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RIGOROUS RESULTS IN STEADY FINGER SELECTION IN VISCOUS FINGERING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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* * * * *

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ABSTRACT

This thesis concerns existence of steadily translating finger solution in a Hele-Shaw cell for non-zero surface tension (ϵ^2). Though there are numerous numerical and formal asymptotic results for this problem, we know of no mathematically rigorous results to date. By showing equivalence of the problem to a certain nonlinear integro differential equation in a complex plane strip, we show that analytic symmetric finger solutions exist for sufficiently small surface tension for a discrete set of values of $(\lambda - \frac{1}{2})/\epsilon^{\frac{1}{3}}$, where λ is the relative finger width.

The methodology consists of proving existence and uniqueness of analytic solutions for a weaker half strip problem for any λ . This is followed by Borel plane analysis near appropriate complex turning points that determine exponentially small terms in ϵ , as $\epsilon \rightarrow 0^+$. We prove that the symmetry condition is satisfied when $(\lambda - \frac{1}{2})/\epsilon^{\frac{1}{3}}$ attains a discrete set of values. When the symmetry condition is satisfied, the weaker half strip problem is shown to be equivalent to the original problem.

To my daughter Kathy

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Finally I thank my wife Changhui for her support, and my daughter Kathy for being my pride and joy.

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CHAPTER 1

INTRODUCTION

1.1 Background Work

A Hele-shaw cell is a pair of parallel plates that are separated by a small gap b (see Figure 1.1). The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw has been studied a lot since 1950's, many review papers have appeared in the literature. Bensimon et al (1986), Saffman (1986), Homsy, Kessler et al (1988), Pelce (1988) summarized the literature as of the 1980's. A more recent review by Tanveer (2000) focusses on the singular effect of surface tension.

A fluid of viscosity μ is being pushed by a less viscous fluid, the more viscous fluid

Figure 1.1: Rectilinear Hele-Shaw flow

Figure 1.2: Steady finger and flow in the lateral plane

is displaced at infinity with velocity V . Usually a single steady finger eventually forms and propagates at constant velocity U with relative finger width λ (Figure 1.2).

The averaged velocity (u, v) in the (x, y) -plane satisfies Darcy's law:

$$(u, v) = -\frac{b^2}{12\mu} \nabla p = \nabla \phi; \quad (1.1)$$

where p is the pressure and ϕ is the velocity potential. By incompressibility:

$$\Delta \phi = 0, \text{ in flow domain } \Omega; \quad (1.2)$$

On side walls:

$$v = 0, \text{ at } y = \pm a. \quad (1.3)$$

In far field:

$$(u, v) = V \hat{x} + O(1), \text{ as } x \rightarrow +\infty; \quad (1.4)$$

where $\hat{x}$ is unit vector on x-axis.

On the interface, the pressure condition is:

$$\phi = \frac{b^2 T}{12\mu} \frac{1}{R}; \quad (1.5)$$

where R is the radius of curvature, T is the surface tension.

The kinematic condition for a steady finger is:

$$\frac{\partial \phi}{\partial n} = U \cos \theta; \quad (1.6)$$

where θ is the angle between the interface normal and the positive x -axis.

There are a number of different but equivalent formulations, starting with the one by Mclean and Saffman (1981). Here we use the formulation by Tanveer (2000) which is valid for fingers that are *a priori* symmetric about the channel centerline.

We can set $a = 1$ and $U = 1$ as this corresponds to nondimensionalizing all lengths by a and velocities by U .

By integrating (1.2) in the domain Ω , the finger width λ is related to U as follows:

$$\lambda = \frac{1}{U}; \quad (1.7)$$

In a frame moving with the steady symmetric finger, the condition (1.6) transforms, without loss of generality, into

$$\psi = 0; \quad (1.8)$$

where ψ is the stream function (harmonic conjugate of ϕ) so that $W = \phi + i\psi$ is an analytic function of $z = x + iy$. The nondimensional pressure condition (1.5) in the moving frame translates into:

$$\phi + \frac{1}{\lambda} x = \frac{B}{R}. \quad (1.9)$$

Figure 1.3: Upper half ξ -plane with a cut on imaginary axis from $\xi = i$ to ∞i

where $\mathcal{B} = \frac{b^2 T}{12\mu V a^2}$.

On the side walls,(1.3) implies that

$$v = \pm \left[\frac{1}{\lambda} - 1 \right] \text{ on } y = -1 \text{ and } y = 1 \text{ respectively} \quad (1.10)$$

while the far field conditions in the finger frame is

$$W \sim - \left[\frac{1}{\lambda} - 1 \right] z + O(1); \quad (1.11)$$

We consider the conformal map $z(\xi)$ of the cut upper half ξ -plane as shown in Figure 1.3.to the flow domain Ω in Figure 1.2.The real ξ axis corresponds to the finger bondary,with $\xi = -\infty, +\infty$ corresponding to the finger tails at $z = -\infty + i\lambda, z = -\infty - \lambda i$ respectively.The two sides of the cut correspond to the two side walls respectively. It is easily seen that the complex potential $W(\xi)$ is given by:

$$W(\xi) = \frac{1-\lambda}{\pi\lambda} \ln(\xi^2 + 1) + c \quad (1.12)$$

where c is some real constant. We define $F(\xi)$ so that

$$z(\xi) = -\frac{1}{\pi} \ln(\xi - i) - \frac{1-2\lambda}{\pi} \ln(\xi + i) - i\lambda - \frac{2-2\lambda}{\pi} F(\xi); \quad (1.13)$$

It follows that such defined $F(\xi)$ is analytic in the entire upper half ξ -plane.(Tanveer 2000). The pressure condition (1.9) translates into requiring that on the real ξ axis:

$$Re F = \frac{\epsilon^2}{|F' + H|} Im \left[\frac{F'' + H'}{F' + H} \right]; \quad (1.14)$$

where

$$H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1}, \text{ with } \gamma = \frac{\lambda}{1-\lambda}, \epsilon^2 = \frac{\pi^2 \lambda \mathcal{B}}{4(1-\lambda)^2}; \quad (1.15)$$

For zero surface tension $\epsilon = 0$, it follows that $F = 0$, this corresponds to the Saffman and Taylor solutions (1958) which are a family of exact solutions for symmetric fingers with arbitrary width of finger $\lambda \in (0, 1)$.

Since the Saffman-taylor solutions

$$z_0(\xi) = -\frac{1}{\pi} \ln(\xi - i) - \frac{1-2\lambda}{\pi} \ln(\xi + i) - i\lambda; \quad (1.16)$$

are univalent, if ϵ and if an appropriate norm of F' as introduced later is sufficiently small enough, then it can be shown that $z(\xi)$ is also univalent.

For non-zero tension, no exact solutions exist. Mclean and Saffman's (1981) numerical work suggested that solutions exist only for some isolated values of finger width. They also found a formal perturbation series solution in powers of surface tension ϵ , however no restriction on λ was obtained from the perturbation series. Romero (1982) and Vander-Broeck (1983) determined a discrete set of branches of steady solutions. So there was an apparent contradiction between numerics and perturbation solution

of Mclean and Saffman. Analytical and numerical work by Kruskal and Sugar (1991) on the simpler problem of geometric model of crystal growth paved the way for resolving the discrepancy between numerics and perturbation for the Saffman-Taylor finger problem. Kessler and Levine (1985) were the first to suggest that exponentially small terms missing in the Mclean-Saffman perturbation expansion were the reason for the discrepancy with numerical calculations. Their numerical calculations supported this contention. This was confirmed by formal analytical work of Shraiman (1986), Combescot et al. (1986, 1988), Hong and Langer (1986), Tanveer (1986, 1987) and Chapman (1998). Though the formal results on exponential asymptotics are generally accepted, some of the conclusions have not been universally accepted (Xu, 1991) (DeGregoria, Schwartz, 1985). Therefore, this makes the need for rigorous results more compelling.

It is to be noted that the problem is not readily accessible to real domain methods. The problem with $\epsilon = 0$ is a structurally unstable system, as shown in this thesis. The solution set for $\epsilon \rightarrow 0^+$ is different from that for $\epsilon = 0$. This makes control of term difficult in the real domain. The importance of a term is determined by its relative size in a neighborhood of the turning point in some complex domain (Tanveer 2000). Small changes in the equation as measured in any real Sobolev norm may result in large changes in the solution set.

1.2 Notations

Following Tanveer 2000, the problem of determining a smooth steady symmetric finger is equivalent to finding function $F(\xi)$ that is analytic in the upper-half ξ plane with continuous derivatives upto the second on $\text{Im } \xi = 0$ and is required to satisfy the following conditions:

Condition (i):

$$\text{Re} F = \frac{\epsilon^2}{|F' + H|} \text{Im} \left[\frac{F'' + H'}{F' + H} \right], \text{ on } \text{Im } \xi = 0; \quad (1.17)$$

where

$$H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1}, \text{ where } \gamma = \frac{\lambda}{1 - \lambda} \in (0, \infty), \quad (1.18)$$

λ is the finger width and $\epsilon^2 > 0$ is a nondimensional parameter proportional to surface tension.

Condition (ii):

$$F(\xi), \xi F'(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \pm\infty; \quad (1.19)$$

Condition (iii)(symmetry condition):

$$\text{Re } F(-\xi) = \text{Re } F(\xi), \text{Im } F(-\xi) = -\text{Im } F(\xi), \text{ for real } \xi; \quad (1.20)$$

Condition (iv):

$$F'(\xi) + H(\xi) \neq 0, \text{Im } \xi \geq 0; \quad (1.21)$$

Comment 1.1: It is easily proved in this case that for sufficiently small F' in an appropriate norm, introduced later, that Condition (iv) follows and that $\frac{1}{2-2\lambda} \ln(\xi - i) + \frac{1-2\lambda}{2-2\lambda} \ln(\xi + i) + F$ is univalent.

Comment 1.2: It is easily seen that if $F(\xi)$ satisfies (1.17), so does $[F(-\xi)]^*$. Hence, if condition (iii) is satisfied, then it is enough to require that equation (1.17) is satisfied for $\operatorname{Re} \xi \leq 0$.

Definition 1.1. Let $\mathcal{R}$ be an open connected (see Figure 1.4) region on complex ξ plane bounded by lines

$$r_u = r_{u,1} \cup r_{u,2} \cup r_{u,3} \cup r_{u,4} \cup r_{u,5} \cup r_{u,6}$$

$$r_l = \{\xi : \xi = -bi + re^{-i(\varphi_0 + \mu)}\} \cup \{\xi : \xi = -bi + re^{i(\pi + \varphi_0 + \mu)}\}$$

where $0 < b < 1, \alpha > 0, 2\alpha < b, 0 < \varphi_0, \mu < \frac{\pi}{2}$ with $\varphi_0 + \mu = \frac{\pi}{3}$ and

$$r_{u,1} = \{\xi : \xi = (\alpha - \nu)i - R + re^{i(\pi - \varphi_0)}, 0 \leq r < \infty\} .$$

$$r_{u,2} = \{\xi : \operatorname{Im} \xi = \alpha - \nu, -R \leq \operatorname{Re} \xi \leq -\nu\} .$$

$$r_{u,3} = \{\xi : \xi = \alpha i + re^{-3\pi i/4}, 0 \leq r \leq \sqrt{2}\nu\} .$$

$$r_{u,4} = \{\xi : \xi = \alpha i + re^{-\pi i/4}, 0 \leq r \leq \sqrt{2}\nu\}$$

$$r_{u,5} = \{\xi : \operatorname{Im} \xi = \alpha - \nu, \nu \leq \operatorname{Re} \xi \leq R\} .$$

$$r_{u,6} = \{\xi : \xi = (\alpha - \nu)i + R + re^{i\varphi_0}, 0 \leq r < \infty\}$$

where $R > 0$ is large enough and $0 < \nu < \alpha$ is small enough so that Lemma 15 and Lemma 20 in Appendix A hold.

Figure 1.4: Region $\mathcal{R}$ in complex ξ plane.

Comment 1.3: Choice of $1-b$ can depend on ϵ while α and φ_0 are chosen independent of ϵ .

Definition 1.2. $\mathcal{R}^- = \mathcal{R} \cap \{Re\xi < 0\}$:

Let $0 < \tau < 1$. We introduce spaces of functions:

Definition 1.3.

$$\mathbf{A}_k = \{F(\xi) : F(\xi) \text{ is analytic in } \mathcal{R} \text{ and continuous in } \bar{\mathcal{R}},$$

$$\text{with } \sup_{\xi \in \bar{\mathcal{R}}} |(\xi - 2i)^{k+\tau} F(\xi)| < \infty\}, k = 0, 1, 2.$$

$$\|F\|_k := \sup_{\xi \in \bar{\mathcal{R}}} |(\xi - 2i)^{k+\tau} F(\xi)|$$

In the above definition, replacing $\mathcal{R}$ with $\mathcal{R}^-$, we get:

Definition 1.4.

$$\mathbf{A}_k^- = \{F(\xi) : F(\xi) \text{ is analytic in } \mathcal{R}^- \text{ and continuous in its closure,}$$

$$\text{with } \sup_{\xi \in \bar{\mathcal{R}}^-} |(\xi - 2i)^{k+\tau} F(\xi)| < \infty\}, k = 0, 1, 2.$$

$$\|F(\xi)\|_k^- := \sup_{\xi \in \bar{\mathcal{R}}^-} |(\xi - 2i)^{k+\tau} F(\xi)|$$

Comment 1.4: If $F(\xi) \in \mathbf{A}_0$, and satisfies symmetry condition (iii) on the real axis, then it follows from a Taylor expansion of $F(\xi)$ near the origin that $Im F(\xi) = 0$ for the imaginary ξ axis segment $-b < Im \xi < \alpha$. By Schwartz reflection principle, $F(-\xi^*) = [F(\xi)]^*$, so $\|F\|_0^- = \|F\|_0$. On the other hand if $F \in \mathbf{A}_0^-$ and

satisfies $Im F(\xi) = 0$ for the imaginary ξ axis segment $-b < Im \xi < \alpha$, then $F(-\xi^*) = [F(\xi)]^*$ extends F to the right half of $\mathcal{R}$ and $\|F\|_0^- = \|F\|_0$.

Comment 1.5: A_k are Banach spaces and $\mathbf{A} \equiv \mathbf{A}_0 \supset \mathbf{A}_1 \supset \mathbf{A}_2$. If $F \in \mathbf{A}$, then F satisfies property :

$$F(\xi) \sim O(\xi^{-\tau}), \text{ as } |\xi| \rightarrow \infty, \xi \in \mathcal{R}. \quad (1.22)$$

Definition 1.5. Let $\mathcal{D}$ be any connected set in complex ξ -plane, we introduce norms:

$$\|F(\xi)\|_{k,D} := \sup_{\xi \in \mathcal{D}} |(\xi - 2i)^{k-\tau} F(\xi)|, \quad k = 0, 1, 2.$$

Finger problem: The problem tackled in this thesis will be to find function F analytic in $\{Im \xi > 0\} \cup \mathcal{R}$ so that $F \in \mathbf{A}$, $F' \in \mathbf{A}_1$ for some τ fixed in $(0, 1)$ and independent of ϵ , satisfying conditions (i) and (iii) on the real axis, with $\|F\|_0, \|F'\|_1$ small enough to guarantee condition (iv). For simplicity, we will restrict ourselves to $\lambda > \frac{1}{2}$, with $\lambda - \frac{1}{2}$ small.

Comment 1.6: The finger problem as defined above implies that conditions (i)-(iv) are satisfied. However, the converse has not been proved in this thesis, though formal reasoning suggests that this is true.

1.3 Main Results

The formal strategy of calculation of finger width (Tanveer 1987, Combescot et al 1986, 1988) involves analytic continuation of equation in an inner neighborhood of turning points in the complex plane and ignoring integral contribution and other terms that are formally small.

Combescot et al (1986,1988) formally reduced the problem of determining a smooth steady finger to determining eigenvalue a so that $G(y)$ is a solution to

$$y \frac{d}{dy} \left(y \frac{dG}{dy} \right) + G^{-1} = y^2 + 2^{-1/3} a; \quad (1.23)$$

satisfying asymptotic matching condition:

$$G(y) \sim y^{-1}, \text{ as } y \rightarrow \infty \text{ for } \arg y \in (-\pi, 0]; \quad (1.24)$$

and in addition satisfying:

$$\text{Im } G = 0, \text{ for large enough } y \text{ on the positive real axis.} \quad (1.25)$$

The finger width λ is related to a through

$$a = \frac{2\lambda - 1}{1 - \lambda} \epsilon^{-4/3}; \quad (1.26)$$

Combescot et al (1986,1988) computed numerically many terms of the formal divergent asymptotic expansion for large y

$$G(y) \sim \frac{1}{y} - \frac{\bar{\delta}_1}{2y^3} + \frac{1}{2y^4} + \frac{3\bar{\delta}_1^2}{5y^5} - \frac{3\bar{\delta}_1}{y^6} + \dots$$

where $\bar{\delta}_1 = 2^{-1/3} a$.

Based on the form of the expansion, they were able to deduce the nature of the singularity in the Borel plane, and conclude that for large y on the positive real axis

$$\text{Im } G(y) \sim \tilde{S}(a) y^{-3/4} e^{-\frac{2^{3/2}}{3} y^{3/2}}; \quad (1.27)$$

where the Stokes multiplier $\tilde{S}(a)$ was determined numerically as a function of a . Numerical calculation revealed $\tilde{S}(a) = 0$ for a discrete set of values of a . The first few of

these discrete set of eigen values $\{a_n\}$ in order of increasing size were found to be $\{1.0278, 3.7168, 7.0934, \dots\}$.

They assume that

$$\tilde{S}(a) = 0 \text{ iff } \text{Im } G(y) = 0 \text{ for large } y \text{ on the positive axis.} \quad (1.28)$$

as necessary for the existence of a symmetric finger.

In this thesis, among the many other results, it will be rigorously proved that the leading order form of $\text{Im}G(y)$ in (1.27) is indeed true and that assumption (1.28) is indeed justified. However, we will not compute the Stokes constants $\tilde{S}(a)$, but rely on Combescot et al.'s numerical computation to deduce that there exist a discrete set $\{a_n\}$ so that $\tilde{S}(a_n) = 0$, $\tilde{S}'(a_n) \neq 0$.

Through a transformation:

$$y = \frac{3^{2/3}}{2} \eta^{2/3}, G(y) = \frac{2}{3^{2/3} \eta^{2/3}} (1 + \psi(\eta)); \quad (1.29)$$

$\psi(\eta)$ satisfies:

$$\frac{d^2 \psi}{d\eta^2} - \frac{1}{3\eta} \frac{d\psi}{d\eta} - \psi = -\frac{4}{9\eta^2} - \frac{4}{9\eta^2} \psi + \frac{2a}{3^{1/3} \eta^{4/3}} - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n \quad (1.30)$$

and matching conditions as $|\eta| \rightarrow \infty$:

$$\psi(\eta, a) \sim \frac{2a}{3^{4/3} \eta^{4/3}}, 0 \leq \arg \eta < \frac{5\pi}{8}; \quad (1.31)$$

Note that

$$\text{Im} \psi(\eta) = \text{Im}(yG(y)) \sim \tilde{S}(a) y^{1/4} e^{-\frac{2^{3/2}}{3} y^{3/2}} \sim \frac{3^{1/6}}{2^{1/4}} \tilde{S}(a) \eta^{1/6} e^{-\eta};$$

One of the main theorems proved in this thesis is

Theorem 1.6. (1) *There exists sufficiently large enough $R > 0$ such that (1.30) with condition (1.31) has unique analytic solution $v_0(\eta, a)$ in the region $|\eta| > R, \arg \eta \in [0, \frac{\pi}{2}]$.*

(2) *Further on the real η axis as $\eta \rightarrow \infty$*

$$\operatorname{Im} v_0(\eta, a) \sim \frac{3^{1/6}}{2^{1/4}} \tilde{S}(a) \eta^{1/6} e^{-\eta}; \quad (1.32)$$

(3) *Further $\operatorname{Im} v_0(\eta, a) = 0$ for $\eta > R$ iff $\tilde{S}(a) = 0$.*

Comment 1.7: Using Combescot et al 's computation, $\tilde{S}(a) = 0$ only if $a \in \{a_n\} = \{1.0278, 3.7168, 7.0934, \dots\}$.

The proof of this Theorem is given in Chapter 4.

However, in order to construct rigorous theory for this problem, it is necessary to control the integral terms and other terms that do not appear in the leading equation of Combescot et al (1986). The primary result of this thesis for the finger problem is the following:

Theorem 1.7. *In the range $\frac{1}{2} < \lambda$ with $\lambda - \frac{1}{2}$ small, for all sufficiently small ϵ ,*

(1) *For each $a_0 \in \{a_n\}$ for which the Stokes constant $\tilde{S}(a_0) = 0$ in Theorem 1.6, there exists an $\beta(\epsilon^{2/3})$ with $\lim_{\epsilon \rightarrow 0} \beta(\epsilon^{2/3}) = a_0$ so that if*

$$\frac{2\lambda - 1}{1 - \lambda} = \epsilon^{1/3} \beta(\epsilon^{2/3}), \quad (1.33)$$

then there exists a solution of the Finger problem $F \in \mathbf{A}_0$.

(2) *Further in addition to (1), if $\tilde{S}'(a_0) \neq 0$, then $\beta(\epsilon^{2/3})$ is analytic function of $\epsilon^{2/3}$ at $\epsilon = 0$, so that for small ϵ ,*

$$\frac{2\lambda - 1}{1 - \lambda} = \epsilon^{1/3} (a_0 + \beta_1 \epsilon^{2/3} + \beta_2 \epsilon^{4/3} + \dots); \quad (1.34)$$

(3) If λ does not satisfy the constraint (1.34), then there is no solution of the Finger problem $F \in \mathbf{A}_0, F' \in \mathbf{A}_1$ with $\|F'\|_1$ small enough.

Our solution strategy consists of two steps:

(a) By relaxing the symmetry condition, we prove the existence and uniqueness of solutions to an appropriate problem in a half strip (Weak Problem) without any restriction on λ .

(b) The symmetry condition is then invoked to determine restriction on λ that will guarantee existence of solution to the Finger problem.

However, in going through the first step, we encounter singularities of integrals of the type $(P) \int_{-\infty}^{\infty} \frac{G(t)}{t-\xi} dt$ since continuity of $G(t)$ at $t = 0$ is not guaranteed when symmetry is relaxed. This makes proof difficult. We circumvent this difficulty by proving the equivalence of the finger problem to a problem where integral quantities are of the type $\int_r \frac{G_1(t)}{t-\xi} dt$, where the path r is outside the domain $\mathcal{R}$ of solutions and $G_1(t)$ can be calculated on r using solutions on the domain $\mathcal{R}$. The singularities in this case are outside the domain of interest and do not affect the regularity argument in $\mathcal{R}$.

In Chapter 2, we prove equivalence of the finger problem to a set of two problems (Problem 1 and Problem 2) in a complex domain. Problem 1 is to solve an integro-differential equation in a Banach spaces of analytic functions in a complex region. By deforming the contour of integration for the integral term in Problem 1, we obtain Problem 2.

In Chapter 3, we derive the Weak problem in left half strip. By contraction argument, the existence and uniqueness of analytic solutions to the Weak problem are proved for any λ .

In Chapter 4, we carry out step (b) in our solution strategy. By rescaling in a neighborhood of the turning point $\xi = -i$, we formulate the inner problem. The leading inner equation is a nonlinear ordinary differential equation with fractional powers. Though the theory of exponential asymptotics for nonlinear ODE is by now well developed largely due to the work of M. Berry, M. Kruskal, J. Ecalle and O. Costin, this theory can not be applied directly to our problem due to the fractional powers in our equation. Using Borel plane analysis which is similar to O. Costin's (1998), we derive the exponentially small term of the leading inner problem. For the full problem, it is argued that for small ϵ , there exists an analytic function $\beta(\epsilon^{2/3})$ so that if $a = \beta(\epsilon^{2/3})$, then $\text{Im } F = 0$ on a part of imaginary ξ -axis intersecting with $\mathcal{R}$. This implies satisfying of symmetry condition, hence the existence of finger solution.

CHAPTER 2

FORMULATION OF EQUIVALENT PROBLEMS

In this chapter, we are going to formulate Problem 1 and Problem 2, which will be proved to be equivalent to Finger Problem defined in Chapter 1.

2.1 Formulation of Problem 1

Definition 2.1. Let $\mathcal{D}$ be an open connected set in complex plane with one or more straight line boundaries. $\mathcal{D}'$ is defined as an angular subset of $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$, $dist(\mathcal{D}', \partial\mathcal{D}) > 0$ and $\mathcal{D}'$ has straight line boundaries that make nonzero angle with respect to $\partial\mathcal{D}$ asymptotically at large distances from the origin (see figure 2.1).

Comment 2.1 : $\mathcal{D}'$ may just be a straight line segment..

Figure 2.1: Angular subset $\mathcal{D}'$ of $\mathcal{D}$.

Lemma 2.2. *If $F(\xi)$ satisfies property (1.22) in $\mathcal{D}$, then*

$$F'(\xi) \sim O(\xi^{-(1+\tau)}), \text{ as } |\xi| \rightarrow \infty, \xi \in \mathcal{D}'.$$

$$F''(\xi) \sim O(\xi^{-(2+\tau)}), \text{ as } |\xi| \rightarrow \infty, \xi \in \mathcal{D}'.$$

where $\mathcal{D}'$ is any angular subset of $\mathcal{D}$

Proof. See Olver (1974)(Theorem 1.4.2). □

Define:

$$\tilde{H}(\xi) = [H(\xi^*)]^* = \frac{\xi - i\gamma}{\xi^2 + 1}; \quad (2.1)$$

$$\tilde{F}(\xi) = [F(\xi^*)]^*; \quad (2.2)$$

Comment 2.2: If $F(\xi)$ is analytic in domain $\mathcal{D}$ containing real axis with property (1.22), then $\tilde{F}(\xi)$ is analytic in $\mathcal{D}^*$ and on real ξ axis $\tilde{F}(\xi) = F^*(\xi)$, and $\tilde{F}(\xi) \sim O(\xi^{-\tau})$, as $|\xi| \rightarrow \infty, \xi \in \mathcal{D}^*$; $\tilde{F}'(\xi) \sim O(\xi^{-(1+\tau)})$, $\tilde{F}''(\xi) \sim O(\xi^{-(2+\tau)})$ as $|\xi| \rightarrow \infty, \xi \in \mathcal{D}^*$, where $\mathcal{D}'$ is any angular subset of $\mathcal{D}$.

Definition 2.3. Let $\mathcal{D}$ be a connected set, for any two functions f, g with second derivative existing in $\mathcal{D}$ and small enough $\|f'\|_{1,\mathcal{D}}$ and $\|g'\|_{1,\mathcal{D}}$ so that $f' + H \neq 0$, $g' + \tilde{H} \neq 0$ in $\mathcal{D}$, we define operator $G(f, g)$ so that

$$G(f, g)(t) := \frac{1}{(f'(t) + H(t))^{1/2}(g'(t) + \tilde{H}(t))^{1/2}} \times \left[\frac{f''(t) + H'(t)}{f'(t) + H(t)} - \frac{g''(t) + \tilde{H}'(t)}{g'(t) + \tilde{H}(t)} \right] \quad (2.3)$$

Comment 2.3: If $F \in \mathbf{A}$ and $F' + H \neq 0$ in $\mathcal{R}$, then $G(F, \tilde{F})(t)$ is analytic in $\mathcal{R} \cap \mathcal{R}^*$, since in that case $\tilde{F}' + \tilde{H} \neq 0$ in $\mathcal{R}^*$.

Lemma 2.4. *Let $\mathcal{D}$ and f, g be as in definition 2.3 .If $\text{dist}(\mathcal{D}, -i) = O(1)$ as $\epsilon \rightarrow 0, 1 - b \rightarrow 0$, then we have:*

$$\|G(f, g)\|_{0, \mathcal{D}} \leq \frac{C(1 + \|f''\|_{2, \mathcal{D}}\|g'\|_{1, \mathcal{D}} + \|f'\|_{1, \mathcal{D}}\|g''\|_{2, \mathcal{D}})}{(K_1 - \|f'\|_{1, \mathcal{D}})^{3/2}(K_1 - \|g'\|_{1, \mathcal{D}})^{3/2}} + \frac{C(\|f''\|_{2, \mathcal{D}} + \|g''\|_{2, \mathcal{D}} + \|f'\|_{1, \mathcal{D}} + \|g'\|_{1, \mathcal{D}})}{(K_1 - \|f'\|_{1, \mathcal{D}})^{3/2}(K_1 - \|g'\|_{1, \mathcal{D}})^{3/2}} \quad (2.4)$$

where

$$\begin{aligned} 0 < H_m &\leq \inf_{\mathcal{D}} \{ |(t - 2i)H(t)|, |(t - 2i)\bar{H}(t)| \}; \\ K &\geq \sup_{\mathcal{D}} |t - 2i|^{-\tau} > 0; \\ K_1 &= \frac{H_m}{K} > 0; \end{aligned} \quad (2.5)$$

Constants C , and K_1 are independent of ϵ and b .

Proof. Without ambiguity, the norms $\|\cdot\|$ denoted in this proof refer to $\|\cdot\|_{\mathcal{D}}$, while $\sup$ is over the domain $\mathcal{D}$. By (2.3):

$$\begin{aligned} G(f, g) &= \frac{[H'(t)\bar{H}(t) - H(t)\bar{H}'(t)]}{(f'(t) + H(t))^{3/2}(g'(t) + \bar{H}(t))^{3/2}} + \\ &\frac{(f''(t)g'(t) - f'(t)g''(t)) + (f''(t)\bar{H}(t) + H'(t)g'(t) - H(t)g''(t) - f'(t)\bar{H}'(t))}{(f'(t) + H(t))^{3/2}(g'(t) + \bar{H}(t))^{3/2}} \end{aligned} \quad (2.6)$$

Using (1.19), (2.1):

$$\begin{aligned} \sup \{|t - 2i|^4 |H'(t)\bar{H}(t) - H(t)\bar{H}'(t)|\} &\leq C \\ \sup |(t - 2i)H| &\leq C, \sup |(t - 2i)^2 H'| \leq C \\ \sup |(t - 2i)\bar{H}| &\leq C, \sup |(t - 2i)^2 \bar{H}'| \leq C \end{aligned}$$

where C is independent of ϵ and b . we also have

$$\begin{aligned}
|(f'(t) + H(t))|^{3/2} &\geq [H_m|t - 2i|^{-1} - \|f'\|_1|t - 2i|^{-1-\tau}]^{3/2} \\
&\geq (H_m - K\|f'\|_1)^{3/2}|t - 2i|^{-3/2} \\
&\geq C(K_1 - \|f'\|_1)^{3/2}|t - 2i|^{-3/2}; \\
|(g'(t) + \tilde{H}(t))|^{3/2} &\geq [H_m|t - 2i|^{-1} - \|g'\|_1|t - 2i|^{-1-\tau}]^{3/2} \\
&\geq (H_m - K\|g'\|_1)^{3/2}|t - 2i|^{-3/2} \\
&\geq C(K_1 - \|g'\|_1)^{3/2}|t - 2i|^{-3/2};
\end{aligned} \tag{2.7}$$

So

$$\begin{aligned}
|G(f, g)(t)| &\leq \frac{C|t - 2i|^{-4}}{(K_1 - \|f'\|_1)^{3/2}(K_1 - \|g'\|_1)^{3/2}|t - 2i|^{-3}} \\
&\quad + \frac{C(\|f''\|_2\|g'\|_1 + \|f'\|_1\|g''\|_2)|t + 2i|^{-4-2\tau}}{(K_1 - \|f'\|_1)^{3/2}(K_1 - \|g'\|_1)^{3/2}|t - 2i|^{-3}} \\
&\quad + \frac{C(\|f''\|_2 + \|g''\|_2 + \|f'\|_1 + \|g'\|_1)|t + 2i|^{-3-\tau}}{(K_1 - \|f'\|_1)^{3/2}(K_1 - \|g'\|_1)^{3/2}|t - 2i|^{-3}} \\
&\leq |t - 2i|^{-\tau} \left\{ \frac{C(1 + \|f''\|_2\|g'\|_1 + \|f'\|_1\|g''\|_2)}{(K_1 - \|f'\|_1)^{3/2}(K_1 - \|g'\|_1)^{3/2}} \right. \\
&\quad \left. + \frac{C(\|f''\|_2 + \|g''\|_2 + \|f'\|_1 + \|g'\|_1)}{(K_1 - \|f'\|_1)^{3/2}(K_1 - \|g'\|_1)^{3/2}} \right\}
\end{aligned}$$

□

Lemma 2.5. *If $F(\xi) \in \mathbf{A}$, then $G(F, \bar{F})(t) = O(t^{-\tau})$, as $|t| \rightarrow \infty$, t in any angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$.*

Proof. By Comment 1.5, Lemma 2.2, Comment 2.2 and Lemma 2.4, the proof follows.

□

Comment 2.4: Let H_m, K, K_1 be as in (2.5) of Lemma 2.4 with $\mathcal{D} = \mathcal{R}$. then H_m, K, K_1 can be chosen to be independent of ϵ and b . If $\|F'\|_1 < \frac{K_1}{2}$ then,

$$|\xi - 2i|\|F' + H\| \geq (H_m - |\xi - 2i|^{-\tau}\|F'\|_1) \geq C(K_1 - \|F'\|_1) > C\frac{K_1}{2} > 0; \quad (2.8)$$

so condition (iv) (1.21) holds in $\mathcal{R}$.

Definition 2.6. Let $\delta > 0, 0 < \delta_1 < \frac{K_1}{2}$ be two constants, define spaces:

$$\mathbf{A}_{0,\delta} = \{f : f \in \mathbf{A}, \|f\|_0 < \delta\}; \mathbf{A}_{1,\delta_1} = \{g : g \in \mathbf{A}_1, \|g\|_1 < \delta_1\};$$

Comment 2.5 : Let $F' \in \mathbf{A}_{1,\delta_1}$. By Comment 2.4, condition (iv) holds in $\mathcal{R}$, hence $\bar{F}' + \bar{H} \neq 0$ in $\mathcal{R}^*$. $G(F, \bar{F})$ is well defined in $\mathcal{R} \cap \mathcal{R}^*$.

Definition 2.7. Let $F \in \mathbf{A}, F' \in \mathbf{A}_{1,\delta_1}$, define operator $I(F)$ so that

$$I(F)(\xi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \bar{F})(t)dt}{t - \xi} \text{ for } \text{Im } \xi < 0, \quad (2.9)$$

Lemma 2.8. Let $F \in \mathbf{A}, F' \in \mathbf{A}_{1,\delta_1}$ and F is analytic in $\text{Im } \xi > 0$ as well. If F satisfies condition (i) on the real axis. Then F satisfies:

$$\epsilon^2 F''(\xi) + L(\xi)F(\xi) = N(F, I, \bar{F})(\xi); \quad (2.10)$$

where

$$L(\xi) = -iH^{3/2}(\xi)\bar{H}^{1/2}(\xi) = -\frac{i\sqrt{\gamma^2 + \xi^2}(\xi + i\gamma)}{(\xi^2 + 1)^2} \quad (2.11)$$

and the operator N is defined as

$$\begin{aligned} N(F, I, \bar{F}) = & \epsilon^2 \left(\frac{\bar{H}'H}{\bar{H}} - H' \right) - i\epsilon^2 (F' + H)^{3/2} (\bar{F}' + \bar{H})^{1/2} I(F) \\ & + iF \left[(F' + H)^{3/2} (\bar{F}' + \bar{H})^{1/2} - H^{3/2} \bar{H}^{1/2} \right] + \epsilon^2 \left[(\bar{F}'' + \bar{H}') \frac{F' + H}{\bar{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}} \right]; \end{aligned} \quad (2.12)$$

Proof. Since $F(\xi)$ is analytic in $\mathcal{R} \cup \{\text{upper half plane}\}$ and satisfies equation (1.18), by Lemma 2.4, using Poisson formula, we have in the upper half plane:

$$\begin{aligned} F(\xi) &= \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{(t - \xi)} \frac{1}{|F'(t) + H(t)|} \text{Im} \left[\frac{F''(t) + H'(t)}{F'(t) + H(t)} \right] \\ &= -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \bar{F})(t) dt}{t - \xi}, \text{Im } \xi > 0; \end{aligned} \quad (2.13)$$

Using Plemelj Formula, analytically extend above equation to the lower half plane:

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i} G(F, \bar{F})(\xi), \text{ for } \text{Im } \xi < 0; \quad (2.14)$$

Multiplying the above by $i(F' - H)^{3/2}(\bar{F}' + \bar{H})^{1/2}$ leads to (2.10) $\square$

Definition 2.9. Let $\alpha_0 > 0$ be a fixed number independent of ϵ so that $b > \alpha_0 > 2\alpha$ (therefore $-\alpha_0 t \in \mathcal{R}$). Define two rays:

$$r_0^+ = \{\xi : \xi = \alpha_0 i + \rho e^{i(\varphi_0 + \frac{1}{2}\mu)}, 0 < \rho < \infty\};$$

$$r_0^- = \{\xi : \xi = \alpha_0 t + \rho e^{i(\pi - \varphi_0 - \frac{1}{2}\mu)}, 0 < \rho < \infty\};$$

Let r_0 be the directed contour along $r_0^- \cup r_0^+$ from left to right.

Definition 2.10. If $F \in \mathbf{A}$, define $\hat{F}(\xi)$ for ξ below r_0 :

$$\hat{F}(\xi) = -\frac{1}{2\pi i} \int_{r_0} \frac{\bar{F}(t)}{t - \xi} dt; \quad (2.15)$$

Comment 2.6: Since $r_0 \subset \mathcal{R}^*$ and $\bar{F}$ satisfies (1.22) in $\mathcal{R}^*$, the above integral is well defined. It is obvious that $\hat{F}(\xi)$ is analytic below r_0 .

Lemma 2.11. If $F \in \mathbf{A}$ and F is analytic also in $\text{Im } \xi > 0$, then $\bar{F}(\xi) = \hat{F}(\xi)$, for $\xi \in \mathcal{R}$.

Proof. Since $F(\xi)$ is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$, then $\bar{F}(\xi)$ is analytic in $\mathcal{R}^* \cup \{\text{Im } \xi < 0\}$. We use property (1.22) and the Cauchy Integral formula to obtain

$$\dot{F}(\xi) = -\frac{1}{2\pi i} \int_{r_0}^{\infty} \frac{\bar{F}(t)}{t - \xi} dt = \bar{F}(\xi), \text{ for } \xi \in \mathcal{R};$$

□

Lemma 2.12. Let $\Gamma = \{t, t = \xi_0 + \rho e^{i\varphi}\}$ be a ray. $\mathcal{D}$ is a region such that if $\xi \in \mathcal{D}$, $|\sin[\arg(\xi - \xi_0) - \varphi]| > m > 0$ for some constant m . Suppose that $\text{dist}(\mathcal{D}, \Gamma) > 0$ and $|\sin(\varphi - \tilde{\phi})| > 0$, where $\tilde{\phi} = \arg(2i - \xi_0)$. Assume $g(\xi)$ be a continuous function on Γ with $\|g\|_{0,\Gamma} < \infty$, then for $k = 0, 1, 2$,

$$\sup_{\mathcal{D}} \left| (\xi - 2i)^{k+\tau} \int_{\Gamma} \frac{g(t)}{(t - \xi)^{k+1}} dt \right| \leq C \|g\|_{0,\Gamma}; \quad (2.16)$$

where constant C depends on $\varphi - \tilde{\phi}$, m and $\text{dist}(\Gamma, \mathcal{D})$.

Proof.

$$\left| (\xi - 2i)^{k+\tau} \int_{\Gamma} \frac{g(t)}{(t - \xi)^{k+1}} dt \right| \leq \|g\|_{0,\Gamma} |\xi - 2i|^{k+\tau} \int_{\Gamma} \frac{|dt|}{|(t - \xi)^{k+1}| |t - 2i|^{\tau}}; \quad (2.17)$$

now on $\Gamma, t - \xi_0 = \rho e^{i\varphi}$ and $|dt| = d\rho$, for $\xi \in \mathcal{D}$, we define

$$\phi_1 = \arg(\xi - \xi_0) - \varphi;$$

see Figure 2.2. Also, from figure, we have

$$|t - \xi| = |\rho - (\xi - \xi_0) e^{-i\varphi}| \geq |\xi - \xi_0| \sin \varphi_1; \quad (2.18)$$

$$|t - \xi| = |\rho - (\xi - \xi_0) e^{-i\varphi}| \geq \rho \sin \varphi_1; \quad (2.19)$$

and

$$|t - 2i| \geq \rho \sin |\varphi - \tilde{\phi}|: \quad (2.20)$$

Breaking up the integral in (2.17) into two parts:

$$\int_{\Gamma} \frac{|dt|}{|(t - \xi)|^{k+1}|t - 2i|^{\tau}} = \int_0^{|\xi - \xi_0|} \frac{d\rho}{|(t - \xi)|^{k+1}|t - 2i|^{\tau}} + \int_{|\xi - \xi_0|}^{\infty} \frac{d\rho}{|(t - \xi)|^{k+1}|t - 2i|^{\tau}}: \quad (2.21)$$

for the first integral in (2.21), use (2.18) and (2.20) and for the second use (2.19) and (2.20) to obtain: (on scaling ρ by $|\xi - \xi_0|$)

$$\int_{\Gamma} \frac{|dt|}{|(t - \xi)|^{k+1}|t - 2i|^{\tau}} = \frac{C}{|\xi - \xi_0|^{k+\tau}} \left(\int_0^1 \frac{d\rho}{\rho^{\tau}} + \int_1^{\infty} \frac{d\rho}{\rho^{k+\tau+1}} \right);$$

□

Comment 2.7: Note if Γ is in $\mathcal{D}'$, an angular subset of $\mathcal{D}_c$ (complement of $\mathcal{D}$), then condition $|\sin[\arg(\xi - \xi_0) - \epsilon\varphi]| > m > 0$ is met.

Definition 2.13.

$$\Omega_0 = \{\xi : \xi \text{ is below the line } \{\xi = \frac{1}{2}\alpha_0 i + \rho e^{i(\pi - \varphi_0 - \frac{1}{3}\mu)}\} \text{ and line } \{\xi = \frac{1}{2}\alpha_0 i + \rho e^{i(\varphi_0 + \frac{1}{3}\mu)}\}\}. \quad (2.22)$$

Comment 2.8: $\mathcal{R} \cup \{\text{Im } \xi < 0\}$ is an angular subset of Ω_0 , and Ω_0 is an angular subset of region $\{\xi \text{ below } r_0\}$.

Lemma 2.14. *Let $F(\xi) \in \mathbf{A}$, then*

$$\sup_{\xi \in \Omega_0} |(\xi - 2i)^{k+\tau} \hat{F}^{(k)}(\xi)| \leq K_2 \|F\|_0; k = 0, 1, 2.$$

where constant K_2 depends only on φ_0 and a_0 .

Proof. Using $\|\hat{F}\|_{0,r_0} \leq \|F\|_0$, (2.15) and applying Lemma 2.12 with $\mathcal{D} = \Omega_0$. $\square$

Comment 2.9: From now on we choose $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$ with

$$\delta < \frac{K_3}{2} \equiv \frac{H_m}{2KK_2}, \delta_1 < \frac{K_1}{2}; \quad (2.23)$$

where H_m, K as in Comment 2.4, and K_2 as in Lemma 2.14, then

$$|\hat{F}' + \hat{H}| \geq C(H_m - K\|\hat{F}'\|_1) \geq C(H_m - KK_2\|F\|_0) \geq C\frac{K_3}{2} > 0; \quad (2.24)$$

so $\hat{F}' + \hat{H} \neq 0$ in $\mathcal{R}$, therefore $G(F, \hat{F})$ is analytic in $\mathcal{R}$.

Comment 2.10: Using Lemmas 2.4 and 2.14, $G(F, \hat{F})(t) = O(t^{-\tau})$ as $|t| \rightarrow \infty$, t in any angular subset of $\mathcal{R} \cap \Omega_0$ containing real axis.

Definition 2.15. If $F(\xi) \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, define operator $I_1(F)$ so that for $\text{Im } \xi < 0$:

$$I_1(F)(\xi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \hat{F})(t)}{t - \xi} dt. \quad (2.25)$$

Define

$$N_1(\xi) \equiv N(F, I_1(F), \hat{F})(\xi) = \epsilon^2 \left(\frac{\bar{H}'H}{\bar{H}} - H' \right) - i\epsilon^2 (F' + H)^{3/2} (\hat{F}' + \bar{H})^{1/2} I_1(F) \\ - i(F' + H)^{3/2} (\hat{F}' + \bar{H})^{1/2} - H^{3/2} \bar{H}^{1/2} \Big] + \epsilon^2 \left[(\hat{F}'' + \bar{H}') \frac{F' + H}{\hat{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}} \right]; \quad (2.26)$$

Lemma 2.16. *If $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$ and $F(\xi)$ is analytic in $\text{Im } \xi > 0$ and also satisfies (1.18), then $F(\xi)$ satisfies*

$$\epsilon^2 F''(\xi) + L(\xi) F(\xi) = N_1(\xi); \quad (2.27)$$

Proof. Since conditions of Lemma 2.8 hold, (2.10) is valid. Also since F is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$ with property (1.22), from Lemma 2.11, $\hat{F} = \bar{F}$, hence $I_1(F)(\xi) = I(\bar{F})(\xi)$, therefore (2.10) implies (2.27). $\square$

Lemma 2.17. *If $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, and $F(\xi)$ satisfies equation (2.27) in $\mathcal{R} \cap \{\text{Im } \xi < 0\}$, then $F(\xi)$ is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$ and satisfies (1.18) on real ξ axis.*

Proof. Note that (2.27) can be rewritten as:

$$F(\xi) = \epsilon^2 I_1(F)(\xi) + \frac{\epsilon^2}{i} G(F, \hat{F})(\xi), \text{ for } \text{Im } \xi < 0; \quad (2.28)$$

Analytically continuing the above equation in the upper half plane, we have:

$$F(\xi) = -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \hat{F}(t)) dt}{t - \xi}, \text{ for } \text{Im } \xi > 0; \quad (2.29)$$

so, $F(\xi)$ is analytic in the upper half plane. Now from Lemma 2.11, $\hat{F}(\xi) = F^*(\xi)$ for real ξ , then equation (2.29) reduces to:

$$F(\xi) = \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - \xi} \frac{1}{|F'(t) + H(t)|^{1/2}} \text{Im} \left[\frac{F''(t) + H'(t)}{F'(t) + H(t)} \right]$$

By the analyticity of $F(\xi)$ and Plemelj formula, the above equation is equivalent to (1.18). $\square$

By Lemma 2.16 and Lemma 2.17, we have proved:

Theorem 2.18. *The finger problem is equivalent to*

Problem 1: *Find function $F(\xi) \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$ satisfying condition (iii), so that equation (2.27) is satisfied in $\mathcal{R} \cap \{\operatorname{Im} \xi < 0\}$.*

2.2 Formulation of Problem 2

Let $\alpha_1 > 0$ be a fixed constant independent of ϵ so that $-\alpha_1 i \in \mathcal{R}$. Define two rays (see Figure 2.3):

Definition 2.19.

$$r_1^+ = \{\xi : \xi = -\alpha_1 i + \rho e^{-i\varphi_0}, 0 < \rho < \infty\};$$

$$r_1^- = \{\xi : \xi = -\alpha_1 i + \rho e^{i(\pi+\varphi_0)}, 0 < \rho < \infty\};$$

$r_1 = r_1^- \cup r_1^+$ is a directed contour from left to right.

Figure 2.3: rays defined in Chapter 2.

Definition 2.20. Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$. Let $\hat{F}(\xi)$ be given by (2.15). For ξ above r_1 , define

$$F_+(\xi) = -\frac{\epsilon^2}{2\pi} \int_{r_1} \frac{G(F, \hat{F})(t)dt}{t - \xi}. \quad (2.30)$$

Comment 2.11: Since $G(F, \hat{F})(t) \sim O(t^{-\tau})$ as $|t| \rightarrow \infty$, t in any angular subset of $\mathcal{R} \cap \Omega_0$ containing r_1 , so F_+ in above definition is well defined and is analytic for ξ above r_1 .

Definition 2.21.

$$\Omega_1 = \left\{ \xi : \xi \text{ above } \left\{ \xi = -\frac{\alpha_1}{2}i + \rho e^{i(\pi + \frac{\epsilon\mu}{2})} \right\} \cup \left\{ \xi = -\frac{\alpha_1}{2}i + \rho e^{-i(\pi + \frac{\epsilon\mu}{2})} \right\} \right\}; \quad (2.31)$$

Comment 2.12: Ω_1 is an angular subset of the region $\{\xi : \xi \text{ above } r_1\}$.

Lemma 2.22. Let $F(\xi) \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, then

$$\sup_{\xi \in \Omega_1} \left| (\xi - 2i)^{k+\tau} F_+^{(k)}(\xi) \right| \leq \epsilon^2 \mathcal{G}_1(\delta, \delta_1); k = 0, 1, 2.$$

where

$$\mathcal{G}_1(\delta, \delta_1) = \frac{C(1 + \delta + \delta^2)}{(K_3 - \delta)^{3/2}(K_1 - \delta_1)^{3/2}};$$

C is constant independent of b and ϵ ; K_1 as in Comment 2.4, K_3 as in Comment 2.9.

Proof. Using Cauchy integral formula, we have:

$$F^{(k)}(t) = \frac{k!}{2\pi i} \int_{\partial\mathcal{R}} \frac{F(s)ds}{(s - t)^{k+1}} \quad (2.32)$$

Applying lemma 2.12, $\|F^{(k)}\|_{k,r_1} \leq C\|F\|_0$, where C depends on α_1, φ_0 ; Now using Lemma 2.4 and $\|\hat{F}^{(k)}\|_k \leq K_2\|F\|_0$ (Lemma 2.14), we have:

$$\|G(F, \hat{F})\|_{0,r_1} \leq \frac{C(1 + \|F\|_0 + \|F\|_0^2)}{(K_3 - \|F\|_0)^{3/2}(K_1 - \|F'\|_1)^{3/2}}$$

where C depends only on α_1, φ_0 and α_0 . Applying Lemma 2.12 again to (2.30) .we complete the proof. $\square$

Comment 2.13: If $\delta < \frac{K_4}{2}, \delta_1 < \frac{K_1}{2}$, then $\mathcal{G}_1(\delta, \delta_1) \leq K_4$ where K_4 is independent of ϵ and b .

Definition 2.23. Let α_2 be a constant independent of ϵ so that $\alpha < \alpha_2 < \frac{1}{2}\alpha_0$, define two rays (see Figure 2.3):

$$r_2^+ = \{\xi : \xi = \alpha_2 t + \rho e^{i\varphi_0 + \frac{1}{4}\mu}, 0 < \rho < \infty\};$$

$$r_2^- = \{\xi : \xi = \alpha_2 t + \rho e^{i(\pi - \varphi_0 - \frac{1}{4}\mu)}, 0 < \rho < \infty\};$$

r_2 is a directed contour from left to right on the path $r_2^- \cup r_2^+$.

Comment 2.14: $r_2 \subset \Omega_0 \cap \Omega_1$ and $\mathcal{R}$ is below r_2 .

Definition 2.24. If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, let $\hat{F}$ be given by (2.15). Define $I_2(F)$ so that

$$I_2(F)(\xi) = -\frac{1}{2\pi} \int_{r_2} \frac{G(F_+, \hat{F})(t) dt}{t - \xi}, \text{ for } \xi \text{ below } r_2; \quad (2.33)$$

Comment 2.15: Because of Lemma 2.4, Lemma 2.14 and Lemma 2.22, $G(F_+, \hat{F})(t) \sim O(t^{-\tau})$ as $|t| \rightarrow \infty$ and analytic for small ϵ for t in any angular subset of $\Omega_0 \cap \Omega_1$ containing r_2 , so $I_2(F)(\xi)$ is well defined and is analytic below r_2 .

Lemma 2.25. Let $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, then $I_2(F)(\xi) \in \mathbf{A}$, and $\|I_2(F)\|_0 \leq \mathcal{G}_2(\delta, \delta_1)$; where

$$\mathcal{G}_2(\delta, \delta_1) = \frac{C(1 + \delta + \epsilon^2 \mathcal{G}_1(\delta, \delta_1) + (\delta + \epsilon^2 \mathcal{G}_1(\delta, \delta_1))^2)}{(K_3 - \delta)^{3/2}(K_1 - \epsilon^2 \mathcal{G}_1(\delta, \delta_1))^{3/2}};$$

C is constant independent of b and ϵ ; K_1 as in Comment 2.4, K_3 as in Comment 2.9

Proof. By Lemma 2.14 and Lemma 2.22, we have:

$$\|\hat{F}^{(k)}\|_{k,r_2} \leq K_2 \|F\|_0, \|F_-^{(k)}\|_{k,r_2} \leq \epsilon^2 \mathcal{G}_1(\delta, \delta_1):$$

Apply Lemma 2.4 and Lemma 2.12 to (2.33) to get the Lemma. $\square$

Comment 2.16: If $\delta < \frac{K_3}{2}, \delta_1 < \frac{K_1}{2}, 0 < \epsilon < \sqrt{\frac{K_1}{K_3}}$, then $\mathcal{G}_2(\delta, \delta_1) \leq K_5$ where K_5 is independent of ϵ and $b : K_4$ as in Comment 2.13.

Definition 2.26.

$$\begin{aligned} N_2(\xi) = N(F, I_2(F), \hat{F})(\xi) &= \epsilon^2 \left[\frac{\bar{H}'H}{\bar{H}} - H' \right] - i\epsilon^2 (F' + H)^{3/2} (\hat{F}' + \bar{H})^{1/2} I_2(F) \\ &+ i F \left[(F' + H)^{3/2} (\hat{F}' + \bar{H})^{1/2} - H^{3/2} \bar{H}^{1/2} \right] + \epsilon^2 \left[(\hat{F}'' + \bar{H}') \frac{F' + H}{\hat{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}} \right]; \end{aligned} \quad (2.34)$$

Lemma 2.27. If $F(\xi) \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, F satisfies (2.27) in $\mathcal{R} \cap \{\text{Im } \xi < 0\}$, then $F(\xi)$ satisfies

$$\epsilon^2 F''(\xi) + L(\xi) F(\xi) = N_2(\xi); \quad (2.35)$$

Proof. If δ is small enough, from Lemma 2.14, $|(\xi - 2i)^{1+\tau} \hat{F}'|$ is small in domain Ω_0 which contains the region between r_2 and the real axis, hence $\hat{F}' + \bar{H} \neq 0$ in that domain. Also from Lemma 2.22, $\sup_{\Omega_1} |(\xi - 2i)^{1+\tau} F'_-|$ is small for small ϵ , but Ω_1 contains the region between r_1 and r_2 , hence $F'_- + H \neq 0$. By Lemma 2.17, $F(\xi)$ is analytic in the upper plane, deforming contour r_1 in (2.30) back to the real axis (it can be done because $G(F, \hat{F})(t) \sim O(t^{-\tau})$), it follows from (2.29) that F_+ and F have the same expression in the upper half plane. So we have $F(\xi) = F_+(\xi)$, for $\text{Im } \xi > 0$. Now by deforming the contour r_2 in (2.33) back to real axis, it follows $I_2(F)(\xi) = I_1(F)(\xi)$, so $N_2(\xi) = N(F, I_2(F), \hat{F})(\xi) = N(F, I_1(F), \hat{F})(\xi) = N_1(\xi)$. $\square$

Definition 2.28.

$$g_1(\xi) = L^{-1/4}(\xi) \exp\left\{-\frac{P(\xi)}{\epsilon}\right\}. \quad (2.36)$$

$$g_2(\xi) = L^{-1/4}(\xi) \exp\left\{\frac{P(\xi)}{\epsilon}\right\}. \quad (2.37)$$

where

$$P(\xi) = i \int_{-\gamma}^{\xi} L^{1/2}(t) dt = i \int_{-\gamma}^{\xi} \frac{(\gamma - it)^{3/4} (\gamma + it)^{1/4}}{(1 + t^2)} dt; \quad (2.38)$$

Many important properties of the function $P(\xi)$ are established in Appendix A.

$g_1(\xi), g_2(\xi)$ are the two WKB solutions of the corresponding homogeneous equation of (2.35), they satisfy the following equation exactly:

$$\epsilon^2 g''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi))g(\xi) = 0; \quad (2.39)$$

where

$$L_1(\xi) = \frac{L''(\xi)}{4L(\xi)} - \frac{5L'^2}{16L^2} \quad (2.40)$$

The Wronskian of g_1 and g_2 is

$$W(\xi) = g_1(\xi)g_2'(\xi) - g_2(\xi)g_1'(\xi) = \frac{2i}{\epsilon}. \quad (2.41)$$

Comment 2.17: By (2.40) and (2.11), the worst singularities of $L_1(\xi)$ at $\xi = -i$ and $\xi = -\gamma$ are poles of second order.

Comment 2.18: Note that equation (2.35) implies

$$\mathcal{V}F(\xi) \equiv \epsilon^2 F''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi))F(\xi) = \tilde{N}_2(\xi); \quad (2.42)$$

where

$$\tilde{N}_2(\xi) = N_2(\xi) + \epsilon^2 L_1(\xi)F(\xi); \quad (2.43)$$

Lemma 2.29. *If $F \in \mathbf{A}$, $F' \in \mathbf{A}_1$, then for $\xi \in \mathcal{R}$.*

$$\left| \frac{\hat{F}'}{\bar{H}} \right| \leq K_6 \|F\|_0; \quad (2.44)$$

$$\left| \frac{F'}{\bar{H}} \right| \leq K_7 \|F'\|_1 \quad (2.45)$$

where K_6, K_7 are constants independent of ϵ and b .

Proof. By (2.5):

$$\begin{aligned} \left| \frac{\hat{F}'}{\bar{H}} \right| &\leq \frac{|\xi - 2i|^{-1-\tau} \|\hat{F}'\|_1}{|\xi - 2i|^{-1} H_m} \leq \frac{K K_2}{H_m} \|F\|_0 \equiv K_6 \|F\|_0; \\ \left| \frac{F'}{\bar{H}} \right| &\leq \frac{|\xi - 2i|^{-1-\tau} \|F'\|_1}{|\xi - 2i|^{-1} H_m} \leq \frac{K}{H_m} \|F'\|_1 \equiv K_7 \|F'\|_1; \end{aligned}$$

□

Lemma 2.30. *Let*

$$G_1(F, \hat{F})(t) = (F'(t) + H(t))^{3/2} (\hat{F}'(t) + \bar{H}(t))^{1/2}; \quad (2.46)$$

If $F \in \mathbf{A}$, $F' \in \mathbf{A}_1$, then in $\mathcal{R}$.

$$|G_1(F, \hat{F})| \leq C \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2} (K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2}; \quad (2.47)$$

where C is independent of ϵ, b and $\gamma; K_6, K_7$ as in Lemma 2.29.

Proof. From (1.19), (2.1):

$$C_1 |\xi - 2i|^{-1} \leq |H| \leq C_2 \frac{|b - \gamma|}{|b - 1|} |\xi - 2i|^{-1}; \quad (2.48)$$

$$C_1 |\xi - 2i|^{-1} \leq |\bar{H}| \leq C_2 |b - 1|^{-1} |\xi - 2i|^{-1}; \quad (2.49)$$

where C_1, C_2 can be chosen independent of independent of ϵ, b and γ . so by Lemma 2.29:

$$\begin{aligned} |G_1(F, \hat{F})| &= |H^{3/2} \bar{H}^{1/2}| \left| \frac{F'}{H} + 1 \right|^{3/2} \left| \frac{\hat{F}'}{\bar{H}} + 1 \right|^{1/2} \\ &\leq C \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2} (K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2}; \end{aligned}$$

□

Lemma 2.31. *Let*

$$G_2(F, \hat{F})(\xi) = \left[(\hat{F}'' + \hat{H}') \frac{F' + H}{\hat{F}' + \bar{H}} - \frac{\hat{H}' H}{\bar{H}} \right] (\xi); \quad (2.50)$$

If $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_1$, then in $\mathcal{R}$

$$\begin{aligned} &|G_2(F, F)| \\ &\leq \frac{C}{(1 - b)(K_3 - \|F\|_0)} |\xi - 2i|^{-2-\tau} \left\{ \|F'\|_1 + |b - \gamma| \|F\|_0 + (1 - b) \|F\|_0 (\|F'\|_1 + 1) \right\}; \end{aligned} \quad (2.51)$$

where C is independent of ϵ, b and γ ; K_3 is as in Comment 2. 13.

Proof. Note By (1.19), (2.1):

$$\bar{H}' = -\frac{(\xi - i\gamma)^2 + (\gamma^2 - 1)}{(\xi^2 + 1)^2}; \quad (2.52)$$

$$\frac{\bar{H}' H}{\bar{H}} = -\frac{[(\xi - i\gamma)^2 + (\gamma^2 - 1)](\xi + i\gamma)}{(\xi^2 + 1)^2(\xi - i\gamma)}; \quad (2.53)$$

$$\begin{aligned} |\hat{F}' + \hat{H}| &\geq |\bar{H}| \left| \frac{\hat{F}'}{\bar{H}} + 1 \right| \geq |\bar{H}| \left(1 - \frac{\|\hat{F}'\|_1 |\xi - 2i|^{-1-\tau}}{H_m |\xi - 2i|^{-1}} \right) \\ &\geq C |\bar{H}| (K_3 - \|F\|_0); \end{aligned} \quad (2.54)$$

$$\begin{aligned}
|G_2(F, \hat{F})| &= \left| F' \frac{\bar{H}'}{\hat{F}' + \bar{H}} - \frac{\bar{H}' H}{\bar{H}} \frac{\hat{F}'}{\hat{F}' + \bar{H}} + \hat{F}'' \frac{F' + H}{\hat{F}' + \bar{H}} \right| \\
&\leq \frac{C \|F'\|_1 |\xi - 2i|^{-2-\tau}}{|b-1| (K_3 - \|F\|_0)} + \frac{C \|F\|_0 |b-\gamma| |\xi - 2i|^{-2-\tau}}{|b-1| (K_3 - \|F\|_0)} \\
&\quad + \frac{C \|F\|_0 (\|F'\|_1 + 1) |\xi - 2i|^{-2-\tau}}{(K_3 - \|F\|_0)}
\end{aligned}$$

□

Lemma 2.32. *Let*

$$G_3(F, \hat{F}) = (F' + H)^{3/2} (\hat{F}' + \bar{H})^{1/2} - H^{3/2} \bar{H}^{1/2}; \quad (2.55)$$

Assume that $F \in \mathbf{A}$, $F' \in \mathbf{A}_1$, then for $\xi \in \mathcal{R}$

$$|G_3(F, \hat{F})(\xi)| \leq C \frac{(\gamma - b)^{3/2}}{(1 - b)^2} |\xi - 2i|^{-2} \left\{ (K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2} - 1 \right\}; \quad (2.56)$$

where C is independent of ϵ and b, K_6, K_7 are as in Lemma 2.29.

Proof. Using (2.48)(2.49) and Lemma 2.29:

$$\begin{aligned}
|G_3(F, \hat{F})| &\leq |H^{3/2} \bar{H}^{1/2}| \left| \left(\frac{F'}{H} + 1 \right)^{3/2} \left(\frac{\hat{F}'}{\bar{H}} + 1 \right)^{1/2} - 1 \right| \\
&\leq C \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2} \left\{ \left(\frac{|F'|}{|H|} + 1 \right)^{3/2} \left(\frac{|\hat{F}'|}{|\bar{H}|} + 1 \right)^{1/2} - 1 \right\} \\
&\leq C \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2} \left\{ (K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2} - 1 \right\};
\end{aligned}$$

□

Lemma 2.33. *Let $F(\xi) \in \mathbf{A}_\delta$, $F'(\xi) \in \mathbf{A}_{1,\delta_1}$, then $\bar{N}_2(\xi) \in \mathbf{A}_2$*

Proof. Note that

$$\bar{N}_2 = \epsilon^2 \left(\frac{\bar{H}'H}{\bar{H}} - H' \right) - i\epsilon^2 G_1(F, \hat{F}) I_2(F) + iFG_3(F, \hat{F}) + \epsilon^2 G_2(F, \hat{F}) + \epsilon^2 FL_1; \quad (2.57)$$

$$\left(\frac{\bar{H}'H}{\bar{H}} - H' \right) = -\frac{2i\gamma}{(\xi^2 + 1)(\xi - i\gamma)}; \quad (2.58)$$

Apply Lemma 2.25 , Lemma 2.30 and Lemma 2.32 to get

$$|I_2(F)G_1(F, \hat{F})| \leq \mathcal{G}_2(\delta, \delta_1) \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2-\tau} (K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2};$$

$$|FG_3(F, \hat{F})| \leq C \|F\|_0 \frac{|b - \gamma|^{3/2}}{|b - 1|^2} |\xi - 2i|^{-2-\tau} \left[(K_7 \|F'\|_1 + 1)^{3/2} (K_6 \|F\|_0 + 1)^{1/2} - 1 \right];$$

From the expression of $L_1(\xi)$, we have:

$$|F| |L_1(\xi)| \leq C |b - 1|^{-2} |\xi - 2i|^{-2-\tau} \|F\|_0; \quad (2.59)$$

$$\epsilon^2 \left| \frac{\bar{H}'H}{\bar{H}} - H' \right| \leq C \epsilon^2 |1 - b|^{-1} |\gamma - b|^{-1} |\xi - 2i|^{-3};$$

By (2.57), we have the lemma. $\square$

Lemma 2.34. *Let $N(\xi) \in \mathbf{A}_2^-$, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt \in \mathbf{A}^-, \text{ and } \|f_1(\xi)\|_0^- \leq C \|N(\xi)\|_2^-;$$

where C is a constant independent of ϵ and b .

Proof. **Case 1:** $|\xi| \geq R$, where R is large enough so that Lemma 15 in Appendix A

holds. On path $\mathcal{P}(-\infty, \xi) = \{t : t = \xi - s, 0 < s < \infty\}$, $\operatorname{Re}(P(\xi) - P(t))$ goes from 0 to $-\infty$ monotonically as $s \rightarrow \infty$.

$$\begin{aligned} |f_1(\xi)| &= \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(-\infty, \xi)} N(t) L^{-1/4}(t) \exp\left\{\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\ &\leq \|N\|_2^- |L^{-1/4}(\xi)| \\ &\quad \times \int_0^1 \frac{|(t(s) - 2i)^{-2-\tau}| |L^{-1/4}(t(s))|}{\left| \frac{d}{ds} \operatorname{Re} P(t(s)) \right|} d \left[\exp\left\{\frac{1}{\epsilon}(\operatorname{Re} P(\xi) - \operatorname{Re} P(t(s)))\right\} \right] \end{aligned}$$

Note that $|L^{-1/4}(\xi)| \leq C|\xi - 2i|^{1/2}$, $|\xi - 2i| \leq |t - 2i|$:

$$|\operatorname{Re}(P'(t)t'(s))| \geq C|L^{1/2}(t)| \geq C|t - 2i|^{-1};$$

So

$$|f_1(\xi)| \leq C\|N\|_2^- |\xi - 2i|^{-\tau}$$

Case 2: for $\xi \in \mathcal{R}_- \cap \{|\xi| \leq R\}$, by Lemma 22 in Appendix A, there exists a path $\mathcal{P}(\xi, -\infty)$ on which $\operatorname{Re} P(t)$ increases as t goes from ξ to $-\infty$, so $\operatorname{Re} P(\xi) - P(t)$ decreases along path $\mathcal{P}(\xi, -\infty)$ as t goes from ξ to $-\infty$. Note that if $|\xi| \leq R$, then $|L^{-1/4}(\xi)| \leq C$, $|\operatorname{Re} P'(t(s))t'(s)| \geq C$, where constant C independent of ϵ and b . Using the same steps as in Case 1, we can estimate $|f_1(\xi)|$. $\square$

Lemma 2.35. *Let $N(\xi) \in \mathbf{A}_2^-$, then there exists $\epsilon_0 > 0$, for $0 < \epsilon < \epsilon_0$, we have*

$$f_2(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{a_1}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt \in \mathbf{A}^-, \text{ and } \|f_2(\xi)\|_0^- \leq C \|N(\xi)\|_2^-.$$

where C is independent of ϵ and b .

Proof. Case 1: For the case $|\xi| \leq R^2$, by Lemma 23 in Appendix A, there is path $\mathcal{P}(\xi, ai)$ on which $\operatorname{Re} P(t)$ is increasing as t goes from ai to ξ .

$$\begin{aligned} |f_2(\xi)| &= \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(\xi, ai)} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\ &\leq C \|N\|_2 |L^{-1/4}(\xi)| \\ &\quad \times \int_{\mathcal{P}(\xi, ai)} |(t - 2i)^{-2-\tau}| |L^{-1/4}(t)| \frac{|d[\exp\{-\frac{1}{\epsilon}(ReP(\xi) - ReP(t))\}]|}{|\frac{d}{ds}P(t(s))|} \end{aligned}$$

when $|\xi| \leq R^2$, $t \in \mathcal{P}(\xi, ai)$, we have:

$$|L^{-1/4}(\xi)| \leq C; \left| \frac{d}{ds} ReP(t(s)) \right| \geq C;$$

where C is independent of b and ϵ . So

$$\begin{aligned} |f_2(\xi)| &\leq C \|N\|_2 \int_{\mathcal{P}(\xi, ai)} \left| d[\exp\{-\frac{1}{\epsilon} Re(P(\xi) - P(t))\}] \right| \\ &\leq C \|N\|_2 \left| \left[1 - \exp\{-\frac{1}{\epsilon} Re(P(\xi) - P(ai))\} \right] \right| \leq C \|N\|_2; \end{aligned}$$

where we used that $\operatorname{Re}(P(\xi) - P(ai)) \geq 0$ (see Lemma 21 in Appendix A).

Case 2: For the case where $|\xi| \geq R^2$, we choose paths

$$\mathcal{P}_1 = \{t : \operatorname{Im} t = \operatorname{Im} \xi, |\operatorname{Re} \xi| \geq |\operatorname{Re} t| \geq \sqrt{|\operatorname{Re} \xi|}\},$$

$$\mathcal{P}_2 = \{t : \operatorname{Im} t = \operatorname{Im} \xi, \sqrt{|\operatorname{Re} \xi|} \geq |\operatorname{Re} t| \geq |\operatorname{Re} \xi^0|\},$$

$$\mathcal{P}_3 = \mathcal{P}(\xi^0, ai);$$

where ξ^0 is the intersection point of line $\{t = \operatorname{Im} \xi\}$ and circle $|t| = R$, and $\mathcal{P}(\xi^0, ai)$ is

chosen as in Case 1 since $|\xi^0| \leq R$. By Lemma 22 in Appendix A, $\operatorname{Re} P(t)$ decreases along $\mathcal{P}_1 \cup \mathcal{P}_2$, we rewrite:

$$\begin{aligned} f_2(\xi) &= \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_1} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \\ &\quad + \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_2} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \\ &\quad + \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(\xi^0, \text{out})} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt; \end{aligned} \quad (2.60)$$

We choose R large enough so that for $t \in \mathcal{P}_1 \cup \mathcal{P}_2$,

$$C_3|t| \leq |\gamma \pm it| \leq C_2|t|, C_3|t| \leq |1 \pm it| \leq C_2|t|;$$

where C_3, C_2 depend only on R . Using (A.1) in Appendix A, we have

$$\operatorname{Re}(P(t) - P(\xi)) \leq C_1 \int_{\operatorname{Re}\xi}^{\operatorname{Re}t} \frac{1}{r} dr \leq C_1 \ln\left(\frac{\operatorname{Re}t}{\operatorname{Re}\xi}\right); \quad (2.61)$$

To estimate the first integral in (2.60):

$$\begin{aligned} &\left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_1} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\ &\leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{\sqrt{|\operatorname{Re}\xi|}}^{|\operatorname{Re}\xi|} |(t - 2i)^{-2-\tau}| |L^{-1/4}(t)| \exp\left\{-\frac{1}{\epsilon}(\operatorname{Re}P(\xi) - \operatorname{Re}P(t))\right\} dt \\ &\leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{\sqrt{|\operatorname{Re}\xi|}}^{|\operatorname{Re}\xi|} t^{-\frac{1}{2}-\tau} \exp\left\{-\frac{1}{\epsilon}(\operatorname{Re}P(\xi) - \operatorname{Re}P(t))\right\} dt \\ &\leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{\sqrt{|\operatorname{Re}\xi|}}^{|\operatorname{Re}\xi|} t^{-\frac{1}{2}-\tau} \exp\left\{\frac{C_1}{\epsilon} \left(\ln\left(\frac{t}{|\operatorname{Re}\xi|}\right)\right)\right\} dt \\ &\leq C \|N\|_2 |\xi - 2i|^{-\tau}; \end{aligned}$$

To estimate the second integral in (2.60):

$$\begin{aligned}
& \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{P_2} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\
& \leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{|Re\xi^{0_1}|}^{\sqrt{|Re\xi|}} |(t - 2i)^{-2-\tau}| L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(ReP(\xi) - ReP(t))\right\} dt \\
& \leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{|Re\xi^{0_1}|}^{\sqrt{|Re\xi|}} \exp\left\{-\frac{1}{\epsilon}(ReP(\xi) - ReP(t))\right\} dt \\
& \leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{|Re\xi^{0_1}|}^{\sqrt{|Re\xi|}} \exp\left\{\frac{C_1}{\epsilon} \left(\ln\left(\frac{t}{|Re\xi|}\right)\right) dt \right\} \\
& \leq C \|N\|_2 |L^{-1/4}(\xi)| |Re\xi|^{-\frac{C_1}{2\epsilon} - \frac{1}{2}} \leq C \|N\|_2 |\xi|^{-\tau};
\end{aligned}$$

The third integral in (2.60) can be estimated as in case 1. $\square$

Lemma 2.36. *Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, $F_{\pm}$ defined by (2.36), we define:*

$$\mathcal{J} = \frac{1}{\epsilon^2} \int_{ai}^t \frac{\epsilon^2 F_{\pm}''(t) + (L(t) + \epsilon^2 L_1(t)) F_{\pm}(t)}{W(t)} g_2(t) dt; \quad (2.62)$$

then

$$|\mathcal{J}| \leq C \epsilon^2 \exp\{ReP(ai)/\epsilon\} \mathcal{G}_1(\delta, \delta_1); \quad (2.63)$$

Proof.

$$\begin{aligned}
|\mathcal{J}| & \leq \frac{2}{\epsilon} \int_a^1 \left| \left(\epsilon^2 F_{\pm}''(it) + (L(it) + \epsilon^2 L_1(it)) F_{\pm}(it) \right) L^{-1/4}(it) \exp\left\{\frac{P(it)}{\epsilon}\right\} \right| dt \\
& \leq \frac{C}{\epsilon} \left\{ \epsilon^2 \sup_t |(it - 2i)^{2+\tau} F_{\pm}''(it)| \int_a^1 |L^{-1/4}(it)| \left| \exp\left\{\frac{P(it)}{\epsilon}\right\} \right| dt \right. \\
& \quad \left. + \sup_t |(it - 2i)^{\tau} F_{\pm}(it)| \int_a^1 (|L(it)| + \epsilon^2 |L_1(it)|) |L^{-1/4}(it)| \left| \exp\left\{\frac{P(it)}{\epsilon}\right\} \right| dt \right\}
\end{aligned}$$

Note $C_3|\gamma - t|^{1/2}|t - 1|^{-2} \leq |L(it)| \leq C_2|t - 1|^{-2}$, $C_3|t - 1|^{-2} \leq |L_1(it)| \leq C_2|t - 1|^{-2}$.

using Lemma 2.22:

$$\begin{aligned} |\mathcal{J}| &\leq \frac{C}{\epsilon} \left\{ \epsilon^4 \mathcal{G}_1(\delta, \delta_1) \int_a^1 (\gamma - t)^{-1/8} (1 - t)^{1/2} \left| \exp\left\{ \frac{P(it)}{\epsilon} \right\} \right| dt \right. \\ &\quad \left. + \epsilon^2 \mathcal{G}_1(\delta, \delta_1) \int_a^1 (1 - t)^{-3/2} \left| \exp\left\{ \frac{P(it)}{\epsilon} \right\} \right| dt \right\} \\ &\leq \frac{C}{\epsilon} \exp\left\{ \frac{\operatorname{Re} P(ai)}{\epsilon} \right\} \left\{ \epsilon^4 \mathcal{G}_1(\delta, \delta_1) \int_a^1 (1 - t)^{3/8} \left| \exp\left\{ \frac{\operatorname{Re}(P(it) - P(ai))}{\epsilon} \right\} \right| dt \right. \\ &\quad \left. + \epsilon^2 \mathcal{G}_1(\delta, \delta_1) \int_a^1 (1 - t)^{-3/2} \left| \exp\left\{ \frac{\operatorname{Re}(P(it) - P(ai))}{\epsilon} \right\} \right| dt \right\} \end{aligned}$$

By equation (A.1) in Appendix A, for $a < t < 1$:

$$\begin{aligned} \operatorname{Re}(P(it) - P(ai)) &= - \int_a^t \frac{|\gamma|^{3/4} |\gamma - t|^{1/4}}{(1 - t)(1 - t)} dt \\ &\leq -C_1 \int_a^t \frac{1}{(1 - t)} dt \leq C_1 \ln \left(\frac{(1 - t)}{(1 - a)} \right); \end{aligned}$$

where C_1 is a positive constant independent of ϵ . So

$$\begin{aligned} &\frac{1}{\epsilon} \int_a^1 (1 - t)^{-3/2} \left| \exp\left\{ \frac{\operatorname{Re}(P(it) - P(ai))}{\epsilon} \right\} \right| dt \\ &\leq \frac{1}{\epsilon} \int_a^1 (1 - t)^{-3/2} \left(\frac{1 - t}{1 - a} \right)^{\frac{C_1}{\epsilon}} dt \leq (1 - a)^{-\frac{C_1}{\epsilon}} \left(\frac{1}{\epsilon} \int_a^1 (1 - t)^{\frac{C_1}{\epsilon} - \frac{1}{2}} dt \right) \leq C; \end{aligned}$$

so we have

$$|\mathcal{J}| \leq C\epsilon^2 \exp\{\operatorname{Re} P(ai)/\epsilon\} \mathcal{G}(\delta, \delta_1);$$

□

Lemma 2.37. *Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, $F_+(\xi)$ defined by (2.36). Let*

$$f(\xi) = \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{\epsilon^2 F_-''(t) + (L(t) + \epsilon^2 L_1(t)) F_-(t)}{W(t)} g_1(t) dt$$

Then $\sup_{\xi \in \{it, 0 < t < 1\}} |f(\xi)| \leq C\epsilon^2 \mathcal{G}_1(\delta, \delta_1)$.

Proof.

$$\begin{aligned} f(\xi) &= \frac{1}{\epsilon^2} g_2(\xi) \int_0^\xi \frac{\epsilon^2 F_+''(t) + (L(t) + \epsilon^2 L_1(t)) F_+(t)}{W(t)} g_1(t) dt \\ &\quad + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^0 \frac{\epsilon^2 F_+''(t) + (L(t) + \epsilon^2 L_1(t)) F_+(t)}{W(t)} g_1(t) dt \\ &=: f_1(\xi) + f_2(\xi); \end{aligned}$$

For $\xi = |\xi|/i \in \{it, 0 < t < 1\}$, note for $0 < t < 1$, $C_3(\gamma - t)^{1/2}|t - 1|^{-2} \leq |L(it)| \leq C_2|t - 1|^{-2}$, $C_3|t - 1|^{-2} \leq |L_1(it)| \leq C_2|t - 1|^{-2}$. Using Lemma 2.22:

$$\sup_t |(it - 2i)^\tau F_+(it)| \leq C\epsilon^2 \mathcal{G}_1(\delta, \delta_1), \sup_t |(it - 2i)^{2+\tau} F_+''(it)| \leq C\epsilon^2 \mathcal{G}_1(\delta, \delta_1);$$

so

$$\begin{aligned} |f_1(\xi)| &\leq \frac{C}{\epsilon} (1 - |\xi|)^{1/2} \left\{ \epsilon^4 \mathcal{G}_1(\delta, \delta_1) \int_0^{|\xi|} (1 - t)^{3/8} \left| \exp\left\{ \frac{\operatorname{Re}(P(i|\xi|) - P(it))}{\epsilon} \right\} \right| dt \right. \\ &\quad \left. + \epsilon^2 \mathcal{G}_1(\delta, \delta_1) \int_a^1 (1 - t)^{-3/2} \left| \exp\left\{ \frac{\operatorname{Re}(P(it) - P(ai))}{\epsilon} \right\} \right| dt \right\} \end{aligned}$$

By equation (A.1) in the Appendix A, for $0 < t < 1$:

$$\begin{aligned} \operatorname{Re}(P(i|\xi|) - P(it)) &= - \int_t^{|\xi|} \frac{|\gamma|^{3/4} |\gamma - t|^{1/4}}{(1 - t)(1 + t)} dt \\ &\leq -C_1 \int_t^{|\xi|} \frac{1}{(1 - t)} dt \leq C_1 \ln \left(\frac{(1 - |\xi|)}{(1 - t)} \right); \end{aligned}$$

where C_1 is a positive constant independent of ϵ . So

$$\begin{aligned} & \frac{1}{\epsilon}(1-|\xi|)^{1/2} \int_0^{|\xi|} (1-t)^{-3/2} \left| \exp\left\{\frac{\operatorname{Re}(P(i|\xi|) - P(it))}{\epsilon}\right\} \right| dt \\ & \leq \frac{1}{\epsilon} \int_0^{|\xi|} (1-t)^{-3/2} \left(\frac{1-|\xi|}{1-t} \right)^{\frac{C_1}{\epsilon}} dt \\ & \leq (1-|\xi|)^{\frac{C_1}{\epsilon} + \frac{1}{2}} \left(\frac{1}{\epsilon} \int_0^{|\xi|} (1-t)^{-\frac{C_1}{\epsilon} - \frac{1}{2}} dt \right) \leq C; \end{aligned}$$

therefore

$$|f_1(\xi)| \leq C\epsilon^2 \mathcal{G}_1(\delta, \delta_1);$$

To estimate $f_2(\xi)$, by lemma 2.34:

$$\begin{aligned} & \left| \frac{1}{\epsilon^2} g_2(0) \int_{-\infty}^0 \frac{\epsilon^2 F_-''(t) + (L(t) + \epsilon^2 L_1(t)) F_-(t)}{W(t)} g_1(t) dt \right| \\ & \leq C \| \epsilon^2 F_-''(t) + (L(t) + \epsilon^2 L_1(t)) F_-(t) \|_2 \leq C\epsilon^2 \mathcal{G}_2(\delta, \delta_1); \end{aligned}$$

Using the fact that $\operatorname{Re}P(\xi)$ is decreasing to $-\infty$ as ξ goes upward along imaginary axis, we have

$$\left| g_2(\xi) g_2^{-1}(0) \right| \leq \left| L^{-1/4}(\xi) L^{1/4}(0) \exp\left\{\frac{1}{\epsilon^2} (\operatorname{Re}P(\xi) - \operatorname{Re}P(0))\right\} \right| \leq C$$

So

$$|f_2(\xi)| \leq C\epsilon^2 \mathcal{G}_1(\delta, \delta_1);$$

□

Definition 2.38. Define operator $\mathcal{U}$: for $N(\xi) \in \mathbf{A}_2$

$$\mathcal{U}N(\xi) := -\frac{1}{\epsilon^2} g_1(\xi) \int_{\alpha}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt; \quad (2.64)$$

Taking derivative in (2.64) and define:

Definition 2.39.

$$\mathcal{U}_1 N(\xi) := -\frac{1}{\epsilon^2} h_1(\xi) g_1(\xi) \int_{a_1}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} h_2(\xi) g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt; \quad (2.65)$$

where

$$h_1(\xi) = -\frac{L'(\xi)}{4L(\xi)} - \frac{1}{\epsilon} P'(\xi), h_2(\xi) = -\frac{L'(\xi)}{4L(\xi)} + \frac{1}{\epsilon} P'(\xi); \quad (2.66)$$

Lemma 2.40.

$$\sup_{\mathbb{R}} |(\xi - 2i) h_k(\xi)| \leq C \left(\frac{1}{\epsilon} \frac{|b - \gamma|^{3/4}}{|b - 1|} - \frac{1}{|b - \gamma|} + \frac{1}{|b - 1|} \right); k = 1, 2 \quad (2.67)$$

where C is a constant independent of ϵ, b, γ .

Proof. The lemma follows from that $P'(\xi) = iL^{1/2}(\xi)$ and equations (2.66) and (2.11). $\square$

Lemma 2.41. *If $F(\xi)$ is a solution of Problem 1, then $F(\xi)$ satisfies the following equation:*

$$F(\xi) = \beta g_1(\xi) + \mathcal{U} \tilde{N}_2(\xi) \quad (2.68)$$

where β is given by (2.62).

Proof. By Thm 2.18, Lemma 2.27 and variation of parameter, $F(\xi)$ can be written as:

$$F(\xi) = \beta_1 g_1(\xi) + \beta_2 g_2(\xi) - \frac{1}{\epsilon^2} g_1(\xi) \int_{\iota}^{\xi} \frac{\tilde{N}_2}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{\tilde{N}_2}{W(t)} g_1(t) dt;$$

In view of Lemmas 2.34 and 2.35, the two integrals in above equation have decay rate $O(\xi^{-\tau})$, $\xi \rightarrow -\infty$. Since $|g_2(\xi)| \rightarrow \infty$ as $\xi \rightarrow -\infty$, property (1.22) implies $\beta_2 = 0$. Using

the fact that $F(\xi)$ is analytic in the upper half plane and $F = F_+, g_1(i) = \infty$, we have $\beta_1 = 0$: So

$$\begin{aligned} F(\xi) &= -\frac{1}{\epsilon^2} g_1(\xi) \int_1^\xi \frac{\tilde{N}_2}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^\xi \frac{\tilde{N}_2}{W(t)} g_1(t) dt \\ &= \frac{1}{\epsilon^2} g_1(\xi) \int_{ai}^\xi \frac{\tilde{N}_2}{W(t)} g_2(t) dt + \mathcal{U} \tilde{N}_2(\xi) \\ &= \frac{1}{\epsilon^2} g_1(\xi) \int_{ai}^\xi \frac{\epsilon^2 F''(t) + (L + \epsilon^2 L_1) F(t)}{W(t)} g_2(t) dt + \mathcal{U} \tilde{N}_2(\xi) \end{aligned}$$

Using $F(\xi) = F_-(\xi)$, $\text{Im} \xi > 0$ (Lemma 2.27):

$$\begin{aligned} F(\xi) &= \frac{1}{\epsilon^2} g_1(\xi) \int_{ai}^\xi \frac{\epsilon^2 F''_+(t) + (L + \epsilon^2 L_1) F_-(t)}{W(t)} g_2(t) dt + \mathcal{U} \tilde{N}_2(\xi) \\ &= \beta g_1(\xi) + \mathcal{U} \tilde{N}_2(\xi): \end{aligned}$$

□

Problem 2: Find function $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$ so that $F(\xi)$ satisfies Symmetry Conditions (iii) (1.20) and equation (2.68).

Theorem 2.42. *Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$. If $F(\xi)$ satisfies the symmetry condition (1.21) and the integral equation (2.68), then for sufficiently small ϵ, δ and δ_1 , $F(\xi)$ satisfies equation (2.27).*

The proof of Theorem will be given later after several lemmas.

Comment 2.19: It is enough to show $F = F_+$ in any subdomain of $\mathcal{R}$ to prove the theorem. For that purpose, b is assumed independent of ϵ in the following lemmas.

Lemma 2.43. *Assume $F(\xi) \in \mathbf{A}$, $F'(\xi) \in \mathbf{A}_1$ and $F(\xi)$ satisfies integral equation (2.68). Let $U(\xi) = F(\xi) - F_-(\xi)$, then*

(1) $\mathcal{U}(\xi)$ is analytic in the lower half plane $\text{Im } \xi < 0$.

(2) $\mathcal{U}(\xi)$ satisfies:

$$\mathcal{U}(\xi) = \mathcal{U} \cdot \tilde{N}(\xi) \quad (2.69)$$

where $\tilde{N} = N(\mathcal{U}, \mathcal{U}') + \epsilon^2 L_1(\xi) \mathcal{U}(\xi)$;

$$\begin{aligned} N(\mathcal{U}, \mathcal{U}') := & -\frac{\epsilon^2 i}{2\pi} G_1(\xi) \int_{r_1} \frac{G_1^{-1}(t) \mathcal{U}''(t) + G_4(t) \mathcal{U}'(t)}{t - \xi} dt \\ & - \epsilon^2 G_4(\xi) G_1(\xi) \mathcal{U}'(\xi) + i G_3(\xi) \mathcal{U}(\xi), \text{ for } \xi \text{ above } r_1; \end{aligned} \quad (2.70)$$

G_1 is given by (2.46), G_3 is given by (2.55):

$$\begin{aligned} G_4(F, \hat{F}, F_-) = & \frac{(F_+'' + H')\{(F' + H)^2 + (F_-' + H)^2 + (F' + H)(F_+' + H)\}}{(\hat{F}' + \hat{H})^{1/2}(F' + H)^{3/2}(F_+' + H)^{3/2}\{(F' + H)^{3/2} + (F_-' + H)^{3/2}\}} \\ & + \frac{(\hat{F}_+'' + \hat{H}')}{(\hat{F}' + \hat{H})^{3/2}(F' + H)^{1/2}(F_-' + H)^{1/2}\{(F' + H)^{1/2} + (F_-' + H)^{1/2}\}}. \end{aligned} \quad (2.71)$$

Proof. Since $F(\xi)$ satisfies (2.68), $F(\xi)$ satisfies equation (2.35). Equation (2.35) can be rewritten as:

$$F(\xi) = \epsilon^2 I_2(\xi) + \frac{\epsilon^2}{i} G(F, \hat{F})(\xi). \quad (2.72)$$

By (2.30), analytically continuing below r_1 :

$$F_- = \epsilon^2 I_3 - i\epsilon^2 G(F, \hat{F}), \text{ for } \xi \text{ below } r_1; \quad (2.73)$$

where

$$I_3(\xi) = -\frac{1}{2\pi} \int_{r_1} \frac{G(F, \hat{F})(t)}{t - \xi} dt, \text{ for } \xi \text{ below } r_1; \quad (2.74)$$

Let $\mathcal{U}(\xi) = F(\xi) - F_-(\xi)$, changing the contour for I_2 to r_1 and for ξ below r_1 , we have:

$$\mathcal{U}(\xi) = \epsilon^2 (I_2 - I_3) = \frac{\epsilon^2}{2\pi} \int_{r_1} \frac{G(F, \hat{F}) - G(F_+, \hat{F})}{t - \xi}; \text{ for } \xi \text{ below } r_1; \quad (2.75)$$

so U is analytic for ξ below r_1 . It is also analytic in $\mathcal{R}$ since both F and F_+ are. Therefore U is analytic in $\mathcal{R} \cup \{Im\xi < 0\}$. Analytically continuing (2.75) to ξ above r_1 :

$$U(\xi) = \frac{\epsilon^2}{2\pi} \int_{r_2}^{\cdot} \frac{G(F, \hat{F}) - G(F_+, \hat{F})}{t - \xi} + \frac{\epsilon^2}{i} [G(F, \hat{F}) - G(F_+, \hat{F})]; \text{ for } \xi \text{ above } r_1; \quad (2.76)$$

Use (2.4):

$$G(F, \hat{F}) - G(F_+, \hat{F}) = G_1^{-1}(F, \hat{F})U''' + G_4(F, \hat{F})U'; \quad (2.77)$$

By (2.76), we can see $U(\xi)$ satisfies the following homogeneous equation for ξ above r_1 :

$$\epsilon^2 \frac{d^2 U}{d\xi^2} + L(\xi)U(\xi) = N(U, U') \quad (2.78)$$

Using $F_- = F - U$ and (2.36), (2.78), $F_-(\xi)$ satisfies the following equation:

$$\epsilon^2 \frac{d^2 F_-}{d\xi^2} + L(\xi)F_-(\xi) = N_2 - N(U, U') \quad (2.79)$$

Using analyticity of F_- at $\xi = i$, it follows from above equation:

$$\begin{aligned} F_-(\xi) &= -\frac{1}{\epsilon^2} g_1(\xi) \int_i^{\xi} \frac{\tilde{N}_2 - \tilde{N}}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{\tilde{N}_2 - \tilde{N}}{W(t)} g_1(t) dt \\ &= \beta g_1(\xi) + \mathcal{U}(\tilde{N}_2 - \tilde{N})(\xi); \end{aligned} \quad (2.80)$$

Subtracting (2.80) from (2.68), we can see that $U(\xi)$ satisfies equation (2.69). $\square$

Lemma 2.44. *Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, then for $\xi \in \mathcal{R}$*

$$|G_1^{-1}(F, \hat{F})(\xi)| \leq C|\xi - 2i|^2; \quad (2.81)$$

where C is independent of ϵ .

Proof. By (2.8),(2.24) and lemma 2.30.we have

$$\begin{aligned}
|G_1^{-1}(F, \hat{F})(\xi)| &\leq \left(-\|F'\|_1|\xi - 2i|^{-1-\tau} + H_m|\xi - 2i|^{-1}\right)^{-3/2} \\
&\quad \times \left(-\|\hat{F}'\|_1|\xi - 2i|^{-1-\tau} + H_m|\xi - 2i|^{-1}\right)^{-1/2} \\
&\leq C(K_1 - \|F'\|_1)^{-3/2}(K_3 - \|F\|_0)^{-1/2}|\xi - 2i|^2 \leq C|\xi - 2i|^2;
\end{aligned}$$

□

Lemma 2.45. *Let $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$, then for $\xi \in \mathcal{R}$*

$$|G_4(F, \hat{F}, F_-)(\xi)| \leq C|\xi - 2i|; \quad (2.82)$$

where C is independent of ϵ .

Proof. By Lemma 2.22,Comment 2.13:

$$|F'_- + H| \geq \left(-|\xi - 2i|^{-1-\tau}\|F'_-\|_1 + H_m|\xi - 2i|^{-1}\right) \geq C(-\epsilon^2 K_4 + K_1)|\xi - 2i|^{-1}; \quad (2.83)$$

Using (2.48),(2.49),Lemma 2.22 and Lemma 2.15:

$$|F''_- + H'| \leq \left(|\xi - 2i|^{-2-\tau}\|F''_+\|_2 + C_1|\xi - 2i|^{-2}\right) \leq (C_2\epsilon^2\mathcal{G}_1(\delta, \delta_1) + C_1)|\xi - 2i|^{-2} \quad (2.84)$$

$$|F'' - \hat{H}'| \leq \left(|\xi - 2i|^{-2-\tau}\|\hat{F}''\|_2 + C_1|\xi - 2i|^{-2}\right) \leq (C_2\delta + C_1)|\xi - 2i|^{-2}$$

the lemma follows from (2.71) (2.8),(2.24) and the above inequalities. □

Lemma 2.46. *let $U(\xi)$ be as in Lemma 2.43, then*

$$\sup_{\xi \in \mathcal{D}} |\xi - 2i|^{k+\tau} |U^{(k)}(\xi)| \leq C\|U\|_0; \quad (2.85)$$

where $\mathcal{D}$ is any angular subset of lower half plane.

Proof. since $U(\xi)$ is analytic in the lower half plane (lemma 2.43),by Cauchy integral formula:

$$U^{(k)}(\xi) = -\frac{k!}{2\pi i} \int_{-\infty}^{\infty} \frac{U(t)}{(t-\xi)^{k+1}} dt; \text{ for } \xi \text{ in } \mathcal{D}$$

then applying Lemma 2.12, we get the lemma. $\square$

Lemma 2.47. *Let $F \in \mathbf{A}_{0,0}$, $F' \in \mathbf{A}_{1,0}$. Let $N(U, U')$ be as defined in (2.70), then*

$$\tilde{N}(U, U') \in \mathbf{A}_2, \|\tilde{N}(U, U')\|_2 \leq C\delta_2 \max(\|U\|_0, \epsilon\|U'\|_1); \quad (2.86)$$

C is some constant independent of ϵ and δ_2 , and

$$\delta_2 = \min \left\{ \epsilon, (K_7\|F'\|_1 + 1)^{3/2} (K_6\|F\|_0 + 1)^{1/2} - 1 \right\} \quad (2.87)$$

where K_6, K_7 as in Lemma 2.29.

Proof. Using Lemma 2.44 ,Lemma 2.45 and Lemma 2.46:

$$|G_1^{-1}U''(t) + G_3U'(t)| \leq C|t - 2i|^{-\tau}\|U\|_0, t \text{ on } r_1; \quad (2.88)$$

applying Lemma 2.12, Lemma 2.32 and Lemma 2.44-46 to (2.70)

$$\begin{aligned} \sup_{\Omega_1 \cap \mathcal{R}} |\xi - 2i|^{2+\tau} |N(U, U')| &\leq C_1\epsilon^2\|U\|_0 + C_2\epsilon^2\|U'\|_1 \\ &+ C_3\|U\|_0 \left\{ (K_7\|F'\|_1 + 1)^{3/2} (K_6\|F\|_0 + 1)^{1/2} - 1 \right\} \leq C\delta_2(\|U\|_0 + \epsilon\|U'\|_1); \end{aligned} \quad (2.89)$$

Now to estimate $N(U, U')$ for $\xi \in \mathcal{R} - \Omega_1$, let

$$r_3 = \left\{ \xi = -\frac{\alpha_1}{3} + \rho e^{i(\pi - \frac{2\theta}{3})} \right\} \cup \left\{ \xi = -\frac{\alpha_1}{3} + \rho e^{i(-\frac{2\theta}{3})} \right\}; \quad (2.90)$$

changing the contour in (2.75) to r_3 :

$$U(\xi) = \frac{\epsilon^2}{2\pi} \int_{r_3} \frac{G_1^{-1}U'' + G_3U'}{t - \xi} dt, \xi \in \mathcal{R} - \Omega_1; \quad (2.91)$$

So

$$U'''(\xi) = \frac{\epsilon^2}{\pi} \int_{r_3} \frac{G_1^{-1}U''' + G_3U''}{(t-\xi)^3} dt, \xi \in \mathcal{R} - \Omega_1; \quad (2.92)$$

Use Lemma 2.44-2.46:

$$\|G_1^{-1}U''' + G_3U''\|_{0,r_3} \leq C\|U''\|_0; \quad (2.93)$$

applying Lemma 2.12 to $U'''(\xi)$ in (2.92) and to $U'(\xi)$ in (2.91):

$$\sup_{\mathcal{R}-\Omega_1} |\xi - 2i|^{2-\tau} |U'''(\xi)| \leq C\epsilon^2\|U''\|_0, \sup_{\mathcal{R}-\Omega_1} |\xi - 2i|^{2-\tau} |U'(\xi)| \leq C\epsilon^2\|U'\|_0, \quad (2.94)$$

so

$$\sup_{\mathcal{R}-\Omega_1} |\xi - 2i|^{2+\tau} |N(U, U')| \leq \sup_{\mathcal{R}-\Omega_1} |\xi - 2i|^{2+\tau} |\epsilon^2 U'''(\xi) + LU'(\xi)| \leq C(\epsilon^4\|U''\|_0 + \epsilon^2\|U'\|_0); \quad (2.95)$$

□

Proof. of Theorem 2.46: Taking derivative in (2.69):

$$U'(\xi) = \mathcal{U}_1 N(U, U'); \quad (2.96)$$

Define the norm of (U, U') by:

$$\|(U, U')\| = \|U\|_0 + \epsilon\|U'\|_1 \quad (2.97)$$

From (2.69), (2.96) and Lemma 2.34, Lemma 2.35 and Lemma 2.40, we have

$$\|(U, U')\| \leq C\|N(U, U')\|_2; \quad (2.98)$$

use (2.86), we have

$$\|(U, U')\| \leq C\delta_2\|(U, U')\|; \quad (2.99)$$

where C is some constant independent of ϵ and δ_2 . when $\epsilon, \|F\|_0, \|F'\|_1$ are small enough, by (2.87) δ_2 is small enough so that $C\delta_2 < 1$ in (2.99), so $\mathcal{U}(\xi) \equiv 0$. Hence $F(\xi) = F_+(\xi)$, $F(\xi)$ is analytic in the upper half plane and $I_2 = I_1, N_2 = N_1$. equation (2.35) reduces to (2.27) □

Theorem 2.48. *Problem 2 is equivalent to Problem 1 and hence to the original Finger problem.*

Proof. It follows from Theorem 2.42 and Theorem 2.18 . □

CHAPTER 3

FORMULATION OF WEAK PROBLEM AND CONTRACTION MAPPING

If $F \in \mathbf{A}$ and satisfies the symmetry condition (1.21) ,by analyticity F satisfies the following equation:

$$F(\xi) = [F(-\xi^*)]^*, \text{ for } \xi \in \mathcal{R}; \quad (3.1)$$

the above equation implies:

$$F'(\xi) = -[F'(-\xi^*)]^*, \text{ for } \xi \in \mathcal{R}; \quad (3.2)$$

$$F''(\xi) = [F''(-\xi^*)]^*, \text{ for } \xi \in \mathcal{R}; \quad (3.3)$$

For $F \in \mathbf{A}^-$,the two pair of rays r_0^+, r_0^- and r_1^+, r_1^- are symmetric about imaginary ξ axis,using (3.1)-(3.3) ,the integral on r_0^+ in (2.15) and the integral over r_1^+ in (2.30) can be evaluated by only the values of $F(\xi)$ in the left half plane,the resulting functions are denoted by $\hat{F}(\xi)$ and $F_-(\xi)$ as before.We now pose the problem with relaxed symmetry condition.

Weak Problem: Find functions $F(\xi) \in \mathbf{A}_\delta^-$, $F'(\xi) \in \mathbf{A}_{1,\delta_1}^-$ that are analytic in $\mathcal{R}^-$, and satisfy equations (2.68) in $\mathcal{R}^-$.

Comment 3.1: Though we use (3.1),(3.2),(3.3) for $\text{Re}\xi > 0$ to evaluate integrals on

r_0^+, r_1^+ involving F, F', F'' : we are not requiring that (3.1)-(3.3) are actually satisfied on $\{Re \xi = 0\} \cap \mathcal{R}$.

Lemma 3.1. *If $F \in \mathbf{A}_\delta^-$, $F' \in \mathbf{A}_{\delta_1}^-$ is a solution of Weak Problem, and $F(\xi)$ satisfies the following condition:*

$$Im F = 0, \text{ on } \{Re \xi = 0\} \cap \mathcal{R}; \quad (3.4)$$

then $F(\xi)$ can be analytically extended to the right half of $\mathcal{R}$ and satisfies (2.68) in $\mathcal{R}$. Therefore $F(\xi)$ is solution of Finger Problem.

Proof. By (3.4) and Schwarz extension theorem, let

$$F(\xi) = [F(-\xi^*)]^*, \text{ for } \xi \in \{Re \xi > 0\} \cap \mathcal{R};$$

then $F(\xi)$ is analytic in $\mathcal{R}$: it is easily checked that if F satisfies (2.68) for $Re \xi < 0$, then $[F(\xi^*)]^*$ satisfies (2.68) for $Re \xi > 0$. Hence $F(\xi)$ is a solution of Problem 2, by Theorem 2.48, $F(\xi)$ is solution of Finger Problem. $\square$

Lemma 3.2. *Let $\mathcal{D}$, f_k, g_k be as in Lemma 2.4. $G_5(f, g)$ is defined by:*

$$G_5(f, g)(t) = (f' + H)^{-3/2} (g' + \bar{H})^{-3/2} \quad (3.5)$$

then

$$|G_5(f_k, g_k)(t)| \leq C |t - 2i|^3; t \in \mathcal{D} \quad (3.6)$$

and

$$|G_5(f_1, g_1) - G_5(f_2, g_2)(t)| \leq C |t - 2i|^3 (\|f'_1 - f'_2\|_{1, \mathcal{D}} + \|g'_1 - g'_2\|_{1, \mathcal{D}}); \quad (3.7)$$

where C is independent of b and ϵ when $\|f'_k, g'_k\|_{1, \mathcal{D}} < K_1/2$.with K_1 as in Lemma 2.4:

Proof. (3.5) follows immediatly from (2.7) in Lemma 2.4. By straight algebra:

$$\begin{aligned} G_5(f_1, g_1) - G_5(f_2, g_2)(t) &= G_5(f_1, g_1)G_5(f_2, g_2) \\ &\times \left\{ \frac{(f'_2 - f'_1)(g'_2 + \bar{H})^{3/2}[(f'_2 + H)^2 + (f'_2 + H)(f'_1 + H) + (f'_1 + H)^2]}{(f'_2 + H)^{3/2} + (f'_1 + H)^{3/2}} \right. \\ &\quad \left. + \frac{(g'_2 - g'_1)(f'_1 + H)^{3/2}[(g'_2 + \bar{H})^2 + (g'_2 + H)(g'_1 + H) + (g'_1 + H)^2]}{(g'_2 + H)^{3/2} + (g'_1 + H)^{3/2}} \right\} \end{aligned}$$

(3.7) follows from the above equation and (2.8) and $\|f', g'\|_{1,D} < K_1/2$. $\square$

Lemma 3.3. *Let $\mathcal{D}, f_k, g_k$ be as in Lemma 2.4. Let*

$$G_6(f, g) := f''(g' + \bar{H}) - g''(f' + H) + H'g' - f'\bar{H}; \quad (3.8)$$

then

$$|G_6(f, g)(t)| \leq C|t - 2i|^{-3-\tau} (\|f''\|_{2,D} + \|g''\|_{2,D} + \|f'\|_{1,D} + \|g'\|_{1,D}) : t \in \mathcal{D}; \quad (3.9)$$

$$\begin{aligned} |G_6(f_1, g_1) - G_6(f_2, g_2)| &\leq C|t - 2i|^{-3-\tau} \left\{ \|f''_1 - f''_2\|_{2,D} + \|g''_1 - g''_2\|_{2,D} \right. \\ &\quad \left. + \|f'_1 - f'_2\|_{1,D} + \|g'_1 - g'_2\|_{1,D} \right\} : t \in \mathcal{D} \end{aligned} \quad (3.10)$$

when $\|f'_k, g'_k\|_{1,D}$ and $\|f''_k, g''_k\|_{2,D}$ are bounded by C , where C is independent of b and ϵ .

Proof.

$$\begin{aligned} |G_6(f, g)(t)| &\leq \|f''\|_2 |t - 2i|^{-2-\tau} (\|g'\|_1 |t - 2i|^{-1-\tau} + C|t - 2i|^{-1}) \\ &\quad + \|g''\|_2 |t - 2i|^{-2-\tau} (\|f'\|_1 |t - 2i|^{-1-\tau} + C|t - 2i|^{-1}) \\ &\quad + C\|g'\|_1 |t - 2i|^{-3-\tau} - C\|f'\|_1 |t - 2i|^{-3-\tau} \\ &\leq C|t - 2i|^{-3-\tau} (\|f''\|_{2,D} + \|g''\|_{2,D} + \|f'\|_{1,D} + \|g'\|_{1,D}) : \end{aligned}$$

(3.10) follows from the following inequality:

$$\begin{aligned}
|G_6(f_1, g_1)(t) - G_6(f_2, g_2)(t)| \leq & \left\{ |(f_1'' - f_2'')g_1'| + |(g_1' - g_2')f_2''| + |(f_1'' - f_2'')\tilde{H}'| \right. \\
& + |(g_1' - g_2')H'| + |(f_1' - f_2')g_1''| + |(g_1'' - g_2'')f_2'| \\
& \left. + |(g_1'' - g_2'')H| + |(f_1' - f_2')\tilde{H}'| \right\}, t \in \mathcal{D};
\end{aligned}$$

□

Lemma 3.4. *Let $G(f, g)$ be defined by (2.4), then*

$$\begin{aligned}
& \|G(f_1, g_1) - G(f_2, g_2)\|_{0, \mathcal{D}} \\
& \leq C (\|f_1' - f_2'\|_{1, \mathcal{D}} + \|g_1' - g_2'\|_{1, \mathcal{D}} + \|f_1'' - f_2''\|_{2, \mathcal{D}} + \|g_1'' - g_2''\|_{2, \mathcal{D}}); \quad (3.11)
\end{aligned}$$

where C is independent of b and ϵ when $\|f_k', g_k'\|_{1, \mathcal{D}} < K_1/2$, K_1 as in Lemma 2.4., $\|f_k'', g_k''\|_{2, \mathcal{D}}$ are bounded by C .

Proof. By (2.6):

$$\begin{aligned}
G(f_1, g_1) - G(f_2, g_2) = & (H' \tilde{H} - H \tilde{H}') (G_5(f_1, g_1) - G_5(f_2, g_2)) \\
& + G_5(f_1, g_1) (G_6(f_1, g_1) - G_6(f_2, g_2)) + (G_5(f_1, g_1) - G_5(f_2, g_2)) G_6(f_2, g_2); \quad (3.12)
\end{aligned}$$

Using above identity and Lemma 3.2-Lemma 3.3, we have (3.11). □

Lemma 3.5. *Let $F(\xi) \in \mathbf{A}$, then*

$$\|F^{(l)}\|_{l, r_1^-} \leq C \|F\|_0, l = 1, 2; \quad (3.13)$$

where C is independent of ϵ and b .

Proof. by Cauchy integral formula,for $\xi \in r_1^-$:

$$F^{(l)}(\xi) = \frac{l!}{2\pi i} \int_{\partial \mathcal{D}} \frac{F(t)}{(t-\xi)^{l+1}} dt;$$

applying Lemma 2.12 with $\mathcal{D} = r_1^-$,we get the Lemma. $\square$

Lemma 3.6. *Let $F_k \in \mathbf{A}_{0,\delta}^-$, $F'_k \in \mathbf{A}_{1,\delta_1}^-$, $k = 1, 2$, then*

$$\|F_{+,1}^{(l)} - F_{-,2}^{(l)}\|_{l,\Omega_1} \leq C\epsilon^2 \|F_1 - F_2\|_0; \quad (3.14)$$

where C is independent of ϵ and b .

Proof. By definition of F_- (see equation(2.30)):

$$F_{+,1}^{(l)} - F_{-,2}^{(l)} = -\frac{\epsilon^2 l!}{2\pi} \int_{r_1} \frac{G(F_1, \hat{F}_1) - G(F_2, \hat{F}_2)}{t - \xi} dt;$$

Applying lemma 2.12 and Lemma 3.4, Lemma 3.5 and Lemma 2.14:

$$\begin{aligned} \|F_{+,1}^{(l)} - F_{-,2}^{(l)}\|_{0,\Omega_1} &\leq C\epsilon^2 \|G(F_1, \hat{F}_1) - G(F_2, \hat{F}_2)\|_{0,r_1} \\ &\leq C\epsilon^2 \left(\|F_1'' - F_2''\|_{2,r_1} + \|\hat{F}_1'' - \hat{F}_2''\|_{2,r_1} + \|F_1' - F_2'\|_{1,r_1} + \|\hat{F}_1' - \hat{F}_2'\|_{1,r_1} \right) \\ &\leq C\|F_1 - F_2\|_0; \end{aligned}$$

$\square$

Lemma 3.7. *Let $F_k \in \mathbf{A}_{0,\delta}^-$, $F'_k \in \mathbf{A}_{1,\delta_1}^-$, $k = 1, 2$, Then*

$$\|I_2(F_1) - I_2(F_2)\|_0 \leq C\|F_1 - F_2\|_0; \quad (3.15)$$

where C is independent of ϵ and b .

Proof. By (2.33)

$$I_2(F_1)(\xi) - I_2(F_2)(\xi) = -\frac{1}{2\pi} \int_{r_2} \frac{G(F_{+,1}, \hat{F}_1)(t) - G(F_{+,2}, \hat{F}_2)(t)}{t - \xi} dt;$$

Applying lemma 2.12 ,Lemma 3.4- Lemma 3.6 and Lemma 2.14:

$$\begin{aligned} \|I_2(F_1) - I_2(F_2)\|_0 &\leq C \|G(F_{+,1}, \hat{F}_1) - G(F_{+,2}, \hat{F}_2)\|_{0,r_2^-} \\ &\leq C \left(\|F_{+,1}'' - F_{+,2}''\|_{2,r_2} + \|\hat{F}_1'' - \hat{F}_2''\|_{2,r_2} + \|F_{+,1}' - F_{+,2}'\|_{1,r_2} + \|F_1'' - F_2''\|_{1,r_2} \right) \\ &\leq C \|F_1 - F_2\|_0; \end{aligned}$$

□

Lemma 3.8. Let $F_k \in \mathbf{A}_{0,\delta}^-$, $F_k' \in \mathbf{A}_{1,\delta_1}^-$, $k = 1, 2$. $G_1(F, \hat{F})$ be defined as in (2.46) in lemma 2.30, then for $\xi \in \mathcal{R}_-$

$$\begin{aligned} &|G_1(F_1, \hat{F}_1)(\xi) - G_1(F_2, \hat{F}_2)(\xi)| \\ &\leq C |\xi - 2i|^{-2} \left(\max\left\{1, \frac{|b - \gamma|^{3/2}}{|b - 1|}\right\} \|F_1 - F_2\|_0 + \frac{|b - \gamma|^{1/2}}{|b - 1|} \|F_1' - F_2'\|_1 \right); \quad (3.16) \end{aligned}$$

Where C is independent of b and ϵ .

Proof. By straight algebra:

$$\begin{aligned} G_1(F_1, \hat{F}_1) - G_1(F_2, \hat{F}_2)(t) &= \frac{(F_2' + H)^{3/2}(\hat{F}_1 - \hat{F}_2)}{(\hat{F}_1' + \hat{H})^{1/2} + (\hat{F}_2' + \hat{H})^{1/2}} \\ &+ \frac{(F_1' - F_2')(\hat{F}_1' + \hat{H})^{1/2}[(F_1' + H)^2 + (F_1' + H)(F_2' + H) + (F_2' + H)^2]}{(F_1' + H)^{3/2} + (F_2' + H)^{3/2}} \end{aligned}$$

applying (2.48),(2.49),(2.54) and Lemma 2.29 to above equation,we obtain the lemma.

□

Lemma 3.9. *Let $F_k \in \mathbf{A}_{0,\delta}^-, F'_k \in \mathbf{A}_{1,\delta_1}^-, k = 1, 2$. $G_2(F, \hat{F})$ be defined as in (2.50) in lemma 2.31, then*

$$\|G_2(F_1, \hat{F}_1) - G_2(F_2, \hat{F}_2)\|_2 \leq C\|F_1 - F_2\|_0 + \frac{C|b - \gamma|}{|b - 1|}\|F_1 - F_2\|_0 + \frac{C}{|b - 1|}\|F'_1 - F'_2\|_1; \quad (3.17)$$

Proof. By Lemma 2.14 and (2.52):

$$\begin{aligned} |(\hat{F}_1'' - \hat{F}_2'')| &\leq C|\xi - 2i|^{-2-\tau}\|F_1 - F_2\|_0; \\ |(\hat{F}'' + \hat{H}')| &\leq |\hat{H}'|(\frac{|\hat{F}''|}{|\hat{H}'|} + 1) \leq |\hat{H}'|(C\|F\|_0 + 1); \end{aligned}$$

By straight algebra:

$$\begin{aligned} G_2(F_1, \hat{F}_1) - G_2(F_2, \hat{F}_2) &= (\hat{F}_1'' - \hat{F}_2'')\frac{\hat{F}'_1 + \hat{H}}{\hat{F}'_1 + \hat{H}} \\ &\quad + \frac{(\hat{F}_2'' + \hat{H}')}{(\hat{F}'_1 + \hat{H})}(F'_1 - F'_2) + \frac{(\hat{F}_2'' + \hat{H}')(F'_2 + \hat{H})}{(\hat{F}'_1 + \hat{H})(\hat{F}'_2 + \hat{H})}(\hat{F}'_2 - \hat{F}'_1); \end{aligned}$$

applying (2.48),(2.49),(2.54) and Lemma 2.29 to above equation,we obtain the lemma. $\square$

Lemma 3.10. *Let $F_k \in \mathbf{A}_{\delta}^-, F'_k \in \mathbf{A}_{\delta_1}^-, k = 1, 2$. then*

$$\begin{aligned} &\|\tilde{N}_2(F_1, I_2(F_1), \hat{F}_1) - \tilde{N}_2(F_2, I_2(F_2), \hat{F}_2)\|_2^- \\ &\leq C \frac{\{\epsilon^2 + (\delta + \delta_1)|b - \gamma|^{3/2}\}}{|1 - b|^2}\|F_1 - F_2\|_0 + C \frac{\epsilon^2 + \delta|b - \gamma|^{1/2}}{|b - 1|}\|F'_1 - F'_2\|_1 \end{aligned} \quad (3.18)$$

Proof. By Definition 2.26 and equation (2.43):

$$\begin{aligned}
& \tilde{N}_2(F_1, I_2(F_1), \dot{F}_1) - \tilde{N}_2(F_2, I_2(F_2), \dot{F}_2) = \\
& -i\epsilon^2 I_2(F_1) \left(G_1(F_1, \dot{F}_1) - G_1(F_2, \dot{F}_2) \right) - i\epsilon^2 G_1(F_2, \dot{F}_2) (I_2(F_1) - I_2(F_2)) \\
& + i(F_1 - F_2) G_3(F_1, \dot{F}_1) + iF_2 \left(G_1(F_1, \dot{F}_1) - G_1(F_2, \dot{F}_2) \right) \\
& + \epsilon^2 L_1(\xi)(F_1 - F_2) + \epsilon^2 \left(G_2(F_1, \dot{F}_1) - G_2(F_2, \dot{F}_2) \right); \quad (3.19)
\end{aligned}$$

Using the above identity ,Lemma 2.30, Lemma 2.32 ,Lemma 3.7-Lemma 3.9

and Lemma 2.25, we have the Lemma. $\square$

Definition 3.11.

$$T(\xi) \equiv \tilde{N}_2(0, I_2(0), 0)(\xi) \equiv \epsilon^2 \left(\frac{\tilde{H}' H}{\tilde{H}} - H' \right) (\xi) - i\epsilon^2 H^{3/2} \tilde{H}^{1/2} I_2(0)(\xi); \quad (3.20)$$

Comment 3.2: By Lemma 2.33, $T(\xi) \in \mathbf{A}_2$.

Lemma 3.12. Let $F \in \mathbf{A}_{0,\delta}^-$, $F' \in \mathbf{A}_{1,\delta_1}^-$, then

$$\begin{aligned}
& \|\tilde{N}_2(F, I_2(F), \dot{F}) - T\|_2^- \\
& \leq C \frac{\{\epsilon^2 + (\delta + \delta_1)|b - \gamma|^{3/2}\}}{|1 - b|^2} \|F\|_0 + C \frac{\epsilon^2 + \delta|b - \gamma|^{1/2}}{|b - 1|} \|F'\|_1; \quad (3.21)
\end{aligned}$$

Proof. In Lemma 3.10, Let $F_1 = F$, $F_2 = 0$, we get the lemma. $\square$

Lemma 3.13. Let $F_k(\xi) \in \mathbf{A}^-$, $k = 1, 2$, $\beta(F_k)$ be defined by (2.62), then

$$|(\beta(F_1) - \beta(F_2))| \leq C\epsilon^2 \exp\left\{\frac{\text{Re}P(ai)}{\epsilon}\right\} \|F_1 - F_2\|_0^-.$$

Proof. By Lemma 2.36 and Lemma 3.6:

$$\begin{aligned}
|\beta(F_1) - \beta(F_2)| &= \left| \frac{1}{\epsilon^2} \int_{at}^t \frac{\epsilon^2(F''_{+,1} - F''_{-,2}) + (L + \epsilon^2 L_1)(F_{+,1} - F_{+,2})}{W} dt \right| \\
&\leq C\epsilon^2 \exp\left\{\frac{\operatorname{Re} P(ai)}{\epsilon}\right\} \|F''_{+,1} - F''_{-,2}\|_2 \\
&\quad + C\|F_{-,1} - F_{-,2}\|_2 \leq C\epsilon^2 \exp\left\{\frac{\operatorname{Re} P(ai)}{\epsilon}\right\} \|F_1 - F_2\|_0
\end{aligned}$$

□

Lemma 3.14. *If b is independent of ϵ , and δ, δ_1 are small enough but independent of ϵ , then if $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$ is a solution of Problem 2, then $\|F\|_0 \leq C\epsilon^2$, $\|F'\|_1 \leq C\epsilon$, with C independent of ϵ .*

Proof. Since

$$F = \beta g_1 + \mathcal{U} \tilde{N}_2(F, I_2(F), \dot{F}), F' = \beta h_1 g_1 + \mathcal{U}_1 \tilde{N}_2(F, I_2(F), \dot{F}); \quad (3.22)$$

Using Lemma 2.34-2.35 Lemma 3.12 and (3.20):

$$\|F\|_0 \leq C\epsilon^2 (1 + \delta + \delta_1) + C(\delta + \delta_1) \|F\|_0 \quad (3.23)$$

so for $C(\delta + \delta_1) \leq \frac{1}{2}$, we have $\|F\|_0 \leq C\epsilon^2$.

Using Lemma 2.34-2.35, Lemma 2.40 :

$$\|F'\|_0 \leq C\epsilon (1 + \delta + \delta_1) + C\frac{1}{\epsilon}(\delta + \delta_1) \|F\|_0 \leq C\epsilon; \quad (3.24)$$

□

We define spaces:

Definition 3.15.

$$\mathbf{E} := \mathbf{A}^- \oplus \mathbf{A}_1^-$$

. For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}$.

$$\|\mathbf{e}\|_{\mathbf{E}} := \|u(\xi)\|_0 + \epsilon \|v(\xi)\|_1$$

.

It is easy to see that $\mathbf{E}$ is Banach space.

Definition 3.16. For $0 \leq \sigma < \frac{1}{5}$, $\bar{\sigma} := \min\{1, 2 - \frac{3\sigma}{2}\}$:

$$\mathbf{B} := \{\mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E} : \|u(\xi)\|_0^- \leq O(\epsilon^{2-\frac{\sigma}{2}}), \|v(\xi)\|_1^- \leq O(\epsilon^{\bar{\sigma}})\}. \quad (3.25)$$

Comment 3.3: For $(F, F') \in \mathbf{B}$, we have $F \in \mathbf{A}_{\delta}^-$, $F' \in \mathbf{A}_{\delta_1}^-$ with $\delta = O(\epsilon^{2-\frac{1}{2}\sigma})$, $\delta_1 = O(\epsilon^{\bar{\sigma}})$, hence for sufficiently small ϵ , $\delta \leq K_3/2$, $\delta_1 \leq K_1/2$.

In (2.64) we replace $F(\xi)$ by $u(\xi)$, $F'(\xi)$ by $v(\xi)$ in the expression of $\tilde{N}_2$:

Definition 3.17. Let

$$\mathbf{O} : \mathbf{E} \longrightarrow \mathbf{E}$$

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathbf{e}) = (\mathbf{O}_1(\mathbf{e}), \mathbf{O}_2(\mathbf{e}))$$

where

$$\mathbf{O}_1(\mathbf{e}) = \mathcal{J}(u(\xi))g_1(\xi) + \mathcal{U}\tilde{N}_2(u, v); \quad (3.26)$$

$$\mathbf{O}_2(\mathbf{e}) = \mathcal{J}(u(\xi))h_1(\xi)g_1(\xi) + \mathcal{U}_1\tilde{N}_2(u, v); \quad (3.27)$$

Lemma 3.18. For sufficiently small ϵ , and

$$|b - 1| = O_s(\epsilon^{\sigma}), |b - \gamma| = O_s(\epsilon^{\sigma}), \text{ where } 0 \leq \sigma < \frac{4}{5}; \quad (3.28)$$

then $(\mathcal{U}T(\xi), \mathcal{U}_1T(\xi)) \in \mathbf{B}$.

Proof. Using integration by parts twice, we have

$$\begin{aligned}\mathcal{U}T(\xi) &= L^{-1}(\xi)T(\xi) + \frac{1}{2i}T_1(ai)e^{P(ai)/\epsilon}g_1(\xi) \\ &\quad - \frac{1}{2i}\epsilon T_2(ai)e^{P(ai)/\epsilon}g_1(\xi) + \epsilon^2\mathcal{U}T_3(\xi);\end{aligned}\tag{3.29}$$

where

$$T_1(\xi) \equiv T(\xi)L^{-3/4}(\xi) \sim C\epsilon^2(\xi+i)^{-5/8}, \text{ as } \gamma-1 \ll \xi+i \ll 1;\tag{3.30}$$

$$T_2(\xi) \equiv T'_1(\xi)L^{-1/2}(\xi) \sim C\epsilon^2(\xi+i)^{-11/8}, \text{ as } \gamma-1 \ll \xi+i \ll 1;\tag{3.31}$$

$$T_3(\xi) \equiv T'_2(\xi)L^{1/4}(\xi) \sim C\epsilon^2(\xi+i)^{-5/2}, \text{ as } \gamma-1 \ll \xi+i \ll 1;\tag{3.32}$$

Using the formula of $T(\xi)$ and $L(\xi)$ and (3.32) ,we have

$$T_3 \in \mathbf{A}_2, \|T_3\| \leq O(\epsilon^{2-\frac{1}{2}\sigma}), \|L^{-1}(\xi)T(\xi)\|_0 \sim O(\epsilon^{2-\frac{1}{2}\sigma})$$

Using that $\text{Re}(P(ai) - P(\xi)) < 0$, we have:

$$\|T_1(ai)e^{P(ai)/\epsilon}g_1(\xi)\| \leq O(\epsilon^{2-\frac{1}{2}\sigma}), \| \epsilon T_2(ai)e^{P(ai)/\epsilon}g_1(\xi) \| \leq O(\epsilon^{2-\frac{1}{2}\sigma});$$

By Lemma 2.34 and Lemma 2.35 , $\|\epsilon^2\mathcal{U}T_3(\xi)\| \sim O(\epsilon^{1-\frac{3}{2}\sigma}) \leq O(\epsilon^{2-\frac{1}{2}\sigma})$; therefore $\|\mathcal{U}T(\xi)\| \leq O(\epsilon^{2-\frac{1}{2}\sigma})$.

Taking derivative in (3.29):

$$\begin{aligned}\mathcal{U}_1T(\xi) &= (L^{-1}(\xi)T(\xi))' + \frac{1}{2i}T_1(ai)e^{P(ai)/\epsilon}g'_1(\xi) \\ &\quad - \frac{1}{2i}\epsilon T_2(ai)e^{P(ai)/\epsilon}g'_1(\xi) + \epsilon^2\mathcal{U}_1T_3(\xi);\end{aligned}\tag{3.33}$$

since

$$\begin{aligned}\|(L^{-1}(\xi)T(\xi))'\|_0 &\sim O(\epsilon^{\bar{\sigma}}), \|T_1(ai)e^{P(ai)/\epsilon}g'_1(\xi)\| \leq O(\epsilon^{\bar{\sigma}}), \\ \|T_2(ai)e^{P(ai)/\epsilon}g'_1(\xi)\| &\leq O(\epsilon^{\bar{\sigma}}), \|\epsilon^2\mathcal{U}_1T_3(\xi)\| \sim O(\epsilon^{3-\frac{5}{2}\sigma}) \leq O(\epsilon^{\bar{\sigma}})\end{aligned}$$

:

we have $\|\mathcal{U}_1 T(\xi)\|_0 \leq O(\epsilon^{\bar{\sigma}})$. □

Lemma 3.19. *Let $(u(\xi), v(\xi)) \in \mathbf{B}$, then $(\mathcal{J}(u)g_1, \mathcal{J}(u)h_1g_1) \in \mathbf{B}$.*

Proof. By Lemma 2.36 :

$$|\mathcal{J}(u)g_1(\xi)| \leq C\epsilon^2 |L^{-1/4}(\xi)| \exp\left\{\frac{1}{\epsilon} \operatorname{Re}(P(ai) - P(\xi))\right\};$$

$$|\mathcal{J}(u)h_1(\xi)g_1(\xi)| \leq C\epsilon^2 |h_1(\xi)L^{-1/4}(\xi)| \exp\left\{\frac{1}{\epsilon} \operatorname{Re}(P(ai) - P(\xi))\right\};$$

Using the fact that $\operatorname{Re} P(\xi) \sim C \ln |\xi|$ as $|\xi| \rightarrow \infty$, and $\operatorname{Re}(P(ai) - P(\xi)) \leq -C' < 0$ as $\xi + i \rightarrow 0$, where C is some positive constant independent of ϵ , we have $\|\mathcal{J}(u)g_1\|_0 \leq O(\epsilon^{2-\frac{1}{2}\sigma})$, $\|\mathcal{J}(u)h_1g_1\|_0 \leq O(\epsilon^{\bar{\sigma}})$. □

Theorem 3.20. *For sufficiently small ϵ , and $|b-1|$ and $|b-\gamma|$ as in (3.28), then the operator $\mathbf{O}$ is a contraction mapping from $\mathbf{B}$ to $\mathbf{B}$. Therefore there exists unique solution $(u(\xi), v(\xi)) \in \mathbf{B}$ to the weak problem.*

Proof. For $(u(\xi), v(\xi)) \in \mathbf{B}$, by lemma 3.12 and (3.25) we have $\|\tilde{N}_2(u, v) - T\|_2 \leq O(\epsilon^{1-\frac{1}{2}\sigma})$; By Lemma 2.34, lemma 2.35, Lemma 3.18 and lemma 3.19 :

$$\begin{aligned} \|\mathbf{O}_1(e)\|_0 &\leq \|\mathcal{J}g_1\|_0 + \|\mathcal{U}T\|_0 + \|\mathcal{U}(\tilde{N}_2 - T)\|_0 \\ &\leq O(\epsilon^{2-\frac{1}{2}\sigma}) + O(\epsilon^{2-\frac{1}{2}\sigma}) + C\|\tilde{N}_2(u, v) - T\|_2 \\ &\leq O(\epsilon^{2-\frac{1}{2}\sigma}); \end{aligned} \tag{3.34}$$

By Lemma 2.34, lemma 2.35, Lemma 3.18, lemma 3.19 and lemma 2.40:

$$\begin{aligned} \|\mathbf{O}_2(e)\|_1 &\leq \|\mathcal{J}h_1g_1\|_1 + \|\mathcal{U}_1T\|_1 + \|\mathcal{U}_1(\tilde{N}_2 - T)\|_1 \\ &\leq O(\epsilon^{\bar{\sigma}}) + O(\epsilon^{\bar{\sigma}}) + C \sup_{\mathcal{R}} |h_k| \|\tilde{N}_2(u, v) - T\|_2 \\ &\leq O(\epsilon^{\bar{\sigma}}); \end{aligned} \tag{3.35}$$

Let $\mathbf{e}_k = (u_k, v_k) \in \mathbf{B}, k = 1, 2$. by Lemma 3.10 and (3.25), we have

$$\|\tilde{N}_2(u_1, v_1) - \tilde{N}_2(u_2, v_2)\|_2 \leq O(\epsilon^{1-\sigma})\|\mathbf{e}_1 - \mathbf{e}_2\|;$$

by Lemma 2.34-2.35 and Lemma 3.13, we have

$$\begin{aligned} \|\mathbf{O}_1(e_1) - \mathbf{O}_1(e_2)\|_0 &\leq \|(\mathcal{J}(u_1) - \mathcal{J}(u_2))g_1\|_0 + \|\mathcal{U}\tilde{N}_2(u_1, v_1) - \tilde{N}_2(u_2, v_2)\|_0 \\ &\leq C\epsilon^2\|u_1 - u_2\|_0 + C\|\tilde{N}_2(u_1, v_1) - \tilde{N}_2(u_2, v_2)\|_2 \\ &\leq O(\epsilon^{1-\sigma})\|\mathbf{e}_1 - \mathbf{e}_2\|; \end{aligned} \quad (3.36)$$

$$\begin{aligned} \epsilon\|\mathbf{O}_2(e_1) - \mathbf{O}_2(e_2)\|_1 &\leq \|(\mathcal{J}(u_1) - \mathcal{J}(u_2))h_1g_1\|_0 + \|\mathcal{U}_1(\tilde{N}_2(u_1, v_1) - \tilde{N}_2(u_2, v_2))\|_0 \\ &\leq C\epsilon\sup_{\mathcal{R}}|h_k|(\epsilon^2\|u_1 - u_2\|_0 + \|\tilde{N}_2(u_1, v_1) - \tilde{N}_2(u_2, v_2)\|_2) \\ &\leq O(\epsilon^{1-\frac{3}{4}\sigma})\|\mathbf{e}_1 - \mathbf{e}_2\|; \end{aligned} \quad (3.37)$$

Therefore, we have proved that $\mathbf{O}$ is a contraction. $\square$

To manifest the dependence on α, b, φ_0 , let $\mathcal{R}_{\alpha, b, \varphi_0}$ be the region in definition 1.1. $\mathbf{A}_{k, \delta}(\mathcal{R}_{\alpha, b, \varphi_0})$ be space $\mathbf{A}_{k, \delta}$ on region $\mathcal{R}_{\alpha, b, \varphi_0}$, the corresponding norm is $\|F_k\|_k(\mathcal{R}_{\alpha, b, \varphi_0})$.

Lemma 3.21. *Let δ be small enough but independent of $\epsilon, |b - 1|$ as in (3.25). If $F_1, F_2 \in \mathbf{A}_{0, \delta}(\mathcal{R}_{\alpha, b, \varphi_0})$ are both solution to Finger problem, then $F_1 \equiv F_2$, and $F_k \in B$.*

Proof. Since $F_1, F_2 \in \mathbf{A}_{0, \delta}(\mathcal{R}_{\alpha, b, \varphi_0})$ are both solution to Finger Problem, then $F_1, F_2 \in \mathbf{A}_{0, \delta}(\mathcal{R}_{\alpha, 2, 1/2, \varphi_0/2})$. $F'_1, F'_2 \in \mathbf{A}_{1, \delta_1}(\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2})$ are solutions to Problem 2 in region $\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2}$, by Lemma 3.14, we have

$$\|F_k\|_0(\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2}) \leq C\epsilon^2, \|F'_k\|_1(\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2}) \leq C\epsilon.$$

Apply uniqueness in Theorem 3.20 with $\sigma = 0$ on region $\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2}$, we have $F_1 \equiv F_2$ on $\mathcal{R}_{\alpha/2, 1/2, \varphi_0/2}$, hence $F_1 \equiv F_2$ on $\mathcal{R}_{\alpha, b, \varphi_0}$. Uniqueness also implies $F_k \in B$.

□

CHAPTER 4

SELECTION OF FINGER WIDTH: INNER PROBLEM

ANALYSIS

4.1 Derivation of Equation Near $\xi = -i$

Rewrite equation (2.35) in Chapter 2:

$$F(\xi) = \epsilon^2 I_2(\xi) - \frac{\epsilon^2}{i(F'(\xi) + H)^{1/2}(\bar{F}'(\xi) + \bar{H})^{1/2}} \left[\frac{F''(\xi) + H'}{F'(\xi) + H} - \frac{\bar{F}''(\xi) + \bar{H}'}{\bar{F}'(\xi) + \bar{H}} \right]; \quad (4.1)$$

Let

$$\gamma = 1 + \epsilon^{4/3}a; \quad (4.2)$$

$$\xi = -i + \epsilon^{4/3}z; \quad (4.3)$$

Note that $\hat{F}(\xi)$ and $I_2(\xi)$ are analytic in a neighborhood of $\xi = -i$.let

$$F(\xi) = \epsilon^{4/3}Q(z); \quad (4.4)$$

then from (2.1):

$$(\hat{F}' + \bar{H})^{1/2} = \frac{1}{\epsilon^{2/3}z^{1/2}} + E_0(\epsilon^{4/3}, \epsilon^{2/3}z^{1/2}); \quad (4.5)$$

where $E_0(\epsilon^{4/3}, \epsilon^{2/3}z^{1/2})$ is analytic in $\epsilon^{4/3}, \epsilon^{2/3}z^{1/2}$.

By (1.19) :

$$(F' + H)^{1/2} = \frac{[zQ'(z) + \frac{i}{2}(z + ia)]^{1/2}}{z^{1/2}} + (\epsilon^{4/3}z)E_1(\epsilon^{4/3}z, Q', z^{-1}); \quad (4.6)$$

where $E_1(\epsilon^{4/3}z, Q', z^{-1})$ is analytic in $\epsilon^{4/3}z, Q', z^{-1}$.

$$\begin{aligned} \frac{F''(\xi) + \bar{H}'}{F'(\xi) + \bar{H}} &= \frac{d}{d\xi} \ln(F' + H) \\ &= \epsilon^{-4/3} \left[-\frac{1}{z} + \frac{[zQ'(z) + \frac{i}{2}(z + ia)]'}{zQ'(z) + \frac{i}{2}(z + ia)} \right] \\ &\quad + E_2(\epsilon^{4/3}z, Q', z^{-1}) + zQ''E_3(\epsilon^{4/3}z, Q', z^{-1}); \end{aligned} \quad (4.7)$$

where $E_2(\epsilon^{4/3}z, Q', z^{-1})$ and $E_3(\epsilon^{4/3}z, Q', z^{-1})$ are analytic in $\epsilon^{4/3}z, Q', z^{-1}$.

$$\frac{\dot{F}''(\xi) + \bar{H}'}{\dot{F}'(\xi) + \bar{H}} = -\frac{1}{\epsilon^{4/3}z} + E_4(\epsilon^{4/3}z, \epsilon^{4/3}z, z^{-1}); \quad (4.8)$$

where $E_4(\epsilon^{4/3}z, \epsilon^{4/3}z, z^{-1})$ is analytic in $\epsilon^{4/3}z, \epsilon^{4/3}z, z^{-1}$. Plug the above equations into (4.1):

$$\begin{aligned} Q''(z) + \frac{(1 - iM_1(z)Q(z))}{z}Q'(z) + \frac{M_1(z)(z + ia)}{2z^2}Q(z) &= -\frac{i}{2z} \\ &\quad + \epsilon^{2/3}E_5(\epsilon^{2/3}, \epsilon^{2/3}z^{1/2}, Q, Q', z^{-1}); \end{aligned} \quad (4.9)$$

where $E_5(\epsilon^{2/3}, \epsilon^{2/3}z^{1/2}, Q, Q', z^{-1})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}z^{1/2}, Q, Q', z^{-1}$, where

$$M_1(z) = \left(zQ'(z) + \frac{i}{2}(z + ia) \right)^{1/2}; \quad (4.10)$$

Let

$$z = -ai\zeta^2, Q(z) = -aD(\zeta); \quad (4.11)$$

(4.9) becomes:

$$\frac{d^2D}{d\zeta^2} + \frac{(1 - i\bar{\delta}MD)}{\zeta} \frac{dD}{d\zeta} + \frac{i\bar{\delta}(\zeta^2 - 1)M}{\zeta^2} D(\zeta) = 2 + \epsilon^{2/3}E_6(\epsilon^{2/3}, \epsilon^{2/3}\zeta, D, D', \zeta^{-2}); \quad (4.12)$$

where $E_6(\epsilon^{2/3}, \epsilon^{2/3}\zeta, D, D', \zeta^{-2})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}\zeta, D, D', \zeta^{-2}$. where

$$M(\zeta) = (\zeta^2 - 1 - \zeta D'(\zeta))^{1/2}; \quad (4.13)$$

$$\bar{\delta} = \sqrt{2}a^{3/2} = \frac{\sqrt{2}}{\epsilon^2} \left(\frac{2\lambda - 1}{1 - \lambda} \right)^{3/2}; \quad (4.14)$$

Note that equation (4.12) is the same as (127) in Tanveer(1987). Let

$$z = i2^{1/3}y^2, G(y) = -i2^{-1/3} \left(zQ'(z) + \frac{i}{2}(z + ia) \right)^{-1/2}; \quad (4.15)$$

then (4.1) becomes:

$$y \frac{d}{dy} \left(y \frac{dG}{dy} \right) + G^{-2} = y^2 + \bar{\delta}_1 + \epsilon^{2/3} E_7(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1}) \quad (4.16)$$

where

$$\bar{\delta}_1 = \left(\frac{\bar{\delta}}{2} \right)^{2/3} = \frac{a}{2^{1/3}}; \quad (4.17)$$

where $E_7(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1}$. Note that the leading order equation of (4.16) is the same as (37) in Combescot et al (1988). Let

$$\zeta' = \frac{y^2}{2^{4/3}}, q(\zeta') = yG(y); \quad (4.18)$$

then (4.16) becomes:

$$\frac{d^2 q}{d\zeta'^2} + \frac{1}{4\zeta'^2} q = \frac{1}{\sqrt{\zeta'}} \left(1 + \frac{a}{2^{5/3}\zeta'} - \frac{1}{q^2} \right) + \epsilon^{2/3} E_8(\epsilon^{2/3}, \epsilon^{2/3}\zeta'^{1/2}, q, q', \zeta'^{-1}), \quad (4.19)$$

where $E_8(\epsilon^{2/3}, \epsilon^{2/3}\zeta'^{1/2}, q, q', \zeta'^{-1})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}\zeta'^{1/2}, q, q', \zeta'^{-1}$.

Let

$$q = 1 + w, \quad (4.20)$$

write

$$\frac{1}{q^2} = 1 - 2\psi + \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n, \quad (4.21)$$

Let

$$\eta = \frac{4\sqrt{2}}{3} \zeta^{3/4}. \quad (4.22)$$

then (4.19) becomes:

$$\begin{aligned} \frac{d^2 \psi}{d\eta^2} - \frac{1}{3\eta} \frac{d\psi}{d\eta} - \psi &= -\frac{4}{9\eta^2} - \frac{4}{9\eta^2} \psi \\ &+ \frac{2^{2/3} a}{3^{4/3} \eta^{4/3}} - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n + \epsilon^{2/3} E_9(\epsilon^{2/3}, \epsilon^{2/3} \eta^{2/3}, \psi, \psi', \eta^{-2/3}). \end{aligned} \quad (4.23)$$

where $E_9(\epsilon^{2/3}, \epsilon^{2/3} \eta^{2/3}, \psi, \psi', \eta^{-2/3})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3} \eta^{2/3}, \psi, \psi', \eta^{-2/3}$ with a series representation convergent for small values of each argument.

Theorem 4.1. *Let $F(\xi), F'(\xi)$ be the solution of the Weak problem as in Theorem 3.20, after change of variable*

$$\xi = -\psi + i\epsilon^{4/3} \frac{3^{4/3}}{2^{5/3}} \eta^{4/3}; \quad (4.24)$$

$$\psi(\eta, \epsilon, a) = \left(1 + \frac{2^{5/3} a}{3^{4/3} \eta^{4/3}} - 2iF'(\xi) \right)^{-1/2} - 1; \quad (4.25)$$

Then $\psi(\eta, \epsilon, a)$ satisfies equation (4.23) for $\eta = O(\epsilon^{-\sigma_1}), \epsilon \rightarrow 0^+$ and the boundary conditions:

$$\psi(\eta, \epsilon, a) \sim \frac{2a}{3^{4/3} \eta^{4/3}}, 0 \leq \arg \eta < \frac{5\pi}{8}, \epsilon \rightarrow 0; \quad (4.26)$$

where σ_1 is a positive number so that

$$1 - \frac{3}{4}\sigma \leq \sigma_1 < \frac{3}{4}(\bar{\sigma}); \quad (4.27)$$

where σ as in Definition 3.16 of Chapter 3.

Proof. As $\eta = O(\epsilon^{-\sigma_1})$, $\epsilon \rightarrow 0^+$, use (4.27) and (4.24), we have $|\xi + i| = O(\epsilon^\sigma)$, applying Theorem 3.20 and transforms (4.24)(4.25) ,(2.35) implies (4.23). For $\xi \in \mathcal{R}$, $\frac{\pi}{2} \leq \arg(\xi + i) < \frac{1\pi}{3}$, using transform (4.24), we have $0 \leq \arg \eta < \frac{5}{3}\pi$: Since $F'(\xi) \sim O(\epsilon^\sigma)$, and $\eta^{-4/3} \sim O(\epsilon^{\frac{4}{3}\sigma_1})$,using (4.25) and (4.27), we have the boundary condition (4.26). $\square$

4.2 Leading Inner problem analysis

Let $\epsilon = 0$ in equation (4.23), we get the leading order equation:

$$\frac{d^2 \psi}{d\eta^2} - \frac{1}{3\eta} \frac{d\psi}{d\eta} - \psi = -\frac{4}{9\eta^2} - \frac{4}{9\eta^2} \psi + \frac{2a}{3^{4/3}\eta^{4/3}} - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n \quad (4.28)$$

and boundary conditions:

$$\psi(\eta, a) \sim \frac{2a}{3^{4/3}\eta^{4/3}}, 0 \leq \arg \eta < \frac{5\pi}{8}; \quad (4.29)$$

We shall show the following Theorem:

Theorem 4.2. *There exists large enough $R > 0$ such that (4.28),(4.29) have a unique solution $\psi_0(\eta, a)$ in the region $|\eta| > R$, $\arg \eta \in [0, \frac{5\pi}{8})$.*

In a more general setting, Costin (1995,1998) discussed the one to one correspondence between actual solutions of differential equations and generalized Borel sums of transseries solutions. However because of the fractional powers in inner equation (4.28), we can not apply Costin's results directly to equation (4.28). Nevertheless Costin's techniques can be modified and similar results can be obtained for our case. let

$$\phi(\eta) = \psi + \frac{2a}{3^{4/3}\eta^{4/3}} \quad (4.30)$$

then

$$\begin{aligned} \frac{d^2\phi}{d\eta^2} - \frac{1}{3\eta} \frac{d\phi}{d\eta} - \phi = & -\frac{4}{9\eta^2} - \frac{4}{9\eta^2}\phi + \frac{8a}{3^{4/3}\eta^{10/3}} \\ & - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \sum_{k=0}^n \binom{n}{k} \phi^k \left(-\frac{2a}{3^{4/3}\eta^{4/3}} \right)^{n-k} \end{aligned} \quad (4.31)$$

Let $\mathcal{L}^{-1}$ be the inverse Laplace transform, define the convolution as

Definition 4.3. $F \star G(p) := \int_0^p F(t)G(p-t)dt.$

then $\mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}F \star \mathcal{L}^{-1}G$, $\mathcal{L}^{-1}(F') = -p\mathcal{L}^{-1}F$. Formally applying $\mathcal{L}^{-1}$ to equation (4.31), we have:

$$p^2\Phi + \frac{1}{3} \int_0^p s\Phi(s)ds - \Phi = F_0(p) + N(\Phi); \quad (4.32)$$

where $\Phi(p) = \mathcal{L}^{-1}(\phi(\eta))$ and

$$F_0(p) = -\frac{4}{9}p + \frac{8a}{3^{4/3}\Gamma(10/3)}p^{7/3}. \quad (4.33)$$

$$\begin{aligned} N(\Phi) = & -\frac{4}{9} \int_0^p (p-s)\Phi(s)ds \\ & - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \sum_{k=0}^n \binom{n}{k} \left(-\frac{2a}{3^{4/3}\Gamma(4/3)} \right)^{n-k} \Phi^{\star k} \star \left(p^{\frac{1}{3}} \right)^{\star(n-k)} \\ = & -\frac{4}{9} \int_0^p (p-s)\Phi(s)ds \\ & - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \left(-\frac{2a}{3^{4/3}} \right)^n \frac{p^{\frac{4n}{3}-1}}{\Gamma(\frac{4n}{3})} \\ & - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \left(\sum_{k=1}^{n-1} \binom{n}{k} \left(-\frac{2a}{3^{4/3}} \right)^{n-k} \Phi^{\star k} \star \frac{p^{\frac{4}{3}(n-k)-1}}{\Gamma(\frac{4}{3}(n-k))} + \Phi^{\star n} \right); \end{aligned} \quad (4.34)$$

In the cut complex p plane, define region $\mathcal{D}$ by

$$\mathcal{D} = C - \{(-\infty, 0) \cup [1, \infty)\} \quad (4.35)$$

Consider the ray segment:

$$\mathbf{r}_{\varphi, R} = \{\rho e^{i\varphi} : 0 < \rho < R, \varphi \in (-\pi, \pi), \varphi \neq 0\}; \quad (4.36)$$

Define Laplace transform $\mathcal{L}_{\varphi}$ along ray $\mathbf{r}_{\varphi, \infty}$ by

$$\mathcal{L}_{\varphi}(\Phi)(\eta) = \int_{\mathbf{r}_{\varphi, \infty}} e^{-\eta p} \Phi(p) dp; \quad (4.37)$$

Let

$$\|f\|_b := \int_{\mathbf{r}_{\varphi, R}} e^{-b|p|} |f(p)| |dp| = \int_0^R e^{-bt} |f(te^{i\varphi})| dt \quad (4.38)$$

and $L_b^1(\mathbf{r}_{\varphi, R}) := \{f : \|f\|_b < \infty\}, b > 0, R \leq \infty$.

Comment 4.1: The Laplace transform $\mathcal{L}_{\varphi}$ is a continuous operator from $L_b^1(\mathbf{r}_{\varphi, R})$ to the space of analytic functions in the half plane $\text{Re}(\eta e^{i\varphi}) > b$ with uniform norm. Let $\mathcal{K}$ be a star-shaped, open set whose closure contains the origin, $\text{diam}(\mathcal{K}) = R < \infty$, let

$$\|f\|_u := R \sup_{p \in \mathcal{K}} \{|f(p)| e^{-b|p|}\}; \quad (4.39)$$

Let $\mathcal{A} := \{f \text{ analytic in } \mathcal{D} - \{0\}, \lim_{p \rightarrow 0} f(p) = 0, \|f\|_u < \infty\}$. Following from Costin (1995) (Proposition 9 and remark 10), we have:

Lemma 4.4. *The spaces L_b^1 and $\mathcal{A}$ are Banach algebras with respect to addition of functions and convolution. Furthermore*

$$\|F * G\|_b \leq \|F\|_b \|G\|_b; \text{ if } F, G \in L_b^1 \quad (4.40)$$

$$\|F * G\|_u \leq \|F\|_u \|G\|_u; \text{ if } F, G \in \mathcal{A} \quad (4.41)$$

$$\|F * G\|_u \leq \|F\|_u \|G\|_b; \text{ if } F \in \mathcal{A}, G \in L_b^1;$$

Lemma 4.5. For $F \in \mathcal{A}$, or $F \in L_b^1$, $\|F\|_{b,u} \rightarrow 0$, as $b \rightarrow \infty$:

Lemma 4.6. For $\varphi \in (0, \pi) \cup (-\pi, 0)$, there is a unique solution $\Phi_0(p)$ of (4.32) in $L_b^1(\mathbf{r}_\varphi, \infty)$.

Proof. (1) Rewrite equation (4.32):

$$\Phi = N_1(\Phi) := \frac{1}{p^2 - 1} \left(F_0(p) + N(\Phi) - \frac{1}{3} \int_0^p s \Phi(s) ds \right); \quad (4.42)$$

Let $L_{b,\delta}^1 = \{f \in L_b^1 : \|f\|_b \leq \delta\}$. Note that

$$\|p^{k/3}\|_b \leq b^{-(k/3+1)} \int_0^\infty t^{k/3} e^{-t} dt = b^{-(k/3+1)} \Gamma(k/3 + 1);$$

$$\|1\|_b = \int_0^R e^{-bt} dt = O(b^{-1}).$$

$$\|F_0\|_b \leq \frac{4}{9} b^{-2} \Gamma(2) + \frac{8a}{3^{4/3} \Gamma(7/3)} b^{-(10/3)} \Gamma(10/3);$$

Use Lemma 4.4.4.5. and equation (4.34). for $\Phi(p) \in L_{b,\delta}^1$.

$$\begin{aligned} \|N(\Phi)\|_b &\leq \frac{1}{2} \sum_{n=2}^\infty |(-1)^n (n+1)| \sum_{k=0}^n \binom{n}{k} \left(\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-k} (b^{-4/3} \Gamma(4/3))^{n-k} \delta^k \\ &\quad + \frac{4}{9} \|p * \Phi\|_b \\ &\leq \frac{1}{2} \sum_{n=2}^\infty |(-1)^n (n+1)| \left(\delta + \frac{2a}{3^{4/3} b^{4/3}} \right)^n + \frac{4}{9} \|p\|_b \|\Phi\|_b \\ &\leq \frac{1}{2} \sum_{n=2}^\infty |(-1)^n (n+1)| \left(\delta + \frac{2a}{3^{4/3} b^{4/3}} \right)^n + O(b^{-2}) \delta; \end{aligned}$$

we used

$$\|p\|_b \leq \int_0^R s e^{-bs} ds = O(b^{-2});$$

Also note

$$\begin{aligned}
\left\| \frac{1}{p^2-1} \int_0^p s \Phi(s) ds \right\|_b &\leq \int_0^R e^{-bt} \frac{1}{|t^2 e^{2i\varphi} - 1|} \int_0^t s |\Phi(se^{i\varphi})| ds dt \\
&\leq \int_0^R (e^{-bt}) \frac{t}{|t^2 e^{2i\varphi} - 1|} \int_0^t |\Phi(se^{i\varphi})| ds dt \\
&\leq -\frac{C}{b} \int_0^R (de^{-bt}) \int_0^t |\Phi(se^{i\varphi})| ds \\
&\leq \frac{C}{b} \int_0^R (e^{-bt}) |\Phi(te^{i\varphi})| dt \leq \frac{C}{b} \|\Phi\|_b;
\end{aligned}$$

Use

$$\|N_1(\Phi)\|_b \leq C \left(\|F_0\|_b + \|N(\Phi)\|_b + \left\| \frac{1}{p^2-1} \int_0^p s \Phi(s) ds \right\|_b \right);$$

so for large enough b , we have $N_1(\Phi) \leq \delta$. Note for $\Phi_1(p), \Phi_2(p) \in L_{b,\delta}^1$, we have: (see remark 15, Costin 1995)

$$\|\Phi_1^{*k} - \Phi_2^{*k}\| \leq k(\|\Phi_1\| + \|\Phi_2 - \Phi_1\|)^{k-1} \|\Phi_1 - \Phi_2\|;$$

so

$$\begin{aligned}
\|N(\Phi_1) - N(\Phi_2)\|_b &\leq \frac{4}{9} \int_0^R e^{-b|p|} \left| \int_0^p (p-s)(\Phi_1(s) - \Phi_2(s)) ds \right| |dp| \\
&\quad + \frac{1}{2} \sum_{n=2}^{\infty} (n+1) \sum_{k=0}^n \binom{n}{k} \left(\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-k} (b^{-4/3} \Gamma(4/3))^{n-k} \|\Phi_1^{*k} - \Phi_2^{*k}\|_b \\
&\leq \frac{1}{2} \sum_{n=2}^{\infty} (n+1) \sum_{k=0}^n k(3\delta)^{k-1} \binom{n}{k} \left(\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-k} (b^{-4/3} \Gamma(4/3))^{n-k} \|\Phi_1 - \Phi_2\|_b \\
&\quad + \frac{4}{9} O(b^{-2}) \|\Phi_1 - \Phi_2\|_b \\
&\quad + \frac{1}{2} \sum_{n=2}^{\infty} n(n+1) \left(3\delta + \frac{2a}{3^{4/3}} b^{-4/3} \right)^{n-1} + \frac{4}{9} O(b^{-2}) \|\Phi_1 - \Phi_2\|_b \\
&\leq C(\delta + b^{-4/3}) \|\Phi_1 - \Phi_2\|_b;
\end{aligned}$$

So for small δ , large b , $N_1(\Phi)$ is a contraction mapping, there exists unique solution $\Phi_0(p)$ in $L_{b,\delta}^1$. $\square$

Lemma 4.7. *The solution $\Phi_0(p)$ in Lemma 4.6 is analytic in $\mathcal{D}$.*

Proof. Let $\mathcal{A}_\delta = \{f \in \mathcal{A} : \|f\|_u \leq \delta\}$ use

$$\|p^{k/3}\|_u \leq C_1 b^{-k/3};$$

$$\|F_0\|_u \leq \frac{4}{9} \frac{R}{b} \sup\{b|p|e^{-b|p|}\} + \frac{8a}{3^{4/3}\Gamma(7/3)b^{3/7}} \sup\{b|p|^{7/3}e^{-b|p|}\};$$

$$\left\| \int_0^p (p-s)\Phi(s)ds \right\|_u = \|p * \Phi\|_u \leq \|p\|_b \|\Phi\|_u = O(b^{-2})\|\Phi\|_u;$$

for $\Phi \in \mathcal{A}_\delta$:

$$\begin{aligned} \|N(\Phi)\|_u &\leq \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \sum_{k=0}^n \binom{n}{k} \left(\frac{2a}{3^{4/3}\Gamma(4/3)} \right)^{n-k} (b^{-1/3}C_1)^{n-k} \|\Phi^{*k}\|_u \\ &\quad + \frac{4}{9} \left\| \int_0^p (p-s)\Phi(s)ds \right\|_u \\ &\leq \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \left(\frac{2a}{3^{4/3}\Gamma(4/3)} \right)^{n-k} (C_1 b^{-1/3} + \|\Phi\|_u)^n + \frac{4}{9} O(b^{-2})\|\Phi\|_u; \end{aligned}$$

$$\begin{aligned} \left\| \int_0^p s\Phi(s)ds \right\|_u &\leq R \sup\{e^{-b|p|} \int_0^{|p|} s|\Phi(se^{t\tau})|ds\} \\ &\leq \sup\{e^{-b|p|} \int_0^{|p|} se^{bs} \|\Phi\|_u ds\} \leq C b^{-1} \|\Phi\|_u; \end{aligned}$$

similarly we can estimate $\|N(\Phi_1) - N(\Phi_2)\|_u$, and we can see $N_1(\Phi)$ is a contraction under $\|\cdot\|_u$. Therefore there exists a unique solution in $\mathcal{A}_\delta$. by uniqueness, this solution must be $\Phi_0(p)$ in Lemma 4.6. $\square$

Lemma 4.8. *The solution $\Phi_0(p)$ in Lemma 4.6 is Laplace transformable along any ray in $\mathcal{D}$ and $\mathcal{L}_\varphi \Phi_0(\eta)$ is a solution of (4.31).*

Proof. The lemma follows from Comment 1 and elementary properties of Laplace transform. $\square$

Comment 4.2: For $0 < \varphi_k < \frac{\pi}{2}$ (or $-\frac{\pi}{2} < \varphi_k < 0$), from the analyticity of $\Phi_0(p)$ and Cauchy integral theorem, we have $\mathcal{L}_{\varphi_1} \Phi_0(\eta) = \mathcal{L}_{\varphi_2} \Phi_0(\eta)$. Let $\mathbf{D}_\nu := \{p : |p| < \nu\}$ where $\nu > 0$ is small, define space

$$\mathcal{F}_0 = \{f(p) \text{ analytic in } \mathcal{D} \cap \mathbf{D}_\nu : f(p) = pA(p) + p^{4/3}B(p) + p^{5/3}C(p)\}. \quad (4.43)$$

$$\|f\| = \sup_{\mathcal{D} \cap \mathbf{D}_\nu} \{|A(p)|, |B(p)|, |C(p)|\}. \quad (4.44)$$

Note that the decomposition in (4.43) is unique, this norm makes $\mathcal{F}_0$ a Banach space. we have the following identities:

$$\begin{aligned} & \int_0^p (p-s)^{k/3} A(p-s) s^{m/3} B(s) ds \\ &= p^{\frac{k+m}{3}+1} \int_0^1 (1-t)^{k/3} A(p(1-t)) t^{m/3} B(pt) dt = p^{\frac{k+m}{3}+1} \text{Analytic}(p); \end{aligned} \quad (4.45)$$

Lemma 4.9. *If $\Phi(p) \in \mathcal{F}_0$, then $\int_0^p s\Phi(s)ds \in \mathcal{F}_0$ and $\|\int_0^p s\Phi(s)ds\| \leq \nu^2 \|\Phi\|$.*

Proof. For $\Phi = pA(p) + p^{4/3}B(p) + p^{5/3}C(p) \in \mathcal{F}_0$,

$$\begin{aligned} \int_0^p s\Phi(s)ds &= \int_0^p s\{sA(s) + s^{4/3}B(s) + s^{5/3}C(s)\}ds \\ &= pA_1(p) + p^{4/3}B_1(p) + p^{5/3}C_1(p); \end{aligned}$$

$$|A_1(p)| = \left| \frac{1}{p} \int_0^p s^2 A(s) ds \right| \leq |p^2| \|A\|_\infty \leq \nu^2 \|A\|_\infty;$$

similarly, we have $|B_1| \leq \nu^2 \|B\|_\infty$; $|C_1| \leq \nu^2 \|C\|_\infty$; so $\|\int_0^p s\Phi(s)ds\| \leq \nu^2 \|\Phi\|$. $\square$

Lemma 4.10. *Let $\Phi, \Psi \in \mathcal{F}_0$, then $\Phi * \Psi(p) \in \mathcal{F}_0$, and $\|\Phi * \Psi\| \leq \nu^2 \|\Phi\| \|\Psi\|$.*

Proof. For $\Psi = pA_1(p) + p^{4/3}B_1(p) + p^{5/3}C_1(p)$, $\Phi = pA(p) + p^{4/3}B(p) + p^{5/3}C(p)$, we have

$$\Phi * \Psi(p) = pA_2(p) + p^{4/3}B_2(p) + p^{5/3}C_2(p);$$

where

$$\begin{aligned} A_2(p) = & \frac{1}{p} \left\{ \int_0^p sA(s)(p-s)A_1(p-s)ds \right. \\ & \left. + \int_0^p s^{4/3}B(s)(p-s)^{5/3}C_1(p-s)ds + \int_0^p s^{5/3}C(s)(p-s)^{4/3}B_1(p-s)ds \right\}; \end{aligned}$$

$$\begin{aligned} B_2(p) = & \frac{1}{p^{4/3}} \left\{ \int_0^p sA(s)(p-s)^{4/3}B_1(p-s)ds \right. \\ & \left. + \int_0^p s^{4/3}B(s)(p-s)A_1(p-s)ds + \int_0^p s^{5/3}C(s)(p-s)^{5/3}C_1(p-s)ds \right\}; \end{aligned}$$

$$\begin{aligned} C_2(p) = & \frac{1}{p^{5/3}} \left\{ \int_0^p sA(s)(p-s)^{5/3}C_1(p-s)ds \right. \\ & \left. + \int_0^p s^{5/3}C(s)(p-s)A_1(p-s)ds + \int_0^p s^{4/3}B(s)(p-s)^{4/3}B_1(p-s)ds \right\}; \end{aligned}$$

Use (4.45), we have

$$|A_2(p)| \leq \nu^2 \|A\|_\infty \|A_1\|_\infty + \nu^3 \|B\|_\infty \|C_1\|_\infty + \nu^3 \|C\|_\infty \|B_1\|_\infty$$

$$|B_2(p)| \leq \nu^2 \|A\|_\infty \|B_1\|_\infty + \nu^2 \|B\|_\infty \|A_1\|_\infty + \nu^3 \|C\|_\infty \|C_1\|_\infty$$

$$|C_2(p)| \leq \nu^2 \|A\|_\infty \|C_1\|_\infty + \nu^2 \|C\|_\infty \|A_1\|_\infty + \nu^3 \|B\|_\infty \|B_1\|_\infty$$

So $\|\Phi * \Psi\| \leq \nu^2 \|\Phi\| \|\Psi\|$. □

Lemma 4.11. For $\Phi_1(p), \Phi_2(p) \in \mathcal{F}_0, k \geq 1$ we have:

$$\|\Phi_1^{*k} - \Phi_2^{*k}\| \leq k\nu^{2(k-1)}(\|\Phi_2\| + \|\Phi_2 - \Phi_1\|)^{k-1}\|\Phi_1 - \Phi_2\|;$$

Proof.

$$\begin{aligned} \|\Phi_1^{*k} - \Phi_2^{*k}\| &= \|((\Phi_1 - \Phi_2) + \Phi_2)^{*k} - \Phi_2^{*k}\| \\ &= \left\| \sum_{m=0}^{k-1} \binom{k}{m} (\Phi_1 - \Phi_2)^{* (k-m)} * \Phi_2^{*m} \right\| \\ &= \|k(\Phi_1 - \Phi_2) * \sum_{m=0}^{k-1} \binom{k-1}{m} (\Phi_1 - \Phi_2)^{* (k-1-m)} * \Phi_2^{*m}\| \\ &\leq k\nu^2 \|(\Phi_1 - \Phi_2)\| \left\| \sum_{m=0}^{k-1} \binom{k-1}{m} (\Phi_1 - \Phi_2)^{* (k-1-m)} * \Phi_2^{*m} \right\| \\ &\leq k\nu^2 \|(\Phi_1 - \Phi_2)\| \sum_{m=0}^{k-1} \binom{k-1}{m} \nu^2 \|(\Phi_1 - \Phi_2)^{* (k-1-m)}\| \|\Phi_2^{*m}\| \\ &\leq k\nu^2 \|(\Phi_1 - \Phi_2)\| \sum_{m=0}^{k-1} \binom{k-1}{m} \nu^2 \nu^{2(k-1-m-1)} \|(\Phi_1 - \Phi_2)\|^{k-1-m} \nu^{2(m-1)} \|\Phi_2\|^m \\ &\leq k\nu^{2(k-1)} \|(\Phi_1 - \Phi_2)\| \sum_{m=0}^{k-1} \binom{k-1}{m} \|(\Phi_1 - \Phi_2)\|^{k-1-m} \|\Phi_2\|^m \\ &\leq k\nu^{2(k-1)} (\|\Phi_2\| + \|\Phi_2 - \Phi_1\|)^{k-1} \|\Phi_1 - \Phi_2\|; \end{aligned}$$

□

Theorem 4.12. In $\mathcal{D} \cap \mathbf{D}_\nu, \Phi_0(p)$ can be written as

$$\Phi_0(p) = pA(p) + p^{4/3}B(p) + p^{5/3}C(p); \quad (4.46)$$

where $A(p), B(p), C(p)$ are analytic functions in $\mathbf{D}_\nu$.

Proof. Rewrite (4.32):

$$\Phi(p) = N_2(\Phi) := -F_0(p) - N(\Phi) + p^2\Phi + \frac{1}{3} \int_0^p s\Phi(s)ds; \quad (4.47)$$

Note $F_0 \in \mathcal{F}_0, \|p^2\Phi\| \leq \nu^2\|\Phi\|$. Now

$$\begin{aligned} p^{1/3} * \Phi &= \int_0^p (s-p)^{1/3} \{sA(s) + s^{4/3}B(s) + s^{5/3}C(s)\} ds \\ &= \int_0^p (s-p)^{1/3} s^{5/3} C(s) ds + \int_0^p (s-p)^{1/3} sA(s) ds + \int_0^p (s-p)^{1/3} s^{4/3} B(s) ds; \\ &= p^3 \int_0^1 (s-1)^{1/3} s^{5/3} C(ps) ds + p^{1+1/3} \int_0^1 (s-1)^{1/3} sA(ps) ds \\ &\quad + p^{1+5/3} \int_0^1 (s-1)^{1/3} s^{4/3} B(ps) ds; \end{aligned}$$

therefore $\|p^{1/3} * \Phi\| \leq \nu\|\Phi\|$. Use (4.34) and Lemma 4.10:

$$\begin{aligned} \|N(\Phi)\| &\leq \frac{4}{9}\nu^2\|\Phi\| + \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \sum_{k=0}^n \binom{n}{k} \left(\frac{2a}{3^{4/3}\Gamma(1/3)} \right)^{n-k} \nu^{n-k} \nu^{2k} \|\Phi^k\| \\ &\leq \frac{4}{9}\nu^2\|\Phi\| + \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \left(\frac{2a\nu}{3^{4/3}\Gamma(1/3)} + \nu^2\|\Phi\| \right)^n \quad (4.48) \end{aligned}$$

$$\|N_2(\Phi)\| \leq \|F_0\| + \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \left(\frac{2a\nu}{3^{4/3}\Gamma(1/3)} + \nu^2\|\Phi\| \right)^n + \nu^2\|\Phi\|;$$

So a ball of radius $2\|F_0\|$ mapped into itself for sufficiently small ν . Now use Lemma

4.11

$$\begin{aligned}
\|N(\Phi_1) - N(\Phi_2)\| &\leq \frac{4}{9}\nu^2\|(\Phi_1(s) - \Phi_2(s))\| \\
&+ \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \sum_{k=1}^n \binom{n}{k} \left(\frac{2a}{3^{4/3}\Gamma(4/3)} \right)^{n-k} \|\Phi_1^{*k} - \Phi_2^{*k}\| \nu^{n-k} \\
&\leq \left\{ \frac{1}{2} \sum_{n=2}^{\infty} |(-1)^n(n+1)| \sum_{k=1}^n k \binom{n}{k} \left(\frac{2a}{3^{4/3}\Gamma(4/3)} \right)^{n-k} \nu^{n-k} \right. \\
&\quad \times \left. \left(\nu^2\|\Phi_2\| + \nu^2\|\Phi_2 - \Phi_1\| \right)^{k-1} \|\Phi_1 - \Phi_2\| \right\} + \frac{4}{9}\nu^2\|\Phi_1 - \Phi_2\| \\
&\leq O(\nu)\|\Phi_1 - \Phi_2\|;
\end{aligned}$$

So $N_2(\Phi)$ is a contraction mapping in space $\mathcal{F}_0$.therefore there is a solution in $\mathcal{F}_0$,by uniqueness,that solution must be $\Phi_0(p)$. $\square$

Lemma 4.13. *For any δ ,there is a constant $K_1 = K_1(\delta, |p|)$ so that for all k ,we have:*

$$\sup_{\mathcal{D} \cap \{|t| \leq |p|\}} |\Phi_0^{*k}(t)| \leq K_1 \delta^k; \quad (4.49)$$

Proof. We use induction,the lemma is true for $k = 1$ since $\Phi_0 \in \mathcal{A}$. Now

$$\begin{aligned}
|\Phi_0^{*(k+1)}| &\leq \left| \int_0^p \Phi_0(p-s) \Phi_0^{*k}(s) ds \right| \\
&\leq \sup |\Phi_0^{*k}(p)| e^{b|p|} \int_0^{|p|} e^{-bs} |\Phi_0(s)| ds \leq K_1 \delta^k e^{b|p|} \|\Phi_0\|_b; \quad (4.50)
\end{aligned}$$

Using the fact $\|\Phi_0\|_b \rightarrow 0$ as $b \rightarrow \infty$,we obtain the lemma. $\square$

Let $\Phi_0(p)$ be the unique solution of (4.32) as in Lemma 4.6.for small ν ,define

$$H(p) = \begin{cases} \Phi_0(p) & \text{for } p \in \mathcal{D}, |p| < 1 - \nu, \\ 0 & \text{otherwise.} \end{cases} \quad (4.51)$$

Let

$$h(1-p) := \Phi_0(p) - H(p) = \begin{cases} 0 & \text{for } p \in \mathcal{D}, |p| < 1-\nu. \\ \Phi_0(p) & \text{otherwise.} \end{cases} \quad (4.52)$$

Lemma 4.14. *For small $\nu > 0$, $H^{\star k}(p)$ extends to an analytic function in the disk $\mathcal{D}_\nu + 1$. Further more, for any δ there is a ν and a constant $K_2 = K_2(\delta, \nu)$ so that*

$$\sup_{p \in \mathcal{D}_\nu} |H^{\star k}(p)| \leq K_1 \delta^k. \quad (4.53)$$

Proof. We use induction. For $k = 1$, by definition, $H(p)$ is analytic at $p = 1$. Suppose that $H^{\star k}(p)$ is analytic at $p = 1$; taking first real $p \in (1-\nu, 1+\nu)$,

$$\begin{aligned} H^{\star(k+1)}(p) &= \int_0^{1-\nu} H(s) H^{\star k}(p-s) ds \\ &= \int_0^{p-(1-\nu)} H(s) H^{\star k}(p-s) ds + \int_{p-(1-\nu)}^{1-\nu} H(p-t) H^{\star k}(t) dt; \end{aligned} \quad (4.54)$$

If $s \in (0, p-(1-\nu))$, then $p-s \in (1-\nu, 1+\nu)$, so $H^{\star k}(p-s)$ is analytic in $(1-\nu, 1+\nu)$; since $p-(1-\nu) \in (0, 2\nu)$, $H(p-(1-\nu))$ is analytic in $(1-\nu, 1+\nu)$; Therefore function $\int_0^{p-(1-\nu)} H(s) H^{\star k}(p-s) ds$ is analytic in $(1-\nu, 1+\nu)$; If $t \in (p-(1-\nu), 1-\nu)$, then $p-t \in (0, 1-\nu)$, so $H(p-t)$ is analytic in $(1-\nu, 1+\nu)$; since $p-(1-\nu) \in (0, 2\nu)$, $H^{\star k}(p-(1-\nu))$ is analytic in $(1-\nu, 1+\nu)$; Therefore function $\int_{p-(1-\nu)}^{1-\nu} H(p-t) H^{\star k}(t) dt$ is analytic in $(1-\nu, 1+\nu)$. For p is complex. Let $p = |p|e^{i\theta}$, then

$$H^{\star(k+1)} = \int_0^{(1-\nu)e^{i\theta}} H(s) H^{\star k}(p-s) ds = e^{i\theta} \int_0^{(1-\nu)} H(ss^{i\theta}) H^{\star k}(|p|-s)e^{i\theta} ds$$

splitting above integral as in (4.54), the same argument follows. To prove the estimate (4.53), we use induction. Fix δ , let

$$\sigma < \delta; M_1 = \max_{|p| < 1/2} |H(p)|; M_2 := \max_{[0, 1-\nu]} |H(p)|; \nu < \frac{\delta}{4M_1};$$

using induction hypothesis to estimate the first integral in (4.54) and using (4.49) with σ in place of δ to estimate the second:

$$\sup |H^{*(k+1)}| \leq (2\nu)M_1K_1\delta^k + M_2K_2\sigma^k \leq K_2\delta^{k+1}; \quad (4.55)$$

□

Let $z := 1 - p$, $|z| < \nu$, plug $\Phi_0(p) = H(p) + h(1 - p)$ in equation (4.32), we have:

$$\begin{aligned} zh(z) + \frac{1}{6} \int_{\nu}^z h(s)ds &= F_1(z) - \frac{5z}{18(z-2)} \int_{\nu}^z h(s)ds \\ &\quad + \frac{1}{9(z-2)} \int_{\nu}^z sh(s)ds + \frac{N(H+h)}{(z-2)}; \end{aligned} \quad (4.56)$$

where

$$F_1(z) = \frac{1}{(z-2)} \left[F_0(1-z) - \frac{1}{3} \int_0^{1-\nu} sH(s)ds - \frac{4}{9} \int_0^{1-\nu} (1-z-s)H(s)ds \right]; \quad (4.57)$$

Lemma 4.15. *The equation (4.56) can be written as*

$$\begin{aligned} zh(z) + \frac{1}{6} \int_{\nu}^z h(s)ds &= F_2(z) - \frac{5z}{18(z-2)} \int_{\nu}^z h(s)ds + \frac{1}{9(z-2)} \int_{\nu}^z sh(s)ds \\ &\quad + \frac{1}{(z-2)} \int_{\nu}^z (h(s)D(s-z))ds; \end{aligned} \quad (4.58)$$

where

$$\begin{aligned}
D(z) &= -\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \sum_{k=1}^n k \binom{n}{k} \left(-\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-k} H^{*(k-1)} * (z^{1/3})^{*(n-k)} \\
&= -b_2 \left(\left(-\frac{2a}{3^{4/3} \Gamma(4/3)} \right) z^{1/3} + H^{*2} \right) \\
&\quad - \frac{1}{2} \sum_{n=3}^{\infty} (-1)^n (n+1) \left\{ n \left(-\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-1} (z^{1/3})^{*(n-1)} \right. \\
&\quad \left. + \sum_{k=2}^{n-1} k \binom{n}{k} \left(-\frac{2a}{3^{4/3} \Gamma(4/3)} \right)^{n-k} H^{*(k-1)} * (z^{1/3})^{*(n-k)} + n H^{*n} \right\} \quad (4.59)
\end{aligned}$$

$$F_2(z) = F_1(z) + \frac{N(H)(1-z)}{z-2}; \quad (4.60)$$

$F_2(z)$ is analytic at $z = 0$.

Proof. Using (4.34).and note that $h * h(1-z) = 0$,for $\nu < \frac{1}{2}$:

$$N(H+h) = N(H) + \int_{\nu}^z h(s) D(s-z) ds; \quad (4.61)$$

since Lemma 4.14 guarantees the uniform convergence of $N(H)(1-z)$:because of $H(p) = \Phi_0(p) \in \mathcal{F}_0$,similar to (4.48),we can see the series in (4.59) converges in $\mathcal{F}_0$. $\square$

Lemma 4.16. In $\mathcal{D} \cap \mathbf{D}_{\nu}$, $D(z)$ can be written as

$$D(z) = zE_1(z) + z^{1/3}E_2(z) + z^{5/3}E_3(z); \quad (4.62)$$

where $E_k(z)$, $k = 1, 2, 3$ are analytic in $\mathbf{D}_{\nu}$.

Proof. By (4.51).and Theorem 4.12, $H(z) = \Phi_0(z)$, $|z| < 1 - \nu$, $H(p) \in \mathcal{F}_0$, $H^{*k} * (z^{1/3})^{*(n-k)} \in \mathcal{F}_0$,so the lemma follows from lemma 4.10 and equation (4.59). $\square$

Lemma 4.17. For $\gamma \geq 0, \beta > -1$, let $B(z), E(z)$ are analytic functions in D_ν , consider the following function:

$$\begin{aligned} g(z) &= \int_\nu^z s^\gamma (s-z)^\beta B(s) E(s-z) ds \\ &= -z^{\gamma+\beta+1} \int_1^{\frac{z}{z}} t^\gamma (t-1)^\beta B(tz) E(z(t-1)) dt; \end{aligned} \quad (4.63)$$

Then function $g(z)$ can be written as

$$g(z) = z^{\gamma+\beta+1} S(z) + U(z); \quad (4.64)$$

where $S(z)$ and $U(z)$ are analytic function in D_ν , and

$$|S(z)| \leq C_1 \|B\|_\infty \|E\|_\infty, |U(z)| \leq C_2 \nu^{\gamma+\beta+1} \|B\|_\infty \|E\|_\infty; \quad (4.65)$$

Proof. It is easy to see that $g(z)$ is analytic in $D_\nu - \{z \leq 0\}$. On t -plane, we use cut $\{t < 0\}, -\pi < \arg t < \pi$ for function t^γ and cut $\{t < 1\}, -\pi < \arg(t-1) < \pi$ for function $(t-1)^\beta$. For $|z| < \nu, |\frac{z}{z}| > 1$, choose the path in integral of (4.63) to be the straight line $\mathcal{P}(1, \frac{z}{z})$ connecting 1 to $\frac{z}{z}$. When z moves around anti-clockwise to $ze^{2i\pi}, \frac{z}{z}$ moves around clockwise along path $\mathcal{P}(\frac{z}{z}, \frac{\nu}{ze^{2i\pi}})$; so

$$\begin{aligned} g(ze^{2i\pi}) &= -e^{2i\pi(\gamma+\beta)} z^{\gamma+\beta+1} \left\{ \int_{\mathcal{P}(1, \frac{z}{z})} t^\gamma (t-1)^\beta B(tz) E(z(t-1)) dt \right. \\ &\quad \left. + \int_{\mathcal{P}(\frac{z}{z}, \frac{\nu}{ze^{2i\pi}})} t^\gamma (t-1)^\beta B(tz) E(z(t-1)) dt \right\}; \end{aligned} \quad (4.66)$$

Now change the contour $\mathcal{P}(\frac{z}{z}, \frac{\nu}{ze^{2i\pi}})$ to

$$-\mathcal{P}(1, \frac{\nu}{z}) \cup \mathcal{P}(1, \frac{\nu}{ze^{2i\pi}}) \cup \mathcal{P}_-(1, 0) \cup \mathcal{P}_+(0, 1);$$

On $\mathcal{P}_-(1, 0)$, $\arg t = 0, \arg(t-1) = -\pi$, while on $\mathcal{P}_+(0, 1)$, $\arg t = -2\pi, \arg(t-1) = -\pi$. See Figure 4.1.

Figure 4.1: The path of integration in Lemma 4.17.

By Cauchy integral formula:

$$\begin{aligned}
& \int_{\mathcal{P}(1, \frac{e}{z}, \frac{e}{ze^{2i\pi}})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt = \int_{-\mathcal{P}(1, \frac{e}{z})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& \quad + \int_{\mathcal{P}(1, \frac{e}{ze^{2i\pi}})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& \quad - \int_{\mathcal{P}_-(1, 0)} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt + \int_{\mathcal{P}_+(1, 0)} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& = - \int_{\mathcal{P}(1, \frac{e}{z})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt + e^{-2i\pi(\gamma+J)} \int_{\mathcal{P}(1, \frac{e}{z})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& \quad + e^{-Ji\pi} \int_1^0 t^\gamma (t-1)^J B(tz) E(z(t-1)) dt + e^{-Ji\pi-2i\pi\gamma} \int_0^1 t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& = - \int_{\mathcal{P}(1, \frac{e}{z})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt + e^{-2i\pi(\gamma+J)} \int_{\mathcal{P}(1, \frac{e}{z})} t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \\
& \quad + e^{-Ji\pi} (e^{-2i\pi\gamma} - 1) \int_0^1 t^\gamma (t-1)^J B(tz) E(z(t-1)) dt; \quad (4.67)
\end{aligned}$$

By (4.66), (4.67):

$$g(ze^{2i\pi}) = g(z) + z^{\gamma+J+1} \left\{ e^{2i\pi\gamma+Ji\pi} (1 - e^{-2i\pi\gamma}) \int_0^1 t^\gamma (t-1)^J B(tz) E(z(t-1)) dt \right\}; \quad (4.68)$$

Let

$$S(z) = \frac{e^{2i\pi\gamma+Ji\pi} (1 - e^{-2i\pi\gamma})}{e^{2i\pi(\gamma+J)} - 1} \int_0^1 t^\gamma (t-1)^J B(tz) E(z(t-1)) dt; \quad (4.69)$$

then $S(z)$ is analytic in $\mathbf{D}_\nu$, and $|S(z)| \leq C|B|_\infty|E|_\infty$. Let

$$U(z) = g(z) - z^{\gamma+J+1} S(z); \quad (4.70)$$

then using (4.68), (4.69) and (4.70):

$$\begin{aligned}
U(ze^{2i\pi}) - U(z) &= (g(ze^{2i\pi}) - g(z)) - (e^{2i\pi(\gamma+J)} z^{\gamma+J+1} S(z) - z^{\gamma+J+1} S(z)) \\
&= (g(ze^{2i\pi}) - g(z)) - (e^{2i\pi(\gamma+J)} - 1) z^{\gamma+J+1} S(z) = 0 \quad (4.71)
\end{aligned}$$

So $U(z)$ is single valued, therefore $U(z)$ is analytic in D_ν , and

$$\begin{aligned}
 |U(z)| &= |g(z) - z^{\gamma+J+1}S(z)| \leq |g(z)| + |z^{\gamma+J+1}S(z)| \\
 &\leq \left| \int_\nu^z s^\gamma (s-z)^J B(s) E(s-z) ds \right| + \nu^{\gamma+J+1} |S(z)| \\
 &\leq \left| \int_\nu^z s^\gamma (s-z)^J ds \right| |B|_\infty |E|_\infty + C \nu^{\gamma+J+1} |B|_\infty |E|_\infty \\
 &\leq C \nu^{\gamma+J+1} |B|_\infty |E|_\infty; \quad (4.72)
 \end{aligned}$$

□

Comment 4.3: If γ is integer, then $S(z) \equiv 0$, so $g(z) = U(z)$ is analytic in D_ν .

Definition 4.18.

$$Q(z) = \int_\nu^z dt \int_\nu^t h(s) ds; \quad (4.73)$$

then $h(z) = Q''(z)$, and (4.58) becomes:

$$\begin{aligned}
 zQ'(z)'' - \frac{1}{6}Q'(z) &= F_2(z) - \frac{5z}{18(z-2)}Q'(z) + \frac{1}{9(z-2)}(zQ'(z) - Q(z)) \\
 &\quad - \frac{1}{z-2} \int_\nu^z D'(s-z)Q'(s) ds \quad (4.74)
 \end{aligned}$$

Integrate the above equation from ν to z :

$$\begin{aligned}
 zQ'(z) - \frac{5}{6}Q(z) &= F_3(z) - \frac{z}{6(z-2)}Q(z) - \frac{1}{3} \int_\nu^z \frac{Q(t)}{(t-2)^2} dt \\
 &\quad - \frac{1}{9} \int_\nu^z \frac{Q(t)}{(t-2)} dt - \frac{1}{z-2} \int_\nu^z D'(s-z)Q(s) ds \\
 &\quad - \int_\nu^z \frac{dt}{(t-2)^2} \int_\nu^t D'(s-t)Q(s) ds; \quad (4.75)
 \end{aligned}$$

where

$$F_3(z) = \int_{\nu}^z F_2(t)dt; \quad (4.76)$$

Rewrite the above equation:

$$\begin{aligned} [z^{-5/6}Q(z)]' &= z^{-11/6} \left\{ F_3(z) - \frac{z}{6(z-2)}Q(z) - \frac{1}{3} \int_{\nu}^z \frac{Q(t)}{(t-2)^2} dt \right. \\ &\quad - \frac{1}{9} \int_{\nu}^z \frac{Q(t)}{(t-2)} dt - \frac{1}{z-2} \int_{\nu}^z D'(s-z)Q(s)ds \\ &\quad \left. - \int_{\nu}^z \frac{dt}{(t-2)^2} \int_{\nu}^t D'(s-t)Q(s)ds \right\}; \quad (4.77) \end{aligned}$$

so

$$\begin{aligned} Q(z) = N_3(Q) &:= z^{5/6} \int_{\nu}^z t^{-11/6} \left\{ F_3(t) - \frac{t}{6(t-2)}Q(t) - \frac{1}{3} \int_{\nu}^t \frac{Q(s)}{(s-2)^2} ds \right. \\ &\quad - \frac{1}{9} \int_{\nu}^t \frac{Q(s)}{(s-2)} ds - \frac{1}{t-2} \int_{\nu}^t D'(s-t)Q(s)ds \\ &\quad \left. - \int_{\nu}^t \frac{d\omega}{(\omega-2)^2} \int_{\nu}^{\omega} D'(s-\omega)Q(s)ds \right\} dt; \quad (4.78) \end{aligned}$$

Consider the space of functions:

$$\begin{aligned} \mathcal{F} := \{Q \text{ analytic in } \mathbf{D}_{\nu} \cap (\mathcal{D}-1) : Q(z) &= z^{5/6}B_1(z) + z^{7/6}B_2(z) \\ &+ z^{3/2}B_3(z) + B_4(z)\}; \quad (4.79) \end{aligned}$$

where $B_k(z)$ are analytic in $\mathbf{D}_{\nu}$. The norm

$$\|Q\| = \sup_{z \in \mathbf{D}_{\nu}} \{|B_1(z)|, \nu^{1/2}|B_2(z)|, \nu^{2/3}|B_3(z)|, \nu^{-2/3}|B_4(z)|\}, \quad (4.80)$$

makes $\mathcal{F}$ a Banach space. From Lemma 14, $D'(z)$ can be written as:

$$D'(z) = z^{-2/3}E_1(z) + z^{2/3}E_2(z) + E_3(z); \quad (4.81)$$

where $E_k(z)$ are analytic in $\mathbf{D}_\nu$. For $A(z)$ analytic in $\mathbf{D}_\nu$, the following identity is useful:

$$\int_\nu^z A(s)s^\gamma ds = \int_\nu^0 A(s)s^\gamma ds + z^{\gamma+1} \int_0^1 A(zt)t^\gamma dt = \text{Constant} + z^{\gamma+1} \text{Analytic}(z); \quad (4.82)$$

Lemma 4.19. For $Q \in \mathcal{F}$, $A(z)$ analytic in $\mathbf{D}_\nu$, let

$$Q_1(z) = z^{5/6} \int_\nu^z t^{-11/6} (tA(t))Q(t)dt$$

then $Q_1 \in \mathcal{F}$, $\|Q_1\| \leq C\nu^{5/6}\|A\|_\infty\|Q\|$.

Proof.

$$\begin{aligned} Q_1(z) &= z^{5/6} \int_\nu^z t^{-11/6} (tA(t)) (t^{5/6}B_1(t) + t^{7/6}B_2(t) + t^{3/2}B_3(t) + B_4(t)) dt \\ &= z^{5/6} \int_\nu^z A(t) (B_1(t) + t^{1/3}B_2(t) + t^{2/3}B_3(t) + t^{-5/6}B_4(t)) dt \\ &= z^{5/6} \int_\nu^0 A(t) (B_1(t) + t^{1/3}B_2(t) + t^{2/3}B_3(t) + t^{-5/6}B_4(t)) dt \\ &\quad + z^{1+5/6} \int_0^1 A(tz)B_1(tz)dt + z^{1+7/6} \int_0^1 t^{1/3}A(zt)B_2(zt)dt \\ &\quad + z^{1+3/2} \int_0^1 t^{2/3}A(zt)B_3(zt)dt + z \int_0^1 t^{-5/6}A(zt)B_4(zt)dt \end{aligned}$$

So

$$\begin{aligned}
\|Q_1(z)\| &\leq \left| \int_{\nu}^0 A(t) (B_1(t) + t^{1/3} B_2(t) + t^{2/3} B_3(t) + t^{-5/6} B_4(t)) dt \right| \\
&\quad + \nu \sup \left| \int_0^1 A(tz) B_1(tz) dt \right| + \nu^{1+1/2} \sup \left| \int_0^1 t^{1/3} A(zt) B_2(zt) dt \right| \\
&\quad + \nu^{1+2/3} \sup \left| \int_0^1 t^{2/3} A(zt) B_3(zt) dt \right| + \nu^{1-2/3} \sup \left| \int_0^1 t^{-5/6} A(zt) B_4(zt) dt \right| \\
&\leq C \left\{ [\nu |B_1|_{\infty} + \nu^{1/3} |B_2|_{\infty} + \nu^{5/3} |B_3|_{\infty} + \nu^{1/6} |B_4|_{\infty}] \right. \\
&\quad \left. + \nu |B_1|_{\infty} + \nu^{1+1/2} |B_2|_{\infty} + \nu^{1+2/3} |B_3|_{\infty} + \nu^{1-2/3} |B_4|_{\infty} \right\} \\
&\leq C \nu^{5/6} \{ |B_1|_{\infty} + \nu^{1/2} |B_2|_{\infty} + \nu^{2/3} |B_3|_{\infty} + \nu^{-2/3} |B_4|_{\infty} \} \\
&\leq C \nu^{5/6} \|Q\|
\end{aligned}$$

□

Lemma 4.20. For $Q \in \mathcal{F}$, $A(z)$ analytic in D_{ν} , let

$$Q_2(z) = z^{5/6} \int_{\nu}^z t^{-11/6} \int_{\nu}^t A(s) Q(s) ds$$

then $Q_2 \in \mathcal{F}$, $\|Q_2\| \leq \nu^{5/6} \|Q\|$.

Proof.

$$\begin{aligned}
Q_2(z) &= z^{5/6} \int_{\nu}^z t^{-11/6} \int_{\nu}^t A(s) Q(s) ds = z^{5/6} \int_{\nu}^z A(s) Q(s) ds \int_s^z t^{-11/6} dt \\
&= -\frac{6}{5} z^{5/6} \int_{\nu}^z z^{-5/6} A(s) Q(s) ds - \frac{6}{5} z^{5/6} \int_{\nu}^z s^{-5/6} A(s) Q(s) ds \\
&=: \dot{Q}_2 + \tilde{Q}_2:
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_2 &= -\frac{6}{5} \int_{\nu}^z A(s)Q(s)ds \\
&= -\frac{6}{5} \int_{\nu}^z A(s)[s^{5/6}B_1(s) + s^{7/6}B_2(s) + s^{3/2}B_3(s) + B_4(s)]ds \\
&= -\frac{6}{5} \int_{\nu}^0 A(s)[s^{5/6}B_1(s) + s^{7/6}B_2(s) + s^{3/2}B_3(s) + B_4(s)]ds \\
&\quad - \frac{6}{5} \int_0^z A(s)[s^{5/6}B_1(s) + s^{7/6}B_2(s) + s^{3/2}B_3(s) + B_4(s)]ds \\
&= -\frac{6}{5} \int_{\nu}^0 A(s)[s^{5/6}B_1(s) + s^{7/6}B_2(s) + s^{3/2}B_3(s) + B_4(s)]ds \\
&\quad - \frac{6}{5} \left\{ z^{1+5/6} \int_0^1 A(zs)B_1(zs)ds + z^{1+7/6} \int_0^1 A(zs)B_2(zs)ds \right. \\
&\quad \left. + z^{1+3/2} \int_0^1 A(zs)B_3(zs)ds + z \int_0^1 B_4(zs)ds \right\}
\end{aligned}$$

So

$$\begin{aligned}
\|\tilde{Q}_2\| &\leq C \left\{ \nu^{-2/3} \sup \left| \int_{\nu}^0 A(s)[s^{5/6}B_1(s) + s^{7/6}B_2(s) + s^{3/2}B_3(s) + B_4(s)]ds \right| \right. \\
&\quad + \left[\nu^1 \sup \left| \int_0^1 A(zs)B_1(zs)ds \right| + \nu^{1+1/2} \sup \left| \int_0^1 A(zs)B_2(zs)ds \right| \right. \\
&\quad \left. + \nu^{1+2/3} \sup \left| \int_0^1 A(zs)B_3(zs)ds \right| + \nu^{1-2/3} \sup \left| \int_0^1 B_4(zs)ds \right| \right] \Big\} \\
&\leq C \left\{ \nu^{-2/3} \left[\nu^{1+5/6} \|B_1(s)\|_{\infty} + \nu^{1+7/6} \|B_2(s)\|_{\infty} + \nu^{1+3/2} \|B_3(s)\|_{\infty} + \nu \|B_4(s)\|_{\infty} \right] \right. \\
&\quad \left. + \left[\nu^1 \|B_1\|_{\infty} + \nu^{1+1/2} \|B_2\|_{\infty} + \nu^{1+2/3} \|B_3\|_{\infty} + \nu^{1-2/3} \|B_4\|_{\infty} \right] \right\} \\
&\leq C \nu \{ \|B_1\|_{\infty} + \nu^{1/2} \|B_2\|_{\infty} + \nu^{2/3} \|B_3\|_{\infty} + \nu^{-2/3} \|B_4\|_{\infty} \} \\
&\leq C \nu \|Q\|;
\end{aligned}$$

Note $\tilde{Q}_2 = \frac{6}{5}Q_1$, using the previous Lemma, we complete the proof. $\square$

For $Q = z^{5/6}B_1(z) + z^{7/6}B_2(z) + z^{3/2}B_3(z) + B_4(z) \in \mathcal{F}$, using (4.81):

$$\int_{\nu}^z D'(t-z)Q(t)dt = \sum_{j=1}^3 \sum_{l=1}^4 g_{j,l}(z); \quad (4.83)$$

where

$$g_{j,l}(z) = \int_{\nu}^z t^{\gamma_j}(t-z)^{\beta_j} E_j(t-z) B_l(t) dt, \quad j = 1, 2, 3; l = 1, 2, 3, 4. \quad (4.84)$$

where

$$\gamma_1 = \frac{5}{6}, \gamma_2 = \frac{7}{6}, \gamma_3 = \frac{3}{2}, \gamma_4 = 0; \beta_1 = \frac{-2}{3}, \beta_2 = \frac{2}{3}, \beta_3 = 0; \quad (4.85)$$

The following lemma follows immediately from Lemma 4.17:

Lemma 4.21. *the function $g_{j,l}$ can be written as:*

$$g_{j,l} = z^{\gamma_j + \beta_j + 1} S_{j,l}(z) + U_{j,l}(z); \quad (4.86)$$

where $S_{j,l}, U_{j,l}$ are analytic in $\mathbf{D}_{\nu}$, and

$$\|S_{j,l}\|_{\infty} \leq C \|E_j\|_{\infty} \|B_l\|_{\infty}, \|U_{j,l}\|_{\infty} \leq C \nu^{\gamma_j + \beta_j + 1} \|E_j\|_{\infty} \|B_l\|_{\infty}. \quad (4.87)$$

Lemma 4.22. *For $Q \in \mathcal{F}$, $A(t)$ is analytic in $\mathbf{D}_{\nu}$, let*

$$Q_{j,l}(z) = z^{5/6} \int_{\nu}^z t^{-11/6} (A(t) t^{\gamma_j + \beta_j + 1} S_{j,l}(t)) dt,$$

$l = 1, 2, 3$; then $Q_{j,l} \in \mathcal{F}$, and $\|Q_{j,l}\| \leq \nu^{1/6} \|Q\|$.

Proof. Illustratively, we only give the proof for $Q_{1,1}$ and $Q_{1,2}$:

$$\begin{aligned} Q_{1,1}(z) &= z^{5/6} \int_{\nu}^z t^{-11/6} \left(A(t) t^{\frac{5}{6} - \frac{2}{3} + 1} S_{1,1}(t) \right) dt \\ &= z^{5/6} \int_{\nu}^0 t^{-2/3} A(t) S_{1,1}(t) dt + z^{5/6} \int_0^z t^{-2/3} A(t) S_{1,1}(t) dt \\ &= z^{5/6} \int_{\nu}^0 t^{-2/3} A(t) S_{1,1}(t) dt + z^{7/6} \int_0^1 t^{-2/3} A(zt) S_{1,1}(zt) dt \end{aligned}$$

$$\begin{aligned}
\|Q_{1,1}(z)\| &\leq \left| \int_{\nu}^0 t^{-2/3} A(t) S_{1,1}(t) dt \right| + \nu^{1/2} \left| \int_0^1 t^{-2/3} A(zt) S_{1,1}(zt) dt \right| \\
&\leq \nu^{1/3} |A(t) S_{1,1}(t)|_{\infty} + \nu^{1/2} |A(t) S_{1,1}|_{\infty} \\
&\leq C \nu^{1/3} |B_1(t)|_{\infty} + C \nu^{1/2} |B_1|_{\infty} \\
&\leq C \nu^{1/3} \|Q\|;
\end{aligned}$$

$$\begin{aligned}
Q_{1,2}(z) &= z^{5/6} \int_{\nu}^z t^{-11/6} \left(A(t) t^{\frac{7}{6}-\frac{1}{3}+1} \right) S_{1,2}(t) dt \\
&= z^{5/6} \int_{\nu}^0 t^{-1/3} A(t) S_{1,2}(t) dt + z^{5/6} \int_0^z t^{-1/3} A(t) S_{1,2}(t) dt \\
&= z^{5/6} \int_{\nu}^0 t^{-1/3} A(t) S_{1,2}(t) dt + z^{3/2} \int_0^1 t^{-1/3} A(zt) S_{1,2}(zt) dt \\
\|Q_{1,2}(z)\| &\leq \left| \int_{\nu}^0 t^{-1/3} A(t) S_{1,2}(t) dt \right| + \nu^{2/3} \left| \int_0^1 t^{-1/3} A(zt) S_{1,2}(zt) dt \right| \\
&\leq \nu^{2/3} |A(t) S_{1,2}(t)|_{\infty} + \nu^{2/3} |A(t) S_{1,2}|_{\infty} \\
&\leq C \nu^{2/3} |B_2(t)|_{\infty} \\
&\leq C \nu^{1/6} \|Q\|;
\end{aligned}$$

□

Lemma 4.23. For $Q \in \mathcal{F} \mathcal{U}_{j,l}$ as in (4.86), $A(t)$ analytic in $\mathbf{D}_{\nu}$, let

$$\dot{Q}_{j,l}(z) = z^{5/6} \int_{\nu}^z t^{-11/6} A(t) \mathcal{U}_{j,l}(t) dt.$$

then $Q_{j,l} \in \mathcal{F}$, and $\|\dot{Q}_{j,l}\| \leq \nu^{1/6} \|Q\|$.

Proof. Illustratively, we only give the proof for $\dot{Q}_{1,1}$. Let

$$A(z) \mathcal{U}_{1,1}(z) = \sum_{k=0}^{\infty} d_k z^k, \quad V_n(z) = \sum_{k=0}^n d_k z^k.$$

then

$$\begin{aligned}
\hat{Q}_{1,4}(z) &= z^{5/6} \int_{\nu}^z t^{-11/6} \sum_{k=0}^{\infty} d_k t^k dt \\
&= -z^{5/6} \nu^{-5/6} \sum_{k=0}^{\infty} \frac{d_k}{k - \frac{5}{6}} \nu^k + \sum_{k=0}^{\infty} \frac{d_k}{k - \frac{5}{6}} z^k \\
&=: -z^{5/6} \tilde{\mathcal{U}}(\nu) + \tilde{\mathcal{U}}(z);
\end{aligned}$$

where

$$\tilde{\mathcal{U}}(z) = \sum_{k=0}^{\infty} \frac{d_k}{k - \frac{5}{6}} z^k; \quad (4.88)$$

Use $d_k z^k = \dot{V}_k(z) - \dot{V}_{k-1}(z)$:

$$\begin{aligned}
\tilde{\mathcal{U}}(z) &= \sum_{k=0}^{\infty} \frac{d_k}{k - \frac{5}{6}} z^k \\
&= \sum_{k=0}^{\infty} \frac{\dot{V}_k(z) - \dot{V}_{k-1}(z)}{k - \frac{5}{6}} \\
&= \sum_{k=0}^{\infty} \left(\frac{\dot{V}_k(z)}{k - \frac{5}{6}} - \frac{\dot{V}_{k-1}(z)}{(k-1) - \frac{5}{6}} \right) \\
&\quad + \sum_{k=0}^{\infty} \dot{V}_{k-1}(z) \left(\frac{1}{((k-1) - \frac{5}{6})} - \frac{1}{k - \frac{5}{6}} \right) \\
&= \sum_{k=0}^{\infty} \dot{V}_{k-1}(z) \left(\frac{1}{((k-1) - \frac{5}{6}) (k - \frac{5}{6})} \right)
\end{aligned}$$

so

$$\begin{aligned}
\|\tilde{\mathcal{U}}(z)\|_{\infty} &\leq \sup |\dot{V}_k(z)| \sum_{k=0}^{\infty} \left(\frac{1}{|((k-1) - \frac{5}{6}) (k - \frac{5}{6})|} \right) \\
&\leq C |\mathcal{U}_{1,4}|_{\infty};
\end{aligned}$$

So

$$\begin{aligned}
\|\dot{Q}_{1,4}\| &\leq \nu^{-5/6}\|\tilde{U}\|_{\infty} + \nu^{-2/3}\|\tilde{U}\|_{\infty} \\
&\leq C\nu^{-5/6}\|U_{1,4}\|_{\infty} + \nu^{-2/3}\|U_{1,4}\|_{\infty} \\
&\leq C\nu^{-5/6}(\nu^{1/3}\|B_4\|_{\infty}) + \nu^{-2/3}(\nu^{1/3}\|B_4\|_{\infty}) \\
&\leq C\nu^{1/6}\|Q\|;
\end{aligned}$$

□

Lemma 4.24. *Let*

$$Q_3(z) = z^{5/6} \int_{\nu}^z t^{-11/5} A(t) \int_{\nu}^t D'(s-t) Q(s) ds dt;$$

then $Q_3(z) \in \mathcal{F}$, $\|Q_3\| \leq C\nu^{1/6}\|Q\|$;

Proof. The lemma follows from (4.83) and Lemmas 4.20-4.22. □

Lemma 4.25. *Let*

$$Q_4(z) = z^{5/6} \int_{\nu}^z t^{-11/5} \int_{\nu}^t A(\omega) \int_{\nu}^{\omega} D'(s-\omega) Q(s) ds d\omega dt;$$

then $Q_4(z) \in \mathcal{F}$, $\|Q_4\| \leq C\nu^{1/6}\|Q\|$.

Proof. Let $\bar{Q}(\omega) = A(\omega) \int_{\nu}^{\omega} D'(s-\omega) Q(s) ds$, then

$$\begin{aligned}
Q_4(z) &= z^{5/6} \int_{\nu}^z t^{-11/5} dt \int_{\nu}^t \bar{Q}(s) ds \\
&= z^{5/6} \int_{\nu}^z \bar{Q}(s) ds \int_s^z t^{-11/5} dt \\
&= -\frac{6}{5} z^{5/6} \int_{\nu}^z s^{-5/6} \bar{Q}(s) ds - \frac{6}{5} \int_{\nu}^z \bar{Q}(s) ds
\end{aligned}$$

the norm of the first term can be bounded by $\nu^{5/6}\|\bar{Q}\|$ as in lemma 4.18, while the norm of the second term can be bounded by $\nu\|\bar{Q}\|$; then the lemma follows from the fact $\|\bar{Q}\| \leq \nu^{-1/2}\|Q\|$. □

Now using (4.78), we have $N_3(Q) = Q_1 + Q_2 + Q_3 + Q_4$, by the above Lemmas, it is easy to see that $N_3(Q)$ is a linear operator on $\mathcal{F}$ with image in $\mathcal{F}$, the norm of the operator is $O(\nu^{1/6})$, therefore it is a contraction, so we have the first part of the following Theorem:

Theorem 4.26. *There exists a unique solution*

$$Q(z) = z^{5/6}B_1(z) + z^{7/6}B_2(z) + z^{3/2}B_3(z) + B_4(z) \in \mathcal{F}$$

for equation (4.74). Furthermore if $B_1(0) = 0$, then

$$B_1(z) \equiv 0, B_2(z) \equiv 0, B_3(z) \equiv 0$$

.

Proof. To prove the second part, let

$$B_l(z) = \sum_{k=0}^{\infty} b_{l,k} z^k; \quad (4.89)$$

Plugging the decomposition of $Q(z)$ in equation (4.74), using (4.81) and Lemma 4.21 and equating the coefficients of powers $z^{5/6+k}$, $z^{7/6+k}$, $z^{3/2+k}$, we have the following recursive equations:

$$k(k + \frac{5}{6})b_{1,k} = \sum_{n \leq k-1} c_{1,1,n,k} b_{1,n} + \sum_{n \leq k-2} c_{1,2,n,k} b_{2,n} + \sum_{n \leq k-1} c_{1,3,n,k} b_{3,n}; \quad (4.90)$$

$$(k + \frac{1}{3})(k + \frac{3}{2})b_{2,k} = \sum_{n \leq k} c_{2,1,n,k} b_{1,n} + \sum_{n \leq k-1} c_{2,2,n,k} b_{2,n} + \sum_{n \leq k-2} c_{2,3,n,k} b_{3,n}; \quad (4.91)$$

$$(k + \frac{2}{3})(k + \frac{7}{6})b_{3,k} = \sum_{n \leq k-1} c_{3,1,n,k} b_{1,n} + \sum_{n \leq k} c_{3,2,n,k} b_{2,n} + \sum_{n \leq k-1} c_{3,3,n,k} b_{3,n}; \quad (4.92)$$

where $c_{l,j,n,k}$ are constants depending only on n, k and coefficients of $E_j(z)$, and we assume $b_{l,-k} = 0, k > 0$. By the above recursive relation, we can see that $b_{1,0} = B_1(0) = 0$ implies $b_{l,k} = 0, l = 1, 2, 3, k \geq 0$. $\square$

Theorem 4.27. *In the neighborhood of $p = 1$, the function $\Phi_0(p)$ in Lemma 4.6 can be written as :*

$$\Phi_0(p) = S(a)(1-p)^{-7/6}A_1(p) + S(a)(1-p)^{-5/6}A_2(p) + S(a)(1-p)^{-1/2}A_3(p) + A_4(p); \quad (4.93)$$

where $A_k(p)$ are analytic in $1 + \mathbf{D}_\nu$ and $A_1(1) = 1$.

Proof. Let $Q(z)$ be as in Theorem 4.26, use $h(z) = Q''(z), p = 1 - z$ and let $S(a) = B_1(0)$, in view of the uniqueness of $\Phi_0(p)$, theorem follows from the previous one. $\square$

Lemma 4.28. *If $\Phi_0(p)$ is analytic in $\mathcal{D} \cup (0, k + \nu), k \geq 1$, then $\Phi_0(p)$ is analytic in $\mathcal{D} \cup (0, k + 1 + \nu)$.*

Proof. Let

$$H_k = \begin{cases} \Phi_0(p) & |p| \leq k + \nu; \\ 0 & |p| > k + \nu; \end{cases} \quad (4.94)$$

and $h(p) := \Phi_0(p) - H_k(p)$ for $k + \nu < |p| < k + 1 + \nu$, then equation (4.32) reads:

$$(p^2 - 1)h(p) + \frac{1}{3} \int_{k+\nu}^p sh(s)ds = F_4(p) + N(H_k + h); \quad (4.95)$$

where

$$F_4(p) = F_0(p) + H_k - p^2 H_k - \frac{1}{3} \int_0^{k+\nu} s H_k(s) ds \quad (4.96)$$

Now use (4.61), equation (4.95) becomes

$$h(p) = N_4(h) := (p^2 - 1)^{-1} \left\{ F_5(p) - \frac{1}{3} \int_{k+\nu}^p sh(s) ds + \int_{k+\nu}^p D(p-s)h(s) ds \right\}; \quad (4.97)$$

where

$$F_5(p) = F_4(p) + N(H_k); \quad (4.98)$$

using Lemma 4.14, $F_5(p)$ is analytic in a neighborhood of $(k+\nu, k+1+\nu)$. $D(p)$ is given by equation (4.59), by Lemma 4.6, $D(p) \in L_b^1$. Let $\mathcal{K}$ is a open small neighborhood of $(k-\nu, k+1+\nu)$ such that

$$-\frac{5\pi}{6} < \varphi_k := \arg(p - (k+\nu)) < \frac{5\pi}{6}, \text{ for } p \in \mathcal{K}; \quad (4.99)$$

$\mathcal{A} = \{F(p) : F \text{ is analytic in } \mathcal{K}\}$ with norm

$$\|F\|_u = \sup_{\mathcal{K}} e^{-b|\operatorname{Re} p|} |F(p)|;$$

Then for $h(p) \in \mathcal{A}$:

$$\begin{aligned} \left\| \int_{k+\nu}^p sh(s) ds \right\|_u &\leq \sup e^{-b|\operatorname{Re} p|} \int_{k+\nu}^p |sh(s)| |ds| \\ &\leq \sup e^{-b|\operatorname{Re} p|} \int_0^{|p-(k+\nu)|} |(k+\nu + te^{i\varphi_k})h(k+\nu + te^{i\varphi_k})| dt \\ &\leq \sup e^{-b|\operatorname{Re} p|} \|h\|_u \int_0^{|p-(k+\nu)|} |(k+\nu + te^{i\varphi_k})e^{b(k+\nu+te^{i\varphi_k} \cos \varphi_k)}| dt \\ &\leq \sup e^{-b(|\operatorname{Re} p| - k - \nu)} \|h\|_u \int_0^{|p-(k+\nu)|} |(k+\nu + te^{i\varphi_k})e^{t \cos \varphi_k}| dt \\ &\leq \frac{k+\nu}{\cos \varphi_k} b^{-1} \|h\|_u \leq \frac{k+\nu}{\cos \frac{\pi}{6}} b^{-1} \|h\|_u; \end{aligned} \quad (4.100)$$

By lemma 4.17 and Lemma 4.18, $\int_{k+\nu}^p D(p-s)h(s)ds$ is analytic in $\mathcal{K}$ and we have:

$$\begin{aligned}
\left\| \int_{k+\nu}^p D(p-s)h(s)ds \right\|_u &\leq \sup e^{-b|Re p|} \int_{k+\nu}^p |D(p-s)h(s)| |ds| \\
&\leq \sup e^{-b|Re p|} \int_0^{|p-(k+\nu)|} |D(t)h(p-t)| |dt| \\
&\leq \sup e^{-b|Re p|} \int_0^{|p-(k+\nu)|} |D(te^{i\varphi_k})h(p-te^{i\varphi_k})| dt \\
&\leq \sup e^{-b|Re p|} \|h\|_u \int_0^{|p-(k+\nu)|} |D(te^{i\varphi_k})| e^{b|Re(p-te^{i\varphi_k})|} dt \\
&\leq \sup e^{-b|Re p|} \|h\|_u \int_0^{|p-(k+\nu)|} |e^{-bt \cos \varphi_k} D(te^{i\varphi_k})| \\
&\quad \times e^{bt \cos \varphi_k + b|Re(p-te^{i\varphi_k})|} dt \\
&\leq \sup_{p \in \mathcal{K}} e^{-b|Re p|} \sup_{t \in (0, |p-k-\nu|)} e^{bt \cos \varphi_k + b|Re(p-te^{i\varphi_k})|} \|D\|_b \|h\|_u \\
&\leq \|D\|_b \|h\|_u;
\end{aligned} \tag{4.101}$$

So the operator N_1 is a contraction in $\mathcal{A}$ since $\|D\|_b \rightarrow 0$ as $b \rightarrow \infty$, therefore there exists a unique $h(p) \in \mathcal{A}$. By uniqueness of Φ_0 , we have the lemma. $\square$

Theorem 4.29. *Let $\Phi_0(p)$ be as in Lemma 4.6, then $\Phi(p) = \Phi_0^*(p^*)$ for complex p . If the Stokes constant $S(a) = 0$, then $\Phi_0(p)$ is analytic in $\mathcal{D} \cup \{p > 0\}$, and $\Phi_0(p)$ is real for real p .*

Proof. Since all coefficients in equation (4.32) are real, taking the complex conjugate of (4.32), $\Phi_0^*(p)$ is also a solution of (4.32). By uniqueness, $\Phi_0(p) = \Phi_0^*(p^*)$. Since $S(a) = 0$, by Theorem 24, Φ_0 is analytic in a neighborhood of $(0, 1 + \nu)$; By above Lemma and induction, $\Phi_0(p)$ is analytic in $\mathcal{D} \cup \{p > 0\}$. $\square$

Proof. of Theorem 4.2.1 Applying Lemma 4.8 and Comment 4.2 .pick $\varphi \in (-\pi/2, 0)$.let

$$\psi_0(\eta, a) = \frac{2a}{3^{4/3}\eta^{4/3}} + \mathcal{L}_\varphi \Phi_0; \quad (4.102)$$

then $\psi_0(\eta, a)$ is a solution of (4.28). By Theorem 4.12. $\Phi_0(p) \sim O(p)$ as $p \rightarrow 0$.from Watson's lemma. $\mathcal{L}_\varphi \Phi_0 \sim O(\eta^{-2})$ as $\eta \rightarrow \infty$, $\arg \eta \in [0, \frac{\pi}{2} + \varphi)$.so $\psi_0(\eta, a) \sim \frac{2a}{3^{4/3}\eta^{4/3}}$. To prove the uniqueness.if $\psi_0^1(\eta)$ and $\psi_0^2(\eta)$ are two solutions of (4.28)(4.29).let $\psi(\eta) = \psi_0^1(\eta) - \psi_0^2(\eta)$.then $\psi(\eta)$ satisfies the following homogeneous equation:

$$\frac{d^2 \psi}{d\eta^2} - \frac{1}{3\eta} \frac{d\psi}{d\eta} - (1 + T(\eta))\psi = 0; \quad (4.103)$$

$$\psi(\eta) \sim o(\eta^{-4/3}), \text{ as } \eta \rightarrow \infty, \arg \eta \in [0, \frac{\pi}{2} + \varphi); \quad (4.104)$$

where $T(\eta) \sim O(\eta^{-2})$ as $\eta \rightarrow \infty$: There are two independent solutions of linear equation (4.103) ψ_1, ψ_2 so that $\psi_1(\eta) = \eta^{1/6}e^\eta(1 + o(1))$, $\psi_2(\eta) = \eta^{1/6}e^{-\eta}(1 + o(1))$.so $\psi(\eta) = c_1\psi_1(\eta) + c_2\psi_2(\eta)$.the boundary condition (4.104) implies $c_1 = 0, c_2 = 0$.so $\psi(\eta) \equiv 0$. $\square$

Theorem 4.30. *Let $\psi_0(\eta)$ be as in Theorem 4.2.then $\text{Im } \psi_0(\eta, a) = 0$ iff $S(a) = 0$:*

Proof. 1)Using the fact $\Phi_0(re^{i\varphi}) = \Phi_0^*(re^{-i\varphi})$ (Theorem4.29) and equation (4.102).we have:

$$\text{Im } \psi_0(\eta, a) = \mathcal{L}_\varphi(\Phi_0) - \mathcal{L}_{-\varphi}(\Phi_0) = \int_{\infty e^{-i\varphi}}^{\infty e^{i\varphi}} \Phi_0(p)e^{-p\eta}dp; \quad (4.105)$$

changing the path of integration in (4.105) to $C_+ \cup C_-$ (see figure 4.2).by Theorem 4.27 and Watson's lemma.we have:

$$\text{Im } \psi_0(\eta, a) \sim 6S(a) \sin \frac{\pi}{6} \Gamma(\frac{\bar{c}}{6}) \eta^{1/6} e^{-\eta}; \quad (4.106)$$

so $\text{Im } \psi_0(\eta, a) = 0$ implies $S(a) = 0$.

2) If $S(a) = 0$, then $\Phi_0(p)$ is analytic in $\text{Re } p > 0$ and $\Phi_0(p)$ is real for real p (Theorem 4.29). By (4.102)

$$\psi_0(\eta, a) = \frac{2a}{3^{4/3}\eta^{4/3}} + \int_0^\infty \Phi_0(p)e^{-\eta p} dp; \quad (4.107)$$

So $\text{Im } \psi_0(\eta, a) = 0$. □

Previous numerical results (Combescot et al, 1988) show that $S(a) = 0$ has discretely infinite many real solutions which are denoted by $a_n, n = 1, 2, \dots$.

Figure 4.2: The path of integration in Lemma 4.30.

4.3 Full Inner Problem Analysis

Now we go back to the full inner equation (4.23). By Theorem 4.1, full equation (4.23) with matching condition (4.26) has unique solution for $\eta \geq \epsilon^{-\sigma_1}$, $\arg \eta \in [0, \frac{5\pi}{8}]$. We shall prove that this solution can be extended to the region

$$\mathcal{R}_1 = \{\eta : \eta \text{ between line segments } r_4 \text{ and } r_5, \arg \eta \in (-\frac{\pi}{8}, \frac{5\pi}{8})\}$$

where r_4 is the line segment connecting $Re^{\frac{5i\pi}{8}}$ to $Re^{-\frac{i\pi}{8}}$, r_5 is the arc connecting $\epsilon^{-\sigma_1}e^{\frac{5i\pi}{8}}$ to $\epsilon^{-\sigma_1}e^{-\frac{i\pi}{8}}$, for some large R independent of ϵ . See Figure 4.3.

Definition 4.31. Let $\psi(\eta)$ be the unique solution in Theorem 4.1 for $\eta \geq \epsilon^{-\sigma_1}$, $\arg \eta \in [0, \frac{5\pi}{8}]$, define $\tilde{\psi}(t) := \psi(t)$, $\tilde{\psi}'(t) := \psi'(t)$ for $t \in r_5$, $\arg \eta \in [0, \frac{5\pi}{8}]$.

Definition 4.32.

$$\psi_1(\eta) = \eta^{1/6}e^{-\eta}, \psi_2(\eta) = \eta^{1/6}e^{\eta}; \quad (4.108)$$

$\psi_1(\eta), \psi_2(\eta)$ satisfy the following equation exactly:

$$\mathcal{L}_1 \psi \equiv \frac{d^2 \psi}{d\eta^2} - \frac{1}{3\eta} \frac{d\psi}{d\eta} - \left(1 - \frac{7}{36\eta^2}\right) \psi = 0; \quad (4.109)$$

Definition 4.33.

$$\eta_0 = \epsilon^{-\sigma_1}, \eta_1 = \epsilon^{-\sigma_1}e^{\frac{5i\pi}{8}}, \eta_2 = i\epsilon^{-\sigma_1}; \quad (4.110)$$

Figure 4.3: Region $\mathcal{R}_1$.

Equation (4.23) can be rewritten as

$$\begin{aligned}\mathcal{L}_1 \psi &= \mathcal{N}(\eta, \epsilon, \psi, \psi') \\ &\equiv -\frac{4}{9\eta^2} - \frac{1}{4\eta^2} \psi + \frac{2^{2/3}a}{3^{4/3}\eta^{4/3}} \\ &\quad - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n + \epsilon^{2/3} E_9(\epsilon^{2/3}, \epsilon^{2/3} \eta^{2/3}, \psi, \psi', \eta^{-2/3});\end{aligned}\tag{4.111}$$

We consider solution ψ of the following integral equation:

$$\begin{aligned}\psi &= \mathcal{L}_2 \psi \equiv -\psi_1(\eta) \int_{\eta_1}^{\eta} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{N}(t) dt + \psi_2(\eta) \int_{\eta_0}^{\eta} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{N}(t) dt \\ &\quad - \psi_1(\eta) \frac{(\psi_2(\eta_1) \tilde{\psi}'(\eta_1) - \psi_2'(\eta_1) \tilde{\psi}(\eta_1))}{2\eta_1^{1/3}} + \psi_2(\eta) \frac{(\psi_1(\eta_0) \tilde{\psi}'(\eta_0) - \psi_1'(\eta_0) \tilde{\psi}(\eta_0))}{2\eta_0^{1/3}};\end{aligned}\tag{4.112}$$

Definition 4.34.

$$\mathbf{B}_1 = \{\psi(\eta) : \psi(\eta) \text{ is analytic in } \mathcal{R}_1, \text{ and continuous on } \bar{\mathcal{R}}_1.\}\tag{4.113}$$

$\mathbf{B}_1$ is Banach space with norm

$$\|\psi\| = \sup_{\bar{\mathcal{R}}_1} R_1 |\psi(\eta)|, \text{ where } R_1 = R^{4/3};\tag{4.114}$$

Let $\mathcal{N}_0(\eta) := \mathcal{N}(\eta, \epsilon, 0, 0)$, then $\|\mathcal{N}_0\| \leq \delta_3$, where δ_3 is some constant which can be chosen to be independent of R and ϵ .

We define space

$$\mathbf{B}_{\delta_3} = \{\psi \in \mathbf{B}_1 : \|\psi - \mathcal{N}_0\| \leq \delta_3\}$$

Lemma 4.35. *Let $\mathcal{N} \in \mathbf{B}_1$, then*

$$\phi_1(\eta) := \psi_1(\eta) \int_{\eta_1}^{\eta} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{N} dt \in \mathbf{B}_1, \phi_2(\eta) := \psi_2(\eta) \int_{\eta_0}^{\eta} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{N} dt \in \mathbf{B}_1;$$

and $\|\phi_1\| \leq C\|\mathcal{N}\|, \|\phi_2\| \leq C\|\mathcal{N}\|$; where C is independent of R, ϵ .

Proof. For $\eta \in \mathcal{R}_1$, we use straight lines to connect η to η_0 (or η_1) so that $\operatorname{Re} t$ is increasing from η to η_0 (from η_1 to η) and $C_1|\eta| \leq |t| \leq C_2|\eta|$.

$$\begin{aligned} |\phi_2(\eta)| &= \left| \eta^{1/6} \int_{\eta_0}^{\eta} \frac{e^{\eta-t}}{2t^{1/6}} \mathcal{N} dt \right| \\ &\leq \frac{C}{R_1} \int_{\eta_0}^{\eta} |e^{\eta-t}| |R_1 \mathcal{N}| |dt| \\ &\leq \frac{C \|\mathcal{N}\|}{R_1} \int_{\eta_0}^{\eta} |e^{\eta-t}| |dt| \leq \frac{C \|\mathcal{N}\|}{R_1} \end{aligned}$$

The same steps work for $\phi_1(\eta)$. □

Lemma 4.36. *If $\psi \in \mathbf{B}_{\delta_3}$, $\phi \in \mathbf{B}_{\delta_3}$, $\psi' \in \mathbf{B}_{\delta_3}$, $\phi' \in \mathbf{B}_{\delta_3}$, then $\mathcal{N}(\eta, \psi, \psi') \in \mathbf{B}_1$ and $\|\mathcal{N}(\eta, \psi, \psi') - \mathcal{N}(\eta, \phi, \phi')\| \leq C R_1^{-1} (\|\phi - \psi\| + \|\phi' - \psi'\|)$:*

Proof. By induction and using $|\psi| \leq C R_1^{-1}$, $|\phi| \leq C R_1^{-1}$, we have

$$|\psi^k - \phi^k| \leq C R_1^{-1} |\psi - \phi|, \text{ for } k \geq 2.$$

Using the above equation and (4.111), we get the lemma. □

Definition 4.37.

$$\mathbf{E}_{\delta_3} := \mathbf{B}_{\delta_3} \oplus \mathbf{B}_{\delta_3}, \|(\psi, \psi')\| = \|\psi\| + \|\psi'\|; \quad (4.115)$$

Theorem 4.38. *There exists a unique solution $\psi \in \mathbf{B}_{\delta_3}$ of equation (4.112).*

Proof. In (4.111)(4.112), replacing ψ with u , ψ' with v , we define

$$\mathcal{O}_1(u, v) := \mathcal{L}_2(u, v), \mathcal{O}_2(u, v) := \frac{d}{d\eta} \mathcal{L}_2(u, v); \quad (4.116)$$

$$\mathcal{O}(u, v) = (\mathcal{O}_1(u, v), \mathcal{O}_2(u, v)) \quad (4.117)$$

Using (4.112), Lemma 4.35 and Lemma 4.36 .it is easily seen that $\mathcal{O}(u, v)$ is a contraction from $\mathbf{E}_1$ to $\mathbf{E}_1$ for small ϵ and large enough R .therefore we have the theorem. $\square$

Lemma 4.39. *Let $\psi(\eta)$ be the solution of (4.112) as in Theorem 4.38 then $\psi(\eta)$ is a solution of (4.111) with $\psi(t) \equiv \tilde{\psi}(t)$ for $|t| = \epsilon^{-1}, \arg t \in [0, \frac{5\pi}{3}]$.*

Proof. Since $\tilde{\psi}(t)$ is a solution of (4.109) for $|t| = \epsilon^{-1}, \arg t \in [0, \frac{5\pi}{3}]$,by variation of parameters:

$$\tilde{\psi}(t) = -\psi_1(t) \int_{\eta_2}^t \frac{\psi_2(s)}{2s^{1/3}} \mathcal{N}(s) ds + \psi_2(t) \int_{\eta_2}^t \frac{\psi_1(s)}{2s^{1/3}} \mathcal{N}(s) ds + A_1 \psi_1(t) + A_2 \psi_2(t); \quad (4.118)$$

Plugging $t = \eta_2$ in the above equation,we have

$$A_1 = \frac{\tilde{\psi}(\eta_2) \psi_2'(\eta_2) - \tilde{\psi}'(\eta_2) \psi_2(\eta_2)}{2\eta_2^{1/3}}; \quad (4.119)$$

$$A_2 = \frac{\tilde{\psi}(\eta_2) \psi_1'(\eta_2) - \tilde{\psi}'(\eta_2) \psi_1(\eta_2)}{2\eta_2^{1/3}}; \quad (4.120)$$

Rewrite equation (4.118):

$$\begin{aligned} \tilde{\psi}(t) &= \mathcal{L}_2 \tilde{\psi} + A_1 \psi_1(\eta) + A_2 \psi_2(\eta) \\ &- \psi_1(t) \int_{\eta_2}^{\eta_1} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{N}(\tilde{\psi})(t) dt + \psi_2(t) \int_{\eta_2}^{\eta_0} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{N}(\tilde{\psi})(t) dt \\ &+ \psi_1(t) \frac{\left(\psi_2(\eta_1) \tilde{\psi}'(\eta_1) - \psi_2'(\eta_1) \tilde{\psi}(\eta_1) \right)}{2\eta_1^{1/3}} - \psi_2(t) \frac{\left(\psi_1(\eta_0) \tilde{\psi}'(\eta_0) - \psi_1'(\eta_0) \tilde{\psi}(\eta_0) \right)}{2\eta_0^{1/3}}; \end{aligned} \quad (4.121)$$

Using integration by parts twice:

$$\begin{aligned}
& -\psi_1(t) \int_{\eta_2}^{\eta_1} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{L}_1 \tilde{\psi}(t) dt + \psi_2(t) \int_{\eta_2}^{\eta_0} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{L}_1 \tilde{\psi}(t) dt \\
& = -\psi_1(t) \frac{\left(\psi_2(\eta_1) \tilde{\psi}'(\eta_1) - \psi_2'(\eta_1) \tilde{\psi}(\eta_1) \right)}{2\eta_1^{1/3}} + \psi_2(t) \frac{\left(\psi_1(\eta_0) \tilde{\psi}'(\eta_0) - \psi_1'(\eta_0) \tilde{\psi}(\eta_0) \right)}{2\eta_0^{1/3}} \\
& = A_1 \psi_1(t) - A_2 \psi_2(t);
\end{aligned} \tag{4.122}$$

So equation (4.121) is reduced to $\tilde{v}(t) = \mathcal{L}_2 \tilde{\psi}$. By equation (4.112), we have

$$v(t) - \tilde{v}(t) = -\psi_1(t) \int_{\eta_1}^t \frac{\psi_2(t)}{2t^{1/3}} \left(\mathcal{N}(v) - \mathcal{N}(\tilde{v}) \right) dt + \psi_2(t) \int_{\eta_0}^t \frac{\psi_1(t)}{2t^{1/3}} \left(\mathcal{N}(v) - \mathcal{N}(\tilde{v}) \right) dt \tag{4.123}$$

Taking derivative in above equation:

$$v'(t) - \tilde{v}'(t) = -\psi_1'(t) \int_{\eta_1}^t \frac{\psi_2(t)}{2t^{1/3}} \left(\mathcal{N}(v) - \mathcal{N}(\tilde{v}) \right) dt + \psi_2'(t) \int_{\eta_0}^t \frac{\psi_1(t)}{2t^{1/3}} \left(\mathcal{N}(v) - \mathcal{N}(\tilde{v}) \right) dt \tag{4.124}$$

Using Lemma 4.35 and Lemma 4.36 ,we have

$$\|(v, v') - (\tilde{v}, \tilde{v}')\| \leq C R_1^{-1} \|(v, v') - (\tilde{v}, \tilde{v}')\|$$

So $v(t) \equiv \tilde{v}(t)$. □

Theorem 4.40. (1) For large enough R , there exists a unique solution $\psi(\eta, \epsilon, a)$ of (4.23)/(4.26) in region $\eta \geq R$ for $\arg \eta \in [0, 5\pi/8]$.

(2) $\psi(R, \epsilon, a)$ of (1) is analytic in $\epsilon^{2/3}$, a.s. as $\epsilon \rightarrow 0$. Furthermore, $\psi(\eta, 0, a) = \psi_0(\eta, a)$ for $\eta > R$.

Proof. Part (1) follows from Theorem 4.1 and Theorem 4.38 .

Note that as $\epsilon \rightarrow 0$, $\epsilon^{2/3}E_9(\epsilon, \eta, \psi) \rightarrow 0$ uniformly. Part (2) follows from the theorem of dependence of solution on parameters.(see Theorem 3.8.5 in Hille 1976)

□

Lemma 4.41. *If*

$$\text{Im } \psi(\eta, \epsilon, a) = 0. \text{ for } \eta \geq R. \quad (4.125)$$

then after change of variable (4.24) (4.25),the symmetry condition (3.4) for $F(\xi)$ in chapter 3 holds on a subinterval of $\{\text{Re}\xi = 0\} \cap \mathcal{R}$.

Proof. The symmetry condition is equivalent to $\text{Re}F'(\xi) = 0$ for $\text{Re}\xi = 0 \cap \mathcal{R}$,the Lemma follows immediately from (4.24),(4.25). □

Lemma 4.42. *If $\text{Im}\psi(R, \epsilon, a(\epsilon)) = 0$,then $\text{Im}\psi(\eta, \epsilon, a(\epsilon)) = 0$ for all $\eta > R$*

Proof. Taking imaginary parts of equation (4.23), $\text{Im}\psi(\eta, \epsilon, a(\epsilon))$ satisfies the following homogeneous equation:

$$\frac{d^2 \text{Im } \psi}{d\eta^2} - \left(\frac{1}{3\eta} + T_1(\eta, \epsilon) \right) \frac{d \text{Im } \psi}{d\eta} - (1 + T_2(\eta, \epsilon)) \text{Im } \psi = 0 \quad (4.126)$$

where functions $T_1(\eta, \epsilon), T_2(\eta, \epsilon)$ satisfy

$$T_1(\eta, \epsilon) \sim \frac{C}{\eta^2}, T_2(\eta, \epsilon) \sim \frac{C}{\eta^2}, \text{ as } \epsilon \rightarrow 0, \eta \gg 1;$$

Two independent solutions of (4.109) are

$$\psi_1 = \eta^{1/6} e^{\eta} (1 + \mathbf{e}(\eta, \epsilon))$$

$$\psi_2 = \eta^{1/6} e^{-\eta} (1 + \mathbf{e}(\eta, \epsilon))$$

where the function $\mathbf{e}(\eta, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0, \eta \gg 1$.(See Olver 1974) So $\text{Im } \psi = c_1 \psi_1 + c_2 \psi_2$. Since $\text{Im}\psi \rightarrow 0$ as $\eta \rightarrow \infty$, c_1 must be 0; c_2 must be 0 since $\text{Im } \psi(R) = 0$. □

Theorem 4.43. Assume $S'(a_n) \neq 0$, then for small enough ϵ and large enough R , there is analytic function $\beta(\epsilon^{2/3})$, such that $\lim_{\epsilon \rightarrow 0} \beta(\epsilon^{2/3}) = a_n$, and $\text{Im } \psi(\eta, \epsilon, \beta(\epsilon^{2/3})) = 0$, for all $\eta \in (R, \infty)$

Proof. For fixed large enough $R > 0$, $S'(a_n) \neq 0$ implies

$$\frac{\partial \text{Im } \psi}{\partial a}(R, a_n) \neq 0. \quad (4.127)$$

Using Lemma 4.40, (4.127) and implicit function theorem, there exists function $\beta(\epsilon^{2/3})$ so that $\text{Im } \psi(R, \epsilon, \beta(\epsilon^{2/3})) = 0$. Then using Lemma 4.42, we complete the proof. $\square$

Lemma 4.44. Let $F(\xi)$ be the solution of the weak problem in Theorem 3.20, we define $q(\xi)$ so that $q(\xi) = \frac{F(\xi) - \overline{F(-\xi^*)}}{2i}$ ($= \text{Im } F$ on $\text{Re } \xi + 0 \cap \mathcal{R}$). Then q satisfies the following homogeneous equation on imaginary axis $\{\xi = i\nu\}$:

$$\epsilon^2 \frac{d^2 q}{d\nu^2} + (L(i\nu) + \tilde{L}(\nu))q = 0; \quad (4.128)$$

where $\tilde{L}(\nu)$ is some real function.

Proof. On imaginary axis $\{\xi = i\nu\}$: Using (2.11), $L(i\nu) = \frac{\sqrt{\gamma^2 - \nu^2}(\nu + \gamma)}{(1 - \nu^2)^2}$ is real. Using (2.58), $\left(\frac{H'H}{H} - H'\right)(i\nu) = \frac{2\gamma}{(1 - \nu^2)(\gamma - \nu)}$ is real. By taking imaginary part in equation (2.42), we have the lemma. $\square$

Comment If $q(i\nu) = 0$ at 2 points, then $q(i\nu) \equiv 0$.

Proof of Theorem 1.7: If λ satisfies restriction (1.35), by Theorem 4.43, $\text{Im } \psi(\eta, \epsilon, \beta(\epsilon^{2/3})) = 0$ on $\eta \in (R, \epsilon^{-\sigma_1})$. By Lemma 4.44 and Theorem 4.1, the corresponding solution $F(\xi)$ for Weak Problem satisfies the symmetry condition (3.4) on $\{\text{Re } \xi = 0\} \cap \mathcal{R}$. By Lemma 3.1, $F(\xi)$ is solution of Finger Problem.

Part (3) follows from Lemma 3.21, Thm 4.1, Theorem 4.40 and Theorem 4.30.

Appendix A

PROPERTIES OF FUNCTION $P(\xi)$

In this appendix, we discuss the properties of the following function:

$$\begin{aligned} P(\xi) &= \int_{-\gamma}^{\xi} i L^{1/2}(t) dt \\ &= i \int_{-\gamma}^{\xi} \frac{(\gamma - it)^{3/4} (\gamma + it)^{1/4}}{(1 + t^2)} dt \end{aligned} \quad (\text{A.1})$$

we choose branch cut $\{\xi : \xi = \rho i, \rho > \gamma\}$, $-\pi \leq \arg(\gamma + i\xi) \leq \pi$ for the function $(\gamma + i\xi)^{1/4}$ and branch cut $\{\xi : \xi = -\rho i, \rho > \gamma\}$, $-\pi \leq \arg(\gamma - i\xi) \leq \pi$ for the function $(\gamma - i\xi)^{3/4}$. For simplicity, we assume $1 < \gamma \leq 2$.

Definition A.1. The curves in complex ξ -plane on which $\text{Re } P(\xi) = \text{constant}$ will be called Stokes lines, whereas curves in complex ξ -plane on which $\text{Im } P(\xi) = \text{constant}$ will be called anti-Stokes lines.

Lemma A.2. 1) $\text{Re } P(\xi)$ decreases along negative $\text{Re } \xi$ axis $(-\infty, 0)$ and increases along positive $\text{Re } \xi$ axis $(0, \infty)$, with $\text{Re } P(\pm\infty) = +\infty$. 2) $\text{Im } P(\xi)$ increases along $\text{Re } \xi$ axis $(-\infty, \infty)$ with $\text{Im } P(-\infty) = -\infty$, $\text{Im } P(\infty) = \infty$.

Proof. Using $\arg(\gamma + it) = -\arg(\gamma - it)$ for t on the real axis, from (1):

$$P(\xi) = i \int_0^{\xi} \frac{(\gamma^2 + t^2)^{1/2}}{(1 + t^2)} \exp\left\{\frac{i}{2} \arg(\gamma - it)\right\} dt + P(0); \quad (\text{A.2})$$

1) for $\xi \in (-\infty, 0)$.

$$\operatorname{Re} P(\xi) = \int_{\xi}^0 \frac{(\gamma^2 + t^2)^{1/2}}{(1 + t^2)} \sin\left\{\frac{1}{2} \arg(\gamma - it)\right\} dt + \operatorname{Re} P(0);$$

Clearly, $\operatorname{Re} P(-\infty) = \infty$ and $\operatorname{Re} P$ decreases since $\arg(\gamma - it) \rightarrow \frac{\pi}{2}$ as $t \rightarrow -\infty$. As for $\xi \in (0, \infty)$, since $\arg(\gamma - it) < 0$ for $t \in (0, \infty)$ it follows that $\operatorname{Re} P$ increases to ∞ as $t \rightarrow \infty$. 2) Now using identity

$$\operatorname{Im} P(\xi) = - \int_{\xi}^0 \frac{(\gamma^2 + t^2)^{1/2}}{(1 + t^2)} \cos\left\{\frac{1}{2} \arg(\gamma - it)\right\} dt + \operatorname{Re} P(0);$$

we have the second part of the lemma. $\square$

Lemma A.3. *$\operatorname{Im} P(\xi) = 0$ and $\operatorname{Re} P(\xi)$ is increasing as ξ goes upward along imaginary ξ axis from $-i\gamma$ to $-i$. Further $\operatorname{Re} P(-i) = +\infty$.*

Proof. For $1 < \rho < \sqrt{1 + \alpha}$, from (1)

$$P(-\rho i) = \int_{\rho}^{\gamma} \frac{(\gamma + s)^{3/4}(\gamma - s)^{1/4}}{(s^2 - 1)} ds,$$

which is real, and increasing as ρ increases. As $\rho \rightarrow 1^+$, $\operatorname{Re} P(-i\rho) \rightarrow +\infty$. $\square$

Lemma A.4. *1) $\operatorname{Re} P(\xi)$ decreases as ξ moves upward along imaginary ξ axis from $-i$ to i , and $\operatorname{Re} P(i) = -\infty$. 2) $\operatorname{Im} P(\rho i) = \text{constant}$, for $\rho \in (-1, 1)$.*

Proof. For $-1 < \rho < 1$, from (1)

$$P(\rho i) = - \int_0^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(1 - t^2)} dt + P(0),$$

so $\operatorname{Im} P(\rho i) = \operatorname{Im} P(0)$, and

$$\operatorname{Re} P(\rho i) = - \int_0^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(1 - t^2)} dt + \operatorname{Re} P(0),$$

On inspection as ρ increases in the interval $(0, 1)$, $\operatorname{Re} P(i\rho)$ decreases, and $\operatorname{Re} P(i) = -\infty$. $\square$

Lemma A.5. 1) $\operatorname{Re} P(\xi)$ increases as ξ moves upward along imaginary ξ axis from i to $i\gamma$. 2) $\operatorname{Im} P(\rho i) = \text{constant}$, for $\rho \in (1, \gamma]$.

Proof. for $\rho \in (1, \gamma]$:

$$P(\rho i) = \int_{\gamma}^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(t^2 - 1)} dt + P(i\gamma),$$

so $\operatorname{Im} P(\rho i) = \operatorname{Im} P(i\gamma)$, and

$$\operatorname{Re} P(\rho i) = \int_{\gamma}^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(t^2 - 1)} dt + \operatorname{Re} P(0).$$

On inspection as ρ increases in the interval $(1, \gamma)$, $\operatorname{Re} P(i\rho)$ increases. $\square$

Lemma A.6. On the left side of the branch cut $\{\xi : \xi = \rho i, \rho > \gamma\}$, we have: 1) $\operatorname{Re} P(i\rho)$ increases in the interval (γ, ∞) and $\operatorname{Re} P(i\infty) = +\infty$. 2) $\operatorname{Im} P(i\rho)$ decreases in the interval (γ, ∞) and $\operatorname{Re} P(i\infty) = -\infty$.

Proof. 1) Note $\arg(\gamma + i\xi) = -\pi$, $\arg(\gamma - i\xi) = 0$ on the left of the branch cut. So

$$\operatorname{Re} P(\rho i) = \int_{\gamma}^{\rho} \frac{(s + \gamma)^{3/4}(s - \gamma)^{1/4} \cos \frac{\pi}{4}}{(s^2 - 1)} ds + \operatorname{Re} P(i\gamma);$$

the integral on the right side of the above equation goes to ∞ as ρ goes to ∞ , so $\operatorname{Re} P(i\infty) = +\infty$. 2) the second part of the lemma follows from:

$$\operatorname{Im} P(\rho i) = - \int_{\gamma}^{\rho} \frac{(s - \gamma)^{3/4}(s + \gamma)^{1/4} \sin \frac{\pi}{4}}{(s^2 - 1)} ds + \operatorname{Im} P(i\gamma);$$

$\square$

Lemma A.7. On the left side of the branch cut $\{\xi : \xi = -\rho i, \rho > \gamma\}$, 1) $\operatorname{Re} P(-i\rho)$ increases in the interval $\rho \in (\gamma, \infty)$ and $\operatorname{Re} P(-i\infty) = +\infty$. 2) $\operatorname{Im} P(-i\rho)$ decreases in the interval $\rho \in (\gamma, \infty)$ and $\operatorname{Im} P(-i\infty) = -\infty$.

Proof. $\arg(\gamma + i\xi) = 0, \arg(\gamma - i\xi) = \pi$ on the left of the branch cut. So

$$\begin{aligned} \operatorname{Re} P(-\rho i) &= \int_{\gamma}^{\rho} \frac{(s - \gamma)^{3/4} (s + \gamma)^{1/4} \cos \frac{\pi}{4}}{(s^2 - 1)} ds \\ \operatorname{Im} P(-\rho i) &= - \int_{\gamma}^{\rho} \frac{(s - \gamma)^{3/4} (s + \gamma)^{1/4} \sin \frac{\pi}{4}}{(s^2 - 1)} ds; \end{aligned}$$

Lemma follows on inspection. $\square$

Lemma A.8. $\operatorname{Im} P(0) > 0$.

Proof. Let C_{δ} be the half circle on the left centered at $\xi = -i$ with radius δ . $p(t) = i \frac{(\gamma - it)^{3/4} (\gamma + it)^{1/4}}{(t^2 - 1)}$; by (1):

$$\begin{aligned} P(0) &= \lim_{\delta \rightarrow 0} \left\{ \int_{-\gamma - i}^{-(1+\delta)i} p(t) dt + \int_{C_{\delta}} p(t) dt + \int_{-(1-\delta)i}^0 p(t) dt \right\} \\ &= -(P) \int_0^{\gamma} \frac{(\gamma - s)^{3/4} (s + \gamma)^{1/4}}{(1 - s^2)} ds + \lim_{\delta \rightarrow 0} \int_{C_{\delta}} p(t) dt; \quad (\text{A.3}) \end{aligned}$$

Now let $p_1(t) = i \frac{(\gamma - it)^{3/4} (\gamma + it)^{1/4}}{(t - i)}$;

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{C_{\delta}} p(t) dt &= \lim_{\delta \rightarrow 0} \int_{C_{\delta}} \frac{p_1(t)}{(t + i)} dt \\ &= \lim_{\delta \rightarrow 0} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{p_1(-i + \delta e^{i\theta})}{(\delta e^{i\theta})} d(\delta e^{i\theta}) = -\pi i p_1(-i) = i \frac{(\gamma - 1)^{3/4} (\gamma + 1)^{1/4}}{2}; \quad (\text{A.4}) \end{aligned}$$

so $\operatorname{Im} P(0) = \frac{(\gamma - 1)^{3/4} (\gamma + 1)^{1/4}}{2} > 0$. $\square$

Lemma A.9. $\operatorname{Im} P(\gamma i) < 0$:

Proof. Let $\hat{C}_\delta$ be the half circle on the left centered at $\xi = i$ with radius δ , $p(t) = i \frac{(\gamma - t)^{3/4}(\gamma + t)^{1/4}}{(t^2 + 1)}$; by (1):

$$\begin{aligned} P(i\gamma) &= \lim_{\delta \rightarrow 0} \left\{ \int_{(1-\delta)i}^{\gamma i} p(t)dt + \int_{\hat{C}_\delta} p(t)dt + \int_0^{(1-\delta)i} p(t)dt \right\} + P(0) \\ &= -(P) \int_0^\gamma \frac{(\gamma - s)^{1/4}(s + \gamma)^{3/4}}{(1 - s^2)} ds + \lim_{\delta \rightarrow 0} \int_{\hat{C}_\delta} p(t)dt + P(0); \quad (\text{A.5}) \end{aligned}$$

Now let $p_2(t) = i \frac{(\gamma - it)^{3/4}(\gamma + it)^{1/4}}{(t - i)}$:

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\hat{C}_\delta} p(t)dt &= \lim_{\delta \rightarrow 0} \int_{\hat{C}_\delta} \frac{p_2(t)}{(t - i)} dt \\ &= \lim_{\delta \rightarrow 0} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{p_2(i + \delta e^{i\theta})}{(\delta e^{i\theta})} d(\delta e^{i\theta}) = -\pi i p_1(i) = -i \frac{(\gamma - 1)^{1/4}(\gamma + 1)^{3/4}}{2}; \quad (\text{A.6}) \end{aligned}$$

so

$$\begin{aligned} \text{Im} P(\gamma i) &= -\frac{(\gamma - 1)^{1/4}(\gamma + 1)^{3/4}}{2} - \text{Im} P(0) \\ &= -\frac{(\gamma - 1)^{1/4}(\gamma + 1)^{3/4} - (\gamma - 1)^{3/4}(\gamma + 1)^{1/4}}{2} < 0. \end{aligned}$$

□

Lemma A.10. *Re $P(0) < 0$.*

Proof. By (3):

$$\text{Re} P(0) = -(P) \int_0^\gamma \frac{(\gamma - t)^{3/4}(\gamma + t)^{1/4}}{(1 - t^2)} dt$$

Let

$$p_3(t) = \frac{(\gamma - t)^{3/4}(\gamma + t)^{1/4}}{(1 - t^2)}$$

then

$$(P) \int_0^\gamma p_3(t)dt = \lim_{\delta \rightarrow 0} \left(\int_0^{1-\delta} p_3(t)dt + \int_{1-\delta}^\gamma p_3(t)dt \right);$$

Use change of variable $t = 2 - s$, we have

$$\int_{1+\delta}^{\gamma} p_3(t)dt = - \int_{2-\gamma}^{1-\delta} \frac{(\gamma - 2 + s)^{3/4}(\gamma + 2 - s)^{1/4}}{(1-s)(3-s)} ds;$$

so

$$(P) \int_0^{\gamma} p_3(t)dt = \lim_{\delta \rightarrow 0} \left(\int_0^{2-\gamma} p(t)dt + \int_{2-\gamma}^{1-\delta} p_4(s)ds \right);$$

where

$$p_4(s) = \frac{(\gamma - s)^{3/4}(\gamma + s)^{1/4}}{(1-s)(1+s)} - \frac{(\gamma - 2 + s)^{3/4}(\gamma + 2 - s)^{1/4}}{(1-s)(3-s)}$$

We want to show that $p_4(s) \geq 0$ for $2 - \gamma \leq s \leq 1$. since

$$(3-s)^4 = [(1+s) + (2-2s)]^4 \geq (1+s)^4 + (1+s)^3(2-2s);$$

we have

$$\begin{aligned} (3-s)^4(\gamma+s) - (1+s)^4(\gamma+2-s) \\ \geq (1+s)^4(2s-2) + (1+s)^3(2-2s)(\gamma+s) \\ = (1+s)^3(2-2s)(\gamma-1) \geq 0; \end{aligned}$$

then using $\gamma - s \geq \gamma - 2 + s$, for $2 - \gamma < s < 1$, we have

$$\begin{aligned} p_4(s) &= \frac{1}{(1-s)(1+s)(3-s)} \times \left\{ (3-s)(\gamma+s)^{1/4}(\gamma-s)^{3/4} \right. \\ &\quad \left. - (1+s)(\gamma+2-s)^{1/4}(\gamma-2+s)^{3/4} \right\} \\ &\geq \frac{1}{(1-s)(1+s)(3-s)} \times \left\{ (3-s)(\gamma+s)^{1/4} - (1+s)(\gamma+2-s)^{1/4} \right\} \geq 0 \end{aligned}$$

□

Lemma A.11. *Re $P(i\gamma) < 0$.*

Proof. By (5)

$$\begin{aligned} \operatorname{Re} P(i\gamma) &= -(P) \int_0^\gamma \frac{(\gamma - s)^{1/4}(\gamma + s)^{3/4}}{(1 - s^2)} ds + \operatorname{Re} P(0) \\ &= -(P) \int_0^\gamma \frac{(\gamma - s)^{1/4}(\gamma + s)^{3/4}}{(1 - s^2)} ds - (P) \int_0^\gamma \frac{(\gamma - s)^{3/4}(\gamma + s)^{1/4}}{(1 - s^2)} ds < 0 \end{aligned}$$

□

Lemma A.12. *Locally in the neighborhood of $\xi = -i\gamma$, there are four Stokes lines $\ell_k, k = 1, 2, 3, 4$, emerging from the point $\xi = -i\gamma$ in the complex sector $-\pi < \arg(\gamma - i\xi) < \pi$.*

Proof. Locally near point $\xi = -i\gamma$

$$P(\xi) = -\frac{4(2\gamma)^{1/4}}{7(\gamma^2 - 1)}(\gamma - i\xi)^{7/4}\{1 + O(\gamma - i\xi)\}, \quad (\text{A.7})$$

$\operatorname{Re} P = 0$ occurs when $\arg(\gamma - i\xi) = \frac{2\pi}{7}, \frac{6\pi}{7}, -\frac{2\pi}{7}, -\frac{6\pi}{7}$; so there are four corresponding Stokes lines $\ell_k, k = 1, 2, 3, 4$, emerging from $y = -i\gamma$. ℓ_1 and ℓ_2 go to the left half plane, the other two go to the right half. □

Lemma A.13. *ℓ_1 is in the third quadrant and intersects with negative $\operatorname{Im} \xi$ axis at some point $\xi_0 i$, where $-1 < \xi_0 < 0$. ℓ_2 is in the left half plane and intersects with negative $\operatorname{Re} \xi$ axis at x_0 where $\operatorname{Re} P(x_0) = 0$; ℓ_2 goes in the second quadrant and then intersects with $\operatorname{Im} \xi$ axis. ℓ_3 and ℓ_4 are symmetric to ℓ_1 and ℓ_2 about $\operatorname{Im} \xi$ axis respectively.*

Proof. First note that none of ℓ_1, ℓ_2 can asymptote to ∞ in the third quadrant since for large ξ , it follows from (1) that $P(\xi) \sim e^{-\frac{i\xi}{4}} \ln \xi$, so $\operatorname{Re} P = 0$ is not possible when

$|\xi|$ is large. Therefore the Stokes line ℓ_1, ℓ_2 must intersect either the negative real or negative imaginary axis. However, since $\operatorname{Re} P(-i\rho)$ is monotonic for $\rho \in (\gamma, \infty)$ and $(1, \gamma)$, the only intersection of ℓ_1, ℓ_2 on the imaginary axis is at $\xi = \xi_0 i$, $-1 < \xi_0 < 0$. We do know there exists such a ξ_0 so that $\operatorname{Re} P(\xi_0 i) = 0$ since $\operatorname{Re} P(-i) = \infty$ and $\operatorname{Re} P(0) < 0$. If ℓ_1 goes through x_0 on the negative real axis, ℓ_2 can not go through the same point since Stokes lines can not intersect except at critical points. Also $\operatorname{Im} P$ is monotonic on ℓ_2 and ℓ_2 can not end at a finite point in the third quadrant, since $\operatorname{Im} P$ does not blow up. Therefore, we must rule out the possibility of ℓ_1 going through x_0 . Similar arguments can be applied to show ℓ_2 can not go through $i\xi_0$. Therefore the only possibility is for ℓ_1 to go up with $i\xi_0$ and for ℓ_2 to join up with x_0 . Once ℓ_2 enters second quadrant through x_0 , it can not go to ∞ because of asymptotic behavior $P(\xi) \sim e^{-i\pi/4} \ln \xi$. It must therefore intersect the positive imaginary axis. It can not do so between 0 and i since $\operatorname{Re} P(i\rho)$ is monotonically decreasing from $\rho = 0$ to $\rho = 1$ where $\operatorname{Re} P(i) = -\infty$. Further $\operatorname{Re} P(i\rho)$ monotonically increases for $\rho \in (1, \gamma)$ and $\operatorname{Re} P(i\gamma) < 0$. So ℓ_2 can not intersect positive imaginary axis between i and $i\gamma$ either. $\square$

The stokes lines are shown in Figure A1.

Figure A.1: Stokes lines defined by $\text{Re } P(\xi) = 0$.

Lemma A.14. *There are three anti-Stokes lines $\mathcal{A}_k$, ($k = 1, 2, 3$) emerging from point $\xi = -i\gamma$. $\mathcal{A}_1$ is on the $\text{Im}\xi$ axis from $\xi = -i\gamma$ to $-i$. $\mathcal{A}_2$ is on the left half ξ -plane .intersects with negative y axis then goes to point i . $\mathcal{A}_3$ is symmetric to $\mathcal{A}_2$ about $\text{Im } \xi$ axis.*

Proof. Use (7) in lemma 12. $\text{Im } P(\xi) = 0$ occurs when $\arg(\gamma - i\xi) = 0, -\frac{4\pi}{7}, \frac{4\pi}{7}$. By Lemma 3, the anti-Stokes line corresponding to $\arg(\gamma - i\xi) = 0$ is $\mathcal{A}_1$. The anti-Stokes line on the left half plane $\mathcal{A}_2$ can not asymptote to ∞ in the third quadrant since $P(\xi) \sim e^{-\frac{4\pi}{7}} \ln \xi$; so it must intersect either negative real axis or negative imaginary axis. by Lemma 4, it can not intersect negative imaginary axis since $\text{Im } P(i\rho) \equiv \text{Im } P(0) > 0$. Therefore $\mathcal{A}_2$ must intersect negative real axis and go to second quadrant. by Lemma 4 and lemma 5, it must end up with the singular point i . $\square$

The anti-Stokes lines are shown in Figure A2.

Figure A.2: Anti-Stokes lines defined by $\text{Im } P(\xi) = 0$.

Now we discuss more properties of function $P(\xi)$.

Lemma A.15. *There exists a constant R so that for $\xi \in \mathcal{R}^-$, $|\xi| > R$, $\operatorname{Re} P(t)$ is increasing along ray $r = \{t : t = \xi + se^{i\varphi}, 0 < s < \infty\} \subset \mathcal{R}^-$ from ξ to $\xi + \infty e^{i\varphi}$ and $C_1|t - 2i|^{-1} \leq \left| \frac{d}{ds} \operatorname{Re} P(t(s)) \right| \leq C_2|t - 2i|^{-1}$, where C_1 and C_2 are constants depending only on R .*

Proof. The lemma follows from the fact that $P(\xi) \sim e^{-\frac{\pi}{4}} \ln \xi$, as $|\xi| \rightarrow \infty$, $\xi \in \mathcal{R}^-$. $\square$

Lemma A.16. *If $10|\gamma - 1| \leq |b - 1|$, then $\operatorname{Re} P(t)$ is increasing along ray $r_l = \{\xi = -bi + se^{i4\pi/3}, 0 \leq s < \infty\}$.*

Proof. We want to show that

$$\frac{d}{ds} \operatorname{Re} P(\xi(s)) = \operatorname{Re}\{P'(\xi)e^{i4\pi/3}\} > 0 \text{ on } r_l. \quad (\text{A.8})$$

Note:

$$P'(\xi) = i \frac{(\xi - i\gamma)}{(\xi + i)} \left(\frac{(\gamma + i\xi)}{(\gamma - i\xi)} \right)^{1/4} \frac{1}{(1 + i\xi)}; \quad (\text{A.9})$$

Let $B(s)$ be the positive angle between $\xi(s) + \gamma i$ and $\xi(s) + i$, then by geometry:

$$\arg P'(\xi) \in (-B + \frac{\pi}{4}, \frac{2\pi}{3}); \quad (\text{A.10})$$

we can see that $B < \frac{\pi}{12}$ implies (8). Let $d_1 = |b - 1| + |\gamma - 1|$, by geometry:

$$\cos B = \frac{(s^2 - 2s|b - 1| \sin \frac{\pi}{3} + |b - 1|^2) + (s^2 - 2sd_1 \sin \frac{\pi}{3} + d_1^2) - |\gamma - 1|^2}{2\sqrt{(s^2 - 2s|b - 1| \sin \frac{\pi}{3} + |b - 1|^2)}\sqrt{(s^2 - 2sd_1 \sin \frac{\pi}{3} + d_1^2)}} \quad (\text{A.11})$$

Let $t = \frac{|\gamma - 1|^2 s}{b - 1| \sin \frac{\pi}{3}}$, $d = \frac{|\gamma - 1|}{|b - 1|}$,

$$\cos B = \frac{(t - 1 - \frac{d}{2})^2 + (1 + d) \cot^2 \frac{\pi}{3} - \frac{1}{4}d^2}{\sqrt{(t - 1)^2 + \cot^2 \frac{\pi}{3}} \sqrt{(t - 1 - d)^2 + (1 + d)^2 \cot^2 \frac{\pi}{3}}} \quad (\text{A.12})$$

The min of the above function over $0 < t < \infty, d \leq 0.1$ is .9688749307, but $\cos \frac{\pi}{12} = 0.9330127$, so $B < \frac{\pi}{12}$. $\square$

Lemma A.17. *If $10|c - 1| \leq |b - 1|$, for any $\xi \in \mathcal{R}_- \cap \{Im\xi < 0\}$, there exists a curve $\mathcal{P}(\xi, \xi_0)$ connecting ξ and some point ξ_0 on real negative axis so that $Re P(t)$ is increasing along the curve from ξ to ξ_0 .*

Proof. By above lemma we have $Re P(t)$ is increasing along ray $r_t = \{\xi = -bt + se^{i(\varphi_0 + \mu)}\}$. Since $Re P(t)$ is decreasing on negative real axis and increasing from origin to $-bt$, let $\eta = P(t)$, then η is one-one mapping on the boundary of $\mathcal{R}_- \cap \{Im\xi < 0\}$, therefore η conformly maps region $\mathcal{R}_- \cap \{Im\xi < 0\}$ to a region $\eta(\mathcal{R}_-)$ in η -plane. (see figure A3). Now for any point $\eta \in \eta(\mathcal{R}_-)$, we can find a point $\eta_0 \in \eta(-\infty, 0)$, so that on line segment $\{\eta_0 + s(\eta - \eta_0)\}$ the real part is increasing. Let $\xi_0 = \eta^{-1}(\eta_0)$, $\mathcal{P}(\xi, \xi_0) = \eta^{-1}(\{\eta_0 + s(\eta - \eta_0)\})$, we have the lemma. $\square$

Figure A.3: Conformal mapping $\eta = P(t)$.

Figure A.4: $\arg P'(t)$ on the boundary of second quadrant.

Lemma A.18.

$$\arg P'(\xi) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \text{ for } \xi \text{ in the second quadrant:} \quad (\text{A.13})$$

Proof. The values of $P'(\xi)$ on the boundary of the second quadrant are shown in figure A.4. Since $\arg P'(\xi)$ is harmonic in the second quadrant, the lemma follows from the strong maximum principle. $\square$

Lemma A.19. For ξ in the second quadrant, $\operatorname{Re} P(t)$ increases along line $\{t = \xi - s, 0 \leq s < \infty\}$ from ξ to $\xi - \infty$.

Proof. $\frac{d}{ds} \operatorname{Re} P(t(s)) = -\operatorname{Re} P'(t(s)) > 0$, since $\arg P'(t(s)) \in (\frac{\pi}{2}, \frac{3\pi}{2})$. $\square$

Lemma A.20. For sufficiently small ν , $\frac{d}{ds} [\operatorname{Re} P(t(s))] > 0$ on line $\{t(s) = b_1 i + s e^{-i\frac{3\pi}{4}}, 0 \leq s \leq \nu\}$, where $0 < b_1 < a$.

Proof. Since $\arg P'(b_1 i) = \frac{\pi}{2}$, for sufficiently small ν , we have $\arg P'(t(s)) \in (\frac{\pi}{2}, \frac{3\pi}{4})$; therefore $\arg(P'(t(s))e^{-i\frac{3\pi}{4}}) \in (-\frac{\pi}{4}, 0)$. Hence

$$\frac{d}{ds} \operatorname{Re} P(t(s)) = \operatorname{Re} (P'(t(s))t'(s)) = \operatorname{Re} (P'(t(s))e^{-i\frac{3\pi}{4}}) > 0$$

$\square$

Lemma A.21. $\operatorname{Re} P(\xi)$ attains its minimum in $\mathcal{R}$ at $\xi = ai$.

Proof. The lemma follows from the Lemma 17, lemma 19 and Lemma 20 and the fact that $\operatorname{Re} P(t)$ is decreasing along the imaginary axis from $-bi$ to ai . $\square$

Lemma A.22. For any $\xi \in \mathcal{R}_-$, there is a path $\mathcal{P}(\xi, -\infty)$ connecting ξ to $-\infty$ so that $\operatorname{Re} P(t)$ is increasing from ξ to $-\infty$.

Proof. i) for $\xi \in \mathcal{R}_-$ and $|\xi| \geq R$, by lemma 16, $\mathcal{P}(\xi, -\infty) = \{t = \xi - s, 0 \leq s < \infty\}$ will work. ii) for $\xi \in \mathcal{R}_- \cap \{Im \xi \geq 0 \text{ and } |\xi| \leq R\}$, by Lemma 19, $\mathcal{P}(\xi, -\infty) = \{t = \xi - s, 0 \leq s < \infty\}$ will work. iii) for $\xi \in \mathcal{R}_- \cap \{Im \xi < 0\}$ and $|\xi| \leq R$, by Lemma 17, $\mathcal{P}(\xi, -\infty) = \mathcal{P}(\xi, \xi_0) \cup (-\infty, \xi_0)$ will work. $\square$

Figure A.5: Path $P(\xi, \alpha i)$ in Lemma A.23.

Lemma A.23. *For any $\xi \in \mathcal{R}_- \cap \{|\xi| \leq R\}$, there is a path $\mathcal{P}(\xi, ai)$ connecting ξ to ai so that $\operatorname{Re} P(t)$ is increasing from ai to ξ , and $\frac{d}{ds} \operatorname{Re} P(t(s)) \geq C > 0$ on $\mathcal{P}(\xi, ai)$; where C is a constant independent of ϵ and b .*

Proof. i) for $\xi \in \mathcal{R}_- \cap \{\operatorname{Im} \xi \geq 0 \text{ and } |\xi| \leq R\}$, by Lemma 19 and Lemma 20, $\mathcal{P}(\xi, ai) = \{t = \xi + s, 0 \leq s < |\operatorname{Re} \xi + \nu| \cup \{t = -\nu + i \operatorname{Im} \xi + se^{i\pi/4}\} \cup (i(\operatorname{Im} \xi + \nu), ai)$ will work. (see figure) iii) for $\xi \in \mathcal{R}_- \cap \{\operatorname{Im} \xi < 0\}$, by anti-stokes picture, the anti-stokes line $\operatorname{Im} P(t) = \operatorname{Im} P(\xi)$ must intersect negative real axis at some point ξ_0 , let $\mathcal{P}(\xi, \xi_0)$ be that part of the antistokes line. $\mathcal{P}(\xi, ai) = \mathcal{P}(\xi, \xi_0) \cup \mathcal{P}(\xi_0, ai)$ will work, where $\mathcal{P}(\xi_0, ai)$ is chosen as in i). $\square$

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