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Eigenfunctions of Composition Operators on Bloch-type Spaces

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the
Doctor of Philosophy Degree in Mathematics

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Suppose φ is a holomorphic self map of the unit disk $\mathbb{D}$, C_φ is a composition operator with the symbol φ which fixes the origin and $0 < |\varphi'(0)| < 1$. In this work, we study the solutions of Schröder equation for composition operators C_φ . More precisely, we study properties of the symbol φ , which ensure all these solutions belong to a Bloch-type space $\mathcal{B}_\alpha$ for $\alpha > 0$. Also, we extend our results from composition operators to weighted composition operators in the second part of Chapter 2.

The lattice of closed invariant subspaces of the shift operator acting on the Hardy space is completely described by Beurling's Theorem [3]. Aleman and Korenblum [1] explicitly found the lattice of Volterra operator acting on the Hardy space with the help of Beurling's Theorem. On the other hand, Sarason [20], [21] found the lattice of Volterra operator and multiplication by x plus Volterra operator acting on $L^2(0, 1)$. We study the lattice of closed invariant subspaces of shift plus complex Volterra operator acting on the Hardy space. We use Korenblum's result from [13] to prove our main theorem. This part of our work is presented in Chapter 3.

I dedicate this dissertation to my mother, father and my wife.

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List of Symbols

$\mathbb{D}$	the open unit disk in the complex plane
$\partial\mathbb{D}$	boundary of $\mathbb{D}$, or the unit circle
$H(\mathbb{D})$	Space of all holomorphic functions on the unit disk $\mathbb{D}$
H^∞	Algebra of bounded holomorphic functions on the unit disk $\mathbb{D}$
φ	Holomorphic self map of $\mathbb{D}$ that fixes the origin and is not an automorphism of $\mathbb{D}$
φ_k	k^{th} iteration of φ
$\varphi^{(h_\alpha)}$	Hyperbolic α -derivative of φ
$\mathcal{B}_\alpha$	Alpha Bloch Space
C_φ	Composition operator
λ	Complex constant
σ	Königs function of φ
u	Holomorphic map of the unit disk
uC_φ	Weighted composition operator
v	principal eigenfunction of uC_φ
$\mathcal{H}^2(\mathbb{D})$	Hardy Space
l^2	Space of square summable sequences
L^2	Space of square integrable functions
$\mathcal{S}^2(\mathbb{D})$	Space of holomorphic functions whose first derivative is in $\mathcal{H}^2(\mathbb{D})$
M_z	Shift operator
V	Volterra operator
$\mathcal{V}$	Complex Volterra operator
T	Shift plus complex Volterra operator

Chapter 1

Preliminaries

In this chapter we give an introduction to composition operators and Bloch-type spaces. We also introduce some notation that will be used throughout this thesis.

We assume $H(\mathbb{D})$ is collection of holomorphic functions defined on the unit disk, $\mathbb{D}$, in the complex plane. The space of bounded holomorphic functions is denoted by H^∞ and defined by

$$H^\infty = \{f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)| < \infty\}.$$

Its norm is given by

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

1.1 Bloch type spaces

The Bloch space, named after a French mathematician André Bloch (1893-1948) and denoted by $\mathcal{B}$, is the space that contains holomorphic functions on the unit disk,

whose derivative growth satisfies certain restriction. More precisely,

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty\}.$$

The norm on the Bloch space is given by

$$\|f\| = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|. \quad (1.1.1)$$

Now, let us generalize the concept of Bloch space to the α -Bloch spaces. For each $\alpha > 0$, we let $\mathcal{B}_\alpha$ denote the space of holomorphic functions f on $\mathbb{D}$ satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

If $f \in \mathcal{B}_\alpha$, the norm is given by

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|. \quad (1.1.2)$$

It is clear that the usual Bloch space $\mathcal{B}$ coincides with the space $\mathcal{B}_1$. From the definition of Bloch-type spaces, it immediately follows that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ for $\alpha \leq \beta$. Now for $f \in \mathcal{B}_\alpha$, we see that

$$|f'(z)| \leq \frac{\|f\|_{\mathcal{B}_\alpha}}{(1 - |z|^2)^\alpha}, \quad \text{for all } z \in \mathbb{D}.$$

Now using above estimate, we get

$$\begin{aligned}
|f(z) - f(0)| &= \left| \int_0^1 z f'(zt) dt \right| \\
&\leq \int_0^1 |z f'(zt)| dt \\
&\leq \|f\|_{\mathcal{B}_\alpha} \int_0^1 \frac{|z|}{(1 - |z|^2 t^2)^\alpha} dt \\
&\leq \|f\|_{\mathcal{B}_\alpha} \int_0^1 \frac{|z|}{(1 - |z|t)^\alpha} dt.
\end{aligned}$$

Suppose $1 - |z|t = u$ then $-|z|dt = du$.

$$\begin{aligned}
\int \frac{|z|}{(1 - |z|t)^\alpha} dt &= - \int \frac{1}{u^\alpha} du \\
&= \begin{cases} \log \frac{1}{u}, & \text{if } \alpha = 1 \\ \frac{1}{(\alpha - 1)} u^{1-\alpha}, & \text{if } \alpha \neq 1 \end{cases},
\end{aligned}$$

which shows that

$$\begin{aligned}
|f(z) - f(0)| &\leq \|f\|_{\mathcal{B}_\alpha} \int_0^1 \frac{|z|}{(1 - |z|t)^\alpha} dt \\
&= \begin{cases} \|f\|_{\mathcal{B}_\alpha} \log \frac{1}{1 - |z|}, & \text{if } \alpha = 1 \\ \|f\|_{\mathcal{B}_\alpha} \frac{1}{(\alpha - 1)} ((1 - |z|)^{1-\alpha} - 1), & \text{if } \alpha \neq 1 \end{cases}. \quad (1.1.3)
\end{aligned}$$

In case $\alpha < 1$ and $f \in \mathcal{B}_\alpha$, from 1.1.3 it follows that $\mathcal{B}_\alpha \subset H^\infty$. It is easy to check that in this case the linear space $\mathcal{B}_\alpha$ is an algebra. In fact, Hardy and Littlewood [8] have shown that for $0 < \alpha < 1$, the space $\mathcal{B}_\alpha$ consists of all functions f holomorphic

on $\mathbb{D}$ satisfying the Lipschitz condition

$$|f(z) - f(w)| \leq M|z - w|^{1-\alpha}, \quad \text{for all } z, w \in \mathbb{D}.$$

If $f \in \mathcal{B}$ then (1.1.3) implies that

$$|f(z) - f(0)| \leq \|f\|_{\mathcal{B}} \log \frac{1}{1 - |z|}. \quad (1.1.4)$$

Functions in the Bloch space have been studied extensively by many authors. See [2], [22] for the theory of Bloch-type functions. The following theorem enables us to view Bloch-type spaces for $\alpha > 1$ in a different way.

Theorem 1.1.1. *Suppose $\alpha > 1$. Then f is in $\mathcal{B}_\alpha$ if and only if*

$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1} |f(z)|$ is bounded in $\mathbb{D}$.

Proof. See [22, Proposition 7]. □

In case $\alpha > 1$, the above theorem suggests that this norm is equivalent to the $\alpha - 1$ Lipschitz-type norm:

$$\|f\|_{\mathcal{B}_\alpha} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1} |f(z)|, \quad f \in \mathcal{B}_\alpha, \alpha > 1. \quad (1.1.5)$$

The following lemma is standard and useful in complex analysis.

Lemma 1.1.2. (*Schwarz Lemma*)

Suppose f is holomorphic self map of $\mathbb{D}$, and $f(0) = 0$. Then

$$|f(z)| \leq |z| \quad (1.1.6)$$

$$|f'(0)| \leq 1. \quad (1.1.7)$$

If equality holds in (1.1.6) for one $z \in \mathbb{D} \setminus \{0\}$, or if equality holds in (1.1.7), then $f(z) = \lambda z$, where λ is a constant, $|\lambda| = 1$.

Proof. See [18, Theorem 12.2]. □

The next theorem is at the heart of the subject.

Theorem 1.1.3. (*Schwarz-Pick theorem*)

Let f be holomorphic self map of $\mathbb{D}$. Then, for all $z_1, z_2 \in \mathbb{D}$,

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|$$

and, for all $z \in \mathbb{D}$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

The space of bounded holomorphic functions on the unit disk, H^∞ , is a subset of the Bloch space that can be shown by using Schwarz-Pick theorem: suppose $f \in H^\infty$ and let that $g = \frac{f}{\|f\|_\infty}$, then $\|g\|_\infty = 1$. So, applying Schwarz-Pick theorem 1.1.3,

we have

$$\frac{|g'(z)|}{1 - |g(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

From this it follows that

$$(1 - |z|^2)|g'(z)| \leq 1 - |g(z)|^2 \leq 1.$$

This shows that $\|f\|_{\mathcal{B}} \leq \|f\|_{\infty}$ and hence $H^{\infty} \subset \mathcal{B}$. On the other hand, H^{∞} is a proper subspace of $\mathcal{B}$. For example, the function

$$f(z) = \log(1 - z)$$

is in $\mathcal{B}$ but not in H^{∞} .

1.2 Composition Operators

Composing functions, in $H(\mathbb{D})$, with any holomorphic self-map φ of $\mathbb{D}$, induces a linear transformation, denoted by C_{φ} and called a composition operator:

$$C_{\varphi}f = f \circ \varphi, \quad \text{for } f \in H(\mathbb{D}).$$

Here φ is called the symbol of C_{φ} .

The concept of a composition operator can be generalized to a weighted composition operator by multiplying the composition operator by a holomorphic function

defined in the unit disk. Suppose $u \in H(\mathbb{D})$ and φ as above then

$$uC_\varphi(f) = (u)(f \circ \varphi) \quad (1.2.1)$$

is known as the weighted composition operator with symbol φ and the weight u . Composition operators are widely studied in operator theory. These operators are usually defined on Banach spaces of holomorphic functions including Hardy, Bergman and Bloch spaces (See, [5], [19], [16]). In our work, we are interested in studying the composition operators acting on Bloch-type spaces. The following theorem provides the necessary and sufficient conditions for the boundedness of a weighted composition operator acting on the Bloch-type spaces.

Theorem 1.2.1. *Let φ and u be analytic on $\mathbb{D}$, φ a self-map of $\mathbb{D}$. Let α and β be positive real numbers.*

1. *If $0 < \alpha < 1$, then uC_φ maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if and only if $u \in \mathcal{B}_\beta$ and*

$$\sup_{z \in \mathbb{D}} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$

2. *The operator uC_φ maps $\mathcal{B}$ boundedly into $\mathcal{B}_\beta$ if and only if*

$$\sup_{z \in \mathbb{D}} |u'(z)| (1 - |z|^2)^\beta \log \frac{1}{1 - |\varphi(z)|^2} < \infty$$

and

$$\sup_{z \in \mathbb{D}} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)} |\varphi'(z)| < \infty.$$

3. If $\alpha > 1$, the uC_φ maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if and only if

$$\sup_{z \in \mathbb{D}} |u'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha-1}} < \infty$$

and

$$\sup_{z \in \mathbb{D}} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$

Proof. See [16, Theorem 2.1] □

This is the most general theorem for boundedness of the weighted composition operators acting on Bloch-type spaces. If we assume $u \equiv 1$ in the theorem, we get corresponding boundedness conditions for the composition operators. Suppose $\alpha = \beta$ and $u \equiv 1$ in the Theorem 1.2.1. Then it gives that the operator C_φ maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\alpha$ for any $\alpha > 0$ if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty. \quad (1.2.2)$$

Since φ is holomorphic self map of the unit disk, Schwarz-Pick Theorem 1.1.3 says that

$$\frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)} \leq 1, \quad \text{for } z \in \mathbb{D},$$

is always true. Hence, for any symbol φ , holomorphic self map of $\mathbb{D}$, the operator C_φ maps $\mathcal{B}$ boundedly into $\mathcal{B}$.

Compactness of composition operators plays important role in the study of eigenfunctions of composition operators. The following theorem provides the necessary and sufficient conditions for the compactness of a weighted composition operator

acting on Bloch-type spaces.

Theorem 1.2.2. *Let u be holomorphic function on $\mathbb{D}$ and let φ be holomorphic self map of $\mathbb{D}$. Let α and β be positive real numbers, and suppose that uC_φ maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$.*

1. *If $0 < \alpha < 1$ then uC_φ maps $\mathcal{B}_\alpha$ compactly into $\mathcal{B}_\beta$ if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$

2. *The operator uC_φ maps $\mathcal{B}$ compactly into $\mathcal{B}_\beta$ if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u'(z)| (1 - |z|^2)^\beta \log \frac{1}{(1 - |\varphi(z)|^2)} = 0$$

and

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)} |\varphi'(z)| = 0.$$

3. *If $\alpha > 1$, then uC_φ maps $\mathcal{B}_\alpha$ compactly into $\mathcal{B}_\beta$ if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u'(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha-1}} = 0$$

and

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$

Proof. See [16, Theorem 3.1] □

If we take u to be the constant function 1, we obtain characterization of compact-

ness of composition operators between Bloch type spaces. Further if we take $\alpha = \beta$, we see that composition operator C_φ maps $\mathcal{B}_\alpha$ compactly into $\mathcal{B}_\alpha$ if and only if

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0. \quad (1.2.3)$$

The following proposition provides a sufficient condition for compactness of weighted composition operators acting on Bloch-type spaces $\mathcal{B}_\alpha$ for some positive number α .

Proposition 1.2.3. *Let $\alpha > 0$ and uC_φ is bounded on $\mathcal{B}_\alpha$. Then, if $\|\varphi\|_\infty < 1$, uC_φ is compact on $\mathcal{B}_\alpha$.*

Proof. It is enough to show that if $f_n \in \mathcal{B}_\alpha$, $\|f_n\|_{\mathcal{B}_\alpha} \leq 1$ for all n and $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|uC_\varphi f_n\|_{\mathcal{B}_\alpha} \rightarrow 0$ as $n \rightarrow \infty$. We know that

$$\begin{aligned} \|uC_\varphi f_n\|_{\mathcal{B}_\alpha} &= \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |(uC_\varphi f_n(w))'| \\ &\leq \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u(w)f'_n(\varphi(w))\varphi'(w)| + \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u'(w)f_n(\varphi(w))|. \end{aligned} \quad (1.2.4)$$

Since uC_φ is bounded on $\mathcal{B}_\alpha$, $u \in \mathcal{B}_\alpha$. Also, since uC_φ is bounded and $\|\varphi\|_\infty < 1$, there exists $M > 0$ such that

$$\sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u(w)\varphi'(w)| < M.$$

Now

$$\begin{aligned} \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u(w) f'_n(\varphi(w)) \varphi'(w)| &\leq \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u(w) \varphi'(w)| \sup_{w \in \mathbb{D}} |f'_n(\varphi(w))| \\ &\leq M \sup_{|w| \leq \|\varphi\|_\infty} |f'_n(w)|. \end{aligned} \quad (1.2.5)$$

Since f_n converges to zero uniformly on any compact subset of $\mathbb{D}$ as $n \rightarrow \infty$, so is true for f'_n . Hence right hand side of (1.2.5) goes to 0 as $n \rightarrow \infty$. On the other hand,

$$\begin{aligned} \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u'(w) f_n(\varphi(w))| &\leq \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |u'(w)| \sup_{w \in \mathbb{D}} |f_n(\varphi(w))| \\ &\leq \|u\|_{\mathcal{B}_\alpha} \sup_{|w| \leq \|\varphi\|_\infty} |f_n(w)| \end{aligned} \quad (1.2.6)$$

Hence right hand side of (1.2.6) goes to 0 as $n \rightarrow \infty$. Using (1.2.5) and (1.2.6), we see that right hand side of (1.2.4) goes to 0 as $n \rightarrow \infty$. This completes the proof of the proposition. $\square$

It is clear from the Proposition 1.2.3 that composition operator C_φ is compact on $\mathcal{B}_\alpha$ for $\alpha > 0$ if $\|\varphi\|_\infty < 1$.

1.3 Eigenfunctions of composition operators

In the early 1870s, Ernst Schröder introduced the eigenvalue equation for composition operators. Suppose symbol φ is a homomorphic self map of $\mathbb{D}$ and $f \in H(\mathbb{D})$ then

$$(C_\varphi)f(z) = (f \circ \phi)(z) = \lambda f(z), \quad (1.3.1)$$

where λ is complex constant and $z \in \mathbb{D}$, is known as Schröder's equation. Königs [12] was first to study this equation in the one complex variable setting.

Theorem 1.3.1. *Königs's Theorem (1884)*

Assume φ is a holomorphic self map of $\mathbb{D}$ such that $\phi(a) = a$ for some $a \in \mathbb{D}$ and $0 < |\varphi'(a)| < 1$. Then the following assertions hold.

i. The sequence of functions

$$\sigma_k(z) := \frac{\varphi_k(z) - a}{\varphi'(a)^k},$$

where φ_k is k^{th} iterates of φ , converges uniformly on a compact subset of $\mathbb{D}$ to a non-constant function σ that satisfies (1.3.1) with $\lambda = \varphi'(a)$.

ii. f and λ satisfy (1.3.1) if and only if there is a positive integer n such that $\lambda = \varphi'(a)^n$ and f is a constant multiple of σ^n .

Proof. See [19, Chapter 6] □

From now on we reserve the symbol σ to denote the unique eigenfunction of Schröder's equation corresponding to eigenvalue $\lambda = \varphi'(a)$. This function is called *Königs function* of φ , or the *principal eigenfunction* of C_φ . The function σ has derivative 1 at a , it is univalent if the symbol φ is univalent, and also it fixes the origin. The assumption $0 < |\varphi'(a)| < 1$ in the theorem is needed to avoid trivial situation, whereas $\varphi(a) = a$ is essential for the proof.

Let us assume $\psi_a(z) = \frac{a - z}{1 - \bar{a}z}$, which is a bijective holomorphic self map of the unit disk. Now assume $\psi = \psi_a \circ \varphi \circ \psi_a$. Clearly ψ fixes the origin and $\psi'(0) =$

$\psi'_a(a)\varphi'(a)\psi'_a(0)$, and by an easy calculation we see that $\psi'(0) = \varphi'(a)$. On the other hand, if F is an eigenfunction of C_ψ with corresponding eigenvalue λ then

$$C_\psi F = \lambda F$$

$$F \circ \psi_a \circ \varphi \circ \psi_a = \lambda F$$

$$F \circ \psi_a \circ \varphi = \lambda (F \circ \psi_a)$$

$$C_\varphi f = \lambda f, \quad \text{where } f = F \circ \psi_a.$$

Hence the eigenfunctions of C_ψ and C_φ are related by $f = F \circ \psi_a$. This shows that it suffices to study eigenfunctions of a composition operator with a symbol that fixes the origin. From now on, we make the standing assumption that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$.

Since Königs theorem characterizes all the eigenfunctions for C_φ acting on $H(\mathbb{D})$, the study of eigenfunctions of C_φ acting on any subspace of $H(\mathbb{D})$ reduces to the study of Königs function σ . Almost 100 years after the Königs' work, Caughran and Schwartz [4] studied the eigenfunctions for the composition operator acting on the Hardy space (see section 3.1) and showed that: if C_φ is compact on the Hardy space, not only σ but also σ^n belong to the Hardy space for all non-negative integers n . In fact this is true for all Banach spaces of holomorphic functions.

Cowen and MacCluer [6] characterized the solutions of the Schröder equation in several variable under some natural hypotheses on the symbol. They gave necessary and sufficient conditions for the existence of an eigenfunction and then described all analytic solutions in the ball.

Now we define Schröder's equation in more general sense. Let us assume u be a holomorphic map on $\mathbb{D}$ and φ as above, then

$$uC_{\varphi}f = \lambda f, \quad (1.3.2)$$

where $f \in H(\mathbb{D})$ and λ is a complex constant, is generalization of equation (1.3.1), and known as “weighted” Schröder's functional equation for weighted composition operator. The following theorem characterizes all the eigenfunctions and eigenvalues of a weighted composition operator under some restriction on the symbol.

Theorem 1.3.2. *Assume φ is a holomorphic self map of $\mathbb{D}$ and u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0, \varphi(0) = 0, 0 < |\varphi'(0)| < 1$. Then, the following statements hold.*

i. *The sequence of functions*

$$v_k(z) = \frac{u(z)u(\varphi(z))\dots u(\varphi_{k-1}(z))}{u(0)^k}$$

where φ_k is k^{th} iterates of φ , converges to a non-constant holomorphic function v of $\mathbb{D}$ that satisfies (1.3.2) with $\lambda = u(0)$.

ii. *f and λ satisfy (1.3.2) if and only if $f = v\sigma^n$, $\lambda = u(0)\varphi'(0)^n$, where n is a non-negative integer and σ is the solution of the Schröder equation (1.3.1) $\sigma \circ \varphi = \varphi'(0)\sigma$.*

Proof. See [10, Lemma 3.2 and Proposition 3.3]. □

From now on, we reserve the symbol v for the principal eigenfunction of uC_φ with corresponding eigenvalue $u(0)$. The function gives value 1 at zero, and it is bounded if u is bounded and $\|\varphi_k\|_\infty < 1$ for some positive integer k . Hosokawa and Nguyen [10] further studied the eigenfunctions of uC_φ acting on the Bloch space and obtained the following theorem. The theorem provides a sufficient condition that ensure $\{v\sigma^n\}$ belongs to the Bloch space for all non-negative integers n .

Theorem 1.3.3. *Assume φ is a holomorphic self map of $\mathbb{D}$ with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$. Let us also assume that uC_φ is bounded on $\mathcal{B}$. For $0 < r < 1$, set*

$$M_r(\varphi) := \sup_{|z|=r} |\varphi(z)| \quad a_r := \sup_{|z|=r} (|u'(z)\varphi(z)| + |u(z)\varphi'(z)|).$$

Suppose that

$$i. \lim_{r \rightarrow 1} \log(1-r) \log M_r(\varphi) = \infty.$$

$$ii. \log |a_r| < \epsilon \log(1-r) \log M_r(\varphi),$$

where $\epsilon > 0$ is a constant satisfying $\epsilon \log \|\varphi\|_\infty > -1$.

Then $v\sigma^n \in \mathcal{B}$ for all non-negative integer n .

Proof. See [10, Theorem 3.5]. □

On the other hand, Hammond in [7] studied spectrum of weighted composition operator in more general spaces. He proved the following theorem.

Theorem 1.3.4. *If uC_φ is a compact operator on any Banach space of holomorphic functions then all the eigenfunctions $\{v\sigma^n\}_0^\infty$ belong to the Banach space.*

Proof. See [7]

□

Chapter 2

Eigenfunctions

In this chapter we study the composition operators and properties of their symbols in order to understand the growth of derivative of Königs function of the symbol. More precisely, we are interested in finding the properties of the symbol which ensure that all the positive integer powers of Königs functions belong to Bloch-type space $\mathcal{B}_\alpha$ for some $\alpha \in (0, \infty)$. Also, we will discuss the similar problem for weighted composition operators.

2.1 Eigenfunctions of Composition Operators

Königs Theorem 1.3.1 is the fundamental work in the study of Schröder's functional equation for composition operators with a symbol that fixes the origin. Let us recall the Schröder's functional equation for composition operators:

$$(C_\varphi)f(z) = (f \circ \varphi)(z) = \lambda f(z), \tag{2.1.1}$$

where $f \in H(\mathbb{D})$ and λ is a complex constant. Further recall that σ is a solution of (2.1.1) called Königs function of symbol φ which fixes origin and $0 < |\varphi'(0)| < 1$. Also from Königs Theorem, we know that $\sigma(0) = 0$ and $\sigma'(0) = 1$.

Definition 2.1.1. Let us define the *Hyperbolic α -derivative* of φ at $z \in \mathbb{D}$ by

$$\varphi^{(h_\alpha)}(z) = \frac{(1 - |z|^2)^\alpha \varphi'(z)}{(1 - |\varphi(z)|^2)^\alpha}.$$

When $\alpha = 1$ then it is simply called the Hyperbolic derivative of φ at z and denoted by $\varphi^{(h)}(z)$.

Definition 2.1.2. Suppose φ is holomorphic self map of $\mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$ and φ_m is the m^{th} iteration of φ for some fixed non-negative integer m . Then we say φ satisfies condition (A) if there exists a non-negative integer m such that

$$|\varphi^{(h_\alpha)}(\varphi_m(z))| = \frac{(1 - |\varphi_m(z)|^2)^\alpha |\varphi'(\varphi_m(z))|}{(1 - |\varphi_{m+1}(z)|^2)^\alpha} \leq |\varphi'(0)|, \quad (\text{A})$$

for all $z \in \mathbb{D}$ and for some fixed $\alpha > 0$.

Remark 2.1.3. If φ satisfies the condition (A) for m then it satisfies the condition for all non-negative integers greater than m .

The following example provides a family of maps that satisfies condition (A).

Example 2.1.4. Consider a map γ that maps the unit disk univalently to the right half plane. This map is given by

$$\gamma(z) = \frac{1+z}{1-z}.$$

For any $t \in (0, 1)$, define

$$\varphi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}.$$

It is well known that φ_t maps the unit disk into the unit disk for each $t \in (0, 1)$, see [19] . These maps are known as *lens map*.

Claim 2.1.5. φ_t satisfies condition (A) for $\alpha = 1$ and $m = 0$. That is to say for all $t \in (0, 1)$, $|\varphi_t^{(h)}(z)| \leq |\varphi_t'(0)|$ for all $z \in \mathbb{D}$.

Proof. Clearly, $\varphi_t(0) = 0$ and

$$|\varphi_t'(z)| = \frac{2t |\gamma(z)^{t-1}| |\gamma'(z)|}{|\gamma(z)^t + 1|^2}.$$

Since $\gamma'(z) = \frac{2}{(1-z)^2}$, we see that $|\varphi'_t(0)| = t$. It is known that image of φ_t touches the boundary of the unit disk non tangentially at 1 and -1 . Now put $w = \gamma(z) = re^{i\theta}$, we see that

$$\begin{aligned} |\varphi_t^{(h)}(z)| &= \frac{1-|z|^2}{1-\left|\frac{w^t-1}{w^t+1}\right|^2} \frac{2t |w^{t-1}| |w'|}{|w^t+1|^2} \\ &= \frac{1-|z|^2}{|w^t+1|^2 - |w^t-1|^2} 2t |w^{t-1}| |w'|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |w^t+1|^2 - |w^t-1|^2 &= (w^t+1)\overline{(w^t+1)} - (w^t-1)\overline{(w^t-1)} \\ &= (w^t+1)(\overline{w}^t+1) - (w^t-1)(\overline{w}^t-1) \\ &= 2(w^t + \overline{w}^t) \\ &= 2r^t(e^{it\theta} + e^{-it\theta}) \\ &= 4r^t \cos t\theta. \end{aligned}$$

And $w' = \gamma'(z) = \frac{2}{(1-z)^2}$

$$|\varphi_t^{(h)}(z)| = \frac{1-|z|^2}{|1-z|^2} \frac{t r^{t-1} |e^{i(t-1)\theta}|}{r^t \cos t\theta}.$$

Using $z = \frac{w-1}{w+1}$, we get

$$\begin{aligned}
|\varphi_t^{(h)}(z)| &= \frac{1 - \left|\frac{w-1}{w+1}\right|^2}{\left|1 - \frac{w-1}{w+1}\right|^2} \frac{t r^{t-1}}{r^t \cos t\theta} \\
&= \frac{|w+1|^2 - |w-1|^2}{4} \frac{t r^{t-1}}{r^t \cos t\theta} \\
&= \frac{4 r \cos \theta}{4} \frac{t r^{t-1}}{r^t \cos t\theta} \\
&= \frac{t \cos \theta}{\cos t\theta}.
\end{aligned}$$

If $z \in (-1, 1)$ then $\gamma(z) \in \mathbb{R}_+$. Therefore $\theta = 0$ and so $|\varphi_t^{(h)}(z)| = t$. On the other hand if $z \in \mathbb{D} \setminus (-1, 1)$ then $|\theta| \in (0, \pi/2)$. Hence $\cos t\theta > \cos \theta > 0$ and so $|\varphi_t^{(h)}(z)| < t$. This completes the proof. $\square$

Remark 2.1.6. From the proof of Claim 2.1.5, we see that $|\varphi_t^{(h)}(z)| \rightarrow 0$ as z approaches the boundary of the unit disk along the real-axis. Hence, using compactness condition provided by (1.2.3), the composition operator with symbol φ_t is a non-compact operator on $\mathcal{B}$.

The following theorem provides the sufficient condition that ensures the Königs function belongs to Bloch-type spaces.

Theorem 2.1.7. *Assume C_φ is bounded on $\mathcal{B}_\alpha$ and φ satisfies the condition (A) for some $\alpha > 0$ and for some fixed non-negative integer m . Then σ belongs to $\mathcal{B}_\alpha$.*

Proof. Since C_φ is bounded on $\mathcal{B}_\alpha$, using (1.2.2) there exists positive number M such that

$$(1 - |z|^2)^\alpha |\varphi'(z)| \leq M(1 - |\varphi(z)|^2)^\alpha, \quad \text{for } z \in \mathbb{D}. \quad (2.1.2)$$

For the m given by the assumption, choose non-negative integer k such that $k > m$.

For $z \in \mathbb{D}$, we have

$$\begin{aligned} (1 - |z|^2)^\alpha |\varphi'_k(z)| &= (1 - |z|^2)^\alpha |\varphi'(\varphi_{k-1}(z))\varphi'(\varphi_{k-2}(z))\dots\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))\dots\varphi'(z)| \\ &= (1 - |z|^2)^\alpha |\varphi'(z)\varphi'(\varphi(z)) \dots\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))\dots\varphi'(\varphi_{k-2}(z)) \varphi'(\varphi_{k-1}(z))|. \end{aligned}$$

By using (2.1.2),

$$(1 - |z|^2)^\alpha |\varphi'_k(z)| \leq M(1 - |\varphi(z)|^2)^\alpha |\varphi'(\varphi(z)) \dots\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))\dots\varphi'(\varphi_{k-2}(z)) \varphi'(\varphi_{k-1}(z))|.$$

Again using (2.1.2) repeatedly, we get

$$(1 - |z|^2)^\alpha |\varphi'_k(z)| \leq M^m (1 - |\varphi_m(z)|^2)^\alpha |\varphi'(\varphi_m(z))\dots\varphi'(\varphi_{k-1}(z))|$$

Now using the condition (A) repeatedly, we get

$$(1 - |z|^2)^\alpha |\varphi'_k(z)| \leq M^m |\varphi'(0)^{k-m}| (1 - |\varphi_k(z)|^2)^\alpha.$$

Thus,

$$\lim_{k \rightarrow \infty} (1 - |z|^2)^\alpha \left| \frac{\varphi'_k(z)}{\varphi'(0)^k} \right| \leq \frac{M^m}{|\varphi'(0)^m|} \overline{\lim}_{k \rightarrow \infty} (1 - |\varphi_k(z)|^2)^\alpha \leq \frac{M^m}{|\varphi'(0)^m|}.$$

This implies that $(1 - |z|^2)^\alpha |\sigma'(z)| \leq \frac{M^m}{|\varphi'(0)^m|}$. Hence, $\sigma \in \mathcal{B}_\alpha$. □

The following corollary provides a sufficient condition that ensures all the integer

powers of the Königs function belong to Bloch-type spaces $\mathcal{B}_\alpha$ for $\alpha < 1$.

Corollary 2.1.8. *Suppose $\alpha < 1$. If C_φ is bounded on $\mathcal{B}_\alpha$ and φ satisfies the condition (A), then $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n .*

Proof. From theorem 2.1.7, we see that $\sigma \in \mathcal{B}_\alpha$. Now let us recall that $\mathcal{B}_\alpha \subset H^\infty$ for $\alpha < 1$, so there exists a positive constant C such that $\|\sigma\|_{H^\infty} \leq C$.

$$\begin{aligned} (1 - |z|^2)^\alpha |(\sigma^n(z))'| &= (1 - |z|^2)^\alpha |n \sigma^{n-1}(z) \sigma'(z)| \\ &\leq \|\sigma\|_{\mathcal{B}_\alpha} n |\sigma^{n-1}(z)| \\ &\leq n \|\sigma\|_{\mathcal{B}_\alpha} C^{n-1}. \end{aligned}$$

Hence, $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n . □

Theorem 2.1.9. *Suppose φ is holomorphic self map of $\mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$. Assume that C_φ is bounded on $\mathcal{B}_\alpha$ for some $\alpha > 0$ and there exists $k \in \mathbb{N}$ such that $\|\varphi_k\|_\infty < 1$. Then $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n .*

Proof. We have $(C_\varphi \sigma)(z) = \sigma(\varphi(z)) = \varphi'(0)\sigma(z)$. Now,

$$\begin{aligned} (C_{\varphi_k} \sigma)(z) &= \sigma(\varphi_k(z)) \\ &= \sigma(\varphi(\varphi_{k-1}(z))) \\ &= \varphi'(0)\sigma(\varphi_{k-1}(z)) \\ &= \varphi'(0)^k \sigma(z). \end{aligned}$$

Thus, σ is the Königs map of φ_k . Since C_φ is bounded on $\mathcal{B}_\alpha$ so is C_{φ_k} , and now

$\|\varphi_k\|_\infty < 1$ implies that C_{φ_k} is compact on $\mathcal{B}_\alpha$ (see Proposition 1.2.3). Hence using the Theorem 1.3.4 (for composition operator), $\sigma^n \in \mathcal{B}_\alpha$. $\square$

The following two examples satisfy the hypothesis of the Theorem 2.1.9.

Example 2.1.10. Consider the map

$$\varphi(z) = \frac{1}{2} z$$

defined on the unit disk $\mathbb{D}$. Then $\varphi(0) = 0$ and $\varphi'(z) = \frac{1}{2}$. Therefore by the Schwarz Lemma, for any $\alpha > 0$

$$\frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < 1, \quad \text{for all } z \in \mathbb{D}.$$

Hence C_φ is bounded on $\mathcal{B}_\alpha$ for any $\alpha > 0$ by the Theorem 1.2.1. Also, clearly $\|\varphi\|_\infty < 1$.

Example 2.1.11. Consider the map

$$\varphi(z) = \frac{iz(z+1)}{2}$$

defined on $\overline{\mathbb{D}}$. We have $\varphi(0) = 0$ so by the Schwarz Lemma for any $\alpha > 0$

$$\frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} < 1, \quad \text{for all } z \in \mathbb{D}$$

and since $\varphi'(z) = \frac{i(2z+1)}{2}$, $|\varphi'(z)| \leq \frac{3}{2}$ therefore

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \frac{3}{2}, \quad \text{for all } z \in \mathbb{D}.$$

Hence C_φ is bounded on $\mathcal{B}_\alpha$ for any $\alpha > 0$ by the Theorem 1.2.1. Now we show that $\|\varphi_2\|_\infty < 1$.

Claim 2.1.12. $|\varphi(z)| = 1$ for some $z \in \overline{\mathbb{D}}$ if and only if $z = 1$.

Proof. Here clearly $\varphi(1) = i$ and hence $|\varphi(1)| = 1$. Conversely, assume that $|\varphi(z)| = 1$. Since $|z| < 1$ implies $\frac{|z(z+1)|}{2} < 1$ so we must have $|z| = 1$ whenever $|\varphi(z)| = \frac{|z(z+1)|}{2} = 1$. Assuming $z = a + bi$ and using the fact $|z| = 1$, we get that

$$\begin{aligned} |z+1| &= 2 \\ \sqrt{a^2 + b^2 + 2a + 1} &= 2 \\ \sqrt{2 + 2a} &= 2 \\ 2 + 2a &= 4 \\ a &= 1. \end{aligned}$$

This shows $z = a = 1$. This completes the proof of the claim. $\square$

From the claim above and the fact that $\varphi(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$, we have $\varphi(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{i\}$. Since $1 \notin \varphi(\overline{\mathbb{D}})$, $\varphi_2(\overline{\mathbb{D}}) \subset \varphi(\mathbb{D} \cup \{i\}) \subset \mathbb{D}$. This shows that the closed set $\varphi_2(\overline{\mathbb{D}})$ is contained in $\mathbb{D}$ and hence $\|\varphi_2\|_\infty < 1$.

The following theorem gives a sufficient condition that ensures all the integer powers of Königs function belong to the Bloch space.

Theorem 2.1.13. *Suppose φ is holomorphic self map of $\mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| <$*

1. *Also, assume that*

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \frac{\log \frac{2}{1 - |z|}}{\log \frac{2}{1 - |\varphi(z)|}} |\varphi'(z)| \leq |\varphi'(0)|, \quad \text{for all } z \in \mathbb{D}. \quad (2.1.3)$$

Then C_φ is bounded on $\mathcal{B}$ and $\sigma^n \in \mathcal{B}$ for all positive integers n .

Proof. Boundedness is consequence of Schwarz-Pick theorem 1.1.3.

From the hypothesis of the theorem, we have

$$(1 - |z|^2) \log \frac{2}{1 - |z|} |\varphi'(z)| \leq |\varphi'(0)| (1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|}, \quad \text{for all } z \in \mathbb{D}. \quad (2.1.4)$$

Suppose k be a positive integer, then

$$\begin{aligned} (1 - |z|^2) |\varphi'_k(z)| \log \frac{2}{1 - |z|} &= (1 - |z|^2) |\varphi'(z) \varphi'(\varphi(z)) \dots \varphi'(\varphi_{k-1}(z))| \log \frac{2}{1 - |z|} \\ &= (1 - |z|^2) \log \frac{2}{1 - |z|} |\varphi'(z) \varphi'(\varphi(z)) \dots \varphi'(\varphi_{k-1}(z))|. \end{aligned}$$

By using (2.1.4), we see that

$$(1 - |z|^2) |\varphi'_k(z)| \log \frac{2}{1 - |z|} \leq |\varphi'(0)| (1 - |\varphi(z)|^2) \log \frac{2}{1 - |\varphi(z)|} |\varphi'(\varphi(z)) \dots \varphi'(\varphi_{k-1}(z))|.$$

And using (2.1.4) repeatedly, we get

$$\begin{aligned} (1 - |z|^2)|\varphi'_k(z)| \log \frac{2}{1 - |z|} &= |\varphi'(0)|^k (1 - |\varphi_k(z)|^2) \log \frac{2}{1 - |\varphi_k(z)|} \\ &\leq 2|\varphi'(0)|^k (1 - |\varphi_k(z)|) \log \frac{2}{1 - |\varphi_k(z)|}. \end{aligned}$$

Since $\log x \leq x$ for $x > 1$,

$$(1 - |z|^2)|\varphi'_k(z)| \log \frac{2}{1 - |z|} \leq 4|\varphi'(0)|^k.$$

Hence,

$$\lim_{k \rightarrow \infty} (1 - |z|^2) \left| \frac{\varphi'_k(z)}{\varphi'(0)^k} \right| \log \frac{2}{1 - |z|} = (1 - |z|^2)|\sigma'(z)| \log \frac{2}{1 - |z|} \leq 4, \quad \forall z \in \mathbb{D}$$

which shows that

$$|\sigma'(z)| \leq \frac{4}{(1 - |z|^2) \log \frac{2}{1 - |z|}}. \quad (2.1.5)$$

Recall that $\sigma(0) = 0$. Now let us get an estimate for σ .

$$\begin{aligned}
|\sigma(z)| &\leq \left| \int_0^1 \sigma'(tz) d(tz) \right| \\
&\leq \int_0^1 |\sigma'(tz) d(tz)| \\
&\leq \int_0^1 \frac{4}{\log \frac{2}{1-|tz|}} \frac{1}{1-|tz|^2} d(t|z|) \\
&\leq 4 \left[\log \left(\log \frac{2}{1-|z|} \right) \right]_0^1 \\
&\leq 4 \left[\log \left(\log \frac{2}{1-|z|} \right) - \log(\log 2) \right]. \tag{2.1.6}
\end{aligned}$$

Now by using (2.1.5) and the estimate above for σ , we get

$$\begin{aligned}
(1-|z|^2)(\sigma^n(z))' &= (1-|z|^2) n |\sigma^{n-1}(z) \sigma'(z)| \\
&\leq 4^n n \left(\log \log \frac{2}{1-|z|} - \log \log 2 \right)^{n-1} \frac{1}{\log \frac{2}{1-|z|}}.
\end{aligned}$$

Taking limit $|z| \rightarrow 1$, it is easy to see that the right hand side of the last expression goes to zero. Hence $\sigma^n \in \mathcal{B}_0 \subset \mathcal{B}$ for all positive integers n .

□

Let us recall the Lipschitz-type norm which is equivalent to the usual norm defined for function $f \in \mathcal{B}_\alpha$, $\alpha > 1$:

$$\|f\|_{\mathcal{B}_\alpha} \equiv \sup_{z \in \mathbb{D}} (1-|z|^2)^{\alpha-1} |f(z)|.$$

Next, we present results for the Bloch-type spaces, $\mathcal{B}_\alpha$ for $\alpha > 1$. Let us start with

the following definition.

Definition 2.1.14. Suppose $f \in \mathcal{B}_\alpha$ for some $\alpha > 0$, then we define the *Bloch number* of f by $b_f = \inf_\alpha \{ \alpha : f \in \mathcal{B}_\alpha \}$.

Theorem 2.1.15. Suppose $\beta > 0$. Then $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n if and only if b_f is at most 1.

Proof. Suppose $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n . Need to show $b_f \leq 1$. On the contrary assume $b_f > 1$. Then there exists a positive integer n_0 such that $1 < 1 + \frac{\beta}{n_0} < b_f$. Now in the view of the Lipschitz-type norm, we see that for any fixed positive integer M there exists $z \in \mathbb{D}$ such that

$$\begin{aligned} M &\leq (1 - |z|^2)^{\beta/n_0} |f(z)| \leq \{(1 - |z|^2)^{\beta/n_0} |f(z)|\}^{n_0} \\ &= (1 - |z|^2)^\beta |f(z)|^{n_0}, \end{aligned}$$

which shows that

$$M \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f(z)|^{n_0} = \|f^{n_0}\|_{\mathcal{B}_{\beta+1}}.$$

Since M is an arbitrary positive integer, $f^{n_0} \notin \mathcal{B}_{\beta+1}$. Which is a contradiction.

Conversely, suppose $b_f \leq 1$. Since $\mathcal{B}_\alpha \subset \mathcal{B}$ for all $\alpha \leq 1$, then clearly $f \in \mathcal{B}$. For any fixed $\beta > 0$ and for any fixed positive integer n ,

$$\begin{aligned} (1 - |z|^2)^{\beta+1} |(f^n)'(z)| &= (1 - |z|^2)^{\beta+1} |n f^{n-1}(z) f'(z)| \\ &\leq n (1 - |z|^2) |f'(z)| (1 - |z|^2)^\beta |f^{n-1}(z)|. \end{aligned}$$

Using the estimate for a function on the Bloch space from (1.1.4), we get

$$\begin{aligned} (1 - |z|^2)^{\beta+1} |(f^n)'(z)| &\leq n \|f\|_{\mathcal{B}} (1 - |z|^2)^{\beta} \left(\|f\|_{\mathcal{B}} \log \frac{1}{1 - |z|} \right)^{n-1} \\ &= n (\|f\|_{\mathcal{B}})^n (1 - |z|^2)^{\beta} \left(\log \frac{1}{1 - |z|} \right)^{n-1}. \end{aligned}$$

The last expression goes to zero as $|z| \rightarrow 0$. This shows that $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n . $\square$

Corollary 2.1.16. *Suppose φ is holomorphic self map of $\mathbb{D}$, $\varphi(0) = 0$, $0 < |\varphi'(0)| < 1$, and also assume $\alpha > 1$. If $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$ then C_{φ} is bounded on $\mathcal{B}_{\alpha}$ and $\sigma^n \in \mathcal{B}_{\alpha}$ for all positive integers n*

Proof. Since $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$, from the Theorem 2.1.7 $\sigma \in \mathcal{B}$. So $b_f \leq 1$. Therefore the result follows from the theorem 2.1.15. $\square$

2.2 Weighted Composition operator

Let us recall that if u is a holomorphic function of the unit disk, and φ is a holomorphic self map of the unit disk then the Schröder equation for weighted composition operator is given by

$$u(z)f(\varphi(z)) = \lambda f(z), \quad (2.2.1)$$

where $f \in H(\mathbb{D})$ and λ is a complex constant.

Let us also recall that if $u(0) \neq 0, \varphi(0) = 0, 0 < |\varphi'(0)| < 1$ then the solutions of (2.2.1) are given by the Theorem 1.3.2. The principal eigenfunction corresponding to the eigenvalue $u(0)$ is denoted by v and all other eigenfunctions are of the form $v\sigma^n$ where Königs function σ of φ and n is a positive integer. Hosokawa and Nguyen [10] studied equation (2.2.1) in the Bloch space and obtained the following theorem.

Theorem 2.2.1. *Assume φ is a holomorphic self map of $\mathbb{D}$ with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$. Let us also assume that uC_φ is bounded on $\mathcal{B}$. For $0 < r < 1$, set*

$$M_r(\varphi) := \sup_{|z|=r} |\varphi(z)| \quad a_r := \sup_{|z|=r} (|u'(z)\varphi(z)| + |u(z)\varphi'(z)|).$$

Suppose that

$$(i) \lim_{r \rightarrow 1} \log(1-r) \log M_r(\varphi) = \infty.$$

$$(ii) \log |a_r| < \epsilon \log(1-r) \log M_r(\varphi),$$

where $\epsilon > 0$ is a constant satisfying $\epsilon \log \|\varphi\|_\infty > -1$.

Then, $v\sigma^n \in \mathcal{B}$ for all non-negative integer n .

In fact a careful analysis of the hypothesis of the above theorem shows that these conditions are strong enough to imply that the weighted composition operator is compact.

Proposition 2.2.2. *Conditions (i) and (ii) of the Theorem 2.2.1 imply that the operator uC_φ is compact on the Bloch space.*

Proof. For all z such that $|z| = r > 0$, we have

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} [|u'(z)\varphi(z)| + |u(z)\varphi'(z)|] \leq \frac{1 - r^2}{1 - M_r(\varphi)^2} a_r.$$

Splitting the expression above and using the fact that $\log x < x$ for $x > 1$, we get the following two equations:

$$(1 - |z|^2)|u'(z)\varphi(z)| \log \frac{1}{1 - |\varphi(z)|^2} \leq \frac{1 - r^2}{1 - M_r(\varphi)^2} a_r \quad (2.2.2)$$

and

$$|u(z)| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| \leq \frac{1 - r^2}{1 - M_r(\varphi)^2} a_r. \quad (2.2.3)$$

Since uC_φ is bounded on $\mathcal{B}$, if $\|\varphi\|_\infty < 1$ then uC_φ is compact (see Proposition 1.2.3). Therefore without loss of generality we may assume that $\|\varphi\|_\infty = 1$. It implies that $\lim_{r \rightarrow 1} M_r(\varphi) = \|\varphi\|_\infty = 1$. This allows us to choose $\epsilon = 1$ in the condition (ii). So,

$$a_r < (1 - r)^{\log M_r(\varphi)}$$

and hence

$$\frac{1-r^2}{1-M_r(\varphi)^2}a_r < \frac{1-r^2}{1-M_r(\varphi)^2}(1-r)^{\log M_r(\varphi)} < \frac{(1-r)^{1+\log M_r(\varphi)}}{1-M_r(\varphi)}. \quad (2.2.4)$$

Since $\lim_{r \rightarrow 1} M_r(\varphi) = \|\varphi\|_\infty = 1$ and $M_r(\varphi)$ is increasing, we can choose c and r_c in $(0, 1)$ such that

$$0 = \log \|\varphi\|_\infty > \log M_r(\varphi) > -c > -1$$

for all $r > r_c > 0$. Clearly $1 \geq 1 + \log M_r(\varphi) > 1 - c > 0$ and so we can see that

$$(1-r)^{1+\log M_r(\varphi)} < (1-r)^{1-c}$$

for $r > r_c$. Using (2.2.4) and the expression above, we see that

$$\frac{1-r^2}{1-M_r(\varphi)^2}a_r < \frac{(1-r)^{1-c}}{1-M_r(\varphi)} \quad (2.2.5)$$

for all $r > r_c$. On the other hand from condition (i), we have that $\lim_{r \rightarrow 1} \log(1-r) \log M_r(\varphi) = \infty$. This allows us to choose $r_0 > 0$ such that

$$\log(1-r) \log M_r(\varphi) > 1$$

for all $r > r_0$. Now let us subtract $\log M_r(\varphi) \log(1-M_r(\varphi))$ on the both sides of above expression and simplify to get

$$\log \frac{(1-r)}{1-M_r(\varphi)} < \frac{1 - \log M_r(\varphi) \log(1-M_r(\varphi))}{\log M_r(\varphi)} \quad (2.2.6)$$

for all $r > r_0$. Let us recall that by the use of L'Hospital's rule, we have

$$\lim_{x \rightarrow 1} \log x \log(1-x) = \lim_{x \rightarrow 1} \frac{x(\log x)^2}{1-x} = 0.$$

By the use of above relation, we see that the right hand side of (2.2.6) goes to $-\infty$ as $r \rightarrow 1$. Hence from (2.2.6), we see that

$$\frac{(1-r)^{1-c}}{1-M_r(\varphi)} = 0 \text{ as } r \rightarrow 1.$$

Now from (2.2.5), we have

$$\frac{1-r^2}{1-M_r(\varphi)^2} a_r = 0 \text{ as } r \rightarrow 1. \quad (2.2.7)$$

Therefore, (2.2.2) and (2.2.3) along with the above expression imply that

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u'(z)|(1-|z|^2) \log \frac{1}{(1-|\varphi(z)|^2)} = 0$$

and

$$\lim_{|\varphi(z)| \rightarrow 1^-} |u(z)| \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)} |\varphi'(z)| = 0.$$

Hence uC_φ is a compact operator on $\mathcal{B}$ by the theorem 1.2.2. $\square$

Now we investigate the properties of weight u and symbol φ of the weighted composition operators uC_φ that ensure $v\sigma^n$ belongs to Bloch-type spaces for all non negative integer n . Let us start with following remark.

Remark 2.2.3. Suppose f is a holomorphic function defined on $\mathbb{D}$. If $\|f'\|_\infty < M$ for some $M > 0$ then $\|f\|_\infty < \infty$.

Proof.

$$\begin{aligned} |f(z) - f(0)| &= \left| \int_0^1 z f'(tz) dt \right| \\ &\leq \int_0^1 |z f'(tz)| dt \\ &\leq M \int_0^1 |z| dt \end{aligned}$$

This shows that $\|f\|_\infty < \infty$. □

Proposition 2.2.4. Assume φ is univalent holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and σ is Königs function of φ . Then, σ is bounded if and only if there is a positive integer k such that $\|\varphi_k\|_\infty < 1$.

Before proving the proof of this proposition, we give the proof of following lemma based on the proof given in [19].

Lemma 2.2.5. Assume φ is holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and σ is Königs function of φ . If σ is univalent and bounded then there is a positive integer k such that $\|\varphi_k\|_\infty < 1$.

Proof. Since σ is univalent, σ^{-1} defined from $\sigma(\mathbb{D})$ to $\mathbb{D}$ is continuous. We have $\sigma(0) = 0$, so by the continuity of σ^{-1} , for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\sigma^{-1}(\sigma(z))| = |z| < \epsilon$ whenever $|\sigma(z)| < \delta$.

Since $0 < |\varphi'(0)| < 1$ and σ is bounded, we can choose a positive integer k such

that $|\varphi'(0)^k \sigma(z)| < \delta$ for all $z \in \mathbb{D}$. Now, $|\sigma(\varphi_k(z))| = |\varphi'(0)^k \sigma(z)| < \delta$, so $|\varphi_k(z)| < \epsilon$ for all $z \in \mathbb{D}$. If we choose $\epsilon = \frac{1}{2}$, then $\|\varphi_k\|_\infty < 1$.

□

Proof of the proposition. Suppose σ is bounded. Since φ is univalent, σ is also univalent, see ([19], page 91). The result now follows from lemma 2.2.5.

Conversely suppose there is a positive integer k such that $\|\varphi_k\|_\infty < 1$. Since we have $\sigma(\varphi(z)) = \varphi'(0)\sigma(z)$,

$$\begin{aligned} \sigma(\varphi_k(z)) &= \sigma(\varphi(\varphi_{k-1}(z))) \\ &= \varphi'(0)\sigma(\varphi_{k-1}(z)) \\ &= \varphi'(0)^k \sigma(z). \end{aligned}$$

Clearly left hand side is bounded and therefore σ is also bounded, which completes the proof. □

Theorem 2.2.6. *Assume φ is univalent holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ satisfying $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$ and for some fixed $\alpha < 1$. If u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$ and $\|u'\|_\infty < \infty$ then uC_φ is bounded on $\mathcal{B}_\alpha$ and $v\sigma^n \in \mathcal{B}_\alpha$ for all non-negative integers n .*

Proof. Since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, from the Theorem 1.2.1, C_φ is bounded on $\mathcal{B}_\alpha$. Further $\|u'\|_\infty$ is bounded so is $\|u\|$ from the Remark 2.2.3. Again using the Theorem 1.2.1, we see that uC_φ is bounded on $\mathcal{B}_\alpha$.

Since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, using the theorem 2.1.7, we see that $\sigma \in \mathcal{B}_\alpha$, $\alpha < 1$ and consequently it is bounded. Since φ is univalent, σ is univalent.

Therefore, Lemma 2.2.5 says that there exists a non-negative integer k such that $\|\varphi_k\|_\infty < 1$. Composing φ_{k-1} on both sides of the Schröder equation (2.2.1) from right,

$$u(\varphi_{k-1}(z))f(\varphi_k(z)) = \lambda f(\varphi_{k-1}(z)). \quad (2.2.8)$$

Since $\|u'\|_\infty$ is bounded so is $\|u\|_\infty$, and since $\|\varphi_k\|_\infty < 1$ it follows that $f \circ \varphi_k$ is bounded. Therefore, left hand side of (2.2.8) is bounded and so is $f \circ \varphi_{k-1}$. Now differentiating both side of (2.2.8), we get that

$$u'(\varphi_{k-1}(z)) \varphi'_{k-1}(z) f(\varphi_k(z)) + u(\varphi_{k-1}(z)) f'(\varphi_k(z)) \varphi'_k(z) = \lambda f'(\varphi_{k-1}(z)) \varphi'_{k-1}(z).$$

Multiplying both sides by $(1 - |z|^2)^\alpha$ and using boundedness of $\|u'\|_\infty$, $\|u\|_\infty$, $f \circ \varphi_k$ and $f' \circ \varphi_k$, we see that there exists a constant M such that

$$(1 - |z|^2)^\alpha |\lambda f'(\varphi_{k-1}(z)) \varphi'_{k-1}(z)| \leq M(1 - |z|^2)^\alpha (|\varphi'_{k-1}(z)| + |\varphi'_k(z)|). \quad (2.2.9)$$

As in the proof of the Theorem 2.1.7, it can be shown that right hand side of the above equation is uniformly bounded and therefore the left hand side is bounded. Again, let us compose φ_{k-2} on (2.2.1), to get

$$u(\varphi_{k-2}(z))f(\varphi_{k-1}(z)) = \lambda f(\varphi_{k-2}(z)).$$

Let us differentiate above expression and then multiply by $(1 - |z|^2)^\alpha$ on both sides. Then, the use of (2.2.8) and (2.2.9) shows that $(1 - |z|^2)^\alpha |f'(\varphi_{k-2}(z)) \varphi'_{k-2}(z)|$ is bounded. Continuing this process, we see that that $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|$ is bounded

and hence $f \in \mathcal{B}_\alpha$. From the Theorem 1.3.2, we know that any holomorphic f satisfying (2.2.1) is of the form $v\sigma^n$ for some positive integer n , so $v\sigma^n \in \mathcal{B}_\alpha$ for all non negative integers n . This completes the proof. □

Theorem 2.2.7. *Assume φ is univalent holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ satisfying $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$ and for some $\alpha < 1$. If u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$ and $\|u\|_\infty < \infty$ then $v\sigma^n \in \mathcal{B}$ for all non-negative integer n .*

Proof. Since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, using the theorem 2.1.7, we see that $\sigma \in \mathcal{B}_\alpha$, $\alpha < 1$ and consequently it is bounded. Since φ is univalent, σ is univalent. Therefore, Lemma 2.2.5 says that there exists a non-negative integer k such that $\|\varphi_k\|_\infty < 1$.

Let us compose φ_{k-1} on both sides of the Schröder equation (2.2.1) from right to get

$$u(\varphi_{k-1}(z))f(\varphi_k(z)) = \lambda f(\varphi_{k-1}(z)).$$

Since $\|u\|_\infty < \infty$ and $\|\varphi_k\|_\infty < 1$, left hand side is uniformly bounded and hence $f \circ \varphi_{k-1}$ is bounded. Next, let us compose φ_{k-2} on both sides of (2.2.1) from right then we get that

$$u(\varphi_{k-2}(z))f(\varphi_{k-1}(z)) = \lambda f(\varphi_{k-2}(z)).$$

Here $f \circ \varphi_{k-1}$ is bounded and u is bounded by the hypothesis so $f \circ \varphi_{k-2}$ is bounded. Now using similar argument we will see that f is uniformly bounded and hence $f \in \mathcal{B}$. From the Theorem 1.3.2, we know that any holomorphic f satisfying (2.2.1) is of the

form $v\sigma^n$ for some positive integer n , so $v\sigma^n \in \mathcal{B}$ for all non negative integers n . This completes the proof. $\square$

The following two theorems give us the sufficient conditions that ensure $v\sigma^n$ belong to Bloch-type spaces $\mathcal{B}_\alpha$ for some $\alpha > 1$ and for all non-negative integers n .

Theorem 2.2.8. *Let φ be holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$. Let β be a fixed positive number and assume*

$$|u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\beta} \leq |u(0)|, \quad \text{for all } z \in \mathbb{D}.$$

Then the following statements are true.

- i. If $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$ and for some $\alpha < 1$, then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all non-negative integers n .*
- ii. If $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathbb{D}$, then $v\sigma^n \in \mathcal{B}_{p+1}$, for some $p > \beta$ and for all non-negative integers n .*

Proof. i. Let $z \in \mathbb{D}$ and use the definition of v_k (see Theorem 1.3.2) and the given

condition

$$\begin{aligned}
(1 - |z|^2)^\beta |v_k(z)| &= (1 - |z|^2)^\beta \frac{|u(z)u(\varphi(z))\dots u(\varphi_{k-1}(z))|}{|u(0)|^k} \\
&\leq (1 - |\varphi(z)|^2)^\beta \frac{|u(\varphi(z))\dots u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}} \\
&\dots \\
&\leq 1.
\end{aligned}$$

Hence $(1 - |z|^2)^\beta |v(z)| = \lim_{k \rightarrow \infty} (1 - |z|^2)^\beta |v_k(z)| \leq 1$. Since z is arbitrary

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |v(z)| < \infty.$$

On the other hand, since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, Theorem 2.1.7 implies that

$$\sigma^n \in \mathcal{B}_\alpha \subset \mathbb{H}^\infty$$

for all non-negative integer n . Hence,

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |v(z)\sigma^n(z)| < \infty$$

for all non-negative integers n . Considering the equivalent norm (see (1.1.5)), we see that $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all non-negative integers n .

ii. From the proof of (i), we see that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |v(z)| < \infty. \tag{2.2.10}$$

On the other hand, since $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$, Theorem 2.1.7 imply that $\sigma \in \mathcal{B}$ so there exists $M > 0$ such that

$$|\sigma(z)| \leq M \log \frac{2}{1 - |z|^2}. \quad (2.2.11)$$

Now using equations (2.2.10) and (2.2.11), we have

$$\begin{aligned} (1 - |z|^2)^p |v(z)\sigma^n(z)| &= \{(1 - |z|^2)^\beta |v(z)|\} \{(1 - |z|^2)^{p-\beta} |\sigma^n(z)|\} \\ &\leq CM(1 - |z|^2)^{p-\beta} \left(\log \frac{2}{1 - |z|^2} \right)^n \end{aligned}$$

for some constant $C > 0$. Now taking limit $|z| \rightarrow 1$, we get that the last expression goes to zero. Hence, $v\sigma^n \in \mathcal{B}_{p+1}$ for all non-negative integers n .

□

Theorem 2.2.9. *Let φ be holomorphic self map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and u is holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$. Suppose that β is a positive integer and*

$$i. \quad |u(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\beta} \frac{\log \frac{2}{(1 - |z|)^\beta}}{\log \frac{2}{(1 - |\varphi(z)|)^\beta}} \leq |u(0)|, \quad \text{for all } z \in \mathbb{D}$$

$$ii. \quad |\varphi^{(h)}(z)| \frac{\log \frac{2}{1 - |z|}}{\log \frac{2}{1 - |\varphi(z)|}} \leq |\varphi'(0)|, \quad \text{for all } z \in \mathbb{D}.$$

Then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all non-negative integers n .

Proof. From definition of v_k (see Theorem 1.3.2) and the condition (i), we have

$$\begin{aligned}
(1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} |v_k(z)| &= (1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} \frac{|u(z)u(\varphi(z))\dots u(\varphi_{k-1}(z))|}{|u(0)|^k} \\
&\leq (1 - |\varphi(z)|^2)^\beta \log \frac{2}{(1 - |\varphi(z)|)^\beta} \frac{|u(\varphi(z))\dots u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}} \\
&\leq (1 - |\varphi_k(z)|^2)^\beta \log \frac{2}{(1 - |\varphi_k(z)|)^\beta} \\
&\leq 2^\beta (1 - |\varphi_k(z)|)^\beta \log \frac{2}{(1 - |\varphi_k(z)|)^\beta}.
\end{aligned}$$

Since $\log x \leq x$ for $x > 1$

$$(1 - |z|^2)^\beta \log \frac{2}{(1 - |z|)^\beta} |v_k(z)| \leq 2^{\beta+1}.$$

So taking limit k approaches to ∞ , we see that

$$(1 - |z|^2)^\beta |v(z)| \leq \frac{2^{\beta+1}}{\log \frac{2}{1-|z|}}. \quad (2.2.12)$$

On the other hand, since φ satisfies condition (ii), equation (2.1.6) of the Theorem 2.1.13 says that there exists $K > 0$ such that

$$|\sigma(z)| \leq K \log \log \frac{2}{1 - |z|}. \quad (2.2.13)$$

Now using (2.2.12) and (2.2.13), we get

$$(1 - |z|^2)^\beta |v(z)\sigma^n(z)| \leq \frac{2^{\beta+1}K^n}{\log \frac{2}{1-|z|}} \left(\log \log \frac{2}{1 - |z|} \right)^n.$$

Clearly right hand side of the above equation goes to 0 as $|z| \rightarrow 1$. Using the norm defined on (1.1.5), $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all non negative integers n .

□

Chapter 3

Invariant Subspaces

3.1 Hardy space

The Hardy space is named after a English mathematician Godfrey Harold Hardy (1877-1947) and it is denoted by $\mathcal{H}^2(\mathbb{D})$. We say that a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk belongs to the Hardy space $\mathcal{H}^2(\mathbb{D})$, if its sequence of power series coefficients is square-summable:

$$\mathcal{H}^2(\mathbb{D}) = \{f \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

We define a norm on $\mathcal{H}^2(\mathbb{D})$ by

$$\|f\|_{\mathcal{H}^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2 \tag{3.1.1}$$

The mapping $\{a_n\}_0^{\infty} \mapsto \sum_{n=0}^{\infty} a_n z^n$ is an isomorphism from l^2 , the space of square-summable complex sequences, onto $\mathcal{H}^2(\mathbb{D})$. Indeed, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{H}^2(\mathbb{D})$

then from (3.1.1), we see that $\{a_n\}_0^\infty$ belongs to l^2 . On the other hand if $\{b_n\}_0^\infty$ belongs to l^2 then it is bounded and the corresponding power series $\sum_{n=0}^\infty b_n z^n$ converges on $\mathbb{D}$ to a holomorphic function that belongs to $\mathcal{H}^2(\mathbb{D})$. Therefore, there is a vector space isomorphism from $\mathcal{H}^2(\mathbb{D})$ onto l^2 . So, $\mathcal{H}^2(\mathbb{D})$ is a Hilbert space and the inner product is given by

$$\langle f, g \rangle_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

A different approach to define the Hardy space is via *integral means*. For this, we write $z = re^{i\theta}$ and define integral means by

$$M_2^2(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad r \in [0, 1).$$

Let us recall that $L^2 = L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ is a Hilbert space of square integrable function on $\partial\mathbb{D}$. The inner product is given by

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta$$

For each n , let $e_n(e^{i\theta}) = e^{in\theta}$, regarded as a function on $\partial\mathbb{D}$. For $n, m \in \mathbb{Z}$, it is well known that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}.$$

This shows that the set $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for L^2 . Now we see that

$$M_2^2(f, r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

This formula shows that $M_2(f, r)$ is increasing function of r and if $f \in \mathcal{H}^2(\mathbb{D})$, then it is bounded by $\|f\|_{\mathcal{H}^2(\mathbb{D})}$. On the other hand, if $\lim_{r \rightarrow 1^-} M_2(f, r) = M < \infty$, it is easy to see that $f \in \mathcal{H}^2(\mathbb{D})$ and $\|f\|_{\mathcal{H}^2(\mathbb{D})} \leq M$. Hence we define $\mathcal{H}^2(\mathbb{D})$ as

$$\mathcal{H}^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \right\}.$$

Hence, an equivalent norm is given by

$$\|f\|_{\mathcal{H}^2(\mathbb{D})}^2 = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Moreover, the Hardy space $\mathcal{H}^2(\mathbb{D})$ can also be viewed as a closed subspace of L^2 .

Let us define $\tilde{L}^2$ as the

$$\tilde{L}^2 = \{f^* \in L^2 : \langle f^*, e_n \rangle = 0, \text{ for } n < 0\}.$$

So, $f^* \in \tilde{L}^2$ has Fourier series in the form

$$f^*(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \text{ with } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Clearly $\tilde{L}^2$ is a closed subspace of L^2 and $\sum_{n=0}^{\infty} a_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} a_n z^n$ defines a natural isomorphism between $\tilde{L}^2$ and $\mathcal{H}^2(\mathbb{D})$. Let $f^* \in \tilde{L}^2$ whose Fourier series is

$\sum_{n=0}^{\infty} a_n e^{in\theta}$ and $f \in \mathcal{H}^2(\mathbb{D})$ have power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For $r \in [0, 1)$, define f_r by

$$f_r(e^{i\theta}) = f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Clearly, $f_r \in L^2$ for every such r . Then it can be shown that

$$\|f_r - f^*\|_{L^2}^2 \rightarrow 0$$

as $r \rightarrow 1^-$, see [14]. It is well known that convergence of L^2 implies that a subsequence converges pointwise almost everywhere, so it follows that for each f in $\mathcal{H}^2(\mathbb{D})$, there exists an increasing sequence $\{r_n\}$ converging to 1 such that

$$\lim_{n \rightarrow \infty} f(r_n e^{i\theta}) = f^*(e^{i\theta})$$

for almost all θ . In fact, $\lim_{r \rightarrow 1} f(re^{i\theta}) = f^*(e^{i\theta})$ for almost all θ , see [14]. In this way we can see that $\mathcal{H}^2(\mathbb{D})$ -norm and inner product coincide with ones in L^2 . Indeed, for $f, g \in \mathcal{H}^2(\mathbb{D})$, we have

$$\|f\|_{\mathcal{H}^2(\mathbb{D})} = \|f^*\|_{L^2}$$

$$\langle f, g \rangle_{\mathcal{H}^2(\mathbb{D})} = \langle f^*, g^* \rangle_{L^2}.$$

Already we know that taking the radial-limit of an $\mathcal{H}^2(\mathbb{D})$ -function gives its L^2 boundary function; also we are able to recover it from its L^2 -boundary function by

the *Poisson Integral Formula*. In fact, for $f \in \mathcal{H}^2(\mathbb{D})$, $0 \leq r < 1$ and $t \in \mathbb{R}$, we have

$$f(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) P_r(\theta - t) d\theta,$$

where

$$P_r(t) := \sum_{-\infty}^{\infty} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r \cos t + r^2} = \operatorname{Re} \left(\frac{1 + re^{it}}{1 - re^{it}} \right).$$

Hence, $P_r(\theta - t)$ can be considered as a function in $z = re^{i\theta} \in \mathbb{D}$ and $e^{it} \in \partial\mathbb{D}$ by

$$P_r(\theta - t) = \operatorname{Re} \left(\frac{e^{it} + z}{e^{it} - z} \right) = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

From the use of Poisson formula along with other things one can deduce the following *Herglotz Representation Theorem*.

Theorem 3.1.1. *Suppose h is analytic on $\mathbb{D}$ and $\operatorname{Re} h(z) > 0$ for all $z \in \mathbb{D}$. Then there exists a finite positive regular Boerl measure μ on $[0, 2\pi]$ such that*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) + i \operatorname{Im} h(0).$$

Proof. See [14, Theorem 2.6.3]. □

Definition 3.1.2. A function $\phi \in \mathbb{H}^\infty$ satisfying $|\phi(e^{i\theta})| = 1$ almost everywhere is called an *inner function*.

Definition 3.1.3. *Blaschke Product* is a function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a_n} z} \frac{|a_n|}{a_n}, \quad z \in \mathbb{D},$$

where m is a non-negative integer, $0 < |a_n| < 1$, and $\sum(1 - |a_n|) < \infty$. The set $\{a_n\}$ may be finite, or even empty.

Balschke product is an inner function with zeros inside the unit disk.

Definition 3.1.4. A non constant inner function that has no zeros in $\mathbb{D}$ is called a *singular inner function*.

One very common example of a singular inner function is

$$S(z) = \exp \left(\frac{z+1}{z-1} \right).$$

Any inner function f can be written as product of two inner functions namely *Blaschke Product* and *singular inner function*, (See, [14], pp. 72).

Definition 3.1.5. A measure μ is *singular measure*, if μ is concentrated on a set of Lebesgue measure zero.

Theorem 3.1.6. *The singular inner functions have the form*

$$S(z) = K \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right)$$

for some singular measure μ and a constant K of modulus 1.

Proof. See [14, Theorem 2.6.5]

□

Herglotz Representation Theorem and Poisson integral formula play important role in the proof of the Theorem 3.1.6. Further if we carefully examine the proof of the theorem then we see that each singular inner function defines a singular measure.

To sum up, each inner function that has singular part defines a singular measure on $[0, 2\pi]$.

3.2 Shift plus complex Volterra Operator

We say that a holomorphic function on the unit disk belongs to the Hardy space $\mathcal{H}^2(\mathbb{D})$, if its sequence of power series coefficients is square-summable. The operator $(M_z f)(z) = zf(z)$ is called shift operator and the operator

$$(\mathcal{V}f)(z) = \int_0^z f(w)dw$$

is known as complex Volterra operator. Lattice of closed invariant subspaces of these operators are well studied in the Hardy space case, see [3] and [1]. Let us define the shift plus complex Volterra operator T on the Hardy space $\mathcal{H}^2(\mathbb{D})$ by

$$(Tf)(z) = zf(z) + \int_0^z f(w)dw, \quad \text{for } f \in \mathcal{H}^2(\mathbb{D}) \text{ and } z \in \mathbb{D}. \quad (3.2.1)$$

Operator T is clearly bounded operator on $\mathcal{H}^2(\mathbb{D})$ since both shift and the complex Volterra operators are bounded on $\mathcal{H}^2(\mathbb{D})$. Here, we are interested to find the lattice of closed invariant subspaces of T . In order to find that, we will use a result from a paper by Korenbljum [13].

Definition 3.2.1. Let us define a subspace of $H(\mathbb{D})$ by

$$\mathcal{S}^2(\mathbb{D}) = \{f \in H(\mathbb{D}) : Df \in \mathcal{H}^2(\mathbb{D})\},$$

where $D = \frac{d}{dz}$ is a differential operator. It is clear that if Df is in $\mathcal{H}^2(\mathbb{D})$, then f belongs to $\mathcal{H}^2(\mathbb{D})$. Now let us define a new norm on $\mathcal{S}^2(\mathbb{D})$ by

$$\|f\|_{\mathcal{S}^2(\mathbb{D})}^2 = \|Df\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|f\|_{\mathcal{H}^2(\mathbb{D})}^2. \quad (3.2.2)$$

Corresponding inner product is given by

$$\langle f, g \rangle_{\mathcal{S}^2(\mathbb{D})} = \langle Df, Dg \rangle_{\mathcal{H}^2(\mathbb{D})} + \langle f, g \rangle_{\mathcal{H}^2(\mathbb{D})}.$$

The following result is known for specialists but we include a proof for the sake of completeness.

Proposition 3.2.2. *The following statements are true:*

- i. $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.
- ii. $\mathcal{S}^2(\mathbb{D})$ is Banach algebra.
- iii. Polynomials are dense in $\mathcal{S}^2(\mathbb{D})$.

Proof. i. Let $f \in \mathcal{S}^2(\mathbb{D})$, then $Df \in \mathcal{H}^2(\mathbb{D})$ and hence

$$|Df(z)| \leq \frac{\|Df\|_{\mathcal{H}^2(\mathbb{D})}}{\sqrt{1-|z|^2}}, \quad \text{for all } z \in \mathbb{D}.$$

Now,

$$\begin{aligned}
|f(z) - f(0)| &= \left| \int_0^1 z Df(tz) dt \right| \\
&\leq \int_0^1 |z Df(tz)| dt \\
&\leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{1 - |tz|^2}} dt \\
&\leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{(1 - |tz|)(1 + |tz|)}} dt.
\end{aligned}$$

Since $1 + t|z| \geq 1$, we have $\frac{1}{\sqrt{1 + t|z|}} \leq 1$.

$$\begin{aligned}
|f(z) - f(0)| &\leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{1 - |tz|}} dt \\
&\leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})}.
\end{aligned} \tag{3.2.3}$$

This clearly shows that f belongs to H^∞ , and hence $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.

- ii. To show $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the above norm, suppose $\{g_n\}_0^\infty$ is Cauchy in $\mathcal{S}^2(\mathbb{D})$ norm. Then, clearly $\{g_n\}$ and $\{Dg_n\}$ are Cauchy on $\mathcal{H}^2(\mathbb{D})$ norm. Since $\mathcal{H}^2(\mathbb{D})$ is a Banach space, $\{g_n\}$ converges to a holomorphic function $g \in \mathcal{H}^2(\mathbb{D})$ and $\{Dg_n\}$ converges to the function $Dg \in \mathcal{H}^2(\mathbb{D})$. Therefore $g \in \mathcal{S}^2(\mathbb{D})$, and hence $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the given norm.

To show pointwise multiplication on $\mathcal{S}^2(\mathbb{D})$ form an algebra, let us assume that

f and g are in $\mathcal{S}^2(\mathbb{D})$.

$$\begin{aligned}
\|fg\|_{\mathcal{S}^2(\mathbb{D})}^2 &= \|D(fg)\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|fg\|_{\mathcal{H}^2(\mathbb{D})}^2 \\
&= \|gDf + fDg\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|fg\|_{\mathcal{H}^2(\mathbb{D})}^2 \\
&\leq \|gDf\|_{\mathcal{H}^2(\mathbb{D})}^2 + 2\|gDf\|_{\mathcal{H}^2(\mathbb{D})}\|fDg\|_{\mathcal{H}^2(\mathbb{D})} + \|fDg\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|fg\|_{\mathcal{H}^2(\mathbb{D})}^2.
\end{aligned} \tag{3.2.4}$$

Using (3.2.3), we see that for any $f \in \mathcal{S}^2(\mathbb{D})$

$$\begin{aligned}
\|f\|_{\infty} &\leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})} + |f(0)| \\
&\leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})} + 2\|f\|_{\mathcal{H}^2(\mathbb{D})} = 2\|f\|_{\mathcal{S}^2(\mathbb{D})}.
\end{aligned}$$

Hence using (3.2.4),

$$\begin{aligned}
\|fg\|_{\mathcal{S}^2(\mathbb{D})}^2 &\leq \|g\|_{\infty}^2 \|Df\|_{\mathcal{H}^2(\mathbb{D})}^2 + 2\|f\|_{\infty}\|g\|_{\infty}\|Df\|_{\mathcal{H}^2(\mathbb{D})}\|Dg\|_{\mathcal{H}^2(\mathbb{D})} \\
&\quad + \|f\|_{\infty}^2 \|Dg\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|g\|_{\infty}^2 \|f\|_{\infty}^2 \\
&\leq 4\|g\|_{\mathcal{S}^2(\mathbb{D})}^2 \|f\|_{\mathcal{S}^2(\mathbb{D})}^2 + 8\|g\|_{\mathcal{S}^2(\mathbb{D})}^2 \|f\|_{\mathcal{S}^2(\mathbb{D})}^2 + 4\|g\|_{\mathcal{S}^2(\mathbb{D})}^2 \|f\|_{\mathcal{S}^2(\mathbb{D})}^2 \\
&\leq 16\|f\|_{\mathcal{S}^2(\mathbb{D})}^2 \|g\|_{\mathcal{S}^2(\mathbb{D})}^2.
\end{aligned}$$

Remark 3.2.3. Clearly $\mathcal{S}^2(\mathbb{D})$ is an algebra with unit and with a norm $\|\cdot\|_{\mathcal{S}^2(\mathbb{D})}$ under which it is a Banach space. Furthermore, we see that the multiplication is continuous in each factor separately. Therefore there exists a norm equivalent to $\|\cdot\|_{\mathcal{S}^2(\mathbb{D})}$, for which $\mathcal{S}^2(\mathbb{D})$ is a Banach algebra (see [11, page 212]).

Keeping in mind the fact above, we do not differentiate the norm $\|\cdot\|_{\mathcal{S}^2(\mathbb{D})}$ and

its equivalent norm that makes $\mathcal{S}^2(\mathbb{D})$ a Banach algebra.

- iii. We want to show polynomials are dense on $\mathcal{S}^2(\mathbb{D})$. Given $f \in \mathcal{S}^2(\mathbb{D})$, let $f_q(z) = f(qz)$ be its dilation and $Df_q(z) = qDf(qz) = q(Df)_q(z)$ be derivative of dilation where $0 < q < 1$. Each function f_q is analytic in a larger disk, so it can be approximated uniformly on $\mathbb{D}$ by a sequence of holomorphic polynomials P_q^n , and hence Df_q can be approximated uniformly on $\overline{\mathbb{D}}$ by holomorphic polynomials DP_q^n . So it will be enough to prove that f can be approximated in $\mathcal{S}^2(\mathbb{D})$ by its dilation. That is to say $\|f - f_q\|_{\mathcal{S}^2(\mathbb{D})} \rightarrow 0$ as $q \rightarrow 1$. This means that $\|f - f_q\|_{\mathcal{H}^2(\mathbb{D})}^2 + \|Df - Df_q\|_{\mathcal{H}^2(\mathbb{D})}^2 \rightarrow 0$ as $q \rightarrow 1$. So finally it is enough to prove that $\|f - f_q\|_{\mathcal{H}^2(\mathbb{D})}^2 \rightarrow 0$ and $\|Df - Df_q\|_{\mathcal{H}^2(\mathbb{D})}^2 \rightarrow 0$ as $q \rightarrow 1$. For this, let us assume

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since $f \in \mathcal{H}^2(\mathbb{D})$, for all $\epsilon > 0$, we can choose a natural number N large enough such that

$$\sum_{n=N+1}^{\infty} |a_n|^2 < \frac{\epsilon}{2}.$$

Now choose $q_\epsilon \in (0, 1)$ such that

$$(1 - q_\epsilon^N)^2 \sum_{n=0}^N |a_n|^2 < \frac{\epsilon}{2}.$$

Then, since

$$\|f - f_q\|_{\mathcal{H}^2(\mathbb{D})}^2 = \left\| \sum_{n=0}^{\infty} a_n z^n (1 - q^n) \right\|^2 = \sum_{n=0}^{\infty} |a_n (1 - q^n)|^2,$$

it follows that for all $q \geq q_\epsilon$

$$\begin{aligned}
\|f - f_q\|_{\mathcal{H}^2(\mathbb{D})}^2 &= \sum_{n=0}^N |a_n(1 - q^n)|^2 + \sum_{n=N+1}^{\infty} |a_n(1 - q^n)|^2 \\
&\leq (1 - q^N) \sum_{n=0}^N |a_n|^2 + \sum_{n=N+1}^{\infty} |a_n|^2 \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This shows that $\|f - f_q\|_{\mathcal{H}^2(\mathbb{D})}^2 \rightarrow 0$ as $q \rightarrow 1$. On the other hand, we have

$$|Df(z) - Df_q(z)| = \left| \sum_{n=1}^{\infty} n a_n z^{n-1} - \sum_{n=1}^{\infty} n a_n q^n z^{n-1} \right| = \left| \sum_{n=1}^{\infty} n a_n (1 - q^n) z^{n-1} \right|.$$

Similarly, we can show that $\|Df - Df_q\|^2 \rightarrow 0$ as q approaches to 1.

□

Definition 3.2.4. Define

$$\mathcal{S}_0^2(\mathbb{D}) = \{f \in \mathcal{S}^2(\mathbb{D}) : f(0) = 0\}$$

Corollary 3.2.5. $\mathcal{S}_0^2(\mathbb{D}) \subset \mathcal{S}^2(\mathbb{D})$ is a Banach algebra with the norm defined for $\mathcal{S}^2(\mathbb{D})$ and $\mathcal{S}^2(\mathbb{D}) = [1] \oplus \mathcal{S}_0^2(\mathbb{D})$, and hence

$$\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{z^n : n \in \mathbb{N}\}.$$

Proof. For any f and g in $\mathcal{S}_0^2(\mathbb{D}) \subset \mathcal{S}^2(\mathbb{D})$, we immediately see that

$$\|fg\|_{\mathcal{S}^2(\mathbb{D})} \leq \|f\|_{\mathcal{S}^2(\mathbb{D})} \|g\|_{\mathcal{S}^2(\mathbb{D})}.$$

Also, we have $f(0) = 0$ and $g(0) = 0$ so $(fg)(0) = 0$ and hence fg belongs to $\mathcal{S}_0^2(\mathbb{D})$. To show $\mathcal{S}_0^2(\mathbb{D})$ is closed subset of $\mathcal{S}^2(\mathbb{D})$, let us take a converging sequence $g_n \in \mathcal{S}_0^2(\mathbb{D})$ for $n \in \mathbb{N}$. So there exists $g \in \mathcal{S}^2(\mathbb{D})$ such that g_n converges to g on $\mathcal{S}^2(\mathbb{D})$ norm. Since $\mathcal{S}^2(\mathbb{D})$ convergence implies $\mathcal{H}^2(\mathbb{D})$ convergence and $g_n(0) = 0$ for all n , so $g(0) = 0$. Therefore $g \in \mathcal{S}_0^2(\mathbb{D})$. Hence $\mathcal{S}_0^2(\mathbb{D})$ is a Banach algebra. The other part of the corollary follows from the Definition 3.2.4. $\square$

The following theorem plays important role in the proof of our main theorem.

Theorem 3.2.6. *Let $\mathcal{V}$ be the Volterra operator on $\mathcal{H}^2(\mathbb{D})$ then the following statements are true:*

- i. *Range of $\mathcal{V} = \mathcal{S}_0^2(\mathbb{D})$.*
- ii. *$\mathcal{V}$ is a bounded isomorphism from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ and its inverse is D .*
- iii. *The operator T acting on $\mathcal{H}^2(\mathbb{D})$ is similar under $\mathcal{V}$ to the multiplication operator M_z acting on $\mathcal{S}_0^2(\mathbb{D})$.*

Proof. i. Let g be in the range of $\mathcal{V}$ then there exists $f \in \mathcal{H}^2(\mathbb{D})$ such that

$$g(z) = (\mathcal{V}f)(z) = \int_0^z f(w)dw$$

then $Dg = f \in \mathcal{H}^2(\mathbb{D})$ and $g(0) = 0$. Hence $g \in \mathcal{S}_0^2(\mathbb{D})$. Conversely, suppose that g belongs to $\mathcal{S}_0^2(\mathbb{D})$, then

$$(\mathcal{V}Dg)(z) = \int_0^z (Dg)(w)dw = g(z) - g(0) = g(z).$$

Therefore g belongs to the range of $\mathcal{V}$.

- ii. First we want to show $\mathcal{V}$ is a bounded operator on $\mathcal{H}^2(\mathbb{D})$. Let us assume f is in the Hardy space.

$$\begin{aligned} \|\mathcal{V}(f)\|_{\mathcal{S}^2(\mathbb{D})} &= \|\mathcal{V}(f)\|_{\mathcal{H}^2(\mathbb{D})} + \|D(\mathcal{V}(f))\|_{\mathcal{H}^2(\mathbb{D})} \\ &\leq \|f\|_{\mathcal{H}^2(\mathbb{D})} + \|f\|_{\mathcal{H}^2(\mathbb{D})} \\ &= 2\|f\|_{\mathcal{H}^2(\mathbb{D})}. \end{aligned}$$

Hence the map $\mathcal{V}$ from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ is bounded. Clearly $\mathcal{V}$ is linear. Now to show $\mathcal{V}$ is one-one, assume that f_1 and f_2 belong the Hardy space, and also assume that

$$\int_0^z f_1(w)dw = \int_0^z f_2(w)dw, \quad \text{for all } z \in \mathbb{D}.$$

Differentiating both sides we see that $f_1 = f_2$ and hence $\mathcal{V}$ is one-one.

From part (i) we have $\mathcal{V}Dg = g$ and clearly $D\mathcal{V}f = f$. This shows that $\mathcal{V}$ is a bounded bijective linear operator from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ and $\mathcal{V}^{-1} = D$.

- iii. Suppose f belongs to $\mathcal{H}^2(\mathbb{D})$ and also suppose $\mathcal{V}f = g$, for some $g \in \mathcal{S}_0^2(\mathbb{D})$.

Therefore we have $f(z) = (\mathcal{V}^{-1}g)(z) = (Dg)(z)$.

$$\begin{aligned}(Tf)(z) &= zf(z) + (\mathcal{V}f)(z) \\ &= z(Dg)(z) + g(z) \\ &= D(zg(z)).\end{aligned}$$

Now applying $\mathcal{V}$ on the both side, we see that

$$\begin{aligned}(\mathcal{V}Tf)(z) &= \mathcal{V}D(zg(z)) \\ &= zg(z) \\ &= z(\mathcal{V}f)(z).\end{aligned}$$

So, $\mathcal{V}T = M_z\mathcal{V}$ and $\mathcal{V}T\mathcal{V}^{-1} = M_z$. That is to say $\mathcal{V}$ transforms the operator T into the operator multiplication by z on $\mathcal{S}_0^2(\mathbb{D})$. $\square$

We can summarize the theorem by the following diagram.

$$\begin{array}{ccc}\mathcal{H}^2(\mathbb{D}) & \xrightarrow{\mathcal{V}} & \mathcal{S}_0^2(\mathbb{D}) \\ \downarrow T & \searrow \mathcal{V}T = M_z\mathcal{V} & \downarrow M_z \\ \mathcal{H}^2(\mathbb{D}) & \xrightarrow{\mathcal{V}} & \mathcal{S}_0^2(\mathbb{D})\end{array}$$

Definition 3.2.7. An element a in Banach algebra $\mathcal{A}$ is called cyclic if the subalgebra generated by a is dense in $\mathcal{A}$.

The next proposition provides connection between invariant subspace of shift operator and closed ideals of a Banach algebra. It is proved in [15] but we include the proof for the sake of completeness.

Proposition 3.2.8. *Let $\mathcal{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are exactly the closed ideals of $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ has a cyclic element, it is commutative. Let a be a cyclic element of $\mathcal{A}$ and let $\mathcal{L}$ be an invariant subspace of multiplication operator $\mathcal{M}_a$. Clearly,

$$\mathcal{M}_{\mathcal{L}} = \{b \in \mathcal{A} : bx \in \mathcal{L} \text{ for all } x \in \mathcal{L}\}$$

is a closed subalgebra of $\mathcal{A}$. Since $\mathcal{L}$ is an invariant subspace of $\mathcal{M}_a$, $\mathcal{M}_a x = ax \in \mathcal{L}$, for all $x \in \mathcal{L}$ and so $a \in \mathcal{M}_{\mathcal{L}}$. Therefore, $\mathcal{M}_{\mathcal{L}}$ contains the subalgebra generated by a . Since $\mathcal{M}_{\mathcal{L}}$ closed and a cyclic, it follows that $\mathcal{M}_{\mathcal{L}} = \mathcal{A}$. This shows that $\mathcal{L}$ is a left ideal and thus, being $\mathcal{A}$ commutative, an ideal of $\mathcal{A}$. On the other hand, each ideal of $\mathcal{A}$ is invariant with respect to $\mathcal{M}_a$, which finishes the proof. $\square$

Lemma 3.2.9. *J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$ if and only if J is an ideal of $\mathcal{S}^2(\mathbb{D})$ contained in $\mathcal{S}_0^2(\mathbb{D})$.*

Proof. Using of Corollary 3.2.5, we see that for any $h \in \mathcal{S}^2(\mathbb{D})$ there exists $h_1 \in \mathcal{S}_0^2(\mathbb{D})$ such that $h = c + h_1$.

Suppose J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$, then for any $h \in \mathcal{S}^2(\mathbb{D})$ and $j \in J$,

$$hj = (c + h_1)j = cj + h_1j \in J.$$

Since norm on both spaces are the same, J is a closed ideal of $\mathcal{S}^2(\mathbb{D})$. On the other hand, if J is an ideal of $\mathcal{S}^2(\mathbb{D})$ contained in $\mathcal{S}_0^2(\mathbb{D})$, then it is obvious that J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$. $\square$

Definition 3.2.10. Let K be a closed subset of the unit circle $\partial\mathbb{D}$ and let G be an inner function. Let us say G is *associated with* K if

- if $a_1, a_2, \dots$ are the zeros of $G(z)$ in the open disc, then all the limit points of $\{a_k\}$ belong to K ;
- the measure determining the singular part of G is supported on K .

The following theorem from Korenbljum [13] characterizes all closed ideals of $\mathcal{S}^2(\mathbb{D})$. This theorem is essential in proving our main theorem.

Theorem 3.2.11. *Suppose G is associated with K . Let $I_{\mathcal{S}^2(\mathbb{D})}(G; K)$ be the set of all $f \in \mathcal{S}^2(\mathbb{D})$ which are divisible by G and which vanish on K . Then $I_{\mathcal{S}^2(\mathbb{D})}(G; K)$ is a closed ideal of $\mathcal{S}^2(\mathbb{D})$. Moreover, every closed ideal of $\mathcal{S}^2(\mathbb{D})$ is obtained in this manner.*

Corollary 3.2.12. *Suppose G is associated with K . Let $I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)$ be the set of all $f \in \mathcal{S}_0^2(\mathbb{D})$ which are divisible by G and which vanish on K . Then $I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)$ is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$. Moreover, every closed ideal of $\mathcal{S}_0^2(\mathbb{D})$ is obtained in this manner.*

Proof. Clearly $I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)$ is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$. Conversely, suppose J be any ideal of $\mathcal{S}_0^2(\mathbb{D})$. From the Lemma 3.2.9, J is also an ideal of $\mathcal{S}^2(\mathbb{D})$ contained in $\mathcal{S}_0^2(\mathbb{D})$. By the Theorem 3.2.11, J is of the form $I_{\mathcal{S}^2(\mathbb{D})}(G; K)$ for some G and K defined in Definition 3. Hence $J = I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)$. $\square$

Theorem 3.2.13 (Main Theorem). *Let T be an operator*

$$(Tf)(z) = zf(z) + \int_0^z f(w)dw$$

defined on $\mathcal{H}^2(\mathbb{D})$. Then the lattice of closed invariant subspaces is

$$\text{Lat } T = \{S \subset \mathcal{H}^2(\mathbb{D}) : S = \{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)\} \text{ for } G, K \text{ as in Definition 3.2.10}\}.$$

Proof. From Corollary 3.2.5, $\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{z^n : n \in \mathbb{N}\}$, so z is a cyclic element of the Banach algebra $\mathcal{S}_0^2(\mathbb{D})$. Thus from the Proposition 3.2.8, closed invariant subspaces of M_z on $\mathcal{S}_0^2(\mathbb{D})$ are exactly the closed ideals of $\mathcal{S}_0^2(\mathbb{D})$. Using Corollary 3.2.12, the lattice of closed invariant subspace of M_z acting on $\mathcal{S}_0^2(\mathbb{D})$ is given by

$$\text{Lat } M_z = \{I_{\mathcal{S}_0^2(\mathbb{D})}(G; K) : I_{\mathcal{S}_0^2(\mathbb{D})}(G; K) \text{ is as in Corollary 3.2.12}\}.$$

Since $\mathcal{V}T\mathcal{V}^{-1} = M_z$, we see that $\mathcal{V}^{-1}(I_{\mathcal{S}_0^2(\mathbb{D})}(G; K))$ is a closed invariant subspace of T . From Theorem 1, we know that $\mathcal{V}^{-1}(f) = Df$. So, $S = \{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)\}$ is a closed invariant subspace of T . Hence,

$$\text{Lat } T = \{S \subset \mathcal{H}^2(\mathbb{D}) : S = \{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G; K)\} \text{ for } G, K \text{ as in Definition 3.2.10}\}.$$

□

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