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CANDELABRA SYSTEMS AND DESIGNS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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* * * * *

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ABSTRACT

Design theory has various applications in other fields including coding theory, statistics, computer science and cryptography. For example, 2-designs have been used from the early years of this century for optimal planning of statistical experiments. This dissertation contributes a method of creating new 3-designs.

A t -(v, k, λ) design is a pair $(X, \mathcal{A})$, where X is a set of v elements (called points) and a family $\mathcal{A}$ of subsets of X , each of size k (called blocks) such that each t -element subset of points of X is contained in exactly λ blocks. A t -(v, k, λ) design is also denoted by $S_\lambda(t, k, v)$.

The problem of finding all parameters (t, v, k, λ) for which t -designs exist is a long standing unsolved problem. It has been proved that if λ is large and the parameters (t, v, k, λ) satisfy the necessary arithmetic conditions then there exists a t -(v, k, λ) design. If $\lambda = 1$, a t -($v, k, 1$) design is called a Steiner t -design (or Steiner system) and is also denoted by $S(t, k, v)$. Steiner designs are particularly important in application problems like statistical experiments and network designs where they are used to minimize construction costs. There are many known Steiner 2-designs, but constructing Steiner t -designs with $t > 2$ is much harder.

Problems that are of interest in design theory are (1) determining all quadruples

(t, v, k, λ) for which a t -(v, k, λ) design exists (the existence problem). (2) economical construction of the blocks of a t -(v, k, λ) design (the construction problem). (3) finding t -designs with large automorphism groups (the symmetry problem) and (4) enumerating the number of distinct t -(v, k, λ) designs on a given point set of size v (the enumeration problem).

This dissertation contributes to a solution of the existence problem for $t > 2$, particularly for $t = 3$ and $\lambda = 1$ (the Steiner 3-designs). Combinatorial structures called candelabra systems can be used in recursive constructions to build new Steiner 3-designs. We introduce a new closure operation on natural numbers involving candelabra systems. Many of the constructions of Steiner designs are based on the “block spreading” construction theorem. The generalization of this theorem is presented in this dissertation. This new closure operation and the generalized “block spreading” construction make it possible to generalize various constructions for Steiner 3-designs and to create new infinite families of Steiner 3-designs as follows: let q be a prime power. Assume that there is a Steiner 3-design 3 -($a + 1, q + 1, 1$), $a > 1$. For every v satisfying certain necessary arithmetic conditions we can construct a Steiner 3-design 3 -($va^d + 1, q + 1, 1$) for every d sufficiently large. In the case of block size 6 this theorem yields new infinite families of Steiner 3-designs: if v is a given positive integer satisfying certain arithmetic conditions, for every non-negative integer m there exists a Steiner 3-design 3 -($v(4 \cdot 5^m + 1)^d + 1, 6, 1$) for sufficiently large d .

This new theory produces new Steiner 2-designs as well: If q and $q + 1$ are both prime powers the existence of the Steiner 2-designs 2 -($a + 1, q + 1, 1$) and 2 -($b + 1, q + 1, 1$) implies the existence of a Steiner 2-design 2 -($ab + b + 1, q + 1, 1$).

Dedicated to my parents and my husband

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CHAPTER 1

INTRODUCTION

Combinatorial designs were introduced by T. P. Kirkman (1806-1895) and independently by Jacob Steiner (1796-1895) in the nineteenth century. Let v be a non-negative integer and k and λ be positive integers, $k \geq 2$. A 2-design on v points with block size k and index λ (or a $2-(v, k, \lambda)$ design) is a pair $(X, \mathcal{A})$, where X is a set of v elements (called points) and $\mathcal{A}$ is a family of subsets of X , each of size k (called blocks) such that each 2-subset of points is contained in exactly λ blocks.

T. P. Kirkman was a minister of the Church of England in the middle of nineteenth century. He educated himself in mathematics and the well-known *15 schoolgirls problem* was proposed by him. The problem is to arrange 15 schoolgirls in five rows of three for seven days' walks such that every two of them walk in a row exactly once. This is equivalent to constructing a $2-(15, 3, 1)$ design whose blocks are partitioned into seven parallel classes. In general, a $2-(v, k, 1)$ design is called resolvable if the blocks can be partitioned into parallel classes.

The method of differences may be used to construct 2-designs. For instance, let $\mathbb{Z}_7$ be the set of points. The blocks are all the sets $B_x = \{x, x + 1, x + 3\}$, where

$x \in \mathbf{Z}_7$. It is not hard to see that $(\mathbf{Z}_7, \{B_x : x \in \mathbf{Z}_7\})$ is a 2-(7,3,1) design. This design is called the *Fano plane* and can be represented by figure 1.1 :

Figure 1.1: Fano plane

The dots represent the point set and the lines and the circle represent the blocks of the design.

The idea of this construction can be generalized. Note that the six differences between the elements of the set $B_0 = \{0, 1, 3\}$ are exactly the non-zero elements of $\mathbf{Z}_7$. This fact makes it possible to construct 2-designs. In other words, $B_0 = \{0, 1, 3\}$ is a difference set on the group $\mathbf{Z}_7$ with parameters (7, 3, 1). It follows easily that the family of blocks $(B_x : x \in \mathbf{Z}_7)$ is a 2-(7,3,1) design with $\mathbf{Z}_7$ as the point set.

R. C. Bose (1901-1987) constructed many families of difference sets. He was a famous statistician and combinatorial mathematician, who contributed many important theorems and constructions to mathematical design theory.

The most well-known designs among the mathematicians are the projective planes and the affine planes. A projective plane of order q (q a prime power) can be viewed as a $2-(q^2 + q + 1, q + 1, 1)$ design. We can construct a $2-(q^2 + q + 1, q + 1, 1)$ design in a simple way. Let $\mathbf{F}_q^3$ denote the three dimensional vector space over the finite field $\mathbf{F}_q$ of q elements. We consider the set of 1-dimensional subspaces as the set of points and the set of 2-dimensional subspaces as the set of blocks of the $2-(q^2 + q + 1, q + 1, 1)$ design. A point is contained in a block if and only if the 1-dimensional subspace is contained in the 2-dimensional subspace. The Fano plane is isomorphic to the projective plane of order 2. There is a very close relationship between projective planes and affine planes. If we delete a block of the projective plane of order q and its points, the resulting design is an affine plane of order q , i.e. a $2-(q^2, q, 1)$ design. This construction is reversible.

If a $2-(v, k, \lambda)$ design exists, then the total number of blocks b is $\lambda \frac{v(v-1)}{k(k-1)}$ and the number of blocks r (*replication number*) containing a given point is $\lambda \frac{v-1}{k-1}$. Since b and r are necessarily integers, the existence of a $2-(v, k, \lambda)$ design implies that the parameters (v, k, λ) must satisfy the following arithmetic conditions:

- (1) $\lambda(v-1) \equiv 0 \pmod{k-1}$
- (2) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

However, these conditions are not always sufficient for the existence of a $2-(v, k, \lambda)$ design. The existence question has been solved for small values of k . For instance, if $k = 3, 4$, then for fixed k and λ , the necessary arithmetic conditions are also sufficient for the existence. More detailed existence results for $2-(v, k, \lambda)$ designs for $k \leq 9$ can be found in [10] on page 41.

Another necessary condition for the existence of a 2-design is the famous *Fisher's inequality*. Namely, if a $2-(v, k, \lambda)$ design exists with $2 \leq k < v$, then $b \geq v$.

A 2-design with $b = v$ is called a symmetric design. The $2-(7, 3, 1)$ design, constructed by using the method of difference sets at the beginning of the introduction, is a symmetric design. More generally, the method of difference sets produces symmetric designs in the following way: let $D = \{d_1, d_2, \dots, d_k\}$ be a k -subset of an abelian group G of cardinality v . If every non-zero element of G occurs exactly λ times among the $k(k-1)$ differences $d_i - d_j$ of D , $i \neq j$ and $i, j = 1, 2, \dots, k$, then D is called a (v, k, λ) -difference set on G . It is easy to see that if D is a (v, k, λ) -difference set on an abelian group G , then $(G, \{D + g : g \in G\})$ is a $2-(v, k, \lambda)$ symmetric design.

If a symmetric $2-(v, k, \lambda)$ design exists, then $\lambda(v-1) = k(k-1)$. The most well known non-existence result for symmetric designs is the celebrated Bruck-Ryser-Chowla theorem.

If v, k, λ are integers such that $\lambda(v-1) = k(k-1)$, then for the existence of a symmetric $2-(v, k, \lambda)$ design it is necessary that:

1. if v is even then $k - \lambda$ is a square;
2. if v is odd, then the equation $z^2 = (k - \lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$ has a solution in integers x, y, z , not all zero.

A symmetric design with $\lambda = 1$ is equivalent to a projective plane. For a symmetric $2-(v, k, 1)$ design, we have

$$\frac{v(v-1)}{k(k-1)} = v \text{ or } v = k^2 - k + 1.$$

Let $n = k - 1$, which is called the order of the projective plane, then the parameters of the projective plane are:

$$v = n^2 + n + 1, \quad k = n + 1, \quad \lambda = 1.$$

As we have already seen, a projective plane of order n exists for any prime power n . The question of whether there are any projective planes of non-prime power order naturally arises. The smallest non-prime power value is $n = 6$. By the Bruck-Ryser-Chowla theorem, if a projective plane of order 6 exists, then the equation $z^2 = 6x^2 - y^2$ has a non-trivial integer solution. It is easy to see that the equation has no non-trivial solution, thus a projective plane of order 6 does not exist. The next questionable value is $n = 10$. In this case, the equation $z^2 = 10x^2 - y^2$ has the non-trivial integer solution $x = 1$, $y = 1$ and $z = 3$ and hence the theorem is inconclusive. The non-existence of a projective plane of order 10 is proved by application of coding theory and an extensive computer search. No projective planes of non-prime power are known and there are no non-existence theorems for non-prime power order projective planes other than the Bruck-Ryser-Chowla theorem.

The Bruck-Ryser-Chowla theorem has the following simple form for $\lambda = 1$:

Let $n \equiv 1$ or $2 \pmod{4}$. If $n = n_1^2 p_1 p_2 \dots p_s$, where n_1 is a positive integer and p_i 's are distinct primes $1 \leq i \leq s$, and for some i , $p_i \equiv 3 \pmod{4}$, then a projective plane of order n do not exist.

For example projective planes (and affine planes) of order 6, 14, 21 ... do not exist. The first unsettled value of n is 12.

Most of the earlier applications of designs were related to statistical experiments

and many of these used 2-designs. The famous statistician R. A. Fisher applied 2-designs for agricultural experiments at the beginning of the century. In the 20th century because of applications in optimal planning of statistical experiments and close connections with coding theory, research on design theory continued with much intensity and it was necessary to generalize the idea of 2-designs. Next we give the definition of a t -design for any positive integer t .

A t -(v, k, λ) design is a pair $(X, \mathcal{A})$, where X is a set of v elements (called points) and a family $\mathcal{A}$ of subsets of X , each of size k (called blocks) such that each t -element subset of points of X is contained in exactly λ blocks. A t -(v, k, λ) design is also denoted by $S_\lambda(t, k, v)$. If $\lambda = 1$, a t -($v, k, 1$) design is called a Steiner t -design (or Steiner system) and is also denoted by $S(t, k, v)$. A design without repeated blocks is called a simple design. There are many known 2-designs, but to construct simple or Steiner t -designs with $t > 2$ is much harder.

Problems that are of interest in design theory include (1) determining all quadruples (t, v, k, λ) for which a t -(v, k, λ) design exists (the existence problem), (2) economical construction of the blocks of a t -(v, k, λ) design (the construction problem), (3) finding t -designs with large automorphism groups (the symmetry problem) and (4) enumerating the number of distinct t -(v, k, λ) designs on a given point set of size v (the enumeration problem).

This dissertation is on the existence problem for $t > 2$, particularly for $t = 3$ and $\lambda = 1$ (the Steiner systems). If a t -(v, k, λ) design exists, the parameters t, v, k, λ

satisfy the arithmetic conditions

$$\lambda \binom{v-i}{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}}, \text{ for } i = 0, 1, \dots, t.$$

A parameter quadruple $t-(v, k, \lambda)$ is called admissible if it satisfies the arithmetic conditions and feasible if a $t-(v, k, \lambda)$ design exists. Dijen K. Ray-Chaudhuri conjectures that for fixed t , k , and λ , if v is sufficiently large, admissibility implies feasibility. This conjecture will be referred to as the " v -large existence conjecture". For $t = 1$, admissibility implies feasibility. Richard M. Wilson [20], [21] and [22] proved that for $t = 2$, fixed k and sufficiently large v , admissibility implies feasibility. There is no similar general existence result yet for $t-(v, k, \lambda)$ designs, when $t > 2$. In the $t = 3$ case, Hanani [14] has shown that there is a $3-(v, 4, 1)$ design (Steiner quadruple system) if and only if the necessary arithmetic conditions are satisfied. In other words the parameter quadruple $3-(v, 4, 1)$ is feasible if and only if it is admissible. Several infinite families of Steiner 3-designs are known:

1. There is a Steiner $3-(q^n + 1, q + 1, 1)$ design for every prime power q and positive integer n (Spherical geometries. If $n=2$, a $3-(q^2 + 1, q + 1, 1)$ -design is called Möbius plane or inversive plane) [1].
2. (Hanani) If q is a prime power and a Steiner $3-(v + 1, q + 1, 1)$ design exists, then a Steiner $3-(vq + 1, q + 1, 1)$ design exists [15].
3. (Hanani) If a Steiner $3-(v + 1, 6, 1)$ design exists, then a Steiner $3-(4v + 2, 6, 1)$ design exists [15].

4. (Blanchard) Let q and u be prime powers. Then, there is a constant $n_0 = n_0(q, u)$ such that there exists a Steiner $3-(u^l q^n + 1, q + 1, 1)$ design for all $l \geq 0$ and $n \geq n_0$ if the parameters are admissible. [10] and [6]
5. (Wilson) Let q be a prime power. If a Steiner $3-(a + 1, q + 1, 1)$ design and a Steiner $3-(b + 1, q + 1, 1)$ design exist, then there exists a Steiner $3-(ab + 1, q + 1, 1)$ design.
(This product theorem and the previous theorem appears in Blanchard's preprint [6] and the product theorem is attributed to Wilson. However, we could not find any published proof of these theorems. Also, these theorems are stated in the CRC Handbook on page 67 [10]. In chapter IV we give an independent proof for Wilson's product theorem.)
6. (Blanchard) Let q be a prime power. For every v satisfying the necessary arithmetic conditions we can construct a Steiner $3-(vq^n + 1, q + 1, 1)$ design for every sufficiently large n [2].

In chapter V, we construct new infinite families of Steiner 3-designs given as follows: let q be a prime power and a a positive integer. Assume that there is a Steiner $3-(a + 1, q + 1, 1)$ design, where $a \geq 2$. Then there exists an integer $d_0 = d_0(q, v, a)$ such that for any positive integer v satisfying the arithmetic conditions

- i. $v - 1 \equiv 0 \pmod{q - 1}.$
- ii. $v(v - 1) \equiv 0 \pmod{q(q - 1)}.$
- iii. $v(v^2 - 1) \equiv 0 \pmod{q(q^2 - 1)}$

there is a Steiner $3-(va^d + 1, q + 1, 1)$ design for all $d \geq d_0$. This theorem generalizes Blanchard's theorem for Steiner 3-designs, which we have stated on page 8 in theorem 6. In particular this generalization yields new infinite families of Steiner 3-designs with block size 6. Let m be a non-negative integer. Then there exists an integer $d_0 = d_0(v, m)$ such that for any positive integer v satisfying the arithmetic conditions

$$\begin{aligned} v &\equiv 1 \pmod{20} \text{ or} \\ v &\equiv 5 \pmod{20}, \end{aligned}$$

there exists a Steiner $3-(v(4 \cdot 5^m + 1)^d + 1, 6, 1)$ design for all $d \geq d_0$.

Only finitely many Steiner t -designs are known for $t = 4$ and $t = 5$ and no Steiner t -designs are known for $t \geq 6$, when $v > k$. However, for fixed t , k and v positive integers with $t \leq k \leq v$, the λ -large theorem (Wilson [1], Graver and Jurkhart [13], Graham, Li and Li [12]) guarantees the existence of a t -(v, k, λ) design for sufficiently large λ whenever the parameters are admissible.

My research project is to make progress toward verifying the v -large existence conjecture for the case $t = 3$.

Group divisible designs are important in the constructions of 2-designs. We generalized the definition of group divisible design for $t \geq 2$. Let v be a non-negative integer, λ a positive integer and K a set of positive integers. A group divisible t -design (or t -GDD) of order v , index λ and block sizes K is a triple $(X, \Gamma, \mathcal{A})$ where,

1. X is a set of size v (called *points*),
2. $\Gamma = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of X , which partition X (called *groups*).

3. $\mathcal{A}$ is a family of subsets of X each of cardinality from K (called *blocks*) such that each block intersects any given group in at most one point.
4. each t -set of points from t distinct groups is in exactly λ blocks.

Note that the parts $G_1, G_2 \dots$ of Γ are not necessarily groups in an algebraic sense. (See figure 1.2. A represents a block of the t -GDD.)

Figure 1.2: Group divisible t -design

A t -GDD with block size k and k groups of uniform group size is called a transversal design and denoted by $TD_\lambda(t, k, m)$ (if $\lambda = 1$, then $TD(t, k, m)$). They are particularly important in the composition methods. (See figure 1.3. Note that a typical block A intersects each group in exactly one point and each group has the same size.)

In Wilson's existence proof, 2-designs are recursively built up from smaller designs and group divisible designs as follows: let v , m and k be positive integers. Bose,

Figure 1.3: Transversal t -design

Shrikhande and Parker [8] prove that, if there is a $2-(v, k, 1)$ design, a $TD(2, k, m)$ and a $2-(m, k, 1)$ design, then there exists a $2-(mv, k, 1)$ design. Wilson [20] generalized this recursive construction greatly.

Our goal is to give a similar fundamental construction for the $t = 3$ case and make significant progress toward the proof of the v -large existence conjecture for the $t = 3$ case. For this purpose, we need to introduce another combinatorial structure, called a *candelabra system*, which will be used in addition to group divisible designs in the recursive constructions in the $t = 3$ case. Our definition of the candelabra system is a generalization of candelabra quadruple system described in the paper of Hartman and Phelps [2, p. 205].

Let v be a non-negative integer, λ and t be positive integers and K be a set of positive integers. A *candelabra t -system* (or *t -CS*) of order v is a quadruple $(X, S, \Gamma, \mathcal{A})$ that satisfies the following properties:

1. X is a set of v elements (called *points*).
2. S is a subset of X of size s (called the *stem* of the candelabra).
3. $\Gamma = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of $X \setminus S$ which partition $X \setminus S$ (called *groups or branches*).
4. $\mathcal{A}$ is a family of subsets of X , each of cardinality from K (called *blocks*).
5. Every t -subset $T \subset X$ with $|T \cap (S \cup G_i)| < t$ for all i , is contained in exactly λ blocks and no t -subsets of $S \cup G_i$ for all i , are contained in any block.

Figure 1.4 shows a candelabra t -system for the $t = 3$ and $\lambda = 1$ case. The shape of the figure justifies the name "candelabra". The dots represent all the three different types of 3-sets satisfying property 5 of the definition. We indicated three different types of blocks.

Figure 1.4: Candelabra t -system for $t = 3$ and $\lambda = 1$

In the $t = 2$ case, a candelabra system becomes a group divisible design since no blocks can intersect the stem. Thus the candelabra system is a possible generalization of group divisible designs for $t > 2$. The following composition theorem is a special case of Hanani's general composition theorem [16]. Many of the new theorems of this dissertation are based on this composition theorem.

Let a , b and k be positive integers and let $s \in \{0, 1, 2\}$. If there exist a Steiner $3-(a+1, k+1, 1)$ design, a Steiner $3-(b+s, k+1, 1)$ design, a 3-CS of group type $(b^k : s)$ with block size $k+1$, (a 3-CS with k groups of size b with stem size s and block size $k+1$), and a $TD(3, k+1, b)$, then there exists a Steiner $3-(ab+s, k+1, 1)$. This is a generalization of the product construction 5 on page 8.

Closure operations are also important tools in the construction of new designs. Closed subsets of natural numbers play a significant role in Wilson's proof of the existence theorem for $t = 2$.

The mapping $K \mapsto B(K) := \{v | v \in \mathbb{N}, 2 - (v, K, 1) \text{ design exists}\}$ satisfies the properties of a closure map (enlarging, monotone and idempotent). Wilson's proof for $t = 2$ depended on the beautiful theorem that closed sets of natural numbers under this closure operation are "almost periodic" and the fact that the set of replication numbers (the number of blocks containing $t - 1$ points) is a closed set for $t = 2$ with respect to this closure operation. This fact is essential in the proof of the existence theorem for $t = 2$. If $t = 3$, the set of replication numbers, however, is not a closed set under this closure operation. This gives the motivation to define a new closure operation involving candelabra systems, such that a constant multiple of the set of replication numbers would be a closed set under this new operation.

For fixed sets K and J of positive integers and for fixed non-negative integers s and t , $C_K^t[J, s]$ will denote the set of positive integers w for which there exists a t -CS $(X, S, \Gamma, \mathcal{A})$ with $|X| = w + s$, $|S| = s$, group sizes from J and block sizes from K . For fixed t , K , and s the mapping $J \mapsto C_K^t[J, s]$ is easily seen to be a closure map on $\mathbb{N}$ (natural numbers).

This new closure operation makes it possible to generalize various constructions for Steiner 3-designs and to create new Steiner 3-designs. This closure operation suggests a potential method to construct new Steiner t -designs for $t \geq 3$.

We define new combinatorial structures, *lattice group divisible designs* and *lattice candelabra systems*, which in addition to their own interesting properties, play an important role in the construction of Steiner 3-designs. Using these new structures and the theory of closed sets we create a new technique to construct designs. This theory gives an independent proof of the product theorem. Specifically, if q is a prime power and a Steiner 3- $(a + 1, q + 1, 1)$ design and a Steiner 3- $(b + 1, q + 1, 1)$ design exist then there exists a Steiner 3- $(ab + 1, q + 1, 1)$ design. Also, we present a theorem on new Steiner 2-designs: if q and $q + 1$ are both prime powers the existence of the Steiner 2-designs 2- $(a + 1, q + 1, 1)$ and 2- $(b + 1, q + 1, 1)$ implies the existence of a Steiner 2-design 2- $(ab + b + 1, q + 1, 1)$.

As a result of this new theory several interesting results are obtained in addition to our main results on designs. One of the most interesting results is the following. If q is a prime power then the existence of a Steiner 3- $(a + 1, q + 1, 1)$ design implies the existence of a transversal 3-design of $q + 1$ groups of size a .

Wilson's "block spreading" method is widely used in the construction of Steiner

2-designs. Blanchard generalizes this method for $t > 2$ [5]. The “block spreading” construction takes a group divisible design with large λ and yields a group divisible design with larger group size and with $\lambda = 1$. This construction is very useful in the construction of designs with $\lambda = 1$ since it is much easier to produce designs with large λ . We generalize Blanchard’s “block spreading” construction [5] for general λ and Blanchard’s construction for Steiner 3-designs [4] to obtain new infinite families of Steiner 3-designs, as we have stated the theorems before on page 8 and 9.

Chapter II covers the basic definitions of the combinatorial structures and their basic properties. Chapter III focuses on candelabra systems and their relations to t -designs. This chapter introduces a new closure operation involving candelabra system and characterizes the closed sets of this closure operation. New combinatorial structures, lattice candelabra systems and lattice group divisible designs, are introduced in chapter IV. Constructions for new Steiner 2 and 3-designs are also presented in this chapter. Finally, in chapter V we present the generalized “block spreading construction” theorem and constructions for new group divisible t -designs. We conclude with the construction of new infinite families of Steiner 3-designs.

All results proved by other authors are attributed to them. New results proved in this dissertation are stated without attribution.

CHAPTER 2

BASIC CONCEPTS

2.1 t -Designs

Let X be a set and let $\mathcal{P}(X)$ denote the set of all subsets of X . A *family* of subsets on a set X with index set I is a mapping $\mathcal{A} : I \rightarrow \mathcal{P}(X)$ written as $(A_i | i \in I)$, where $A_i = \mathcal{A}(i)$ for all $i \in I$. If $I = \emptyset$, we call $\mathcal{A}$ the empty family.

Definition 2.1.1 *Let v be a non-negative integer, let λ and t be positive integers and let K be a set of positive integers. A t -(v, K, λ) design is a pair $(X, \mathcal{A})$, where*

1. X is a set of v elements (called points),
2. $\mathcal{A}$ is a family of subsets of X , each of cardinality from K (called blocks),
3. every t -subset of distinct points occurs in exactly λ blocks.

A t -(v, K, λ) design is also denoted by $S_\lambda(t, K, v)$. We also say $(X, \mathcal{A})$ is a t -design of order v with block sizes from K and index λ . A subset $F \subseteq X$ is called a *subdesign*, if for any t -subset of points of F , F contains completely the λ blocks of $\mathcal{A}$ passing through the given points. If K contains a single element k , then we

write k instead of $\{k\}$. When $\lambda = 1$, then we omit λ and write $S(t, K, v)$. If $(X, \mathcal{A})$ is an $S(t, K, v)$ (with $\lambda = 1$), then every block occurs exactly once in the family $\mathcal{A}$ and hence the family of blocks $\mathcal{A}$ can be viewed as a subset of $\mathcal{P}(X)$. A *Steiner system* $S(t, k, v)$ is a t -design with uniform block size k and with $\lambda = 1$. Let $B_t[K, \lambda]$ denote the set of positive integers v , for which there exists an $S_\lambda(t, K, v)$, that is $B_t[K, \lambda] = \{v | S_\lambda(t, K, v) \text{ exists}\}$. When $\lambda = 1$, we use the notation $B_t[K]$. For any set K of positive integers, $(X, \emptyset)$ with $0 < |X| < t$ is an $S(t, K, |X|)$, where $\emptyset$ is the empty family. Thus, for any set K of positive integers, $v \in B_t[K]$ where $0 < v < t$.

Let R_k^t denote the set of replication numbers of an $S(t, k, v)$, that is $R_k^t = \{r | r(k - t + 1) + t - 1 \in B_t[k]\}$. Note that r is the number of blocks containing a given set of $t - 1$ points.

Necessary conditions for the existence of an $S_\lambda(t, k, v)$ are:

$$\lambda \binom{v-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad \text{for } s = 0, 1, \dots, t-1$$

In the introduction we discussed in detail those cases, when these conditions are sufficient for the existence of an $S_\lambda(t, k, v)$.

Since the terms *derived design* and *residual design* will occur several times in this dissertation, we will give the definitions at this time. The derived design and the residual design are constructed from a given existing design. Let $\mathcal{D} = (X, \mathcal{A})$ be an $S_\lambda(t, k, v)$, let $\mathcal{A} = (A_i | i \in I)$ and let x be a point of X . We define the *derived design* $\mathcal{D}_x$ of $\mathcal{D}$ at x , to be the subfamily of blocks containing x and the set of all points of

$\mathcal{D}$ except x , that is $\mathcal{D}_x = (X', \mathcal{A}')$, where $X' = X \setminus x$ and $\mathcal{A}' = (A_i \setminus x | i \in I, x \in A_i)$. It is clear that $\mathcal{D}_x$ is an $S_\lambda(t-1, k-1, v-1)$. $\mathcal{D}$ is called an extension of $\mathcal{D}_x$.

We can define the *residual design* $\mathcal{D}^x$ of $\mathcal{D}$ at x as the subfamily of blocks not containing x and the set of points of $\mathcal{D}$ except x . Thus, $\mathcal{D}^x = (X', \mathcal{A}')$ where $X' = X \setminus x$ and $\mathcal{A}' = (A_i | i \in I, x \notin A_i)$. It is not hard to see that, $\mathcal{D}^x$ is an $S_{\lambda'}(t-1, k, v-1)$, where $\lambda' = \lambda \frac{v-k}{k-t+1}$.

We need to define the concept of “copies of a design” which occurs in section 5.3. First we give an alternative equivalent definition for t -designs. Let X be a finite set and let $\mathcal{P}_k(X)$ denote the set of k -subsets of the set X . For any set of positive integers K let $\mathcal{P}_K(X) = \bigcup_{k \in K} (\mathcal{P}_k(X))$.

Definition 2.1.2 Let X be a set of v points, let K be a set of positive integers and let $\mathcal{A}$ be a mapping, $\mathcal{A} : \mathcal{P}_K(X) \longrightarrow \mathbb{N}_0$ ($\mathbb{N}_0$ set of non-negative integers).

Then $(X, \mathcal{A})$ is called a t -(v, K, λ) design, written $S_\lambda(t, K, v)$, if for every t -set $T \in \mathcal{P}_t(X)$

$$\sum_{T \subseteq A, A \in \mathcal{P}_K(X)} \mathcal{A}(A) = \lambda$$

If $\mathcal{A}(A) \in \{0, 1\}$ for all $A \in \mathcal{P}_K(X)$, then the design is called a simple design (a design without repeated blocks).

We say a design $\mathcal{D} = (X, \mathcal{A})$ consists of c copies of the design $\mathcal{D}' = (X, \mathcal{A}')$, if for all $A \in \mathcal{P}_K(X)$ $\mathcal{A}(A) = c \cdot \mathcal{A}'(A)$, and we write $\mathcal{D} = c \cdot \mathcal{D}'$. Therefore $\mathcal{D}$ is referred to as c copies of $\mathcal{D}'$.

2.2 Group Divisible t -Designs

The concept of group divisible designs plays an important role in the construction of 2-designs.

Definition 2.2.1 *Let v be a non-negative integer, let λ be a positive integer and let K be a set of positive integers. A group divisible design (or GDD) of order v , index λ and block sizes from K is a triple $(X, \Gamma, \mathcal{A})$ where,*

1. X is a set of size v (called points).
2. $\Gamma = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of X , which partition X (called groups, but they are not necessarily groups in an algebraic sense),
3. $\mathcal{A}$ is a family of subsets of X each of cardinality from K (called blocks) such that each block intersects any given group in at most one point.
4. every pair of points from distinct groups is in exactly λ blocks.

GDD's can be constructed recursively as it is stated in the next theorem.

A mapping $s : X \rightarrow \mathbb{N}$ is called a positive integral weight on X and written as $s_x = s(x)$ for $x \in X$.

Theorem 2.2.2 (Wilson [20], The fundamental construction for *GDD*'s) . Let $(X, \Gamma, \mathcal{A})$ be a *GDD* and let s_x be a positive integral weight assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation

$$S_Y = \bigcup_{x \in Y} S_x$$

for $Y \subseteq X$, put $X^* = S_X$ and $\Gamma^* = \{S_G : G \in \Gamma\}$. Suppose that for each block $A \in \mathcal{A}$, a *GDD* $(S_A, \{S_x : x \in A\}, \mathcal{B}_A)$ exists and denote $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$. Then $(X^*, \Gamma^*, \mathcal{A}^*)$ is a *GDD*.

The principal of Wilson's fundamental construction for *GDD* will be used for constructing other combinatorial structures as in section 3.2.

The next definition is a generalization of the concept of *GDD*'s for $t > 2$.

Definition 2.2.3 Let v be a non-negative integer, let λ and t be positive integers and let K be a set of positive integers. A group divisible t -design (or t -*GDD*) of order v , index λ and block sizes from K is a triple $(X, \Gamma, \mathcal{A})$ where,

1. X is a set of size v (called points).
2. $\Gamma = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of X which partition X (called groups).
3. $\mathcal{A}$ is a family of subsets of X each of cardinality from K (called blocks) such that each block intersects any given group in at most one point.
4. each t -set of points from t distinct groups is in exactly λ blocks.

Note that a 2-*GDD* is called a *GDD*. (Figure 2.1 represents a *t-GDD* with a typical block *A*.)

Figure 2.1: Group divisible *t*-design

A *list* over a set S is a mapping $l : S \rightarrow \mathbf{N}_0$, ($\mathbf{N}_0$ the set of non-negative integers), that is, by a list over the set S we mean a collection of elements s of S in which each element s occurs with non-negative multiplicity $l(s)$. A list can be informally written as $(s_1, s_2, \dots)$, where s_i 's are not necessarily distinct. By the group type of a *t-GDD* $(X, \Gamma, \mathcal{A})$ we mean the list $(|G| | G \in \Gamma)$ of group sizes. If a *t-GDD* has n_i groups of size g_i , $1 \leq i \leq r$, then the list contains each g_i counted n_i times. In this case we denote its group type by $(g_1^{n_1} g_2^{n_2} \dots g_r^{n_r})$.

Wilson's fundamental construction (theorem 2.2.2) can be generalized easily for the $t > 2$ case:

Theorem 2.2.4 (The fundamental construction for t -GDD's) . Let $(X, \Gamma, \mathcal{A})$ be a t -GDD and let s_x be a positive integral weight assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation

$$S_Y = \bigcup_{x \in Y} S_x$$

for $Y \subseteq X$, put $X^* = S_X$ and $\Gamma^* = \{S_G : G \in \Gamma\}$. Suppose that for each block $A \in \mathcal{A}$, a t -GDD $(S_A, \{S_x : x \in A\}, \mathcal{B}_A)$ exists and denote $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$. Then $(X^*, \Gamma^*, \mathcal{A}^*)$ is a t -GDD.

Using Wilson's terminology, we refer to a t -GDD with a weighting as recipe and the additional t -GDD's for each block as ingredients.

Proof: Let $\{x_1^*, x_2^*, \dots, x_t^*\}$ be a t -set of points of X^* from t distinct groups such that $x_i^* \in S_{x_i}$, where $x_i \in X$ for $i = 1, 2, \dots, t$. The t -set $\{x_1, x_2, \dots, x_t\}$ is in different groups of the t -GDD $(X, \Gamma, \mathcal{A})$, thus there is a unique block A of $\mathcal{A}$ containing it. Since $x_1^*, x_2^*, \dots, x_t^*$ belong to t different groups of the t -GDD $(S_A, \{S_x : x \in A\}, \mathcal{B}_A)$, there is a unique block of $\mathcal{B}_A$ containing $\{x_1^*, x_2^*, \dots, x_t^*\}$ and no other blocks of $\mathcal{A}^*$ can contain the t -set $\{x_1^*, x_2^*, \dots, x_t^*\}$.

Particularly, the t -GDD's with block size k and k groups of uniform group size m , besides their own interesting properties, play very important role in constructions of new designs. They are called *transversal designs* and denoted by $TD_\lambda(t, k, m)$. As in our previous notations, if $\lambda = 1$, then we use $TD(t, k, m)$. (See figure 2.2. Note that a typical block A intersects each group in exactly one point and each group has the same size m .)

Figure 2.2: Transversal t -design

The following two well-known results are stated in Hanani's paper [15]:

Theorem 2.2.5 (Hanani's paper [15]) *Let q be a prime power. If $t \leq q$, then there is a $TD(t, q + 1, q)$.*

Theorem 2.2.6 (Hanani's paper [15]) *If $q = 2^r$, then there is a $TD(3, q + 2, q)$.*

The next theorem immediately follows from the fundamental theorem and was proved by MacNeish when $t = 2$ and the proof is mentioned in a paper of Bose [7] for $t > 2$.

Theorem 2.2.7 (Bose's paper [7]) *Let t, v, u and k be positive integers with $1 \leq t \leq k$. If there exist a $TD(t, k, v)$ and a $TD(t, k, u)$, then there exists a $TD(t, k, uv)$.*

Blanchard [4] has proved the existence of a $TD(t, k, m)$ for large group size, when $t > 2$. The $t = 2$ case is the well-known result of Chowla-Erdős-Strauss [1]. They used different terminology, which we will discuss in section 2.4.

Theorem 2.2.8 (Blanchard [4]) *For any positive integers t and k , $1 \leq t \leq k$, there is an integer $m_0(t, k) = m_0$ such that for any integer $m \geq m_0$ there is a $TD(t, k, m)$.*

2.3 Candelabra t -Systems

In Wilson's existence proof [20], [21] and [22], 2-designs are recursively built up from "smaller" designs composed with a GDD . Our goal is to give a similar kind of fundamental construction for the $t = 3$ case. For this purpose, we need to introduce another combinatorial structure, called candelabra systems, which will be used in addition to the group divisible designs in the recursive constructions in the $t = 3$ case. The following definition of candelabra systems is a generalization of candelabra quadruple systems described in the paper of Hartman and Phelps [11].

Definition 2.3.1 *Let v be a non-negative integer, let λ and t be positive integers and let K be a set of positive integers. A candelabra t -system (or t -CS) of order v is a quadruple $(X, S, \Gamma, \mathcal{A})$ that satisfies the following properties:*

1. X is a set of v elements (called points).
2. S is a subset of X of size s (called the stem of the candelabra).

3. $\Gamma = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of $X \setminus S$, which partition $X \setminus S$ (called groups or branches).
4. $\mathcal{A}$ is a family of subsets of X , each of cardinality from K (called blocks).
5. Every t -subset $T \subset X$ with $|T \cap (S \cup G_i)| < t$ for all i , is contained in λ blocks and no t -subsets of $S \cup G_i$ for all i , are contained in any block.

Figure 2.3 shows a candelabra t -system for the $t = 3$ and $\lambda = 1$ case. The dots represent all the three different types of 3-sets satisfying property 5 of the definition. We indicated three different types of blocks.

Figure 2.3: Candelabra t -system for $t = 3$ and $\lambda = 1$

If $t = 1$, the family $\mathcal{A}$ of blocks must be the empty family. Thus, if Y_i , $i \in I$, and S are mutually disjoint sets, then $(\sum_{i \in I} Y_i \cup S, S, \{Y_i : i \in I\}, \emptyset)$ is a 1-CS. In the $t = 2$ case, if the family $\mathcal{A}$ of blocks is not the empty family, the set S (stem) can be neglected since no blocks of the candelabra system can intersect the stem and

$(X, \Gamma, \mathcal{A})$ is a 2-*GDD*. Thus, the candelabra system is a possible generalization of the group divisible design for $t > 2$.

By the group type of a t -*CS* $(X, S, \Gamma, \mathcal{A})$ we mean the list $(|G||G \in \Gamma : |S|)$ of group sizes and stem size. The stem size is separated from the group sizes by a colon. If a t -*CS* has n_i groups of size g_i , $1 \leq i \leq r$ and stem size s , then we use the notation $(g_1^{n_1} g_2^{n_2} \dots g_r^{n_r} : s)$ to denote the group type.

A candelabra system will be called uniform if and only if all groups have the same size. $NG_t[m, K, s]$ will denote the set of positive integers n for which there exists a uniform candelabra t -system of group type $(m^n : s)$ with block sizes from K and with $\lambda = 1$. The notation NG stands for "number of groups."

Candelabra systems were first introduced by Hanani [16], who used different terminology. He investigated uniform candelabra systems in the $t = 3$ and $k = 4$ case. In his paper he used the notation $P_g[4, 1, ng + s]$ to denote a uniform 3-*CS* of group type $(g^n : s)$ with block size 4. Mills [18] also discusses uniform candelabra systems with $t = 3$, $k = 4$ and $s = 0$, calling them *G*-designs.

Let Y and S be two mutually disjoint sets of points. For any positive integer t and for any set of positive integers K , $(Y \cup S, S, \{Y\}, \emptyset)$ is a candelabra t -system, where $\emptyset$ is the empty family. We call this candelabra system *trivial candelabra system* with block sizes from K . Thus, $1 \in NG_t[m, K, s]$ for any positive integers t and m , for any non-negative integer s and for any set of positive integers K .

For fixed sets K and J of positive integers and for fixed non-negative integers s and t , $C_K^t[J, s]$ will denote the set of positive integers w for which there exists a candelabra t -system $(X, S, \Gamma, \mathcal{A})$ with $|X| = w + s$, $|S| = s$, group sizes from J and

block sizes from K . If $t = 2$ and $S = 0$, then $C_K^2[J, 0] = G_K[J]$, where $G_K[J]$ defined by Wilson [20] denotes the set of positive integers v for which there exists a 2-GDD of order v , group sizes from J and block sizes from K .

Ray-Chaudhuri's and Singhi's result [19] guarantees the existence of a $TD_\lambda(t, k, m)$ for large values of λ .

Theorem 2.3.2 (Ray-Chaudhuri and Singhi [19]) *For any triple of positive integers (t, k, m) with $1 \leq t \leq k$, there is an integer $\lambda(t, k, m)$ such that for any integer $\lambda \geq \lambda(t, k, m)$ there exists a $TD_\lambda(t, k, m)$.*

2.4 Orthogonal Arrays and Ordered Designs

Definition 2.4.1 *Given t, k, v and λ positive integers, with $1 \leq t \leq k$. A t -(v, k, λ) orthogonal array of order v , strength t and index λ , denoted by $OA_\lambda(t, k, v)$, is a quadruple (R, C, S, O) where,*

R is a set of k elements (called rows),

C is a set of λv^k element (called columns),

S is a set of v element (called symbols) and

O is a map $O : R \times C \longrightarrow S$ such that

for every given set of t distinct elements $(r_1, r_2 \dots r_t)$ of R and for every ordered t -tuple $(s_1, s_2 \dots s_t)$ of the set of symbols S the system of equations

$$\begin{aligned} O(r_1, y) &= s_1 \\ O(r_2, y) &= s_2 \\ &\vdots \\ O(r_t, y) &= s_t \end{aligned}$$

has exactly λ solutions.

We note that in Beth, Jungnickel, Lenz [1], $OA_\lambda(2, k, v)$ is denoted as $OA(v, k, \lambda)$.

For any positive integer s let $I_s = \{1, 2 \dots s\}$. An $OA_\lambda(t, k, v)$ can be viewed as a $k \times \lambda v^t$ -matrix with entries from I_v , such that in each set of t rows, every ordered t -tuple of I_v occurs as a column exactly λ times.

When the index $\lambda = 1$, we write $OA(t, k, v)$. An $OA_\lambda(2, k, v)$ is called idempotent, if it contains every constant column $(x, x \dots x)$ exactly λ times.

The existence of an $OA_\lambda(t, k, v)$ implies the existence of a $TD_\lambda(t, k, v)$ and vice versa.

Let A be an $OA_\lambda(t, k, v)$ on the set I_v . We can construct a $TD_\lambda(t, k, v)$ on the set $I_k \times I_v$, where the groups of the TD are the sets $\{i\} \times I_v$, $i = 1, 2 \dots k$, and the blocks of the TD can be obtained by replacing each column $(x_1, x_2 \dots x_t)$ of A by the k -set $\{(1, x_1), (2, x_2) \dots (k, x_k)\}$.

Conversely, if $(X, \Gamma, \mathcal{A})$ is a $TD_\lambda(t, k, v)$, then without the loss of generality we can assume that, each group G_i of Γ is $\{i\} \times I_v$, $i = 1, 2 \dots k$. The columns of the

OA can be obtained by replacing each block $\{(1, x_1), (2, x_2) \dots (k, x_k)\}$ of the TD by the column $(x_1, x_2 \dots x_k)$.

With this correspondence, theorems 2.2.8 and 2.3.2 have the following forms.

Theorem 2.4.2 (Blanchard [4]) *For any positive integers t and k , $1 \leq t \leq k$, there is an integer $m_0(t, k) = m_0$ such that for any integer $m \geq m_0$ there is an $OA(t, k, m)$.*

Theorem 2.4.3 (Ray-Chaudhuri and Singhi [19]) *For any triple of positive integers (t, k, m) with $1 \leq t \leq k$, there is an integer $\lambda(t, k, m)$ such that for any integer $\lambda \geq \lambda(t, k, m)$ there exists an $OA_\lambda(t, k, m)$.*

Next we give the definition of another type of array, called ordered design.

Definition 2.4.4 *Given t, k, v and λ positive integers, with $1 \leq t \leq k$. An ordered design of order v , denoted by $OD_\lambda(t, k, v)$, is a quadruple (R, C, S, O) where,*

R is a set of k elements (called rows),

C is a set of $\lambda v(v-1) \dots (v-t+1)$ element (called columns),

S is a set of v element (called symbols) and

O is a map $O : R \times C \longrightarrow S$ such that

for every given set of t distinct elements $(r_1, r_2 \dots r_t)$ of R and for every ordered t -subset of distinct elements $\{s_1, s_2 \dots s_t\}$ of S the system of equations

$$O(r_1, y) = s_1$$

$$O(r_2, y) = s_2$$

$$\vdots$$

$$O(r_t, y) = s_t$$

has exactly λ solutions.

Intuitively, an $OD_\lambda(t, k, v)$ can be viewed as a $k \times \lambda v(v-1) \cdots (v-t+1)$ -matrix with entries from I_v , whose columns contain k distinct elements, such that in each set of t rows, every ordered t -subset of I_v occurs as a column exactly λ times.

Similarly to our previous notations, if $\lambda = 1$ we write $OD(t, k, v)$. There is one-to-one correspondence between the $OD_\lambda(2, k, v)$ and the idempotent $OA_\lambda(2, k, v)$, which follows from the definitions easily.

Ordered designs are equivalent to another type of combinatorial structure defined below:

Definition 2.4.5 Let t, v, k and λ be positive integers with $1 \leq t \leq k$. A lattice t -design, denoted by $LD_\lambda(t, k, v)$, is a family of k -sets, called blocks, over the set $X = I_k \times I_v$ satisfying the following properties:

1. Each block intersects each of the k -rows of X in exactly one point.
2. Each block intersects each of the columns in at most one point.

3. *Each t -set of points, which meets each row and column in at most one point is contained in exactly λ blocks.*

Note that every block is transversal to the rows and partial transversal to the columns.

This concept was introduced by K. Chang for the $t = 2$ case. In his dissertation [9] a lattice design, when $t = 2$ and $\lambda = 1$, is essentially a 2-design with two distinguished parallel classes.

Theorem 2.4.6 *Let t, v, k and λ be positive integers with $1 \leq t \leq k$. There is an $OD_\lambda(t, k, v)$ if and only if there is an $LD_\lambda(t, k, v)$.*

Proof. The proof is similar to the proof of the correspondence between the orthogonal arrays and transversal designs.

Now we mention two important infinite families of ordered designs ([10], p.397) for future references.

Theorem 2.4.7 (CRC Handbook [10], p.397) *Let q be a prime power.*

1. *There is an $OD(2, q, q)$.*
2. *There is an $OD(3, q + 1, q + 1)$.*

CHAPTER 3

CANDELABRA SYSTEMS AND RELATED RESULTS

3.1 Candelabra t -Systems and t -Designs

Candelabra systems and designs are very closely related. As we will see in the following theorem, one can construct candelabra systems from t -designs and conversely t -designs from candelabra systems.

From now on we will assume $\lambda = 1$ unless another value is specified.

Theorem 3.1.1 *Let t and v be positive integers and let K be a set of positive integers. There exists an $S(t, K, v)$ if and only if there exists a t -CS with block sizes from K and group type $(k - t + 1 | k \in K : t - 1)$.*

Proof. X will denote the set of points of the $S(t, K, v)$. Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_r$ denote the blocks of the $S(t, K, v)$, which pass through the given $t - 1$ points $\theta_1, \theta_2, \dots, \theta_{t-1}$ and $\mathcal{A}'$ denote the set of blocks, which do not contain all the given $t - 1$ points simultaneously. Since $\lambda = 1$, the sets $\mathcal{A}_i \setminus \{\theta_1, \theta_2, \dots, \theta_{t-1}\}$ for $i = 1, 2, \dots, r$ partition the set $X \setminus \{\theta_1, \theta_2, \dots, \theta_{t-1}\}$. Then the quadruple $(X, S = \{\theta_1, \theta_2, \dots, \theta_{t-1}\}, \Gamma = \{\mathcal{A}_i \setminus \{\theta_1, \theta_2, \dots, \theta_{t-1}\} | i = 1, 2, \dots, r\}, \mathcal{A}')$ is a t -CS with block sizes from K and group type $(k - t + 1 | k \in K : t - 1)$.

Conversely, if there exists a candelabra system $(X, S = \{\theta_1, \theta_2, \dots, \theta_{t-1}\}, \Gamma = \{G_1, G_2, \dots, G_r\}, \mathcal{A}')$ of group type $(k - t + 1 | k \in K : t - 1)$ with block sizes from K , then $(X, \mathcal{A}' \cup \{G_i \cup S : i = 1, 2, \dots, r\})$ is a t -design $S(t, K, v)$, where $|X| = v$.

Corollary 3.1.2 *Let t, k and v be positive integers, $1 \leq t \leq k$. There is an $S(t, k, v)$ if and only if there is a t -CS of group type $((k - t + 1)^r : t - 1)$ with block size k , where r is the number of blocks of an $S(t, k, v)$ containing a given set of $t - 1$ points.*

Figure 3.1: The construction of a candelabra system from a Steiner t -design.

Proof. It is an immediate consequence of theorem 3.1.1. (See figure 3.1)

It follows from the previous corollary that, $r \in R_k^t$ if and only if there exists a t -CS of group type $(k - t + 1)^r : t - 1$ with block size k .

Thus the following theorem has been proved:

Theorem 3.1.3 $VG_t[(k - t + 1), \{k\}, (t - 1)] = R_k^t$.

The $t = 2$ case of theorem 3.1.3 is Hanani's theorem ([20] pg.231) on the number of groups of group divisible designs of group size $k - 1$ and block size k . Thus theorem 3.1.3 generalizes Hanani's theorem for $t > 2$.

The idea of the construction in theorem 3.1.1 can be generalized.

Theorem 3.1.4 *Let t, s, k and v be positive integers, $1 \leq s < t \leq k$, and let K be a set of positive integers. Let $(X, \mathcal{A})$ be an $S(t, K, v)$ and $\{\theta_1, \theta_2, \dots, \theta_s\}$ be a given set of points of X . If the blocks of $\mathcal{A}$ containing the given set of points $\{\theta_1, \theta_2, \dots, \theta_s\}$ partition the points $X \setminus \{\theta_1, \theta_2, \dots, \theta_s\}$, then there is a t -CS with block sizes from K and group type $(k - s | k \in K : s)$. Conversely, if there is a t -CS of group type $(k - s | k \in K : s)$ with block sizes from K , then there is a t -design $S(t, K, v)$.*

Proof. The argument is similar to the proof of theorem 3.1.1. If the blocks containing the given set of points partition the rest of the points, then roughly speaking, the "tails" of these blocks, as groups, and the given set of points, as stem, form a candelabra system.

Let $A_1, A_2, \dots, A_{\lambda_s}$ denote the blocks of the $S(t, K, v)$, which pass through the given set of s points $\theta_1, \theta_2, \dots, \theta_s$ and $\mathcal{A}'$ denote the set of blocks, which do not contain all the given s points simultaneously. By the hypothesis of the theorem the sets $A_i \setminus \{\theta_1, \theta_2, \dots, \theta_s\}$ for $i = 1, 2, \dots, \lambda_s$ partition the set $X \setminus \{\theta_1, \theta_2, \dots, \theta_s\}$, thus the quadruple $(X, S = \{\theta_1, \theta_2, \dots, \theta_s\}, \Gamma = \{A_i \setminus \{\theta_1, \theta_2, \dots, \theta_s\} | i = 1, 2, \dots, \lambda_s\}, \mathcal{A}')$ is a t -CS with block sizes from K and group type $(k - s | k \in K : s)$.

If there is a t -CS $(X, S = \{\theta_1, \theta_2, \dots, \theta_s\}, \Gamma = \{G_1, G_2, \dots, G_{\lambda_s}\}, \mathcal{A}')$ of group type $(k - s | k \in K : s)$ with block sizes from K , then $(X, \mathcal{A}' \cup \{G_i \cup S : i = 1, 2, \dots, \lambda_s\})$ is a t -design $S(t, K, v)$, with $|X| = v$.

For instance, if the derived design at x , for $x \in X$, of a Steiner t -design $(X, \mathcal{A})$ admits a parallel class of blocks Π , then $(X, \{x\}, \Pi, \mathcal{A} \setminus (\bigcup_{B \in \Pi} (B \cup x)))$ is a candelabra system of order v , $s = 1$, with block sizes from K and group sizes from $K - 1$. Thus we can state the following theorem:

Theorem 3.1.5 *Let k be a positive integer, $k \geq 2$. If there is an $S(3, k + 1, k^2 + 1)$, then there exists a 3-CS of group type $(k^k : 1)$ with $\lambda = 1$ and block size $k + 1$. Conversely, if there is a 3-CS of group type $(k^k : 1)$ with block size k , then there exists an $S(3, k + 1, k^2 + 1)$.*

Proof. The derived design of an inversive plane $S(3, k + 1, k^2 + 1)$ is an affine plane $S(2, k, k^2)$ of order k , which admits a parallelism.

Corollary 3.1.6 *Let q be a prime power. There exists a 3-CS of group type $(q^q : 1)$ with $\lambda = 1$ and block size $q + 1$. (See figure 3.2.)*

Proof. There is an inversive plane $S(3, q + 1, q^2 + 1)$ for any prime power q .

This structure was constructed by Hartman [17] as well, who used different terminology. Hanani also used this construction in his existence proof for Steiner quadruple systems [14].

Figure 3.2: The construction of a 3-CS of group type $(q^q : 1)$ with blocks size $q + 1$.

The converse of theorem 3.1.1 can be generalized also. Since we will use the generalized version of this theorem several times in the future, we state the theorem at this time.

Theorem 3.1.7 (Adjunction Theorem) *Let t, v be positive integers and K be a set of positive integers. Let $(X, S, \Gamma, \mathcal{A})$ be a t -CS of order v with block sizes from K . If for each $G \in \Gamma$ there exists an $S(t, K, |G \cup S|)$ containing a subdesign $S(t, K, |S|)$, then there exists an $S(t, K, v)$.*

Proof. Let $(S, \mathcal{M}_S)$ be an $S(t, K, |S|)$. For each $G \in \Gamma$ we can construct a t -design $(G \cup S, \mathcal{M}_G)$ with block sizes from K containing the subdesign $(S, \mathcal{M}_{S_G})$. Let us define the following set of blocks: $\mathcal{M}'_G = \mathcal{M}_G \setminus \mathcal{M}_{S_G}$. Then $(X, \mathcal{A} \cup (\cup_{G \in \Gamma} \mathcal{M}'_G) \cup \mathcal{M}_S)$ is the required $S(t, K, v)$. To check that a t -element subset T of X is contained in exactly one block, we consider the following cases:

Case 1. If $T \subseteq S$ then T is contained in unique block of $\mathcal{M}_S$.

Case 2. If $T \not\subseteq S$ but it is contained in some $G \cup S$, then T is covered by a unique block of $\mathcal{M}'_G$.

Case 3. In all other cases, T is contained in a unique block of $\mathcal{A}$.

There is another way to construct candelabra systems from an $S(t, K, v)$. Let $(X, \mathcal{A})$ be an $S(t, k, v)$ and let $X = S \cup (\cup_{i \in I} G_i)$ be a partition of points of a Steiner t -design, such that $|S \cup G_i| < t$ for $i \in I$. This partition forms a t -CS, considering S as the stem, G_i for $i \in I$ as the groups and $\mathcal{A}$ as the set of blocks of the candelabra system. Thus, if $(X, \mathcal{A})$ is an $S(t, K, v)$, then quadruple $(X, \{x_0\}, \{\{x\} | x \in X \setminus x_0\}, \mathcal{A})$ for any $x_0 \in X$ is a uniform t -CS of group type $(1^{v-1} : 1)$ with block sizes from K and the stem and all the groups are singletons.

3.2 Fundamental Construction of Candalebra Systems

The next theorem generalizes Hanani's and Hartman's constructions for candelabra systems ([16] and [17]).

Theorem 3.2.1 (Fundamental construction for 3-CS's) *Let $(X, S, \Gamma, \mathcal{A})$ be a 3-CS with $S = \{s_0\}$ and let K be a set of positive integers. Let w be a non-negative integral weighting of X , where by weighting of a set H we mean a map from H into the set of non-negative integers. There exists a 3-CS with block sizes from K and group type $(\sum_{x \in G} w(x) | G \in \Gamma : w(s_0))$ if the following conditions hold:*

1. *For every block A of 3-CS not containing s_0 , there is a 3-GDD with block sizes from K and group type $(w(x) | x \in A, w(x) \neq 0)$.*

2. For every block A of 3-CS passing through s_0 , there is a 3-CS with block sizes from K and group type $(w(x)|x \in A, w(x) \neq 0, x \neq s_0 : w(s_0))$.

Using Wilson's terminology, we will refer to a 3-CS with a weighting as a recipe and the 3-GDD's and the 3-CS's, given in (1) and (2), as ingredients.

Proof. For each $x \in X$, let Y_x be a set of $w(x)$ points such that the sets Y_x are mutually disjoint. Let $Y = \bigcup_{x \in X} Y_x$. For each block A of $\mathcal{A}$ not containing s_0 we can construct a 3-GDD $(\bigcup_{x \in A} Y_x, \{Y_x | x \in A, Y_x \neq \emptyset\}, \mathcal{M}_A)$ with block sizes from K . For each block A' of $\mathcal{A}$ passing through s_0 construct a 3-CS $(\bigcup_{x \in A'} Y_x, Y_{s_0}, \{Y_x | x \in A', x \neq s_0, Y_x \neq \emptyset\}, \mathcal{M}_{A'})$ with block sizes from K . Define $\Sigma = \{\bigcup_{x \in G} Y_x | G \in \Gamma\}$ and $\mathcal{M} = \bigcup_{A \in \mathcal{A}} \mathcal{M}_A$. We claim that $(Y, Y_{s_0}, \Sigma, \mathcal{M})$ is a 3-CS with block sizes from K . Let a, b and c be three distinct points of Y . There are three cases:

Case 1: Let $a, b \in Y_{x_1}$ and $c \in Y_{x_2}$, where $x_1 \neq x_2$ and $x_i \in X \setminus s_0$ for $i = 1, 2$. There is a unique block A of $\mathcal{A}$ passing through the triple $\{x_1, x_2, s_0\}$. Since a (or b) and c belong to different groups of the 3-CS $(\bigcup_{x \in A} Y_x, Y_{s_0}, \{Y_x | x \in A, x \neq s_0, Y_x \neq \emptyset\}, \mathcal{M}_A)$, they are contained in some block of $\mathcal{M}_A$. Furthermore, this is the only block of $\mathcal{M}$ that covers the given triple.

Case 2: Let $a \in Y_{x_1}$, $b \in Y_{x_2}$ and $c \in Y_{x_3}$, where x_1, x_2, x_3 are different points and $x_i \in X \setminus s_0$ for $i = 1, 2, 3$. Then $\{x_1, x_2, x_3\}$ are contained in a unique block A of $\mathcal{A}$. If A intersects s_0 , since $\{a, b, c\}$ belong to different groups of the 3-CS $(\bigcup_{x \in A} Y_x, Y_{s_0}, \{Y_x | x \in A, x \neq s_0, Y_x \neq \emptyset\}, \mathcal{M}_A)$, there is a unique block of $\mathcal{M}_A$ covering $\{a, b, c\}$ and no other blocks of $\mathcal{M}$. If A does not intersect s_0 , they are contained

in a unique block of the 3-GDD $(\bigcup_{x \in A} Y_x, \{Y_x | x \in A, Y_x \neq \emptyset\}, \mathcal{M}_A)$, since a, b, c are in different groups of the 3-GDD. Again, no other block of $\mathcal{M}$ can contain $\{a, b, c\}$.

Case 3: Let $a \in Y_{s_0}$, $b \in Y_{x_1}$ and $c \in Y_{x_2}$, where $x_1 \neq x_2$ and $x_i \in X \setminus s_0$ for $i = 1, 2$. In this case, the unique block A of $\mathcal{A}$ containing the given triple, contains s_0 as well. Since a is in Y_{s_0} and b, c are in two different groups of the 3-CS $(\bigcup_{x \in A} Y_x, Y_{s_0}, \{Y_x | x \in A, x \neq s_0, Y_x \neq \emptyset\}, \mathcal{M}_A)$, they are covered by a unique block of $\mathcal{M}_A$. No other block of $\mathcal{M}$ could contain them simultaneously.

Corollary 3.2.2 *Let a, b, k and l be positive integers, $k \geq 2$, and let s be a non-negative integer. If there exist a 3-CS of group type $(l^a : 1)$ with block size $k + 1$, a 3-CS of group type $(b^k : s)$ with block size $k + 1$ and a $TD(3, k + 1, b)$, then there is a 3-CS of group type $((lb)^a : s)$ with block size $k + 1$.*

Proof. Let the 3-CS of group type $(l^a : 1)$ with block size $k + 1$ be the recipe with the following weighting: weight the point of the stem by s and all the other points by the uniform weight b . For each block passing through the point of the stem we have the ingredient 3-CS of group type $(b^k : s)$ with block size $k + 1$ and for each block avoiding the stem we have a $TD(3, k + 1, b)$. Thus, the resulting 3-CS has a group type $((lb)^a : s)$ with block size $k + 1$.

The next theorem is a generalization of Hanani's general theorem (Proposition 8 in [16]).

Theorem 3.2.3 *Let w be a non-negative weighting of the points set X of a 3-design $(X, \mathcal{A})$ with $\lambda = 1$. Let x_0 be a given point of X with $w(x_0) = s$. There exists an $S(3, K, v)$, where $v = \sum_{x \in X} w(x)$ if the following conditions are satisfied:*

1. For each $x \in X \setminus x_0$, there is an $S(3, K, w(x) + s)$ containing a subdesign $S(3, K, s)$.
2. For every block A of $\mathcal{A}$ not containing x_0 , there is a 3-GDD with block sizes from K and group type $(w(x)|x \in A, w(x) \neq 0)$.
3. For every block A of $\mathcal{A}$ passing through x_0 , there is a 3-CS with block sizes from K and group type $(w(x)|x \in A, w(x) \neq 0, x \neq x_0 : s)$.

Proof. As we mentioned at the end of the previous section a 3-design $(X, \mathcal{A})$ can be considered a 3-CS of group type $(1^{(|X|-1)} : 1)$ with block sizes from K . Now, we use the fundamental construction (theorem 3.2.1) with this 3-CS as recipe and the ingredients given in (2) and (3). Thus, we can construct a 3-CS with block sizes from K and group type $(w(x)|x \in X, w(x) \neq 0, x \neq x_0 : s)$. For each $x \in X \setminus x_0$, we can construct a $S(3, K, w(x) + s)$ containing a subdesign $S(3, K, s)$. Now, applying the adjunction theorem (theorem 3.1.7) the statement follows.

Corollary 3.2.4 *Let a, b and k be positive integers, $k \geq 2$, and let $s \in \{0, 1, 2\}$. If there exist an $S(3, k + 1, a + 1)$, an $S(3, k + 1, b + s)$, a 3-CS of group type $(b^k : s)$ with block size $k + 1$ and a $TD(3, k + 1, b)$, then there is an $S(3, k + 1, ab + s)$.*

Proof. Let ∞ be a given point of the Steiner 3-design $S(3, k + 1, a + 1)$. Weight ∞ by s and weight all the other points by the uniform weight b . It is easy to check that the conditions of theorem 3.2.3 are satisfied, thus we can construct a Steiner 3-design $S(3, k + 1, ab + s)$.

Corollary 3.2.4 is a generalization of Wilson's product theorem mentioned in the introduction.

3.3 Applications

Theorem 3.3.1 *Let q be a prime power. There exists a 3-CS of group type $((q^m)^{q^n} : 1)$ with block size $q + 1$ for all n, m non-negative integers.*

Proof. We proceed by induction on m . If $m=0$, then a Steiner 3-design $S(3, q + 1, q^n + 1)$ can be considered a 3-CS of group type $(1^{(q^n)} : 1)$ with block size $q + 1$. The existence of an $S(3, q + 1, q^n + 1)$ is well known [1]. Assume we have a 3-CS of group type $((q^{m-1})^{q^n} : 1)$ with block size $q + 1$. We will denote the single point of the stem by ∞ . Weight ∞ by 1 and weight all the other points by the uniform weight q . We construct a 3-CS of group type $((q^m)^{q^n} : 1)$ with block size $q + 1$ using the 3-CS of group type $((q^{m-1})^{q^n} : 1)$ as the recipe. For each block of the 3-CS not containing ∞ , we have the ingredient $TD(3, q + 1, q)$ and for each block passing through ∞ , we can construct the ingredient 3-CS of group type $(q^q : 1)$ with block size $q + 1$. The existence of a $TD(3, q + 1, q)$ and a 3-CS of group type $(q^q : 1)$ with block size $q + 1$ follows from theorem 2.2.5 and corollary 3.1.6. Now, applying the fundamental construction of candelabra systems we obtain a 3-CS of group type $((q^m)^{q^n} : 1)$ with block size $q + 1$.

Theorem 3.3.2 (Hanani [15]) *Let q be a prime power. If there is a Steiner 3-design $S(3, q + 1, v + 1)$, then there is a 3-CS of group type $(q^v : 1)$ with block size $q + 1$.*

Proof. Consider the Steiner 3-design as a 3-CS of group type $(1^v : 1)$ with block size $q + 1$. Weight the point of the stem by 1 and all other points by the uniform weight q . Let the recipe be this 3-CS with the given weighting. The necessary ingredients of the fundamental construction here are a $TD(3, q + 1, q)$ and a 3-CS of group type $(q^q : 1)$ with block size $q + 1$, which exist by theorem 2.2.5 and corollary 3.1.6. Therefore, by the fundamental construction we can construct a 3-CS of group type $(q^v : 1)$ with block size $q + 1$.

Corollary 3.3.3 (Hanani [15]) *Let q be a prime power. If there exists a Steiner 3-design $S(3, q + 1, v + 1)$, then there exists a Steiner 3-design $S(3, q + 1, vq + 1)$.*

Proof. Use theorem 3.3.2 to obtain a 3-CS of group type $(q^v : 1)$ with block size $q + 1$ and use theorem 3.1.7 to complete the 3-CS to an $S(3, q + 1, vq + 1)$.

Theorem 3.3.4 (Hanani [15]) *If there is an $S(3, 6, v + 1)$, then there is a 3-CS of group type $(4^v : 2)$ with block size 6.*

Proof. By theorem 3.1.1, we can obtain a 3-CS of group type $(4^5 : 2)$ with block size 6 from a $S(3, 6, 22)$. From the given $S(3, 6, v + 1)$, we get a 3-CS of group type $(1^v : 1)$ with block size 6, which will be the recipe. Weight the point of the stem by 2 and weight all the other points by 4. Again, we use the notation ∞ to denote

the point in the stem. If a block of the recipe 3- CS contains ∞ , then construct the ingredient 3- CS of group type $(4^5 : 2)$ with block size 6. If a block of the 3- CS does not contain the point ∞ , then construct the ingredient $TD(3, 6, 4)$ given in theorem 2.2.6. Now, apply the fundamental construction to obtain a 3- CS of group type $(4'' : 2)$ with block size 6.

Corollary 3.3.5 (Hanani [15]) *If there is an $S(3, 6, v+1)$, then there is an $S(3, 6, 4v+2)$.*

Proof. The proof follows from theorem 3.3.4 and theorem 3.1.7.

3.4 Closure Operation and Closed Sets

Closure operations are important tools in the construction of new designs. Closed subsets of natural numbers play a significant role in the proof of the existence theorem for $t = 2$. In this section we will give the definition of a new closure operation involving candelabra systems. The closed sets obtained in this way will be useful in constructing Steiner 3-designs.

Definition 3.4.1 *By a closure operation on the subsets A of a set X , we mean a map $A \mapsto \bar{A}$ from the class $\mathcal{P}(X)$ of all subsets of X into $\mathcal{P}(X)$ satisfying the following properties:*

1. $A \subseteq \bar{A}$ (extensive),

2. $\bar{\bar{A}} = \bar{A}$ (idempotent).

3. if $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$ (isotone).

A subset $A \subseteq X$ is said to be closed with respect to the given closure operation if and only if $A = \bar{A}$.

The following theorem was introduced by Ray-Chaudhuri for general t , which is an extension of Wilson's results [20] for $t = 2$. Recall that, for a subset $K \subseteq N$ $B_t[K] = \{v | S(t, K, v) \text{ exists}\}$.

Theorem 3.4.2 *For a fixed positive integer t , the map $K \mapsto B_t[K]$ is a closure operation on the subsets of positive integers. A set K will be called B -closed with parameter t if and only if $B_t[K] = K$.*

Proof. (i) (extensive) For any $k \in K$, $k \geq t$, $(I_k, \{I_k\})$ is a $S(t, K, k)$. If $k < t$, then $(I_k, \emptyset)$ is a $S(t, K, k)$.

(ii) (isotone) $K_1 \subseteq K_2$ obviously implies $B_t[K_1] \subseteq B_t[K_2]$.

(iii) (idempotent) Let $v \in B_t[B_t[K]]$. Then there is a t -design $(X, \mathcal{B})$ of order v with block sizes from $B_t[K]$. For each block $B \in \mathcal{B}$ we can construct a t -design $(B, \mathcal{M}_B)$ with block sizes from K . Let $\mathcal{M} = \cup_{B \in \mathcal{B}} \mathcal{M}_B$. We claim that the design $(X, \mathcal{M})$ is an $S(t, K, v)$. For a given t -set $\{x_1, x_2, \dots, x_t\} \subseteq X$, there is a unique block B of $\mathcal{B}$, which contains it. Thus, $\{x_1, x_2, \dots, x_t\}$ is contained in unique block of $\mathcal{M}_B$.

This implies that $B_t[B_t[K]] \subseteq B_t[K]$ and the proof is complete.

Theorem 3.4.3 (Hanani, theorem 10.1 in [20].) *If $t = 2$, then R_k (the set of replication numbers) is a B -closed set.*

The previous theorem is important in the proof of the existence theorem for pairwise balanced designs. Unfortunately the analogous theorem for $t > 2$ is not true. However, the set $(k - t + 1)R_k^t$ will be a closed set under the following closure operation. Recall that, for fixed sets K and J of positive integers and for fixed non-negative integers s and t , $C_K^t[J, s]$ will denote the set of positive integers w for which there exists a candelabra t -system $(X, S, \Gamma, \mathcal{A})$ with $|X| = w + s$, $|S| = s$, group sizes from J and block sizes from K .

Theorem 3.4.4 (Breaking up groups) *For a fixed set of positive integers K , fixed non-negative integer s and fixed positive integer t , the map $J \mapsto C_K^t[J, s]$ is a closure operation on the set of positive integers. A set J is called C -closed with parameters (t, K, s) if and only if $C_K^t[J, s] = J$.*

Proof. (i) (extensive) Take any positive integer $j \in J$. Let $I_j = \{1, 2, \dots, j\}$ and let S be a set of size s . Take the trivial t -CS $(I_j \cup S, S, \{I_j\}, \emptyset)$, which is a t -CS with group sizes from J and block sizes from K . Thus $J \subseteq C_K^t(J, s)$.

(ii) (isotone) Clearly $J_1 \subseteq J_2$ implies $C_K^t(J_1, s) \subseteq C_K^t(J_2, s)$.

(iii) (idempotent) Let $w \in C_K^t[C_K^t[J, s], s]$. Then there exists a t -CS $(X, S, \Gamma, \mathcal{A})$ of order $w + s$ with group sizes from $C_K^t[J, s]$ and block sizes from K . For each group $G \in \Gamma$ let $(G \cup S, S, \Sigma_G, \mathcal{M}_G)$ be a t -CS with group sizes from J and block sizes from K . Let $\Sigma = \bigcup_{G \in \Gamma} \Sigma_G$ and $\mathcal{M} = \mathcal{A} \cup (\bigcup_{G \in \Gamma} \mathcal{M}_G)$. We claim $(X, S, \Sigma, \mathcal{M})$ is a t -CS of order $w + s$, with group sizes from J and block sizes from K .

Let T be a t -set of X such that $|T \cap (S \cup H)| \leq t - 1$ for any $H \in \Sigma$.

Case 1. If $T \subset G \cup S$ for some $G \in \Gamma$, then T is contained in a unique block of $\mathcal{M}_G$ and no other blocks.

Case 2. If $T \not\subset G \cup S$ for any $G \in \Gamma$, then there is a unique block of $\mathcal{A}$ which contains T and this is the only one.

This proves that $C_K^t[C_K^t[J, s], s] \subseteq C_K^t[J, s]$.

The closure operation defined in the previous theorem is a generalization of Wilson's closure map $J \mapsto G_K[J]$ given in [20]. As we have mentioned in section 2.3, if $t = 2$ and $s = 0$, then $C_K^2[J, 0] = G_K[J]$.

Theorem 3.4.5 *Let m and t be positive integers, let s be a non-negative integer and let K be a set of positive integers. The set $m.NG_t[m, K, s]$ is a C -closed set with parameters (t, K, s) .*

Proof. $C_K^t[m, s]$ is a closed set with parameters (t, K, s) and $m.NG_t[m, K, s] = C_K^t[m, s]$.

Corollary 3.4.6 *Let t and k be positive integers, $1 \leq t \leq k$. The set $(k - t + 1)R_k^t$ is a C -closed set with parameters $(t, k, t - 1)$.*

Proof. This corollary follows from theorem 3.4.5, since $NG_t[(k - t + 1), \{k\}, (t - 1)] = R_k^t$ by theorem 3.1.3.

3.5 Applications of Closed Sets in Constructions of 3-Designs

As we mentioned earlier, the closed subsets of natural numbers can be used to construct designs. In this section we will give a new, more general construction for Steiner 3-designs applying the theory of closed sets.

Theorem 3.5.1 *Let m and k be positive integers, $k \geq 2$. $NG_2(m, k, 0)$ is a C -closed set with parameters $(2, k, 0)$ if and only if there is a 2- CS of group type $(m^k : 0)$ with block size k , in other words if there is a $TD(2, k, m)$.*

Proof. If $NG_2[m, k, 0]$ is a C -closed set with parameters $(2, k, 0)$, then $C_k^2[NG_2[m, k, 0], 0] = NG_2[m, k, 0]$. By the isotone property of the closure operation, we have $C_k^2[1, 0] \subseteq C_k^2[NG_2[m, k, 0], 0]$. Obviously $k \in C_k^2[1, 0]$ and hence $k \in NG_2[m, k, 0]$, that is there is a $TD(2, k, m)$.

Conversely, assume there is a $TD(2, k, m)$. If $v \in C_k^2[NG_2[m, k, 0], 0]$, then there is a 2- CS (or a 2- GDD) of order v , $|S| = 0$, with block size k and group sizes from $NG_2[m, k, 0]$. Let this 2- GDD be denoted by $(X, \Gamma, \mathcal{A})$. Weight each point of X by the uniform weight m . By the fundamental construction theorem for GDD 's (theorem 2.2.2) we can construct a 2- GDD $(X', \Gamma', \mathcal{A}')$ of order mv , with block size k and group sizes from $m \cdot NG_2[m, k, 0]$. For each group $G' \in \Gamma'$ let $(G', \Sigma_{G'}, \mathcal{M}_{G'})$ be a 2- GDD with block size k and group size m . Let $\Sigma = \bigcup_{G' \in \Gamma'} \Sigma_{G'}$ and $\mathcal{M} = \mathcal{A}' \cup (\bigcup_{G' \in \Gamma'} \mathcal{M}_{G'})$. Then $(X', \Sigma, \mathcal{M})$ is a 2- GDD of group type (m^v) with block size k (or a 2- CS with $|S| = 0$), which implies $v \in NG_2[m, k, 0]$.

Corollary 3.5.2 *Let q be a prime power. $NG_2[q^n, q, 0]$ is a C -closed set with parameters $(2, q, 0)$ for every positive integer n .*

Proof. In theorem 3.5.1. we take $m = q^n$ and $k = q$. The existence of a $TD(2, q, q^n)$ follows from theorem 2.2.5 and theorem 2.2.7.

Corollary 3.5.3 *Let k and m be positive integers, $k \geq 2$. There exists an integer $m_0 = m_0(k)$ such that $NG_2[m, k, 0]$ is a C -closed set with parameters $(2, k, 0)$ for all $m \geq m_0$.*

Proof. We apply theorem 3.5.1. The existence of a $TD(2, k, m)$ for $m \geq m_0(k)$ was previously stated in theorem 2.2.8.

Theorem 3.5.4 *Let k, m be positive integers, $k \geq 2$ and let s be a non-negative integer. If $NG_3[m, k+1, s]$ is a C -closed set with parameters $(3, k+1, 1)$, then there is a 3-CS of group type $(m^k : s)$ with block size $k+1$.*

Proof. The statement is an immediate consequence of

$$k \in C_{k+1}^3[1, 1] \subseteq C_{k+1}^3[NG_3[m, k+1, s], 1] = NG_3[m, k+1, s].$$

$C_{k+1}^3[NG_3[m, k+1, s], 1] = NG_3[m, k+1, s]$, since $NG_3[m, k+1, s]$ is a C -closed set with parameters $(3, k+1, 1)$. Since $1 \in NG_3[m, k+1, s]$, we have $C_{k+1}^3[1, 1] \subseteq C_{k+1}^3[NG_3[m, k+1, s], 1]$ by the isotone property of the closure operation. Clearly, $k \in C_{k+1}^3[1, 1]$. Hence, we obtain $k \in NG_3[m, k+1, s]$, which is equivalent to the existence of a 3-CS of group type $(m^k : s)$ with block size $k+1$.

Corollary 3.5.5 *If for some positive integer k , $NG_3[k, k+1, 1]$ is a C -closed set with parameters $(3, k+1, 1)$, then there is an*

- (i) $S(3, k+1, k^2+1)$ (a Möbius plane of order k).
- (ii) $S(2, k, k^2)$ (an affine plane of order k).
- (iii) $S(2, k+1, k^2+k+1)$ (a projective plane of order k).

proof. (i) follows from theorem 3.5.4 with $s = 1$ and $m = k$, and from the adjunction theorem (theorem 3.1.7).

(ii) If an $S(3, k+1, k^2+1)$ exists, then an $S(2, k, k^2)$ exists.

(iii) There is an $S(3, k, k^2)$ if and only if there is an $S(2, k+1, k^2+k+1)$.

Theorem 3.5.6 *Let k, m be positive integers, $k \geq 2$, and let s be a non-negative integer. If there exist a $TD(3, k+1, m)$ and a 3-CS of group type $(m^k : s)$ with block size $k+1$, then $NG_3[m, k+1, s]$ is a C -closed set with parameters $(3, k+1, 1)$.*

Proof. Let $v \in C_{k+1}^3[NG_3[m, k+1, s], 1]$. Then there exists a 3-CS $(X, S, \Gamma, \mathcal{A})$ of order $v+1$, $|S| = 1$, with block size $k+1$ and group sizes from $NG_3[m, k+1, s]$. Weight each point of $X \setminus S$ with the uniform weight m and weight the single point of the stem by s . By theorem 3.2.1 (the fundamental construction for 3-CS's), we can construct a 3-CS $(X', S', \Gamma', \mathcal{A}')$ of order $mv+s$, $S' = s$, with block size $k+1$ and group sizes from $m \cdot NG_3[m, k+1, s]$. For each group $G' \in \Gamma'$ let $(G' \dot{\cup} S', S', \Sigma_{G'}, \mathcal{M}_{G'})$ be a 3-CS with group size m and block size $k+1$. Let $\Sigma = \bigcup_{G' \in \Gamma'} \Sigma_{G'}$ and $\mathcal{M} = \mathcal{A}' \cup (\bigcup_{G' \in \Gamma'} \mathcal{M}_{G'})$. Then $(X', S', \Sigma, \mathcal{M})$ is a 3-CS of group type $(m^v : s)$ with block size $k+1$. Hence $v \in NG_3[m, k+1, s]$. This implies that $C_{k+1}^3[NG_3[m, k+1, s], 1] \subseteq NG_3[m, k+1, s]$ and the proof is complete.

Corollary 3.5.7 *Let q be a prime power. Then $NG_3[q^n, q+1, 1]$ is a C -closed with parameters $(3, q+1, 1)$ for every positive integer n .*

Proof. Use theorem 3.5.6 with $m = q^n$, $k = q$ and $s = 1$. By theorems 2.2.5 and 2.2.7 there is a $TD(3, q+1, q^n)$ for $n \geq 1$. The existence of a 3-CS of group type $((q^n)^q : 1)$ with block size $q+1$ follows from theorem 3.3.1.

Corollary 3.5.8 *$NG_3[4, 6, 2]$ is C -closed with parameters $(3, 6, 1)$.*

Proof. We apply theorem 3.5.6 with $m = 4$, $k = 5$ and $s = 2$. We can construct a 3-CS of group type $(4^5 : 2)$ with block size 6 from an $S(3, 6, 22)$. By theorem 2.2.6 there is a $TD(3, 6, 4)$.

Corollary 3.5.9 *Let k, m be positive integers, $k \geq 2$, and let s be a non-negative integer. There exists an integer $m_0 = m_0(k)$ such that for any 3-CS of group type $(m^k : s)$ with block size $k+1$ satisfying $m \geq m_0$, the set $NG_3[m, k+1, s]$ is C -closed with parameters $(3, k+1, 1)$.*

Proof. The existence of a $TD(3, k+1, m)$ for $m \geq m_0$ follows from theorem 2.2.8. Now we apply theorem 3.5.6.

An alternative way to prove theorems 3.3.2 and 3.3.4 (Hanani's theorems) using the theory of closed sets is the following:

Theorem 3.5.10 *Let q be a prime power. If there is a Steiner 3-design $S(3, q+1, v+1)$, then there is a 3-CS of group type $((q^n)^v : 1)$ with block size $q+1$, consequently there is an $S(3, q+1, vq^n+1)$ for any positive integer n .*

Proof. $NG_3(q^n, q+1, 1)$ is C -closed with parameters $(3, q+1, 1)$ for any positive integer n and prime power q . If there is an $S(3, q+1, v+1)$ then

$$v \in C_{q+1}^3[1, 1] \subseteq C_{q+1}^3[NG_3[q^n, q+1, 1], 1] = NG_3[q^n, q+1, 1].$$

Hence there exists an $S(3, q+1, vq^n+1)$.

Theorem 3.5.11 *Let v be a positive integer. If there is an $S(3, 6, v+1)$, then there is a 3-CS of group type $(4^v : 2)$ with block size 6, consequently there is an $S(3, 6, 4v+2)$.*

Proof. From corollary 3.5.8, $NG_3[4, 6, 2]$ is C -closed with parameters $(3, 6, 1)$. Thus,

$$v \in C_6^3[1, 1] \subseteq C_6^3[NG_3[4, 6, 2], 1] = NG_3[4, 6, 2]$$

which implies the existence of an $S(3, 6, 4v+2)$.

We can use similar argument to give a second proof for corollary 3.2.4.

Theorem 3.5.12 *Let a, b, k be positive integers, $k \geq 2$, and let $s \in \{0, 1, 2\}$. If there exist an $S(3, k+1, a+1)$, an $S(3, k+1, b+s)$, a 3-CS of group type $(b^k : s)$ with block size $k+1$ and a $TD(3, k+1, b)$, then there exists a 3-CS of group type $(b^a : s)$ with block size $k+1$ and hence there exists an $S(3, k+1, ab+s)$.*

Proof. Theorem 3.5.6 asserts that $NG_3[b, k+1, s]$ is a C -closed set with parameters $(3, k+1, 1)$. Thus,

$$a \in C_{k+1}^3[1, 1] \subseteq C_{k+1}^3[NG_3[b, k+1, s], 1] = NG_3[b, k+1, s]$$

and the statement follows.

CHAPTER 4

LATTICE CANDELABRA SYSTEMS AND LATTICE GROUP DIVISIBLE DESIGNS

4.1 Definitions and Basic Facts

Let us consider the set $I_n \times I_m$. The subsets $R_i = \{i\} \times I_m$ for $i = 1, 2, \dots, n$ are called the rows and the subsets $C_j = I_n \times \{j\}$ for $j = 1, 2, \dots, m$ are called the columns of the set $I_n \times I_m$.

Definition 4.1.1 *Let t, k, n and m be positive integers. $1 \leq t \leq k$. Let $(X, \Gamma, \mathcal{A})$ be a uniform group divisible t -design of group type (m^n) constructed over the set $X = I_n \times I_m$, where $\Gamma = \{R_i : i = 1, 2, \dots, n\}$. $(X, \Gamma, \mathcal{A})$ is called a lattice group divisible t -design (or t -LGDD) if for any block $A \in \mathcal{A}$ and for any column C_j , $j = 1, 2, \dots, m$, either $|A \cap C_j| < t$ or $A \subseteq C_j$. A lattice group divisible design with block size k and k groups of size m is called a lattice transversal design and denoted by $LTD(t, k, m)$. (See figure 4.1.)*

If $t = 1$, a $TD(1, k, m)$ is an $LTD(1, k, m)$. Consider a $TD(1, k, m)$ on the set $I_k \times I_m$. The set of m blocks of a $TD(1, k, m)$ forms a parallel class. We can rearrange

Figure 4.1: Lattice group divisible t -design

the points within the groups of the $TD(1, k, m)$ such that the blocks of the parallel class coincide with the columns of the set $I_k \times I_m$. Thus, we obtain an $LT D(1, k, m)$.

In the $t = 2$ case, a lattice transversal design is equivalent to an idempotent orthogonal array.

A straightforward consequence of definition 4.1.1 is the following theorem:

Theorem 4.1.2 *Let t, k, n and m be positive integers, $1 \leq t \leq k$. The existence of a t -LGDD of group type (m^n) implies the existence of m Steiner t -designs $S(t, k, n)$.*

Proof. Let $(X, \Gamma, \mathcal{A})$ be a t -LGDD of group type (m^n) with $X = I_n \times I_m$. For each $j \in I_m$, the set of points C_j and the set of blocks of $\mathcal{A}$ contained in C_j determine an $S(t, k, n)$.

Definition 4.1.3 Let t, k, n and m be positive integers, $1 \leq t \leq k$. Let $(X, S, \Gamma, \mathcal{A})$ be a uniform candelabra t -system of group type $(m^n : |S|)$ constructed over the set $X = I_n \times I_m \cup S$ with $S = \emptyset$ or $S = \{\infty\}$, where $\Gamma = \{R_i : i = 1, 2, \dots, n\}$. $(X, S, \Gamma, \mathcal{A})$ is called a lattice candelabra t -system (or t -LCS) if for any block $A \in \mathcal{A}$ and for any C_j , either $|A \cap (C_j \cup S)| < t$ or $A \subseteq (C_j \cup S)$. (See figure 4.2.)

Figure 4.2: Lattice candelabra t -system.

If $t = 1$, every uniform 1-CS with stem size 0 or 1 is a 1-LCS since the family of blocks of a 1-CS must be the empty family.

As we discussed in section 2.3, in the $t = 2$ case a candelabra systems becomes a group divisible design. Thus, a 2-LCS essentially is a 2-LGDD.

The next theorem follows from definition 4.1.3:

Theorem 4.1.4 *Let t, k, n and m be positive integers, $1 \leq t \leq k$. The existence of a t -LCS of group type $(m^n : s)$, $s = 0$ or $s = 1$, implies the existence of an $S(t, k, n + s)$.*

Proof. Let $(X, S, \Gamma, \mathcal{A})$ be a t -LCS of group type $(m^n : s)$, where $X = (I_n \times I_m) \cup S$. For each $j \in I_m$, the set of points $C_j \cup S$ and the set of blocks of $\mathcal{A}$ contained in $C_j \cup S$ determines an $S(t, k, n + s)$.

An interesting property of t -LCS's is described in the next theorem:

Theorem 4.1.5 *Let t, k, n, m , be positive integers, $2 < t \leq k$, let $s \in \{0, 1\}$ and let K be a set of positive integers. If there exist a t -LCS of group type $(m^n : s)$ with block sizes from K and an $S(t, K, m + s)$, then there exists a t -LCS of group type $(n^m : s)$ with block sizes from K .*

Proof. Let $(X, S, \Gamma, \mathcal{A})$ denote the given t -LCS of group type $(m^n : s)$ with block sizes from K , where $X = I_n \times I_m$. Let $\mathcal{A}' = \{A \mid A \in \mathcal{A}, A \not\subseteq C_j \cup S \text{ for any } j, 1 \leq j \leq m\}$, that is $\mathcal{A}'$ denotes the blocks of $\mathcal{A}$, which are not contained in any of the sets $C_j \cup S$ for $j \in I_m$. For each group $G_i = R_i$, $1 \leq i \leq n$, of the t -LCS together with S (the stem) construct an $S(t, K, m + s)$ denoted by $(G_i \cup S, \mathcal{B}_{G_i})$. Let $\mathcal{B} = \cup_{i=1}^n \mathcal{B}_{G_i}$, then the quadruple $(X, S, \{C_j : j \in I_m\}, \mathcal{A}' \cup \mathcal{B})$ is a t -LCS of group type $(n^m : s)$ with block sizes from K . Any t -set contained in $R_i \cup S$ for some $i \in I_n$ is covered by a unique block B of $\mathcal{B}$, such that $B \subseteq (R_i \cup S)$. Any t -set not contained in $R_i \cup S$ for all $i \in I_n$ and not contained of $C_j \cup S$ for all $j \in I_m$ is covered by a unique block of $\mathcal{A}'$.

According to our previous theorem, from a t -LCS, $t > 2$, of group type $(m^n : s)$ with block sizes from K and from an $S(t, K, m + s)$ we can construct a t -LCS of group type $(n^m : s)$ with block sizes from K by interchanging the role of the rows and columns of the given t -LCS. In the sequel we will refer to this property as “row-column interchangeable property” of the lattice candelabra systems.

If $t = 1$, for any positive integer n and m and any set of positive integers K there is a 1-LCS of group type $(m^n : s)$, $s = 0$ or $s = 1$, with block sizes from K since the family of blocks must be the empty family.

In the $t = 2$ case, a 2-LCS is equivalent to a 2-LGDD, thus theorem 4.1.5 can be stated in the following way:

Theorem 4.1.6 *Let n, m be positive integers and let K be a set of positive integers. If there exist a 2-LGDD of group type (m^n) with block sizes from K and an $S(2, K, m)$, then there exists a 2-LGDD of group type (n^m) with block sizes from K .*

Proof. Let $(X, \Gamma, \mathcal{A})$ denote the given 2-LGDD of group type (m^n) with block sizes from K , where $X = I_n \times I_m$. Let $\mathcal{A}' = \{A \mid A \in \mathcal{A}, A \not\subseteq C_j \text{ for any } j, 1 \leq j \leq m\}$, that is $\mathcal{A}'$ denotes the blocks of $\mathcal{A}$, which are not contained in any of the sets C_j for $j \in I_m$. For each group $G_i = R_i$, $1 \leq i \leq n$, of the 2-LGDD construct an $S(2, K, m)$ denoted by $(G_i, \mathcal{B}_{G_i})$. Let $\mathcal{B} = \cup_{i=1}^n \mathcal{B}_{G_i}$, then the triple $(X, \{C_j : j \in I_m\}, \mathcal{A}' \cup \mathcal{B})$ is a 2-LGDD of group type (n^m) with block sizes from K . Any 2-set contained in a row R_i for some $i \in I_n$ is covered by a unique block B of $\mathcal{B}$, such that $B \subseteq R_i$. Any 2-set contained in distinct rows and distinct columns is covered by a unique block of $\mathcal{A}'$.

Note that theorem 4.1.6 is not true for $t > 2$.

4.2 The Fundamental Construction for t -LGDD's and for 3-LCS's

The fundamental construction of t -GDD can be extended for t -LGDD's.

Theorem 4.2.1 (The fundamental construction for t -LGDD's) *Let t, n, a and b be positive integers and let K be a set of positive integers. If for every block A of a t -LGDD of group type (a^n) there is a t -LGDD of group type $(b^{|A|})$ with block sizes from K , then there exists a t -LGDD of group type $(ab)^n$ with block sizes from K .*

Proof. Let $(X = I_n \times I_a, \Gamma = \{\{i\} \times I_a : i = 1, 2, \dots, n\}, \mathcal{A})$ be the recipe t -LGDD of group type (a^n) . Weight each point of X by the uniform weight b . For each block A of $\mathcal{A}$, we can construct a t -LGDD $(A \times I_b, \{\{a\} \times I_b : a \in A\}, \mathcal{B}_A)$ with block sizes from K . Let $X^* = I_n \times (I_a \times I_b)$, $\Gamma^* = \{\{i\} \times (I_a \times I_b) : i = 1, 2, \dots, n\}$ and $\mathcal{A}^* = \cup_{A \in \mathcal{A}} \mathcal{B}_A$. From the fundamental construction for t -GDD (theorem 2.2.4), $(X^*, \Gamma^*, \mathcal{A}^*)$ is a t -GDD of group type $((ab)^n)$ with block sizes from K . We claim that the t -GDD obtained this way is a t -LGDD as well. That is, if $\{(n_1, (u, v)), (n_2, (u, v)) \dots (n_t, (u, v))\}$ is a t -set of points from t different groups, where (u, v) is a fixed pair of $I_a \times I_b$, then the unique block containing the given t -set is contained in the set $I_n \times \{(u, v)\}$.

There is a unique block A of the recipe t -LGDD of group type (a^n) that contains the t -set $\{(n_1, u), (n_2, u) \dots (n_t, u)\}$ such that $A \subset I_n \times \{u\}$. Since the t -set $\{(n_1, (u, v)), (n_2, (u, v)) \dots (n_t, (u, v))\}$ belongs to the set $A \times \{v\}$ of the t -LGDD $(A \times I_b, \{\{i\} \times I_b : i \in A\}, \mathcal{B}_A)$ for $v \in I_b$, it is contained in a unique block B of $\mathcal{B}_A$ such that $B \subset A \times \{v\}$. Thus $B \subset A \times \{v\} \subset I_n \times \{(u, v)\}$ and the proof is complete.

3-*LCS*'s can be constructed recursively also. The following theorem gives the fundamental construction for 3-*LCS*'s.

Theorem 4.2.2 (The fundamental construction for 3-*LCS*'s) *Let n , a and b be positive integers. Let $(X, \{\infty\}, \Gamma, \mathcal{A})$ be a 3-*LCS* of group type $(a^n : 1)$ and let K be a set of positive integers. For any given positive integer b , there exists a 3-*LCS* of group type $((ab)^n : 1)$ with block sizes from K , if the following conditions hold:*

1. *For every block A of 3-*LCS* not containing ∞ , there is a 3-*LGDD* of group type $(b^{|A|})$ with block sizes from K .*
2. *For every block A of 3-*LCS* passing through ∞ , there is a 3-*LCS* of group type $(b^{|A|-1} : 1)$ with block sizes from K .*

Proof. Let $(X = I_n \times I_a \cup \{\infty\}, \{\infty\}, \Gamma = \{\{i\} \times I_a : i = 1, 2, \dots, n\}, \mathcal{A})$ be the recipe 3-*LCS* of group type (a^n) . Weight each point of $X \setminus \{\infty\}$ by the uniform weight b and weight ∞ by 1. For each block A of $\mathcal{A}$ not containing ∞ we can construct a 3-*LGDD* $(A \times I_b, \{\{a\} \times I_b : a \in A\}, \mathcal{B}_A)$ with block sizes from K . For each block A' of $\mathcal{A}$ passing through ∞ construct a 3-*LCS* $((A' \setminus \infty) \times I_b \cup \{\infty^*\}, \{\infty^*\}, \{\{a\} \times I_b : a \in A' \setminus \{\infty\}\}, \mathcal{B}_{A'})$ with block sizes from K . Let $X^* = I_n \times (I_a \times I_b) \cup \{\infty^*\}$, $\Gamma^* = \{\{i\} \times (I_a \times I_b) : i = 1, 2, \dots, n\}$ and $\mathcal{A}^* = \cup_{A \in \mathcal{A}} \mathcal{B}_A$. From the fundamental construction for 3-*CS* (theorem 3.2.1), $(X^*, \{\infty^*\}, \Gamma^*, \mathcal{A}^*)$ is a 3-*CS* of group type $((ab)^n)$ with block sizes from K . We claim that the 3-*CS* obtained this way is a 3-*LCS* as well.

Case 1. Let $\{(n_1, (u, v)), (n_2, (u, v)), (n_3, (u, v))\}$ be a 3-set of points, where (u, v) is a fixed pair of $I_a \times I_b$. We need to show that the unique block containing the given 3-set is contained in the set $I_n \times \{(u \times v)\} \cup \{\infty^*\}$.

There is a unique block A of the recipe 3-*LCS* of group type $(a^n : 1)$ that contains the 3-set $\{(n_1, u), (n_2, u), (n_3, u)\}$ such that $A \subset I_n \times u \cup \{\infty\}$.

If A does not contain ∞ , then $\{(n_1, (u, v)), (n_2, (u, v)), (n_3, (u, v))\}$ is covered by a unique block B of the 3-*LGDD* $(A \times I_b, \{(i \times I_b : i \in A), \mathcal{B}_A\})$. Since the 3-set $\{(n_1, (u, v)), (n_2, (u, v)), (n_3, (u, v))\}$ belongs to the set $A \times \{v\}$ for $v \in I_b$, $B \subset A \times \{v\}$. Thus $B \subset A \times \{v\} \subset I_n \times \{(u \times v)\}$.

If A contains ∞ , then $\{(n_1, (u, v)), (n_2, (u, v)), (n_3, (u, v))\}$ is covered by a unique block B of the 3-*LCS* $((A \setminus \{\infty\}) \times I_b \cup \{\infty^*\}, \{\infty^*\}, \{(i \times I_b : i \in A \setminus \{\infty\}), \mathcal{B}_A\})$. Since the 3-set $\{(n_1, (u, v)), (n_2, (u, v)), (n_3, (u, v))\}$ belongs to the set $(A \setminus \{\infty\}) \times \{v\}$ for $v \in I_b$, $B \subset (A \setminus \{\infty\}) \times \{v\} \cup \{\infty^*\}$. Thus $B \subset (A \setminus \{\infty\}) \times \{v\} \cup \{\infty^*\} \subset I_n \times \{(u \times v)\} \cup \{\infty^*\}$.

Case 2. Let $\{(n_1, (u, v)), (n_2, (u, v)), \infty^*\}$ be a 3-set of points, where (u, v) is a fixed pair of $I_a \times I_b$.

There is a unique block A of the 3-*LCS* of group type $(a^n : 1)$ that contains the 3-set $\{(n_1, u), (n_2, u), \infty\}$ such that $A \subset I_n \times \{u\} \cup \{\infty\}$. Then the 3-set $\{(n_1, (u, v)), (n_2, (u, v)), \infty^*\}$ is covered by a unique block B of the 3-*LCS* $((A \setminus \{\infty\}) \times I_b \cup \{\infty^*\}, \{\infty^*\}, \{(i \times I_b : i \in A \setminus \{\infty\}), \mathcal{B}_A\})$. The 3-set $\{(n_1, (u, v)), (n_2, (u, v)), \infty^*\}$ belongs to the set $(A \setminus \{\infty\}) \times v \cup \{\infty^*\}$ for $v \in I_b$, hence $B \subset (A \setminus \{\infty\}) \times \{v\} \cup \{\infty^*\}$. Thus $B \subset (A \setminus \{\infty\}) \times \{v\} \cup \{\infty^*\} \subset I_n \times \{(u \times v)\} \cup \{\infty^*\}$ and the proof is complete.

4.3 Constructions for Lattice Transversal t -Designs

Theorem 4.3.1 *Let k be a positive integer, $k \geq 2$. If there is an $S(2, k, k^2)$, then there is an $LTD(2, k, k)$.*

Proof. Let $(X, \mathcal{A})$ denote an affine plane of order k , that is an $S(2, k, k^2)$, and let $\mathcal{P}_r$ and $\mathcal{P}_c$ denote two distinguished parallel classes of $(X, \mathcal{A})$. Let us construct $(X, \mathcal{A})$ on the point set $X = I_k \times I_k$, whose rows and columns are the parallel classes $\mathcal{P}_r$ and $\mathcal{P}_c$ respectively. Then $(X, \{\{i\} \times I_k : i = 1, 2, \dots, k\}, \mathcal{A} \setminus \mathcal{P}_r)$ is an $LTD(2, k, k)$.

Corollary 4.3.2 *There is an $LTD(2, q, q)$ for any prime power q .*

Proof. Since there is an affine plane of order q for any prime power q , the corollary is straightforward consequence of theorem 4.3.1.

Before we give our next construction for LTD 's, we present the proof of theorem 2.2.5, which is given in [15]. For any positive integer s let $I_s^o = \{0, 1, 2, \dots, s\}$.

Theorem 4.3.3 *Let q be a prime power. If $t \leq q + 1$, then there exists a $TD(t, q + 1, q)$.*

Proof. Let $\mathbf{F}_q$ denote the finite field of size q . Let $X = I_q^o \times \mathbf{F}_q$ be the set of points. Now we construct the blocks $P_{\mathbf{a}}$ for all $\mathbf{a} = (a_0, a_1, \dots, a_{t-1}) \in (\mathbf{F}_q)^t$ by $P_{\mathbf{a}} = \{(q-1, a_0), (q, a_{t-1}), (i, \sum_{j=0}^{t-1} a_j x^{ij}) : i = 0, 1, \dots, q-2\}$, where x is a fixed generator element of $\mathbf{F}_q$. Let $\mathcal{P} = \{P_{\mathbf{a}} : \mathbf{a} \in (\mathbf{F}_q)^t\}$ denote the set of blocks. then $(X, \{\{i\} \times \mathbf{F}_q : i = 0, 1, \dots, q\}, \mathcal{P})$ is a $TD(t, q + 1, q)$.

First note that there are q^t blocks. We have to show that for any t -set

$\{(i_1, b_{i_1}), (i_2, b_{i_2}), \dots, (i_t, b_{i_t})\}$, $i_1 < i_2 < \dots < i_t$, there is exactly one block of $\mathcal{P}$ containing it.

Case 1. If $i_l \neq q$ and $i_l \neq q - 1$ for $l = 1, 2, \dots, t$, then we need to solve the system of t equations and t unknowns

$$\sum_{j=0}^{t-1} a_j (x^u)^j = b_{i_l} \quad l = 1, 2, \dots, t$$

for a unique $\mathbf{a} = (a_0, a_1, \dots, a_{t-1}) \in (\mathbf{F}_q)^t$. Since the coefficient matrix of the system is Vandermonde type, which is non-singular, $\mathbf{a}$ is uniquely determined.

Case 2. If $i_t = q - 1$, then $b_{i_t} = a_0$. We need to solve the system of equations

$$\sum_{j=0}^{t-1} a_j (x^u)^j = b_{i_l} \quad l = 1, 2, \dots, t - 1$$

or

$$\sum_{j=1}^{t-1} a_j (x^u)^j = b_{i_l} - b_{i_t} \quad l = 1, 2, \dots, t - 1$$

for a unique $(a_1, a_2, \dots, a_{t-1})$. This system has $t - 1$ equations, $t - 1$ unknowns and non-singular coefficient matrix, thus $(a_1, a_2, \dots, a_{t-1})$ are uniquely determined.

Case 3. If $i_t = q$ but $i_{t-1} \neq q - 1$, in this case $b_{i_t} = a_{t-1}$ and we need to solve the system

$$\sum_{j=0}^{t-1} a_j (x^u)^j = b_{i_l} \quad l = 1, 2, \dots, t - 1$$

or

$$\sum_{j=0}^{t-2} a_j (x^u)^j = b_{i_l} - b_{i_t} (x^u)^{t-1} \quad l = 1, 2, \dots, t - 1$$

for a unique $(a_0, a_1 \dots a_{t-2})$. As in the previous case, there is a unique solution $(a_0, a_1 \dots a_{t-2})$, since the system has $t - 1$ equations with the same number of unknowns and a non-singular coefficient matrix.

Case 4. If $i_t = q$ and $i_{t-1} = q - 1$, then $b_{i_t} = a_{t-1}$ and $b_{i_{t-1}} = a_0$. The system of $t - 2$ equations with the same number of variables

$$\sum_{j=0}^{t-1} a_j (x^{i_l})^j = b_{i_l} \quad l = 1, 2 \dots t - 2$$

or

$$\sum_{j=1}^{t-2} a_j (x^{i_l})^j = b_{i_l} - b_{i_{t-1}} - b_{i_t} (x^{i_l})^{t-1} \quad l = 1, 2 \dots t - 2$$

has a unique solution $(a_1, a_2 \dots a_{t-2})$, since the coefficient matrix of the system is non-singular.

Theorem 4.3.4 *Let q be a prime power. If $t \leq q + 1$ and $\gcd(q - 1, t - 1) > 1$, then there exists an $LTD(t, q + 1, q)$.*

Proof. We can obtain an $LTD(t, q + 1, q)$ from the $TD(t, q + 1, q)$ constructed in the previous theorem. It is not hard to see that it is sufficient to find a parallel class of q blocks. In this case, enumerate the blocks of the parallel class by the elements of $\mathbf{F}_q$. Since any permutation of the elements of $\mathbf{F}_q$ within a group leads to an isomorphic $TD(t, q + 1, q)$, we can rearrange the elements within groups in the following way: in each group, let α be the point, at which the α -th member of the parallel class intersects the group. Obviously, in this way we obtain the required $LTD(t, q + 1, q)$.

Now, we need only to find a parallel class of q blocks. If $\gcd(q-1, t-1) > 1$, then choose an element γ of $\mathbf{F}_q \setminus 0$, which is not a $(t-1)$ th-power in $\mathbf{F}_q$. Let us consider the set of q blocks in the form of $\{(q-1, \gamma a), (q, -a), (i, \gamma a - ax^{(t-1)i}) : i = 0, 1, \dots, q-2\}$ for $a \in \mathbf{F}_q$. We claim that they form a parallel class of q blocks. If two of these blocks intersect, then there are two distinct elements of $\mathbf{F}_q$, such that $\gamma a - ax^{(t-1)i} = \gamma b - bx^{(t-1)i}$ for some $i = 0, 1, \dots, q-2$. In this case $\gamma = x^{(t-1)i}$, which contradicts the choice of γ .

A parallel class of q blocks in a $TD(t, q+1, q)$ can be constructed in different ways also. In the previous proof, to construct a parallel class of q blocks we use the fact that the polynomial function $\gamma - x^{(t-1)}$ has no roots over the finite field $\mathbf{F}_q$. More generally, if there is a polynomial function over $\mathbf{F}_q$ of degree $t-1$, which has no roots over $\mathbf{F}_q$, then we can construct a parallel class of q blocks in a $TD(t, q+1, q)$. We will use this idea later, but first we prove a lemma.

Lemma 4.3.5 *Let q be a prime power. If t is an odd positive integer, then there exists a polynomial function of degree $t-1$ over $\mathbf{F}_q$, which has no roots over the finite field $\mathbf{F}_q$.*

Proof. If q is even, then there are $\frac{q}{2}$ elements and if q is odd, then there are $\frac{q-1}{2}$ elements of $\mathbf{F}_q$, which are in the form of $\beta + \beta^{-1}$ for some $\beta \in \mathbf{F}_q \setminus 0$. Thus, if q is even, $\frac{q}{2}$ elements and if q is odd, $\frac{q-1}{2}$ elements can not be written in the previously given form. Choose such an element α of $\mathbf{F}_q$. Then we claim that the polynomial function $1 - \alpha x^{\frac{t-1}{2}} + x^{t-1}$ has no roots over $\mathbf{F}_q$. Otherwise, if x_0 is a root of $1 - \alpha x^{\frac{t-1}{2}} + x^{t-1}$, then $\alpha = (x_0^{\frac{t-1}{2}})^{-1} + x_0^{\frac{t-1}{2}}$, which is a contradiction.

Theorem 4.3.6 *Let q be a prime power. If t is an odd positive integer and $t \leq q + 1$, then there exists an $LTD(t, q + 1, q)$.*

Proof. We use the construction of theorem 4.3.4 to obtain a $TD(t, q, q + 1)$. Choose α as in lemma 4.3.5. The following set of q blocks $\{(q - 1, a), (q, a), (i, a - \alpha\alpha x^{i(\frac{t-1}{2})} + ax^{i(t-1)}) : i = 0, 1, \dots, q - 2\}$ for $a \in \mathbf{F}_q$ forms a parallel class. Thus, we can rearrange the elements within groups to get the required $LTD(t, q + 1, q)$.

Corollary 4.3.7 *Let q be a prime power and let $t \leq q + 1$. If t is an odd positive integer or if $\gcd(q - 1, t - 1) > 1$, then there exists an $LTD(t, q + 1, q^n)$ for all positive integers n .*

Proof. We prove the result by induction on n . If $n = 1$, then the statement is true. Assume we have an $LTD(t, q + 1, q^{n-1})$. For each block of the $LTD(t, q + 1, q^{n-1})$, there is a $LTD(t, q + 1, q)$. Using the fundamental construction theorem for $LGDD$'s (theorem 4.2.1), we obtain an $LTD(t, q + 1, q^n)$.

Theorem 4.3.8 *For any positive integers t and k , $1 \leq t \leq k$, there is a number $v_0(k, t)$ such that if $v \geq v_0$, then there exists an $LTD(t, k, v)$.*

Proof. Deleting groups of a $TD(t, k, v)$ yields a $TD(t, k', v)$ for any $1 \leq k' \leq k$. Thus, if we omit $t - 1$ groups of a $TD(t, k + t - 1, v)$, we obtain a $TD(t, k, v)$. Furthermore, we claim that the $TD(t, k, v)$ obtained this way is an $LTD(t, k, v)$ as well. Let $(X, \{G_1, \dots, G_{t-1}, G_t, \dots, G_{k+t-1}\}, \mathcal{M})$ be a $TD(t, k + t - 1, v)$ and let $\{x_1, x_2, \dots, x_{t-1}\}$ be a given $t - 1$ -set of points from $t - 1$ distinct groups such that

$x_i \in G_i$ for $i = 1, 2, \dots, t-1$. Set $X^* = X \setminus (\cup_{i=1}^{t-1} G_i)$, $\Gamma^* = \{G_t, \dots, G_{k+t-1}\}$ and $\mathcal{M}^* = \{M \setminus \{y_1, y_2, \dots, y_{t-1}\} : M \in \mathcal{M} \text{ and } y_i = M \cap G_i \text{ for } i = 1, 2, \dots, t-1\}$. Then $(X^*, \Gamma^*, \mathcal{M}^*)$ is an $LTD(t, k, v)$ since the set of blocks of the $TD(t, k+t-1, v)$ containing the $t-1$ points $x_1, x_2, \dots, x_{t-1}$ forms a parallel class of v -blocks in the transversal design obtained by deleting the first $t-1$ groups of the $TD(t, k+t-1, v)$.

By Blanchard's theorem (theorem 2.2.8), there is a constant $v_1 = v_1(k+t-1)$ such that if $v \geq v_1$, then there exists a $TD(t, k+t-1, v)$. It follows that if $v \geq v_0(k, t) = v_1(k+t-1)$, then there exists an $LTD(t, k, v)$.

4.4 More Lattice Transversal t -Designs

In this section we use Steiner designs to construct more LTD 's. These will be used in the recursive constructions to build new designs in the following section and in chapter 5.

Theorem 4.4.1 *Let k and a be positive integers, $k \geq 2$. If there exist an $S(2, k, a)$ and an $LTD(2, k, k)$, then there exists an $LTD(2, k, a)$ or equivalently there exists an idempotent $OA(2, k, a)$.*

Proof. Weight each point of the Steiner 2-design $S(2, k, a)$ by the uniform weight k . For each block of the Steiner 2-design there is an $LTD(2, k, k)$. Hence, by the fundamental construction theorem for $LGDD$'s we can construct a 2- $LGDD$ of group type (k^a) with block size k . We apply the "row and column interchangeable" property of 2- $LGDD$'s (theorem 4.1.6) to obtain an $LTD(2, k, a)$.

Corollary 4.4.2 *Let q be a prime power and a be a positive integer. If there is an $S(2, q, a)$, then there is an $LTD(2, q, a)$ or equivalently there exists an idempotent $OA(2, q, a)$.*

Proof. By corollary 4.3.2 there is an $LTD(2, q, q)$ for any prime power q . Therefore the result follows from theorem 4.4.1.

Theorem 4.4.3 *Let k and a be positive integers, $k \geq 2$. If there exist an $S(3, k+1, a+1)$, an $LTD(3, k+1, k)$ and an $OD(3, k+1, k+1)$, then there exists an $LTD(3, k+1, a)$.*

Proof. The definition of ordered design (OD) is given in section 2.4. By deleting a single point P of the $S(3, k+1, a+1)$ we obtain an $S(3, \{k, k+1\}, a)$. Any two blocks of size k intersect in at most one point and any two points of $S(3, \{k+1, k\}, a)$ are covered by a unique block of size k . The number of blocks of the $S(3, \{k+1, k\}, a)$ of size k is $\frac{a(a-1)}{k(k-1)}$ and the number of blocks of size $k+1$ is $\frac{a(a-1)(a-k)}{k(k-1)(k+1)}$.

Now, we will construct the blocks of the $LTD(3, k+1, a)$ on the set $I_{k+1} \times I_a$. We construct an $S(3, \{k+1, k\}, a)$, $(I_a, \mathcal{A})$, with the properties given before.

For each block A of size k of $\mathcal{A}$ we construct an $LTD(3, k+1, k)$ with point set $I_{k+1} \times A$ and block set $\mathcal{K}_A \cup \mathcal{C}_A$, where $\mathcal{C}_A = \bigcup_{j \in A} (I_{k+1} \times \{j\})$. Without loss of generality, we can assume that each $I_{k+1} \times \{j\}$, $j \in A$ is a block of the $LTD(3, k+1, k)$.

For a block A of $\mathcal{A}$ of size $k+1$, we construct an $OD(3, k+1, k+1)$ on the point set $I_{k+1} \times A$ and with block set $\mathcal{L}_A$.

Let $\mathcal{K} = \bigcup_{A \in \mathcal{A}, |A|=k} \mathcal{K}_A$ and $\mathcal{L} = \bigcup_{A \in \mathcal{A}, |A|=k+1} \mathcal{L}_A$. Finally, we define the block set $\mathcal{M} = \mathcal{K} \cup \mathcal{L} \cup_{j=1}^a (I_{k+1} \times \{j\})$. We claim that $(I_{k+1} \times I_a, \{\{i\} \times I_a : i = 1, 2, \dots, k+1\}, \mathcal{M})$ is an $LTD(3, k+1, a)$.

First note that we have the right number of blocks:

$$\frac{a(a-1)}{k(k-1)}(k^3 - k) + \frac{a(a-1)a-k}{k(k-1)(k+1)}(k^3 - k) + a = a^3.$$

Let $(i_1, a_1), (i_2, a_2)$ and (i_3, a_3) be three points that belong to three distinct groups.

Case 1. If $a_1 = a_2 = a_3$, then they are contained in the unique block $I_{k+1} \times a_1$. No other blocks can contain the given three points since any blocks of $\mathcal{K} \cup \mathcal{L}$ intersect each column in less than three points.

Case 2. If $a_1 = a_2$, but $a_1 \neq a_3$, then there is a unique block A of $\mathcal{A}$ of size k passing through $a_1 = a_2$ and a_3 . Since $(i_1, a_1), (i_2, a_2)$ and (i_3, a_3) belong to different groups of the $LTD(3, k+1, k)$ $(I_{k+1} \times A, \{\{i\} \times A : i = 1, 2, \dots, k+1\}, \mathcal{K}_A)$, there is a unique block of $\mathcal{K}_A$ containing the triple. The blocks of $\mathcal{M}$ intersect the columns in at most one point, thus no blocks of $\mathcal{M}$ can contain a triple $\{(i_1, a_1), (i_2, a_2), (i_3, a_3)\}$ with $a_1 = a_2$. Any two blocks A and B of $\mathcal{A}$ of size k intersect in at most one point, thus no other block of $\mathcal{K}$ can cover the given triple.

Case 3. If a_1, a_2 and a_3 are distinct, then there is a unique block A of $\mathcal{A}$ containing $\{a_1, a_2, a_3\}$. The block A can have size either k or $k+1$. If the size is k , as in case 2, $(i_1, a_1), (i_2, a_2)$ and (i_3, a_3) are contained in a unique block of the $LTD(3, k+1, k)$ constructed for A and contained in no other blocks. If $\{a_1, a_2, a_3\}$ is covered by a unique block of size $k+1$ of $\mathcal{A}$, then $(i_1, a_1), (i_2, a_2)$ and (i_3, a_3) contained in a unique block of the $OD(3, k+1, k+1)$ constructed on the point set $I_{k+1} \times A$ and with block

set $\mathcal{L}_A$. Since any two blocks of $\mathcal{A}$ intersect in less than three points this is the only block, which contains the given three points.

Corollary 4.4.4 *Let q be a prime power and let a be a positive integer. If there is an $S(3, q+1, a+1)$, then there is an $LTD(3, q+1, a)$.*

Proof. For any prime power q , there exist an $LTD(3, q+1, q)$ by theorem 4.3.6 and an $OD(3, q+1, q+1)$ by theorem 2.4.7. Thus, the corollary follows from theorem 4.4.3.

In the $t = 2$ case, we can state the following theorem:

Theorem 4.4.5 *Let k and a be positive integers. If there exist an $S(2, k+1, a+1)$, a $TD(2, k+1, k)$ and an $OD(2, k+1, k+1)$, then there exists a $TD(2, k+1, a)$.*

Proof. The proof goes the same way as the proof of theorem 4.4.3.

By deleting a single point P of the $S(2, k+1, a+1)$ we obtain an $S(2, \{k, k+1\}, a)$. No two blocks of size k intersect and any point of $S(2, \{k, k+1\}, a)$ is covered by a unique block of size k . The number of blocks of the $S(2, \{k, k+1\}, a)$ of size k is $\frac{a}{k}$ and the number of blocks of size $k+1$ is $\frac{a(a-k)}{k(k+1)}$.

Now, we will construct the blocks of the $TD(2, k+1, a)$ on the set $I_{k+1} \times I_a$. We construct an $S(2, \{k+1, k\}, a)$, $(I_a, \mathcal{A})$, with the properties given before.

For each block A of size k of $\mathcal{A}$ we construct a $TD(2, k+1, k)$ with point set $I_{k+1} \times A$ and block set $\mathcal{K}_A$.

For a block A of $\mathcal{A}$ of size $k+1$, we construct an $OD(2, k+1, k+1)$ on the point set $I_{k+1} \times A$ and with block set $\mathcal{L}_A$.

Let $\mathcal{K} = \bigcup_{A \in \mathcal{A}, |A|=k} \mathcal{K}_A$ and $\mathcal{L} = \bigcup_{A \in \mathcal{A}, |A|=k+1} \mathcal{L}_A$. Finally, we define the block set $\mathcal{M} = \mathcal{K} \cup \mathcal{L}$. We claim that $(I_{k+1} \times I_a, \{\{i\} \times I_a : i = 1, 2, \dots, k+1\}, \mathcal{M})$ is a $TD(3, k+1, a)$.

First note that we have the right number of blocks:

$$\frac{a}{k}k^2 + \frac{a(a-k)}{k(k+1)}k(k+1) = a^2.$$

Let (i_1, a_1) and (i_2, a_2) be two points that belong to two distinct groups.

Case 1. If $a_1 = a_2$, then there is a unique block A of $\mathcal{A}$ of size k passing through $a_1 = a_2$. Since (i_1, a_1) and (i_2, a_2) belong to different groups of the $TD(2, k+1, k)$ $(I_{k+1} \times A, \{\{i\} \times A : i = 1, 2, \dots, k+1\}, \mathcal{K}_A)$, there is a unique block of $\mathcal{K}_A$ containing the pair of points. The blocks of $\mathcal{M}$ intersect the columns in at most one point, thus no blocks of $\mathcal{M}$ can contain the pair $\{(i_1, a_1), (i_2, a_2)\}$ with $a_1 = a_2$. Since any two blocks A and B of $\mathcal{A}$ of size k can not intersect, no other block of $\mathcal{K}$ can cover the given pair.

Case 2. If a_1 and a_2 are distinct, then there is a unique block A of $\mathcal{A}$ containing $\{a_1, a_2\}$. The block A can have size either k or $k+1$. If the size is k , as in case 1, (i_1, a_1) and (i_2, a_2) are contained in a unique block of the $TD(2, k+1, k)$ constructed for A and contained in no other blocks. If $\{a_1, a_2\}$ is covered by a unique block of size $k+1$ of $\mathcal{A}$, then (i_1, a_1) and (i_2, a_2) are contained in a unique block of the $OD(2, k+1, k+1)$ constructed on the point set $I_{k+1} \times A$ and with block set $\mathcal{L}_A$. Since any two blocks of $\mathcal{A}$ intersect in less than two points, this is the only block which contains the given pair.

Corollary 4.4.6 *Let q be a prime power such that $q+1$ is a prime power as well and let a be a positive integer. If there is an $S(2, q+1, a+1)$, then there is a $TD(2, q+1, a)$.*

Proof. Note that, since q and $q+1$ are prime powers, it is necessary that either q or $q+1$ be a power of 2. We have a $TD(2, q+1, q)$ for any power q by theorem 4.3.3. Since $q+1$ is a prime power, theorem 2.4.7 asserts the existence of an $OD(2, q+1, q+1)$. Hence, theorem 4.4.5 implies the corollary.

4.5 Constructions for LCS 's and Applications

Theorem 4.5.1 *Let k be a positive integer, $k \geq 2$. If there is an $S(3, k+1, k^2+1)$, then there exists a 3- LCS of group type $(k^k : 1)$ with block size $k+1$.*

Proof. By theorem 3.1.5, we can construct a 3- CS of group type $(k^k : 1)$ with block size $k+1$ from an $S(3, k+1, k^2+1)$. It is sufficient to exhibit a set of k blocks, which intersect at only ∞ (the single point of the stem). Since the derived design of an $S(3, k+1, k^2+1)$ is an affine plane $S(2, k, k^2)$, we can choose another parallel class of k blocks of the affine plane. (One class was used to form the groups of the 3- CS .) Obviously, these blocks intersect only at ∞ . Now, as before, permute the points within groups to obtain the 3- LCS of group type $(k^k : 1)$ with block size $k+1$.

Corollary 4.5.2 *If q is a prime power, then there exists a 3- LCS of group type $(q^q : 1)$ with block size $q+1$.*

Proof. For every prime power q , there exists a Möbius plane $S(3, q + 1, q^2 + 1)$. Hence the statement follows from theorem 4.5.1.

Theorem 4.5.3 *If q is a prime power, then there exists a 3-LCS of group type $(q^m)^{q^n} : 1$ with block size $q + 1$ for all n and m non-negative integers.*

Proof. We proceed by induction on m . If $m = 0$, then $S(3, q + 1, q^n + 1)$ is a 3-LCS of group type $(1^{q^n} : 1)$ with block size $q + 1$. Assume that there is a 3-LCS of group type $((q^{m-1})^{q^n} : 1)$ with block size $q + 1$. By theorems 4.3.6 and 4.5.2, there exist an $LTD(3, q + 1, q)$ and a 3-LCS of group type $(q^q : 1)$ with block size $q + 1$. Thus, the conditions of the fundamental construction theorem for 3-LCS's (theorem 4.2.2) are satisfied and hence we can construct a 3-LCS of group type $(q^m)^{q^n} : 1$ with block size $q + 1$.

Theorem 4.5.4 *Let k and v be positive integers, $k \geq 2$. If there exist an $S(3, k + 1, v + 1)$, a 3-LCS of group type $(k^k : 1)$ with block size $k + 1$ and an $LTD(3, k + 1, k)$, then there exist*

1. 3-LCS of group type $(k^v : 1)$ with block size $k + 1$ and
2. 3-LCS of group type $(v^k : 1)$ with block size $k + 1$.

Proof. Let the Steiner 3-design $S(3, k + 1, v + 1)$ be a 3-LCS of group type $(1^v : 1)$ with block size $k + 1$. For every block not containing the point in the stem, we construct an $LTD(3, k + 1, k)$ and for every block passing through the point in the

stem, we construct a 3-*LCS* of group type $(k^k : 1)$ with block size $k + 1$. Applying the fundamental construction for 3-*LCS*'s (theorem 4.2.2), we obtain a 3-*LCS* of group type $(k^v : 1)$ with block size $k + 1$.

To prove the second part of the theorem, interchange the role of the rows and columns of the 3-*LCS* obtained in the first part, by theorem 4.1.5 the resulting 3-*LCS* is of group type $(v^k : 1)$ with block size $k + 1$.

Corollary 4.5.5 *Let q be a prime power and v be a positive integer. If there is an $S(3, q + 1, v + 1)$, then there exist*

1. 3-*LCS* of group type $(q^v : 1)$ with block size $q + 1$ and
2. 3-*LCS* of group type $(v^q : 1)$ with block size $q + 1$.

Proof. If q is a prime power, then there exist a 3-*LCS* of group type $(q^q : 1)$ with block size $q + 1$ by corollary 4.5.2, and a *LTD*(3, $q + 1$, q) by theorem 4.3.6. Therefore the statement follows from theorem 4.5.4.

Corollaries 4.4.4 and 4.5.5 can be used to construct new C -closed sets and as a result new Steiner designs.

Theorem 4.5.6 *Let q be a prime power and let a be a positive integer. If there is an $S(3, q + 1, a + 1)$, then $NG_3(a, q + 1, 1)$ is a C -closed set with parameters $(3, q + 1, 1)$.*

Proof. If there exists an $S(3, q + 1, a + 1)$, then we can construct a 3-*LCS* of group type $(a^q : 1)$ with block size $q + 1$ by corollary 4.5.5 and an *LTD*(3, $q + 1$, a)

by corollary 4.4.4. The existence of a 3-*LCS* of group type $(a^q : 1)$ with block size $q + 1$ and an *LTD* $(3, q + 1, a)$ implies that $NG_3[a, q + 1, 1]$ is a C -closed set with parameters $(3, q + 1, 1)$ by theorem 3.5.6.

In the next theorem we give an independent proof for Wilson's product theorem.

Theorem 4.5.7 *Let q be a prime power and let a and b be positive integers. If there exist an $S(3, q + 1, a + 1)$ and an $S(3, q + 1, b + 1)$, then there exists a 3-*CS* of group type $(b^a : 1)$ with block size $q + 1$ and consequently there exists an $S(3, q + 1, ab + 1)$.*

Proof. Since $NG_3[b, q + 1, 1]$ is C -closed with parameters $(3, q + 1, 1)$ by theorem 4.5.6, we have

$$a \in C_{q+1}^3[1, 1] \subseteq C_{q+1}^3[NG_3[b, q + 1, 1], 1] = NG_3[b, q + 1, 1]$$

and the proof is complete.

The same idea can be used to construct 2-designs.

Theorem 4.5.8 *Let q be a prime power and let a be a positive integer. If there exists an $S(2, q, a)$, then $NG_2[a, q, 0]$ is C -closed with parameters $(2, q, 0)$.*

Proof. From corollary 4.4.2, the existence of an $S(2, q, a)$ implies the existence of an *LTD* $(2, a, q)$. Thus by theorem 3.5.1, $NG_2[a, q, 0]$ is a C -closed set with parameters $(2, q, 0)$.

Theorem 4.5.9 *Let q be a prime power and let a and b be positive integers. If there exist an $S(2, q, a)$ and an $S(2, q, b)$, then there exist a 2-GDD of group type b^a with block size q and consequently there exists an $S(2, q, ab)$.*

Proof. The previous theorem states that $NG_2[b, q, 0]$ is a C -closed set with parameters $(2, q, 0)$ and hence

$$a \in C_q^2[1, 0] \subseteq C_q^2[NG_2[b, q, 0], 0] = NG_2[b, q, 0]$$

and the theorem follows.

Theorem 4.5.10 *Let q be a prime power such that $q + 1$ is a prime power as well and let a be a positive integer. If there exists an $S(2, q + 1, a + 1)$, then $NG_2[a, q + 1, 0]$ is C -closed with parameters $(2, q + 1, 0)$.*

Proof. The proof is similarly to the proofs of theorems 4.5.6 and 4.5.8. From theorem 4.4.6, if there is an $S(2, q + 1, a + 1)$, then there is a $TD(2, q + 1, a)$. Hence from theorem 3.5.1, $NG_2[a, q + 1, 0]$ is a C -closed set parameters $(2, q + 1, 0)$.

Theorem 4.5.11 *Let q be a prime power such that $q + 1$ is a prime power as well. Let a and b be positive integers. If there exist an $S(2, q + 1, a + 1)$ and an $S(2, q + 1, b + 1)$, then there is a 2-GDD of group type $b^{(a+1)}$ with block size $q + 1$ and consequently there exists an $S(2, q + 1, ab + b + 1)$.*

Proof. By theorem 4.5.6, $NG_2[b, q + 1, 0]$ is C -closed with parameters $(2, q + 1, 0)$. Hence,

$$a + 1 \in C_{q+1}^2[1, 0] \subseteq C_{q+1}^2[NG_2[b, q + 1, 0], 0] = NG_2[b, q + 1, 0].$$

which implies the existence of a 2-*GDD* of group type $b^{(a+1)}$ with block size $q+1$. Next we adjoin a point to the points set of this 2-*GDD* to construct an $S(2, q+1, ab+b+1)$.

Note that if $a = q$ and $b = q$ in theorem 4.5.9, then the Steiner 2-design obtained is the affine plane of order q and if $a = q$ and $b = q$ in theorem 4.5.11, then the resulting Steiner 2-design is the projective plane of order q .

CHAPTER 5

NEW INFINITE FAMILIES OF 3-DESIGNS

5.1 Partial designs

The concept of partial designs was introduced by J. L. Blanchard [5]. For any positive integer n , let $I_n = \{1, 2, \dots, n\}$ and for $t \leq n$, let $\mathcal{P}_t(I_n)$ denote the set of t -subsets of I_n .

Definition 5.1.1 *Let $\mathcal{D}$ be a family of k -sets with possible multiplicities from the set I_u and let $\mathcal{T}$ be a subset of $\mathcal{P}_t(I_u)$. We say $\mathcal{D}$ is a partial design, written $P_\lambda(t, k, u)$, covering $\mathcal{T}$ if and only if the following holds:*

1. *each t -set in $\mathcal{T}$ is contained in exactly λ blocks of $\mathcal{D}$, counting multiplicities and*
2. *each t -set not belonging to $\mathcal{T}$ is not contained in any block of $\mathcal{D}$.*

Let $q_1, q_2, \dots, q_n$ be different prime powers and let $X = I_u \times \prod_{i=1}^n I_{q_i}$.

Let Φ be the projection map, where $\Phi : I_u \times \prod_{i=1}^n I_{q_i} \rightarrow I_u$ and $\Phi(i, (x_1, x_2, \dots, x_n)) = i$.

for all $i \in I_u$ and $(x_1, x_2 \dots x_n) \in \prod_{i=1}^n I_{q_i}$. The projection map Φ can be naturally extended to the subsets of X . For $T' \subseteq X$, $\Phi(T') = \{a | a \in I_u, \text{ such that there is an } x \in T', \Phi(x) = a\}$. For a t -subset T of I_u let

$$X_T = \{T' | T' \text{ is a } t\text{-subset of } I_u \times \prod_{i=1}^n I_{q_i} \text{ and } \Phi(T') = T\}.$$

Let $\mathcal{T} \subseteq \mathcal{P}_t(I_u)$.

$$X_{\mathcal{T}} := \bigcup_{T \in \mathcal{T}} X_T.$$

In other words, $X_{\mathcal{T}}$ denotes the set of t -sets of $I_u \times \prod_{i=1}^n I_{q_i}$, whose projection onto the first coordinate is a t -set in $\mathcal{T}$.

The next theorem generalizes Blanchard's block spreading construction [5] for general lambda, which is itself a generalization of Wilson's results [22]. First we state a lemma due to Blanchard [2], which will be used in the proof of this generalized block spreading construction.

Let q be a prime power and t and u be positive integers satisfying $t \leq u \leq q$ and let $\mathbf{F}_q$ denote the finite field of size q . For any r , $0 \leq r < t$, and any r -subset $E = \{e_1, e_2 \dots e_r\} \subseteq \mathbf{F}_q$ and $t - r$ -subset $S = \{s_{r+1}, \dots s_t\} \subseteq \mathbf{F}_q$ define the $t \times t$ matrix

$$M(E; S) = \begin{pmatrix} 1 & e_1 & \cdots & e_1^{t-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & e_r & \cdots & e_r^{t-2} & 0 \\ 1 & s_{r+1} & \cdots & s_{r+1}^{t-2} & s_{r+1}^{t-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_t & \cdots & s_t^{t-2} & s_t^{t-1} \end{pmatrix}$$

Lemma 5.1.2 *Given t and u , with $2 \leq t \leq u$, there is a number $q_0 = q_0(t, u)$ such that . for any prime power $q \geq q_0$ there are two u -subsets of $\mathbf{F}_q$, $\mathcal{E}$ and φ , such that for any r , $0 \leq r < t$, and any r -subset $E \subseteq \mathcal{E}$ and $(t - r)$ -subset $S \subseteq \varphi$ the matrix $M(E; S)$ is non-singular.*

If $a = q_1 q_2 \dots q_n$ is the prime power decomposition of a , we say q_i is a prime power factor a .

Theorem 5.1.3 *Let u and t be positive integers, $2 \leq t \leq u$, and let $\mathcal{T} \subseteq \mathcal{P}_t(I_u)$. Let $q^* = q_0(t, u)$, where $q_0(t, u)$ is as in lemma 5.1.2. Then for any partial design $P_\lambda(t, k, u)$ on the point set I_u covering $\mathcal{T}$ with prime power decomposition $\lambda = q_1 q_2 \dots q_n$ satisfying $q_i \geq q^*$, $i \in I_n$, there exists a partial design $P(t, k, u \lambda^d)$ on the point set $X = I_u \times \prod_{i=1}^n I_{q_i^d}$ that covers $X_{\mathcal{T}}$ for all $d \geq |\mathcal{T}|$.*

Proof. Let $\mathcal{D}$ denote a partial design $P_\lambda(t, k, u)$ on I_u covering the t -sets of $\mathcal{T}$, where $\lambda = q_1 q_2 \dots q_n$ is the prime power decomposition of λ satisfying $q_i \geq q^*$, $i \in I_n$.

Let $\mathbf{F}_{\mathbf{q}_i}$ be a finite fields of size q_i and let $V_i = (\mathbf{F}_{\mathbf{q}_i})^d$ be the d -dimensional vector space over the finite field $\mathbf{F}_{\mathbf{q}_i}$ for $i \in I_n$. Since $q_i \geq q^*$, $i \in I_n$, there are u -subsets of $\mathbf{F}_{\mathbf{q}_i}$, $\mathcal{E}_i = \{e_{i1}, e_{i2} \dots e_{iu}\}$ and $\varphi_i = \{s_{i1}, s_{i2} \dots s_{iu}\}$ for $i \in I_n$, such that for any r , $0 \leq r < t$ and any r -subsets $E_i \subseteq \mathcal{E}_i$ and $t - r$ -subset $S_i \subseteq \varphi_i$ the matrix $M(E_i; S_i)$ is non-singular.

Since $d \geq |\mathcal{T}|$, we can set $d' = d - |\mathcal{T}|$ and index the coordinates of a vector $\vec{x}_i$ of V_i the following way:

$$\vec{x}_i = ((x_{iT} : T \in \mathcal{T}) | (x_{il} : 1 \leq l \leq d')).$$

Thus, the first $|\mathcal{T}|$ coordinates are indexed by the t -sets T of $\mathcal{T}$ and the remaining $d' = d - |\mathcal{T}|$ coordinates are indexed by the integers from 1 to d' . Let $H_i \subset V_i$, $i \in I_n$, be the hyper-plane of all vectors $\vec{y}_i$ satisfying

$$\sum_{T \in \mathcal{T}} y_{iT} + \sum_{l=1}^{d'} y_{il} = 0.$$

For any $(\vec{x}_i^{(1)}, \vec{x}_i^{(2)} \dots \vec{x}_i^{(t-1)}) \in V_i^{t-1}$ we use the notation $\bar{x}_i = (\vec{x}_i^{(1)}, \vec{x}_i^{(2)} \dots \vec{x}_i^{(t-1)})$, where $i \in I_n$. For each $j \in I_u$, $i \in I_n$, we define the linear map $\Gamma_{ij} : V_i^{t-1} \times H_i \rightarrow V_i$ the following way: let $(\bar{x}_i, \vec{y}_i) \in V_i^{t-1} \times H_i$. The image of $(\bar{x}_i, \vec{y}_i)$ will be defined coordinate-wise:

$$\Gamma_{ij}(\bar{x}_i, \vec{y}_i)_T = \begin{cases} \sum_{k=1}^{t-1} e_{ij}^{k-1} x_{iT}^{(k)} & \text{if } j \in T \\ \sum_{k=1}^{t-1} s_{ij}^{k-1} x_{iT}^{(k)} + s_{ij}^{t-1} y_{iT} & \text{if } j \notin T \end{cases}$$

and for $l = 1, 2 \dots d'$

$$\Gamma_{ij}(\bar{x}_i, \vec{y}_i)_l = \sum_{k=1}^{t-1} s_{ij}^{k-1} x_{il}^{(k)} + s_{ij}^{t-1} y_{il}.$$

For each t -set $T \in \mathcal{T}$, there are $\lambda = q_1 q_2 \cdots q_n$ blocks from $\mathcal{D}$ containing T . Thus, for each $T \in \mathcal{T}$ there exists a bijection

$$\mathcal{N}_T : \{A \in \mathcal{D} : T \subset A\} \rightarrow \prod_{i=1}^n \mathbf{F}_{\mathbf{q}_i} = \mathbf{F}_{\mathbf{q}_1} \times \mathbf{F}_{\mathbf{q}_2} \times \cdots \times \mathbf{F}_{\mathbf{q}_n}$$

between the blocks containing T and the set $\prod_{i=1}^n \mathbf{F}_{\mathbf{q}_i}$.

For each $A \in \mathcal{D}$ and for $i \in I_n$ we define a map $f_{iA} : I_u \rightarrow V_i$ as follows: again, we define the image coordinate-wise. For each t -set $T \in \mathcal{T}$

$$f_{iA}(j)_T = \begin{cases} \mathcal{N}_T(A)|_{\mathbf{F}_{\mathbf{q}_i}} & \text{if } j = \max T \text{ and } T \subset A. \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{N}_T(A)|_{\mathbf{F}_{\mathbf{q}_i}}$ denotes the projection of $\mathcal{N}_T A$ onto $\mathbf{F}_{\mathbf{q}_i}$, and for any $l = 1, 2, \dots, d'$

$$f_{iA}(j)_l = 0.$$

Now, we define the set of blocks of the partial design $P(t, k, u\lambda^d)$ covering $X_{\mathcal{T}}$ on the point set $X = I_u \times \prod_{i=1}^n V_i$. For each $((\bar{x}_1, \bar{x}_2 \dots \bar{x}_n), (\vec{y}_1, \vec{y}_2 \dots \vec{y}_n), A) \in \prod_{i=1}^n V_i^{t-1} \times \prod_{i=1}^n H_i \times \mathcal{D}$ construct the block

$$B((\bar{x}_1, \bar{x}_2 \dots \bar{x}_n), (\vec{y}_1, \vec{y}_2 \dots \vec{y}_n), A) =$$

$$\{(j, (z_1, z_2, \dots, z_n)) : j \in A, z_i = \Gamma_{ij}(\bar{x}_i, y_i) + f_{iA}(j), \text{ for } i \in I_n\}.$$

Note that we have the right number of blocks. The number of blocks of a $P(t, k, u\lambda^d)$ covering the t -sets of $X_{\mathcal{T}}$ are

$$\frac{|\mathcal{T}| \cdot q_1^{dt} q_2^{dt} \cdots q_n^{dt}}{\binom{k}{t}}.$$

On the other hand the number of blocks of the form

$B((\tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n), (\vec{y}_1, \vec{y}_2 \dots \vec{y}_n), A)$ are

$$\underbrace{(q_1^d)^{t-1} (q_2^d)^{t-1} \dots (q_n^d)^{t-1}}_{\text{the number of choices for } (\tilde{x}_1 \dots \tilde{x}_n)} \cdot \underbrace{(q_1)^{d-1} (q_2)^{d-1} \dots (q_n)^{d-1}}_{\text{the number of choices for } (\vec{y}_1 \dots \vec{y}_n)} \cdot \underbrace{\frac{|\mathcal{T}| \cdot q_1 q_2 \dots q_n}{\binom{k}{t}}}_{\text{the number of choices for } A \text{ in } \mathcal{D}} =$$

$$\frac{|\mathcal{T}| \cdot q_1^{dt} q_2^{dt} \dots q_n^{dt}}{\binom{k}{t}}.$$

We note that, each of the defined blocks intersects the set $\{\{j\} \times \prod_{i=1}^n V_i\}$ for each $j \in I_u$ in at most one point and the projection of each block onto the first coordinate is a block in $P_\lambda(t, k, u)$, which covers only the t -sets of $\mathcal{T}$. Therefore the defined blocks do not cover any t -set that does not belong to $X_{\mathcal{T}}$.

We claim that these blocks form a partial design $P(t, k, u\lambda^d)$ covering the t -sets of $X_{\mathcal{T}}$. We need to show that for any t -set $T' = \{r_1, r_2 \dots r_t\} \in \mathcal{T}$ and any $\{(z_1^{(r_1)} \dots z_n^{(r_1)}), (z_1^{(r_2)} \dots z_n^{(r_2)}) \dots (z_1^{(r_t)} \dots z_n^{(r_t)})\}$ there is exactly one block of the given type covering the t -set $\{(r_1, (z_1^{(r_1)} \dots z_n^{(r_1)})), (r_2, (z_1^{(r_2)} \dots z_n^{(r_2)})) \dots (r_t, (z_1^{(r_t)} \dots z_n^{(r_t)}))\}$.

In other words for any t -set $T' = \{r_1, r_2 \dots r_t\} \in \mathcal{T}$ and any $\{(z_1^{(r_1)} \dots z_n^{(r_1)}), (z_1^{(r_2)} \dots z_n^{(r_2)}) \dots (z_1^{(r_t)} \dots z_n^{(r_t)})\}$ the system of nt equations

$$\begin{aligned} z_i^{(r_1)} &= \Gamma_{ir_1}(\tilde{x}_i, \vec{y}_i) + f_{iA}(r_1) \\ z_i^{(r_2)} &= \Gamma_{ir_2}(\tilde{x}_i, \vec{y}_i) + f_{iA}(r_2) \\ &\vdots \\ z_i^{(r_t)} &= \Gamma_{ir_t}(\tilde{x}_i, \vec{y}_i) + f_{iA}(r_t) \end{aligned} \quad \text{for } i \in I_n$$

can be solved simultaneously for a unique

$(\tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n) \in \prod_{i=1}^n V_i^{t-1}$, $(\vec{y}_1, \vec{y}_2 \dots \vec{y}_n) \in \prod_{i=1}^n H_i$, $A \in \mathcal{D}$ containing T' . More

concretely, we have to find the unique elements $x_{iT}^{(k)}$, $T \in \mathcal{T}$, $1 \leq k \leq t-1$, $x_{il}^{(k)}$, $1 \leq l \leq d'$, $1 \leq k \leq t-1$ and y_{iT} , $T \in \mathcal{T}$, y_{il} , $1 \leq l \leq d'$ for $i \in I_n$ and the block A of $\mathcal{D}$ containing T' .

This system can be solved coordinate-wise. Without loss of generality we can assume $T' = \{1, 2, \dots, t\}$. We have to distinguish three cases:

Case 1: The system for the l -th coordinate, where $1 \leq l \leq d'$:

$$\begin{pmatrix} z_{il}^{(1)} \\ \vdots \\ z_{il}^{(t)} \end{pmatrix} = \begin{pmatrix} 1 & s_{i1} & \cdots & s_{i1}^{t-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_{it} & \cdots & s_{it}^{t-1} \end{pmatrix} \begin{pmatrix} x_{il}^{(1)} \\ \vdots \\ x_{il}^{(t-1)} \\ y_{il} \end{pmatrix} + \begin{pmatrix} f_{iA}(1)_l \\ \vdots \\ f_{iA}(t)_l \end{pmatrix} \text{ for } i \in I_n.$$

By definition, $f_A(i)_l = 0$ for all $i \in I_n$ and since the matrix is Vandermonde type, it is non-singular. Thus, $x_{il}^{(k)}$, $1 \leq k < t$, and y_{il} are uniquely determined and $1 \leq l \leq d'$ and $i \in I_n$.

Case 2: The system for the T -th coordinate, where $T = T'$:

$$\begin{pmatrix} z_{iT}^{(1)} \\ \vdots \\ z_{iT}^{(t)} \end{pmatrix} = \begin{pmatrix} 1 & e_{i1} & \cdots & s_{i1}^{t-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & e_{it} & \cdots & s_{it}^{t-2} & 0 \end{pmatrix} \begin{pmatrix} x_{iT}^{(1)} \\ \vdots \\ x_{iT}^{(t-1)} \\ y_{iT} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ N_T(A)|F_{q_i} \end{pmatrix} \text{ for } i \in I_n.$$

By definition, $f_{iA}(j)_T = 0$ for $1 \leq j < t$ and $f_{iA}(t)_T = N_T(A)|F_{q_i}$, since $\max T = \max T' = t$. For each i , the first $(t-1)$ equations in each system uniquely determine $x_{iT}^{(k)}$, $1 \leq k < t$, and the t -th equation determines $N_T(A)|F_{q_i}$. Thus, the n bottom equations determine $N_T(A)$, which, by the definition of $N_T(A)$, determines a unique

block A containing T . The coordinates y_{iT} 's, $i \in I_n$, are not yet determined. They will be determined later as in case 3.

Case 3: The system for the T -th coordinate, when $T \neq T'$. Let $|T \cap T'| = r$, $0 \leq r < t$. Without loss of generality we can assume that $T \cap T' = \{1, 2, \dots, r\}$. Then the system is

$$\begin{pmatrix} z_{iT}^{(1)} \\ \vdots \\ z_{iT}^{(t)} \end{pmatrix} = \begin{pmatrix} 1 & e_{i1} & \cdots & e_{i1}^{t-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & e_{ir} & \cdots & e_{ir}^{t-2} & 0 \\ 1 & s_{i(r+1)} & \cdots & s_{i(r+1)}^{t-2} & s_{i(r+1)}^{t-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_{it} & \cdots & s_{it}^{t-2} & s_{it}^{t-1} \end{pmatrix} \begin{pmatrix} x_{iT}^{(1)} \\ \vdots \\ x_{iT}^{(t-1)} \\ y_{iT} \end{pmatrix} + \begin{pmatrix} f_{iA}(1)_T \\ \vdots \\ f_{iA}(t)_T \end{pmatrix}$$

for $i \in I_n$.

From lemma 5.1.2 the matrix is non-singular for $i \in I_n$. Since A has been found, the vector

$$\begin{pmatrix} f_{iA}(1)_T \\ \vdots \\ f_{iA}(t)_T \end{pmatrix}$$

is known and can be transferred to the left-hand side. Thus, we can solve the system uniquely for $x_{iT}^{(k)}$, $1 \leq k \leq t-1$ and y_{iT} , for $i \in I_n$. The coordinates of $\vec{y}_i$ for $i \in I_n$ have been determined for all l , $1 \leq l \leq d'$, and $T \in \mathcal{T}$ except when $T = T'$. Now, for all $i \in I_n$ we can choose $y_{iT'}$ from the defining equation of the hyper-plane H_i .

This shows that the system of nt equations can be solved for a unique $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in \prod_{i=1}^n V_i^{t-1}$ and $(\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n) \in \prod_{i=1}^n H_i$ and $A \in \mathcal{D}$ containing T' , that is T'

covered by a unique block in the form of $B((\bar{x}_1, \bar{x}_2 \dots \bar{x}_n), (\bar{y}_1, \bar{y}_2 \dots \bar{y}_n), A)$. Thus, the blocks defined this way form a partial design $P(t, k, u\lambda^d)$ on X covering the t -sets of $X_{\mathcal{T}}$.

5.2 A Construction for t -GDD's

The “block spreading” method (theorem 5.1.3) is very useful in the construction of designs with $\lambda = 1$. We start with a t -GDD with large λ and the construction yields a t -GDD with $\lambda = 1$. Blanchard [5] generalized Wilson's construction for $t > 2$ and the next theorem is a further generalization of Blanchard's construction.

Theorem 5.2.1 *Let $t, k, u, t \leq k \leq u$ be positive integers satisfying the arithmetic conditions*

$$\binom{u-s}{t-s} \equiv 0 \quad \text{mod} \binom{k-s}{t-s} \quad \text{for } s = 0, 1 \dots t-1.$$

Then there is an integer $d_0 = d_0(t, k, u)$ such that for any integer $a \geq 2$, there is a t -GDD of group type $(a^d)^u$ with block size k , $\lambda = 1$ for $d \geq d_0$.

The proof of theorem 5.2.1 uses Wilson's λ -large theorem, given in [1] on pg. 377.

Theorem 5.2.2 *Let t, k, u with $t \leq k \leq u$ be positive integers. Then there is an integer $\lambda_0(t, k, u)$ such that an $S_\lambda(t, k, u)$ exists, whenever the conditions*

$$i. \quad \lambda \binom{u-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad \text{for } s = 0, 1, \dots, t-1$$

$$ii. \quad \lambda \geq \lambda_0(t, k, u)$$

are satisfied.

Proof of theorem 5.2.1. Choose d' so that $2^{d'} > \lambda_0(t, k, u)$, where $\lambda_0(t, k, u)$ is as in theorem 5.2.2. Since $a \geq 2$, $a^{d'} \geq \lambda_0(t, k, u)$. The necessary arithmetic conditions of theorem 5.2.2 are obviously satisfied, thus there exists an $S_{a^{d'}}(t, k, u)$. Let $a = q_1 q_2 \cdots q_n$ be the prime power decomposition of a . We can choose d' large enough to satisfy $q_i^{d'} > q_0(t, u)$ for $i = 1, 2, \dots, n$ as well, where $q_0(t, u)$ is as in theorem 5.1.3. Note that d' depends on t, k and u and not on the choice of a .

Let $d_1 = d' \binom{u}{t}$. By theorem 5.1.3, we can construct a partial design $P(t, k, ua^{d_1})$ on the point set $X = I_u \times \prod_{i=1}^n I_{q_i^{d_1}}$ covering the t -sets of $X_{\mathcal{T}}$, where $\mathcal{T} = \mathcal{P}_t(I_u)$ and $X_{\mathcal{T}}$ is defined in section 5.1. This partial design is a t -GDD of group type $(a^{d_1})^u$ with index unity, with block size k . The groups of the t -GDD are $\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}}$, $i \in I_u$.

By theorem 2.2.8, there is an integer $m_0(t, k)$ such that if $m \geq m_0(t, k)$, then there exists a $TD(t, k, m)$. We can choose $l_0 = l_0(t, k)$ such that $2^{l_0} \geq m_0(t, k)$. Thus, if $l \geq l_0(t, k)$, then there is a $TD(t, k, a^l)$ for any given $a \geq 2$.

Weight each point of the previously constructed t -GDD of group type $(a^{d_1})^u$ by the uniform weight a^l , where $l \geq l_0(t, k)$. For each block of the t -GDD there is a $TD(t, k, a^l)$, thus from the fundamental construction theorem for t -GDD (theorem 2.2.2) we can construct a t -GDD of group type $(a^{d_1+l})^u$ with block size k for all $l \geq l_0(t, k)$. It follows that, if $d \geq d_0(t, k, u) = d_0$, where $d_0 = d_1 + l_0 = d' \binom{u}{t} + l_0$, then there exists a t -GDD of group type $(a^d)^u$ for any given $a \geq 2$.

The following theorem is straightforward consequence of theorem 5.2.1.

Theorem 5.2.3 *Let $t, k, u, t \leq k \leq u$ be positive integers satisfying the arithmetic conditions*

$$\binom{u-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad \text{for } s = 0, 1, \dots, t-1.$$

Then there is an integer $d_1 = d_1(t, k, u)$ such that for any positive integer g , there is a t -GDD of group type $(g)^u$ with block size k and $\lambda = 1$ if $g = p^d \cdot g'$, where p is a prime, $d \geq d_1$ and there is a $TD(t, k, g')$.

Proof. Let $d_1 = d_1(t, k, u) = d_0(t, k, u)$, where $d_0(t, k, u)$ is as in theorem 5.2.1. Hence, if $d \geq d_1$ then there exists a t -GDD of group type $(p^d)^u$ with block size k . Now, we apply the fundamental construction for t -GDD's. As a recipe we use the t -GDD of group type $(p^d)^u$ and we weight each point by the uniform weight g' . For each block of the t -GDD there is an ingredient $TD(t, k, g')$, thus we can construct a t -GDD of group type $(g)^u$ with block size k , where $g = p^d \cdot g'$.

Corollary 5.2.4 *Let $t, k, u, t \leq k \leq u$ be positive integers satisfying the arithmetic conditions*

$$\binom{u-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad \text{for } s = 0, 1, \dots, t-1.$$

Then there is an integer $d_2 = d_2(t, k, u)$ such that for any positive integer g , there is a t -GDD of type $(g)^u$ with block size k and $\lambda = 1$ if $g = p^d \cdot g'$, where p is a prime, $d \geq d_2$ and $g' \geq d_2$.

Proof. By theorem 2.2.8, there is an integer $m_0 = m_0(t, k)$ such that if $m \geq m_0(t, k)$, there exists a $TD(t, k, m)$. Let $d_2 = \max(d_1, m_0)$, where d_1 is as in theorem 5.2.3. Since the conditions of theorem 5.2.3 are satisfied, we can construct a t -GDD of group type $(g)^u$ with block size k .

Corollary 5.2.5 *Let $t, k, u, t \leq k \leq u$ be positive integers satisfying the arithmetic conditions*

$$\binom{u-s}{t-s} \equiv 0 \pmod{\binom{k-s}{t-s}} \quad \text{for } s = 0, 1, \dots, t-1.$$

Then there is an integer $d_3 = d_3(t, k, u)$ such that for any positive integer g , there is a t -GDD of group type $(g)^u$ with block size k and $\lambda = 1$ if the exponent of every prime power divisor p of g is at least d_3 .

Proof. Let $d_3 = d_3(t, k, u) = d_2(t, k, u)$, where $d_2(t, k, u)$ is as in corollary 5.2.4. Let $g = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ be the prime decomposition of g and let $g' = p_2^{\alpha_2} \dots p_n^{\alpha_n}$. Thus, if $\alpha_i \geq d_3$ for all $i \in I_n$, then obviously $g' \geq d_3$. Hence, by corollary 5.2.4, there exists a t -GDD of group type $(g)^u$ with block size k .

5.3 A Construction for Steiner 3-Designs

The following theorem is a generalization of Blanchard's construction for Steiner 3-designs [2]. We use "the block spreading construction" (theorem 5.1.3) to construct new Steiner 3-designs.

Theorem 5.3.1 *Let k and a be positive integers, $k \geq 2$, $a \geq 2$. Assume that there exist a Steiner system $S(3, k+1, a+1)$, a $TD(3, k+1, a)$ and a 3-CS of group type $(a^k : 1)$ with block size $k+1$. Then there exists an integer $d_0 = d_0(k, v, a)$ such that for any positive integer v satisfying the arithmetic conditions*

1. $v - 1 \equiv 0 \pmod{k-1}$,
2. $v(v-1) \equiv 0 \pmod{k(k-1)}$,
3. $v(v^2-1) \equiv 0 \pmod{k(k^2-1)}$

there is a Steiner 3-design $S(3, k+1, va^d+1)$ for all $d \geq d_0$.

We use the following two results to prove theorem 5.3.1.

Theorem 5.3.2 *Let k and a be positive integers, $k \geq 2$, $a \geq 2$. Assume that there exist a $TD(3, k+1, a)$ and a 3-CS of group type $(a^k : 1)$ with block size $k+1$. Then there exist integers $v_1 = v_1(k)$ and $d_1 = d_1(k, v)$ such that for any Steiner system $S(2, k, v)$ satisfying $v \geq v_1(k)$ and the arithmetic condition*

$$\binom{v+1}{3} \equiv 0 \pmod{\binom{k+1}{3}}$$

there is a 3-CS of group type $((a^d)^v : 1)$ with block size $k + 1$ for all $d \geq d_1$. Here $d_1 = d_1(k, v)$ does not depend on a .

Theorem 5.3.3 *Let k and a be positive integers, $k \geq 2$, $a \geq 2$. Assume that there exist a Steiner system $S(3, k + 1, a + 1)$, a $TD(3, k + 1, a)$ and a 3-CS of group type $(a^k : 1)$ with block size $k + 1$. Then there exist integers $v_1 = v_1(k)$ and $d_1 = d_1(k, v)$ such that for any Steiner system $S(2, k, v)$ satisfying $v \geq v_1(k)$ and the arithmetic condition*

$$\binom{v+1}{3} \equiv 0 \pmod{\binom{k+1}{3}}$$

there is a Steiner 3-design $S(3, k + 1, va^d + 1)$ for all $d \geq d_1$. (Note that $d_1 = d_1(k, v)$ does not depend on a .)

The following extension theorem given by Blanchard [3] is needed in the proof of theorem 5.3.2.

Theorem 5.3.4 *Let k and v be positive integers, $k \geq 2$. There exist integers $v_0 = v_0(k)$ and $\lambda_0 = \lambda_0(k, v)$ such that for any Steiner system $S(2, k, v)$ $\mathcal{D}'$, with $v \geq v_0(k)$ and any $\lambda \geq \lambda_0(k, v)$ satisfying the necessary arithmetic condition*

$$\lambda \binom{v+1}{3} \equiv 0 \pmod{\binom{k+1}{3}}$$

there is an extension of λ copies of the Steiner system $\mathcal{D}'$; in other words, there is an $S_\lambda(3, k + 1, v + 1)$ $\mathcal{D}$, such that $\lambda \cdot \mathcal{D}'$ is a derived design of $\mathcal{D}$.

(We defined the concept of “copies of a design” in section 2.1.)

Proof of theorem 5.3.2. Let $v_1(k) = v_0(k)$, where $v_0(k)$ is as in theorem 5.3.4. Let $\mathcal{D}'$ be a Steiner 2-design $S(2, k, v)$ with $v \geq v_1(k)$. We can choose c_o so that $2^{c_o} > \lambda_0(k, v)$, where $\lambda_0(k, v)$ is as in theorem 5.3.4. Since $a \geq 2$, $a^{c_o} > \lambda_0(k, v)$ and c_o depends on only k and v and not on a .

The necessary arithmetic conditions of theorem 5.3.4

$$a^{c_o} \binom{v+1}{3} \equiv 0 \pmod{\binom{k+1}{3}}$$

are obviously satisfied, thus there exists an $S_{a^{c_o}}(3, k+1, v+1)$ $\mathcal{D}$, such that $a^{c_o}\mathcal{D}'$ is a derived design of $\mathcal{D}$. We can assume that the Steiner 3-design $\mathcal{D}$ is constructed on the point set $I_v \cup \infty'$ and $\mathcal{D}_{\infty'}$ is $a^{c_o}\mathcal{D}'$, where $\mathcal{D}_{\infty'}$ denotes the derived design of $\mathcal{D}$ at ∞' .

Let $\mathcal{T}$ be a family of 3-sets that are contained in the blocks of $\mathcal{D}^{\infty'}$, where $\mathcal{D}^{\infty'}$ is the residual design of $\mathcal{D}$ at ∞' . Since the derived design of $\mathcal{D}$ at ∞' consists of a^{c_o} copies of the Steiner system $\mathcal{D}'$, any 3-set covered by the derived design is not in $\mathcal{T}$ and any 3-set in $\mathcal{T}$ is not covered by any block of the derived design. Thus, the 3-sets covered by the derived design $\mathcal{D}_{\infty'}$ are complementary to the family $\mathcal{T}$. It follows that the residual design $\mathcal{D}^{\infty'}$, is a partial design $P_{a^{c_o}}(3, k+1, v)$ that covers the 3-sets of $\mathcal{T}$.

Let $a = q_1 q_2 \cdots q_n$ be the prime power decomposition of a . We can choose c_o large enough to satisfy $q_i^{c_o} > q_0(v)$ for $i = 1, 2, \dots, n$ as well, where $q_0(v)$ is as in theorem 5.1.3. Let $d_1 = c_o \binom{v}{3}$ and $X = I_v \times \prod_{i=1}^n I_{q_i^{d_1}}$. Next we construct a 3-CS

of group type $((a^{d_1})^v : 1)$ with block size $k + 1$ on $X \cup \infty$, where the groups of the candlebra system are $\{\{i\} \times \prod_{i=1}^n q_i^{d_1}, i \in I_v\}$ and the stem is $\{\infty\}$.

First note that, by applying corollary 3.2.2 repeatedly on a 3-CS of group type $(a^k : 1)$ with block size $k + 1$ and a $TD(3, k + 1, a)$, we get a 3-CS of group type $((a^l)^k : 1)$ with block size $k + 1$ for every positive integer l . For each block A of $\mathcal{D}$. ($\mathcal{D}$ is an $S_{a^{c_0}}(3, k + 1, v + 1)$), that passes through the point ∞' , we can construct the 3-CS $((A \setminus \infty') \times \prod_{i=1}^n I_{q_i^{d_1}} \cup \infty, \{\infty\}, \{\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}} : i \in A \setminus \infty'\}, \mathcal{M}_A)$ of group type $((a^{d_1})^k : 1)$ with block size $k + 1$, which exists by our previous remark by taking $l = d_1$. Note that, if A is a block of $\mathcal{D}$ that passes through ∞' , then A is repeated a^{c_0} times.

From the blocks of the residual design $\mathcal{D}^{\infty'}$, which is a partial design $P_{a^{c_0}}(3, k + 1, v)$, by theorem 5.1.3 we obtain a partial design $P(3, k + 1, va^{d_1})$ on $X = I_v \times \prod_{i=1}^n I_{q_i^{d_1}}$ with $\lambda = 1$ and with block set $\mathcal{N}$, that covers the 3-sets of $X_{\mathcal{T}}$, where $X_{\mathcal{T}}$ is defined in section 5.1.

Let $\mathcal{M} = (\bigcup_{(\infty \in A)} \mathcal{M}_A) \bigcup \mathcal{N}$. We claim that $(X \cup \infty, \{\infty\}, \{\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}} : i \in I_v\}, \mathcal{M})$ is a 3-CS of group type $((a^{d_1})^v : 1)$ with block size $k + 1$.

Case 1. If $T = \{(v_1, a), (v_2, b), (v_3, c)\}$ is a 3-set of X such that the projection $\{v_1, v_2, v_3\}$ onto I_v is a 3-set of $\mathcal{T}$, then $T \in X_{\mathcal{T}}$. The 3-sets of $X_{\mathcal{T}}$ are covered by the partial design $P(3, k + 1, va^{d_1})$ obtained by the block spreading construction, thus T is contained in a unique block of $\mathcal{N}$. Since the 3-sets covered by the derived design $\mathcal{D}_{\infty'}$ are complementary to the family $\mathcal{T}$, no other blocks can contain the given triple T .

Case 2. If $T = \{(v_1, a), (v_2, b), (v_3, c)\}$ is a 3-set of X such that the projection onto I_v is a 3-set not in $\mathcal{T}$, then $\{v_1, v_2, v_3\}$ is covered by unique block A of $\mathcal{D}$, which passes through ∞' . We can say that the block A is unique, since each block of $\mathcal{D}$ which passes through ∞' is repeated a^{c_0} times. Since T consists of 3 points from 3 different groups of the 3-CS $((A \setminus \infty') \times \prod_{i=1}^n I_{q_i^{d_1}} \cup \infty, \{\infty\}, \{\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}} | i \in A \setminus \infty'\}, \mathcal{M}_A)$, T is contained in a unique block of $\mathcal{M}_A$ and no other blocks of $\mathcal{M}$ can contain T .

Case 3. If $T = \{(v_1, a), (v_1, b), (v_2, c)\}$ is a 3-set of X such that the projection onto I_v is a 2-set $\{v_1, v_2\}$, then $\{\infty', v_1, v_2\}$ is covered by a unique block A of $\mathcal{D}$ which contains ∞' . In this case T consists of points from two different groups of the 3-CS $((A \setminus \infty) \times \prod_{i=1}^n I_{q_i^{d_1}} \cup \infty, \{\infty\}, \{\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}} | i \in A \setminus \infty'\}, \mathcal{M}_A)$, thus T is contained in a unique block of $\mathcal{M}_A$ and this is the only block of $\mathcal{M}$ which can contain T .

Case 4. If $T = \{\infty, (v_1, a), (v_2, b)\}$ is a 3-set of $X \cup \infty$ containing ∞ , then the projection of $T \setminus \infty$ onto I_v is the 2-set $\{v_1, v_2\}$. Hence, the 3-set of $I_v \cup \infty' \setminus \{\infty', v_1, v_2\}$ is covered by a unique block A of $\mathcal{D}$ containing ∞' . Thus, as in the two previous cases, T is contained in a unique block of the 3-CS $((A \setminus \infty') \times \prod_{i=1}^n I_{q_i^{d_1}} \cup \infty, \{\infty\}, \{\{i\} \times \prod_{i=1}^n I_{q_i^{d_1}} | i \in A \setminus \infty'\}, \mathcal{M}_A)$. Again, no other blocks of $\mathcal{M}$ can contain T .

Now, we use the fundamental construction for candelabra systems (theorem 3.2.1) to obtain a 3-CS of group type $((a^d)^v : 1)$ with block size $k+1$ for all $d \geq d_1 = d_1(k, v)$. We take the previously constructed 3-CS of group type $((a^{d_1})^v : 1)$ with block size $k+1$ as recipe. From our hypothesis, for each block passing through the point of the stem we have the ingredient 3-CS of group type $(a^k : 1)$ with block size $k+1$ and for each block avoiding the point of the stem we have the ingredient $TD(3, k+1, a)$.

Applying the fundamental construction theorem repeatedly we can obtain a 3-*CS* of group type $((a^d)^v : 1)$ with block size $k + 1$ for every positive integer $d \geq d_1 = d_1(k, v)$.

Proof of theorem 5.3.3. Note that the hypotheses of theorem 5.3.2 hold. thus we can construct a 3-*CS* of group type $((a^d)^v : 1)$ with block size $k + 1$ for $d \geq d_1(k, v)$. Since a Steiner 3-design $S(3, k + 1, a + 1)$, a $TD(3, k + 1, a)$ and a 3-*CS* of group type $(a^k : 1)$ with block size $k + 1$ exist by hypothesis, there is a Steiner 3-design $S(3, k + 1, a^d + 1)$ for all positive integer d . This statement follows from the repetitive application of theorem 3.2.4. Thus, for each group of the 3-*CS* of group type $((a^d)^v : 1)$ together with ∞ , we can construct a Steiner 3-design $S(3, k + 1, a^d + 1)$ for all positive integers $d \geq d_1(k, v)$. Hence, by the adjunction theorem, the 3-*CS* of group type $((a^d)^v : 1)$ can be completed to a $S(3, k + 1, va^d + 1)$ for all positive integers $d \geq d_1(k, v)$.

Proof of theorem 5.3.1. Assume that v satisfies the arithmetic conditions

1. $v - 1 \equiv 0 \pmod{k - 1}$,
2. $v(v - 1) \equiv 0 \pmod{k(k - 1)}$,
3. $v(v^2 - 1) \equiv 0 \pmod{k(k^2 - 1)}$.

The existence of $S(3, k + 1, a + 1)$ implies that a^l satisfies the arithmetic conditions

1. $a^l - 1 \equiv 0 \pmod{k - 1}$,
2. $a^l(a^l - 1) \equiv 0 \pmod{k(k - 1)}$,
3. $a^l(a^{2l} - 1) \equiv 0 \pmod{k(k^2 - 1)}$

for all $l \geq 1$. Then it is easy to check that va^l satisfies the congruences

1. $va^l - 1 \equiv 0 \pmod{k-1},$
2. $va^l(va^l - 1) \equiv 0 \pmod{k(k-1)},$
3. $va^l(v^2a^{2l} - 1) \equiv 0 \pmod{k(k^2-1)}$

for any $l \geq 1$. Relations (1) and (2) are the necessary arithmetic conditions for the existence of an $S(2, k, va^l)$. From Wilson's existence theorem, [20], [21] and [22] there is an integer $v'(k)$ such that for any $w \geq v'(k)$ satisfying the necessary arithmetic conditions there exists an $S(2, k, w)$. We can choose l large enough to satisfy $va^l \geq v'(k)$. Therefore there exists an $S(2, k, va^l)$. We can choose l such that $va^l \geq v_1(k)$ as well, where $v_1(k)$ is as in theorem 5.3.3. Note that l depends only on k . With relation (3), the conditions of theorem 5.3.3 are satisfied and we can conclude that there is an $S(3, k+1, va^d+1)$ for $d \geq d_0(k, v, a)$, where $d_0(k, v, a) = d_1(k, va^l)$.

5.4 Application, New Infinite Families of Steiner 3-Designs

Finally, we conclude with constructions of new infinite families of Steiner 3-designs.

Theorem 5.4.1 *Let q be a prime power and let a be a positive integer, $a \geq 2$. Assume that there is a Steiner system $S(3, q+1, a+1)$. Then there exists an integer $d_0 = d_0(q, v, a)$ such that for any positive integer v satisfying the arithmetic conditions*

- i. $v - 1 \equiv 0 \pmod{q-1},$
- ii. $v(v-1) \equiv 0 \pmod{q(q-1)},$
- iii. $v(v^2-1) \equiv 0 \pmod{q(q^2-1)}$

there is a Steiner 3-design $S(3, q + 1, va^d + 1)$ for all $d \geq d_0$.

Proof. By theorems 4.4.4 and 4.5.5, the existence of an $S(3, q + 1, a + 1)$ implies the existence of a $TD(3, q + 1, a)$ and a 3-CS of group type $(a^q : 1)$ with block size $q + 1$. Since the conditions of theorem 5.3.1 are satisfied, we can construct a Steiner 3-design $S(3, q + 1, va^d + 1)$ for all $d \geq d_0(q, v, a)$.

Corollary 5.4.2 *Let m be a non-negative integer. Then there exists an integer $d_0 = d_0(v, m)$ such that for any positive integer v satisfying the arithmetic conditions*

$$v \equiv 1 \pmod{20} \text{ or}$$

$$v \equiv 5 \pmod{20}$$

there exists an $S(3, 6, v(4 \cdot 5^m + 1)^d + 1)$ for all $d \geq d_0$.

As far as we know, they are new infinite families of Steiner 3-designs with block size 6.

Proof. By corollary 3.3.5, if there is an $S(3, 6, v + 1)$, then there exists an $S(3, 6, 4v + 2)$. Thus, from the Steiner 3-design $S(3, 6, 5^m + 1)$, we can construct an $S(3, 6, 4 \cdot 5^m + 2)$ for all $m \geq 0$. We use theorem 5.4.1 with $q = 5$ and $a = 4 \cdot 5^m + 1$ to obtain the desired Steiner 3-designs.

Theorem 5.4.3 *Let q be a prime power and let a be a positive integer, $a \geq 2$. Assume that there is a Steiner system $S(3, q + 1, a + 1)$. Then there exist integers*

$v_2 = v_2(q)$ and $d_2 = d_2(q, v)$ such that for any positive integer v satisfying $v \geq v_2(q)$ and the necessary arithmetic conditions

- i. $v - 1 \equiv 0 \pmod{q - 1}.$
- ii. $v(v - 1) \equiv 0 \pmod{q(q - 1)}.$
- iii. $v(v^2 - 1) \equiv 0 \pmod{q(q^2 - 1)}$

there is a Steiner 3-design $S(3, q + 1, va^d + 1)$ for all $d \geq d_2$, where $d_2 = d_2(q, v)$ does not depend on a .

Proof. The relations (i) and (ii) are the necessary arithmetic conditions for the existence of an $S(2, q, v)$. From Wilson's existence theorem, [20], [21] and [22] there is an integer $v'(q)$ such that if $v \geq v'(q)$, then there is an $S(2, q, v)$. Let $v_2 = \max(v'(q), v_1(q))$ and let $d_2 = d_1(q, v)$, where $v_1(q)$ and $d_1(q, v)$ are as in theorem 5.3.3. Since the conditions of theorem 5.3.3 are satisfied, we can construct a Steiner 3-design $S(3, q + 1, va^d + 1)$ for all $d \geq d_2(q, v)$, where $d_2 = d_2(q, v)$ does not depend on a .

Corollary 5.4.4 *Let m be a non-negative integer. Then there are integers v_2 and $d_2 = d_2(v)$ such that for any positive integer v satisfying $v \geq v_2$ and the arithmetic conditions*

$$\begin{aligned} v &\equiv 1 \pmod{20} \text{ or} \\ v &\equiv 5 \pmod{20} \end{aligned}$$

there exists an $S(3, 6, v(4 \cdot 5^m + 1)^d + 1)$ for all $d \geq d_2$, where $d_2 = d_2(v)$ does not depend on m .

Proof. We use theorem 5.4.3 with $q = 5$ and $a = 4 \cdot 5^m + 1$.

Appendix A

NOTATION AND SYMBOLS

General symbols

$A \times B$	the cartesian product of sets A and B
$\prod_{i=1}^n A_i$	the cartesian product of the sets A_i , $i = 1, 2, \dots, n$
$ A $	the cardinality of the set A
$A \subseteq B$	A is a subset of B
$A \cup B$	the union of the sets A and B
$A \cap B$	the intersection of the sets A and B
$\mathcal{P}(X)$	the set of all subsets of the set X
$\mathcal{P}_k(X)$	the set of k subsets of the set X
$\mathcal{P}_K(X)$	for any set of positive integers K ,
	$\mathcal{P}_K(X) = \bigcup_{k \in K} \mathcal{P}_k(X)$
$\binom{n}{k}$	the number of k -subsets of an n -set
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of non-negative integers
$\mathbb{F}_q$	the finite field of order q , q is a prime power
I_s	$I_s = \{1, 2, \dots, s\}$
I_s^0	$I_s = \{0, 1, 2, \dots, s\}$

Specific symbols

t -(v, K, λ) design, t -(v, k, λ) design	t -design, section 2.1
$S_\lambda(t, K, v)$, $S_\lambda(t, k, v)$, $S(t, K, v)$	t -design, section 2.1
$S(t, k, v)$	
GDD , t - GDD	group divisible design, section 2.2
$TD_\lambda(t, k, m)$, $TD(t, k, m)$	transversal design, section 2.2
t - CS	candelabra t -system section 2.3
$OA_\lambda(t, k, v)$, $OA(t, k, v)$	orthogonal array, section 2.4
$OD_\lambda(t, k, v)$, $OD(t, k, v)$	ordered design, section 2.4
$LD_\lambda(t, k, v)$, $LD(t, k, v)$	lattice design, section 2.4
t - $LGDD$	lattice group divisible design, section 4.1
$LTD_\lambda(t, k, v)$, $LTD(t, k, v)$	lattice transversal design, section 4.1
t - LCS	lattice candelabra system, section 4.1
$P_\lambda(t, k, u)$	partial design, section 5.1
$\mathcal{D}_x$	derived design at x , section 2.1
$\mathcal{D}^x$	residual design at x , section 2.1
$B_t[K, \lambda]$, $B_t[K]$	section 2.1
R_k^t	the set of replication numbers, section 2.1
$NG_t[m, K, s]$	the set of number of groups, section 2.3
$C_K^t[J, s]$	section 2.3

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