 1 and \((1 + x)\) multiplies the payoffs in row 2. The coefficient \( \delta \in (0, 1) \) is a measure of how much the new quarterback influences the running game. In comparing mobile quarterbacks and pocket passers of the same caliber it is reasonable to suggest that mobile quarterbacks have a higher \( \delta \) value, since they are contributing to both phases of the game. However, instead of assigning a \( \delta \) value to each quarterback, we suppose that \( \delta \) is a general measure of how much improvement in a team’s passing game affects that team’s running game.

Table 3.1: The Improvement in Passing Model

<table>
<thead>
<tr>
<th>Offense</th>
<th>Defense</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Protect Run</td>
<td>Protect Pass</td>
<td></td>
</tr>
<tr>
<td>Run</td>
<td>( a_{r,r}(1 + \delta x) )</td>
<td>( a_{r,p}(1 + \delta x) )</td>
<td></td>
</tr>
<tr>
<td>Pass</td>
<td>( a_{p,r}(1 + x) )</td>
<td>( a_{p,p}(1 + x) )</td>
<td></td>
</tr>
</tbody>
</table>

For our simulations we generally suppose \( \delta = \frac{1}{2} \). This implies that as the passing game improves, the running game improves by half as much as the passing game does. An examination of the choice of \( \delta \) is given in Figure 5.1 following the example.
To explore how the mixed strategy changes as the passing game improves we must iteratively increase $x$, and solve the resulting game once for each $x$. It is important to mention that each quarterback corresponds to exactly one $x$ rating. This rating may increase or decrease over the course of a player’s career, however at any given moment it is fixed. Next, we solve the model.

3.2 Solving the Model

The objective is to construct functions that optimally solve the games. There are a few known methods for solving games, most notably the simplex method [10]. This can be a powerful method when dealing with large systems, but here we are solving only two-by-two systems. We solve the model analytically, and in doing so, we find running and passing Nash equilibria, both depending on the influence of the new quarterback.

First, assume that both the offense and defense are playing optimally. Let $R$ be the probability the offense calls a running play. Consider the following equality

$$a_{r,r}(1 + \delta x)R + a_{p,r}(1 + x)(1 - R) = a_{r,p}(1 + \delta x)R + (1 + x)a_{p,p}(1 - R). \quad (3.1)$$

The left and right hand side of (3.1) are expected values of the first and second columns of Table 3.1, respectively. Both are expressions for the value of the game, hence they are equivalent. From here, we desire to solve for $R$ to obtain a Nash equilibrium function of quarterback influence. Following some algebraic manipulation
we obtain
\[
R(\delta, x) = \frac{(a_{p,p} - a_{p,r})(1 + x)}{(a_{r,r} - a_{r,p})(1 + \delta x) + (a_{p,p} - a_{p,r})(1 + x)}.
\tag{3.2}
\]

Since \( R(\delta, x) + P(\delta, x) = 1 \), we have that
\[
P(\delta, x) = 1 - R(\delta, x)
\tag{3.3}
\]
is the passing function. From (3.2) we see that the probability of running is related to the difference of the passing payoffs. If this difference is large, then it is likely that there is a passing entry that has a relatively small initial payoff, thus causing more running plays to be called. As a notational convenience, let
\[
D(\delta, x) = (a_{r,r} - a_{r,p})(1 + \delta x) + (a_{p,p} - a_{p,r})(1 + x).
\]
To find the defensive Nash equilibrium functions, we perform a similar calculation to (3.1). The defensive Nash equilibrium functions are
\[
N(\delta, x) = \frac{(a_{p,p} - a_{r,p}) + x(a_{p,p} - a_{r,p}\delta)}{D}, \quad (N \text{ stands for defend run})
\tag{3.4}
\]
and
\[
S(\delta, x) = 1 - N(\delta, x), \quad (S \text{ stands for defend pass}).
\tag{3.5}
\]

There are several important points to make. First, the functions in (3.2), (3.3), (3.4), and (3.5) provide optimal mixed strategy solutions to the game for each \( x \). Second, when considering quarterback influence, we limit the amount of influence, \( x \), to the closed interval \([0, 1]\). This is because if \( x > 1 \), then expected yards per play increases unrealistically for reasonable choices of \( a_{r,r}, a_{r,p}, a_{p,r}, \) and \( a_{p,p} \). Lastly, since \((1 + x) > (1 + \delta x)\) there may exist some \( x_0 > 0, x_0 \in (0, 1] \) such that the passing row
dominates the running row for all \( x \geq x_0 \). We call this value the breaking point, since it is the point where the offense switches from a mixed strategy Nash equilibrium to a pure strategy Nash equilibrium. We note that breaking points do not always exist in \((0, 1]\), and in these cases, the game is a mixed strategy game for all \( x \in (0, 1]\). This is realistic, since in most cases an offense would not instantly pursue a pure pass strategy upon improving in passing. However, for those cases where breaking points do exist in \((0, 1]\), the following development is necessary. Since it is reasonable to assume that the payoff corresponding to Run/Defend Run is less than the payoff corresponding to Pass/Defend Run and the payoff corresponding to Pass/Defend Pass is less than the payoff corresponding to Run/Defend Pass, we suppose that \( a_{r,r} < a_{p,r} \) and \( a_{r,p} > a_{p,p} \).

Proposition 1.1

If \( a_{r,r} < a_{p,r} \) and \( a_{r,p} > a_{p,p} \), then the breaking point is

\[
x = \frac{(a_{r,p} - a_{p,p})}{(a_{p,p} - a_{r,p} \delta)}.
\]

To prove Proposition 1.1, we assume that \( a_{r,r} < a_{p,r} \) and \( a_{r,p} > a_{p,p} \). If \( a_{r,r} < a_{p,r} \), then \( a_{r,r}(1 + \delta x) < a_{p,r}(1 + x) \) for all positive \( x \) and \( \delta \in (0, 1) \). Thus, the breaking point must satisfy \( a_{p,p}(1 + x) = a_{r,p}(1 + \delta x) \). After solving for \( x \), we find that

\[
x = \frac{(a_{r,p} - a_{p,p})}{(a_{p,p} - a_{r,p} \delta)},
\]

which is the value of the breaking point. Since we have assumed that \( a_{r,p} > a_{p,p} \), then it must hold that \( a_{p,p} > a_{r,p} \delta \), for the breaking point to be positive. Otherwise, there exists no positive breaking point, and we solve a mixed strategy game for all \( x > 0 \).
Thus, the breaking point is a function of $\delta$ and the initial payoffs. As one would expect, the defense converges to a pure strategy at the breaking point, and

$$S(\delta, (a_{r,p} - a_{p,p})/(a_{p,p} - a_{r,p} \delta)) = 1.$$ Intuitively this makes sense, because if the passing row is increasing faster than the running row, the game ought to converge to Pass/Defend Pass, which it does. Thus, the breaking point is the value $x$ for which both the offense and defense converge to a pure strategy. To account for the defensive change from a mixed strategy to a pure strategy at the breaking point, we propose the discontinuous Nash equilibrium running function:

$$R^*(\delta, x) = \begin{cases} \frac{(a_{p,p} - a_{p,r})(1+x)}{D}, & \text{if } 0 \leq x < \frac{(a_{r,p} - a_{p,p})}{(a_{p,p} - a_{r,p} \delta)} \\ 0, & \text{if } x \geq \frac{(a_{r,p} - a_{p,p})}{(a_{p,p} - a_{r,p} \delta)} > 0. \end{cases}$$ (3.6)

First note that $P^*(\delta, x) = 1 - R^*(\delta, x)$. $R^*$ suggests that there is a mixed strategy solution prior to the breaking point, and that after the breaking point has been reached, quarterback play has become so influential that the offense only passes. $R^*$ completely describes the offensive strategy for every two-by-two game with our assumed initial payoff structure. We now have all the functions we need to optimally solve the game from the offense’s perspective. For our analysis in Chapter 4, we need only one more function, and that is the value of the game function.

To find the value of the game $V$ we take an expected value of the first column of the payoff matrix using $R$ and $P$. This gives

$$V(\delta, x) = a_{r,r}(1 + \delta x)R(\delta, x) + a_{p,r}(1 + x)P(\delta, x).$$ (3.7)
After simplifying, we obtain the following expression for the value of the game

$$V(\delta, x) = \frac{(1 + \delta x)(1 + x)\det A}{D}, \text{ where } \det A \neq 0. \quad (3.8)$$

Observe that in (3.8) it must hold that \( \det A \neq 0 \), since if \( \det A = 0 \), then \( V(\delta, x) = 0 \) for all \( x \in [0, 1] \). However, if \( \det A = 0 \), the rows and columns of \( A \) are linearly dependent, which implies that \( A \) has dominant rows and columns. Thus, if \( A \) has no dominant rows and no dominant columns, then \( \det A \neq 0 \). Therefore, for any game with payoff matrix \( A \) that has only a mixed strategy Nash equilibrium solution, it holds that \( \det A \neq 0 \). We note that the procedure for calculating the value of the game depends on whether or not there exists a pure strategy Nash equilibrium solution. If there does not exist a pure strategy Nash equilibrium solution, then (3.8) is used to find the value of the game. Otherwise, the value of the game is the value of the pure strategy Nash equilibrium. Since \( V \) depends on the nature of the game, it is discontinuous, with its jump occurring at the breaking point. To define \( V \), we assume that the initial payoff corresponding to Pass/Defend Run is larger than the payoff corresponding to Pass/Defend Pass. This implies that \( a_{p,r}(1+x) > a_{p,p}(1+x) \) for all \( x \in [0, 1] \). Thus, the value of the game function is

$$V^*(\delta, x) = \begin{cases} 
\frac{(1+\delta x)(1+x)\det A}{D}, & \text{if } 0 \leq x < \frac{(a_{r,p}-a_{p,p})}{(a_{p,p}-a_{r,p})}, \\
a_{p,p}(1+x), & \text{if } x \geq \frac{(a_{r,p}-a_{p,p})}{(a_{p,p}-a_{r,p})} > 0.
\end{cases} \quad (3.9)$$

Hence, for a large improvement in passing, the value of the game is the value of the Pass/Defend Pass entry of the matrix. Before exploring the example, we examine a few important properties of \( R^* \).
We are in a position to examine how the optimal mix changes with an improvement in passing. Remember, improvement in passing corresponds to an increase in $x$. So as $x$ increases, how does $R^*$ respond? To answer this question we analyze $\frac{\partial R^*}{\partial x}$. Recall that for values of $x$ less than the breaking point

$$R^*(\delta, x) = \frac{(a_{p,p} - a_{p,r})(1 + x)}{D},$$

where $D = (a_{r,r} - a_{r,p})(1 + \delta x) + (a_{p,p} - a_{p,r})(1 + x)$.

Differentiating, we obtain

$$\frac{\partial R^*}{\partial x} = \frac{(a_{r,r} - a_{r,p})(a_{p,p} - a_{p,r})(1 - \delta)}{D^2}.$$  \hspace{1cm} (3.10)

We show that $\frac{\partial R^*}{\partial x} > 0$. Suppose that $\frac{\partial R^*}{\partial x} \leq 0$. This implies that $(a_{r,r} - a_{r,p})(a_{p,p} - a_{p,r}) \leq 0$. So it must be that either $a_{r,r} > a_{r,p}$ and $a_{p,p} < a_{p,r}$ or $a_{r,r} < a_{r,p}$ and $a_{p,p} > a_{p,r}$ or $a_{r,r} = a_{r,p}$ or $a_{p,p} = a_{p,r}$. Any of these possibilities contradicts our assumption that $A$ has no dominant rows and no dominant columns. Thus, $\frac{\partial R^*}{\partial x} > 0$.

As simple as that is to show, it is an important conclusion that suggests if a team improves in passing, they should actually pass less than they did previously. This is because as quarterback influence increases, the passing game increases more than the running game. However, since $\frac{\partial R^*}{\partial x} > 0$ the optimal share of running plays is increasing causing the optimal share of passing plays to decrease.

There is another interesting property of $\frac{\partial R^*}{\partial x}$. We observe that

$$\frac{\partial^2 R^*}{\partial x^2} < 0,$$  \hspace{1cm} (3.11)

which means that $\frac{\partial R^*}{\partial x}$ is decreasing. Thus, $R^*$ is increasing in $x$ at a diminishing rate. This implies that the greatest change in mixed strategy occurs for sufficiently small
positive values of $x \in [0, 1]$. Thus, only a small quarterback influence is necessary to incite a change in mixed strategy. This proves the existence of a correlation between player personnel and balance for our model given in Table 3.1.

Now that we have established the theoretical machinery needed to solve our games, we are ready to put it to use. We explore the relationship between run/pass balance and an improvement in passing by examining how the Chicago Bears optimal mix may change with the addition of Jay Cutler.
CHAPTER IV

PREDICTING THE IMPACT OF JAY CUTLER ON THE MIX OF RUN AND PASS FOR THE 2009 CHICAGO BEARS

In his fourth NFL season, Jay Cutler is among the more promising young quarterbacks in the league. Last season, he finished 3rd in the NFL in total passing yardage. This past offseason he was traded from Denver to Chicago. For Bears fans, this is good news, since in the past five seasons the Chicago Bears have finished no better than 14th in the NFL in total passing yardage. With an improvement in passing, Chicago figures to be competitive in the NFC. The following analysis aims to examine how the acquisition of Jay Cutler may alter the run/pass balance of the Chicago Bears for the upcoming season. We first construct the payoff matrix for the Chicago Bears using empirical data from the 2008 season. This data determines the values of the initial payoffs for the improvement in passing model outlined in Chapter 3. Then, to predict how the mixed strategies change for Chicago, we explore how Jay Cutler influenced the expected yardage per play in his first season as a starter in Denver. Once this influence is known, we project it onto the 2009 Chicago Bears improvement in passing model.
4.1 Finding the Chicago Bears Payoff Matrix

Since there are no explicit statistics quantifying our initial payoffs, we must do a little work to find them. We begin by providing Table 4.1 which lists statistics that will be useful in our effort to find the payoffs. To find the payoffs we use equation (2.1). All of the empirical data used in this paper is courtesy of http://www.NFL.com.

Table 4.1: 2008 Chicago Bears Statistics

<table>
<thead>
<tr>
<th></th>
<th>Rushing</th>
<th>Passing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yards</td>
<td>1673</td>
<td>3061</td>
<td>4734</td>
</tr>
<tr>
<td>Expected Value</td>
<td>2.92</td>
<td>4.36</td>
<td>3.73</td>
</tr>
<tr>
<td>Plays</td>
<td>434</td>
<td>557</td>
<td>991</td>
</tr>
<tr>
<td>Percentage of Plays</td>
<td>0.44</td>
<td>0.56</td>
<td>1</td>
</tr>
</tbody>
</table>

To find the payoffs, first, let $A = \begin{pmatrix} a_{r,r} & a_{r,p} \\ a_{p,r} & a_{p,p} \end{pmatrix}$ be the matrix of initial payoffs. Next, we assume that NFL defenses know how often Chicago runs and passes, and that when they play Chicago, they defend the run and pass equally as often. Then we know from Table 4.1 that the following equations (using approximate values) hold

$$0.44a_{r,r} + 0.56a_{r,p} = 2.92 \quad (4.1)$$

$$0.44a_{p,r} + 0.56a_{p,p} = 4.36.$$
To understand how an improvement in the passing game will affect the current play selection ratio of the Chicago Bears, we assume that the Bears utilize their optimal mixed strategy. Also, we assume that the expected value of defending the run is the same as the expected value of defending the pass. This is reasonable since there exist run defenses that match-up well against certain types of pass plays and pass defenses that match-up well against certain types of run plays. This assumption yields the equation

\[ 0.44(a_{r,r} - a_{r,p}) + 0.56(a_{p,r} - a_{p,p}) = 0. \tag{4.2} \]

Combining (4.1) and (4.2) produces a dependent system of three equations in four unknowns where \( a_{p,p} \) is the free variable. These equations are approximately

\[ A \approx \begin{pmatrix}
1.65a_{p,p} - 4.27 & -1.28a_{p,p} + 8.52 \\
-1.28a_{p,p} + 9.97 & a_{p,p}
\end{pmatrix}. \tag{4.3} \]

Since we assume the defense plays a mixed strategy, the offensive strategy must also be mixed. Therefore, \( A \) has no dominant rows and no dominant columns. To ensure \( A \) has our preferred payoff structure, we must choose \( a_{p,p} \in [2.59, 4.36] \). An exposition of the choice of \( a_{p,p} \) is given in the Appendix. Choosing \( a_{p,p} \) in this way forces \( a_{p,p} < a_{p,r} \).

If \( a_{p,p} < a_{p,r} \), then \( a_{r,p} > a_{p,p} \), otherwise \( a_{p,p} \) is a pure strategy Nash equilibrium.

Also, if \( a_{r,p} > a_{p,p} \), then \( a_{r,r} < a_{r,p} \), otherwise \( a_{r,p} \) is a pure strategy Nash equilibrium.

Continuing in this way produces our desired payoff structure: \( a_{p,p} < a_{p,r}, a_{r,p} > a_{p,p}, a_{r,r} < a_{r,p} \), and \( a_{r,r} < a_{p,r} \). For our analysis we choose \( a_{p,p} = \frac{10}{3} \). In choosing \( a_{p,p} = \frac{10}{3} \), we suppose that since the offense’s goal is to obtain at least \( \frac{10}{3} \) yards per pass and the defense’s goal is to limit the offense’s gain to less than \( \frac{10}{3} \) yards per pass, competitive
nature creates a balance, for which $a_{p,p} = \frac{10}{3}$. This gives that $A \approx \begin{pmatrix} 1.22 & 4.25 \\ 5.69 & 3.33 \end{pmatrix}$.

Now we define the approximate payoff matrix in Table 4.2. As before, we let $\delta = \frac{1}{2}$.

Table 4.2: The 2009 Chicago Bears Improvement in Passing Model

<table>
<thead>
<tr>
<th>Defense</th>
<th>Defend Run</th>
<th>Defend Pass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offense</td>
<td>Run 1.22(1 + $\delta x$)</td>
<td>4.25(1 + $\delta x$)</td>
</tr>
<tr>
<td></td>
<td>Pass 5.69(1 + $x$)</td>
<td>3.33(1 + $x$)</td>
</tr>
</tbody>
</table>

4.2 Jay Cutler and the 2009 Chicago Bears Run/Pass Balance

Table 4.3 gives optimal solutions to the game for selected values of $x$ and for $\delta = \frac{1}{2}$.

First, note that when $x = 0$, $R^*$ gives the percentage of run plays the Chicago Bears called in 2008. Also, when $x = 0$, $V$ gives their average yards per play last year, which was also calculated in Table 4.1. Next, observe that a 0.1 increase in $x$ results in a 0.29 increase in $V$. Hence, $V$ is sensitive to an increase in $x$. Thus, the results given in Table 4.3 support the restriction $x \in [0, 1]$. Observe that in Table 4.3, the number of running plays called increases as passing improves. As a sidenote, the breaking point is approximately 0.759, which implies that to be justified in optimally playing a pure pass strategy, the expected yards per play would have to increase by more than 2 yards due to an improvement in passing. Since this is a large increase, it is unrealistic to anticipate that Chicago would ever pursue a pure strategy.
Table 4.3: How an Improvement in Passing Changes the Mix of Run and Pass for the 2009 Chicago Bears

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R*</td>
<td>0.438</td>
<td>0.449</td>
<td>0.459</td>
<td>0.468</td>
<td>0.476</td>
</tr>
<tr>
<td>∂R*/∂x</td>
<td>0.123</td>
<td>0.107</td>
<td>0.094</td>
<td>0.083</td>
<td>0.074</td>
</tr>
<tr>
<td>P*</td>
<td>0.562</td>
<td>0.551</td>
<td>0.541</td>
<td>0.532</td>
<td>0.524</td>
</tr>
<tr>
<td>V</td>
<td>3.733</td>
<td>4.022</td>
<td>4.308</td>
<td>4.590</td>
<td>4.870</td>
</tr>
</tbody>
</table>

Now to examine how Jay Cutler impacts the run/pass balance of the Chicago Bears, we first find an x rating for Jay Cutler. To do so, we perform computations akin to (4.1) and (4.2), and set up the 2007 Denver Broncos improvement in passing model. This is given in Table 4.4. Remember, the 2007 Denver Broncos improvement in passing model is found by collecting data from the 2006 season. As before, we define a dependent system for a_{r,r}, a_{r,p}, and a_{p,r} with a_{p,p} = \frac{10}{3}. Table 4.5 gives the solutions for some selected values of x.

Table 4.4: The 2007 Denver Broncos Improvement in Passing Model

<table>
<thead>
<tr>
<th></th>
<th>Defense</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Defend Run</td>
<td>Defend Pass</td>
</tr>
<tr>
<td>Offense</td>
<td>Run</td>
<td>2.73(1 + δx)</td>
</tr>
<tr>
<td></td>
<td>Pass</td>
<td>4.86(1 + x)</td>
</tr>
</tbody>
</table>
Table 4.5: How an Improvement in Passing Changes the Mix of Run and Pass for the 2007 Denver Broncos

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0.502</td>
<td>0.513</td>
<td>0.523</td>
<td>0.532</td>
<td>0.540</td>
</tr>
<tr>
<td>∂R/∂x</td>
<td>0.125</td>
<td>0.108</td>
<td>0.094</td>
<td>0.083</td>
<td>0.074</td>
</tr>
<tr>
<td>P</td>
<td>0.498</td>
<td>0.487</td>
<td>0.477</td>
<td>0.468</td>
<td>0.460</td>
</tr>
<tr>
<td>V</td>
<td>3.793</td>
<td>4.075</td>
<td>4.354</td>
<td>4.629</td>
<td>4.901</td>
</tr>
</tbody>
</table>

Next, we find the expected yardage per play in 2007, which is given in Table 4.6, and set $V^*$ from (3.9) equal to this value and solve for $x$. The $V^*$ we use is the one corresponding to the 2007 Denver Broncos improvement in passing model. Thus,

Table 4.6: 2007 Denver Broncos Statistics

<table>
<thead>
<tr>
<th></th>
<th>Rushing</th>
<th>Passing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yards</td>
<td>1957</td>
<td>3584</td>
<td>5541</td>
</tr>
<tr>
<td>Plays</td>
<td>429</td>
<td>547</td>
<td>976</td>
</tr>
<tr>
<td>Expected Value</td>
<td>3.62</td>
<td>5.32</td>
<td>4.57</td>
</tr>
</tbody>
</table>

the $x$ value we seek is a solution to $V^*(\frac{1}{2}, x) = 4.57$. This gives $x \approx 0.279$.

Therefore, if we assume that Jay Cutler has a similiar influence in Chicago to the one he had in Denver, then substituting $x \approx 0.279$ into the 2008 Chicago Bears improvement in passing model, Table 4.2, gives that the Bears should expect
an increase of about 0.799 yards per play, regardless of run or pass. Also, because of this anticipated improvement in passing, the Bears should call about 3% more running plays this season. While this number is not overwhelmingly significant, from a coaching standpoint it is valuable. It suggests that a strong emphasis should be placed on developing the running game in training camp and during the pre-season, since, it should be a central focus on Sunday afternoons. To highlight the importance of this result, let us take a moment to discuss a hypothetical example.

Suppose that a team acquires a proven quarterback. Also, suppose that because of an anticipated improvement in passing, the coaching staff plans to call 5% more pass plays this season than they did last season. Now suppose that we solve the improvement in passing model, and find that they should call 3% more running plays this season than they did last season. Already, the coaching staff is beginning the season operating at 8% inefficiency. Thus, the importance of this paper is not to merely suggest that if a team experiences an increase in passing they should pass less, instead it is to define a methodology for which the coaching staff can implement to maintain play-calling efficiency in a game where player personnel is frequently changing.
CHAPTER V

SUMMARY AND EXTENSIONS

This paper discusses how player personnel changes the optimal mix for an offense. To examine this notion, we suppose that an offense has acquired a proven quarterback that will positively influence both the passing game and the running game. After establishing a theoretical framework using a basic model, we explore how the Chicago Bears optimal strategies change this season with an anticipated increase in passing production.

These conclusions have far-reaching implications. This study suggests that if a team drafts a potential franchise quarterback that will improve their expected returns in passing, they should pass less than they did a season ago. Thus, providing such a player with an adequate running attack is necessary for his development and franchise success, since we have shown that the offensive coordinator should call more running plays this year than he did last year. Some of the quarterbacks that have been drafted by teams that have provided them with successful running games are Ben Roethlisberger, Eli Manning, Matt Ryan, and Joe Flacco. All of these quarterbacks have played in the postseason, with Ben Roethlisberger and Eli Manning sharing three Super Bowl victories.
This analysis also extends to different phases of the game, offensively and defensively. If a team trades for a lock-down cornerback, the defensive coordinator should blitz more. If the offense signs a talented running back, they should pass more. This analysis describes the value of players, since if a team signs John Doe, then they are able to say “he allots us freedom in this way.” Of course, this can become complicated when a team drafts several players. However, it is reasonable to suppose that there is a positive correlation between the order in which the players are drafted and the immediate utility they contribute to the organization. Thus, given a draft ordering of players, and their comparative strengths and weaknesses, it is possible to understand how the optimal mix changes on offense and defense using these game-theoretic arguments.

We note the limitations of the model. First of all, the choice of $\delta$ is ad hoc, however its presence is unavoidable. In Figure 5.1, we give the effect of the choice of $\delta$ on the breaking point using the 2006 Denver Broncos data. Interestingly, a smaller $\delta$ increases $R^*$ faster for initial values of $x$, while at the same time decreases the value of the breaking point. Therefore, if $\delta$ is believed to be smaller than $\frac{1}{2}$ and the anticipated quarterback influence is prior to the breaking point, then the coach should call more running plays than if $\delta$ were $\frac{1}{2}$. Additionally, Figure 5.2 and Figure 5.3 give all values of $R^*$ using 2008 Chicago Bears data and 2006 Denver Broncos data, respectively. We note that black or white regions in the square correspond to pure strategy situations, and gray regions correspond to mixed strategy situations. One observes that as $\delta \to 1$, the game approaches a mixed strategy game for all $x$. 

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The Effect of the Choice of $\delta$ on the Breaking Point

Figure 5.1: An Examination of the Choice of $\delta$ on the Breaking Point Using 2006 Denver Broncos Data
Figure 5.2: Optimal Mixed Strategies for the 2009 Chicago Bears

Secondly, the projection of $x$, found by examining past Denver Bronco’s data, onto the 2008 Chicago Bears improvement in passing model is debatable. Clearly, the increase in expected yardage per play for Denver from the 2006 season to the 2007 season was not due solely to Jay Cutler’s production. With that said, it is always true that a quarterback rating reflects both the statistical makeup of a quarterback and the quality of his supporting cast. Thus, this conjunction is inescapable. Since the apparent talent disparity between Chicago and Denver is small (last season their records were 9-7 and 8-8, respectively), we argue that this projection is a reasonable estimate of his influence.

Lastly, there are many ways to extend this argument into a more detailed analysis. First, finding the value of $\delta$ would improve the accuracy of the model.
A legitimate bound on $\delta$ could be found by examining offenses that have recently experienced an improvement in passing. Another way to extend this analysis would be to run the model after finding the exact initial values of a two-by-two matrix for a large data sample. To do this, one would have to distinguish between defensive plays, classifying each one as defend run or defend pass. To reduce the subjectivity of these assignments, one would have to introduce an algorithm that partitioned the defensive plays. This algorithm would have to take into consideration defensive coverages, blitzes, alignments, and situational factors. And finally, the vision for game-theoretic analysis in sports supports the construction of large payoff matrices that consist of all offensive and defensive strategies, where trends, strengths, and weaknesses are considered. As difficult as this sounds, it can be done by breaking
down film of NFL games. All of these extensions would aid in the development of coaching strategy, which in turn, would evolve the game. For other extensions, Schatz gives a list of potential research problems in this area [11]. Here are a few problems that Schatz suggests:

1. How does weather effect expected returns on offense and on special teams?

2. Is it possible to compare the production of two players that play different positions?


[5] Duane W. Rockerbie. The passing premium puzzle revisited. *Journal of Quan-

2008.


a game-theoretic stochastic dynamic programming approach. *Journal of Sport

1951.


Sports*, 1, 2005.

28
APPENDIX

To find the bounds on $a_{p,p}$ we proceed as follows. Let $q$ be the probability the defense defends the run. The following system consists of the general form of equations (4.1) and (4.2):

$$A = \begin{pmatrix}
qa_{r,r} & (1-q)a_{r,p} & 0 & 0 & \mu_r \\
0 & 0 & qa_{p,r} & (1-q)a_{p,p} & \mu_p \\
qa_{r,r} & -qa_{r,p} & (1-q)a_{p,r} - (1-q)a_{p,p} & 0 & 0
\end{pmatrix}.$$

We use Gaussian elimination to find equations for $a_{r,r}$, $a_{r,p}$, and $a_{p,r}$ depending on $a_{p,p}$. These equations are

$$a_{r,r} = (1 - \frac{1}{q})^2 a_{p,p} - \mu_p (\frac{1}{q} - 1)^2 + \mu_r, \quad (A.1)$$

$$a_{r,p} = (1 - \frac{1}{q}) a_{p,p} + \mu_p (\frac{1}{q} - 1) + \mu_r, \quad \text{and} \quad (A.2)$$

$$a_{p,r} = (1 - \frac{1}{q}) a_{p,p} + \frac{\mu_p}{q}. \quad (A.3)$$

Since we assume that the initial payoffs are positive we have that

$$a_{p,p} > \frac{\mu_p (\frac{1}{q} - 1)^2 - \mu_r}{(1 - \frac{1}{q})^2}, \quad (A.4)$$

$$a_{p,p} < \frac{\mu_p (\frac{1}{q} - 1) - \mu_r}{1 - \frac{1}{q}}, \quad \text{and} \quad (A.5)$$

$$a_{p,p} < \frac{\mu_p}{1 - q}. \quad (A.6)$$
To be sure that the initial payoff matrix $A$ has the desired payoff structure we must force $a_{p,p} < a_{p,r}$. To do this we assume that $a_{p,p} < \mu_p$. We now show that if $a_{p,p} < \mu_p$, then $a_{p,p} < a_{p,r}$. Again, let $q$ be the probability that the defense defends the run. Note that each of the following equations hold:

\[ a_{p,r}q + a_{p,p}(1 - q) = \mu_p, \quad (A.7) \]
\[ a_{p,p}(1 - q) < \mu_p(1 - q), \quad \text{and} \quad (A.8) \]
\[ \mu_p q + \mu_p(1 - q) = \mu_p. \quad (A.9) \]

Using equations A.7, A.8, and A.9 we have that

\[ \mu_p q + a_{p,p}(1 - q) < \mu_p(1 - q) + \mu_p q = \mu_p = a_{p,r}q + a_{p,p}(1 - q) \quad (A.10) \]

Which implies that $a_{p,p} < \mu_p < a_{p,r}$. Thus, to ensure that $a_{p,p} < a_{p,r}$ we choose $a_{p,p} < \mu_p$. We now show that $\mu_p$ is less than the prescribed upper bounds for $a_{p,p}$ found in equations A.5 and A.6. Clearly, $\mu_p < \frac{\mu_p}{1-q}$. To show that $\mu_p < \frac{\mu_p(1 - \frac{1}{q}) - \mu_r}{1 - \frac{1}{q}}$, we assume that $\mu_p > \frac{\mu_p(1 - \frac{1}{q}) - \mu_r}{1 - \frac{1}{q}}$. This implies that $\mu_r < 0$ which is a contradiction. Thus, we choose $a_{p,p}$ from the interval $\left[ \frac{\mu_p(1 - \frac{1}{q})^2 - \mu_r}{(1 - \frac{1}{q})^2}, \mu_p \right]$. 

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