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Operational Modal Parameter Estimation from Short Time-Data Series

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Operational Modal Parameter Estimation from Short Time-Data Series

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Abstract

Operational Modal Analysis (OMA) is a technique of extracting modal parameters of a system from output responses only. It is an emerging field in structural dynamics and has been applied to complex structures that are often difficult to analyze using traditional Experimental Modal Analysis (EMA) techniques. Since the input information is unavailable in OMA, the technique makes use of certain assumptions about the input excitation to the system and the data processing methods used by current OMA algorithms are based on these assumptions. However, in some real-world scenarios not only the forcing function to the system violates these assumptions but also the time data series of system response is short in length and buried under noise. In such cases, ensemble averaging techniques for filtering out noise are rendered ineffective and the current OMA algorithms which utilize power spectrum and correlation functions result in inconsistent modal parameters, especially modal damping.

This research develops a time domain operational modal parameter estimation (MPE) method based on describing the total system response as a Nonlinear Auto Regressive process with eXogenous input (NARX) wherein the linear part of the model describes the response of the structure and the nonlinear terms fit the noise present in data series. The method is developed in line with the concept of Unified Matrix Polynomial Approach (UMPA) and utilizes a set of least squares solutions to compute modal parameters.

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1 Introduction

1.1 Operational Modal Analysis

The field of Operational Modal Analysis (OMA) has recently become an emerging research interest. Operational Modal Analysis is also known as Response-Only Modal Analysis and as the name suggests, the modal parameters are extracted by processing only the system response data. Such a technique becomes especially useful in situations where measurement of input forces to the system is either impossible or extremely difficult. Therefore this method has been applied to large-scale structures where application of traditional EMA methods poses difficulty.

The modal parameter estimation (MPE) from output-only data requires two major assumptions to be met, namely, broadband, random and smooth nature of input in the frequency range of interest and uniform spatial distribution of the forcing excitation. In the past decade or so, several algorithms and signal processing techniques have been developed to extract modal parameters from output data only. Many of these methods have been shown to be analogous to EMA algorithms and several investigations have been made in their capabilities of estimating modal parameters ^{(Chauhan, 2008), (Martell, 2010)}. However, in the face of conditions when the measured data does not absolutely comply with the assumptions on the forcing function regarding its spatial and temporal nature, the current techniques fail to provide a conclusive estimate of modal parameters, especially modal damping ^{(Chauhan, 2008), (Martell, 2010)}. Further, if the system's output data series is short in length and is buried under noise, which in the case of self-excitation problems, like flight flutter or machining chatter, is also correlated with the system's output, the

averaging of the noise from true data becomes rather difficult, therefore yielding inconsistent modal parameter estimates. This issue provides the necessary motivation for the development of a NARX model based approach for estimating modal parameters using output time data series. In such an approach, nonlinear terms are added to a linear Auto-Regressive with eXogenous input (ARX) model to describe the noise and essentially filter out the system's true output data from the noisy data series. Since an ARX model describes the true output of the system, its modal parameters or the roots of its characteristic polynomial are computed by calculating the coefficients of the linear ARX terms.

1.2 Thesis Outline

The thesis is divided into five chapters.

Chapter One introduces the reader to the field of OMA and the goal of research work done, i.e., an accurate estimation of modal parameters from output-only data. It also gives an outline of the work presented in the following chapters of the thesis.

Chapter Two provides a brief overview of the theoretical background of OMA and the literature survey of present OMA algorithms. The problem statement at hand is described in detail and the criticality of an accurate estimation of modal damping is also discussed in this chapter.

Chapter Three focuses on the development of the NARX model based approach to estimate modal parameters from output-only data. The theoretical development of the NARX model for describing Single Input Single Output (SISO) systems and Multiple Input Multiple Output (MIMO) systems to compute their modal parameters is illustrated. This chapter also discusses the q -domain or shift- domain NARX model identification and briefly reviews four different

information criteria and their application in the selection of a parsimonious NARX model for MPE from structural data.

Chapter Four implements the NARX model based approach on analytically generated datasets. Three different test cases are discussed in detail and their results are presented. Also, since recently researchers have suggested the use of δ - operator in identification of a NARX model, a brief investigation of the effect of δ -domain processing on MPE is done by means of a test case. A comparison between processing in the two domains is shown in this test case and the results are presented in the last section of Chapter Four.

Chapter Five summarizes the thesis and draws conclusions from the work done. It also proposes recommendations for future work in this area.

2 Literature Survey

2.1 Theoretical Background

The system response due to an input excitation in the frequency domain can be written as:

$$\{X(\omega)\}_{N_o \times 1} = \{H(\omega)\}_{N_o \times N_i} \{F(\omega)\}_{N_i \times 1} \quad 2.1$$

Equation 2.1 describes the relationship between the system's output response, its characteristics in terms of its poles and zeros buried in the frequency response function (FRF) and input excitation to the system. Post multiplying Equation 2.1 with $\{X(\omega)\}^H$ yields:

$$\{X(\omega)\}_{N_o \times 1} \{X(\omega)\}_{1 \times N_o}^H = \{H(\omega)\}_{N_o \times N_i} \{F(\omega)\}_{N_i \times 1} \{F(\omega)\}_{1 \times N_i}^H \{H(\omega)\}_{N_i \times N_o}^H \quad 2.2$$

$$\{G_{XX}(\omega)\}_{N_o \times N_o} = \{H(\omega)\}_{N_o \times N_i} \{G_{FF}(\omega)\}_{N_i \times N_i} \{H(\omega)\}_{N_i \times N_o}^H \quad 2.3$$

Equation 2.3 can be reduced to a form independent of input excitation if the following two assumptions are fulfilled

1. The forcing function is broadband, random and has no poles and zeros in the frequency range of interest.
2. The input to the system is spatially sufficient, i.e., $N_i \geq N_o$.

therefore resulting in:

$$\{G_{XX}(\omega)\}_{N_o \times N_o} = C * \{H(\omega)\}_{N_o \times N_i} \{I\}_{N_i \times N_i} \{H(\omega)\}_{N_i \times N_o}^H \quad 2.4$$

It is to be noted that since the input functions are not measured, modal scaling cannot be solved for directly. Neglecting the multiplication with C , Equation 2.4 relates output power spectrum (OPS) to FRF as:

$$\{G_{XX}(\omega)\}_{N_o \times N_o} \approx \{H(\omega)\}_{N_o \times N_i} \{H(\omega)\}_{N_i \times N_o}^H \quad 2.5$$

Since modal parameters are characteristics of the FRF matrix, Equation 2.5 shows that modal parameters can also be estimated from the OPS matrix. The partial fraction model for OPS can be written as ^{(Chauhan, 2008), (Peeters & Van der Auweraer, 2005)}:

$$\{G_{XX}(\omega)\} = \sum_{r=1}^N \frac{[R_{XX}]_r}{j\omega - \lambda_r} + \frac{[R_{XX}^*]_r}{j\omega - \lambda_r^*} + \frac{[S_{XX}]_r}{j\omega - (-\lambda_r)} + \frac{[S_{XX}^*]_r}{j\omega - (-\lambda_r^*)} \quad 2.6$$

It should be noted from the partial fraction model shown in Equation 2.6 that the roots of the characteristic equation obtained from processing the OPS data are λ_r , λ_r^* , $-\lambda_r$ and $-\lambda_r^*$.

2.2 OMA Algorithms

There are several algorithms for modal parameter estimation from output-only data in literature. Most of these are either extensions or modifications of traditional EMA algorithms to account for the experimental discrepancies between OMA and EMA. These algorithms can be divided into three classes on the basis of the domain they work in, namely, time domain, frequency domain and spatial domain.

The time domain algorithms use a time series model of the response of the system to estimate its modal parameters. Most algorithms process auto and cross correlation functions of the output to formulate the characteristic equation and compute the modal poles. Some popular examples of such methods are Natural Excitation Technique ^(James, Carne, & Lauffer, 1995), Prediction Error Method ^(Ljung, 1999) and Instrument Variable Method ^(Peeters & De Roeck, 2001). These algorithms essentially set

up a system of least squares equation from correlation functions of output data based on an Auto Regressive Moving Average with eXogenous input (ARMAX) process or an ARX model for describing a linear system. Traditional time domain EMA algorithms like Least Squares Complex Exponential (LSCE) ^(Brown, Allemang, Zimmerman, & Mergeay, 1979), Ibrahim Time Domain (ITD) ^(Ibrahim & Mikulcik, 1977), Polyreference Time Domain (PTD) ^(Vold & Rocklin, 1982) and Eigensystem Realization Algorithm (ERA) ^(Juang & Pappa, 1985) can therefore be extended to estimate modal information from the output correlation data.

The spatial domain algorithms decompose spatial information present in the Input-Output plane at each discrete point on the temporal axis. Such algorithms use the concept of linear superposition and the method of expansion theorem. The response of a system at a given frequency is a linear summation of the response due to each mode. Since the response near a modal frequency is largely dominated by the corresponding modal vector, a decomposition of the frequency domain data in the spatial plane yields the approximate modal vector. The modal frequencies and damping are computed at a later stage after approximating the mode shapes. Some examples of such algorithms are Complex Mode Indicator Function (CMIF) ^(Allemang & Brown, 2006), Enhanced Mode Indicator Function (EMIF) ^{(Phillips, Allemang, & Fladung, 1998), (Fladung, 2001), (Allemang & Brown, 2006)}, Operational Modal Analysis *EMIF* (OMA-EMIF) ^(Chauhan, 2008), Frequency Domain Decomposition (FDD) ^(Brincker, Zhang, & Andersen, 2000) and enhanced Frequency Domain Decomposition (eFDD) ^(Brincker, Ventura, & Andersen, 2000).

Frequency domain algorithms as compared to the time domain algorithms use auto and cross power spectrum to estimate a system's modal parameters. A model similar to the Rational Fraction Polynomial (RFP) ^(Richardson & Formenti, 1982) model shown in Equation 2.7 is fit to the OPS data in a least squares sense and modal poles are computed from the characteristic equation.

$$\{H(\omega)\} = \frac{\sum_{k=0}^n [b_k](j\omega)^k}{\sum_{k=0}^m [a_k](j\omega)^k} \quad 2.7$$

All of the traditional EMA algorithms mentioned above can be explained by a consistent mathematical formulation as described by UMPA^{(Allemang & Brown, 1998), (Allemang & Phillips, 2004)}. The same approach of solving a sequential least squares problem has been extended to formulate the OMA MPE algorithms^{(Chauhan, 2008), (Chauhan, Martell, Allemang, & Brown, 2006)}. It is shown in Equation 2.6 that OPS data has four poles for each structural mode. Due to the high model order of the problem, numerical ill-conditioning of the information matrix when solving in the frequency domain often results in inconclusive results^{(Allemang & Phillips, 2004), (Chauhan, Martell, Allemang, & Brown, 2006)}. To overcome such a difficulty the concept of Positive Power Spectrum (PPS) when solving in frequency domain is proposed^(Chauhan, 2008).

2.3 Problem Definition

It is important to note that the algorithms mentioned in Section 2.2 are based on the two critical assumptions made regarding the nature of input. In order to meet the requirements of these assumptions, various signal processing techniques like Random Decrement Averaging^(Ibrahim, 1977), Root Mean Square (RMS) Averaging^(Phillips, Allemang, & Zucker, 1998) and Cyclic Averaging have been implemented in OMA^(Chauhan, 2008). All of these methods require the measured data series to be long enough in length to separate sufficient ensembles (time blocks) to perform ensemble averaging (for computing expectations). However, in some real-world scenarios not only the forcing function violates the assumptions but also the time data series of system response is short in length and buried under noise. In such cases, signal processing becomes more challenging and the above mentioned averaging techniques become difficult to implement.

One such application is estimation of modal parameters from in-flight structural response data wherein the structure is spatially excited by unmeasured operational aerodynamic forces and the response is measured at various output locations. This process of estimating modal parameters poses the following issues ^(Vecchio, Peeters, & Van der Auweraer, 2002).

1. The time data series is small in length.
2. The structural response is buried under measurement noise due to the aerodynamic turbulence.
3. Input forcing function (aerodynamic forces) of the system cannot be measured.
4. The structure is assumed to be linear in the operating frequency range.

In the face of these issues in the data measurement process, the parameters describing structural response, i.e., modal frequency, modal damping and mode shapes need to be extracted from the data. Of these three parameters, modal damping is often the most important. Modal damping in this case is plotted against wind speed and near-zero damping suggests the onset of flutter (critical velocity) ^(Dimitridis, 2001). Therefore, an accurate estimate of modal damping is required to correctly predict the critical velocity and design a safe flight envelope.

Current UMPA based OMA algorithms utilize the OPS and PPS data in the frequency domain and the correlation functions in the time domain. Although these methods provide a good estimate of the modal frequency and mode shape, they fail to provide a conclusive estimate of modal damping ^{(Chauhan, 2008), (Martell, 2010)}. This can be attributed to the presence of noise which cannot be averaged out due to the small length of time data series, thus making the estimates of damping inaccurate. These shortcomings provide the necessary motivation for investigating other techniques to process the measured output data for extracting modal parameters.

In the literature, the Nonlinear Auto Regressive Moving Average with eXogenous input (NARMAX) and NARX process based model identification methods have been shown to work well with short time data series and under the presence of noise ^{(Dimitridis, 2001), (Kukreja, 2008)}. Also, it has been shown that the structural response of an aero-elastic system can be well described by a process of NARMAX class ^{(Kukreja, 2008), (Dimitridis, 2001)}. Therefore, based on these results, the initial investigation into the NARX process based approach was focused on estimating the modal parameters from in-flight data. However, the thesis develops a general time domain Operational MPE method in line with the concept of UMPA to extract modal information from noisy and short output data series.

3 NARX Approach

3.1 NARMAX and NARX Models

The NARMAX and NARX models ^(Leontaritis & Billings, 1985) are nonlinear counterparts of the ARMAX and ARX processes and have been an interesting research topic in the field of system identification for more than two decades. A number of physical phenomena have been successfully modeled and explained by the NARMAX and NARX model based approaches.

A general NARMAX process can be written as ^{(Billings & Chen, 1989), (Billings & Coca, 2002)}:

$$y(t) = f\left(y(t-1), \dots, y(t-n_y), u(t-d), \dots, u(t-n_u), e(t-1), \dots, e(t-n_e)\right) + e(t) \quad 3.1$$

Equation 3.1 relates output at a given time instant to the past outputs, inputs, noise terms and the current measurement error $e(t)$. The function $f(\cdot)$ is an unknown nonlinear mapping function and n_y, n_u, n_e are the maximum output, input and noise lags respectively.

In a similar fashion, an ARX model can also be extended to account for the nonlinearity thus resulting in the NARX model shown in Equation 3.2 ^{(Chen & Billings, 1989), (Billings & Chen, 1989)}.

$$y(t) = f\left(y(t-1), \dots, y(t-n_y), u(t-d), \dots, u(t-n_u)\right) + e(t) \quad 3.2$$

Since the function $f(\cdot)$ is unknown, the NARMAX or NARX model identification involves estimating an appropriate structure of $f(\cdot)$ and computing the parameters of the model. In the literature, there are various implementations of the NARMAX models of which polynomial and rational representations, neural networks and wavelets are the most common ones. A polynomial type nonlinear map $f(\cdot)$ can be written as ^(Billings & Coca, 2002):

$$f_k(x) = \sum_k^n \theta_k g_k(\mathbf{x}) \quad 3.3$$

where θ_k are the coefficients of the polynomial, g_k is the polynomial of a chosen order and $\mathbf{x}$ is the vector denoted as

$$\mathbf{x} = [y(t-1), \dots, y(t-n_y), u(t-d), \dots, u(t-n_u), e(t-1), \dots, e(t-n_e)] \quad 3.4$$

The polynomial implementation of the NARMAX process shown in Equation 3.3 can be reduced to a linear-in-parameter NARX model by excluding the past noise terms from the vector $\mathbf{x}$ shown in Equation 3.4.

Several algorithms have been developed by researchers to identify the terms of a linear-in-parameter polynomial type NARX model. The key idea in all these algorithms is to evaluate each term from a pool of candidate terms and select the ones which best fit the data. Researchers have utilized both, the least squares and the maximum likelihood approaches, for identification of the model terms.

The algorithms utilizing the least squares approach minimize the cost function or the Mean Square Error (MSE) of the model fit and select the terms which result in maximum reduction of the MSE ^{(Haber & Keviczky, 1976), (Billings & Leontaritis, 1982), (Billings, 1989)}. Orthogonal Least Squares (OLS) algorithm and its variants, Fast Orthogonal Search and Robust Orthogonal Search algorithms, use orthogonal polynomial functions to reduce the computation complexity of matrix inversion. In these algorithms, from each candidate term a function orthogonal to all the previously selected terms is computed and the reduction in MSE associated with the term is computed by calculating the norm of the orthogonal function and its coefficient ^{(Korenberg, 1985), (Korenberg, 1987), (Korenberg, 1989), (Billings & Chen, 1989), (Billings, Chen, & Korenberg, 1988)}. The terms with maximum Error Reduction Ratio

(ERR) are selected and added to the model. Fast Recursive Algorithm (FRA) and its two-stage variant, unlike the OLS, solve the least squares problem recursively and are well-defined under the regression context. These algorithms have been shown to be computationally more efficient and numerically stable compared to the OLS and its variants ^{(Li, Peng, & Irwin, 2005), (Li, Peng, & Bai, 2006)}.

3.2 NARX Implementation on SISO System

The main objective of taking a NARX process based approach is to describe the colored noise present in the measured data series. In this study, the system is assumed to be largely linear in the frequency range of interest and hence the dynamic response of the system can be modeled as an ARMAX process. The transfer function of a linear SISO system can be described in discrete form (z-domain) as an ARMAX model as follows:

$$H(z) = \frac{b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} \quad 3.5$$

Since it is only required to find the natural frequencies, the characteristic equation in the denominator (or the Auto Regressive part of the linear model shown in Equation 3.5) is to be solved for its roots. Also, because the natural frequencies of a system are global properties and do not depend upon where the sensors are mounted on the structure, this approach can be used for any placement of transducers on the system as long as the system is sufficiently observable in the frequency range of interest. The characteristic polynomial equation in z-domain for a SISO system can be written as:

$$a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n} = 0 \quad 3.6$$

Let $y(k)$ be the response of the system at time k . Equation 3.6 therefore results in:

$$a_0 y(k) + a_1 y(k-1) + a_2 y(k-2) + \dots + a_n y(k-n) = 0 \quad 3.7$$

Since the measured output is known at discrete time points, Equation 3.7 can be solved by the method of least squares to compute coefficients a_i which can be used to compute the z-domain roots from Equation 3.6.

It is to be noted that if the excitation to the system is purely impulsive in nature, the input after time $k = 0$ is zero. Therefore with a suitable choice of k , Equation 3.7 can be thought of as the characteristic equation of system response due to impulse excitation in the time domain. Furthermore, the Fourier transform of an impulse function is a flat broadband signal in the frequency domain. Hence, if the input excitation complies with the assumptions mentioned in Section 2.1, Equation 3.6 and Equation 3.7 should yield good estimates of modal poles.

However, if the measured output data is buried under noise which is correlated with the system's output and cannot be averaged out due to short length of the time series, the above approach to estimating modal frequency is rendered inefficient. Furthermore, since the input to the system is not measured, it can so happen that the input forcing function is smooth but colored over the frequency range of interest. Therefore, to account for the measurement noise and the response due to unmeasured forces, linear-in-parameter non-linear functional terms are added to the ARX process thus resulting in a linear-in-parameter NARX process. The resulting discrete time difference equation describing such a process can be written as:

$$a_0 y(k) + a_1 y(k-1) + a_2 y(k-2) + \dots + a_n y(k-n) + \sum_{i=1}^l \beta_i \theta_i = 0 \quad 3.8$$

where,

$y(k)$ = measured response at time $t = k$

a_i = coefficient of characteristic polynomial equation

θ_i = non-linear functional terms of the form $\{[y(k-1)]^c[y(k-2)]^{c_2} \dots [y(k-d)]^{c_d}\}$

β_i = coefficient of non-linear monomial term

d = total output lag to describe non-linear part of the process

n = model order

c_i = power of the monomial term where $i = 1, 2, \dots, d$

Equation 3.8 can be reduced to a set of linear least squares equation by normalizing one of the a_i coefficients and moving the corresponding term to the right hand side. Equation 3.9 shows the normalization of the higher order coefficient a_0 :

$$Y = AX + \epsilon \quad 3.9$$

where,

$$Y = - \begin{Bmatrix} y(k) \\ \vdots \\ y(k+N) \end{Bmatrix} \quad 3.10$$

$$A = \begin{bmatrix} y(k-1) & \dots & y(k-n) & \theta_1(k) & \dots & \theta_l(k) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y(k-1+N) & \dots & y(k-n+N) & \theta_1(k+N) & \dots & \theta_l(k+N) \end{bmatrix} \quad 3.11$$

also denoted as:

$$A = \begin{bmatrix} | & | & | & | & | \\ p_1 & p_2 & \dots & p_k & p_{k+1} & \dots & p_s \\ | & | & | & | & | \end{bmatrix} \quad 3.12$$

$$X = \begin{Bmatrix} a_1 \\ \vdots \\ a_m \\ \beta_1 \\ \vdots \\ \beta_l \end{Bmatrix} \quad 3.13$$

$$\epsilon = \begin{Bmatrix} \epsilon(k) \\ \vdots \\ \epsilon(k + N) \end{Bmatrix} \quad 3.14$$

N = number of equations at successive discrete time points

$\epsilon(k)$ = residual or measurement noise at time $t = k$

s = total number of selected monomial terms

3.3 NARX Implementation on MIMO System

UMPA^{(Allemang & Brown, 1998), (Allemang & Phillips, 2004)} extends an ARX model describing a SISO system (LSCE type algorithm) to one describing a MIMO system (PTD or ERA type algorithm) by converting the scalar coefficient polynomial characteristic equation into a matrix coefficient polynomial characteristic equation. Therefore, the characteristic polynomial equation in z -domain for a MIMO system can be written in as:

$$[a_0]_{N_o \times N_o} z^0 + [a_1]_{N_o \times N_o} z^{-1} + [a_2]_{N_o \times N_o} z^{-2} + \dots [a_n]_{N_o \times N_o} z^{-n} = 0 \quad 3.15$$

This approach of converting a scalar coefficient polynomial characteristic equation to a matrix coefficient polynomial characteristic equation is utilized to extend the single output NARX model shown in Equation 3.8 to a multiple output NARX model. In traditional EMA, since both the input and output functions are measured, the time domain MPE algorithms are classified into low and high order algorithms based on the size of the coefficient matrix, which in turn is governed by the number of input and output degrees of freedom (DOFs). However, in OMA since the output of the system at the response locations is the only measurement being made, the size of matrix coefficients is governed by the number of output DOFs only. Therefore, a NARX model describing a MIMO system can be written as:

$$[a_0]_{N_o \times N_o} \{y(k)\}_{N_o \times 1} + [a_1]_{N_o \times N_o} \{y(k-1)\}_{N_o \times 1} + \dots [a_n]_{N_o \times N_o} \{y(k-n)\}_{N_o \times 1} + \sum_{i=1}^l [\beta_i]_{N_o \times N_o} \{\theta_i\}_{N_o \times 1} = \{0\}_{N_o \times 1} \quad 3.16$$

where,

$\{y(k)\}_{N_o \times 1}$ = measured response vector at time $t = k$ denoted as:

$$\{y(k)\}_{N_o \times 1} = [y_1(k), y_2(k) \dots, y_{N_o}(k)]^T \quad 3.17$$

$[a_i]_{N_o \times N_o}$ = matrix coefficient of characteristic polynomial equation

$\{\theta_i\}_{N_o \times 1}$ = column vector of non-linear functional terms denoted as:

$$\{\theta_i\}_{N_o \times 1} = \begin{Bmatrix} (y_1(k-1))^{c_1} (y_1(k-2))^{c_2} \dots (y_1(k-d))^{c_d} \\ (y_2(k-1))^{c_1} (y_2(k-2))^{c_2} \dots (y_2(k-d))^{c_d} \\ \vdots \\ (y_{N_o}(k-1))^{c_1} (y_{N_o}(k-2))^{c_2} \dots (y_{N_o}(k-d))^{c_d} \end{Bmatrix} \quad 3.18$$

$[\beta_i]_{N_o \times N_o}$ = matrix coefficient of vector of non-linear monomial terms

n = model order

d = total output lag to describe non-linear part of the process

c_i = power of the monomial term where $i = 1, 2, \dots, d$

N_o = number of output DOFs or response locations

Equation 3.16 is a linear-in-parameter NARX model which describes the response of the system at multiple response locations in terms of the past responses at the given output DOFs. The resemblance between the SISO NARX model described by Equation 3.8 and the MIMO NARX

model described by Equation 3.16 can be readily noticed with the only difference of scalar and matrix coefficients of the characteristic polynomial equation in the respective models.

As shown in Equation 3.9 for the SISO NARX model, Equation 3.16 can also be reformulated as a set of linear least squares equations by normalizing one of the matrix coefficients $[a_i]$ to $[I]$ and moving the corresponding term to the right hand side. Equation 3.19 shows the normalization of the higher order coefficient $[a_0]$:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \epsilon \quad 3.19$$

where,

$$\mathbf{Y} = - \begin{bmatrix} \{y(k)\}^T \\ \vdots \\ \{y(k+N)\}^T \end{bmatrix} \quad 3.20$$

$$\mathbf{A} = \begin{bmatrix} \{y(k-1)\}^T & \dots & \{y(k-n)\}^T & \{\theta_1(k)\}^T & \dots & \{\theta_l(k)\}^T \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \{y(k-1+N)\}^T & \dots & \{y(k-n+N)\}^T & \{\theta_1(k+N)\}^T & \dots & \{\theta_l(k+N)\}^T \end{bmatrix} \quad 3.21$$

also denoted as:

$$\mathbf{A} = \left[\begin{pmatrix} | \\ \mathbf{p}_1 \\ | \end{pmatrix} \begin{pmatrix} | \\ \mathbf{p}_2 \\ | \end{pmatrix} \dots \begin{pmatrix} | \\ \mathbf{p}_k \\ | \end{pmatrix} \begin{pmatrix} | \\ \mathbf{p}_{k+1} \\ | \end{pmatrix} \dots \begin{pmatrix} | \\ \mathbf{p}_s \\ | \end{pmatrix} \right] \quad 3.22$$

$$\mathbf{X} = \begin{Bmatrix} [a_1]_{N_o \times N_o} \\ \vdots \\ [a_n]_{N_o \times N_o} \\ [\beta_1]_{N_o \times N_o} \\ \vdots \\ [\beta_l]_{N_o \times N_o} \end{Bmatrix} \quad 3.23$$

$$\epsilon = \begin{bmatrix} \{\epsilon(k)\}^T \\ \vdots \\ \{\epsilon(k+N)\}^T \end{bmatrix} \quad 3.24$$

$\{\epsilon(k)\}$ = residual error vector of size $N_o \times 1$ at time $t = k$ denoted as:

$$\{\epsilon(k)\}_{N_o \times 1} = [\epsilon_1(k), \epsilon_2(k) \dots, \epsilon_{N_o}(k)]^T \quad 3.25$$

In case of the MIMO NARX model, it is also important to note the structure of the vector θ_i which is comprised of the nonlinear terms that describe the noise present in the output data series. It is shown in Equation 3.18 that although each row of the vector θ_i is a non-linear monomial function of previous responses at the corresponding output DOF, the structure of the non-linearity is same throughout the column vector. Hence, similar to the identification of a non-linear monomial term in the SISO NARX model, a complete column vector of non-linear monomial terms of the same structure of non-linearity is selected in the MIMO NARX model. A detailed discussion of the selection of the NARX model terms is presented in Section 3.4.

3.4 Selection of NARX Model Terms

3.4.1 SISO System

Since, the linear terms associated with the scalar coefficients a_i form the system's characteristic equation which has the modal parameters buried in it, the selection of these terms is based on the model order which in turn is based on the number of poles to be computed. For a SISO system with scalar coefficients, this relationship is straight forward and the number of modal poles

computed is equal to the model order n . Therefore, the number of linear terms retained in the model should be equal to the number of modal poles that need to be computed.

For a chosen model order, the selection of the appropriate nonlinear monomial terms of the SISO NARX model from a candidate pool is based on minimization of the Sum Squared Errors (SSE) or the cost function of the complete model fit as mentioned in Section 3.1. The cost function J of a SISO model fit with m linear-in-parameter model terms is given as ^{(Haykin, 2002), (Li, Peng, & Irwin, 2005), (Li, Peng, & Bai, 2006)}.

$$J(A_m) = Y^T Y - Y^T A_m (A_m^T A_m)^{-1} A_m^T Y \quad 3.26$$

A_m in Equation 3.26 is the intermediate information matrix with m terms, including both the linear and non-linear monomials, having been selected and is denoted as:

$$A_m = [p_1, p_2, \dots, p_m] \quad 3.27$$

If an additional term is added to the information matrix, the resulting decrease in the cost function is given as ^(Li, Peng, & Bai, 2006).

$$\Delta J_{m+1} (A_{m+1}) = J(A_m) - J([A_m, p_{m+1}]) \quad 3.28$$

Since the cost function $J(A_m)$ is an indicator of the quality of the model fit with m terms having already been selected, an addition of a term in the model should result in reduction of the cost function. Therefore, in the selection process each new term is selected such that the chosen term, among all other terms in the candidate pool, maximizes the contribution to the model fit. i.e., $\Delta J_{m+1} (A_{m+1})$ is maximum.

3.4.2 MIMO System

As in the case of a SISO system, the linear monomial term vectors associated with the matrix coefficients $[a_i]$ form the system's characteristic equation for a MIMO model. Therefore, the selection of these vectors for a MIMO NARX model is also based on the model order, which in turn is based on the number of modal poles to be computed. For a MIMO system with matrix coefficients, the number of poles computed is equal to the product of the size of the coefficients matrix N_o and model order n (Allemang & Brown, 1998), (Allemang & Phillips, 2004). Since these poles also include the computational noise poles which are not the system's structural poles, for a given size of the coefficients matrix, a model order is chosen such that,

$$\text{Total number of poles} \geq N_o \times n$$

Once a model order is chosen based on the above mentioned relationship, the non-linear monomial term vectors of the NARX MIMO model from a candidate pool are selected such that the cost function of the model fit is minimized. For a MIMO system, the cost function of the model fit with m linear-in-parameter model term vectors is given as:

$$J(\mathbf{A}_m) = \text{trace} (\mathbf{Y}^T \mathbf{Y} - \mathbf{Y}^T \mathbf{A}_m (\mathbf{A}_m^T \mathbf{A}_m)^{-1} \mathbf{A}_m^T \mathbf{Y}) \quad 3.29$$

$\mathbf{A}_m$ in Equation 3.29 is the intermediate information matrix with m vectors, including both the linear and non-linear monomials, having been selected and is denoted as:

$$\mathbf{A}_m = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m] \quad 3.30$$

Since the selection of each new vector should minimize the cost function, a vector is selected such that the chosen vector, among all other monomial term vectors in the candidate pool, maximizes the contribution to the model fit. i.e., $\Delta J_{m+1} (\mathbf{A}_{m+1})$ is maximum. The constructional

form of such a vector is shown in Equation 3.18 and $\Delta J_{m+1}(\mathbf{A}_{m+1})$ is computed in the same manner as shown in Equation 3.28.

It is to be noted that the above formulation of cost function involving the trace of the resulting matrix inside the parenthesis in Equation 3.29 ensures that the reduction in cost function of the model fit is a collective reduction in the cost function of each output DOF, i.e.,

$$\text{trace}(\mathbf{Y}^T \mathbf{Y}) = Y_1^T Y_1 + Y_2^T Y_2 + \dots + Y_{N_o}^T Y_{N_o} \quad 3.31$$

Equation 3.31 also suggests that an output DOF with largest response magnitude is weighed the highest in the reduction of cost function while one with smallest response magnitude is weighed the lowest, as in the case of a least squares problem.

3.4.3 Parsimonious Model Selection

At this point it is also important to note that in both the cases of SISO and MIMO systems, with each new selected term, although the model fit improves, the complexity of the model is also increased. Therefore a balance between the model fit and the model complexity is needed to achieve the most optimal model for describing a given time data series. In order to obtain a parsimonious model, an information criterion is used which weighs the model fit against its complexity. A brief study of four information criteria existing in the literature and their implementation in identifying the NARX model terms for estimating the modal parameters from operational data is presented in the following subsections.

3.4.3.1 Akaike Information Criterion

Akaike Information Criterion (AIC) was first introduced as a method for the identification of models (Akaike, 1974). However, since the use of such a criterion for identifying models from a large

candidate pool is computationally very expensive, researchers have used it to obtain parsimony in the model selected by step-wise approaches^(Li, Peng, & Bai, 2006). AIC value is given as^(Hu, 2007):

$$AIC\ value = -2 \log f(y|\hat{\theta}) + 2k \quad 3.32$$

for Maximum Likelihood Estimation and

$$AIC\ value = n \log \left(\frac{RSS}{n} \right) + 2k \quad 3.33$$

for Least Squares Estimation

where,

$f(.)$ = maximized value of likelihood of the estimated model

n = number of observations or equations to form an over-determined set

k = number of parameters of the estimated model

RSS = residual sum of squared errors

While the first term in the formula for AIC value determines the goodness of the model fit, the second term is a penalty term on the complexity of the model. AIC gives an approximately unbiased estimate of the expected Kullback distance between the true model and the fitted model. Kullback distance is essentially a measure of the information lost when a fitted model of finite dimension is used to approximate the true model of an infinite dimension. AIC is also an asymptotically efficient criterion, i.e., AIC will asymptotically select the fitted candidate model which minimizes the mean squared error of prediction without considering a true model when large sample data is available^{(Burnham & Anderson, 2004), (Cavanaugh, 2009)}.

3.4.3.2 Bayesian Information Criterion

Bayesian Information Criterion (BIC) is also known as Schwarz Information Criterion after the name of the author of the paper who first presented it. BIC is a method of model selection from a candidate pool on the basis of Bayesian posterior probability of the model instead of using Bayes factor to compare only two models ^{(Kass & Raftery, 1995), (Schwarz, 1978)}. BIC value can be computed as ^(Cavanaugh & Neath, 2012).

$$BIC\ value = -2 \log f(y|\hat{\theta}) + k \log n \quad 3.34$$

for Maximum Likelihood Estimation and

$$BIC\ value = n \log \left(\frac{RSS}{n} \right) + k \log n \quad 3.35$$

for Least Squares Estimation.

As in the case of AIC, the first term in the formula for BIC indicates the goodness of model fit and the second term penalizes for the complexity of the model. Unlike AIC, BIC is a consistent criterion, i.e., if the true model is of a finite dimension and is represented in the candidate pool under consideration, BIC will essentially select the model having the correct structure with probability one ^(Cavanaugh, 2009). Although BIC is a large-sample estimator in Bayesian analyses, since it selects more parsimonious models, it is often chosen over AIC in frequentist analyses.

3.4.3.3 Deviance Information Criterion

Deviance Information Criterion (DIC) was introduced directly as a method to compare the optimality of models by quantifying their parsimony ^(Spiegelhalter, Best, Carlin, & Van der Linde, 2002). DIC value is computed as:

$$DIC\ value = -2 \log f(y|\hat{\theta}) + p_D \quad 3.36$$

for Maximum Likelihood Estimation and

$$DIC\ value = n \log \left(\frac{RSS}{n} \right) + p_D \quad 3.37$$

for Least Squares Estimation.

p_D is the effective number of parameters and is computed by subtracting the deviance calculated at posterior mean of the parameters from the posterior mean deviance ^(Helser, Lai, & Black, 2012). The goodness of fit term is the same as in the case of AIC and BIC, however, the complexity of the model is evaluated by the effective number of parameters. Therefore, DIC is usually implemented when the number of parameters is more than the number of observation points. Non-hierarchical models are best identified with this method. Further, DIC, unlike BIC but like AIC, is an asymptotically efficient criterion and does not consider any true model, therefore making it good for short-term predictions ^(Spiegelhalter, 2006).

3.4.3.4 Hannan-Quinn Information Criterion

Hannan-Quinn Information Criterion (HQIC) was proposed as an alternative criterion to AIC and BIC, wherein, HQIC value is given as ^(Hannan & Quinn, 1979):

$$HQIC\ value = -2 \log f(y|\hat{\theta}) + 2k \log(\log n) \quad 3.38$$

for Maximum Likelihood Estimation and

$$HQIC\ value = n \log \left(\frac{RSS}{n} \right) + 2k \log(\log n) \quad 3.39$$

for Least Squares Estimation.

While AIC considers only the estimation uncertainty and BIC considers both the estimation and parameter uncertainty, HQIC considers the estimation uncertainty and the logarithm of parameter uncertainty. Therefore, HQIC penalizes the complexity of the model more than AIC but less than BIC. It is also noted that HQIC, like BIC, is not an estimator of the Kullback discrepancy, but considers a true model, hence making it a consistent criterion ^(Burnham & Anderson, 1998).

3.4.3.5 Implementation of AIC and BIC in NARX Model Identification

Since in all the above discussed information criteria the key idea is to penalize the complexity of the model, the selected model should have the minimum information criterion value. It is also to be noted that AIC and BIC, apart from being asymptotically efficient and consistent respectively, can also be classified as predictive and descriptive criterions ^(Cavanaugh, 2009) and since DIC and HQIC have similar properties as AIC and BIC, they can be considered as derived variants of AIC and BIC respectively. Further, AIC and BIC have been more widely mentioned in the literature for identification of the NARX models. Therefore, for the purpose of this thesis, only AIC and BIC are implemented in identification of the NARX model terms.

For the NARX SISO model, the AIC or BIC values can be readily computed as shown in Equation 3.33 and Equation 3.35 by substituting k with the total number of terms in the NARX model, n with the number of discrete time points used to formulate Equation 3.9 and RSS with the cost function of the intermediate model fit A_m computed in Equation 3.26. These values are calculated for each intermediate model and once a term p_{m+1} is chosen from the candidate pool by maximizing $\Delta J_{m+1}(A_{m+1})$, its contribution to the model fit is weighed against the added complexity of the model and the term is retained if the following relationship is satisfied.

$$(\text{AIC or BIC})_{A_{m+1}} < (\text{AIC or BIC})_{A_m} \quad 3.40$$

Since the NARX MIMO model has matrix coefficients, although RSS can be substituted with $J(\mathbf{A}_m)$, there is not a clear choice for the variables k and n to compute AIC/BIC values for the model fit. Therefore, four statistical cases are considered as shown in Table 3.1.

Case Number	Number of parameters, k	Number of observations, n
Case I	Total no. of monomial term vectors $\times 1$ Coefficient matrix is considered as one parameter	N Number of discrete time points used to formulate Equation 3.19
Case II	Total no. of monomial term vectors $\times N_o$ Number of parameters in equation of one DOF	N Number of discrete time points used to formulate Equation 3.19
Case III	Total no. of monomial term vectors $\times N_o^2$ All individual elements of coefficient matrix	$N \times N_o$ Total number of equations for all output DOFs
Case IV	Total no. of monomial term vectors $\times \frac{N_o(N_o+1)}{2}$ Non-symmetric elements of coefficient matrix	$N \times N_o$ Total number of equations for all output DOFs

Table 3.1 Possible variable substitutions for computing AIC/BIC values (NARX MIMO)

Case II is found to work well for both AIC and BIC in selection of the NARX MIMO model and also best justifies the formulation of the cost function as a collective cost function of each output DOF as shown in Equation 3.31. Therefore, the AIC and BIC values are computed for each intermediate model fit $\mathbf{A}_m$ by substituting the variables k and n as mentioned in Case II and as in the case of the NARX SISO model, the decision regarding the retention of chosen monomial term vector $\mathbf{p}_{m+1}$ is made on the basis of the relationship shown in Equation 3.40.

3.5 Computation and Selection of Modal Parameters

Having selected the linear-in-parameter NARX model and identified the model terms as shown in the previous section, the next step towards computing modal parameters is to compute the

coefficients or the parameters of the constructed model which are scalars in the case of the single output system and matrices in case of the multiple output system. For a system of equations as shown in Equation 3.9 and Equation 3.19, the parameters of the model are computed by the method of least squares as ^(Haykin, 2002).

$$X = (A^T A)^{-1} A^T Y \quad 3.41$$

for a SISO system and

$$X = (A^T A)^{-1} A^T Y \quad 3.42$$

for a MIMO system.

It is important to note that although the parameter vector X contains both a_i and β_i coefficients only the a_i coefficients are associated with the characteristic polynomial equation of the system. Therefore, modal parameters are computed by forming a companion matrix using the a_i coefficients and an Eigenvalue Decomposition (ED) is performed on the companion matrix to obtain the eigenvalues and eigenvectors from which the modal poles and, in case of multiple outputs, the associated mode shapes are extracted ^(Allemang & Phillips, 2004).

As in case of traditional EMA UPMA methods, an estimate of the number of system poles present in the frequency range of interest is obtained by the power spectrum based CMIF or Singular Value Percentage Contribution (SVPC) plot ^(Chauhan, 2008). However, since the true model order is unknown due to the presence of noise on the measured data, the model order is iterated over a range and pole-consistency and pole-density diagrams are generated to select the system poles and the associated mode shapes ^(Phillips, Allemang, & Brown, 2011). The same approach of using the

pole-consistency and pole-density diagrams is implemented in this method, which are generated by performing following iterations:

1. As in the traditional EMA UMPA methods, the model order is iterated over a range based on the estimate obtained from CMIF or SVPC plot of OPS matrix.
2. Since in a NARX model the accuracy of the model depends on the nonlinear terms selected from the candidate pool, the sufficiency of the candidate pool of the nonlinear monomial terms or vectors (in case of MIMO systems) is an important concern ^(Dimitridis, 2001). For the linear-in-parameter polynomial type NARX model shown in Equation 3.8 and Equation 3.16, the sufficiency of the candidate pool is determined by the maximum output lag d and the power of the nonlinear monomial term, i.e., $\sum c_i$. Although an estimate of the above quantities can be obtained through *apriori* knowledge of the system to be analyzed, since the true values are unknown, an iterative approach is taken. Therefore, for a chosen model order, the candidate pool is varied by iterating the maximum output lag d and the power of the nonlinear monomial term $\sum c_i$.
3. Since the normalization of the high or low order coefficient of the characteristic polynomial has an effect on the location of computational poles due to presence of noise ^(Allemang, 2008), normalization of both the high and low order coefficients is performed for identification of the nonlinear model terms. In other words, the least squares equations are developed for both the normalizations to form $\mathbf{A}_m$ and compute the cost function $J(\mathbf{A}_m)$ and difference in the cost function $\Delta J_{m+1}(\mathbf{A}_{m+1})$. Further, both AIC and BIC are utilized for the selection of terms and poles are computed from both the selected models.
4. Finally, the least squares solution for computing the parameters of the model after the selection of terms is also performed for both the normalizations. Equation 3.43 shows the

low order coefficient normalization for the SISO system from which the set of linear equations can be formulated as shown in Equation 3.9.

$$a_0 y(k) + a_1 y(k-1) + \dots + a_{n-1} y(k-n+1) + \sum_{i=1}^l \beta_i \theta_i = -y(k-n) \quad 3.43$$

Equation 3.44 shows the low order coefficient normalization for the MIMO system from which the set of linear equations can be formulated as shown in Equation 3.19.

$$[a_0]_{N_o \times N_o} \{y(k)\}_{N_o \times 1} + \dots + [a_{n-1}]_{N_o \times N_o} \{y(k-n+1)\}_{N_o \times 1} + \sum_{i=1}^l [\beta_i]_{N_o \times N_o} \{\theta_i\}_{N_o \times 1} = -\{y(k-n)\}_{N_o \times 1} \quad 3.44$$

The above iterations can be represented algorithmically in a flowchart as shown in Figure 3.1.

Figure 3.1 Flowchart showing the iterations performed to generate pole-stability diagrams

Having computed all the poles and the associated vectors (in case of multiple outputs) for all the above iterations in the sequence shown in Figure 3.1, the pole-density and pole-consistency diagrams are generated. It is to be noted that the system is assumed to be largely linear in the frequency range of interest, therefore, the true structural poles of the system excluding the computational poles due to noise, should remain consistent irrespective of any of the above mentioned iterations performed. Hence, the clusters of consistent system poles can easily be identified on a pole-density plot presented in the complex plane. Further, to select poles with consistent modal vectors, a matrix of pole-weighted state vectors is assembled and an SVD is performed on the matrix to ensure the consistency of cluster by observing the number of significant singular values and the associated right singular vectors. Also, MAC (Modal Assurance Criterion) values are computed for all pole-weighted state vector pairs and the pole-vector pairs with the MAC values above a predetermined threshold are selected. The modal parameters are then extracted from these pole-weighted state vectors. Readers interested in the details of this procedure of generating clear pole-consistency diagrams and selecting modal parameters can refer to (Phillips, Allemang, & Brown, 2011).

4 Test Cases

To check the effectiveness of the NARX model based operational MPE method as developed in Chapter 3, the algorithm is implemented on three sets of data generated analytically with M, C and K matrices and the results are presented in this chapter. These datasets are generated keeping in mind the different physical conditions in which the NARX model based approach may be utilized such as in a machine-tool vibration problem and an in-flight flutter data analysis.

4.1 Test Case I

The first test data set is generated keeping in mind the conditions wherein, the nonlinearity results from the closed loop interaction of the system with the ambience. To simulate such a condition, an impulse response of a system with nonlinearity is computed analytically for a chosen set of initial conditions as shown in Equation 4.1.

$$\{y(k)\} = [a_1]_{6 \times 6} \{y(k-1)\}_{6 \times 1} + [a_2]_{6 \times 6} \{y(k-2)\}_{6 \times 1} + [\beta_1]_{6 \times 6} \{(y(k-2))^2\}_{6 \times 1} \quad 4.1$$

where, $\{y(k)\} = [0.1, 0.01, 0, 0, 0.01, 0.1]^T$ for $k \leq 0$

The coefficients $[a_1]$ and $[a_2]$ are computed from M, C and K matrices of a light to moderately damped six DOF system. The coefficient $[\beta_1]$ is generated by trial and error such that the overall system is stable and the response of system does not grow exponentially. Also, to simulate real-world measurement, white random noise of the signal-to-noise Ratio (SNR) 10 dB is added to the resulting data series. Then an FFT is applied on the final time series and only the data in frequency range of 0-128 Hz is inverse Fourier transformed to the time domain from which the modal parameters are estimated using a NARX model.

The resulting response data of the system at three output DOFs both with and without the inclusion of the term associated with the coefficient $[\beta_1]$ is shown in Figure 4.1. A considerable difference in the two datasets at each output DOF can be readily noticed. These three output DOFs are further used to estimate the modal parameters from the time data series.

As mentioned in Section 3.5, all the model iterations are performed and poles from all the models selected at each iteration stage are computed to generate the raw pole-density diagram. By a raw pole-density diagram it is meant that no poles are rejected from the plot on the basis of any consistency criteria. In other words, it can be called an “unclear” pole-density diagram.

Figure 4.2 shows the resulting pole-density diagram wherein, the real part of the pole is plotted on the Y-axis and imaginary part of the pole is plotted on the X-axis. The clusters of poles around the true pole values can be easily noticed from the figure. These clusters around each mode are zoomed in and plotted in Figure 4.3. The poles from these clusters can be further found on the basis of higher levels of parameter consistency according to the procedure mentioned in (Phillips, Allemang, & Brown, 2011).

Having found a set of consistent modal poles, to get an indication of the confidence level in the estimated modal parameters, a statistical evaluation of the set is performed and the results are shown in Table 4.1. From the table it can be noticed that the estimates of modal frequency and damping are not only consistent but also reasonably accurate.

Figure 4.1 Impulse response of system at three output DOFs for Test Case I

Figure 4.2 Raw pole-density diagram generated utilizing NARX model for Test Case I

Figure 4.3 Zoomed pole-density diagram for Test Case I

	True Pole (Real), (Hz)	Mean of Est. Poles (Real), (Hz)	Variance of Est. Poles (Real), (Hz)	True Pole (Imag), (Hz)	Mean of Est. Poles (Imag), (Hz)	Variance of Est. Poles (Imag), (Hz)
Mode 1	-0.059	-0.066	0.0053	9.674	9.661	1.070
Mode 2	-0.613	-0.603	0.081	28.735	28.787	2.372
Mode 3	-1.421	-1.434	0.1971	42.958	42.935	1.619

Table 4.1 Statistical evaluation of consistent modal poles estimated in Test Case I

4.2 Test Case II

The second test data set is generated to simulate the conditions wherein the measurement noise present on the data is correlated with system output at previous time instants. Therefore, the data

is generated by first computing the response of a very lightly damped, six DOF system to random broadband input excitation as shown in Equation 4.2.

$$\{y(k)\} = \mathcal{F}^{-1} [\{H(\omega)\}_{6 \times 6} \{F(\omega)\}_{6 \times 1}] \quad 4.2$$

where,

$$H(\omega) = [Ms^2 + Cs + K]^{-1} \text{ for } s = j\omega$$

$$F(\omega) = \text{Random broadband excitation}$$

Second, correlated noise is added to the response data as shown in Equation 4.3.

$$\{y'(k)\} = \{y(k)\} + [\beta_1]\{(y(k-1))^2\} \quad 4.3$$

The coefficient $[\beta_1]$ is chosen at random and scaled such that, along with additional random broadband noise, the resulting SNR of the time series $y'(k)$ is 10 dB. Further, as done in the previous test case, the FFT is applied on the final time series $y'(k)$ and only the data in the frequency range of 0-128 Hz is inverse Fourier transformed for the purpose of MPE using the NARX model.

The resulting response data of the system at the three output DOFs both, with and without the inclusion of the term associated with the coefficient $[\beta_1]$ is shown in Figure 4.4. Again a considerable difference in the two datasets at each output DOF can be readily noticed. These three output DOFs are further used to estimate the modal parameters from the time data series.

Figure 4.4 Response of system at three output DOFs for Test Case II

Figure 4.5 Raw pole-density diagram generated utilizing NARX model for Test Case II

Figure 4.6 Zoomed pole-density diagram for Test Case II

Figure 4.5 shows the resulting raw pole-density diagram wherein, the real part of the pole is plotted on the Y-axis and imaginary part of the pole is plotted on the X-axis. The clusters of poles around the true pole values are further zoomed in and plotted in Figure 4.6. A set of consistent modal parameters is found as mentioned in the previous test case and the statistical evaluation of the set of consistent modal poles is presented in Table 4.2.

	True Pole (Real), (Hz)	Mean of Est. Poles (Real), (Hz)	Variance of Est. Poles (Real), (Hz)	True Pole (Imag), (Hz)	Mean of Est. Poles (Imag), (Hz)	Variance of Est. Poles (Imag), (Hz)
Mode 1	-0.0029	-0.0051	0.00041	21.633	21.636	2.952
Mode 2	-0.0261	-0.0248	0.0069	64.487	63.482	1.266
Mode 3	-0.0586	-0.0562	0.0033	96.541	96.539	0.989

Table 4.2 Statistical evaluation of consistent modal poles estimated in Test Case II

Figure 4.7 Comparison of use of AIC and BIC in NARX model based MPE

Further, to justify the use of both AIC and BIC in identification of the NARX model terms and computation of modal parameters from the structural data, the consistency of modal poles computed using both the criteria is evaluated by means of a pole-density diagram. Figure 4.7 shows the resulting pole-density diagram. It can be noticed that although the poles computed using both AIC and BIC spread approximately the same space in the complex plane, AIC shows denser clusters for the first two modes.

4.3 Test Case III

The third test data set is generated to simulate conditions wherein the input, although broadband, random and smooth in the frequency range of interest, also has a certain color characteristic. Further, to check the effect of insufficient spatial excitation on MPE using the NARX model, the excitation is provided at only one input DOF and the response of the system is measured at six

output DOFs. The system used is the same light to moderately damped system used in the first test case and the response of the system is computed as shown in Equation 4.4.

$$\{y(k)\} = \mathcal{F}^{-1} [\{H(\omega)\}_{6 \times 6} \{F(\omega)\}_{6 \times 1}] \quad 4.4$$

where, $F(\omega) = [f_1(\omega), 0, 0, 0, 0, 0]$ for all ω

The magnitude and phase of the excitation signal $f_1(\omega)$ in the frequency range of 0-128 Hz is shown in Figure 4.8 and the resulting response of the system at three output DOFs with an addition of random broadband noise is shown in Figure 4.9. These three output DOFs are further used to estimate the modal parameters from the time data series.

Figure 4.8 Excitation signal provided to the system at one input DOF for Test Case III

Figure 4.9 System response at three output DOFs due to colored input for Test Case III

Figure 4.10 Raw pole-density diagram generated utilizing NARX model for Test Case III

Figure 4.11 Zoomed pole-density diagram for Test Case III

	True Pole (Real), (Hz)	Mean of Est. Poles (Real), (Hz)	Variance of Est. Poles (Real), (Hz)	True Pole (Imag), (Hz)	Mean of Est. Poles (Imag), (Hz)	Variance of Est. Poles (Imag), (Hz)
Mode 1	-0.059	-0.068	0.0098	9.674	9.516	0.629
Mode 2	-0.613	-0.884	0.472	28.735	28.605	1.951
Mode 3	-1.421	-2.003	0.987	42.958	43.490	2.016

Table 4.3 Statistical evaluation of consistent modal poles estimated in Test Case III

Figure 4.10 shows the resulting raw pole-density diagram where the real part of the pole is plotted on the Y-axis and imaginary part of the pole is plotted on the X-axis. The region near the poles clusters around each mode is zoomed in and plotted in Figure 4.11. It can be noticed from the figure that the clusters around the true pole values are not as dense as in the previous test

cases and hardly any modal poles are found in the left half plane with the low order coefficient normalization. Therefore, a less stringent consistency criterion is followed to select the modal parameters from these clusters and the consequence on the accuracy of their estimation can be readily noticed from the statistical evaluation of this set presented in Table 4.3.

4.4 NARX Model Identification in δ -domain

The NARX model discussed in Chapter Three consists of monomial terms which are both linear and nonlinear functions of the lagged outputs. Therefore, if a shift operator called q - operator is defined such that,

$$q [y(t)] = y(t + 1) \quad 4.5$$

the model shown in Equation 3.16 is referred to as a shift- domain or q - domain model. It has been shown that due to the numerical similarity of the data values at successive time points at high sampling rates, the structure selection of a polynomial type NARX model poses numerical conditioning problems when identified in q -domain ^(Billings & Aguirre, 1995). To this effect of solving the problem of numerical ill-conditioning, the use of δ -operator for identification of the NARX model is proposed and a one-to-one mapping equivalence of a polynomial type NARX model between the δ - and the q -domain is proved ^(Anderson & Kadiramanathan, 2007).

The delta- operator is defined as:

$$\delta = \frac{q-1}{\Delta t} \quad 4.6$$

and an equivalent δ -domain model of the NARX model shown in Equation 3.2 is written as:

$$\delta^n y(t) = f(y(t), \delta y(t), \dots, \delta^{n-1} y(t), u(t), \delta u(t), \dots, \delta^{n-1} u(t)) + \delta^n e(t) \quad 4.7$$

For the purpose of this thesis, a brief investigation into the effect of model identification in δ -domain on estimation of the modal parameters is made and the results are discussed in this section. For this case study, the response of a single degree of freedom (SDOF) system to an impulse excitation is generated at various sampling rates and no kind of nonlinearity or noise is added to the data. This dataset is then processed, in both the δ - and the q -domain, to compute the modal poles.

Even though the model structure is known in this case and the equivalent δ -domain model of an ARX model describing an SDOF system can be written as shown in Equation 4.8, the model identification process is carried out in both the domains to compare their effects on the estimation of modal parameters.

$$\alpha_2 \delta^2 y(t) + \alpha_1 \delta y(t) + \alpha_0 y(t) = 0 \quad 4.8$$

Case	True pole (Hz)	Est. pole, q -domain (Hz)	Est. pole, δ -domain (Hz)
$F_{\text{samp}} = 256 \text{ Hz}$	$-6.39 + 44.72j$	$-6.33 + 45.04j$	$-6.78 + 45.32j$
$F_{\text{samp}} = 256 \text{ Hz}$	$-11.64 + 90.38j$	$-11.51 + 89.97j$	$-11.35 + 89.66j$
$F_{\text{samp}} = 1024 \text{ Hz}$	$-25.56 + 178.88j$	$-25.81 + 177.65j$	$-25.61 + 178.72j$
$F_{\text{samp}} = 1024 \text{ Hz}$	$-46.56 + 361.52j$	$-47.01 + 363.46j$	$-45.72 + 362.50j$
$F_{\text{samp}} = 4196 \text{ Hz}$	$-51.12 + 357.76j$	$-53.94 + 360.23j$	$-50.53 + 358.83j$
$F_{\text{samp}} = 4196 \text{ Hz}$	$-93.12 + 1446.08j$	$-97.37 + 1454.71j$	$-94.20 + 1448.99j$
$F_{\text{samp}} = 4196 \text{ Hz}$	$-106.67 + 1821.47j$	$-113.31 + 1806.19j$	$-108.81 + 1827.75j$

Table 4.4 MPE results for model identification in δ - and q -domain

Having identified the model, the coefficients of the model are computed by solving a set of deterministic linear equations. Then using these coefficients, the roots or the modal poles of the system's characteristic polynomial are estimated. A comparison of the results obtained in the two cases with the true value of modal poles is presented in Table 4.4. It can be seen from the table that q -domain processing affects the accuracy of pole estimation at high sampling rates if the pole is located near the Nyquist frequency of the measurement.

5 Conclusions and Future Work

5.1 Summary and Conclusions

The NARX model based approach towards estimating modal parameters of a system from a short and noisy time series of the output-only data is presented in this thesis. The method is developed in line with the concept of UMPA and has been successfully extended to the MIMO systems. To check the performance of the method, it is tested on three sets of analytically generated data and the modal frequency and modal damping estimates are obtained. In this process, a comparative study of various information criteria is also conducted to understand the differences in their application to structural data buried under noise.

Based on the test cases discussed in Chapter 4, the following conclusions can be made regarding the NARX model based approach:

- The method is successfully able to describe the colored or correlated or biased noise present in the data series of system's response. Therefore the method yields good estimates of the modal parameters even from short data records on which the previously discussed averaging techniques cannot be performed effectively. This ability of the NARX model based approach to work with short and noisy data series makes it applicable to problems like analysis of in-flight data for the prediction of flutter.
- When the system is spatially insufficiently excited by a colored input signal, the accuracy of the estimation of modal damping does not change considerably as suggested by the mean value of the estimated poles. However, the precision of estimation is adversely affected and there is a noticeable increase in the variance of the damping estimates.

- AIC and BIC are extended to the identification of the NARX model for a MIMO system and in spite of the differences between the two criterion, both have been shown to work well with the structural data buried under noise. Also, since the use of a descriptive or a predictive information criterion does not affect the consistency of the modal parameters of a system as shown for the second test case, iterating the information criterion is justified for generating the pole-stability diagram.
- The identification of NARX model in δ -domain with respect to estimating the modal parameters is briefly investigated. The difference in the results of MPE by processing in δ -domain and q -domain can be seen only at high sampling rates therefore suggesting the sufficiency of processing in q -domain for operational MPE of large structures which have their modes at low frequencies.

5.2 Recommendations for Future Work

The approach presented in this thesis to estimate the modal parameters from noisy and short time series is still in a nascent stage and although, it has been tested on analytically generated datasets, the first step towards consolidating the method is to check its ability to accurately estimate modal damping by intensive testing on experimental data obtained by in-laboratory testing. Further, it can be applied to real-world testing problems like analyzing in-flight data to extract the operational modal characteristics of the system for predicting the flutter velocity.

In order to improve the computation speed of the algorithm, a fast and recursive method which is also well-defined within the regression context could be developed for the case of the multiple output DOFs by extending the work shown in (Li, Peng, & Irwin, 2005). An in-depth investigation into identifying the NARX model in δ -domain for the purpose of operational MPE

can be done and the technique can be extended to systems with multiple output DOFs. Also, a correlation analysis based approach to identify the nonlinear models with input nonlinearities as shown in (Lang, Futterer, & Billings, 2005) can be extended to extract modal parameters from the output-only data with colored or biased noise present in the time series.

For completely developing this method as a time domain nonlinear UMPA algorithm, a low order NARX model based approach can be devised wherein, instead of iterating the model order as in the case of a high order UMPA algorithm, the size of coefficient matrix is condensed and iterated over a range of values to generate the pole-density diagram. Since there are nonlinear terms in the NARX model to describe the noise, it is important to investigate if techniques like SVD and ED, as in the case of traditional EMA algorithms, can be used for condensing the size of coefficient matrix of the NARX model.

Since an estimate of the model order of the linear ARX part of the NARX model can be obtained from a CMIF plot, the iteration of the model order can be avoided by choosing a sufficiently large but fixed model order. An approach similar to the approach of adding generalized residuals in the frequency domain shown in (Fladung, 2001) could be implemented on the NARX model. In such an approach, the nonlinear terms of the NARX model can be considered as noise residual terms which are iterated for generating a pole-density diagram and the model order is kept constant.

Also, it would be an interesting and challenging exercise to develop a similar approach for MPE in the frequency domain wherein the nonlinear polynomial terms can be added to the numerator of the RFP or RFP-z (for better numerical conditioning) model to describe the input noise terms. Since the basis of such an approach would be an RFP model, as in the case of the frequency

domain UMPA algorithms, residuals terms can also be added to account for the contribution of the modes outside the bandwidth being processed.

6 References

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