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DIRAC INDUCTION FOR UNIMODULAR LIE GROUPS

Dissertation

Presented in Partial Fulfillment of the Requirements for
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School of The Ohio State University

By

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* * * * *

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1996

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Introduction

Let G be a connected unimodular Lie group, G_K be a maximal compact subgroup. Assume $\dim G/G_K = 2n$, G/G_K admits an invariant spin-structure, and $S^\pm$ are the standard semi-spinor representations of $Spin(2n)$. Then, the geometric realization of square-integrable representations for semisimple Lie groups and nilpotent Lie groups on the spaces of the harmonic spinors suggests the following question :

Conjecture 1 (*Kasparov [13]*) *Each irreducible square-integrable representation of a connected unimodular Lie group G can be realized on the Kernel of the closure of the operator*

$$\mathbf{D}_\pi = \mathbf{D}_\pi^+ \oplus \mathbf{D}_\pi^- : \mathbf{L}^2(\mathcal{V}_\pi^+ \oplus \mathcal{V}_\pi^-) \longrightarrow \mathbf{L}^2(\mathcal{V}_\pi^- \oplus \mathcal{V}_\pi^+), \quad (1)$$

where π is an irreducible representation of the group G_K , $\mathcal{V}_\pi^\pm$ are the homogeneous vector bundles over G/G_K associated with the spinor representations $S^\pm$ and G_K -module π , and $\mathbf{D}_\pi$ is the Dirac operator associated with π . Moreover, either $\text{Ker} \mathbf{D}_\pi^- = 0$ or $\text{Ker} \mathbf{D}_\pi^+ = 0$.

There had been two evidences for the conjecture.

When G is a semisimple Lie group and $G_K = K$ is a maximal compact subgroup of G , we can assume $\text{rank} G = \text{rank} K$ because of the well-known criterion of Harish-Chandra for the existence of discrete series for real reductive groups. In this case, Atiyah and Schmid applied the Atiyah-Singer covering index theorem to the

Dirac operators on the symmetric space G/G_K , and they obtained the geometric realization for most of the discrete series for G on $KerD_\pi^+$ ([4], theorem 9.3), provided by the Atiyah-Schmid vanishing theorem ([4], theorem 5.20) which asserts that $KerD_\pi^- = 0$ for suitable irreducible representation π of K . This is the first evidence for Conjecture 1.

The second evidence was provided by the geometric realization of the square-integrable representations for nilpotent Lie groups. If N is a connected and simply-connected nilpotent Lie group having square-integrable representations *mod* the center Z , λ is a regular element in $\mathfrak{n}^*$, and $\pi = \epsilon_\lambda = e^{2i\pi\lambda}$ is the character of Z associated with λ . By applying their L^2 -index theorem, Connes and Moscovici showed that $KerD_\pi^+ - KerD_\pi^- = \pi_\lambda$, the square-integrable representation of N associated with λ via the Kirillov correspondence ([8], 7.1.A), and $KerD_\pi^- = 0$ if λ is sufficiently positive ([8], 7.1.B). Thus, $KerD_\pi^+ = \pi_\lambda$ for this type λ .

According to a theorem of Anh ([2]), the class of all connected unimodular Lie groups having square-integrable irreducible representations modulo their centers is not much more general than the set of groups $N \ltimes G$, the semi-direct product of a reductive Lie group G and a nilpotent Lie group N having square-integrable representations modulo its center Z , and G acts on Z trivially. In particular, if G is semisimple, then the semi-direct product $N \ltimes G$ of G and N has square-integrable representation modulo its center if and only if $rankG = rankK$, where K is a maximal compact subgroup of G ([1]).

The purpose of this paper is to give a way to attack Conjecture 1 for the semi-direct product $N \ltimes G$ of a connected and simply connected nilpotent Lie group N having square-integrable irreducible representations modulo its center

Z and a semisimple Lie group G with a maximal compact subgroup K , with $\text{rank} K = \text{rank} G$. Moreover, we assume Z is central in $N \ltimes G$. Here, the square-integrable representation of $N \ltimes G$ (in the conjecture) will be understood as the square-integrable representation modulo the center of $N \ltimes G$, and the above G_K will be replaced by $Z \times K$.

There are three steps in our approach to Conjecture 1 which will be described briefly now. To do that, we assume $D_c^+ = D_\lambda^+$ is the Dirac operator over N/Z (endowed with an invariant Riemannian metric) associated with a character $\epsilon_\lambda = e^{2i\pi\lambda}$ for Z , where $\lambda \in \mathfrak{n}_{reg}^*$, and D_μ^+ is the Dirac operator over G/K associated with an irreducible K -module τ having μ as a highest weight. Moreover, we let $S_1^\pm$ be the standard semi-spinor representations associated with N/Z and $S_2^\pm$ be the standard semi-spinor representations associated with G/K . If $S^\pm$ are the standard semi-spinor representations associated with $N \ltimes G/Z \times K$, then $S^+ = S_1^+ \otimes S_2^+ + S_1^- \otimes S_2^-$, $S^- = S_1^- \otimes S_2^+ + S_1^+ \otimes S_2^-$.

We denote $\text{Ker} D_{\lambda,\mu}^{++}$ as the space of L^2 -harmonic spinors over $N \ltimes G/Z \times K$ with values in the direction $S_1^+ \otimes S_2^+$, then it is easy to see that $\text{Ker} D_{\lambda,\mu}^{++} \subseteq \text{Ker} D_{\lambda,\mu}^+$. Here, $D_{\lambda,\mu}^+$ is the Dirac operator over $N \ltimes G/Z \times K$ associated with $Z \times K$ -module $\epsilon_\lambda \otimes \tau$.

Now, we start to discuss these steps.

- (1) In step 1, we would like to prove $\text{Ker} D_{\lambda,\mu}^{++} = \text{Ker} D_\lambda^+ \otimes \text{Ker} \dot{D}_{\mu'}^+$ as representations (Theorem 1.1, Chapter I), where $\dot{D}_{\mu'}^+$ is the Dirac operator over $\dot{G}/\dot{K}$ associated with an irreducible $\dot{K}$ -module having a highest weight μ' which is related to μ , and $\dot{G}$ and $\dot{K}$ are the double coverings of G and K , respectively. The first thing we need to do is to extend the action of N on

$Ker D_c^+$ to the action of $N \ltimes G$ on $Ker D_c^+$. The expected difficulty comes from the action of G on N . Since G is not compact, for $g \in G$, g acts on N/Z not by isometry in general. In other words, the Riemannian metric on N/Z varies under the action of G , so does the Dirac operator D_c^+ . We overcome this difficulty by the following way. Let D_g^+ be the Dirac operator on N/Z endowed with a new Riemannian metric so that g acts on N/Z by an isometry. If λ is sufficiently positive, it follows from Connes-Moscovici's theorems ([8], 7.1) that $Ker D_c^+$ and $Ker D_g^+$ are all irreducible ϵ_λ -representations of N , and $Ker D_g^- = Ker D_c^- = 0$. Thus, there is an intertwining operator $U_g : Ker D_c^+ \rightarrow Ker D_g^+$. On the other hand, we can find a *canonical* isomorphism $l_c(g) : Ker D_c^+ \rightarrow Ker D_g^+$ between the Hilbert spaces, and this isomorphism is resulted from the change of the Riemannian metric on N/Z under the action of G . We define the action of G on $Ker D_c^+$ by $U_g^{-1} \circ l_c(g) : Ker D_c^+ \rightarrow Ker D_c^+$, for any $g \in G$. Then one has a projective unitary representation of $N \ltimes G$ on $Ker D_c^+$. To replace this projective representation of $N \ltimes G$ on $Ker D_c^+$ by ordinary one, we may pass to the double covers $\dot{G}$ and $\dot{K}$ of G and K , respectively, which are provided by Satake's condition (C3). Finally, in order to prove $Ker D_{\lambda,\mu}^{++} = Ker D_\lambda^+ \otimes Ker D_\mu^+$ as representations, Satake's harmonic condition (C4) is needed.

- (2) In the second step, we shall prove a vanishing theorem for $Ker D_{\lambda,\mu}^-$ extending Moscovici's vanishing theorem ([17]) and the Atiyah-Schmid vanishing theorem ([4]), provided λ and μ being sufficiently positive (Theorem 2.1, Chapter II). Our proof is a combination of the proof for Moscovici's vanishing theorem in the case of nilpotent Lie groups and Atiyah-Schmid's proof for their

vanishing theorem in the case of semisimple Lie groups. As usual, our vanishing theorem plays the same rule as Moscovici's vanishing theorem and the Atiyah-Schmid vanishing theorem did in the geometric realization of discrete series for nilpotent Lie groups and semisimple Lie groups, respectively.

- (3) In step 3, first of all, we will give a sufficient and necessary condition for $Ker D_{\lambda, \mu}^+$ being irreducible as a unitary representation of $N \ltimes G$, which is the main result of this paper. Since the irreducible representation $Ker D_{\lambda, \mu}^{++} = Ker D_{\lambda}^+ \otimes Ker D_{\mu}^+$ is included in $Ker D_{\lambda, \mu}^+$, in order to prove the irreducibility for $Ker D_{\lambda, \mu}^+$, it is sufficient to have $Ind D_{\lambda, \mu}^+ = Ind D_{\lambda}^+ \cdot Ind D_{\mu}^+$ = the formal degree of $Ker D_{\lambda}^+ \otimes Ker D_{\mu}^+$, provided $Ker D_{\lambda, \mu}^- = 0$ and Satake's conditions (C1)–(C4). Our theorem 3.1 (concerning above condition) in Chapter III is a product of Connes-Moscovici's L^2 -index theorem for the Dirac operator $D_{\lambda, \mu}^+$ over $N \ltimes G/Z \times K$ and the above observation. The similar ideas had been used by Atiyah and Schmid in [4], and by Connes and Moscovici in [8].

As a main application of the above theorem, we will verify that condition for $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$, the semidirect product of the Heisenberg group $\mathbf{H}_n$ and real symplectic group $\mathbf{Sp}(n, \mathbb{R})$. It is well-known that groups $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$ are the most typical unimodular Lie groups having square-integrable representations modulo their centers ([1], [2]). Thus, we obtain an affirmative answer to Conjecture 1 (for generic λ and μ), which is Theorem 4.1 of Chapter IV. Moreover, Theorem 4.1 strongly indicates that Baum-Connes-Kasparov's Conjecture is true for the reduced group C^* -algebra of $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$ (Dirac Induction on the level of K -Theory. See Baum-Connes-Higson [5] or Kasparov [13] for the statement of Baum-Connes-Kasparov's Conjecture), we shall discuss this in

another paper.

The paper is organized as follows. In Chapter I, we consider the variation of the Dirac operator under the change of the Riemannian metric on N/Z , Satake's conditions, and the relation among $Ker D_{\lambda,\mu}^{++}$, $Ker D_{\lambda}^{+}$ and $Ker \dot{D}_{\mu}^{+}$. The main result (Theorem 1.1) is the isomorphism $Ker D_{\lambda,\mu}^{++} = Ker D_{\lambda}^{+} \otimes Ker \dot{D}_{\mu}^{+}$. In fact, this also gives a geometric realization of the discrete series representations of $N \ltimes G$ on the space $Ker D_{\lambda,\mu}^{++}$ of the *restricted* harmonic spinors on $N \ltimes G/Z \times K$. In Chapter II, we will prove $Ker D_{\lambda,\mu}^{-} = 0$ for sufficiently positive λ and μ . This result is independent of Satake's conditions (C1)–(C4). In Chapter III, by applying Connes-Moscovici's L^2 -index theorem for $D_{\lambda,\mu}^{+}$ and the vanishing theorem in Chapter II, we get a powerful tool for attacking Conjecture 1, which is Theorem 3.1. Finally, in Chapter IV, Conjecture 1 will be proved for $H_n \ltimes Sp(n, \mathbb{R})$ (Theorem 4.1) for generic λ and μ by applying theorem 3.1 in Chapter III. The computation for the curvature forms for some homogeneous vector bundles over $Sp(n, \mathbb{R})/K$ and twisted $\hat{a}$ -polynomial is needed in order to verify the condition in theorem 3.1 of Chapter III. Our computation is similar to the ones in Borel-Hirzebruch [7] and Griffiths-Schmid [10].

Chapter I

Discrete Series For The Group $N \ltimes G$

We start with introducing some notations that we are going to use through the paper.

Let G be a connected noncompact semisimple Lie group with finite center, $\mathfrak{g}$ be the lie algebra of the left invariant vector fields on G . Let K be a maximal compact subgroup of G such that $\text{rank} K = \text{rank} G$, with lie algebra $\mathfrak{k}$. We put $\mathfrak{p} = \{Y \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } X \in \mathfrak{k}\}$, where B denotes the Killing form of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} \cap \mathfrak{p} = \{0\}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$. Since the restriction of B to $\mathfrak{p}$ is a positive definite real bilinear form, so we can let $So(\mathfrak{p})$ be the rotation group of $\mathfrak{p}$ under this positive definite bilinear form. Under the adjoint action of K on $\mathfrak{g}$, $\mathfrak{p}$ is stable and we get a homomorphism $Ad : K \rightarrow So(\mathfrak{p})$. Thus, we get a G -invariant Riemannian metric on the homogeneous space G/K .

First, we can assume by replacing G if necessary by a suitable double covering that there exists a homomorphism $\tilde{Ad} : K \rightarrow Spin(\mathfrak{p})$ satisfying $\Psi_2 \circ \tilde{Ad} = Ad$, where $\Psi_2 : Spin(\mathfrak{p}) \rightarrow So(\mathfrak{p})$ is the connected two-fold covering of $So(\mathfrak{p})$. Let $\sigma_2 : Spin(\mathfrak{p}) \rightarrow GL(S_2)$ be the canonical complex spinor representation of $Spin(\mathfrak{p})$. Since $\text{rank} G = \text{rank} K$, so the dimension of $\mathfrak{p}$ is even and then it is well-known that $S_2 = S_2^+ \oplus S_2^-$ and it is stable under σ_2 . Let $\tilde{\sigma}_2 = \sigma_2 \circ \tilde{Ad} : K \rightarrow GL(S_2)$, then it is a representation of K on S_2 . If V_τ is an irreducible unitary K -module, there are two vector bundles $G \times_K (S_2^\pm \otimes V_\tau) \rightarrow G/K$ associated with the representations $\tilde{\sigma}_2 \otimes \tau$

(for K) on the spaces $S_2^\pm \otimes V_\tau$. Let $C^\infty(G/K, S_2^\pm \otimes V_\tau) = (C^\infty(G) \otimes S_2^\pm \otimes V_\tau)^K$ be the sets of sections of the vector bundles defined above, where $(C^\infty(G) \otimes S_2^\pm \otimes V_\tau)^K = \{u \in C^\infty(G) \otimes S_2^\pm \otimes V_\tau \mid k \cdot u = u \text{ for all } k \in K\}$ (k acts on $C^\infty(G)$ by the right translation and on $S_2^\pm \otimes V_\tau$ in the obvious manner).

Now, let $\{Y_j\}$ be an orthonormal basis of $\mathfrak{p}$. Each Y_j determines a left-invariant vector field $r(Y_j)$ by infinitesimal right translation. Then the G -invariant Dirac operator $D_\tau = D_\tau^+ \oplus D_\tau^-$ over G/K associated with τ is defined by

$$\begin{aligned} D_\tau^\pm &: (C^\infty(G) \otimes S_2^\pm \otimes V_\tau)^K \rightarrow (C^\infty(G) \otimes S_2^\mp \otimes V_\tau)^K \\ D_\tau^\pm &= \sum_j r(Y_j) \otimes c(Y_j) \otimes 1, \end{aligned} \quad (2)$$

where $c(Y_j)$ is the Clifford multiplication by $Y_j \in \mathfrak{p}$ on Clifford module S_2 . And it follows that $\text{Ker } D_\tau^\pm = L^2\text{-Kernel of } D_\tau^\pm$ are G -modules, on which G acts by the left regular representations.

Next, let's discuss the Dirac operator associated with a nilpotent Lie group. Starting from now on, N will denote a connected and simply connected nilpotent Lie group having square-integrable irreducible representations modulo its center Z . For $\lambda \in \mathfrak{n}_{reg}^*$, $\epsilon_\lambda = e^{2i\pi\lambda}$ is the character of Z associated with λ . The corresponding homogeneous line bundle over N/Z will be denoted by $\mathcal{E}_\lambda = (N \times_Z \mathbb{C} \rightarrow N/Z)$. Since the isotropy representation is trivial, any inner product on $\mathfrak{n}/\mathfrak{z}$ gives a N -invariant Riemannian metric on the space N/Z , and we will fix one, once and for all (the choice of this inner product will be made clear later). On the Riemannian manifold N/Z , there is a unique N -invariant spin-structure, namely that given by the trivial homomorphism $\tilde{Ad} : Z \rightarrow \text{Spin}(\mathfrak{z}^\perp)$, where $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{z}^\perp$. We assume $\dim N/Z$ is even and $S_1 = S_1^+ \oplus S_1^-$ is the standard irreducible $\text{Spin}(\mathfrak{z}^\perp)$ representation. If $(C^\infty(N))^Z \otimes S_1^\pm$ are the sets of C^∞ -sections of the vector bundles

associated with Z -modules $\mathbb{C} \otimes S_1^\pm$, where Z acts on $\mathbb{C}$ by the character ϵ_λ , and $\{X_i\}$ is an orthonormal basis of $\mathfrak{z}^\perp$, then the Dirac operators $D_\lambda^\pm$ with coefficients in $\mathcal{E}_\lambda$ are given by

$$\begin{aligned} D_\lambda^\pm & : (C^\infty(N))^Z \otimes S_1^\pm \rightarrow (C^\infty(N))^Z \otimes S_1^\mp \\ D_\lambda^\pm & = \sum_i r(X_i) \otimes c(X_i). \end{aligned} \quad (3)$$

We define $\text{Ker} D_\lambda^\pm$ as the L^2 -Kernels of $D_\lambda^\pm$, which are N -modules and N acts by the left regular representations.

Finally, we can talk about twisted Dirac operators over $N \ltimes G/Z \times K$.

As we mentioned in the introduction, $N \ltimes G$ stands for a semidirect product of N and G (as above), the center Z of N is central in $N \ltimes G$. Now, we fix an $\text{Ad}(Z \times K)$ -invariant inner product on $\mathfrak{n}$ (this inner product will be fixed in the paper), the lie algebra of N . Consider the $\text{Ad}(Z \times K)$ -invariant decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{z}^\perp$. Since G is semisimple and G acts on Z trivially, it follows that $\mathfrak{z}^\perp$ is G -invariant as a whole. Moreover, there is an $N \ltimes G$ -invariant Riemannian metric on the space $N \ltimes G/Z \times K$ determined by the inner product on $\mathfrak{n}$ and the inner product on $\mathfrak{g}$. Assume there is a homomorphism $\tilde{A}d : K \rightarrow \text{Spin}(\mathfrak{z}^\perp)$ satisfying $\Psi_1 \circ \tilde{A}d = Ad$, where $\Psi_1 : \text{Spin}(\mathfrak{z}^\perp) \rightarrow \text{So}(\mathfrak{z}^\perp)$ is the connected two-fold covering of $\text{So}(\mathfrak{z}^\perp)$ and Ad is the homomorphism $Ad : K \rightarrow \text{So}(\mathfrak{z}^\perp)$. Thus, we obtain a invariant Spin-structure on the Riemannian manifold $N \ltimes G/Z \times K$, which is resulted from the homomorphism $\tilde{A}d : Z \times K \rightarrow \text{Spin}(\mathfrak{z}^\perp) \times \text{Spin}(\mathfrak{p})$, where $\tilde{A}d$ should be understood as the product of the homomorphisms $\tilde{A}d : K \rightarrow \text{Spin}(\mathfrak{z}^\perp)$ and $\tilde{A}d : K \rightarrow \text{Spin}(\mathfrak{p})$. Then, the Dirac operators $D_{\lambda, \tau}^\pm$ over $N \ltimes G/Z \times K$

associated with the irreducible $Z \times K$ -module $\epsilon_\lambda \otimes \tau$ are defined by

$$\begin{aligned} D_{\lambda, \tau}^\pm &: (C^\infty(N \ltimes G) \otimes S^\pm \otimes V_\tau)^{Z \times K} \rightarrow (C^\infty(N \ltimes G) \otimes S^\mp \otimes V_\tau)^{Z \times K} \\ D_{\lambda, \tau}^\pm &= \sum_i r(X_i) \otimes c(X_i) \otimes 1 + \sum_j r(Y_j) \otimes c(Y_j) \otimes 1, \end{aligned} \quad (4)$$

where $S^\pm$ are the standard semi-spinor representations of $Spin(\mathfrak{p} \oplus \mathfrak{z}^\perp)$ and $S^+ = S_1^+ \otimes S_2^+ + S_1^- \otimes S_2^-$, $S^- = S_1^- \otimes S_2^+ + S_1^+ \otimes S_2^-$. We denote the L^2 -Kernels of $D_{\lambda, \tau}^\pm$ by $Ker D_{\lambda, \tau}^\pm$, and denote the space of L^2 -solutions for the equation $D_{\lambda, \tau}^+ \omega = 0$ with values in $S_1^+ \otimes S_2^+ \otimes V_\tau$ by $Ker D_{\lambda, \tau}^{++}$. It is easy to see that $Ker D_{\lambda, \tau}^{++} \subseteq Ker D_{\lambda, \tau}^+$ is a closed subspace, and they are all $N \ltimes G$ -invariant subspaces of the left regular representation of $N \ltimes G$. The following lemma was given in Rosenberg [20].

Lemma 1. 1 *The group $N \ltimes G$ is unimodular.*

Proof. Since N and G are unimodular, it is sufficient to show that the conjugation action of G on N preserves the Haar measure. Since G acts on Z trivially, so $N \ltimes G$ will be unimodular if and only if the action of G on $\mathfrak{n}/\mathfrak{z}$ is unimodular. For $\lambda \in \mathfrak{n}_{reg}^*$, λ defines a nondegenerate alternating bilinear form B_λ on $\mathfrak{n}/\mathfrak{z}$. If $g \in G$, then $g \cdot \lambda$ has the same restriction to $\mathfrak{z}$ as λ , hence lies in $N \cdot \lambda$. Therefore, changing g by an element in N if necessary, we may assume g stabilizes λ . It follows that g acts on $\mathfrak{n}/\mathfrak{z}$ by a symplectic transformation for B_λ , hence acts unimodularly. $\square$

Corollary 1. 1 *If $f \in L^2(N)$ and $g \in G$, then*

$$\int_N |f(x)|^2 dx = \int_N |f(gxg^{-1})|^2 dx. \quad (5)$$

Proof. This comes from the proof of lemma 1.1 : the Haar measure dx on N is invariant under the action of g . $\square$

Now, we begin to study the action of G on the space $\text{Ker} D_\lambda^+$. We start with considering the change of the Dirac operator D_λ^+ on N/Z under the action of G . For the identity element $e \in G$, let $\langle, \rangle_e$ be the inner product on $\mathfrak{z}^\perp$ we fixed above, $S_e^\pm = S_1^\pm$ and write D_λ^+ as D_e^+ associated with the inner product $\langle, \rangle_e$ on $\mathfrak{z}^\perp$. We define the new inner product $\langle, \rangle_g$ on $\mathfrak{z}^\perp$ for $g \in G$ by setting $\langle \text{Ad}(g)X_i, \text{Ad}(g)X_j \rangle_g = \langle X_i, X_j \rangle_e$ for all i and j , where $\{X_i\}$ is an orthonormal basis of $\mathfrak{z}^\perp$ with respect to the inner product $\langle, \rangle_e$ on $\mathfrak{z}^\perp$. Then we got the following isometry

$$\text{Ad}(g) : (\mathfrak{z}^\perp, \langle, \rangle_e) \rightarrow (\mathfrak{z}^\perp, \langle, \rangle_g). \quad (6)$$

In turn, this isometry induces an isomorphism of the corresponding Clifford algebras

$$\text{Ad}(g) : \text{Cliff}(\mathfrak{z}^\perp, \langle, \rangle_e) \rightarrow \text{Cliff}(\mathfrak{z}^\perp, \langle, \rangle_g). \quad (7)$$

If we see the $\text{Spin}(\mathfrak{z}^\perp)$ -module S_e as a minimum left ideal in the complexification of the Clifford algebra $\text{Cliff}(\mathfrak{z}^\perp, \langle, \rangle_e)$ constructed from the basis $\{X_i\}$, then $S_g = \text{Ad}(g)S_e$ is the standard $\text{Spin}(\mathfrak{z}^\perp, \langle, \rangle_g) = \text{Spin}_g(\mathfrak{z}^\perp)$ representation, which is compatible with the $\mathbb{Z}_2$ gradings of the corresponding Spin-modules.

For the inner product $\langle, \rangle_g$ on $\mathfrak{z}^\perp$, there is a N -invariant Spin-structure on the Riemannian manifold N/Z associated with $\langle, \rangle_g$, which is determined by the trivial homomorphism $\tilde{\text{Ad}} : Z \rightarrow \text{Spin}_g(\mathfrak{z}^\perp)$. Again, for the Z character $\epsilon_\lambda = e^{2i\pi\lambda}$, we define a new Dirac operator D_g^+ associated with ϵ_λ for $g \in G$ by

$$\begin{aligned} D_g^+ & : (C^\infty(N))^{\mathbb{Z}} \otimes S_g^+ \rightarrow (C^\infty(N))^{\mathbb{Z}} \otimes S_g^- \\ D_g^+ & = \sum_i r(\text{Ad}(g)X_i) \otimes c(\text{Ad}(g)X_i). \end{aligned} \quad (8)$$

Let l_g be the left regular representation of N on $\text{Ker} D_g^+$.

Definition 1. 1 For $\xi \in Ker D_{g_0}^+$, we define $l_{g_0}(g)\xi$ by

$$[l_{g_0}(g)\xi](x) = Ad(g)\xi(g^{-1}xg) \in S_{gg_0}^+, \quad (9)$$

for $g \in G$, $x \in N$ and $g_0 \in G$, where $Ad(g)$ is the isomorphism $Ad(g) : Cliff(\mathfrak{z}^\perp, <, >_{g_0}) \rightarrow Cliff(\mathfrak{z}^\perp, <, >_{gg_0})$.

Lemma 1. 2 The map $l_{g_0}(g) : Ker D_{g_0}^+ \rightarrow Ker D_{gg_0}^+$ is a well-defined isomorphism of Hilbert spaces. Moreover, we have $l_{g_0}(g) \circ l_{g_0}(x) = l_{gg_0}(gxg^{-1}) \circ l_{g_0}(g)$ for $x \in N$.

Proof. First of all, let's prove that $l_{g_0}(g) : Ker D_{g_0}^+ \rightarrow Ker D_{gg_0}^+$ is well-defined.

By the definition of $D_{gg_0}^+$, one has

$$\begin{aligned} D_{gg_0}^+[l_{g_0}(g)\xi] &= \sum_i r(Ad(gg_0)X_i) \otimes c(Ad(gg_0)X_i)[l_{g_0}(g)\xi] \\ &= \sum_i r[Ad(g)(Ad(g_0)X_i)] \otimes c[Ad(g)(Ad(g_0)X_i)][Ad_g\xi(g^{-1} \cdot g)], \end{aligned} \quad (10)$$

where $\xi(g^{-1} \cdot g)$ means the function on N with values in $S_{g_0}^+$ so that its value at $x \in N$ is $\xi(g^{-1}xg)$.

Since $Ad(g)$ is an isomorphism of algebras, we have

$$D_{gg_0}^+[l_{g_0}(g)\xi] = Ad(g) \sum_i r(Ad(gg_0)X_i) \otimes c(Ad(g_0)X_i) \cdot \xi(g^{-1} \cdot g). \quad (11)$$

Now, we write $\xi \in Ker D_{g_0}^+$ as

$$\xi = \sum_\alpha \xi^\alpha \otimes Ad(g_0)s_\alpha^+, \quad (12)$$

where $\{s_\alpha^+\}$ is an orthonormal basis of S_e^+ , and $\{Ad(g_0)s_\alpha^+\}$ is an orthonormal basis of $S_{g_0}^+$.

Note that

$$\begin{aligned}
(r[Ad(g)(Ad(g_0)X_i)]\xi^\alpha(g^{-1} \cdot g))(x) &= \frac{d}{dt}\xi^\alpha[g^{-1}x \cdot e^{tAd(g)(Ad(g_0)X_i)}g] \big|_{t=0} \\
&= \frac{d}{dt}\xi^\alpha[g^{-1}xg \cdot e^{tAd(g_0)X_i}g^{-1}g] \big|_{t=0} \\
&= \frac{d}{dt}\xi^\alpha[g^{-1}xg \cdot e^{tAd(g_0)X_i}] \big|_{t=0} \\
&= r(Ad(g_0)X_i)\xi^\alpha(g^{-1}xg). \tag{13}
\end{aligned}$$

It follows that

$$\begin{aligned}
D_{gg_0}^+[l_{g_0}(g)\xi](x) &= Ad(g) \sum_{i,\alpha} r(Ad(g_0)X_i)\xi^\alpha(g^{-1}xg) \otimes c(Ad(g_0)X_i) \cdot (Ad(g_0)s_\alpha^+) \\
&= Ad(g)(D_{g_0}^+\xi)(g^{-1}xg) \\
&= 0. \tag{14}
\end{aligned}$$

Thus, $l_{g_0}(g) : Ker D_{g_0}^+ \rightarrow Ker D_{gg_0}^+$ is well-defined.

Now, let's prove that $l_{g_0}(g) \circ l_{g_0}(x) = l_{gg_0}(g x g^{-1}) \circ l_{g_0}(g)$.

For $\xi \in Ker D_{g_0}^+$, we have

$$\begin{aligned}
[l_{g_0}(g) \circ l_{g_0}(x)\xi](y) &= (l_{g_0}(g)[l_{g_0}(x)\xi])(y) \\
&= Ad(g)\xi(x^{-1}g^{-1}yg), \tag{15}
\end{aligned}$$

where $y \in N$.

On the other hand,

$$\begin{aligned}
[l_{gg_0}(g x g^{-1}) \circ l_{g_0}(g)\xi](y) &= l_{g_0}(g)\xi(g x^{-1}g^{-1}y) \\
&= Ad(g)\xi(x^{-1}g^{-1}yg). \tag{16}
\end{aligned}$$

Thus, $l_{g_0}(g) \circ l_{g_0}(x)\xi = l_{gg_0}(g x g^{-1}) \circ l_{g_0}(g)\xi$.

□

Now, we assume the first condition of Satake.

Condition 1 *The unitary representation $(Ker D_e^+, l_e)$ of N is non-trivial and irreducible.*

Remark 1. 1 *According to theorems of Connes-Moscovici ([8], 7.1), $(Ker D_e^+, l_e)$ is non-trivial and irreducible if λ is sufficiently positive. So, Condition 1 is satisfied for this type λ .*

Condition 1 means that $(Ker D_e^+, l_e)$ is in the discrete series of N . Therefore, a continuous deformation of it given by $(Ker D_g^+, l_g)$ (for $g \in G$) should be trivial, i.e., $(Ker D_e^+, l_e) \cong (Ker D_g^+, l_g)$ for any $g \in G$. Consider the Iwasawa decomposition for G , namely $G = (N'A)K$.

Definition 1. 2 *Let $U_{g_0} : (Ker D_e^+, l_e) \rightarrow (Ker D_{g_0}^+, l_{g_0})$ be an intertwining operator between these two irreducible N -modules for $g_0 \in (N'A)$, define the operator $U_{g_0 k}$ by*

$$\begin{aligned} U_{g_0 k} &: Ker D_e^+ \rightarrow Ker D_{g_0 k}^+ \\ U_{g_0 k} \xi &= Ad(g_0)[Ad(k)(c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1})U_{g_0} \xi)], \end{aligned} \quad (17)$$

for any $\xi \in Ker D_e^+$, $k \in K$, where $Ad(g_0)$, $Ad(k)$, and $Ad(g_0^{-1})$ mean the isomorphisms of the Clifford algebras we mentioned above, and $c(\tilde{Ad}(k^{-1})) \cdot$ means the Clifford multiplication of $\tilde{Ad}(k^{-1}) \in Spin(\mathfrak{z}^\perp)$ on the Clifford module S_e^+ . (The importance of formula (17) will be seen in the proof of Proposition 1.4.)

Lemma 1. 3 *The operator $U_{g_0 k} : Ker D_e^+ \rightarrow Ker D_{g_0 k}^+$ is a well-defined isomorphism of Hilbert spaces.*

Proof. It is sufficient to check that $U_{g_0 k} \xi \in Ker D_{g_0 k}^+$ for any $\xi \in Ker D_e^+$.

In fact,

$$\begin{aligned}
D_{g_0k}^+(U_{g_0k}\xi) &= \sum_i r(Ad(g_0k)X_i) \otimes c(Ad(g_0k)X_i) \{ Ad(g_0)[Ad(k)(c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1})U_{g_0}\xi)] \} \\
&= Ad(g_0) \{ \sum_i r(Ad(g_0k)X_i) \otimes c(Ad(k)X_i) [Ad(k)(c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1})U_{g_0}\xi)] \} \\
&= Ad(g_0)Ad(k) \{ \sum_i r(Ad(g_0k)X_i) \otimes c(X_i) \cdot c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1})U_{g_0}\xi \}.
\end{aligned}$$

Note that $c(\tilde{Ad}(k)) \cdot c(X_i) \cdot c(\tilde{Ad}(k^{-1})) = Ad(k)X_i$, it follows that

$$\begin{aligned}
D_{g_0k}^+(U_{g_0k}) &= Ad(g_0)Ad(k) \{ \sum_i r(Ad(g_0k)X_i) \otimes c(\tilde{Ad}(k^{-1})) \cdot c(Ad(k)X_i) \cdot Ad(g_0^{-1})U_{g_0}\xi \} \\
&= Ad(g_0)Ad(k) \{ c(\tilde{Ad}(k^{-1})) \cdot \sum_i r(Ad(g_0k)X_i) \otimes c(Ad(k)X_i) \cdot Ad(g_0^{-1})U_{g_0}\xi \} \\
&= Ad(g_0)Ad(k) \{ c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1}) \sum_i r(Ad(g_0)Ad(k)X_i) \\
&\quad \otimes c(Ad(g_0)Ad(k)X_i) \cdot U_{g_0}\xi \} \\
&= Ad(g_0)Ad(k) \{ c(\tilde{Ad}(k^{-1})) \cdot Ad(g_0^{-1})D_{g_0}^+U_{g_0}\xi \} = 0.
\end{aligned}$$

Here, we have used the fact that $\{Ad(k)X_i\}$ is also an orthonormal basis of $(\mathfrak{g}^\perp, \langle \cdot, \cdot \rangle_c)$, and so $\{Ad(g_0)Ad(k)X_i\}$ is an orthonormal basis of $(\mathfrak{g}^\perp, \langle \cdot, \cdot \rangle_{g_0})$. Thus, $D_{g_0}^+ = \sum_i r(Ad(g_0)Ad(k)X_i) \otimes c(Ad(g_0)Ad(k)X_i)$. $\square$

By Lemma 1.3 and the fact that $U_{g_0} : Ker D_c^+ \rightarrow Ker D_{g_0}^+$ is an intertwining operator, we have

Proposition 1. 1 $U_{g_0k} : Ker D_c^+ \rightarrow Ker D_{g_0k}^+$ is an intertwining operator between $(Ker D_c^+, l_c)$ and $(Ker D_{g_0k}^+, l_{g_0k})$. Thus, for each $g \in G$, we have a well defined intertwining operator $U_g : Ker D_c^+ \rightarrow Ker D_g^+$.

The following lemma will be needed later.

Lemma 1. 4 For any $g \in G$ and $k \in K$, we have

$$U_{gk}\xi = Ad(g)[Ad(k)(c(\tilde{Ad}(k^{-1})) \cdot Ad(g^{-1})U_g\xi)]. \quad (18)$$

Proof. It is sufficient to verify

$$U_{(g_0 k_1) k_2} \xi = Ad(g_0 k_1) [Ad(k_2) (c(\tilde{Ad}(k_2^{-1})) \cdot Ad((g_0 k_1)^{-1}) U_{g_0 k_1} \xi)], \quad (19)$$

for g_0 as above and $k_1, k_2 \in K$.

Note that

$$U_{(g_0 k_1) k_2} \xi = Ad(g_0) Ad(k_1 k_2) \{c(\tilde{Ad}(k_2^{-1} k_1^{-1})) \cdot Ad(g_0^{-1}) U_{g_0} \xi\}. \quad (20)$$

On the other hand,

$$\begin{aligned} & Ad(g_0 k_1) [Ad(k_2) (c(\tilde{Ad}(k_2^{-1})) \cdot Ad((g_0 k_1)^{-1}) U_{g_0 k_1} \xi)] \\ &= Ad(g_0) Ad(k_1 k_2) \{c(\tilde{Ad}(k_2^{-1})) \cdot Ad((g_0 k_1)^{-1}) Ad(g_0) Ad(k_1) [c(\tilde{Ad}(k_1^{-1})) \cdot Ad((g_0^{-1}) U_{g_0} \xi)]\} \\ &= Ad(g_0) Ad(k_1 k_2) \{c(\tilde{Ad}(k_2^{-1} k_1^{-1})) \cdot Ad(g_0^{-1}) U_{g_0} \xi\}. \end{aligned} \quad (21)$$

Thus, we have completed the proof. $\square$

Lemma 1. 5 *If we put $U_{g, g'} = U_g \circ (U_{g'})^{-1} : Ker D_g^+ \rightarrow Ker D_{g'}^+$, then*

$$U_{gg', g} \circ l_e(g) = \alpha(g, g') l_{g'}(g) \circ U_{g', e}, \quad (22)$$

where $\alpha : G \times G \rightarrow \mathbb{T}$ is a function.

Proof. from the *Uniqueness* of $U_{gg'}$, it is sufficient to prove that $l_{g'}(g) \circ U_{g'} \circ l_e^{-1}(g) \circ U_g$ is also an intertwining operator from $Ker D_e^+$ to $Ker D_{gg'}^+$.

Indeed, for any $x \in N$, by using Lemma 1.2, we have

$$\begin{aligned} [l_{g'}(g) \circ U_{g'} \circ l_e^{-1}(g) \circ U_g] \circ l_e(x) &= l_{g'}(g) \circ U_{g'} \circ l_e^{-1}(g) \circ l_g(x) \circ U_g \\ &= l_{g'}(g) \circ U_{g'} \circ l_e(g^{-1} x g) \circ l_e^{-1}(g) \circ U_g \\ &= l_{gg'}(x) \circ [l_{g'}(g) \circ U_{g'} \circ l_e^{-1}(g) \circ U_g]. \end{aligned} \quad (23)$$

Thus, $l_{g'}(g) \circ U_{g'} \circ l_e^{-1}(g) \circ U_g$ is also an intertwining operator from $Ker D_e^+$ to $Ker D_{gg'}^+$. $\square$

Corollary 1. 2 *The function $\alpha : G \times G \rightarrow \mathbb{T}$ is a multiplier, i.e.,*

1. $\alpha(e, g) = \alpha(g, e) = 1$, for $g \in G$;
2. $\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$, for $x, y, z \in G$.

Proof. From the proof for above lemma, we see that

$$\begin{aligned} U_g &= \alpha(g, e)l_e(g) \circ U_{e,e} \circ l_e^{-1}(g) \circ U_g \\ &= \alpha(g, e)l_e(g) \circ l_e^{-1}(g) \circ U_g \\ &= \alpha(g, e)U_{g,e}. \end{aligned}$$

It follows that $\alpha(g, e) = 1$. Similarly, we can prove that $\alpha(e, g) = 1$.

Now, let's prove part 2 of the lemma. First of all, it is easy to see that

$$\begin{aligned} U_{(xy)z} &= \alpha(xy, z)l_z(xy) \circ U_z \circ l_e^{-1}(xy)\alpha(x, y)l_y(x) \circ U_y \circ l_e^{-1}(x) \circ U_x \\ &= \alpha(xy, z)\alpha(x, y)l_z(xy) \circ U_z \circ l_e^{-1}(xy) \circ l_y(x) \circ U_y \circ l_e^{-1}(x) \circ U_x. \end{aligned}$$

On the other hand,

$$\begin{aligned} U_{x(yz)} &= \alpha(x, yz)l_{yz}(x) \circ U_{yz} \circ l_e^{-1}(x) \circ U_x \\ &= \alpha(x, yz)\alpha(y, z)l_{yz}(x) \circ l_z(y) \circ U_z \circ l_e^{-1}(y) \circ U_y \circ l_e^{-1}(x) \circ U_x. \end{aligned}$$

Since $U_{(xy)z} = U_{x(yz)}$, we only need to have $l_z(xy) = l_{yz}(x) \circ l_z(y)$ and $l_e^{-1}(xy) \circ l_y(x) = l_e^{-1}(y)$ in order to have part 2. For $\xi \in Ker D_z^+$, we have

$$\begin{aligned} ([l_{yz}(x) \circ l_z(y)]\xi)(x_0) &= Ad(x)(l_z(y)\xi)(x^{-1}x_0x) \\ &= Ad(x)Ad(y)\xi(y^{-1}x^{-1}x_0xy) \\ &= [l_z(xy)\xi](x_0), \forall x_0 \in N. \end{aligned}$$

Thus, $l_z(xy) = l_{yz}(x) \circ l_z(y)$.

Next, by noting that $l_y(x) \circ l_e(y) = l_e(xy)$, we get $l_e^{-1}(xy) \circ l_y(x) = l_e^{-1}(y)$.

□

Now, we can define the action of G on $\text{Ker} D_c^+$.

Proposition 1. 2 *Define the action of G on $\text{Ker} D_c^+$ by setting $l(g) = U_g^{-1} \circ l_e(g) : \text{Ker} D_c^+ \rightarrow \text{Ker} D_c^+$, for any $g \in G$. Then*

$$l(g)l(g') = \alpha(g, g')l(gg'), \quad (24)$$

for $g, g' \in G$. Moreover, $(\text{Ker} D_c^+, l)$ is a unitary α -representation of G with the multiplier α .

Proof. By the definition of l , we see that

$$l(g)l(g') = U_g^{-1} \circ l_e(g) \circ U_{g'}^{-1} \circ l_e(g'). \quad (25)$$

Lemma 1.5 gives

$$U_{gg', g} \circ l_e(g) = \alpha(g, g')l_{g'}(g) \circ U_{g'}. \quad (26)$$

It follows that

$$\begin{aligned} l(g)l(g') &= \alpha(g, g')U_g^{-1} \circ U_g \circ U_{gg'}^{-1} \circ l_{g'}(g) \circ l_e(g') \\ &= \alpha(g, g')U_{gg'}^{-1} \circ l_e(gg') \\ &= \alpha(g, g')l(gg'). \end{aligned}$$

□

In the following, we always assume the system of intertwining operators $\{U_g\}$ for $g \in G$ can be chosen so that the following *continuity* condition is satisfied, which is Satake's condition 2.

Condition 2 *The α -representation $(Ker D_e^+, l)$ of G is continous.*

Now, we extend the action of G on $Ker D_e^+$ to the action of $N \ltimes G$.

Definition 1.3 *For any $g \in G$ and $x \in N$, the action of xg on $Ker D_e^+$ is given by $\tilde{l}(xg)$*

$$\tilde{l}(xg) = l_e(x)l(g) : Ker D_e^+ \rightarrow Ker D_e^+. \quad (27)$$

Then, it follows the proposition

Proposition 1.3 *$(\tilde{l}, N \ltimes G)$ is a continous multiplier representation of $N \ltimes G$ on $Ker D_e^+$.*

Proof. Let $w_1 = x_1g_1, w_2 = x_2g_2$, then

$$\tilde{l}(w_1)\tilde{l}(w_2) = l_e(x_1)l(g_1)l_e(x_2)l(g_2) = l_e(x_1) \circ U_{g_1}^{-1} \circ l_e(g_1) \circ l_e(x_2) \circ l(g_2).$$

By Lemma 1.3, we have

$$\begin{aligned} \tilde{l}(w_1)\tilde{l}(w_2) &= l_e(x_1) \circ U_{g_1}^{-1} \circ l_{g_1}(g_1x_2g_1^{-1}) \circ l_e(g_1) \circ l(g_2) \\ &= l_e(x_1) \circ l_e(g_1x_2g_1^{-1}) \circ U_{g_1}^{-1} \circ l_e(g_1) \circ l(g_2) \\ &= l_e(x_1g_1x_2g_1^{-1}) \circ l(g_1) \circ l(g_2) \\ &= \alpha(g_1, g_2)l_e(x_1g_1x_2g_1^{-1}) \circ l(g_1g_2). \end{aligned}$$

On the other hand,

$$\tilde{l}(w_1w_2) = \tilde{l}(x_1g_1x_2g_2) = \tilde{l}(x_1g_1x_2g_1^{-1} \cdot g_1g_2) = l_e(x_1g_1x_2g_1^{-1}) \circ l(g_1g_2).$$

Thus, $\tilde{l}(w_1)\tilde{l}(w_2) = \alpha(g_1, g_2)\tilde{l}(w_1w_2)$, and the function $\tilde{\alpha} : (N \ltimes G) \times (N \ltimes G) \rightarrow \mathbb{T}$ defined by $\tilde{\alpha}(w_1, w_2) = \alpha(g_1, g_2)$ is a multiplier since α is a multiplier.

We have finished the proof of the proposition. $\square$

Next, we study the relation among the left regular representations $((L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K}, N \ltimes G)$, $(L^2(G/K, S_2^+ \otimes V_\tau), G)$ and $(Ker D_c^+, N)$.

Proposition 1. 4 *For any $\xi \in Ker D_c^+$ and any $\eta \in L^2(G/K, S_2^+ \otimes V_\tau)$, define ω as $\omega(x, g) = Ad(g^{-1})U_g\xi(x) \otimes \eta(g)$ for $x \in N$ and $g \in G$. Then the map Φ*

$$\begin{aligned} \Phi : Ker D_c^+ \otimes L^2(G/K, S_2^+ \otimes V_\tau) &\longrightarrow (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K} \\ \xi \otimes \eta &\rightarrow \omega, \end{aligned} \quad (28)$$

is an isometry, where ω is defined as above.

Proof. The fact that $\omega \in (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K}$ could be seen in the following way.

First of all, let's prove ω is k -invariant for all $k \in K$. Consider the action of k on ω

$$(k \cdot \omega)(x, g) = c(\tilde{Ad}(k))[Ad((gk)^{-1})U_{gk}\xi(x)] \otimes (k \cdot \eta)(g) = c(\tilde{Ad}(k)) \cdot Ad((gk)^{-1})U_{gk}\xi(x) \otimes \eta(g).$$

this is because that η is k -invariant.

From Lemma 1.4, we have $U_{gk}\xi = Ad(g)[Ad(k)(c(\tilde{Ad}(k^{-1})) \cdot Ad(g^{-1})U_g\xi)]$,

it follows that

$$\begin{aligned} (k \cdot \omega)(x, g) &= c(\tilde{Ad}(k))[c(\tilde{Ad}(k^{-1})) \cdot Ad(g^{-1})U_g\xi(x)] \otimes \eta(g) \\ &= Ad(g^{-1})U_g\xi(x) \otimes \eta(g) \\ &= \omega(x, g). \end{aligned}$$

Thus, ω is k -invariant.

It is very easy to see that ω is also z -invariant for $z \in Z$, we have that ω is $Z \times K$ -invariant.

Next, by considering Corollary 1.1, we have

$$\begin{aligned} \|\omega\|^2 &= \int_{N \ltimes G} \langle \omega(x, g), \omega(x, g) \rangle dx dg \\ &= \int_N \langle U_g \xi(x), U_g \xi(x) \rangle dx \cdot \int_G \langle \eta(g), \eta(g) \rangle dg \\ &= \|\xi\|^2 \cdot \|\eta\|^2. \end{aligned}$$

It follows that Φ is an isometry. $\square$

Let $\mathcal{L}_0 = \text{Im} \Phi \subseteq (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K}$ and l_2 be the left regular representation of G on $L^2(G/K, S_2^+ \otimes \mathcal{V}_\tau)$. Put $l'_2(g)$ as $(l'_2(g)\eta)(g') = \alpha(g, g^{-1}g')^{-1} [l_2(g)\eta](g')$, for $\eta \in L^2(G/K, S_2^+ \otimes \mathcal{V}_\tau)$. Then we have

Lemma 1. 6 *(l'_2, G) is a multiplier unitary representation of G on $L^2(G/K, S_2^+ \otimes \mathcal{V}_\tau)$ with the multiplier α^{-1} .*

Proof. First of all, we have to check that $l'_2(g)\eta \in L^2(G/K, S_2^+ \otimes \mathcal{V}_\tau)$ (for $g \in G$), i.e., $l'_2(g)\eta$ is k invariant for any $k \in K$. By the definition of $l'_2(g)\eta$, it is sufficient to verify that

$$[k \cdot l'_2(g)\eta](g') = [l'_2(g)\eta](g'),$$

i.e.,

$$\alpha(g, g^{-1}g'k) = \alpha(g, g^{-1}g').$$

Equivalently, we only need to show $\alpha(g_1, g_2k) = \alpha(g_1, g_2)$ for all $g_1, g_2 \in G$ and $k \in K$.

Claim : $\alpha(g, k) = 1$ for $\forall g \in G$ and $\forall k \in K$.

Indeed, by Lemma 1.4 and Lemma 1.5, we have

$$U_{gk,g} \circ l_c(g) = \alpha(g, k) l_k(g) \circ U_{k,c},$$

where

$$\begin{aligned} U_{gk,g} \circ l_c(g) &= U_{gk} \circ U_g^{-1} \circ l_c(g) \\ &= Ad(gk)[c(\tilde{Ad}(k^{-1})) \cdot Ad(g^{-1})U_g] \circ U_g^{-1} \circ l_c(g) \\ &= Ad(gk)(\tilde{Ad}(k^{-1})) \cdot Ad(gkg^{-1}) \circ l_c(g). \end{aligned}$$

Thus,

$$\begin{aligned} [U_{gk,g} \circ l_c(g)\xi](x) &= Ad(gk)(\tilde{Ad}(k^{-1})) \cdot Ad(gkg^{-1})Ad(g)\xi(g^{-1}xg) \\ &= Ad(gk)(\tilde{Ad}(k^{-1})) \cdot Ad(gk)\xi(g^{-1}xg). \end{aligned}$$

On the other hand,

$$\begin{aligned} [l_k(g) \circ U_k\xi](x) &= Ad(g)(U_k\xi)(g^{-1}xg) \\ &= Ad(gk)[c(\tilde{Ad}(k^{-1})) \cdot \xi(g^{-1}xg)] \\ &= Ad(gk)(\tilde{Ad}(k^{-1})) \cdot Ad(gk)\xi(g^{-1}xg). \end{aligned}$$

It follows that $\alpha(g, k) = 1$. Since

$$\alpha(g_1g_2, k)\alpha(g_1, g_2) = \alpha(g_1, g_2k)\alpha(g_2, k),$$

we get $\alpha(g_1, g_2k) = \alpha(g_1, g_2)$ for any $g_1, g_2 \in G$.

Next, let's check α^{-1} is the multiplier of l'_2 . Consider

$$\begin{aligned} [l'_2(g_1)l'_2(g_2)\eta](g) &= \alpha^{-1}(g_1, g_1^{-1}g)[l'_2(g_2)\eta](g_1^{-1}g) \\ &= \alpha^{-1}(g_1, g_1^{-1}g)\alpha^{-1}(g_2, g_2^{-1}g_1^{-1}g)\eta(g_2^{-1}g_1^{-1}g) \\ &= \alpha^{-1}(g_1, g_1^{-1}g)\alpha^{-1}(g_2, g_2^{-1}g_1^{-1}g)\alpha(g_1g_2, g_2^{-1}g_1^{-1}g)[l'_2(g_1g_2)\eta](g). \end{aligned}$$

Note that

$$\alpha(g_1 g_2, g_2^{-1} g_1^{-1} g) \alpha(g_1, g_2) = \alpha(g_1, g_1^{-1} g) \alpha(g_2, g_2^{-1} g_1^{-1} g),$$

It follows that $l'_2(g_1) l'_2(g_2) = \alpha^{-1}(g_1, g_2) l'_2(g_1 g_2)$.

We have finished the proof for this Lemma. $\square$

Now, let's consider the left regular representation $(L, N \ltimes G)$ on the space $\mathcal{L}_0 = \text{Im} \Phi$. Then it is not difficult to see

Proposition 1.5 *Under the correspondence in Proposition 1.4, one has the following correspondence (which is an isomorphism : $\text{Ker} D_e^+ \otimes L^2(G/K, \mathcal{S}_2 \otimes \mathcal{V}_\tau) \rightarrow \mathcal{L}_0$)*

$$\begin{aligned} \Phi : \text{Ker} D_e^+ \otimes L^2(G/K, \mathcal{S}_2 \otimes \mathcal{V}_\tau) &\longrightarrow \mathcal{L}_0 \\ \tilde{l}(xg) \otimes l'_2(g) &\rightarrow L(xg), \end{aligned} \tag{29}$$

where $L(xg)$ is the action of $xg \in N \ltimes G$ on the space $\mathcal{L}_0 \subseteq (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes \mathcal{V}_\tau)^{Z \times K}$.

Proof. If $\omega \in \mathcal{L}_0$ is given by $\omega(x, g) = \text{Ad}(g^{-1}) U_g \xi(x) \otimes \eta(g)$ for $\xi \in \text{Ker} D_e^+$ and $\eta \in L^2(G/K, \mathcal{S}_2 \otimes \mathcal{V}_\tau)$, then it is easy to see that

$$\begin{aligned} [L(x_0) \omega](x, g) &= \omega(x_0^{-1} x, g) \\ &= \text{Ad}(g^{-1}) U_g \xi(x_0^{-1} x) \otimes \eta(g) \\ &= \text{Ad}(g^{-1}) [U_g l_e(x_0) \xi](x) \otimes \eta(g) \\ &= \tilde{\omega}(x, g), \end{aligned}$$

where $\tilde{\omega}(x, g) = \text{Ad}(g^{-1}) [U_g l_e(x_0) \xi](x) \otimes \eta(g)$ and $\tilde{\omega} \in \mathcal{L}_0$. Thus, $L(x_0)$ is corresponding to $l_e(x_0)$ for $x_0 \in N$.

Now, for $g_0 \in G$, we have

$$\tilde{l}(g_0)\xi \otimes l'_2(g_0)\eta = l(g_0)\xi \otimes l'_2(g_0)\eta.$$

Thus,

$$\begin{aligned} Ad(g^{-1})U_g[l(g_0)\xi](x) \otimes [l'_2(g_0)\eta](g) &= Ad(g_0)U_gU_{g_0}^{-1}l_c(g_0)\xi(x) \otimes [l'_2(g_0)\eta](g) \\ &= Ad(g^{-1})U_{g,g_0}l_c(g_0)\xi(x) \otimes [l'_2(g_0)\eta](g) \\ &= \alpha(g_0, g_0^{-1}g)Ad(g^{-1})l_{g_0^{-1}g}(g_0)U_{g_0^{-1}g}\xi(x) \otimes [l'_2(g_0)\eta](g) \\ &= \alpha(g_0, g_0^{-1}g)Ad(g^{-1}g_0)U_{g_0^{-1}g}\xi(g_0^{-1}xg_0) \otimes [l'_2(g_0)\eta](g) \\ &= Ad(g_0^{-1}g)^{-1}U_{g_0^{-1}g}\xi(g_0^{-1}xg_0) \otimes \eta(g_0^{-1}g). \end{aligned}$$

On the other hand,

$$\begin{aligned} [L(g_0)\omega](x, g) &= \omega(g_0^{-1}xg) \\ &= \omega(g_0^{-1}xg_0, g_0^{-1}g) \\ &= Ad(g_0^{-1}g)^{-1}U_{g_0^{-1}g}\xi(g_0^{-1}xg_0) \otimes \eta(g_0^{-1}g). \end{aligned}$$

Finally, we get

$$\tilde{l}(g_0) \otimes l'_2(g_0) \rightarrow L(g_0).$$

□

Remark 1. 2 *It follows from the above proposition that the representation $\tilde{l} \otimes l'_2$ of $N \ltimes G$ on $Ker D_e^+ \otimes L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_\tau)$ is a (ordinary) unitary representation. In fact, this is our motivation to define the representation (l'_2, G) on the space $L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_\tau)$.*

Next, to conclude our observation, we need two further assumptions which are Satake's condition 3 and condition 4. The first one enable us to replace the multiplier representations l and l'_2 of G by ordinary ones.

Condition 3 *There exists a (at most) double covering group $(\dot{G}, \psi)$ of G trivializing the multiplier α , i.e., one has a continuous map $\chi : \dot{G} \rightarrow \mathbb{T}$ such that*

$$\alpha(\psi(\dot{g}_1), \psi(\dot{g}_2)) = \chi(\dot{g}_1)\chi(\dot{g}_2)\chi(\dot{g}_1\dot{g}_2)^{-1}, \quad (30)$$

for all $\dot{g}_1, \dot{g}_2 \in \dot{G}$.

Then, the lie algebra of $\dot{G}$ may be identified with the lie algebra of G . Put $\dot{K} = \psi^{-1}(K)$. It follows from Condition 3 and the fact $\alpha(g, k) = 1$ (in the proof of Lemma 1.6) that

Lemma 1. 7 *The restriction of χ to $\dot{K}$ gives a character of $\dot{K}$.*

Lemma 1. 8 *Define the action of $\dot{G}$ on $Ker D_e^+$ by setting*

$$\begin{aligned} \dot{l}(\dot{g}) &: Ker D_e^+ \longrightarrow Ker D_e^+ \\ \dot{l}(\dot{g}) &= \chi(\dot{g})^{-1}l(\psi(\dot{g})), \end{aligned} \quad (31)$$

for any $\dot{g} \in \dot{G}$, then $(\dot{l}, \dot{G})$ is a (ordinary) unitary representation of $\dot{G}$ on the space $Ker D_e^+$.

Proof. For any $\dot{g}_1, \dot{g}_2 \in \dot{G}$, we have

$$\begin{aligned} \dot{l}(\dot{g}_1)\dot{l}(\dot{g}_2) &= \chi(\dot{g}_1)^{-1}\chi(\dot{g}_2)^{-1}l(\psi(\dot{g}_1))l(\psi(\dot{g}_2)) \\ &= \chi^{-1}(\dot{g}_1\dot{g}_2)\alpha^{-1}(\psi(\dot{g}_1), \psi(\dot{g}_2))l(\psi(\dot{g}_1))l(\psi(\dot{g}_2)) \\ &= \chi^{-1}(\dot{g}_1\dot{g}_2)l(\psi(\dot{g}_1)\psi(\dot{g}_2)) \\ &= \dot{l}(\dot{g}_1\dot{g}_2). \end{aligned}$$

Thus, $\dot{l}$ is a (ordinary) unitary representation of $\dot{G}$ on the space $Ker D_e^+$. $\square$

From this Lemma, it is easy to see that

Corollary 1. 3 *If we define $(\tilde{l})(xg) : Ker D_e^+ \rightarrow Ker D_e^+$ by setting $(\tilde{l})(xg) = l_e(x) \cdot \dot{l}(g)$, then $(\tilde{l}, N \ltimes \dot{G})$ is a (ordinary) representation of $N \ltimes \dot{G}$ on the space $Ker D_e^+$.*

Proof. For the simplicity of the writing, we assume that $\dot{G} = G$. From above lemma we know that $\dot{l}$ is a (ordinary) unitary representation of G on $Ker D_e^+$. Thus, in order to prove $(\tilde{l})$ is a (ordinary) unitary representation of $N \ltimes G$ on $Ker D_e^+$, it is sufficient to check that

$$\dot{l}(g) \circ l_e(x) = l_e(gxg^{-1}) \circ \dot{l}(g).$$

Indeed,

$$\begin{aligned} \dot{l}(g) \circ l_e(x) &= \chi^{-1}(g) l(g) \circ l_e(x) \\ &= \chi^{-1}(g) U_g^{-1} \circ l_e(g) \circ l_e(x) \\ &= \chi^{-1}(g) U_g^{-1} \circ l_g(gxg^{-1}) \circ l_e(g) \\ &= \chi^{-1}(g) l_g(gxg^{-1}) \circ U_g^{-1} \circ l_e(g) \\ &= \chi^{-1}(g) l_g(gxg^{-1}) \circ l(g) \\ &= l_g(gxg^{-1}) \circ \dot{l}(g). \end{aligned}$$

This is what we want. $\square$

Now, it is time to make the multiplier representation (l'_2, G) of G on $L^2(G/K, \mathcal{S}_2' \otimes \mathcal{V}_\tau)$ to be a (ordinary) representation. To do that, we define a irreducible unitary representation $\dot{\tau}$ of $\dot{K}$ as $\dot{\tau}(\dot{k}) = \chi(\dot{k}) \tau(\psi(\dot{k}))$, for $\dot{k} \in \dot{K}$.

Lemma 1. 9 For any $\eta \in L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_\tau)$, define $\dot{\eta} \in L^2(\dot{G}/\dot{K}, \mathcal{S}_2^+ \otimes \mathcal{V}_{\dot{\tau}})$ as $\dot{\eta}(\dot{g}) = \chi(\dot{g})^{-1} \eta(\psi(\dot{g}))$. Then the following correspondence

$$\begin{aligned} L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_\tau) &\longrightarrow L^2(\dot{G}/\dot{K}, \mathcal{S}_2^+ \otimes \mathcal{V}_{\dot{\tau}}) \\ \eta &\longrightarrow \dot{\eta} \end{aligned} \quad (32)$$

is an isomorphism of the Hilbert Spaces. Moreover, $\chi(\dot{g})l_2''(\psi(\dot{g}))$ becomes $\dot{l}_2(\dot{g})$ for any $\dot{g} \in \dot{G}$, where $\dot{l}_2$ is the left regular representation of $\dot{G}$ on the space $L^2(\dot{G}/\dot{K}, \mathcal{S}_2^+ \otimes \mathcal{V}_{\dot{\tau}})$.

Proof. Again, we assume $\dot{G} = G$. First of all, let's prove that $\dot{\eta} \in L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_{\dot{\tau}})$.

In fact, for any $k \in K$,

$$(k \cdot \dot{\eta})(g) = \chi(k)\chi((gk)^{-1})(k \cdot \eta)(g) = \chi(k)\chi((gk)^{-1})\eta(g) = \chi(g)^{-1}\eta(G) = \dot{\eta}(g).$$

It is also not difficult to see that the correspondence in the Lemma is an isomorphism of Hilbert spaces.

Finally, let's prove that $\dot{l}_2(g)$ becomes $\chi(g)l_2''(g)$ under above correspondence. To see that, consider $\dot{\eta} \in L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_{\dot{\tau}})$ and $g \in G$,

$$\begin{aligned} [\dot{l}_2(g)\dot{\eta}](g') &= \dot{\eta}(g^{-1}g') \\ &= \chi(g^{-1}g')^{-1}\eta(g^{-1}g') \\ &= \chi(g^{-1}g')^{-1}[l_2(g)\eta](g') \\ &= \chi(g')^{-1}\chi(g')\chi(g^{-1}g')^{-1}\alpha(g, g^{-1}g')\alpha^{-1}(g, g^{-1}g')[l_2(g)\eta](g') \\ &= \chi(g')^{-1}\chi(g)[l_2''(g)\eta](g'). \end{aligned}$$

Thus, $\dot{l}_2(g)$ becomes $\chi(g)l_2''(g)$ under above correspondence. $\square$

Now, it follows from Proposition 1.5, Lemma 1.8, and Lemma 1.9 that

Proposition 1. 6 *There is an one-to-one correspondence*

$$\begin{aligned} \text{Ker } D_c^+ \otimes L^2(\dot{G}/\dot{K}, S_2^+ \otimes V_\tau) &\longrightarrow \mathcal{L}_0 \\ \xi \otimes \dot{\eta} &\longrightarrow \omega, \end{aligned} \quad (33)$$

$$(\tilde{l})(x\dot{g}) \otimes \dot{l}_2(\dot{g}) \longrightarrow L(x\psi(\dot{g})), \quad (34)$$

for $x \in N$ and for $\dot{g} \in \dot{G}$, where $\dot{\eta}(\dot{g}) = \chi(\dot{g})^{-1} \eta(\psi(\dot{g}))$ and $\omega(x, g) = \text{Ad}(g^{-1})U_g \xi(x) \otimes \eta(g)$ for $g = \psi(\dot{g})$.

Remark 1. 3 *The importance of this proposition is that both $(\tilde{l})$ and $\dot{l}_2$ are ordinary unitary representations and $\dot{l}_2$ leaves $\text{Ker } D_\tau^+$ invariant, where D_τ^+ is the Dirac operator over $\dot{G}/\dot{K}$ associated with the irreducible $\dot{K}$ module $\dot{\tau}$. Next, we shall prove $\text{Ker } D_c^+ \otimes \text{Ker } D_\tau^+ \cong \text{Ker } D_{\lambda, \tau}^{++}$ under the correspondence in Proposition 1.6, provided Satake's condition 4.*

Finally, we start to discuss the *harmonic* condition of Satake (Condition 4) and the proof of the equality we mentioned in Remark 1.3. Let's begin with some more discussion on the Dirac operator $D_{\lambda, \tau}^+$

Definition 1. 4 *Define operators $d_1^\pm$ by setting*

$$\begin{aligned} d_1^\pm &: (L^2(N \ltimes G) \otimes S_1^\pm \otimes S_2^+ \otimes V_\tau)^{Z \times K} \longrightarrow (L^2(N \ltimes G) \otimes S_1^\mp \otimes S_2^+ \otimes V_\tau)^{Z \times K} \\ d_1^\pm &= \sum_i r(X_i) \otimes c(X_i) \otimes 1, \end{aligned} \quad (35)$$

and define operators $d_2^\pm$ as

$$\begin{aligned} d_2^\pm &: (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^\pm \otimes V_\tau)^{Z \times K} \longrightarrow (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^\mp \otimes V_\tau)^{Z \times K} \\ d_2^\pm &= \sum_j r(Y_j) \otimes c(Y_j) \otimes 1, \end{aligned} \quad (36)$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{g}^\perp$ and $\{Y_j\}$ is an orthonormal basis of $\mathfrak{p}$.

Then, it is not hard to see that $d_1^\pm$ and $d_2^\pm$ are densely defined closed operators (see Wolf [22]). Moreover, for any $\omega \in (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K}$, $D_{\lambda, \tau}^+ \omega$ is well-defined if and only if $d_1^+ \omega$ and $d_2^+ \omega$ are well-defined. And when that is so, we have $D_{\lambda, \tau}^+ \omega = d_1^+ \omega + d_2^+ \omega$. In particular, $D_{\lambda, \tau}^+ \omega = 0$ if and only if $d_1^+ \omega = 0$ and $d_2^+ \omega = 0$.

Now, we give the **harmonic** condition of Satake.

Condition 4 For every $\xi \in \text{Ker } D_c^+$, and every $x \in N$, if we see the correspondence

$$\dot{g} \rightarrow \chi(\dot{g}) \text{Ad}(\psi(\dot{g})^{-1}) U_{\psi(\dot{g})} \xi(x), \quad (37)$$

as a function $\mathcal{T} : \dot{G} \rightarrow S_1^+$ on the group $\dot{G}$ (S_1^+ -valued), then $(r(Y)\mathcal{T})(\dot{g}) = 0$ for all $Y \in \mathfrak{p} \subseteq \mathfrak{g}$ (the lie algebra of G and $\dot{G}$), i.e.

$$\frac{d}{dt}(\mathcal{T}(\dot{g}e^{tY}))|_{t=0} = 0, \quad (38)$$

for all $\dot{g} \in \dot{G}$.

To avoid complicated notations, **without loss of generality**, we assume $\dot{G} = G$, $\dot{K} = K$, and $\psi = i_d$ in the rest of this chapter.

Proposition 1. 7 Let $W_0 = \text{the image of } \text{Ker } D_e^+ \otimes \text{Ker } D_\tau^+ \text{ under the correspondence in Proposition 1.6. Then } W_0 \subseteq \text{Ker } D_{\lambda, \tau}^+$

Proof. Since $\mathcal{L}_0 \subseteq (L^2(N \ltimes G) \otimes S_1^+ \otimes S_2^+ \otimes V_\tau)^{Z \times K}$ is closed, it is sufficient to prove: if $\omega(x, g) = \chi(g) \text{Ad}(g^{-1}) U_g \xi(x) \otimes \dot{\eta}(g)$ for $\xi \in \text{Ker } D_e^+$ and $\dot{\eta} \in \text{Ker } D_\tau^+ \subseteq L^2(G/K, S_2^+ \otimes \mathcal{V}_\tau)$, then $D_{\lambda, \tau}^+ \omega = 0$.

However, by the discussion after the definition for $d_1^\pm$ and $d_2^\pm$, we need only to prove that $d_1^+ \omega = 0$ and $d_2^+ \omega = 0$. Indeed, Consider $d_1^+ \omega$. Let $U_g \xi(x) =$

$\sum_{\alpha} f_{\alpha}(x, g) \otimes Ad(g)s_{\alpha}^{+}$, where $\{s_{\alpha}^{+}\}$ is an orthonormal basis of S_1^{+} , then

$$\begin{aligned}
d_1^{+}\omega(x, g) &= \sum_i \sum_{\alpha} \frac{d}{dt} (\chi(g) f_{\alpha}(x g e^{tX_i} g^{-1}, g))|_{t=0} \otimes c(X_i) s_{\alpha}^{+} \otimes \dot{\eta}(g) \\
&= \chi(g) \sum_i \sum_{\alpha} \frac{d}{dt} f_{\alpha}(x e^{tAd(g)X_i}, g)|_{t=0} \otimes c(X_i) s_{\alpha}^{+} \otimes \dot{\eta}(g) \\
&= \chi(g) Ad(g^{-1}) \sum_i \sum_{\alpha} \frac{d}{dt} f_{\alpha}(x e^{tAd(g)X_i}, g)|_{t=0} \otimes c(Ad(g)X_i) Ad(g)s_{\alpha}^{+} \otimes \dot{\eta}(g) \\
&= \chi(g) Ad(g^{-1}) D_g^{+}(U_g \xi)(x) \otimes \dot{\eta}(g) \\
&= 0.
\end{aligned} \tag{39}$$

Here, the last equality follows from the fact that $U_g : Ker D_c^{+} \rightarrow Ker D_g^{+}$ is an isomorphism and the definition of D_g^{+}

Next, let's consider $d_2^{+}\omega$,

$$\begin{aligned}
d_2^{+}\omega(x, g) &= [(\sum_j r(Y_j) \otimes c(Y_j) \otimes 1)\omega](x, g) \\
&= \sum_j \frac{d}{dt} [\chi(g e^{tY_j}) Ad(g e^{tY_j})^{-1} U_{g e^{tY_j}} \xi(x)]|_{t=0} \otimes c(Y_j) \dot{\eta}(g) \\
&\quad + \chi(g) Ad(g^{-1}) U_g \xi(x) \otimes [D_{\tau}^{+} \dot{\eta}](g) \\
&= 0.
\end{aligned} \tag{40}$$

Here, we have used Condition 4. Thus, $W_0 \subseteq Ker D_{\lambda, \tau}^{++}$ $\square$

Next, we want to prove $Ker D_{\lambda, \tau}^{++} \subseteq W_0$. To do that, we need the following preliminaries. Let $\{s_{\beta}^{+}\}$ be an orthonormal basis of S_2^{+} , $\{s_{\alpha}^{+}\}$ be an orthonormal basis of S_1^{+} (as in the proof of Proposition 1.7), and $\{v_{\gamma}\}$ be an orthonormal basis of V_{τ} . For any $\omega \in (L^2(N \ltimes G) \otimes S_1^{+} \otimes S_2^{+} \otimes V_{\tau})^{\mathbb{Z} \times K}$, we can write $\omega(x, g)$ as

$$\omega(x, g) = \sum_{\alpha, \beta, \gamma} f_{\alpha, \beta, \gamma}(x, g) s_{\alpha}^{+} \otimes s_{\beta}^{+} \otimes v_{\gamma}. \tag{41}$$

Define $\omega_{\beta,\gamma}^g$ for all β, γ and $g \in G$ as

$$\omega_{\beta,\gamma}^g(x) = \sum_{\alpha} f_{\alpha,\beta,\gamma}(x, g) Ad(g) s_{\alpha}^+, \quad (42)$$

for $x \in N$. Then it is easy to see that $\omega_{\beta,\gamma}^g$ is a $S_g^+ = Ad(g)S_1^+$ valued function on the group N and

$$\omega(x, g) = \sum_{\beta,\gamma} Ad(g^{-1}) \omega_{\beta,\gamma}^g(x) \otimes s_{\beta}^+ \otimes v_{\gamma}. \quad (43)$$

Moreover, we have

Lemma 1. 10 $\omega_{\beta,\gamma}^g \in (L^2(N))^Z \otimes Ad(g)S_1^+$ for all β and γ , and almost all $g \in G$.

Proof. This follows from the fact that

$$\begin{aligned} \|\omega\|^2 &= \sum_{\beta,\gamma} \int_G dg \int_N \sum_{\alpha} |f_{\alpha,\beta,\gamma}(x, g)|^2 dx \\ &= \sum_{\beta,\gamma} \int_G \|\omega_{\beta,\gamma}^g\|^2 dg < \infty. \end{aligned}$$

□

Lemma 1. 11 If $\omega \in Ker D_{\lambda,\tau}^{++}$, then $\omega_{\beta,\gamma}^g \in Ker D_g^+$ for all β and γ , and almost all $g \in G$.

Proof. Since $\omega \in Ker D_{\lambda,\tau}^{++}$, so ω is smooth and $d_1^+ \omega = 0$. Thus,

$$\begin{aligned} 0 &= d_1^+ \omega(x, g) \\ &= \sum_{\beta,\gamma} [\sum_i \sum_{\alpha} r(X_i) f_{\alpha,\beta,\gamma} \otimes c(X_i) s_{\alpha}^+](x, g) \otimes s_{\beta}^+ \otimes v_{\gamma} \\ &= \sum_{\beta,\gamma} [Ad(g^{-1}) \sum_i \sum_{\alpha} \frac{d}{dt} f_{\alpha,\beta,\gamma}(x e^{tAd(g)X_i}, g) |_{t=0} \otimes c(Ad(g)X_i) Ad(g) s_{\alpha}^+] \otimes s_{\beta}^+ \otimes v_{\gamma}. \end{aligned} \quad (44)$$

It follows that

$$\sum_i \sum_\alpha \frac{d}{dt} f_{\alpha, \beta, \gamma}(x e^{t \text{Ad}(g) X_i}, g) \big|_{t=0} \otimes c(\text{Ad}(g) X_i) \text{Ad}(g) s_\alpha^+ = 0 \quad (45)$$

for all β and γ , and all $g \in G$ and $x \in N$.

On the other hand, it follows from the smoothness of $\omega_{\beta, \gamma}^g$ (in "x") and the above discussion (before this lemma) that: for $g \in G$ so that $\omega_{\beta, \gamma}^g \in (L^2(N))^Z \otimes \text{Ad}(g) S_1^+$, we have

$$\begin{aligned} (D_g^+ \omega_{\beta, \gamma}^g)(x) &= \sum_i \sum_\alpha \frac{d}{dt} f_{\alpha, \beta, \gamma}(x e^{t \text{Ad}(g) X_i}, g) \big|_{t=0} \otimes c(\text{Ad}(g) X_i) \text{Ad}(g) s_\alpha^+ \\ &= 0. \end{aligned} \quad (46)$$

It follows that $\omega_{\beta, \gamma}^g \in \text{Ker } D_g^+$. $\square$

Now, assume $\{\xi_\nu\}$ is an orthonormal basis of $\text{Ker } D_c^+$, then $\{U_g \xi_\nu\}$ is an orthonormal basis of $\text{Ker } D_g^+$.

Lemma 1. 12 *If $\omega \in \text{Ker } D_{\lambda, \tau}^{++}$, then ω could be written as*

$$\omega(x, g) = \sum_\nu \text{Ad}(g^{-1}) U_g \xi_\nu(x) \otimes \chi(g) \dot{\eta}_\nu(g), \quad (47)$$

where $\dot{\eta}_\nu \in L^2(G/K, \mathcal{S}_2^+ \otimes \mathcal{V}_i)$.

Proof. Since $\omega(x, g) = \sum_{\beta, \gamma} \text{Ad}(g^{-1}) \omega_{\beta, \gamma}^g(x) \otimes s_\beta^+ \otimes v_\gamma$ and $\omega_{\beta, \gamma}^g \in \text{Ker } D_g^+$ for all β and γ , and almost all $g \in G$, it follows that

$$\begin{aligned} \omega(x, g) &= \sum_{\beta, \gamma} \sum_\nu \text{Ad}(g^{-1}) \langle \omega_{\beta, \gamma}^g, U_g \xi_\nu \rangle U_g \xi_\nu(x) \otimes s_\beta^+ \otimes v_\gamma \\ &= \sum_\nu \text{Ad}(g^{-1}) U_g \xi_\nu(x) \otimes \left[\sum_{\beta, \gamma} \langle \omega_{\beta, \gamma}^g, U_g \xi_\nu \rangle \otimes s_\beta^+ \otimes v_\gamma \right]. \end{aligned} \quad (48)$$

Let

$$\dot{\eta}_\nu(g) = \chi(g)^{-1} \sum_{\beta, \gamma} \langle \omega_{\beta, \gamma}^g, U_g \xi_\nu \rangle \otimes s_\beta^+ \otimes v_\gamma = \chi(g)^{-1} \eta_\nu(g),$$

then the fact that $\dot{\eta}_\nu \in L^2(G/K, S_2^+ \otimes \mathcal{V}_i)$ could be seen as follows.

First of all, let's prove that η_ν is k -invariant for any $k \in K$.

Indeed,

$$\begin{aligned}
& [k \cdot \eta_\nu](g) \\
&= \sum_{\beta, \gamma} \langle \omega_{\beta, \gamma}^{gk}, U_{gk} \xi_\nu \rangle \otimes c(\tilde{A}d(k)) \cdot s_\beta^+ \otimes \tau(k) v_\gamma \\
&= \sum_{\beta, \gamma} \langle Ad(gk) \sum_{\alpha} f_{\alpha, \beta, \gamma}(\cdot, gk) \otimes s_\alpha^+, Ad(gk)[c(\tilde{A}d(k^{-1})) \cdot Ad(g^{-1}) U_g \xi_\nu] \rangle \\
&\quad \otimes c(\tilde{A}d(k)) \cdot s_\beta^+ \otimes \tau(k) v_\gamma \\
&= \sum_{\beta, \gamma} \langle \sum_{\alpha} f_{\alpha, \beta, \gamma}(\cdot, gk) \otimes s_\alpha^+, c(\tilde{A}d(k^{-1})) \cdot Ad(g^{-1}) U_g \xi_\nu \rangle \otimes c(\tilde{A}d(k)) \cdot s_\beta^+ \otimes \tau(k) v_\gamma \\
&= \sum_{\beta, \gamma} \langle Ad(g) \sum_{\alpha} f_{\alpha, \beta, \gamma}(\cdot, gk) \otimes c(\tilde{A}d(k)) \cdot s_\alpha^+, U_g \xi_\nu \rangle \otimes c(\tilde{A}d(k)) \cdot s_\beta^+ \otimes \tau(k) v_\gamma.
\end{aligned}$$

Note that $k \cdot \omega = \omega$, i.e.,

$$\begin{aligned}
& \sum_{\beta, \gamma} \sum_{\alpha} f_{\alpha, \beta, \gamma}(x, gk) \otimes c(\tilde{A}d(k)) \cdot s_\alpha^+ \otimes c(\tilde{A}d(k)) \cdot s_\beta^+ \otimes \tau(k) v_\gamma \\
&= \sum_{\beta, \gamma} \sum_{\alpha} f_{\alpha, \beta, \gamma}(x, g) \otimes s_\alpha^+ \otimes s_\beta^+ \otimes v_\gamma,
\end{aligned}$$

it follows that

$$k \cdot \eta_\nu = \eta_\nu$$

for all γ and all $g \in G$.

Finally, we can see that η_ν is square integrable from the fact

$$\|\omega\|^2 = \sum_{\nu} \|\dot{\eta}_\nu\|^2.$$

□

From this Lemma, we see that $\text{Ker} D_{\lambda, \tau}^{++} \subseteq \mathcal{L}_0$. Next, we will show that

$$\text{Ker} D_{\lambda, \tau}^{++} \subseteq W_0.$$

Proposition 1. 8 $Ker D_{\lambda, \tau}^{++} \subseteq W_0(= \text{the image of } Ker D_c^+ \otimes Ker D_{\tau}^+)$ under the correspondence in Proposition 1.6.

Proof. From Lemma 1.12, it is sufficient to prove $\dot{\eta}_{\nu} \in Ker D_{\tau}^+$ for any ν . Now, for any $\dot{\eta}_c \in C_c^{\infty}(G/K, \mathcal{S}_2^- \otimes \mathcal{V}_{\tau})$, define $\omega_{\nu} \in \mathcal{L}_0$ as $\omega_{\nu}(x, g) = \chi(g) Ad(g^{-1}) U_g \xi_{\nu}(x) \otimes \dot{\eta}_c(g)$ for any $x \in N$ and any $g \in G$. Then it is easy to see that ω_{ν} is smooth in both "x" and "g", and it has compact support in the direction of G .

From the proof of Proposition 1.7 and the harmonic condition (Condition 4), we have

$$\begin{aligned} d_2^- \omega_{\nu}(x, g) &= [(\sum_j r(Y_j) \otimes c(Y_j) \otimes 1) \omega_{\nu}](x, g) \\ &= \chi(g) Ad(g^{-1}) U_g \xi(x) \otimes [D_{\tau}^+ \dot{\eta}_c](g). \end{aligned} \quad (49)$$

Also from the fact that $d_2^+ \omega = 0$ ($D_{\lambda, \tau}^+ \omega = 0$), we get

$$\begin{aligned} 0 &= \langle \omega, d_2^- \omega_{\nu} \rangle \\ &= \langle \dot{\eta}_{\nu}, D_{\tau}^- \dot{\eta}_c \rangle, \end{aligned} \quad (50)$$

it follows that

$$D_{\tau}^+ \dot{\eta}_{\nu} = 0 \quad (51)$$

for any ν . $\square$

Theorem 1. 1 Let $N \ltimes G$ be a semidirect product of N and G , where N is a connected and simply connected nilpotent Lie group having square-integrable irreducible representations modulo its center Z and G is a connected noncompact semisimple Lie group with finite center having a maximal compact subgroup K , with $rank K = rank G$. Assume Z is central in $N \ltimes G$, V_{τ} is an irreducible unitary

K -module, and $\epsilon_\lambda = e^{2i\pi\lambda}$ is the character of Z associated with $\lambda \in \mathfrak{n}_{reg}^*$. If $D_{\lambda,\tau}^+$ is the Dirac operator over $N \ltimes G/Z \times K$ associated with the irreducible $Z \times K$ -module $\epsilon_\lambda \otimes \tau$, $D_c^+ = D_\lambda^+$ is the Dirac operator over N/Z associated with the Z -character ϵ_λ , $D_{\dot{\tau}}^+$ is the Dirac operator over $\dot{G}/\dot{K}$ associated with the irreducible $\dot{K}$ -module $\dot{\tau}$ ($\dot{G}$ and $\dot{K}$ are the (at most) double coverings of G and K , respectively; $\dot{\tau}$ is given in Condition 3), and the conditions of Satake (Condition 1–Condition 4) are satisfied, then one has an isomorphism of Hilbert spaces

$$\text{Ker} D_\lambda^+ \otimes \text{Ker} D_{\dot{\tau}}^+ \cong \text{Ker} D_{\lambda,\tau}^{++}, \quad (52)$$

which is given in Proposition 1.6.

Moreover, if one denotes the left regular representation of $N \ltimes G$ on the space $\text{Ker} D_{\lambda,\tau}^{++}$ by L , the left regular representation of $\dot{G}$ on the space $\text{Ker} D_{\dot{\tau}}^+$ by $\dot{l}_2$, the left regular representation of N on $\text{Ker} D_\lambda^+$ by l_e , and the action of $\dot{G}$ on $\text{Ker} D_\lambda^+$ by $\dot{l}$, then, under above isomorphism of Hilbert spaces, one has the following correspondence

$$l_e(x) \cdot \dot{l}(\dot{g}) \otimes \dot{l}_2(\dot{g}) \rightarrow L(xg), \quad (53)$$

for any $\dot{g} \in \dot{G}$, where $g = \psi(\dot{g})$.

Proof. It follows from Proposition 1.6, Proposition 1.7 and Proposition 1.8. $\square$

From this theorem, it is not difficult to see:

Corollary 1. 4 *The unitary representation of $N \ltimes G$ on the space $\text{Ker} D_{\lambda,\tau}^{++}$ is irreducible if and only if the left regular representation of $\dot{G}$ on the space $\text{Ker} D_{\dot{\tau}}^+$ is irreducible.*

Remark 1. 4 By the conditions similar to Condition 1–Condition 4, we can prove

$$Ker D_{\lambda}^{+} \otimes Ker D_{\tau}^{-} \cong Ker D_{\lambda, \tau}^{+-}; \quad (54)$$

$$Ker D_{\lambda}^{-} \otimes Ker D_{\tau}^{+} \cong Ker D_{\lambda, \tau}^{-+}; \quad (55)$$

$$Ker D_{\lambda}^{-} \otimes Ker D_{\tau}^{-} \cong Ker D_{\lambda, \tau}^{--} \quad (56)$$

as representations.

Remark 1. 5 Recall that Atiyah and Schmid constructed most of discrete series representations on the spaces of harmonic spinors for semisimple Lie groups, and Connes and Moscovici obtained a similar result for nilpotent Lie groups. Combine our Theorem 1.1 with the corresponding theorems in Atiyah-Schmid [4] and Connes-Moscovici [8] . we get that

the left regular representation of $N \ltimes G$ on the space $Ker D_{\lambda, \tau}^{++}$ is irreducible provided λ being sufficiently positive (in the sense of [8]) and the highest weight of τ being sufficiently positive (in the sense of [4]).

Thus, Theorem 1.1 gives a geometric realization of the discrete series representations of $N \ltimes G$ on the space $Ker D_{\lambda, \tau}^{++}$ of the restricted harmonic spinors on $N \ltimes G/Z \times K$. The importance of this result is evidenced by a theorem of Anh that says every discrete series representation of $N \ltimes G$ is in the tensor product form likes the one in our theorem.

Chapter II

A Vanishing Theorem For $\text{Ker} D_{\lambda, \mu}^-$

In this chapter, we shall prove a vanishing theorem for $\text{Ker} D_{\lambda, \mu}^-$, extending Moscovici's vanishing theorem ([17]) and Atiyah-Schmid's vanishing theorem ([4]). First, by modifying Atiyah and Schmid's proof for their vanishing theorem of $\text{Ker} D_{\mu}^-$, we will prove an important inequality for the spinor Laplacian $D_{\mu}^+ D_{\mu}^-$. Then we will combine Moscovici's proof for his vanishing theorem of $\text{Ker} D_{\lambda}^-$ and the above inequality to prove our vanishing theorem. Since we shall be making use of the Plancherel decomposition of $L^2(G)$, we start by reviewing some basic facts here.

Throughout this chapter, we freely use the notations of Chapter I. For a connected semisimple Lie group G , the Plancherel decomposition of $L^2(G)$ is concerned with a direct integral decomposition of $L^2(G)$

$$L^2(G) = \int_{\hat{G}} H_l \otimes H_l^* dl. \quad (57)$$

To explain the notation, $\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G , l the symbol for a generic element of $\hat{G}$, H_l the G -module corresponding to l , H_l^* its dual, and dl the Plancherel measure on $\hat{G}$. The tensor product $H_l \otimes H_l^*$ is not to be understood in the algebraic sense, but rather as the completion of the algebraic tensor product with respect to the natural unitary structure. We shall denote the action of G on H_l by π_l , and its contragredient by π_l^* . The left and right regular representations of G decompose as $\int_{\hat{G}} \pi_l \otimes 1 dl$ and

$\int_{\hat{G}} 1 \otimes \pi_l^* dl$. In our applications, the space $L^2(G)$ will be replaced by $L^2(G/K, \mathcal{F})$ (the space of L^2 -sections of a homogeneous vector bundle $\mathcal{F}$ over the symmetric space G/K associated with a K -module F). If F is a finite dimensional unitary K -module, then $\mathcal{F} = G \times_K F \rightarrow G/K$ is a homogeneous vector bundle over G/K , and $L^2(G/K, \mathcal{F})$ may be identified with the space of right K -invariants in $L^2(G) \otimes F$. Because of the direct integral decomposition of $L^2(G)$, this gives

$$L^2(G) \otimes F = \int_{\hat{G}} H_l \otimes H_l^* \otimes F dl. \quad (58)$$

Since right regular representation equals $\int_{\hat{G}} 1 \otimes \pi_l^* dl$, K acts on $L^2(G) \otimes F$ as $\int_{\hat{G}} 1 \otimes \pi_l^* \otimes \tau dl$, where τ is the representation of K on F . Thus

$$L^2(G/K, \mathcal{F}) \cong \int_{\hat{G}} H_l \otimes W_l dl, \quad (59)$$

where W_l is the K -invariant part of $H_l^* \otimes F$ by the action $\pi_l^* \otimes \tau$.

Because the canonical action of G on $L^2(G/K, \mathcal{F}) \subseteq L^2(G) \otimes F$ is given by $l_G \otimes 1$, where l_G is the left regular representation of G on $L^2(G)$, then, according to the decomposition of $L^2(G/K, \mathcal{F})$, the left regular representation of G on $L^2(G/K, \mathcal{F})$ is given by $\int_{\hat{G}} \pi_l \otimes 1 dl$.

Remark 2. 1 *Since the Casimir operator Ω of $\mathfrak{g}^{\mathbb{C}}$ acts on H_l by scalar, it defines a function on $\hat{G}$. It could be shown that this function is measurable and*

$$l_G(\Omega) = \int_{\hat{G}} \pi_l(\Omega) \otimes 1 dl, \quad (60)$$

according to the decomposition of $L^2(G/K, \mathcal{F})$ and the left regular representation l_G of G (see Atiyah-Schmid [4]).

Now, let's introduce more notations we are going to use. Note that $\text{rank} G = \text{rank} K$, we can assume H to be a compact Cartan subgroup of K

and G , and $\mathfrak{h}$ be the lie algebra of H . Let $\mathfrak{h}^{\mathbb{C}}$ be the complexification of $\mathfrak{h}$. For the root system $\Phi = \Phi(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, a root $\alpha \in \Phi$ is said to be compact or noncompact, depending on whether or not it is a root of the pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Thus Φ decomposes into a disjoint union $\Phi = \Phi^c \cup \Phi^n$ of the set Φ^c , Φ^n of all compact and noncompact roots, respectively. From now on, we fix a particular system of positive roots Ψ in Φ . ρ_c and ρ_n stand for the half-sums of all positive compact and noncompact positive roots, respectively; $\rho = \rho_c + \rho_n$ is the half-sum of all positive roots.

Let μ be a highest weight of a irreducible K -module $(\tau, V_\tau = V_\mu)$ with respect to the system of positive roots Ψ . Assume $D_\mu = D_\mu^+ \oplus D_\mu^- = D_\tau = D_\tau^+ \oplus D_\tau^-$ is the Dirac operator associated with K -module (τ, V_τ) , according to the Parthasarathy Theorem (see [19] or [4]), the spinor Laplacian

$$D_\mu^2 = -\Omega + (\mu - \rho_n, \mu - \rho_n + 2\rho), \quad (61)$$

where Ω represents the Casimir operator acting on $L^2(G/K, \mathcal{S}_2^\pm \otimes \mathcal{V}_\mu)$ (it is $l_G(\Omega)$ in Remark 2.1). It follows from Remark 2.1 that

Lemma 2. 1 *According to the Plancherel decomposition of $L^2(G/K, \mathcal{S}_2^\pm \otimes \mathcal{V}_\mu)$, one has*

$$D_\mu^2 = \int_{\hat{G}} [-\pi_l(\Omega) + (\mu - \rho_n, \mu - \rho_n + 2\rho)] \otimes 1 dl. \quad (62)$$

The following lemma (section 5, [4]) is based on the algebraic version of Parthasarathy's formula.

Lemma 2. 2 *Let π be an irreducible unitary representation of G , and V_ν an irreducible K -module, of highest weight ν , such that*

- (1) V_ν occurs in the restriction of π to K ;

(2) $(\nu - \rho_n, \alpha) \geq 0$, if $\alpha \in \Psi^c$, the set of all positive compact roots.

Then

$$\pi(\Omega) \leq (\mu - \rho_n + \rho_c, \mu - \rho_n + \rho_c) - (\rho, \rho). \quad (63)$$

Now, we are in the position of proving the following inequality.

Proposition 2. 1 *For a given constant $M > 0$, there exists a constant $C_M > 0$ such that if $(\mu, \alpha) > C_M$ for any $\alpha \in \Psi$, then*

$$\langle D_\mu^+ D_\mu^- \phi, \phi \rangle \geq M \|\phi\|^2, \quad (64)$$

for any $\phi \in \text{Dom}(D_\mu^-) \subseteq L^2(G/K, \mathcal{S}_2^- \otimes \mathcal{V}_\mu)$.

Proof. Let C_M be chosen so that

- (a) $(\mu + \rho_c - B, \alpha) > 0$, for any $\alpha \in \Psi$;
- (b) $2(\mu + \rho_c, B) - (B, B) > M$, for $B \neq 0$;

where B is any sum of distinct positive non-compact roots.

Recall that the Plancherel theorem gives the direct integral decomposition

$$L^2(G/K, \mathcal{S}_2^- \otimes \mathcal{V}_\mu) = \int_{\hat{G}} H_l \otimes V_l^- dl, \quad (65)$$

where $V_l^- = K$ -invariant part of $H_l^* \otimes \mathcal{S}_2^- \otimes \mathcal{V}_\mu \cong \text{Hom}_K(H_l, \mathcal{S}_2^- \otimes \mathcal{V}_\mu)$.

Note that in the tensor product $V_\nu \otimes W$ of an irreducible K -module V_ν , of highest weight ν , with an arbitrary finite dimensional K -module W , every irreducible constituent has a highest weight which is the sum of ν and some weight ν_W of W . Moreover, $\nu + \nu_W$ occurs as highest weight in $V_\nu \otimes W$ at most as often as the multiplicity of the weight ν_W in W . Since every weight of $\mathcal{S}_2^+ \oplus \mathcal{S}_2^-$ can be expressed

as $\rho_n - B$, where B stands for a sum of distinct positive, non-compact roots; weight ρ_n has multiplicity one and occurs in S_2^+ . Thus, V_l^- can be nonzero only if H_l contains an irreducible K -module with a highest weight of the form $\mu + \rho_n - B$, $B \neq 0$. Because of the condition (a), we have $(\mu + \rho_n - B - \rho_n, \alpha) = (\mu - B, \alpha) \geq 0$, for any $\alpha \in \Psi^c$. Apply Lemma 2.2, we have

$$\pi_l(\Omega) \leq (\mu + \rho_c - B, \mu + \rho_c - B) - (\rho, \rho). \quad (66)$$

In turn this gives

$$\begin{aligned} D_\mu^+ D_\mu^- |_{H_l} &= -\pi_l(\Omega) + (\mu - \rho_n, \mu - \rho_n + 2\rho) \\ &\geq -(\mu + \rho_c - B, \mu + \rho_c - B) + (\rho, \rho) \\ &\quad + (\mu - \rho_n, \mu - \rho_n + 2\rho) \\ &= 2(\mu + \rho_c, B) - (B, B) \\ &> M, \end{aligned} \quad (67)$$

here, we have used the fact that $B \neq 0$.

Now, for any $\phi \in \text{Dom}(D_\mu^-) \subseteq L^2(G/K, \mathcal{S}_2^- \otimes \mathcal{V}_\mu)$, we have

$$\phi = \int_{\hat{G}} \phi_l dl, \quad (68)$$

and

$$\|\phi\|^2 = \int_{\hat{G}} \|\phi_l\|^2 dl. \quad (69)$$

where $\phi_l \in H_l \otimes V_l^-$ (according to the formula (65)).

It follows from Lemma 2.1 that

$$\begin{aligned} \langle D_\mu^+ D_\mu^- \phi, \phi \rangle &= \int_{\hat{G}} \langle -\pi_l(\Omega) + (\mu - \rho_n, \mu - \rho_n + 2\rho) \phi_l, \phi_l \rangle dl \\ &\geq \int_{\hat{G}} M \|\phi_l\|^2 dl \\ &= M \|\phi\|^2. \end{aligned} \quad (70)$$

We have finished the proof of this proposition. $\square$

Remark 2. 2 *It follows from above proposition that: if μ is sufficiently far away from all root hyperplanes, then $\text{Ker } D_{\mu}^{-} = 0$, which is Atiyah-Schmid's vanishing theorem.*

From now on, we focus on the proof for the vanishing theorem about $\text{Ker } D_{\lambda,\mu}^{-}$. As in Chapter I, we let $\{Y_j\}$ be an orthonormal basis of $\mathfrak{p}$, $\{X_i\}$ be an orthonormal basis of $\mathfrak{z}^{\perp}$, $d_1 = \sum_i r(X_i) \otimes c(X_i) \otimes 1$, and $d_2 = \sum_j r(Y_j) \otimes c(Y_j) \otimes 1$. Moreover, for any $\phi \in (C_c^{\infty}(N \ltimes G) \otimes S^{-} \otimes V_{\mu})^{Z \times K} = (C_c^{\infty}(N \ltimes G) \otimes [S_1^{+} \otimes S_2^{-} \oplus S_1^{-} \otimes S_2^{+}] \otimes V_{\mu})^{Z \times K}$, we can write it as $\phi = \phi_{+-} + \phi_{-+}$, where $\phi_{+-} \in (C_c^{\infty}(N \ltimes G) \otimes S_1^{+} \otimes S_2^{-} \otimes V_{\mu})^{Z \times K}$, and $\phi_{-+} \in (C_c^{\infty}(N \ltimes G) \otimes S_1^{-} \otimes S_2^{+} \otimes V_{\mu})^{Z \times K}$. An easy computation shows the following

Lemma 2. 3 *For any $\phi \in (C_c^{\infty}(N \ltimes G) \otimes S^{-} \otimes V_{\mu})^{Z \times K}$, one has*

$$\begin{aligned} \langle D_{\lambda,\mu}^{+} D_{\lambda,\mu}^{-} \phi, \phi \rangle &\geq \langle d_1^2(\phi_{-+}), \phi_{-+} \rangle + \langle d_2^2(\phi_{+-}), \phi_{+-} \rangle \\ &\quad + \langle R(\phi_{-+}), \phi_{-+} \rangle + \langle \phi_{+-}, R(\phi_{-+}) \rangle, \end{aligned} \quad (71)$$

where $R = \sum_{i,j} r[X_i, Y_j] \otimes c(X_i) \otimes c(Y_j) \otimes 1$.

Proof. First of all, by a direct computation, we have

$$D_{\lambda,\mu}^{+} D_{\lambda,\mu}^{-} \phi = d_1^2(\phi_{+-}) + d_1^2(\phi_{-+}) + d_2^2(\phi_{+-}) + d_2^2(\phi_{-+}) + R(\phi_{+-}) + R(\phi_{-+}).$$

Note that

$$d_1^2(\phi_{+-}) \in C^{\infty}(N \ltimes G) \otimes (S_1^{+} \otimes S_2^{-}) \otimes V_{\mu},$$

$$d_1^2(\phi_{-+}) \in C^{\infty}(N \ltimes G) \otimes (S_1^{-} \otimes S_2^{+}) \otimes V_{\mu},$$

$$d_2^2(\phi_{+-}) \in C^\infty(N \ltimes G) \otimes (S_1^+ \otimes S_2^-) \otimes V_\mu,$$

$$d_2^2(\phi_{-+}) \in C^\infty(N \ltimes G) \otimes (S_1^- \otimes S_2^+) \otimes V_\mu,$$

$$R(\phi_{+-}) \in C^\infty(N \ltimes G) \otimes (S_1^- \otimes S_2^+) \otimes V_\mu,$$

$$R(\phi_{-+}) \in C^\infty(N \ltimes G) \otimes (S_1^+ \otimes S_2^-) \otimes V_\mu.$$

Thus,

$$\begin{aligned} \langle D_{\lambda,\mu}^+ D_{\lambda,\mu}^- \phi, \phi \rangle &= [\langle d_1^2(\phi_{+-}), \phi_{+-} \rangle + \langle d_2^2(\phi_{+-}), \phi_{+-} \rangle + \langle R(\phi_{-+}), \phi_{+-} \rangle] \\ &+ [\langle d_1^2(\phi_{-+}), \phi_{-+} \rangle + \langle d_2^2(\phi_{-+}), \phi_{-+} \rangle + \langle R(\phi_{+-}), \phi_{-+} \rangle]. \end{aligned}$$

Since

$$\langle d_1^2(\phi_{+-}), \phi_{+-} \rangle \geq 0,$$

$$\langle d_2^2(\phi_{-+}), \phi_{-+} \rangle \geq 0,$$

$$\langle R(\phi_{+-}), \phi_{-+} \rangle = \langle \phi_{+-}, R(\phi_{-+}) \rangle.$$

The lemma follows. $\square$

In the following, we assume the center Z of N is a circle group, by factoring out $\text{Ker}(\epsilon_\lambda = e^{2i\pi\lambda})$. Fix an element $z_0 \in \mathfrak{z}$ so that $\lambda(z_0) = p \in \mathbb{Z}$.

Proposition 2. 2 *There exists $p_0 \in \mathbb{Z}^+$ such that if $p \geq p_0$, then*

$$\begin{aligned} &\langle d_1^2(\phi_{-+}), \phi_{-+} \rangle + \langle R(\phi_{-+}), \phi_{+-} \rangle + \langle \phi_{+-}, R(\phi_{-+}) \rangle \\ &\geq C_1 \|\phi_{-+}\|^2 - C_2 \|\phi_{+-}\|^2, \end{aligned} \tag{72}$$

where $C_1 > 0$ and $C_2 > 0$ are constants.

Proof. Since $\lambda \in \mathfrak{n}_{rcg}^*$, we can choose an orthonormal basis $\{X'_1, X''_1, X'_2, X''_2, \dots, X'_n, X''_n\}$ for $\mathfrak{z}^\perp$ such that $[X'_1, X''_1] = a_1 z_0 \mod \mathfrak{z}^\perp$, $[X'_2, X''_2] = a_2 z_0 \mod \mathfrak{z}^\perp, \dots, [X'_n, X''_n] = a_n z_0 \mod \mathfrak{z}^\perp$; $[X'_i, X''_j] = 0 \mod \mathfrak{z}^\perp$ for $i \neq j$; $[X'_i, X'_j] = 0 \mod \mathfrak{z}^\perp$; and $[X''_i, X''_j] = 0 \mod \mathfrak{z}^\perp$. Here, $a_1, a_2, \dots, a_n$ are all positive numbers (make a suitable choice of the orientation of N/Z).

Since $d_1 = \sum_{i=1}^n r(X'_i) \otimes c(X'_i) \otimes 1 + \sum_{i=1}^n r(X''_i) \otimes c(X''_i) \otimes 1$, thus

$$\begin{aligned} d_1^2 &= \sum_{i=1}^n r([X'_i, X''_i]) \otimes c(X'_i)c(X''_i) \otimes 1 + \sum_{i \neq j} r[X'_i, X''_j] \otimes c(X'_i)c(X''_j) \otimes 1 \\ &\quad - \sum_{i=1}^n [r(X'_i)^2 + r(X''_i)^2] \otimes 1 + \sum_{i < j} r([X'_i, X'_j]) \otimes c(X'_i)c(X'_j) \otimes 1 \\ &\quad + \sum_{i < j} r([X''_i, X''_j]) \otimes c(X''_i)c(X''_j) \otimes 1. \end{aligned} \quad (73)$$

From the choice of $\{X'_1, X''_1, X'_2, X''_2, \dots, X'_n, X''_n\}$, we got

$$\begin{aligned} d_1^2 &= \sum_{i=1}^n a_i r(z_0) \otimes c(X'_i)c(X''_i) \otimes 1 - \sum_{i=1}^n [r(X'_i)^2 + r(X''_i)^2] \otimes 1 \\ &\quad + F_0, \end{aligned} \quad (74)$$

where F_0 is a first order differential operator of the form

$$F_0 = \sum_i r(X'_i) \otimes A'_i + \sum_i r(X''_i) \otimes A''_i, \quad (75)$$

with $A'_i, A''_i \in \text{End}(S_1)$.

Let's define $z_i \in (\mathfrak{z}^\perp)^\mathbb{C}$ by $z_i = X'_i + \sqrt{-1}X''_i$ for $1 \leq i \leq n$. Then it follows that

$$\begin{aligned} \sum_{i=1}^n [r(X'_i)^2 + r(X''_i)^2] \otimes 1 &= -\sqrt{-1} \sum_{i=1}^n r([X'_i, X''_i]) \otimes 1 + \sum_{i=1}^n r(\overline{z_i})r(z_i) \otimes 1 \\ &= -\sqrt{-1} \sum_{i=1}^n a_i r(z_0) \otimes 1 + \sum_{i=1}^n r(\overline{z_i})r(z_i) \otimes 1 \\ &\quad + F_1, \end{aligned} \quad (76)$$

where F_1 is also of the form (75).

Note that the action of $r(z_0)$ on the space $(C^\infty(N \ltimes G) \otimes S \otimes V_\mu)^{Z \times K}$ is as the same as the multiplication by $-2\pi p\sqrt{-1}$, thus

$$\begin{aligned} d_1^2 &= 2\pi p \sum_{i=1}^n a_i \otimes 1 - 2\pi p \sum_{i=1}^n a_i \otimes \sqrt{-1}c(X_i' X_i'') \otimes 1 \\ &\quad - \sum_{i=1}^n r(\bar{z}_i)r(z_i) \otimes 1 + F. \end{aligned} \quad (77)$$

Now, let's choose a convenient basis of $S_1^\pm$. Take J to be the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i = \pm 1$, and set $J^\pm = \{\alpha \in J \mid \alpha_1 \alpha_2 \dots \alpha_n = \pm(-1)^n\}$. It is not difficult to see that there exists an orthonormal basis $\{u_\alpha \mid \alpha \in J\}$ in $S_1 = S_1^+ \oplus S_1^-$, with $\{u_\alpha \mid \alpha \in J^\pm\}$ an orthonormal basis for $S_1^\pm$, and such that

$$c(X_i' X_i'')u_\alpha = \sqrt{-1}\alpha_i u_\alpha, i = 1, 2, \dots, n. \quad (78)$$

Moreover, we assume $\{v_\beta\}$ is an orthonormal basis of $S_2^+ \otimes V_\mu$. Then for any $\phi_{-+} \in (C_c^\infty(N \ltimes G) \otimes S_1^- \otimes S_2^+ \otimes V_\mu)^{Z \times K}$, it can be written as

$$\phi_{-+} = \sum_{\beta} \left(\sum_{\alpha \in J^-} f_{\beta, \alpha} \otimes u_\alpha \right) \otimes v_\beta, \quad (79)$$

where $f_{\beta, \alpha} \in [C_c^\infty(N \ltimes G)]^Z$.

Then,

$$\begin{aligned} \langle d_1^2(\phi_{-+}), \phi_{-+} \rangle &= 2\pi p \sum_{i=1}^n a_i \sum_{\beta, \alpha} \|f_{\beta, \alpha}\|^2 + 2\pi p \sum_{i=1}^n \alpha_i a_i \sum_{\beta, \alpha} \|f_{\beta, \alpha}\|^2 \\ &\quad + \sum_i \sum_{\beta, \alpha} \|r(z_i)f_{\beta, \alpha}\|^2 + \langle F\phi_{-+}, \phi_{-+} \rangle \\ &= 2\pi p \sum_{\beta} \sum_{\alpha \in J^-} \sum_{i=1}^n (a_i + \alpha_i a_i) \|f_{\beta, \alpha}\|^2 \\ &\quad + \sum_i \sum_{\beta, \alpha} \|r(z_i)f_{\beta, \alpha}\|^2 + \langle F\phi_{-+}, \phi_{-+} \rangle. \end{aligned} \quad (80)$$

Let $a_0 = \min\{a_1, a_2, \dots, a_n\} > 0$, since $\alpha \in J^-$ in above sum, thus

$$\langle d_1^2(\phi_{-+}), \phi_{-+} \rangle \geq 2\pi p a_0 \|\phi_{-+}\|^2 + \sum_i \sum_{\beta, \alpha} \|r(z_i) f_{\beta, \alpha}\|^2 + \langle F \phi_{-+}, \phi_{-+} \rangle. \quad (81)$$

Since F is of the form (75), one has

$$|\langle F \phi_{-+}, \phi_{-+} \rangle| \leq C_0 \sum_i \sum_{\beta} \sum_{\alpha, \alpha' \in J^-} \|r(z_i) f_{\beta, \alpha}\| \cdot \|f_{\beta, \alpha'}\|, \quad (82)$$

for some $C_0 > 0$, independent of ϕ_{-+} .

Consequently,

$$\begin{aligned} \langle d_1^2(\phi_{-+}), \phi_{-+} \rangle &\geq \{2\pi p a_0 \|\phi_{-+}\|^2 + \frac{1}{2} \sum_i \sum_{\beta, \alpha} \|r(z_i) f_{\beta, \alpha}\|^2 \\ &\quad - C_0 \sum_i \sum_{\beta} \sum_{\alpha, \alpha' \in J^-} \|r(z_i) f_{\beta, \alpha}\| \cdot \|f_{\beta, \alpha'}\|\} \\ &\quad + \frac{1}{2} \sum_i \sum_{\beta, \alpha} \|r(z_i) f_{\beta, \alpha}\|^2 \\ &\geq (2\pi p a_0 - C'_0) \|\phi_{-+}\|^2 + \frac{1}{2} \sum_i \sum_{\beta, \alpha} \|r(z_i) f_{\beta, \alpha}\|^2 \\ &= (2\pi p a_0 - C'_0) \|\phi_{-+}\|^2 + \frac{1}{2} \sum_i \|r(z_i) \phi_{-+}\|^2, \end{aligned} \quad (83)$$

where $C'_0 > 0$ is a constant related to C_0 .

Take $p_0 \in \mathbb{Z}^+$ so that if $p > p_0$ then $C_1 = 2\pi p a_0 - C'_0 > 0$. Thus, for this kind $p \in \mathbb{Z}^+$, we have

$$\langle d_1^2(\phi_{-+}), \phi_{-+} \rangle \geq C_1 \|\phi_{-+}\|^2 + \frac{1}{2} \sum_{i=1}^n \|r(z_i) \phi_{-+}\|^2, \quad (84)$$

for any ϕ_{-+} .

Finally, consider

$$R = \sum_{i,j} r[X'_i, Y_j] \otimes c(X'_i) \otimes c(Y_j) \otimes 1 + \sum_{i,j} r[X''_i, Y_j] \otimes c(X''_i) \otimes c(Y_j) \otimes 1. \quad (85)$$

Since $[X'_i, Y_j], [X''_i, Y_j] \in \mathfrak{z}^\perp$ for any i and j , it follows that R is of the form

$$R = \sum_i r(X'_i) \otimes E'_i + \sum_i r(X''_i) \otimes E''_i, \quad (86)$$

with $E'_i, E''_i \in \text{End}(S)$ for $i = 1, 2, \dots, n$. So there exists a constant $C''_0 > 0$ such that

$$|< R(\phi_{-+}), \phi_{+-} > + < \phi_{+-}, R(\phi_{-+}) >| \leq C''_0 \sum_i \| r(z_i)\phi_{-+} \| \cdot \| \phi_{+-} \|. \quad (87)$$

Then, one has

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^n \| r(z_i)\phi_{-+} \|^2 + < R(\phi_{-+}), \phi_{+-} > + < \phi_{+-}, R(\phi_{-+}) > \\ & \geq \frac{1}{2} \sum_{i=1}^n (\| r(z_i)\phi_{-+} \|^2 - 2C''_0 \| r(z_i)\phi_{-+} \| \cdot \| \phi_{+-} \|) \\ & \geq -\frac{n}{2}(C''_0)^2 \| \phi_{+-} \|^2. \end{aligned} \quad (88)$$

Let $C_2 = \frac{n}{2}(C''_0)^2$, it follows from (84) and (88) that

$$\begin{aligned} & < d_1^2(\phi_{-+}), \phi_{-+} > + < R(\phi_{-+}), \phi_{+-} > + < \phi_{+-}, R(\phi_{-+}) > \\ & \geq C_1 \| \phi_{-+} \|^2 - C_2 \| \phi_{+-} \|^2. \end{aligned} \quad (89)$$

□

Lemma 2. 4 *For a given constant $M > 0$, there exists a constant $C_M > 0$ such that if $(\mu, \alpha) > C_M$ for any $\alpha \in \Psi$, then*

$$< d_2^2\phi_{+-}, \phi_{+-} > \geq M \| \phi_{+-} \|^2, \quad (90)$$

for any ϕ_{+-} .

Proof. First of all, one has

$$(C_c^\infty(N \ltimes G) \otimes S_1^+ \otimes S_2^- \otimes V_\mu)^{Z \times K} \subseteq (C_c^\infty(N \ltimes G) \otimes S_1^+ \otimes S_2^- \otimes V_\mu)^K. \quad (91)$$

Thus, for any $\phi_{+-} \in (C_c^\infty(N \ltimes G) \otimes S_1^+ \otimes S_2^- \otimes V_\mu)^{Z \times K}$, we have $\phi_{+-}(x, \cdot) \in (C_c^\infty(G) \otimes S_1^+ \otimes S_2^- \otimes V_\mu)^K$, for every $x \in N$. So, we can see $\phi_{+-}(x, \cdot)$ as a section of the twisted spinor bundle $G \times_K [S_1^+ \otimes S_2^- \otimes V_\mu] \rightarrow G/K$ associated with the K -module $S_1^+ \otimes V_\mu$. Since the highest weights of all irreducible constituents of $S_1^+ \otimes V_\mu$ are in the form $\mu + \nu$, where ν is a weight of the K -module S_1^+ .

Thus, by Proposition 2.1, one has: For a given constant $M > 0$, there exists a constant $C_M > 0$ such that if $(\mu, \alpha) > C_M$ for any $\alpha \in \Psi$, then

$$\langle d_2^2 \phi_{+-}(x, \cdot), \phi_{+-}(x, \cdot) \rangle \geq M \|\phi_{+-}(x, \cdot)\|^2, \quad (92)$$

for any $x \in N$, where the norm is the L^2 -norm on the sections of above bundle over G/K .

It follows that

$$\begin{aligned} \langle d_2^2 \phi_{+-}, \phi_{+-} \rangle &= \int_N \langle d_2^2 \phi_{+-}(x, \cdot), \phi_{+-}(x, \cdot) \rangle dx \\ &\geq M \int_N \|\phi_{+-}(x, \cdot)\|^2 \\ &= M \|\phi_{+-}\|^2. \end{aligned} \quad (93)$$

□

Now, our main result of this section follows from Lemma 2.3, Proposition 2.2 and Lemma 2.4.

Theorem 2. 1 *There exists $p_0 \in \mathbb{Z}^+$ and $C > 0$ such that if $p \geq p_0$ and $(\mu, \alpha) > C$ for any $\alpha \in \Psi$, then $\text{Ker} D_{\lambda, \mu}^- = 0$.*

Proof. By Proposition 2.2, there exists $p_0 \in \mathbb{Z}^+$ such that if $p \geq p_0$, then

$$\begin{aligned} & \langle d_1^2(\phi_{-+}), \phi_{-+} \rangle + \langle R(\phi_{-+}), \phi_{+-} \rangle + \langle \phi_{+-}, R(\phi_{-+}) \rangle \\ & \geq C_1 \|\phi_{-+}\|^2 - C_2 \|\phi_{+-}\|^2, \end{aligned} \quad (94)$$

where $C_1 > 0$ and $C_2 > 0$ are constants.

For this $C_1 + C_2 > 0$, by Lemma 2.4, we can find $C > 0$ so that if $(\mu, \alpha) > C$ for any $\alpha \in \Psi$, then

$$\langle d_2^2 \phi_{+-}, \phi_{+-} \rangle \geq (C_1 + C_2) \|\phi_{+-}\|^2. \quad (95)$$

Thus, by Lemma 2.3, we have

$$\begin{aligned} \langle D_{\lambda,\mu}^+ D_{\lambda,\mu}^- \phi, \phi \rangle & \geq C_1 \|\phi_{-+}\|^2 - C_2 \|\phi_{+-}\|^2 + (C_1 + C_2) \|\phi_{+-}\|^2 \\ & = C_1 \|\phi\|^2. \end{aligned} \quad (96)$$

It follows that $\text{Ker} D_{\lambda,\mu}^- = 0$. $\square$

Chapter III

Connes-Moscovici's L^2 -Index Theorem And Dirac Induction

In this chapter we shall relate Connes-Moscovici's L^2 -index theorem with Conjecture 1. To start with, we need recall the definition of the topological index of a twisted Dirac operator over a noncompact homogeneous space (full details can be found in [8]).

Let W be a unimodular Lie group with Haar measure dw , L be a compact subgroup of W with the normalized Haar measure dl . Assume $V = \{\xi \in \mathfrak{w}^* \mid \xi|_{\mathfrak{l}} = 0\} = (\mathfrak{w}/\mathfrak{l})^*$ has an $Ad^*(L)$ -invariant metric and fix a L -invariant volume element $\omega \in \wedge^{2n} V$ ($\dim V = 2n$), which determines the orientation of V . Moreover, assume the Riemannian manifold W/L admits a W -invariant Spin-structure i.e., the co-isotropy representation $Ad^* : L \rightarrow So(V)$ lifts to $Spin(V)$ (NOTE: before we have used the isotropy representation Ad of L to define the W -invariant Spin-structure on W/L . Here, in order to have the same notations as the ones in [8], we use the co-isotropy representation Ad^* to define the W -invariant Spin-structure and Dirac operator in the rest of the paper.). Let D_ϵ^+ be a twisted Dirac operator over W/L associated with a unitary representation $\epsilon : L \rightarrow GL(E)$ ($\dim W/L = \dim V = 2n$). Recall that there is a canonical connection (exactly the connection we used to define the Dirac operators) on the principal bundle $L \rightarrow W \rightarrow W/L$, which is determined by an $Ad(L)$ -invariant splitting of $\mathfrak{w} : \mathfrak{w} = \mathfrak{l} \oplus \mathfrak{m}$. The connection form of this connection is given by the projection $\theta : \mathfrak{w} \rightarrow \mathfrak{l}$ parallel to $\mathfrak{m}$, and whose

curvature form Θ is given by

$$\Theta(X, Y) = -\frac{1}{2}\theta([X, Y]), X, Y \in \mathfrak{m}, \quad (97)$$

and then we define $\Theta_\epsilon \in \wedge^2 \mathfrak{m}^* \otimes \mathfrak{gl}_\mathbb{C}(E)$ by the relation

$$\Theta_\epsilon(X, Y) = \frac{1}{2\pi\sqrt{-1}}\epsilon(\Theta(X, Y)), X, Y \in \mathfrak{m}. \quad (98)$$

As explained in [8] that the pullback of $Tr(exp(\Theta_\epsilon)) \in \wedge \mathfrak{m}^*$, via the projection $I - \theta : \mathfrak{w} \rightarrow \mathfrak{m}$, defines an element $ch(\epsilon) \in H^*(\mathfrak{w}, L, \mathbb{R})$, the relative lie algebra cohomology with trivial coefficients. The cohomology class $ch(\epsilon)$ is called the Chern character of the unitary representation $\epsilon : L \rightarrow GL(E)$.

Next, we need an analogue of the Hirzebruch $\hat{a}$ -polynomial. To define it, start from the real L -module V ($Ad^* : L \rightarrow So(V)$), then we can construct $\Theta_V \in \wedge^2 \mathfrak{m}^* \otimes \mathfrak{gl}_\mathbb{C}(V)$ as above. The pullback of the following element

$$det^{1/2} \frac{\Theta_V}{exp(\frac{1}{2}\Theta_V) - exp(-\frac{1}{2}\Theta_V)} \quad (99)$$

to $\wedge \mathfrak{w}^*$ defines an element $\hat{a}(\mathfrak{w}, L) \in H^*(\mathfrak{w}, L, \mathbb{R})$, the Hirzebruch polynomial.

Note that $H^{2n}(\mathfrak{w}, L, \mathbb{R}) \cong \wedge^{2n} V^*$. For $\Omega = \sum_q \Omega^q \in H^*(\mathfrak{w}, L, \mathbb{R})$, we define the scalar $\Omega[V]$ by the relation $\Omega^{2n} = (\Omega[V])\omega$, where ω is the volume form we fixed before. The following definition is in section 4 of [8].

Definition 3. 1 *The topological index $Ind_t D_\epsilon^+$ of the Dirac operator D_ϵ^+ is defined by*

$$Ind_t D_\epsilon^+ = \{ch(\epsilon) \cdot \hat{a}(\mathfrak{w}, L)\}[V]. \quad (100)$$

Now, let $D_{\lambda, \mu}^+$ be the Dirac operator over $N \ltimes G/Z \times K$ defined before.

We identify $(\mathfrak{z}^\perp)^*$ with $V_1 = \{\xi \in \mathfrak{n}^* \mid \xi|_{\mathfrak{z}} = 0\}$ and $\mathfrak{p}^*$ with $V_2 = \{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{k}} = 0\}$.

Assume ω_1 and ω_2 are the corresponding invariant volume forms of $(\mathfrak{z}^\perp)^*$ and $\mathfrak{p}^*$, respectively. Since $\wedge^{\dim V_1 + \dim V_2}(V_1 \oplus V_2) = \wedge^{\dim V_1}(V_1) \wedge \wedge^{\dim V_2}(V_2)$, we can take $\omega = \omega_1 \wedge \omega_2$ to be the $Z \times K$ -invariant volume form on $V_1 \oplus V_2$.

Lemma 3. 1 *If we denote the pullbacks of $ch(\epsilon_\lambda) \in H^*(\mathfrak{n}, Z, \mathbb{R})$ and $ch(\tau) \in H^*(\mathfrak{g}, K, \mathbb{R})$ to $\wedge^*(V_1 \oplus V_2)$ by the same letters, respectively, then the pullbacks $ch(\epsilon_\lambda)$ and $ch(\tau)$ are in the group $H^*(\mathfrak{n} \oplus \mathfrak{g}, Z \times K, \mathbb{R})$. Moreover*

$$ch(\epsilon_\lambda \otimes \tau) = ch(\epsilon_\lambda) \cdot ch(\tau). \quad (101)$$

Proof. First of all, it is easy to see that the pullback of $ch(\tau) \in H^*(\mathfrak{g}, K, \mathbb{R})$ to $\wedge^*(V_1 \oplus V_2)$ is in the group $H^*(\mathfrak{n} \oplus \mathfrak{g}, Z \times K, \mathbb{R})$. In order to prove the pullback of $ch(\epsilon_\lambda) \in H^*(\mathfrak{n}, Z, \mathbb{R})$ to $\wedge^*(V_1 \oplus V_2)$ is in the group $H^*(\mathfrak{n} \oplus \mathfrak{g}, Z \times K, \mathbb{R})$, we only need to verify that $ch(\epsilon_\lambda)$ is K -invariant.

Indeed,

$$\begin{aligned} \Theta_{\epsilon_\lambda}(Ad(k)X, Ad(k)Y) &= \frac{1}{2\pi\sqrt{-1}}\epsilon_\lambda(\Theta(Ad(k)X, Ad(k)Y)) \\ &= \frac{1}{2\pi\sqrt{-1}}\epsilon_\lambda(-\frac{1}{2}\theta(Ad(k)[X, Y])) \\ &= \frac{1}{2\pi\sqrt{-1}}\epsilon_\lambda(-\frac{1}{2}Ad(k)(\theta[X, Y])), \end{aligned} \quad (102)$$

where $X, Y \in \mathfrak{z}^\perp$ and $k \in K$.

It follows that $ch(\epsilon_\lambda) = Tr(exp(\Theta_{\epsilon_\lambda})) \in \wedge(\mathfrak{z}^\perp)^*$ is K -invariant.

Finally, since $[X, Y] \in \mathfrak{z}^\perp$ if $X \in \mathfrak{z}^\perp$ and $Y \in \mathfrak{p}$, it follows that

$$\Theta_{\epsilon_\lambda \otimes \tau}(X, Y) = \begin{cases} \Theta_{\epsilon_\lambda}(X, Y) \otimes 1 & \text{if } X, Y \in \mathfrak{z}^\perp \\ 1 \otimes \Theta_\tau(X, Y) & \text{if } X, Y \in \mathfrak{p} \\ 0 & \text{if } X \in \mathfrak{z}^\perp \text{ and } Y \in \mathfrak{p} \end{cases}.$$

Thus,

$$Tr(exp(\Theta_{\epsilon_\lambda \otimes \tau})) = Tr(exp(\Theta_{\epsilon_\lambda} \otimes 1) \cdot exp(1 \otimes \Theta_\tau))$$

$$= ch(\epsilon_\lambda) \cdot ch(\tau). \quad (103)$$

□

Now, consider the action of K on the space $V_1 \cong (\mathfrak{g}^\perp)^*$, $Ad^* : K \rightarrow So(V_1) \cong So((\mathfrak{g}^\perp)^*)$. Let Θ_0 be the $so(V_1)$ valued K -invariant two form defined as

$$\Theta_0(X, Y) = \frac{1}{2\pi\sqrt{-1}} Ad^*\left(-\frac{1}{2}[X, Y]_{\mathfrak{k}}\right), X, Y \in \mathfrak{p}. \quad (104)$$

Proposition 3. 1 *The topological index $Ind_t D_{\lambda, \mu}^+$ of the Dirac operator $D_{\lambda, \mu}^+$ is given by*

$$Ind_t D_{\lambda, \mu}^+ = Ind_t D_\lambda^+ \cdot \{ch(\tau) \cdot det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)\}[\mathfrak{p}^*]. \quad (105)$$

Proof. Consider the action of $Z \times K$ on the space $V_1 \oplus V_2$, $Ad^* : Z \times K \rightarrow So(V_1) \oplus So(V_2) \subseteq So(V_1 \oplus V_2)$. Since the action of Z is trivial, the action of $Z \times K$ on the space $V_1 \oplus V_2$ is determined by the homomorphism $Ad^* : K \rightarrow So(V_1) \oplus So(V_2)$, the differential of the later is in the form

$$Ad^*(X) = \begin{pmatrix} Ad^*(X)|_{V_1} & 0 \\ 0 & Ad^*(X)|_{V_2} \end{pmatrix},$$

for $X \in \mathfrak{k}$.

Then, it follows that

$$\Theta_{V_1 \oplus V_2}(X, Y) = \frac{1}{2\pi\sqrt{-1}} \begin{pmatrix} Ad^*\{-\frac{1}{2}[X, Y]_{\mathfrak{k}}\}|_{V_1} & 0 \\ 0 & Ad^*\{-\frac{1}{2}[X, Y]_{\mathfrak{k}}\}|_{V_2} \end{pmatrix},$$

for any $X, Y \in \mathfrak{g}^\perp \oplus \mathfrak{p}$.

In turn, this gives

$$\Theta_{V_1 \oplus V_2} = \begin{pmatrix} \Theta_0 & 0 \\ 0 & \Theta_{V_2} \end{pmatrix},$$

where Θ_{V_2} is the curvature two form associated with the K -module $Ad^* : K \rightarrow So(V_2)$.

By Definition 3.1, Lemma 3.1 and the definition of Θ_0 , we see that

$$\begin{aligned}
Ind_t D_{\lambda, \mu}^+ &= ch(\epsilon_\lambda \otimes \tau) \hat{a}(\mathfrak{n} \oplus \mathfrak{g}, Z \times K)[V_1 \oplus V_2] \\
&= ch(\epsilon_\lambda) \cdot ch(\tau) \cdot det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)[V_1 \oplus V_2] \\
&= \{ch(\epsilon_\lambda)[V_1]\} \cdot \{ch(\tau) \cdot det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)\}[V_2].
\end{aligned} \tag{106}$$

Here, we have used the fact that the homomorphism $Ad^* : Z \rightarrow So((\mathfrak{z}^\perp)^*)$ is trivial, so $\hat{a}(\mathfrak{n}, Z) = 1$. $\square$

Next, let's compute the formal degree of the representation $Ker D_\lambda^+ \otimes Ker D_{\dot{\tau}}^+ \cong Ker D_{\lambda, \tau}^{++}$ of the group $N \ltimes G$ (see Theorem 1.1 in Chapter I). As in Chapter I (after Condition 4), to avoid complicated notations, **without loss generality**, we assume $\dot{G} = G$, $\dot{K} = K$, and $\psi = i_d$ (see Chapter I). According to the definition of $\dot{\tau}$ (after Condition 3 and before Lemma 1.9), the highest weight of $\dot{\tau}$ is $\mu + \chi$.

Proposition 3. 2 *Under the same assumptions as Theorem 1.1 plus the condition that μ being far away from all of the root hyperplanes such that $(Ker D_{\dot{\tau}}^+, G)$ is irreducible (see Theorem (9.3) of [4]), one has*

$$d(Ker D_{\lambda, \tau}^{++}) = d(Ker D_\lambda^+) \cdot d(Ker D_{\dot{\tau}}^+), \tag{107}$$

where $d(Ker D_{\lambda, \tau}^{++})$ is the formal degree of the left regular representation $(Ker D_{\lambda, \tau}^{++}, N \ltimes G)$, $d(Ker D_\lambda^+)$ is the formal degree of the left regular representation $(Ker D_\lambda^+, N)$ and $d(Ker D_{\dot{\tau}}^+)$ is the formal degree of the left regular representation $(Ker D_{\dot{\tau}}^+, G)$.

Proof. According to the isomorphism in Theorem 1.1 and the definition of the formal degree, we have

$$\int_{N \ltimes G/Z} | (L(n) \phi \otimes \psi, \phi \otimes \psi) |^2 dndg = \frac{1}{d(Ker D_{\lambda, \tau}^{++})} (\phi, \phi)^2 (\psi, \psi)^2, \quad (108)$$

where $\phi \in Ker D_{\lambda}^{+}$ and $\psi \in Ker D_{\tau}^{+}$.

Note that $L(n) \phi \otimes \psi = \chi(g)^{-1} l_e(n) U_g^{-1} l_e(g) \phi \otimes i_2(g) \psi$, it follows that

$$\begin{aligned} \int_{N \ltimes G/Z} | (L(n) \phi \otimes \psi, \phi \otimes \psi) |^2 dndg &= \int_{N \ltimes G/Z} | (l_e(n) U_g^{-1} l_e(g) \phi, \phi) |^2 \cdot | (i_2(g) \psi, \psi) |^2 dndg \\ &= \int_G | (i_2(g) \psi, \psi) |^2 dg \int_{N/Z} | (l_e(n) U_g^{-1} l_e(g) \phi, \phi) |^2 dn \end{aligned} \quad (109)$$

Since $U_g^{-1} l_e(g) \phi \in Ker D_{\lambda}^{+}$, thus

$$\begin{aligned} \int_{N/Z} | (l_e(n) U_g^{-1} l_e(g) \phi, \phi) |^2 dn &= \frac{1}{d(Ker D_{\lambda}^{+})} (U_g^{-1} l_e(g) \phi, U_g^{-1} l_e(g) \phi) (\phi, \phi) \\ &= \frac{1}{d(Ker D_{\lambda}^{+})} (\phi, \phi)^2. \end{aligned} \quad (110)$$

Then, it is easy to see that

$$\begin{aligned} \int_{N \ltimes G/Z} | (L(n) \phi \otimes \psi, \phi \otimes \psi) |^2 dndg &= \frac{1}{d(Ker D_{\lambda}^{+})} (\phi, \phi)^2 \int_G | (i_2(g) \psi, \psi) |^2 dg \\ &= \frac{1}{d(Ker D_{\lambda}^{+}) \cdot d(Ker D_{\tau}^{+})} (\phi, \phi)^2 (\psi, \psi)^2 \end{aligned}$$

and it follows that

$$d(Ker D_{\lambda, \tau}^{++}) = d(Ker D_{\lambda}^{+}) \cdot d(Ker D_{\tau}^{+}). \quad (112)$$

□

Now, we can state the main result of this paper.

Theorem 3. 1 *Let $N \ltimes G$ be a semidirect product of N and G , where N is a connected and simply connected nilpotent Lie group having square-integrable irreducible representations modulo its center Z and G is a connected noncompact semisimple Lie group with finite center having a maximal compact subgroup K , with $\text{rank} K = \text{rank} G$. Assume Z is central in $N \ltimes G$, V_τ is an irreducible unitary K -module having the highest weight μ , and $\epsilon_\lambda = e^{2i\pi\lambda}$ is the character of Z associated with $\lambda \in \mathfrak{n}_{\text{reg}}^*$. If $D_{\lambda,\tau}^+$ is the Dirac operator over $N \ltimes G/Z \times K$ associated with the irreducible $Z \times K$ -module $\epsilon_\lambda \otimes \tau$, $D_c^+ = D_\lambda^+$ is the Dirac operator over N/Z associated with the Z -character ϵ_λ , D_τ^+ is the Dirac operator over $\dot{G}/\dot{K}$ associated with the irreducible $\dot{K}$ -module $\dot{\tau}$ ($\dot{G}$ and $\dot{K}$ are the (at most) double coverings of G and K , respectively; $\dot{\tau}$ is given after Condition 3 and before Lemma 1.9), the conditions of Satake (Condition 1–Condition 4) are satisfied, and λ and μ satisfy the condition of λ being sufficiently positive and μ being far away from all of root hyperplanes such that*

(a) $\text{Ker} D_{\lambda,\tau}^- = 0$ (see Theorem 2.1 of Chapter II);

(b) $\text{Ker} D_\lambda^- = 0$ and $\text{Ker} D_\lambda^+$ is an irreducible representation of N (see [8]). In fact, this means that Condition 1 is satisfied;

(c) $\text{Ker} D_\tau^- = 0$ and $\text{Ker} D_\tau^+$ is an irreducible representation of $\dot{G}$ (see [4]).

Then, the left regular representation $\text{Ker} D_{\lambda,\tau}^+$ of $N \ltimes G$ is irreducible if and only if

$$\text{Ind}_t D_\tau^+ = \{ch(\tau) \cdot \det^{1/2} \frac{\Theta_0}{\exp(\frac{1}{2}\Theta_0) - \exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)\}[\mathfrak{p}^*]. \quad (113)$$

Moreover, when that is so, we have $\text{Ker} D_{\lambda,\tau}^+ = \text{Ker} D_{\lambda,\tau}^{++} \cong \text{Ker} D_\lambda^+ \otimes \text{Ker} D_\tau^+$.

Proof. Under the conditions in the theorem ((b) and (c)), we have

$$(1) \ d(Ker D_{\lambda}^+) = Ind_t D_{\lambda}^+ \text{ (Connes-Moscovici)};$$

$$(2) \ d(Ker D_{\tau}^+) = Ind_t D_{\tau}^+ \text{ (Atiyah-Schmid)}.$$

It follows from Connes-Moscovici's L^2 -index theorem, condition (a), Proposition 3.1, Proposition 3.2 and the fact that $Ker D_{\lambda, \tau}^{++} \subseteq Ker D_{\lambda, \tau}^+$ is irreducible (Theorem 1.1) that the left regular representation $Ker D_{\lambda, \tau}^+$ of $N \ltimes G$ is irreducible if and only if

$$Ind_t D_{\tau}^+ = \{ch(\tau) \cdot det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)\}[\mathfrak{p}^*]. \quad (114)$$

□

Remark 3. 1 *From our Theorem 2.1, we see that : for generic λ and μ , we do have $Ker D_{\lambda, \tau}^- = 0$. This is an affirmative answer to the second part of Conjecture 1. As the complement, Theorem 3.1 gives a way to determine the irreducibility of the left regular representation $Ker D_{\lambda, \tau}^+$ of $N \ltimes G$, which is the first part of Conjecture 1.*

Chapter IV

Dirac Induction For The Group $H_n \ltimes Sp(n, \mathbb{R})$

In this chapter, we are going to show that Conjecture 1 is true for the group $H_n \ltimes Sp(n, \mathbb{R})$ by verifying the conditions in Theorem 3.1 of Chapter III.

We begin with our discussion by recalling the action of the real symplectic group $Sp(n, \mathbb{R})$ on the Heisenberg group H_n . The multiplication of the group $H_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + x \cdot y'), \quad (115)$$

where $x \cdot y'$ is the natural scalar product of x and y' in $\mathbb{R}^n$. It is easy to see that $Z = \{(0, 0, z) \mid z \in \mathbb{R}\}$ is the center of H_n and the lie algebra $\mathfrak{h}_n$ of H_n is generated by the elements

$$\zeta = (0, 0, 1), \xi_i = (e_i, 0, 0), \eta_i = (0, e_i, 0), \quad (116)$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Moreover, the elements ζ, ξ_i and η_i satisfy the following commutation relations:

$$[\xi_i, \zeta] = [\eta_i, \zeta] = 0, 1 \leq i \leq n; [\xi_i, \xi_j] = [\eta_i, \eta_j] = 0, 1 \leq i, j \leq n; [\xi_i, \eta_j] = \delta_{ij}\zeta, 1 \leq i, j \leq n;$$

where δ_{ij} is the usual Kronecker symbol.

The action of the real symplectic group $Sp(n, \mathbb{R})$ on the Heisenberg group H_n is defined in the following way. Let B be the skew-symmetric bilinear form on $\mathbb{R}^n \times \mathbb{R}^n \times 0$ defined by $B((x_1, x_2), (y_1, y_2)) = x_1 \cdot y_2 - x_2 \cdot y_1$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$.

Then we see that for any $\xi, \eta \in \mathbb{R}^n \times \mathbb{R}^n \times 0 \subseteq \mathfrak{h}_n$, we have $[\xi, \eta] = B(\xi, \eta)\zeta$. It follows that

$$[s\xi, s\eta] = B(s\xi, s\eta)\zeta = B(\xi, \eta)\zeta, \quad (117)$$

for any $s \in \mathbf{Sp}(n, \mathbb{R})$. If we extend $s : \mathbb{R}^n \times \mathbb{R}^n \times 0 \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times 0$ to $s : \mathfrak{h}_n \rightarrow \mathfrak{h}_n$ given by $s|_{\mathfrak{z}=\mathbb{R}} = i_d$, then

$$[s\xi, s\eta] = B(\xi, \eta)\zeta = B(\xi, \eta)s\zeta = s[\xi, \eta], \quad (118)$$

for any $\xi, \eta \in \mathbb{R}^n \times \mathbb{R}^n \times 0$. Thus, $s : \mathfrak{h}_n \rightarrow \mathfrak{h}_n$ is an isomorphism of lie algebras which reduces to the identity on $\mathfrak{z}$ and leave $\mathbb{R}^n \times \mathbb{R}^n \times 0$ invariant as a whole.

Via the exponential map, we got the action of $\mathbf{Sp}(n, \mathbb{R})$ on the group $\mathbf{H}_n$. We denote the semi-direct product of $\mathbf{H}_n$ and $\mathbf{Sp}(n, \mathbb{R})$ (according to the above action) by $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$.

Next, let's consider the invariant spin-structure on the space $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R}) / \mathbf{Z} \times \mathbf{K}$.

Recall that the maximal compact subgroup K of $\mathbf{Sp}(n, \mathbb{R})$ is given by

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in Sp(n, \mathbb{R}) \mid a + \sqrt{-1}b \in U(n) \right\} \cong U(n).$$

According to the decomposition $\mathfrak{sp}(n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, it is well-known that there is an invariant spin-structure on the space $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}$ given by the lifting $\tilde{Ad}^* : K \rightarrow Spin(\mathfrak{p}^*)$ of the homomorphism $Ad^* : K \rightarrow So(\mathfrak{p}^*)$.

Similarly, consider the decomposition $\mathfrak{h}_n = \mathfrak{z}^\perp \oplus \mathfrak{z}$ with respect to the canonical inner product on $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ ($\mathfrak{z}^\perp = \mathbb{R}^n \times \mathbb{R}^n \times 0$). Since $K \subseteq So(2n)$, it follows that above decomposition is the orthogonal decomposition with respect to the K action $Ad : K \rightarrow So(\mathfrak{h}_n)$. It is obvious that there is a

canonical invariant spin-structure on the space $\mathbf{H}_n/\mathbf{Z}$ associated with this inner product.

Note that there exists an lifting $\tilde{Ad}^* : K \rightarrow Spin((\mathfrak{g}^\perp)^*)$ of the homomorphism $Ad^* : K \rightarrow So((\mathfrak{g}^\perp)^*)$, it follows that there is an invariant spin-structure on the space $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$ as we explained in Chapter I.

From above discussions, we can talk about Dirac operators D_λ , D_τ and $D_{\lambda, \tau}$.

Remark 4. 1 *There is an obvious invariant complex structure J_1 on $\mathbf{H}_n/\mathbf{Z}$ (compatible with above invariant spin-structure), which is also K -invariant. Thus, this invariant complex structure is the one used in Satake's paper [21]. Moreover, there is also an invariant complex structure J_2 on $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}$, which is defined by $J_2 : \mathfrak{p} \rightarrow \mathfrak{p}$*

$$J_2 = Ad \left(\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \right),$$

see Helgason [11] for details. It follows that $J_1 + J_2$ is the invariant complex structure on the space $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$ (see [21]).

Now, we briefly discuss the relation between the Dirac operator D and the operator $\bar{\partial} + \bar{\partial}^*$, which will be needed in order to verify the conditions of Satake (Condition 1-Condition 4). It had been shown that if X is a compact Kahler manifold with a spin-structure, then the Dirac operator D is as the same as the operator $\bar{\partial} + \bar{\partial}^*$ with coefficients in the square-root of the canonical line bundle over X (Hitchin [12], Lawson-Michelsohn [16]). It is interesting to find out what it reads for the spaces $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}$, $\mathbf{H}_n/\mathbf{Z}$ and $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$. Let $V = \mathbb{C}^n = \mathbb{R}^{2n}$ with the natural structures. We extend the canonical $Spin(2n)$ -modules $S^\pm$ to

the irreducible $Spin_c(2n) = Spin(2n) \times S^1/\mathbb{Z}_2$ -modules $S_c^\pm$ in the natural way. Considering the covering $\Psi \times \gamma : Spin_c(2n) \rightarrow So(2n) \times S^1$ ($\gamma(x) = x^2$ for any $x \in S^1$), there is a natural lifting $\Pi : U(n) \rightarrow Spin_c(2n)$ of the homomorphism $1 \times det : U(n) \rightarrow So(2n) \times S^1$. Then it was proved in Atiyah-Bott-Shapiro [3] (see also Gilkey [9]) that there is a natural isomorphism between $\Lambda^{0,\pm}$ and $S_c^\pm$ which defines an equivalence of these two representations of $U(n)$ and preserves the Clifford multiplication of V on the left. It follows that the action of $u \in U(n)$ on the $Spin(2n)$ -module $S^\pm$ is the same as the action of $\Pi(u) \cdot det^{-1/2}(u)$ on $\Lambda^{0,\pm}$. From all of these, we have

Lemma 4. 1 *According to above invariant spin-structures and invariant complex structures, one has*

- (a) *The Dirac operator D_τ is the same as $\bar{\partial} + \bar{\partial}^*$ operator over $\mathbf{Sp}(\mathbf{n}, \mathbb{R})/\mathbf{K}$ associated with K -module $\tau \otimes det^{-1/2}$, where $det^{-1/2}$ is the map $det^{-1/2} : K \rightarrow U(\mathfrak{p}^*) \rightarrow S^1$;*
- (b) *The Dirac operator D_λ is the same as $\bar{\partial} + \bar{\partial}^*$ operator over $\mathbf{H}_n/\mathbf{Z}$ associated with Z -character λ ;*
- (c) *The Dirac operator $D_{\lambda,\tau}$ is the same as $\bar{\partial} + \bar{\partial}^*$ operator over $\mathbf{H}_n \ltimes \mathbf{Sp}(\mathbf{n}, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$ associated with $Z \times K$ -module $\epsilon_\lambda \otimes \tau \otimes det^{-1/2}$, where $det^{-1/2}$ is the map $det^{-1/2} : K \rightarrow \{U((\mathfrak{z}^\perp)^*) \oplus U(\mathfrak{p}^*)\} \rightarrow S^1$.*

Now, we consider Satake's conditions. In [21], Satake has verified his Conditions (Condition 1-Condition 4) for $\bar{\partial} + \bar{\partial}^*$ operator over $\mathbf{H}_n \ltimes \mathbf{Sp}(\mathbf{n}, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$. By the close relation between the Dirac operator $D_{\lambda,\tau}$ and $\bar{\partial} + \bar{\partial}^*$ operator given in the above lemma, we see the following

Proposition 4. 1 *The conditions of Satake (Condition 1-Condition 4) for the Dirac operator $D_{\lambda,\tau}$ are satisfied for positive λ (in the sense of [18]). In particular, Condition 3 reads that there is a unique double covering group $(\widehat{Sp(n, \mathbb{R})}, \psi)$ and a function $\chi : \widehat{Sp(n, \mathbb{R})} \rightarrow \mathbb{T}$ trivializing the factor set α . Moreover, $\chi|_{\widehat{K}} = 1$.*

Now, we turn to compute curvature forms of homogeneous vector bundles over symmetric spaces. The similar computation had been given in [7] and [10].

Let G be a semisimple Lie group and K be a maximal compact subgroup of G ($\text{rank} K = \text{rank} G$). As in Chapter III, let $\theta : \mathfrak{g} \rightarrow \mathfrak{k}$ be the connection form of the canonical connection on the principal bundle $K \rightarrow G \rightarrow G/K$, and Θ be the curvature form of this connection. Assume $\{Y_1, Y_2, \dots, Y_r, \dots, Y_m\}$ is a basis of $\mathfrak{g}$ such that $\{Y_1, Y_2, \dots, Y_r\}$ is a basis of $\mathfrak{k}$ and $\{Y_{r+1}, \dots, Y_m\}$ is a basis of $\mathfrak{p}$. Let $\{\phi^1, \dots, \phi^r, \dots, \phi^m\}$ be a dual basis for the left-invariant Maurer-Cartan forms over G . Then Griffiths and Schmid proved that

$$\Theta = -\frac{1}{2} \sum_{i,j=r+1}^m [Y_i, Y_j] \otimes \phi^i \wedge \phi^j. \quad (119)$$

Moreover, this expression for Θ remains true if $\{Y_1, Y_2, \dots, Y_r, \dots, Y_m\}$ is a basis of the complexification $\mathfrak{g}^{\mathbb{C}}$.

Next, let H be a compact Cartan subgroup of K and G , $\mathfrak{h}$ be the lie algebra of H . Let $\mathfrak{h}^{\mathbb{C}}$ be the complexification of $\mathfrak{h}$. For the root system $\Phi = \Phi(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, we fix a particular system of positive roots Ψ , let e_{α} be a root vector for the root $\alpha \in \Phi$, and $\omega^{\alpha} \in (\mathfrak{g}^{\mathbb{C}})^*$ be the dual of $e_{\alpha} \in \mathfrak{g}^{\mathbb{C}}$.

Lemma 4. 2 *If $\varepsilon : K \rightarrow GL(E)$ is a unitary representation of K and the symmetric space G/K is Hermitian, then*

$$\Theta_{\varepsilon} = \frac{-1}{2\pi\sqrt{-1}} \sum_{\alpha, \beta \in \Psi_n} \varepsilon([e_{\alpha}, e_{-\beta}]) \otimes \omega^{\alpha} \wedge \bar{\omega}^{\beta}, \quad (120)$$

where Θ_ϵ is the curvature form associated with K -module E (see Chapter III), Ψ_n is the set of all non-compact positive roots, and $\bar{\omega}^\beta = \omega^{-\beta}$.

Proof. This follows from formula (119) for Θ and the fact that $[c_\alpha, c_\beta] = 0$ for $\alpha, \beta \in \Psi_n$ (Proposition 7 of [11]). $\square$

From now on, we will restrict ourself on the group $\mathbf{Sp}(\mathbf{n}, \mathbb{R})$. Our purpose is to compute $\det^{1/2} \frac{\Theta_0}{\exp(\frac{1}{2}\Theta_0) - \exp(-\frac{1}{2}\Theta_0)}$, where Θ_0 is the curvature form associated with real K -module $(\mathfrak{z}^\perp)^*$ (given by $Ad^* : K \rightarrow \mathbf{So}((\mathfrak{z}^\perp)^*)$ as above).

Recall from [14] and [15]: the complexification lie algebra of $\mathfrak{sp}(n, \mathbb{R})$ is $\mathfrak{sp}(n, \mathbb{C})$, and $(\mathfrak{h}')^\mathbb{C}$ is a Cartan subalgebra given by

$$(\mathfrak{h}')^\mathbb{C} = \left\{ \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \mid h \in \mathfrak{gl}(n, \mathbb{C}) \text{ is a diagonal matrix} \right\},$$

this Cartan subalgebra is the complexification of the lie algebra of the canonical maximal torus inside $\mathbf{Sp}(\mathbf{n})$ (but, not in K). The positive roots for the pair $(\mathfrak{sp}(n, \mathbb{C}), (\mathfrak{h}')^\mathbb{C})$ are given by

$$\Psi = \{e_i - e_j, e_i + e_j \mid i < j\} \cup \{2e_k\}, \quad (121)$$

and the corresponding root vectors are given by

$$\begin{aligned} E_{e_i - e_j} &= E_{i,j} - E_{j+n,i+n}, E_{e_i + e_j} = E_{i,j+n} + E_{j,i+n}, E_{2e_k} = E_{k,k+n} \\ E_{-e_i - e_j} &= E_{i+n,j} + E_{j+n,i}, E_{-2e_k} = E_{k+n,k}, \end{aligned} \quad (122)$$

where $E_{i,j}$ is the matrix with 1 at (i, j) position and 0 at the other positions.

Moreover, set $\Psi_n = \{e_i + e_j \mid i < j\} \cup \{2e_k\}$.

Recall that another Cartan subalgebra, which is the complexification of the lie algebra of a maximal torus inside K , is $\mathfrak{h}^\mathbb{C} \subseteq \mathfrak{sp}(n, \mathbb{C})$, which is defined by

$$\mathfrak{h}^\mathbb{C} = \left\{ \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix} \mid h \in \mathfrak{gl}(n, \mathbb{C}) \text{ is a diagonal matrix} \right\}.$$

The lie algebras $\mathfrak{h}^{\mathbb{C}}$ and $(\mathfrak{h}')^{\mathbb{C}}$ are related by an isomorphism $Ad(w) : (\mathfrak{h}')^{\mathbb{C}} \rightarrow \mathfrak{h}^{\mathbb{C}}$, where $w \in \mathbf{Sp}(\mathfrak{n})$ (Cayley transform).

Then it is not difficult to see that

Corollary 4. 1 *The curvature form Θ_0 associated with real K -module $(\mathfrak{z}^{\perp})^*$ (given by $Ad^* : K \rightarrow So((\mathfrak{z}^{\perp})^*)$ as above) is given by*

$$\Theta_0 = w \left\{ \frac{-1}{2\pi\sqrt{-1}} \sum_{\alpha, \beta \in \Psi_n} [c_{\alpha}, c_{-\beta}] \otimes \omega^{\alpha} \wedge \bar{\omega}^{\beta} \right\} w^{-1}, \quad (123)$$

where $\Psi_n = \{c_i + c_j \mid i < j\} \cup \{2c_k\}$ as we defined above.

Proof. It follows from Lemma 4.2 and the above observations. $\square$

Proposition 4. 2 *For any $m \in \mathbb{Z}^+, m > 0$, we have*

$$Tr \Theta_0^{2m} = 0. \quad (124)$$

Proof. First of all, one sees that $w^{-1}\Theta_0 w$ is in the form

$$w^{-1}\Theta_0 w = \frac{-1}{2\pi\sqrt{-1}} \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega^{tr} \end{pmatrix}$$

So, in order to prove $Tr \Theta_0^{2m} = 0$, it is sufficient to prove $Tr \Omega^{2m} = 0$. In the following, we see $E_{c_i - c_j} = E_{i,j} - E_{j+n,i+n}$ as E_{ij} for computing $Tr \Omega^{2m}$.

Note that

$$\Omega^{2m} = \sum_{\alpha_i, \beta_i \in \Psi_n} \Pi_{i=1}^{2m} [e_{\alpha_i}, e_{-\beta_i}] \otimes \omega^{\alpha_i} \wedge \bar{\omega}^{\beta_i}. \quad (125)$$

It follows that

$$Tr \Omega^{2m} = \sum_{\alpha_i, \beta_i \in \Psi_n} Tr \Pi_{i=1}^{2m} [e_{\alpha_i}, e_{-\beta_i}] \otimes \omega^{\alpha_i} \wedge \bar{\omega}^{\beta_i}. \quad (126)$$

By noting that $E_{ij} \cdot E_{jk} = E_{ik}$, $E_{ij} \cdot E_{j'k} = 0$ if $j \neq j'$ and $[e_{\alpha_i}, e_{-\beta_i}] = e_{\alpha_i - \beta_i}$, we have

$$Tr \Pi_{i=1}^{2m} [e_{\alpha_i}, e_{-\beta_i}] \otimes \omega^{\alpha_i} \wedge \bar{\omega}^{\beta_i} \neq 0 \quad (127)$$

only α_i and β_i satisfy

$$\alpha_1 = e_{j_1} + e_{j'_1}, \beta_1 = e_{j'_1} + e_{j_2}, [e_{\alpha_1}, e_{-\beta_1}] = E_{j_1 j_2};$$

$$\alpha_2 = e_{j_2} + e_{j'_2}, \beta_2 = e_{j'_2} + e_{j_3}, [e_{\alpha_2}, e_{-\beta_2}] = E_{j_2 j_3};$$

.....

$$\alpha_{2m-1} = e_{j_{2m-1}} + e_{j'_{2m-1}}, \beta_{2m-1} = e_{j'_{2m-1}} + e_{j_{2m}}, [e_{\alpha_{2m-1}}, e_{-\beta_{2m-1}}] = E_{j_{2m-1} j_{2m}};$$

$$\alpha_{2m} = e_{j_{2m}} + e_{j'_{2m}}, \beta_{2m} = e_{j'_{2m}} + e_{j_1}, [e_{\alpha_{2m}}, e_{-\beta_{2m}}] = E_{j_{2m} j_1}.$$

Here, we see $e_{e_i - e_i}$ as E_{ii} .

And in above case, we have

$$\begin{aligned} & Tr \Pi_{i=1}^{2m} [e_{\alpha_i}, e_{-\beta_i}] \otimes \omega^{\alpha_i} \wedge \bar{\omega}^{\beta_i} \\ &= Tr(E_{j_1 j_1}) \omega^{\alpha_1} \wedge \bar{\omega}^{\beta_1} \wedge \omega^{\alpha_2} \wedge \bar{\omega}^{\beta_2} \wedge \dots \wedge \omega^{\alpha_{2m}} \wedge \bar{\omega}^{\beta_{2m}} \\ &= \omega^{\alpha_1} \wedge \bar{\omega}^{\beta_1} \wedge \omega^{\alpha_2} \wedge \bar{\omega}^{\beta_2} \wedge \dots \wedge \omega^{\alpha_{2m}} \wedge \bar{\omega}^{\beta_{2m}}. \end{aligned}$$

This is a term in $Tr \Omega^{2m}$.

Now, we consider another term related to above term in $Tr \Omega^{2m}$. Note that

$$[e_{\alpha_1}, e_{-\beta_{2m}}] = e_{\alpha_1 - \beta_{2m}} = E_{j'_1 j'_{2m}};$$

$$[e_{\alpha_{2m}}, e_{-\beta_{2m-1}}] = e_{\alpha_{2m} - \beta_{2m-1}} = E_{j'_{2m} j'_{2m-1}};$$

.....

$$[e_{\alpha_2}, e_{-\beta_1}] = e_{\alpha_2 - \beta_1} = E_{j'_2 j'_1}.$$

It follows that

$$\begin{aligned}
& Tr[[e_{\alpha_1}, e_{-\beta_{2m}}][e_{\alpha_{2m}}, e_{-\beta_{2m-1}}] \dots [e_{\alpha_2}, e_{-\beta_1}](\omega^{\alpha_1} \wedge \bar{\omega}^{\beta_{2m}}) \wedge (\omega^{\alpha_{2m}} \wedge \bar{\omega}^{\beta_{2m-1}}) \wedge \dots \wedge (\omega^{\alpha_2} \wedge \bar{\omega}^{\beta_1}) \\
&= (\omega^{\alpha_1} \wedge \bar{\omega}^{\beta_{2m}}) \wedge (\omega^{\alpha_{2m}} \wedge \bar{\omega}^{\beta_{2m-1}}) \wedge \dots \wedge (\omega^{\alpha_2} \wedge \bar{\omega}^{\beta_1}) \\
&= -(\omega^{\alpha_1} \wedge \bar{\omega}^{\beta_1}) \wedge (\omega^{\alpha_2} \wedge \bar{\omega}^{\beta_2}) \wedge \dots \wedge (\omega^{\alpha_{2m}} \wedge \bar{\omega}^{\beta_{2m}}).
\end{aligned}$$

Thus $Tr\Omega^{2m} = 0$. $\square$

Now, we can state our main theorem in this section.

Theorem 4. 1 *Let $D_{\lambda, \tau}^+$ be the Dirac operator over $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})/\mathbf{Z} \times \mathbf{K}$ associated with the irreducible $Z \times K$ -module $\epsilon_\lambda \otimes \tau$, where τ has a highest weight μ . Assume $D_c^+ = D_\lambda^+$ is the Dirac operator over $\mathbf{H}_n/\mathbf{Z}$ associated with the Z -character ϵ_λ , and $D_{\dot{\tau}}^+$ is the Dirac operator over $\widehat{Sp}(n, \mathbb{R})/\widehat{K}$ associated with the irreducible $\widehat{K}$ module $\dot{\tau}$ ($\dot{\tau}$ is given in Condition 3). Furthermore, suppose λ and μ satisfy the condition of λ being sufficiently positive and μ being far away from all of root hyperplanes such that*

(a) $Ker D_{\lambda, \tau}^- = 0$ (see Theorem 2.1 in Chapter II);

(b) $Ker D_{\dot{\tau}}^- = 0$ and $Ker D_{\dot{\tau}}^+$ is an irreducible representation of $\widehat{Sp}(n, \mathbb{R})$ (see [4]).

Then, the left regular representation $Ker D_{\lambda, \tau}^+$ of $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$ is irreducible. Moreover, $Ker D_{\lambda, \tau}^+ = Ker D_{\lambda, \tau}^{++} \cong Ker D_\lambda^+ \otimes Ker D_{\dot{\tau}}^+$.

Proof. From Proposition 4.2 and the definition of $det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)}$ (see [6]), one sees that

$$det^{1/2} \frac{\Theta_0}{exp(\frac{1}{2}\Theta_0) - exp(-\frac{1}{2}\Theta_0)} = 1. \quad (128)$$

Thus

$$\{ch(\tau) \cdot \det^{1/2} \frac{\Theta_0}{\exp(\frac{1}{2}\Theta_0) - \exp(-\frac{1}{2}\Theta_0)} \cdot \hat{a}(\mathfrak{g}, K)\}[\mathfrak{p}^*] = Ind_t D_\tau^+. \quad (129)$$

On the other hand, $Ind_t D_{\dot{\tau}}^+ = Ind_t D_\tau^+$, which follows from Proposition 4.1 and the definition of $\dot{\tau}$.

It follows from Theorem 3.1 that the left regular representation $Ker D_{\lambda, \tau}^+$ of $\mathbf{H}_n \ltimes \mathbf{Sp}(n, \mathbb{R})$ is irreducible. Moreover, $Ker D_{\lambda, \tau}^+ = Ker D_{\lambda, \tau}^{++} \cong Ker D_\lambda^+ \otimes Ker D_\tau^+$.

□

Remark 4. 2 *Theorem 4.1 shows that Conjecture 1 is true for generic λ and μ .*

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