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MUSCL TYPE LIMITERS FOR FLUX SPLITTING METHODS AND APPLICATIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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1995

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1995

**This is dedicated to My parents,
My Wife Jo-Tsao and Lovely Bo Bo**

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TABLE OF CONTENTS

DEDICATION	ii
ACKNOWLEDGMENTS	iii
VITA	v
LIST OF TABLES	ix
LIST OF FIGURES	x
CHAPTER	PAGE
I Introduction	1
1.1 Background	1
1.2 Objectives of the Present Study	6
1.3 Contents and Organization	7
II Numerical Algorithm and Model Descriptions	10
2.1 Governing Equations	10
2.2 Curvilinear Transformation	13
2.3 Flux Splitting	16
2.3.1 Steger-Warming Splitting	18
2.3.2 van Leer Splitting	20
2.4 Numerical Schemes	21
2.4.1 MUSCL Type Spatial differencing	23
2.4.2 Solution Algorithm	27
2.4.3 Boundary Conditions	28
2.5 Turbulence Model	29

III	Theoretical Study of of MUSCL Methods	32
3.1	TVD conditions for MUSCL methods	33
3.2	Vector Conservation Law for Gas Dynamics	37
3.3	Monotonicity Conditions	39
3.4	NVD analysis	45
3.5	Comparison of Existing Limiters	50
3.5.1	Shock Tube Problem	50
3.5.2	Blunt Body Problem	53
3.6	Development of New Limiters	69
3.7	Test of New Limiters	78
IV	Numerical Applications	84
4.1	Steady Transonic Airfoil	84
4.2	Oscillating Transonic Airfoil	85
4.3	Steady Viscous Airfoil	91
4.4	High Lift Systems	97
4.4.1	Multi-Element Airfoils	97
4.4.2	Oscillating Viscous Airfoils	126
V	Conclusions and Suggestions	140
5.1	Conclusions	140
5.2	Suggestions for future works	143
APPENDICES		
A	Scaling of the Governing Equations	145
B	Stability Analysis for MUSCL Methods	150
BIBLIOGRAPHY		158

LIST OF TABLES

TABLE		PAGE
1	MUSCL type scheme without limiters	48
2	MUSCL type scheme with limiters	49
3	Existing and New limiters	75
4	Lift Coefficient for Racing Car Wings	109

LIST OF FIGURES

FIGURE		PAGE
1	Diagram for MUSCL type spatial distributions in cells	26
2	(a) Monotonicity region(shaded), where, $\alpha \in [-2, 0]$ and M is finite, (b) 2nd order TVD condition(bounded by the dash lines) is in mono- tonicity region with $M=2$ and $\alpha=0$	44
3	A class of limiter functions	46
4	nodal variables in cells (a) original variables (b) normalized variables	47
5	Normalized variable diagrams for (a) MUSCL type interpolation with- out limiters, (b) MUSCL type interpolation with 4 limiters and $\eta = 1/3$	51
6	Computation of the shock tube problem by non-limited schemes, (a) MacCormack scheme+Steger-Warming Splitting, (b) 1st order upwind scheme+van Leer splitting, (c) 2nd order upwind scheme+ Steger- Warming splitting, (d) 2nd order upwind scheme+van Leer splitting, dashed line—computed solutions and solid line—exact solutions	54
7	Computation of the shock tube problem by first-order time accurate Steger-Warming Splitting, dashed line—computed solutions and solid line—exact solutions	55
8	Computation of the shock tube problem by first-order time accurate van Leer Splitting, dashed line—computed solutions and solid line—exact solutions	56

9	Computation of the shock tube problem by second-order time accurate Steger-Warming Splitting, dashed line—computed solutions and solid line—exact solutions	57
10	Computation of the shock tube problem by second-order time accurate van Leer Splitting, dashed line—computed solutions and solid line—exact solutions	58
11	O-type 22x31 grid points for Blunt body flow	60
12	Mach number contours for blunt body flow with Steger-Warming splitting	61
13	Mach number contours for blunt body flow with van Leer splitting . .	62
14	Convergence histories of blunt body problem by first-order time-accurate splittings	63
15	Convergence histories of blunt body problem by second-order time-accurate splittings	64
16	Convergence histories of blunt body problem by first-order time-accurate splittings with new β limiters	65
17	Convergence histories of blunt body problem by second-order time-accurate splittings with new β limiters	66
18	Pressure distribution for blunt body problem along the centerline by first-order time accuracy, (A) new van Albada limiter (B) new β limiter	67
19	Pressure distribution for blunt body problem along the centerline by second-order time accuracy, (A) new van Albada limiter (B) new β limiter	68
20	(a) A general TVD region (bounded by dotted lines), Warming-Beam schemes(solid line), Lax-Wendroff schemes(dashed line) (b) Second order TVD region (bounded by solid lines), superbee limiter(upper bound), minmod limiter (lower bound)	72

21	Normalized variable diagrams of the new β limiters	76
22	The new β limiter functions	77
23	Computation of the shock tube problem by first-order time-accurate Steger-Warming Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions	80
24	Computation of the shock tube problem by first-order time-accurate van Leer Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions	81
25	Computation of the shock tube problem by 2-order time-accurate Steger-Warming Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions	82
26	Computation of the shock tube problem by 2-order time-accurate van Leer Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions	83
27	C-type 259x49 grid points for NACA 0012 airfoils	86
28	Mach number contours for $M_\infty = 0.8$, Angle of Attack of 1.25° , the upper plot is computed using the new β limiter and the lower plot is computed using the new van Albada limiter	87
29	Pressure coefficient distribution for $M_\infty = 0.8$, Angle of Attack of 1.25° , the upper plot is computed using the new β limiter and the lower plot is computed using the new van Albada limiter	88
30	Mach number contours in the first half oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack	92
31	Mach number contours in the second half oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack	93
32	Pressure coefficient distribution in an oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack	94

33	Lift and moment coefficient distributions in an oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack	95
34	O-type grid 249X64 points for NACA 0012 airfoils	96
35	Mach number contours of subsonic flow over two NACA 0012 airfoils, M is Mach number and AoA is Angle of Attack	98
36	Pressure coefficient distributions on two airfoil surfaces, (A) $M_\infty = 0.5$, Angle of Attack of 0° , and (B) $M_\infty = 0.6$, Angle of Attack of 6.48° .	99
37	Geometry of a two-element racing car airfoil	103
38	Grid mesh 391x80 for the multi-element airfoil with a Gurney flap . .	105
39	Grid mesh distributions around the Gurney flap (the lower plot), Grid mesh distributions around the gap between the two-element airfoil (the upper plot)	106
40	Mach number contours for a two-dimensional racing car airfoil with a Gurney flap with Mach number=0.2	107
41	Effect of the overlap size on the lift coefficient of a two-dimensional racing car airfoil with a Gurney flap	108
42	Surface grid meshes 392x80x10 for 3-D racing car airfoil with a Gurney flap	111
43	Surface grid meshes 391x80x10 for 3-D racing car airfoil with a Gurney flap and Side fin	112
44	Mach number contours of the three-dimensional uniform incoming flow case	113
45	Incoming velocity along spanwise direction of the three-dimensional uniform incoming flow case	114

46	Surface Cp distribution in spanwise direction of the three-dimensional uniform incoming flow case	115
47	Surface Cp distribution for the 2D and 3D racing car wings	116
48	Incoming velocity along spanwise direction of the three-dimensional small amplitude non-uniform incoming flow case	117
49	Surface Cp distribution in spanwise direction of the three-dimensional small amplitude non-uniform incoming flow case	118
50	Mach number contours of a racing car wing surface at 0.01 chord length height for the three-dimensional large amplitude non-uniform incoming flow case	119
51	Mach number contours of cross section plane for the three-dimensional large amplitude non-uniform incoming flow case	120
52	Mach number contours of cross section plane of the three-dimensional large amplitude non-uniform incoming flow case	121
53	Mach number contours of cross section plane for the three-dimensional large amplitude non-uniform incoming flow case	122
54	Surface U profiles of the three-dimensional large amplitude non-uniform incoming flow case	123
55	Surface Cp distribution in spanwise direction for the three-dimensional large amplitude non-uniform incoming flow case	124
56	Spanwise loading on the three-dimensional racing car wings, AoA is Angle of Attack	125
57	Mach number contours in an oscillation for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=10+5 \sin(\omega t)$, $k=0.2$, AoA is Angle of Attack	133
58	Surface pressure coefficient distribution for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=10+5 \sin(\omega t)$, $k=0.2$, AoA is Angle of Attack	134

59	Lift and moment coefficient hysteresis vs angle of attack.	135
60	Mach number contours in an first half oscillating for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack	136
61	Mach number contours in an second half oscillating for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack	137
62	Surface pressure coefficient distribution for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack	138
63	Lift and moment coefficient hysteresis vs angle of attack.	139

CHAPTER I

Introduction

1.1 Background

Recently, much effort has been directed toward the development of high resolution algorithms for the conservation form of hyperbolic equations which permit discontinuities to be captured as weak solutions of the governing equations. The goal of these efforts is to improve resolution of discontinuities without spurious numerical oscillations. A systematic analysis of high resolution schemes has been conducted by many researchers. Previously, Lax [1] showed that weak solutions of hyperbolic conservation laws can be obtained if both the governing equations and the numerical approximations are written in conservation form. Lax and Wendroff [2] further proved that the solutions of the discretized equations in the conservation form can satisfy the Rankine-Hugoniot equations across a discontinuity automatically. However, weak solutions are not uniquely determined by their initial values and not all discontinuous solutions have physical meanings. With the assumptions of inviscid and non heat-conducting flow, the Euler equations admit non-physically correct expansion shock solutions with a negative jump in entropy in reversible flow. According to the second law of the thermodynamics, the entropy decrease in expansion waves cannot occur in

real flow. Lax [3] suggested that an additional condition called an entropy condition is needed in numerical schemes to select the physically relevant solutions for all inviscid flow models. However, spurious oscillatory solutions are widely seen in second-order numerical approximation even if the physically correct discontinuities are selected. A criterion to prevent unwanted numerical oscillations is the monotonicity criterion suggested by van Leer [4] and total variation diminishing (TVD) condition introduced by Harten[5]. It is well known that a first-order monotonic upwind scheme will converge to the correct solutions without oscillations arising from large numerical dissipation and monotonicity effects. For the simulation of complex flowfield structures, large dissipation errors of first-order upwind schemes tend to smear out the discontinuities with a reasonable mesh spacing. It is essential that high resolution algorithms required in the current simulation of fluid dynamic problems maintain higher-order accuracy in spatial differencing. These schemes should also contain monotonicity and TVD conditions.

As discussed by Yee [6], the development of high-order shock capturing schemes can be divided into two classes. The traditional one is to use linear numerical dissipation terms with adjustable parameters. In the utilization of linear dissipation terms one cannot avoid spurious oscillations and numerical instability when the discontinuities are encountered which cause nonphysical solutions. In the development of the modern high resolution schemes, the nearly parameter-free nonlinear numerical dissipation is used to allow the correct discontinuities to be predicted without any spurious oscillation and loss of accuracy. Van Leer [4] has developed non-linear correc-

tion factors, called limiters, which control the gradients of computed solutions. The resulting algorithms are second-order monotonicity preserving schemes. To maintain the monotonicity properties of time evolution of the discrete solutions, Harten[7] first proposed the concept of total variation diminishing conditions. A numerical solution is said to be of bounded total variation or total variation stable if the total variation is uniformly bounded in time and space. All the first-order TVD schemes are monotonicity preserving and generate monotonic shock profiles. Harten also designed a second-order accurate TVD scheme incorporating limiters, which are based on the ratios of consecutive solutions, to control spurious oscillations around the shock waves. However, not all the TVD schemes satisfy entropy conditions.

The methodology of Harten for the development of the second-order TVD schemes is summarized as followings;

- (1) Extend a first-order monotonic numerical flux to second-order accuracy.
- (2) Restrict the amplitude of the flow properties by means of limiters to obey the TVD conditions.
- (3) Add the additional entropy condition, if necessary.

As noted by Yee [8, 9], one class of successful high-resolution schemes includes the MUSCL(monotonic upstream-centered schemes for conservation laws) type schemes. The MUSCL schemes can be considered as projection-evolution schemes [10] which are comprised of a numerical stage(projection) and a physical stage(evolution). In the numerical stage, the variables in numerical fluxes are interpolated at cell surfaces by a linear or parabolic polynomial upwind-biased differencing from the neighboring nodal

values. A limiter is usually used to impose constraints on the slope of the polynomial to reduce spurious numerical oscillations. TVD conditions of Sweby [11, 12] and monotonicity criterion developed by Spekreijse [13] for limiters are considered in this stage. The monotonicity conditions of the limiter functions derived by Spekreijse show that there is no contradiction between monotonicity and second-order accuracy in one or in more dimensions. The TVD conditions of Sweby for limiters in one dimension are contained in the criterion of monotonicity of Spekreijse. In the physical stage, the numerical flux can be evaluated by either Flux-Difference Splitting(FDS) of Roe [14] and Osher-Chakaravarthy [15] or Flux Vector Splitting(FVS) of Steger-Warming [16] and van Leer [17].

The most successful flux difference splitting is Roe's scheme which captures any local discontinuity in each mesh level without numerical dissipation. However, an entropy condition is needed to avoid non-physical expansion shocks. In addition, a strong shock wave in multi-dimensional calculations may cause numerical instability in Roe's scheme even if the entropy condition is applied. However, the entropy condition is not necessary in the efficient FVS schemes. Steger and Warming introduced upwinding into systems of hyperbolic conservation laws which are based on the representation of the flux vector as the sum of the positive and negative components such that one can apply forward- and backward-differencing to the Jacobian matrices, respectively. They made use of the flux-vector splitting for a second-order upwind scheme without limiters. The drawback of the Steger-Warming splitting is that the split flux vectors are not differentiable when the eigenvalues pass through zero, such

as at sonic points. This leads to the formation of small glitches or oscillations. Later, van Leer constructed a flux-vector splitting for a perfect gas in terms of low-order polynomials of the local one-dimensional Mach number, which gives continuous eigenvalues at sonic points or stationary points. The splitting of van Leer has the desirable property that one of the eigenvalues of the split-flux Jacobian is identically zero. Numerical results of van Leer show that there are noticeably better results around sonic points and sharper shock transitions than can be obtained with the Steger-Warming splitting. The MUSCL approach with limiters used in the FVS schemes to avoid oscillations around sonic points was suggested by Anderson, et al [18] and has been widely used in current CFD applications. The MUSCL type flux-split algorithms have led to several comparisons including inviscid flows studied by Anderson et al. and Kroll et al. [19], and viscous flows investigated by van Leer et al. [20], von Lavante[21] and Gaitonde [22]. These comparisons have shown that a disadvantage of flux vector splitting in viscous flow codes is the excessive numerical dissipation near discontinuities and shear layers. The numerical dissipation significantly degrades resolution of surface flow properties. Hanel [23, 24] modified the van Leer splitting to preserve total enthalpy for the steady-state solutions. Hanel's modification of the van Leer splitting obtains accurate total-enthalpy values in the boundary layer. This has been observed for the low-speed flow simulation in Hanel's studies. Hanel's modification has been extended by van Leer for use in one-sided upwinding for the energy flux in multi-dimensional high speed calculations. The pressure oscillations around strong shock waves are not avoided in this modification. In this study, only the low-speed

viscous flow simulation is studied. Hanel's modification of the van Leer splitting is used in the current Navier-Stokes codes.

1.2 Objectives of the Present Study

In the study of the influence of limiters for high resolution schemes, only Sweby [11] compared TVD schemes with different limiters and Yee [9] discussed the choices of limiters used in several high resolution schemes for high-speed perfect and real gases. Less extensive investigation of the influence of limiters on the accuracy and stability of MUSCL type flux splitting methods has been conducted by other researchers. A systematic study of the accuracy and stability of limiters is necessary for high resolution schemes.

First, Leonard's NVD (normalized variable diagram) [25], which is a plot of the locally normalized convected control volume face variable with respect to the normalized adjacent upstream node variable, is used to investigate stability and accuracy of limiters for MUSCL methods with the splitting of Steger-Warming and van Leer. Numerical comparisons of the effects of limiters for the resolution of discontinuities and convergence rate are performed on the one-dimensional shock tube and the two-dimensional high speed blunt body flow problems.

Second, new formulation of limiters for MUSCL methods are developed using the TVD conditions of Sweby [11, 12], monotonicity criterion developed by Spekreijse [13] and NVD analysis. Modifications to existing limiters are developed in which the averaging function like that in the van Albada limiter is used in the smooth regions of the solution. A first order upwind differencing is used in discontinuity regions.

Numerical validation of the new limiters are tested on one-dimensional shock tube and two-dimensional blunt body flow problems.

Finally, the MUSCL type upwind scheme with Hanel's modification of the van Leer splitting is applied to several numerical applications. A second order explicit Runge-Kutta scheme developed by Jameson [26] is used for the temporal discretization. Numerical validation includes application to steady inviscid transonic and viscous subsonic airfoil problems. Numerical applications are also extended to aerodynamic analysis of high lift systems including two and three-dimensional stationary multi-element racing car wings and an oscillating NACA 0012 airfoil. The prediction of surface loading is the main issue in these problems.

1.3 Contents and Organization

The remaining chapters of this thesis are organized as follows;

Chapter II describes the governing equations for time-dependent two-dimensional flow of an ideal gas formulated in conservation and nondimensional form. The normalization of governing equations are derived in Appendix A. The transformation of the governing equations from Cartesian coordinates to general curvilinear coordinates is introduced. A finite volume version of the second-order explicit Runge-Kutta time-marching schemes is adopted to keep time-accuracy. In spatial differencing, the MUSCL method with limiters is used to determine variables in numerical fluxes. Numerical functions are evaluated by Steger-Warming and van Leer splitting. Hanel's modification of the van Leer splitting is used to avoid excessive numerical dissipation in viscous flow problems. The numerical boundary conditions for inviscid and vis-

cous flow are also discussed. The Baldwin-Lomax model is chosen in turbulent flow simulation.

Chapter III introduces the development of TVD conditions , NVD analysis and monotonicity criterion for high resolution schemes. The TVD conditions derived by Harten and formulated by Sweby for limiters of non-MUSCL methods are applied to the stability analysis of the MUSCL method of van Leer. The stability requirement of MUSCL type limiters in the explicit Euler scheme is proposed for a scalar hyperbolic conservation law. The derivation is shown in Appendix B. The requirement derived for stability of MUSCL-type limiters is also suitable for a system of hyperbolic conservation laws. An important property for the limiter functions is that the second-order TVD condition is contained in monotonicity criterion of Spekreijse for multi-dimensional MUSCL type finite volume methods.

A comparative study of the influence of limiter functions on the resolution of discontinuous solutions and convergence rate to steady-state solutions is shown. New limiters are developed using the second-order TVD condition, NVD analysis. These are tested in numerical experiments for a one-dimensional shock tube problem and a two-dimensional high speed blunt body flow problem.

Chapter IV describes numerical applications of MUSCL type upwind codes on several fluid dynamics problems. Numerical validation is performed for steady inviscid transonic and viscous subsonic airfoil problems. The prediction of surface pressure coefficients is compared with experimental data or previously validated computations. Numerical applications are also extended to aerodynamic analysis of high lift sys-

tems including stationary multi-element racing car airfoils and an oscillating NACA 0012 airfoil. Numerical prediction of lift and moment coefficients is one of the main objectives in these studies.

Chapter V gives conclusions and suggestions for future works.

CHAPTER II

Numerical Algorithm and Model Descriptions

2.1 Governing Equations

In two dimensions, the equations of fluid motion in conservative nondimensional form are

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \left(\frac{\partial F_v}{\partial x} + \frac{\partial G_v}{\partial y} \right) \quad (2.1)$$

where,

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u (e + p) \end{bmatrix}, \quad G = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v v + p \\ \rho v (e + p) \end{bmatrix}$$

Euler equations are recovered by eliminating the viscous fluxes. For a viscous flow, the viscous flux vector F_v, G_v are represented as:

$$F_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ R \end{bmatrix},$$

$$G_v = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ S \end{bmatrix}$$

with

$$\tau_{xx} = \frac{1}{Re}((\lambda + 2\mu)u_x + \lambda v_y) \quad (2.2)$$

$$\tau_{xy} = \frac{1}{Re}\mu(u_x + v_y) \quad (2.3)$$

$$\tau_{yy} = \frac{1}{Re}((\lambda + 2\mu)v_y + \lambda u_x) \quad (2.4)$$

$$R = u\tau_{xx} + v\tau_{xy} + \alpha \frac{1}{Re} \frac{\partial a^2}{\partial x} \quad (2.5)$$

$$S = u\tau_{xy} + v\tau_{yy} + \alpha \frac{1}{Re} \frac{\partial a^2}{\partial y} \quad (2.6)$$

$$\alpha = \mu Pr^{-1}(\gamma - 1)^{-1} \quad (2.7)$$

where ρ , u , v , p , a and e are the density, velocity components along x -, y - direction, static pressure and sound of speed, respectively;

The total energy per unit volume is given by

$$e = \frac{p}{(\gamma - 1)} + \frac{\rho(u^2 + v^2)}{2} \quad (2.8)$$

This system is completed with an equation of state

$$p = p(\rho, \epsilon) \quad (2.9)$$

For the case of a perfect gas,

$$p = (\gamma - 1)\rho\epsilon \quad (2.10)$$

where ϵ is the internal energy per unit mass and γ is the ratio of specific heats. μ , γ , a , Pr , Re are the molecular viscosity, ratio of specific heat, speed of sound, Prandtl

number and Reynolds number, respectively. Stokes hypothesis $\lambda + \frac{2}{3}\mu = 0$ is assumed.

The molecular viscosity is determined by Sutherland's law as follows:

$$\mu = \left(\frac{T}{T_\infty}\right)^{3/2} \left(\frac{T_\infty + C_T}{T + C_T}\right) \quad (2.11)$$

where T is temperature and C_T for air is 110.4K.

An important and useful technique in numerical approximations is the scaling of the governing equations. Generally speaking, scaling means reformulating the mathematical model with nondimensional variables. This is especially important when comparing variables of different order of magnitude. The ideal nondimensional variables should be of the same order, which can reduce truncation errors when performing calculations. The characteristic parameters such as Mach number, Reynolds number and Prandtl number can be varied independently.

The choice of scaling parameters is arbitrary. In this study, the variables are scaled as:

$$\begin{aligned} \bar{x} &= \frac{x}{L} \quad \bar{y} = \frac{y}{L} \quad \bar{t} = \frac{at}{L} \\ \bar{\rho} &= \frac{\rho}{\rho_\infty} \quad \bar{u} = \frac{u}{a_\infty} \quad \bar{v} = \frac{v}{a_\infty} \\ \bar{T} &= \frac{T}{T_\infty} \quad \bar{e} = \frac{e}{\rho_\infty a_\infty^2} \quad \bar{p} = \frac{p}{\rho_\infty a_\infty^2} \\ \bar{\tau} &= \frac{\tau}{\rho_\infty a_\infty^2} \quad \bar{\mu} = \frac{\mu}{\mu_\infty} \quad \bar{Re} = \frac{\rho_\infty l a_\infty}{\mu_\infty} \end{aligned} \quad (2.12)$$

where ∞ refers to free stream quantities. Assuming L is the reference length. Details of the nondimensionalization can be found in Appendix A. For simplicity, the * has been dropped in (2.1) to (2.8).

2.2 Curvilinear Transformation

In order to solve flow problems with arbitrary geometries, it is convenient to transform the governing equations from Cartesian coordinates to general curvilinear coordinates where

$$\xi = \xi(x, y, t)$$

$$\eta = \eta(x, y, t)$$

$$\tau = t \tag{2.13}$$

The transformation is chosen so that the grid spacing in curvilinear space is uniform. This produces a computational space ξ and η which is a rectangular domain and which has a regular uniform mesh. There will be a one to one correspondence between a point in physical space and in the computational domain.

Using the chain rule, we obtain

$$\begin{bmatrix} \partial_t \\ \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} 1 & \xi_t & \eta_t \\ 0 & \xi_x & \eta_x \\ 0 & \xi_y & \eta_y \end{bmatrix}, \begin{bmatrix} \partial_\tau \\ \partial_\xi \\ \partial_\eta \end{bmatrix} \tag{2.14}$$

Then the curvilinear derivatives in terms of Cartesian derivatives yield:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \xi_t \frac{\partial}{\partial \xi} + \eta_t \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} \quad (2.15)$$

where

$$\xi_x = J y_\eta \quad \xi_y = -J x_\eta \quad \xi_t = -x_\tau \xi_x - y_\tau \xi_y$$

$$\eta_x = -J y_\xi \quad \eta_y = J x_\xi \quad \eta_t = -x_\tau \eta_x - y_\tau \eta_y \quad (2.16)$$

The inverse Jacobian is

$$J^{-1} = (x_\xi y_\eta - x_\eta y_\xi) \quad (2.17)$$

Using the above transformation, the governing equations can be written in conservation law form in curvilinear coordinates as

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0. \quad (2.18)$$

where

$$F = F_{in} - \frac{1}{Re} F_v \quad (2.19)$$

$$G = G_{in} - \frac{1}{Re} G_v \quad (2.20)$$

and the flux vector Q , the inviscid flux vector F_{in} , G_{in} and the viscous flux vector F_v, G_v are represented as

$$Q = J^{-1} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix},$$

$$F = J^{-1} \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ \rho u(e + p)U - \xi_t p \end{bmatrix},$$

$$G = J^{-1} \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ \rho v(e + p)V - \eta_t p \end{bmatrix}$$

With the contravariant velocities:

$$U = \xi_t + \xi_x u + \xi_y v$$

and

$$V = \eta_t + \eta_x v + \eta_y v \quad (2.21)$$

the viscous terms are

$$F_v = J^{-1}(\xi_x \bar{F}_v + \xi_y \bar{G}_v) \quad (2.22)$$

$$G_v = J^{-1}(\eta_x \bar{F}_v + \eta_y \bar{G}_v) \quad (2.23)$$

$$\bar{F}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ R \end{bmatrix},$$

$$\bar{G}_v = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ S \end{bmatrix}$$

with

$$\tau_{xx} = \mu(4(\xi_x u_\xi + \eta_x u_\eta) - 2(\xi_y v_\xi + \eta_y v_\eta))/3$$

$$\tau_{xy} = \mu(\xi_y u_\xi + \eta_y u_\eta + \xi_x v_\xi + \eta_x v_\eta)$$

$$\tau_{yy} = \mu(-2(\xi_x u_\xi + \eta_x u_\eta) + 4(\xi_y v_\xi + \eta_y v_\eta))/3$$

$$R = u\tau_{xx} + v\tau_{xy} + \alpha(\xi_x \frac{\partial a^2}{\partial \xi} + \eta_x \frac{\partial a^2}{\partial \eta})$$

$$S = u\tau_{xy} + v\tau_{yy} + \alpha(\xi_y \frac{\partial a^2}{\partial \xi} + \eta_y \frac{\partial a^2}{\partial \eta})$$

$$\alpha = \mu Pr^{-1}(\gamma - 1)^{-1} \quad (2.24)$$

2.3 Flux Splitting

A hyperbolic system of conservation law equations can be written in quasi-linear form

as

$$\frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} + B \frac{\partial Q}{\partial y} = 0 \quad (2.25)$$

where A and B are the Jacobian Matrices

$$A = \frac{\partial F}{\partial Q} \quad (2.26)$$

$$B = \frac{\partial G}{\partial Q} \quad (2.27)$$

With the perfect gas assumption (2.10), the inviscid equations of gasdynamics have the rather remarkable property that the nonlinear flux vector $F(U)$ is a homogeneous function of degree one in U ; ie. $F(aU) = a F(u)$ for any value a . Because F satisfies the homogeneous property and A has a complete set of linearly independent eigenvectors, then the flux vector F can be split into subvectors. Each subvector is associated with a set of eigenvalues. The eigenvalues associated with a subvector can be positive or negative, and similarly for the flux G .

The flux vector F can be split into subvectors as

$$F = F^+ + F^- \quad (2.28)$$

where F^+ corresponds to the subvector associated with the positive eigenvalues of A , and F^- corresponds to the subvector associated with the positive eigenvalues of A .

The splitting can be derived as

$$F = AU = PAP^{-1} \quad (2.29)$$

where the diagonal elements of Λ are called eigenvalues λ

$$\lambda = \lambda^+ + \lambda^- \quad (2.30)$$

$$\lambda^+ = \frac{\lambda + |\lambda|}{2}, \lambda^- = \frac{\lambda - |\lambda|}{2} \quad (2.31)$$

Using the above formula, we split the diagonal matrix

$$\Lambda = \Lambda^+ + \Lambda^- \quad (2.32)$$

where Λ^+ and Λ^- have the diagonal elements λ^+ and λ^- , respectively. Hence F can be split as

$$\begin{aligned} F &= P(\Lambda^+ + \Lambda^-)P^{-1}U \\ &= (A^+ + A^-)U \\ &= F^+ + F^- \end{aligned} \quad (2.33)$$

and similarly for the flux G .

Several splitting procedures are possible. Two well known flux vector splitting approaches are described in the following sections.

2.3.1 Steger-Warming Splitting

The generalized flux-vector form of F is written in terms of the eigenvalues, λ_i , as

$$F = \rho \begin{bmatrix} 2(\gamma - 1)\lambda_1 + \lambda_3 + \lambda_4 \\ 2(\gamma - 1)\lambda_1 u + \lambda_3(u + a) + \lambda_4(u - a) \\ 2(\gamma - 1)\lambda_1 v + \lambda_3 v + \lambda_4 v \\ 2(\gamma - 1)\lambda_1(u^2 + v^2) + \frac{\lambda_3}{2}[(u + a)^2 + v^2] + \frac{\lambda_4}{2}[(u - a)^2 + v^2] + W \end{bmatrix} \quad (2.34)$$

where

$$W = \frac{(3 - \gamma)(\lambda_3 + \lambda_4)a^2}{2(\gamma - 1)}$$

with

$$\lambda_1 = u, \lambda_3 = u + a, \lambda_4 = u - a$$

The determination of the forward and backward flux vector components F^+ and F^- can be obtained from (2.34) by inserting $\lambda_i^\pm$ for subsonic flow $|M_\infty| < 1$

Following suggestions of Anderson, et al [18], oscillations caused by eigenvalues which pass through zero can be overcome by the addition of a small parameter ϵ , where $10^{-7} \leq \epsilon \leq 10^{-6}$, to redefine positive and negative eigenvalues as

$$\lambda_i^\pm = \frac{\lambda_i \pm (\lambda_i^2 + \epsilon^2)^{1/2}}{2} \quad (2.35)$$

For sonic and supersonic flow,

$$F^+ = F, F^- = 0 \text{ for } M_\infty \geq 1 \quad (2.36)$$

For subsonic flow,

$$F^- = F, F^+ = 0 \text{ for } M_\infty \leq 1 \quad (2.37)$$

and similarly for the flux G.

2.3.2 van Leer Splitting

For sonic and supersonic flow, the evaluation of $F^\pm$ remained unchanged. For subsonic flow, the $F^\pm$ values suggested by van Leer are given in terms of the local one-dimensional Mach number $M_\infty = u/a$ as

$$F^\pm = \begin{bmatrix} f_\infty^\pm \\ f_\infty^\pm [(\gamma - 1)u \pm 2a]/\gamma \\ f_\infty^\pm v \\ f_\infty^\pm [(\gamma - 1)u \pm 2a^2/2(\gamma^2 - 1) + v^2/2] \end{bmatrix} \quad (2.38)$$

where

$$f_\infty^\pm = \pm \rho a \left[\frac{1}{2} (M_\infty \pm 1) \right]^2$$

Fluxes in van Leer splitting are continuously differentiable at the sonic and stagnation points. There are noticeably better results around sonic points and sharper shock transitions than can be obtained with the Steger-Warming splitting. The splitting of G can be obtained in a similar manner in terms of the local one-dimensional Mach number $M_y = v/a$

A drawback of the prescribed flux vector splitting noted by van Leer [20] is excessive numerical dissipation in the boundary layer. The numerical dissipation significantly deteriorates the prediction of surface quantities. Hanel [23, 24] modified

the van Leer splitting to preserve total enthalpy for the steady-state solutions. Hanel claims the modification can give accurate total-enthalpy values in the boundary layer. This has been observed for the low-speed flow in Hanel's studies. The modified van Leer splitting has been further extended by van Leer by using one-sided upwinding for the energy flux in multi-dimensional high speed calculations. The pressure oscillations around the strong shock waves cannot be avoided in this modification. In this study, only the low-speed viscous flow simulation is studied. The modifications suggested by Hanel for the van Leer splitting can be summarized as

$$f_x^\pm = \pm \rho a \left[\frac{1}{4} (M_x \pm 1)^2 - \frac{1}{4} (M_x^2 - 1)^2 \right] \quad (2.39)$$

$$\begin{aligned} f_{mom}^\pm &= f_x^\pm v + \frac{\rho a^2}{r} \left[\frac{1}{4} (M_x \pm 1)^2 (2 \mp M_x) \right. \\ &\quad \left. \mp \frac{3}{16} (M_x^2 - 1)^2 \right] \end{aligned} \quad (2.40)$$

$$f_{energy}^\pm = f_x^\pm H \quad (2.41)$$

2.4 Numerical Schemes

The governing equations have been transformed for use on a general curvilinear coordinate system. The Euler/Navier-Stokes equations are solved numerically by a finite volume approach in this study. The computational domain in a curvilinear system is discretized by contiguous cells of unit length. The definition of a conservation law requires that the net flux across all the faces be balanced by the time rate of change

of the properties inside the cell. Assume Q to be uniform over the entire cell, and the fluxes F and G to be uniform across each corresponding side of cells.

The governing equations can be discretized as

$$\begin{aligned} \frac{\partial Q_{i,j}}{\partial \tau} \Delta \xi \Delta \eta + (F_{i+1/2,j} - F_{i-1/2,j}) \Delta \eta \\ + (G_{i,j+1/2} - G_{i,j-1/2}) \Delta \xi = 0 \end{aligned} \quad (2.42)$$

Since $\Delta \xi = \Delta \eta = 1$, the discretization further becomes

$$\begin{aligned} \frac{\partial Q}{\partial \tau} + (F_{i+1/2,j} - F_{i-1/2,j}) \\ + (G_{i,j+1/2} - G_{i,j-1/2}) = 0 \end{aligned} \quad (2.43)$$

The numerical approximations of uniform flux across each corresponding side of cells can be estimated by the spatial differencing suggested by Anderson et al. [18] such as

$$\begin{aligned} F_{i+1/2} &= F_{i+1/2}^+ + F_{i+1/2}^- \\ &= F^+(Q_{i+1/2,j}^L) + F^-(Q_{i+1/2,j}^R) \end{aligned} \quad (2.44)$$

F^+ and F^- at the interfaces are obtained by extrapolating nodal point values of Q toward the interfaces. This leads to the following approximation:

$$\frac{\partial F}{\partial x} = \frac{1}{\Delta x} [F^+(Q_{i+1/2,j}^L) - F^+(Q_{i-1/2,j}^L) + F^-(Q_{i+1/2,j}^R) - F^-(Q_{i-1/2,j}^R)] \quad (2.45)$$

where $F^\pm$ are evaluated at Q^R and Q^L . The values of Q^R and Q^L are determined by MUSCL type interpolation. The splitting of Steger-Warming and van Leer are used to evaluate the numerical fluxes $F_{i+1/2,j}, F_{i-1/2,j}, G_{i,j+1/2}, G_{i,j-1/2}$.

2.4.1 MUSCL Type Spatial differencing

One of the successful high-order schemes with TVD properties is the MUSCL approach of van Leer [27]. High-order spatial accuracy can be achieved by the introducing five points in the schemes. van Leer observed that the high-order spatial accuracy in the Godunov's schemes [28] can be obtained by replacing the piecewise constant initial data of the Riemann problem with the piecewise linear initial data. It is sufficient to modify the first projection stage without modifying the Riemann solver to generate higher spatial approximation. The state variables at the interfaces can be obtained from an extrapolation between the neighboring cell averages.

The spatial interpolation can be chosen as first, second or even third-order accuracy. The different interpolations contain the different dissipation and dispersion properties. If monotonicity is not conserved in spatial differencing, the interpolated values at cell interfaces may cause spurious numerical solutions around discontinuities in the flowfield. Thus an analysis of the accuracy of the interpolation functions is necessary.

The MUSCL method of van Leer is chosen in this study. The interpolated discrete variables are representative of the average state within the cells. The piecewise linear or quadratic distributions are usually used to describe these values. The general local representation in a cell i is shown in Figure 1 and is expressed as

$$Q(x) = Q_i + \frac{1}{\Delta x}(x - x_i)\delta_i Q + \frac{3\kappa}{2\Delta x^2}[(x - x_i)^2 - \frac{\Delta x^2}{12}]\delta_i^2 Q \quad (2.46)$$

where, $x_{i-1/2} < x < x_{i+1/2}$, Q_i is the average value defined by

$$Q_i = \frac{1}{\Delta x} \int_{i-1/2}^{i+1/2} Q(x) dx \quad (2.47)$$

and $\delta_i Q$, $\delta_i^2 Q$ are the estimates of the first and second derivatives within cell i . For $3\kappa=1$, (2.46) becomes a Taylor expansion up to third-order accuracy and the parabolic representation then generates a third-order accurate spatial discretization. With other values of the parameter κ , the above representation is considered to be linear with various truncation terms.

In order to define completely the representation, the derivatives $\delta_i Q$ and $\delta_i^2 Q$ have to be estimated. Because these gradients are required to rely on quantities of adjacent cells, a central difference of averaged values is used as follows

$$\delta_i Q = \frac{Q_{i+1} - Q_{i-1}}{2} \quad (2.48)$$

$$\delta_i^2 Q = Q_{i+1} - 2Q_i + Q_{i-1} \quad (2.49)$$

and setting $x = x_i \pm \Delta x/2$ in (2.46), a widely used MUSCL-type spatial differencing is obtained. The Q values at the cell interfaces are given by

$$Q_{i+1/2,j}^R = Q_{i+1,j} - \frac{1}{4}[(1 - \eta)\tilde{\Delta}_- + (1 + \eta)\tilde{\Delta}_+]_{i+1,j}$$

$$Q_{i+1/2,j}^L = Q_{i,j} + \frac{1}{4}[(1-\eta)\bar{\Delta}_+ + (1+\eta)\bar{\Delta}_-]_{i,j} \quad (2.50)$$

and similarly for $Q_{i-1/2,j}$, $Q_{i,j+1/2}$ and $Q_{i,j-1/2}$.

Here the spatial order of accuracy is determined by the value of η :

$\eta = -1$, fully upwind scheme

$\eta = 0$, Fromm scheme

$\eta = 1/3$, third-order upwind-biased scheme

$\eta = 1$, three-point central-difference scheme

In general, the basic idea of designing a high-order TVD scheme is to introduce non-linear limiting function which imposes constraints on the gradients of the dependent variable Q . The addition of limiters are widely used in MUSCL methods. Several well known limiters chosen in this study are:

(a) minmod limiter of Roe [12]

$$\bar{\Delta}_+ = \text{minmod}(\Delta_+, \Delta_-)_{i,j} \quad (2.51)$$

$$\bar{\Delta}_- = \text{minmod}(\Delta_-, \Delta_+)_{i+1,j} \quad (2.52)$$

where

$$\text{minmod}(x, \omega y) = \max\{0, \min[|x|, \omega y]\} \quad (2.53)$$

Figure 1: Diagram for MUSCL type spatial distributions in cells

where $s=\text{sgn}(x)$ denotes the sign of the variable x . The minmod function produces zero when x and y have different signs and chooses the smaller one from x and y if they have the same sign.

(b) van Albada's limiter [29]:

$$\tilde{\Delta}_+ = \frac{\Delta_-[\Delta_+^2 + \rho] + \Delta_+[(\Delta_-^2 + \rho)]}{\Delta_+^2 + \Delta_-^2 + 2\rho} \quad (2.54)$$

where $10^{-7} \leq \rho \leq 10^{-6}$

(c) superbee limiter of Roe [12]:

$$\begin{aligned} \tilde{\Delta}_+ &= s \max[0, \min(2 |\Delta_-|, s\Delta_-), \\ &\quad \min(|\Delta_+|, 2s\Delta_-)]; \\ s &= \text{sgn}(\Delta_+) \end{aligned} \quad (2.55)$$

where

$$(\Delta_+)_{i,j} = Q_{i+1,j} - Q_{i,j}$$

$$(\Delta_-)_{i,j} = Q_{i,j} - Q_{i-1,j}$$

2.4.2 Solution Algorithm

The solution scheme used here is a finite volume version of the second-order explicit Runge-Kutta time-marching scheme suggested by Jameson [26] as follows:

$$\begin{aligned}\frac{Q_{i,j}^{n+1} - Q_{i,j}^n}{\Delta\tau_{i,j}} &= [(F_{i+1/2,j}^n - F_{i-1/2,j}^n) + (G_{i,j+1/2}^n - G_{i,j-1/2}^n)] \\ &= P(Q^n)\end{aligned}$$

$$Q^1 = Q^n - \Delta\tau P(Q^n)$$

$$Q^2 = Q^n - \frac{\Delta\tau}{2}(P(Q^n) + P(Q^1))$$

$$Q^{n+1} = Q^n - \frac{\Delta\tau}{2}(P(Q^n) + P(Q^2)) \quad (2.56)$$

where

$$\Delta\tau_{i,j} = \frac{CFL}{\max |\lambda_{i,j}|} \quad (2.57)$$

Following Jameson's stability criteria, the CFL value is less than 2.

2.4.3 Boundary Conditions

One of the most important factors in determining the success or failure of computations is proper specification of initial data and boundary conditions. Once the PDE is approximated by numerical methods, it may be higher order than the PDE. The numerical approximation will require additional numerical boundary conditions. Basically, boundary conditions (BC) are divided into two categories; surface BC's and

far field BC's. For solid surface BC's, no flow is permitted across the surface. Along the body surface $\eta(x, y, t) = 0$, the condition of tangency is used in inviscid flow as $V=0$; or

$$\begin{pmatrix} u \\ v \end{pmatrix} = J^{-1} \begin{bmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{bmatrix} \begin{pmatrix} U - \xi_t \\ -\eta_t \end{pmatrix}$$

while in viscous flow, the no slip condition is specified, $U=0$ as well. For fixed solid surfaces, ξ_t and η_t terms vanish. The pressure on the body surface can be obtained from the normal momentum equation

$$\begin{aligned} & \rho[\partial_\tau \eta_t + u \partial_\tau \eta_x + v \partial_\tau \eta_y] - \rho U(\eta_x u_\xi + \eta_y v_\xi) \\ & = (\eta_y \xi_x + \xi_x \eta_y) p_\xi + (\eta_x^2 + \eta_y^2) p_\eta = P_n \sqrt{\eta_x^2 + \eta_y^2} \end{aligned} \quad (2.58)$$

The far field boundaries are usually set to freestream values.

2.5 Turbulence Model

In this study, the Baldwin-Lomax [30] turbulence model is used. This is a two-layer, zero equation model. It is based on a modification of Cebeci-Smith's mixing-length model [31] which uses the velocity and length scale from algebraic relations. It is one of the most commonly used turbulence models in CFD applications and is used only for wall-bounded flows. Inner and outer layer formulations are used to determine the turbulent viscosity over the flowfield. The model divides the eddy viscosity into two parts:

$$\mu_t = \begin{pmatrix} (\mu_t)_{inner} & \text{for } y \leq y_{crossover} \\ (\mu_t)_{outer} & \text{for } y \geq y_{crossover} \end{pmatrix} \quad (2.59)$$

where y is the normal distance from the wall and $y_{crossover}$ is the smallest value of y such that $(\mu_t)_{inner} = (\mu_t)_{outer}$. In the inner layer, the length scale is evaluated by Prandtl's mixing-length model and van Driest's damping function as

$$(\mu_t)_{inner} = \rho l^2 \omega \quad (2.60)$$

with

$$l = ky[1 - \exp(-y^+/A^+)]$$

and

$$|\omega| = [(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})^2]^{0.5}$$

and

$$y^+ = \frac{(\rho_w \tau_w)^{1/2} y}{\mu_w}$$

where τ_w is the shear stress at the wall and $A^+=26$.

In the outer layer, the eddy viscosity is given by

$$(\mu_t)_{outer} = KC_{cp} \rho F_{wake} F_{kleb}(y) \quad (2.61)$$

with

$$F_{wake} = \text{Min}(y_{max} F_{max}, C_{wk} y_{max} U_{diff}^2 / F_{max})$$

The value of F_{max} is determined from the function $F(y)$ as

$$F(Y) = Y|\omega|(1 - e^{-Y^{+}/28})$$

and y_{max} is the value of y at which F_{max} occurs.

The Klebanoff intermittency correction is defined as

$$F_{kleb}(y) = [1 + 5.5(\frac{C_{kleb}^6}{y_{max}})]^{-1}$$

and

$$U_{diff} = (u^2 + v^2)_{max}^{1/2} - (u^2 + v^2)_{min}^{1/2}$$

Where $C_{cp} = 1.6$, $C_{kleb} = 0.25$, $C_{\omega k} = 0.25$, $k=0.4$ and $K=0.0168$.

CHAPTER III

Theoretical Study of of MUSCL Methods

It has been seen that the evaluation of cell interface values such as $Q_{i\pm 1/2}^R$ and $Q_{i\pm 1/2}^L$ in the MUSCL type spatial differencing could be first-, second- or third-order accurate. However, MUSCL methods may not be sufficient to avoid the generation of over- and undershoots in resolution of discontinuities. The addition of limiter functions in MUSCL methods has been used for removing oscillations, but the use of limiters may severely hamper the convergence to steady state solutions. The accuracy and stability of limiters has not been adequately addressed in the current literature. In this section, a systematic investigation of stability and accuracy of MUSCL type limiters is performed using TVD conditions, monotonicity and NVD analysis. Improving limiters is also studied.

It is known that the first-order upwind schemes, which contain monotonicity and TVD properties, have been widely used success in simple gas dynamics problems. Large dissipative errors of the first-order upwind schemes tend to smear out discontinuities in complicated fluid dynamics problems. Therefore the development of second-order upwind schemes is necessary in the present numerical simulations. One of the successful efforts to design a high-order numerical scheme is based on the monotonicity criterion for second-order accurate finite volume upwind methods. This

monotonicity criterion of Spekreijse shows that there is no contradiction between monotonicity and second-order accuracy in one or more dimensions. A comparative study of existing limiters and the development of new limiters based on TVD conditions, monotonicity criterion of Spekreijse and NVD analysis are described in the following sections.

3.1 TVD conditions for MUSCL methods

First, the TVD schemes developed by Harten[5] is introduced for the scalar hyperbolic conservation law.

In one spatial dimension, a scalar hyperbolic equation in conservation form is

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad (3.1)$$

The spatial coordinate is denoted by x , the dependent variable is u . Time is denoted by t , the flux is represented by f and the characteristic speed is $a(u) = \frac{\partial f}{\partial u}$.

A general three-point explicit difference scheme for (3.1) is

$$u_i^{n+1} = u_i^n - \lambda(h_{i+1/2}^n - h_{i-1/2}^n) \quad (3.2)$$

where λ is the mesh ratio $\frac{\Delta t}{\Delta x}$. $\Delta x = x_{i+1} - x_i$ is assumed to be uniform, and $u_i = u(i\Delta x, t)$. $h_{i+1/2}$ and $h_{i-1/2}$ are Lipschitz continuous, and

$$h_{i+1/2}^n = h(u_i, u_{i+1}) \quad (3.3)$$

where

$$h(u, u) = f(u)$$

A general form suggested by Harten can be rewritten as

$$u_i^{n+1} = u_i^n - \lambda(C_{i+1/2}^- \delta u_{i+1/2} - C_{i-1/2}^+ \delta u_{i-1/2}) \quad (3.4)$$

where, $\delta u_{i+1/2} = u_{i+1} - u_i$ and $\delta u_{i-1/2} = u_i - u_{i-1}$. $C_{i+1/2}^-$ and $C_{i-1/2}^+$ are assumed to be bounded functions and represent the contributions from the waves with positive and negative wave speed, respectively.

In Harten's theory of TVD, the numerical solution u_i^n is said to be bounded total variation stable if the total variation is uniformly bounded in t and Δx as follows;

$$TV(u^{n+1}) \leq TV(u^n) \quad (3.5)$$

or

$$\frac{d}{dt} TV(u^n) \leq 0 \quad (3.6)$$

where the total variation (TV) in x of a discrete solution to a scalar conservation law is defined by

$$TV(u) = \sum |u_{i+1} - u_i| \quad (3.7)$$

The TVD conditions for (3.2) and (3.4), which have been derived by Harten [7] and generalized by Jameson and Lax [32] for explicit three-point schemes, to satisfy

(3.5) or (3.6) are

$$C_{i+1/2}^+ \geq 0$$

$$C_{i+1/2}^- \geq 0 \quad (3.8)$$

Furthermore, the development of TVD theory by Harten can be further extended to a one-parameter family of five-point difference schemes in conservation form as

$$u_i^{n+1} + \lambda\theta(h_{i+1/2}^{n+1} - h_{i-1/2}^{n+1}) = u_i^n - \lambda(1 - \theta)(h_{i+1/2}^n - h_{i-1/2}^n) \quad (3.9)$$

and

$$h_{i+1/2}^n = h(u_{i-1}, u_i, u_{i+1}) \quad (3.10)$$

where

$$h(u, u, u) = f(u)$$

This one-parameter family of schemes contains implicit as well as explicit schemes. When $\theta = 0$, (3.9) is an explicit method. When $\theta = 1/2$, (3.9) is the trapezoidal implicit formula. When the $\theta = 1$, (3.9) is the backward Euler method. In order to check the TVD property of (3.9), this one-parameter family of schemes can be rewritten in another form.

$$\begin{aligned} & u_i^{n+1} - \lambda\theta(C_{i+1/2}^- \delta u_{i+1/2} - C_{i-1/2}^+ \delta u_{i-1/2})^{n+1} \\ = & u_i^n + \lambda(1 - \theta)(C_{i+1/2}^- \delta u_{i+1/2} - C_{i-1/2}^+ \delta u_{i-1/2})^n \end{aligned} \quad (3.11)$$

with the same definitions of $C_{i+1/2}^-$, $C_{i-1/2}^+$ and λ . in (3.4).

If (3.9) is called a TVD scheme, Harten gives sufficient TVD conditions as

$$\begin{aligned}\lambda(1 - \theta)C_{i+1/2}^\pm &\geq 0 \\ \lambda(1 - \theta)(C_{i+1/2}^- + C_{i+1/2}^+) &\leq 1\end{aligned}\tag{3.12}$$

The additional CFL-like TVD constraint for the explicit Euler method ($\theta=0$) is

$$\lambda(C_{i+1/2}^+ - C_{i+1/2}^-) \leq 1\tag{3.13}$$

As discussed by Shu and Osher [33, 34], (3.8) and (3.13) can be used for the stability criterion of not only the Euler explicit method($\theta = 0$) but also the Runge-Kutta methods. These TVD conditions are chosen for the stability analysis of MUSCL type limiters in this study.

In general, the basic idea of designing a high-order TVD scheme is to introduce a non-linear limiting function which imposes constraints on the gradients of the dependent variable u . In Sweby's work [11, 12] for TVD methods, a limiter $\Psi(r)$ is assumed to be a function of the ratio of consecutive gradients in the scalar hyperbolic equation and defined in Appendix B. By means of $\Psi(r)$, a stability analysis can be performed on the linear wave equation in one dimension, which is a model for the Euler equations, as

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0\tag{3.14}$$

where x is the spatial coordinate, u is velocity. t is time and a is the constant wave speed which is non-negative.

In the stability requirements for the MUSCL method with limiters shown in Appendix B, a TVD condition for the limiter is given by

$$0 \leq (\Psi(r)/r, \Psi(r)) \leq 2 \quad (3.15)$$

The stability requirement for explicit Euler type MUSCL methods using TVD conditions is

$$\begin{aligned} CFL &= \lambda|a| \\ &\leq \frac{2}{2+\delta} \end{aligned} \quad (3.16)$$

where δ is the upper bound of the TVD condition for each limiter and $0 \leq \delta \leq 2$.

3.2 Vector Conservation Law for Gas Dynamics

In one spatial dimension, a hyperbolic system of gas dynamic equations in the conservation form is

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad (3.17)$$

where,

$$U = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u(e + p) \end{bmatrix}$$

Time is denoted by t , the spatial coordinate is denoted by x , and the primitive variables are velocity u , pressure p and density ρ . The total energy per unit volume is denoted by e and the internal energy per unit mass ϵ is given by

$$e = p\epsilon + \frac{\rho u^2}{2} \quad (3.18)$$

This system is completed with an equation of state

$$p = p(\rho, \epsilon) \quad (3.19)$$

For the case of a perfect gas,

$$p = (\gamma - 1)\rho\epsilon \quad (3.20)$$

where γ is the ratio of specific heats.

This system can be written in quasi-linear form as

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \quad (3.21)$$

where A is the Jacobian Matrix:

$$A = \frac{\partial F}{\partial U} \quad (3.22)$$

(3.21) is hyperbolic at the point (x, t, U) if there exists a similarity transformation such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_l & \\ & & \ddots \\ 0 & & & \lambda_m \end{bmatrix} \quad (3.23)$$

where Λ is a diagonal matrix, the eigenvalues λ_l of A are real, and the norms of P and P^{-1} are uniformly bounded.

Assuming the coefficient matrix A is constant. (3.21) can be transformed into the uncoupled system

$$\frac{\partial u_l}{\partial t} + \lambda_l \frac{\partial u_l}{\partial x} = 0 \quad (3.24)$$

with $l=1,2,3,\dots,m$ which is in the form of a linear wave equation for each l . It is thus seen that the stability requirements (3.15) can be applied to the system hyperbolic conservation law.

3.3 Monotonicity Conditions

The main contribution of Spekreijse [13] is a monotonicity criterion for limiter functions used in second-order-accurate flux-split upwind schemes. It is shown that there is no contradiction between monotonicity and second-order accuracy in one or more dimensions. In the present study, the monotonicity criterion derived by Spekreijse can be expressed in the same form as the TVD conditions of Sweby for limiters in one-dimensional problems. The stability criterion of Spekreijse for second-order monotonic upwind schemes for multi-dimension problems are described as follows.

The nonlinear hyperbolic conservation law in two dimensions is given by

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} = 0 \quad (3.25)$$

Suppose that the fluxes F and G can be split in positive and negative parts, which are defined in (2.34), with the positive derivatives of F^+ and G^+ and the negative derivatives of F^- and G^- .

Then (3.25) can be discretized as

$$\begin{aligned} Q_{i,j}^{n+1} = & Q_{i,j}^n + \Delta\tau_{i,j} [\\ & A_{i+1/2,j}^n (Q_{i+1,j}^n - Q_{i,j}^n) + \\ & B_{i,j+1/2}^n (Q_{i,j+1}^n - Q_{i,j}^n) + \\ & C_{i-1/2,j}^n (Q_{i,j}^n - Q_{i-1,j}^n) + \\ & D_{i,j-1/2}^n (Q_{i,j}^n - Q_{i,j-1}^n)] \end{aligned} \quad (3.26)$$

where $\Delta\tau_{i,j}$ is defined in (2.57).

A scheme is called monotonic if it has positive, uniformly bounded coefficients, ie.,

$$A_{i+1/2,j}^n, B_{i,j+1/2}^n, C_{i-1/2,j}^n, D_{i,j-1/2}^n \geq 0 \quad (3.27)$$

and there exists a bound $G \geq 0$ such that

$$A_{i+1/2,j}^n, B_{i,j+1/2}^n, C_{i-1/2,j}^n, D_{i,j-1/2}^n \leq G \quad (3.28)$$

Therefore, a scheme is called monotonic if its solutions satisfy

$$\text{Min}(Q_{i-1,j}, Q_{i+1,j}, Q_{i,j-1}, Q_{i,j+1}) \leq Q_{i,j} \leq \text{Max}(Q_{i-1,j}, Q_{i+1,j}, Q_{i,j-1}, Q_{i,j+1}) \quad (3.29)$$

From (3.26), we obtain

$$Q_{i,j} = \frac{A_{i+1/2,j}Q_{i+1,j} + B_{i,j+1/2}Q_{i,j+1} + C_{i-1/2,j}Q_{i-1,j} + D_{i,j-1/2}Q_{i,j-1}}{A_{i+1/2,j} + B_{i,j+1/2} + C_{i-1/2,j} + D_{i,j-1/2}} \quad (3.30)$$

and if the coefficients $A_{i+1/2,j}$, $B_{i,j+1/2}$, $C_{i-1/2,j}$, $D_{i,j-1/2}$ are positive and bounded, the monotonicity of numerical solutions is proved.

In the same manner, a finite volume discretization of (3.25) is expressed as

$$\begin{aligned} Q_{i,j}^{n+1} = & Q_{i,j}^n + \Delta\tau_{i,j}[(F^+(Q_{i+1/2,j}^L) - F^+(Q_{i-1/2,j}^R)) \\ & + (F^-(Q_{i+1/2,j}^L) - F^-(Q_{i-1/2,j}^R)) \\ & + (G^+(Q_{i+1/2,j}^L) - G^+(Q_{i-1/2,j}^R)) \\ & + (G^-(Q_{i+1/2,j}^L) - G^-(Q_{i-1/2,j}^R))] \end{aligned} \quad (3.31)$$

where

$$Q_{i+1/2,j}^L = Q_{i,j} + 0.5\Psi(R_{i,j})(Q_{i,j} - Q_{i-1,j})$$

$$Q_{i-1/2,j}^R = Q_{i,j} + 0.5\Psi\left(\frac{1}{R_{i,j}}\right)(Q_{i,j} - Q_{i-1,j})$$

$$Q_{i,j+1/2}^L = Q_{i,j} + 0.5\Psi(S_{i,j})(Q_{i,j} - Q_{i,j-1})$$

$$Q_{i,j-1/2}^R = Q_{i,j} + 0.5\Psi\left(\frac{1}{S_{i,j}}\right)(Q_{i,j} - Q_{i,j+1})$$

with Ψ is a continuous function called the limiter.

and

$$R_{i,j} = \frac{(Q_{i+1,j} - Q_{i,j+1})}{(Q_{i,j} - Q_{i-1,j})}$$

and

$$S_{i,j} = \frac{(Q_{i,j+1} - Q_{i,j})}{(Q_{i,j} - Q_{i,j-1})}$$

(3.31) can be rewritten like (3.26) by taking

$$A_{i+1/2,j} = -\Delta\tau_{i,j} \frac{F^-(Q_{i+1/2,j}^R) - F^-(Q_{i-1/2,j}^R)}{Q_{i+1/2,j}^R - Q_{i-1/2,j}^R} \frac{Q_{i+1/2,j}^R - Q_{i-1/2,j}^R}{Q_{i+1,j} - Q_{i,j}} \quad (3.32)$$

$$C_{i-1/2,j} = \Delta\tau_{i,j} \frac{F^+(Q_{i+1/2,j}^L) - F^+(Q_{i-1/2,j}^L)}{Q_{i+1/2,j}^L - Q_{i-1/2,j}^L} \frac{Q_{i+1/2,j}^L - Q_{i-1/2,j}^L}{Q_{i,j} - Q_{i-1,j}} \quad (3.33)$$

$$B_{i,j+1/2} = -\Delta\tau_{i,j} \frac{G^-(Q_{i,j+1/2}^R) - G^-(Q_{i,j-1/2}^R)}{Q_{i,j+1/2}^R - Q_{i,j-1/2}^R} \frac{Q_{i,j+1/2}^R - Q_{i,j-1/2}^R}{Q_{i,j+1} - Q_{i,j}} \quad (3.34)$$

$$D_{i,j-1/2} = \Delta\tau_{i,j} \frac{G^+(Q_{i,j+1/2}^L) - G^+(Q_{i,j-1/2}^L)}{Q_{i,j+1/2}^L - Q_{i,j-1/2}^L} \frac{Q_{i,j+1/2}^L - Q_{i,j-1/2}^L}{Q_{i,j} - Q_{i,j-1}} \quad (3.35)$$

To obtain positive coefficients $A_{i+1/2,j}$, etc; it is sufficient that

$$\begin{aligned}
\frac{Q_{i+1/2,j}^R - Q_{i-1/2,j}^R}{Q_{i+1,j} - Q_{i,j}} &\geq 0 & \frac{Q_{i+1/2,j}^L - Q_{i-1/2,j}^L}{Q_{i,j} - Q_{i-1,j}} &\geq 0 \\
\frac{Q_{i,j+1/2}^R - Q_{i,j-1/2}^R}{Q_{i,j+1} - Q_{i,j}} &\geq 0 & \frac{Q_{i,j+1/2}^L - Q_{i,j-1/2}^L}{Q_{i,j} - Q_{i,j-1}} &\geq 0
\end{aligned} \tag{3.36}$$

Inequality in (3.36) is fulfilled by taking $\Delta\tau_{i,j}$ very small, and assuming uniform bounded derivatives of F^+, F^-, G^+, G^- . Spekreijse derived the following conditions on the limiter $\Psi(R)$ to meet the positivity of bounded coefficients given by (3.27) and (3.28).

The monotonicity condition is satisfied if

$$\alpha \leq \Psi(R) \leq M \tag{3.37}$$

$$-M \leq \frac{\Psi(R)}{R} \leq 2 + \alpha \tag{3.38}$$

where $\alpha \in [-2, 0]$ and M is finite.

The monotonicity region given by (3.37) and (3.38) is depicted in Figure 2 (a).

The $\Psi(R)$ function is derived in Appendix B and is repeated as follows;

Minmod limiter is

$$\Psi(R) = \text{Max}(0, \text{Min}(R, 1)) \tag{3.39}$$

Superbee limiter is

$$\Psi(R) = \text{Max}(0, \text{Min}(2R, 1), \text{Min}(R, 2)) \tag{3.40}$$

Figure 2: (a) Monotonicity region(shaded), where, $\alpha \in [-2, 0]$ and M is finite, (b) 2nd order TVD condition(bounded by the dash lines) is in monotonicity region with $M=2$ and $\alpha=0$

van Albada's limiter is

$$\Psi(R) = \frac{R^2 + R}{R^2 + 1} \quad (3.41)$$

These various $\Psi(R)$ are shown in Figure 3.

3.4 NVD analysis

First, the accuracy and stability of the MUSCL method with limiters are investigated using Leonard's NVD approach. The NVD is normalized variables diagram which is a plot of functional relation between a normalized convected control surface variable such as $\tilde{Q}_f$ and a normalized upstream node variable such as $\tilde{Q}_c$ defined in the following section.

A criterion of accuracy and stability for spatial differencing schemes is shown in the NVD plot. Following the notation used by Leonard, in Figure 4, the backward-biased differencing Q_f is $Q_{i+1/2,j}^L$, (Q_u, Q_c, Q_d) are (Q_{i-1}, Q_i, Q_{i+1}) and the nodal values downstream(d), central(c) and upstream(u) and the face value(f). In the current NVD analysis, only the backward-biased interpolation is considered. In order to equate the MUSCL type interpolation and the NVD method, all variables in the MUSCL type spatial differencing are normalized by

$$\tilde{Q} = \frac{Q - Q_u}{Q_d - Q_u}. \quad (3.42)$$

The functional relation between $\tilde{Q}_f$ and $\tilde{Q}_c$ is represented as a characteristic curve in NVD. The stability and accuracy of the spatial differencing are observed by its

Figure 3: A class of limiter functions

Figure 4: nodal variables in cells (a) original variables (b) normalized variables

Table 1: MUSCL type scheme without limiters

numerical methods	$\tilde{Q}_f$ versus $\tilde{Q}_c$
2-order upwind scheme	$\tilde{Q}_f = 1.5 \tilde{Q}_c$
Fromm scheme	$\tilde{Q}_f = \tilde{Q}_c + 0.25$
3-order upwind scheme	$\tilde{Q}_f = \frac{5}{3} \tilde{Q}_c + \frac{1}{3}$
central scheme	$\tilde{Q}_f = \frac{1}{2} \tilde{Q}_c + \frac{1}{2}$

characteristic curve distribution. The functional relations for MUSCL methods (2.50), in terms of the normalized convected face value $\tilde{Q}_f$ and the normalized adjacent upstream node value $\tilde{Q}_c$, are given in Tables I, II and plotted in Figure 5 (a) and 5 (b).

In Leonard's criteria, it is noted that

- (i) the characteristic curve of a second-order accurate spatial differencing scheme should pass through (0.5,0.75) in the normalized variables diagram.
- (ii) the characteristic curve of a third-order accurate spatial differencing scheme should pass through (0.5,0.75) with a slope of 0.75 in the normalized variables diagram.
- (iii) a spatial differencing scheme may produce nonphysical oscillations if its characteristic curve passes through the second quadrant or above (1,1).
- (iv) a spatial differencing scheme may give artificially dissipative results if its characteristic curve passes the fourth quadrant or below (1,1).
- (v) the non-linear characteristic curve of a third-order accurate spatial differencing

Table 2: MUSCL type scheme with limiters

$\eta = \frac{1}{3}$	$\tilde{Q}_f = \tilde{Q}_c + \frac{ss(\tilde{Q}_c)}{4} (1 + \frac{ss(\tilde{Q}_c)}{3} (1 - 2\tilde{Q}_c))$
minmod limiter	$ss(\tilde{Q}_c) = \minmod(\tilde{Q}_c, 1 - \tilde{Q}_c)$
van Albada's limiter	$ss(\tilde{Q}_c) = \frac{\tilde{Q}_c[(1-\tilde{Q}_c)^2 + \rho] + (1-\tilde{Q}_c)[\tilde{Q}_c^2 + \rho]}{\tilde{Q}_c^2 + (1-\tilde{Q}_c)^2 + 2\rho}$
superbee limiter	$ss(\tilde{Q}_c) = s \max[0, \min(2 \tilde{Q}_c , s(1-\tilde{Q}_c)), \min(\tilde{Q}_c , 2s(1-\tilde{Q}_c))]$; $s = \text{sgn}(\tilde{Q}_c)$

scheme should pass through (0,0), (1,1) and (0.5,0.75) with a slope of 0.75 at (0.5,0.75) in the normalized variables diagram.

In Figure 5 (a), the characteristic curve distributions of the non-limited MUSCL type methods are shown to satisfy criteria (i) and (ii) of Leonard. However, these curves pass through the second quadrant or above (1,1) which means numerical oscillations cannot be avoided in calculations. Obviously, the upwind-biased scheme without limiters is not a good choice for the interpolation function.

In Figure 5 (b), It is shown that the non-linear NVD curves of MUSCL methods with different limiters obey the Leonard's accuracy criteria. A careful examination of the NVD plot reveals the accuracy and stability of each limiter. In particular, in nonmonotonic regions (for $\tilde{Q}_c$ less than 0 or greater than 1), the interpolation functions with minmod type limiters become first-order upwind schemes resulting in increased dissipation effects. This is the same idea as in the TVD type scheme for the treatment of oscillatory regions. Only the van Albada limiter shows slightly different

spatial differencing from the first-order upwind scheme. In the monotonic region of the NVD plot, it is seen that only the averaging function of the van Albada limiter is consistent with Leonard's criteria (v).

In addition, for the superbee limiter, all the values of $\tilde{Q}_f$ in some monotonic regions (around $\tilde{Q}_c$ is greater than 0.75 and less than 1) are equal to 1. It is shown that the convected face value Q_f equals the downstream node value Q_d in the monotonic regions. The superbee limiter loses third-order accuracy in smooth regions. This may explain convergence problems and the small time step limitation in the superbee limiter.

From the NVD analysis, it is seen that the choice of a limiter is important in MUSCL methods. Numerical comparisons of the effects of limiters for the resolution of discontinuities and convergence rate are performed on the one-dimensional shock tube and the two-dimensional high speed blunt body flow problems.

3.5 Comparison of Existing Limiters

3.5.1 Shock Tube Problem

Numerical solutions of a one-dimensional shock-tube flow are chosen to judge the validity of the existing limiters. In the unsteady shock tube problem, a tube of large extent contains a diaphragm separating a perfect gas at rest with different pressures but at uniform temperature. With the rupture of the diaphragm, an expansion wave propagates into the high-pressure gas, while a shock wave, followed by a contact discontinuity, propagates into the low-pressure gas. In the current calculations, the initial pressure and density ratios across the diaphragm are taken as 10 to 1. The

Figure 5: Normalized variable diagrams for (a) MUSCL type interpolation without limiters, (b) MUSCL type interpolation with 4 limiters and $\eta = 1/3$

initial location of the diaphragm is taken at $x=0.5$. Numerical results from the splitting schemes with the limiters are compared with exact solutions from Liepmann and Roshko [35].

In Figures 6 to 10, the density ratio distributions from different methods at $t=0.24$ with $\lambda=0.24$ and CFL number of 0.5 are shown. Numerical solutions of first- and second-order time-accurate upwind and MacCormack type non-limited flux splittings are compared in Figure 6. In Figure 6 (a), the resolution of the MacCormack type scheme contains a large spike in density. Figure 6 (b) shows that the first-order explicit upwind scheme smears out the discontinuities with monotonic profiles. For the results using second-order explicit upwind schemes shown in 6 (c) and 6 (d), numerical accuracy seems to be acceptable with moderate overshoots around the shock and contact discontinuity. The solutions of flux splitting without limiters are not acceptable compared with exact solutions.

In Figures 7 to 10, the density ratio distributions show that the addition of limiters reduces numerical oscillations around contact surfaces and shock waves in first- or second-order time-accurate calculations except for the results for the superbee limiter. The results shown in Figures 7 and 8 are obtained using third-order accuracy in spatial differencing and first-order in time accuracy. The solutions shown in Figures 8 to 10 are obtained using third-order spatial accuracy and second-order time accuracy. It is observed that the superbee limiter is sensitive to the time accuracy. Resolution of the superbee limiter for first-order time accurate results is improved by upgrading time accuracy to second order. However, the solutions of the superbee limiter are

still not monotonic in either the Steger-Warming or the van Leer splitting. The primary influence on the accuracy of the schemes is found to be the limiter function, not the splitting methods. The differences in the results for this one-dimensional case caused by the flux-split algorithms are not obvious. As noted by Yee [6], the minmod limiter is diffusive, the van Albada limiter is less diffusive and the superbee limiter is compressive. From numerical comparisons, we see that the van Albada limiter enhances the resolution of contact discontinuities and shock waves. The results using the minmod limiter are somewhat smeared compared to the exact solutions. In general, upwind methods are considered to be naturally dissipative. Thus the diffusive minmod type limiters are generally not good choices. Less compressive limiters like the van Albada are much better.

3.5.2 Blunt Body Problem

In the high-speed blunt body flow simulation, the freestream Mach number is 8.03. The perfect gas assumption is used and only steady-state solutions are considered in this study. A CFL number of 0.5 is chosen. A 31×22 grid shown in Figure 11 is used for this test. In this study, no special adjustment is used to speed up convergence in the explicit Runge-Kutta schemes.

Mach number contour plots in Figure 12 and Figure 13 show that a bow shock wave in front of the blunt body is captured in the numerical results. However, the discrepancy in the numerical results using different limiters is not obvious in these Mach number contour plots. The influence of the limiters on the stability and convergence is shown in Figure 14 and Figure 15. The results of first-order accurate

Figure 6: Computation of the shock tube problem by non-limited schemes, (a) MacCormack scheme+Steger-Warming Splitting, (b) 1st order upwind scheme+van Leer splitting, (c) 2nd order upwind scheme+ Steger-Warming splitting, (d) 2nd order upwind scheme+van Leer splitting, dashed line-computed solutions and solid line-exact solutions

Figure 7: Computation of the shock tube problem by first-order time accurate Steger-Warming Splitting, dashed line-computed solutions and solid line-exact solutions

Figure 8: Computation of the shock tube problem by first-order time accurate van Leer Splitting, dashed line—computed solutions and solid line—exact solutions

Figure 9: Computation of the shock tube problem by second-order time accurate Steger-Warming Splitting, dashed line-computed solutions and solid line-exact solutions

Figure 10: Computation of the shock tube problem by second-order time accurate van Leer Splitting, dashed line-computed solutions and solid line-exact solutions

calculations using the superbee limiter do not converge using the Steger-Warming or the van Leer splitting. The convergence problem of the superbee limiter is not improved in the second-order time-accurate calculations. The superbee limiter gives remarkable resolution for linear problems, but it results in poor stability properties for the non-linear problems. The minmod-type limiters including minmod and superbee limiters are shown to degrade convergence to steady-state. Only the van Albada limiter results in rapid convergence to steady-state solutions. In a high speed flow simulation, it is seen that the van Albada limiter improves the convergence rate with any order of time accuracy. The stable convergence and good resolution of the van Albada limiter result from its smooth averaging function, which has been observed in the NVD analysis.

In Figures 18 and 19, steady-state pressure distributions along the centerline show the capturing of the bow shock is poor using the minmod limiter, especially in Steger-Warming methods. As discussed by Gaitonde [22], the Steger-Warming scheme contains more numerical dissipation effects than any other splitting does. The van Albada limiter gives consistent shock standoff distances. In addition, it is found that the bow shock is resolved through four grid points using the minmod limiter in the Steger-Warming splitting, but only three points are required using the other limiters. In the current numerical tests, the less diffusive van Albada limiter is seen to be a good choice for MUSCL type approaches.

Figure 11: O-type 22x31 grid points for Blunt body flow

Figure 12: Mach number contours for blunt body flow with Steger-Warming splitting

Figure 13: Mach number contours for blunt body flow with van Leer splitting

Figure 14: Convergence histories of blunt body problem by first-order time-accurate splittings

Figure 15: Convergence histories of blunt body problem by second-order time-accurate splittings

Figure 16: Convergence histories of blunt body problem by first-order time-accurate splittings with new β limiters

Figure 17: Convergence histories of blunt body problem by second-order time-accurate splittings with new β limiters

Figure 18: Pressure distribution for blunt body problem along the centerline by first-order time accuracy, (A) new van Albada limiter (B) new β limiter

Figure 19: Pressure distribution for blunt body problem along the centerline by second-order time accuracy, (A) new van Albada limiter (B) new β limiter

3.6 Development of New Limiters

In the previous section, it is shown that the less diffusive van Albada limiter achieves satisfactory resolution of discontinuities and stable convergence to steady-state solutions. The high resolution of the van Albada limiter resulting from its averaging function is observed by its non-linear characteristic curve obeying Leonard's stability criteria in the monotonic region of the normalized variable diagram(NVD).

In Leonard's stability criteria, in the non-monotonic region, there are no conditions for accuracy and stability in the NVD analysis. In Figure 3, it is observed that the curve of the van Albada limiter is different from the curves of other limiters in the non-monotonic region ($\tilde{Q}_c \leq 0$ or $\tilde{Q}_c \geq 1$). The simple and apparently robust strategy always used in TVD schemes is to set

$$\tilde{Q}_f = \tilde{Q}_c \quad (3.43)$$

for $\tilde{Q}_c \leq 0$ or $\tilde{Q}_c \geq 1$.

This approach is called a first-order upwinding. It does not degrade the accuracy order of the overall algorithm. The order of accuracy, based on the Taylor series, has meaning only for the smooth behavior in the monotonic region which is the region where $0 \leq \tilde{Q}_c \leq 1$. First-order upwinding is widely used in minmod type limiters. Keeping the smooth non-linear characteristic curve of the van Albada limiter in the monotonic region, a modified overall nonlinear NVD characteristic is obtained by continuing with the linear NVD characteristic curve of a first-order upwind scheme (3.43) in the nonmonotonic region. The modified van Albada limiter is given in Table

III and is shown in Figure 3. The new averaging function of the modified van Albada limiter is given by

$$\Delta_+ = \frac{[(\Delta_- \Delta_+ + |\Delta_- \Delta_+|)/2. + \rho](\Delta_+ + \Delta_-)}{\Delta_+^2 + \Delta_-^2 + 2\rho} \quad (3.44)$$

where, $10^{-7} \leq \rho \leq 10^{-6}$ in contrast to that given in (2.54). This gives the modified van Albada limiter which is

$$\Psi(R) = \frac{1/2(|R| + R)(1 + R)}{R^2 + 1}.$$

As mentioned in the Introduction, the high resolution algorithms should contain higher-order spatial accuracy and monotonicity properties. The NVD analysis may not be sufficient to prove the accuracy and stability of the modified limiter. Criteria such as monotonicity and TVD conditions for the development of high resolution schemes are used to examine the modified limiters. These criteria are also expected to improve the accuracy and convergence rate of the minmod type limiters.

The TVD condition for limiters given in (3.15) is

$$0 \leq (\Psi(R)/R, \Psi(R)) \leq 2 \quad (3.45)$$

with $\Psi(R)=0$ for $R \leq 0$, which means that the second-order accuracy must be lost at extrema of the solution. This condition is represented as a dotted line region in Figure 20 (a). The TVD condition (3.45) is a general formulation for any order numerical scheme. Sweby proposed that any second-order scheme relying on the points $(Q_{i-2}, Q_{i-1}, Q_i, Q_{i+1}, Q_{i+2})$ must use a weighted average of the Lax-Wendroff

scheme [2] and the Warming and Beam upwind scheme [36]. A general formulation of the second-order schemes is expressed in terms of limiters as

$$\Psi(R) = (1 - \theta(R))\Psi(R)^{LW} + \theta(R)\Psi(R)^{WB} \quad (3.46)$$

where $0 \leq \theta(R) \leq 1$.

Since

$$\Psi(R)^{LW} = 1, \Psi(R)^{WB} = R \quad (3.47)$$

(3.46) becomes a second-order TVD condition as

$$\bar{\Psi}(R) = 1 + \theta(R)(R - 1), \quad (3.48)$$

and the corresponding region bounded by the solid line is shown in Figure 20 (b). Notice that $\Psi(1)=1$ is contained in this condition which is a general requirement for second-order accuracy. From Figure 2 (b), which is obtained by comparing Figures 2 (a) and 20 (b), an important property of limiter functions can be observed; namely, that the second-order TVD condition is contained in the monotonicity criterion of Spekreijse with $M=2$ and $\alpha \in [-2, 0]$. It is seen that a limiter obeys the second-order accurate TVD condition which also contains monotonicity in multi dimensions. However, just because a limiter satisfies the monotonicity condition does not necessarily mean that the second order TVD condition is obeyed. In Figure 20 (b), a class of limiters can be defined in the second-order TVD region in terms of $\Psi(R)$. These are called β limiters by Sweby, and defined as

Figure 20: (a) A general TVD region (bounded by dotted lines), Warming-Beam schemes (solid line), Lax-Wendroff schemes (dashed line) (b) Second order TVD region (bounded by solid lines), superbee limiter (upper bound), minmod limiter (lower bound)

$$\Psi^\beta(R) = \text{Max}(0, \text{Min}(SR, 1), \text{Min}(R, S)) \quad (3.49)$$

where $1 \leq S \leq 2$.

Sweby showed that the superbee limiter is represented by the β limiter with $S=2$ and the minmod limiter is defined by the β limiter with $S=1$. This general formulation of $\Psi^\beta(R)$ covers all existing MUSCL type limiters as well as the modified van Albada limiter which is shown in Figure 3. As discussed by Sweby, for the β limiters, a higher value of parameter S gives smaller time step requirement and sharper profiles of discontinuities; a lower value of S produces larger numerical dissipation and stable convergence.

From Figure 3, it is seen that the van Albada limiter does not lie in the second-order TVD region when $R \leq 0$. Hence, in a modified van Albada limiter, $\Psi(R) = 0$, for $R \leq 0$ is assumed to force $\Psi(R)$ to satisfy the second-order TVD condition. The modified van Albada limiter which satisfies the TVD conditions and monotonicity with second-order accuracy can be clearly seen in Figure 3.

Numerical experiments show the van Albada limiter improves resolution and gives stable convergence. The less-diffusive property of the van Albada limiter is caused by its smoothness. This means the van Albada function tends to $1/2(\Delta_+ + \Delta_-)$ if Δ_+ and Δ_- are adjacent finite differences of a smooth solution, but tends to the smallest value where the solution is not smooth. This averaging in the van Albada limiter avoids central differencing across a discontinuity in the solution or in its derivative, thus preventing numerical oscillations. The smooth averaging function of the van

Albada limiter has been retained in the modified van Albada limiter. A modification to the β limiters is to add smooth averaging like that in the van Albada limiter. The creation of a new $\Psi(R)$ may be accomplished by fitting the van Albada limiter in Figure 3. This new limiter, different from the the van Albada limiter, has averaging function is given by

$$\tilde{\Delta}_+ = \frac{2\Delta_- \Delta_+ + \rho}{\Delta_+ + \Delta_- + \rho} \quad (3.50)$$

or

$$\Psi^{vl}(R) = \frac{|R| + R}{|R| + 1} \quad (3.51)$$

where, $10^{-7} \leq \rho \leq 10^{-6}$, (3.50) is called the van Leer limiter [11]. Its limitation of small time step and divergence problems have been discussed by Sweby [11, 12]. After combining with the β limiter (3.49), a new formulation of limiters is obtained as

$$\Psi(R) = \text{Min}(\Psi^{vl}, \text{Max}(0, (\text{Min}(SR, 1), \text{Min}(R, S)))) \quad (3.52)$$

where $1 \leq S \leq 2$.

In the form of the MUSCL type spatial differencing (2.50), the new β limiter is expressed as

$$\begin{aligned} \tilde{\Delta}_+ &= sg \text{ Min}(Vl, \text{max}[0, \text{min}(S |\Delta_-|, sg\Delta_-), \\ &\quad \text{min}(|\Delta_+|, Ssg\Delta_-)]) \end{aligned} \quad (3.53)$$

Table 3: Existing and New limiters

limiter	$\Psi(R)$
minmod	$\text{Max}(0, \text{Min}(R, 1))$
superbee	$\text{Max}(0, \text{Min}(2R, 1), \text{Min}(R, 2))$
van Albada	$\frac{R^2+R}{R^2+1}$
new van Albada	$\frac{1/2(R +R)(1+R)}{R^2+1}$
new β	$\text{Min}(\Psi^{\text{vl}}, \text{Max}(0, (\text{Min}(S R, 1), \text{Min}(R, S)))$ with $1 \leq S \leq 2$ and $\Psi^{\text{vl}}(R) = \frac{ R +R}{ R +1}$

where $1 \leq S \leq 2$ and $\text{sg} = \text{sign}(\Delta_+)$

with

$$Vl = \frac{\Delta_- \Delta_+ + |\Delta_- \Delta_+| + \rho}{\Delta_+ + \Delta_- + \rho}$$

where $10^{-7} \leq \rho \leq 10^{-6}$.

In this study, two modified limiters are suggested. Equation (3.44) is modified from the van Albada limiter and (3.53) is obtained by reformulating the van Leer limiter and β limiters with adjustable parameters. The accuracy and stability of new limiters must be investigated. In the Ψ^β limiter functions, there is a parameter S between 1 and 2 to be determined. Numerical experiments are performed on the one-dimensional shock tube and two-dimensional blunt body flow problems, and the results compared with those for existing limiters. The new limiters are shown in terms of the NVD curves and $\Psi(R)$ in Figures 21 and 22. The NVD analysis and

Figure 21: Normalized variable diagrams of the new β limiters

Figure 22: The new β limiter functions

the plots of $\Psi(R)$ show that the curves of the new Ψ^{α} and the new Ψ^{β} with $S=1.2$ are close to the curve for the original van Albada limiter. This represents the new Ψ^{α} and the new Ψ^{β} with $S=1.2$ maintaining the accuracy and stability of the van Albada limiters in the monotonic region. Note that a value of S near 1.2 in the new Ψ^{β} , whose characteristic curve is close to that of $S=1.2$ in the NVD plot, will have the same accuracy and stability properties as $S=1.2$. In the numerical experiments, values of 1., 1.2, 1.5, 1.7 are chosen for S . The $\Psi(R)$ function for various limiters is given in Table 3.

3.7 Test of New Limiters

In the unsteady shock tube problem, computations using the new limiters are shown in Figures 23 to 26. The density distribution computed by the new Ψ^{α} maintains good resolution of discontinuities as does the van Albada limiter for any order of time accuracy. The deterioration of expansion fans and shock waves caused by the superbee limiter is not evident in the results of the class of new Ψ^{β} limiters. For the new β limiters, choosing 1.2 and 1.5 for S gives sharper resolution of discontinuities in the computations using the Steger-Warming and van Leer splitting for any order time-accuracy. It can be seen that the new limiters improve resolution of contact discontinuities, expansion fans and shock waves in the one-dimensional shock tube problem. The results of new Ψ^{β} limiters show improved resolution with S near 1.2.

In the numerical tests of new limiters on the two-dimensional high-speed blunt body flow problem, Mach number contour plots in Figures 12 and 13 show that the bow shock wave in front of the blunt body is properly captured. However, the dis-

crepancy in the numerical results using different limiters is not obvious in these Mach number contour plots. The impressive result is that computations using the new Ψ^{α} and new Ψ^{β} with S of 1.2 and 1.5 reach steady-state solutions with a faster convergence rate than computations with any existing limiter. The convergence histories are shown in Figures 16 and 17. The most rapid convergence is obtained using Ψ^{β} with $S=1.2$ in the Steger-Warming splitting. The effect on convergence rate using the modified van Albada limiter is not as obvious in the results of the van Leer splitting. The smooth convergence process with the modified van Albada limiters is seen in the computations using the Steger-Warming splitting. The addition of the new β limiters clearly improves the calculations of the Steger-Warming splitting and the van Leer splitting. In addition, in Figures 18 and 19, the pressure distribution along the centerline shows most of the bow shock waves resolved by three grid points. The new limiters contain reduced diffusive errors and satisfactory resolution of shock waves. The bow shock is resolved by four grid points in the first-order time-accurate computations using the van Leer splitting. The addition of the new limiters improves calculations with the Steger-Warming splitting more than those with the van Leer splitting. The weaker diffusive property of the two new limiters reduces excessive numerical dissipation effects in the Steger-Warming case. The modified van Albada limiter and the new β limiter with S near 1.2 are shown to have good stability and accuracy with both splitting procedures in the blunt body flow problems.

Figure 23: Computation of the shock tube problem by first-order time-accurate Steger-Warming Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions

Figure 24: Computation of the shock tube problem by first-order time-accurate van Leer Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions

Figure 25: Computation of the shock tube problem by 2-order time-accurate Steger-Warming Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions

Figure 26: Computation of the shock tube problem by 2-order time-accurate van Leer Splitting with new β limiters, dashed line-computed solutions and solid line-exact solutions

CHAPTER IV

Numerical Applications

In order to validate the current MUSCL type upwind codes, several applications are performed on steady and unsteady inviscid and viscous fluid dynamics problems.

4.1 Steady Transonic Airfoil

In the steady-state inviscid flow case, the transonic flow over a NACA 0012 airfoil is studied. The mesh distribution shown in Figure 27 is a 259x49 C grid with no special clustering or adapting on the upper or lower surface near the shock waves. All outer boundaries are 20 chord lengths away from the body. For inviscid flow, on airfoil surfaces, the flow tangency condition is used, ie., the velocity in the normal direction is zero. The pressure on the body surface is obtained from the normal momentum equation. Since there is a steady uniform incoming flow, free stream conditions are specified for the outer boundary conditions. The flow properties of the trailing edge line are obtained by a simple average from the properties of the neighboring lines.

The freestream conditions, which is Mach number of 0.8 and angle of attack of 1.25° , are used as initial conditions. For verification, the numerical results of Pulliam and Steger [37] are used to compare with the current results for the new β limiter with $S=1.2$ and the modified van Albada limiter. Hanel's modification of the van

Leer splitting is used in these calculations. The Mach number contours and C_p distributions shown in Figures 28 and 29. A comparison of C_p distributions shows that the current prediction of the shock waves on the upper and lower surfaces is in good agreement with the numerical results of Pulliam and Steger. The shock wave resolution using the new β limiter is slightly sharper than the result with the modified van Albada limiter.

4.2 Oscillating Transonic Airfoil

In computational aerodynamic analysis, a particularly complicated flow phenomena is encountered at transonic speeds when the airfoil undergoes an unsteady motion of large amplitude, due either to aeroelastic forces or a maneuver. In such a motion, shock waves on the airfoil may change in strength and location. The flow over the airfoil may separate due to angle-of-attack effects or shock strength, and may separate and reattach in a time-dependent fashion. The complexity of such flows requires modeling based on the full Navier-Stokes equations which are difficult to solve. Therefore, it is desirable to look for simplifications. When the flow is well attached, the dominant nonlinearity is due to the shock waves and the viscous effects can be neglected. A simplification is possible if the shock wave is relatively simple across a wing, since a two-dimensional wing prediction may give adequate results. Thus, in the unsteady transonic flow, the simplest modeling is based on two-dimensional attached flow with shock waves. Typically, in flutter calculations, a two-dimensional section model is analyzed. Hence the problem of transonic flow past an oscillating airfoil is a fundamental one and is chosen for this study. The governing equations can

Figure 27: C-type 259x49 grid points for NACA 0012 airfoils

Figure 28: Mach number contours for $M_\infty = 0.8$, Angle of Attack of 1.25° , the upper plot is computed using the new β limiter and the lower plot is computed using the new van Albada limiter

Figure 29: Pressure coefficient distribution for $M_\infty = 0.8$, Angle of Attack of 1.25° , the upper plot is computed using the new β limiter and the lower plot is computed using the new van Albada limiter

be simplified to unsteady Euler equations with the assumption of small amplitude oscillations. The time-dependent Euler equations for transonic flow about oscillating airfoils has been the focus of several studies. Magnus and Yoshihara [38] used a finite difference scheme of the Lax-Wendroff type. Lerat and Sides [39] adopted MacCormack type finite volume method with dissipative terms. Steger [40] used an implicit finite difference algorithm to solve both inviscid and viscous airfoil flows. In addition, Steger and Bailey [41] extended the research to transonic aileron buzz. Chyu, Davis and Chang [42] and Chyu [43] calculated unsteady transonic flow and its aeroelastic applications. Belk[44] computed time-dependent inviscid flow for a NASA RSW model with upwind schemes and a block-grid strategy. Anderson, Thomas and Rumsey [45] extended flux-vector splitting to unsteady flow on dynamic meshes. Batina [46] performed unsteady Euler solutions on unstructured dynamic meshes.

In order to compare with the experimental data used by Batina [46], in transonic oscillating airfoil flow simulation, the movement of a NACA 0012 airfoil is assumed to be a sinusoidal oscillation. The angle of attack varies according to

$$\theta = \theta_m + \theta_0 \sin(\omega t) \quad (4.1)$$

where,

$$k = \frac{\omega C}{2U_\infty}, t = \frac{\tau U_\infty}{a_\infty}$$

θ_m = mean pitching angle of attack

θ_0 = amplitude of the sinusoidal oscillation

τ = characteristic time

t =real time

ω =frequency of oscillation

k =reduced frequency

C =chord length of a airfoil

U_{∞} =free stream velocity

a_{∞} = free stream sonic speed

Hanel's modification of the van Leer splitting in conjunction with the efficient modified van Albada limiter are chosen for this unsteady flow simulation. The parameters used in the calculations are a Mach number of 0.755, reduced frequency of 0.0814, angle of attack of $0.016 + 2.5 \sin(\omega t)$ with an oscillation mode pitching at one quarter chord length. The unsteady calculations are started from steady solutions at the mean pitching angle of attack. The calculations are performed on the 259x49 C grid shown in Figure 27. Figures 30 and 31 show the Mach number contours of results using the modified van Albada limiter. These results are obtained by computing two cycles of the oscillation. The calculated instantaneous pressure distribution is compared with experimental data shown in Figure 32. The pressure and Mach number contour plots show that there is a shock wave on the upper surface of the airfoil in the first half of the cycle, and the flow over the lower surface is predominantly subcritical. During the second half of the cycle, the shock wave moves to the lower surface and the flow over the upper surface becomes subcritical. The pressure distribution indicates that the shock position oscillates over 25 percent of the chord

with the variation of angle of attack. The numerical results show that the pressure distribution in the second half of the cycle is slightly lower than the experimental data in front of the shock waves. Lift and moment coefficient distributions in an oscillation cycle are presented in Figure 33. The results computed by the original van Leer splitting are also shown to compare with the results of the Hanel splitting. Both lift and moment coefficient distributions computed by the van Leer splitting and Hanel's modification are in satisfactory agreement with experimental data. Only the estimated lift coefficient is lower than experimental results in the downstroke. The discrepancy between numerical results and experimental data can be overcome by the consideration of viscosity effects which is neglected in the unsteady Euler equations.

4.3 Steady Viscous Airfoil

Before proceeding to the case of high-lift system simulation, the current upwind code has been run for two stationary airfoil to verify its operations.

In steady-state flow, the analysis of the flowfield around a NACA 0012 airfoil is computed to demonstrate the current Navier-Stokes code. The two sets of parameters for the computation are a Mach number of 0.5, Reynolds number of 3×10^6 with angle of attack of 0° and Mach number of 0.6, Reynolds number of 3×10^6 with angle of attack of 6.48° . The steady-state solutions are calculated using the 249×64 O-type grid shown in Figure 34. The numerical results are compared with experimental data obtained in a subsonic wind tunnel at The Ohio State University Aeronautical and Astronautical Research Laboratory.

In this study, only the low-speed flow simulation is studied. Shock capturing is

Figure 30: Mach number contours in the first half oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack

Figure 31: Mach number contours in the second half oscillating cycle for $M_\infty = 0.755$, $\theta=0.016+2.51\sin(\omega t)$, $k=0.0814$, AoA is Angle of Attack

Figure 32: Pressure coefficient distribution in an oscillating cycle for $M_\infty = 0.755$, $\theta = 0.016 + 2.51 \sin(\omega t)$, $k = 0.0814$, AOA is Angle of Attack

Figure 33: Lift and moment coefficient distributions in an oscillating cycle for $M_\infty = 0.755$, $\theta = 0.016 + 2.51 \sin(\omega t)$, $k = 0.0814$, AoA is Angle of Attack

Figure 34: O-type grid 249X64 points for NACA 0012 airfoils

not important and the accurate simulation of physical properties in boundary-layer flow is of primary interest. The efficient and accurate simulation of Hanel splitting for the low speed flow is used in the current viscous flow simulations. Figures 35 and 36 show the Mach number contours and pressure coefficient distributions on the airfoil surface after the calculations have converged. The comparison of pressure coefficient distributions shows that the current scheme accurately predicts surface quantities observed in experimental data in both high angle of attack and zero angle of attack cases. Only a slight underestimate of the C_p distribution near the leading edge of the airfoil with a high angle of attack is found in the numerical results. Excessive numerical dissipation is reduced in the current flux vector splitting codes. A symmetric Mach number contour around the airfoil in the case of zero angle of attack is captured. The validation of the current Navier-Stokes equations code is seen in these steady-state solutions.

4.4 High Lift Systems

4.4.1 Multi-Element Airfoils

A study of low-speed aerodynamics, especially high-lift system performance has a major influence on the design and safety of any airplane configuration. An accurate simulation of the flow physics over high lift systems is essential. Previously, the Stratford [47, 48] criterion was applied by Smith [49], Smith and Leibeck [50] and Leibeck [51] to design multi-element airfoils for achieving maximum lift. The design of the shape of each element in high-lift airfoils requires the maximum pressure difference between the upper and lower surfaces of each element. The shape of a high lift airfoil

Figure 35: Mach number contours of subsonic flow over two NACA 0012 airfoils, M is Mach number and AoA is Angle of Attack

Figure 36: Pressure coefficient distributions on two airfoil surfaces, (A) $M_\infty = 0.5$, Angle of Attack of 0° , and (B) $M_\infty = 0.6$, Angle of Attack of 6.48°

is designed to delay the pressure minimum on the suction surface as far back as possible to coincide with the pressure recovery region and avoid flow separation. Until now, most performance optimization of high-lift systems has been conducted in two-dimensional wind tunnel studies of multi-element airfoils. A full-scale aircraft analysis is difficult and expensive. Reliable computational methods reduce the cost of wind tunnel testing required. However, an accurate calculation of the flow over multi-element airfoils is an unsolved problem, even though much progress has been made by many researchers. Multi-element slotted airfoils require complicated grid geometries as well as complicated boundary treatment in numerical solution procedures. Several flow regions from low-speed flow to transonic flow may appear on the highly loaded multi-element airfoils. Compressibility effects have to be considered in numerical approximations. Laminar boundary layers on the downstream element may become turbulent when merging with wakes from the upstream element. These merging shear layers are called confluent boundary layers. Strong pressure gradients frequently cause the development of viscous wakes. Large flow separation is observed if pressure gradients are adverse. The physics of confluent boundary layers, separated flows and boundary layer transition are among the main difficulties which high-lift airfoil codes may not model accurately.

The numerical studies of low-speed high-lift systems have varied from potential methods to the Navier-Stokes solvers [52, 53, 54, 55, 56] on structured or unstructured grids. So far, even for the two-dimensional high-lift cases, adequate simulation of such complicated physics has not been achieved. The objective of this study is to

develop the capability of Navier-Stokes analysis for lift enhancement on the multi-element airfoils. In addition, three-dimensional effects, though much more complex and computationally more expensive, are of great significance in the flow physics of high-lift systems. The current upwind code is extended to the analysis of three-dimensional effects on the production of maximum lift.

One of the high-lift systems is a high speed ground vehicle, such as a racing car, for which performance is enhanced by increasing the downforce from multi-element wings mounted on the vehicle. The maximum negative lift can be produced by adding a short, 90 degree inverted flat plate at the trailing edge of the airfoils to increase aerodynamic downforce. This is called a Gurney flap. As Leibeck first hypothesized [51], a Gurney flap can increase the lift by the deflecting the flow at the trailing edge. In fact, tabs like the Gurney flap can be mounted on any element in the multi-element high lift systems to increase the loading of the main element and delay flow separation on the flap. Generation of maximum downforce is predicted for a two-element racing car airfoil with a Gurney flap mounted on the trailing edge of the second element as shown in Figure 37. The formation of maximum downforce depends mostly on the size of the gap between neighboring airfoil elements and on flow velocity conditions including angle of attack, Reynolds number and Mach number. Influence of overlap sizes between the main element and the second element on the maximum downforce is also studied. In the extension of the analysis to three-dimensional effects, a uniform incoming flow over a three-dimensional two-element airfoil with a Gurney flap, one small and one large amplitude nonuniform incoming flow over a three-dimensional

two-element wing with a Gurney flap and a side fin mounted on the wing tip are chosen. The flow conditions for the three-dimensional cases are the same as those used in the two-dimensional studies. The prediction of C_l and C_p distributions in the three-dimensional calculations are compared with the results of the two-dimensional cases.

The configuration of a two-element racing car airfoil and its 392×80 O type grid are shown in Figures 37 and 38. Free stream conditions for this study are Mach number of 0.2, Reynolds number of two million and angle of attack of 0° . From the Mach number contours shown in Figure 40, it is seen that small regions of Mach number near 0.6 appeared on the leading edge of these highly loaded flapped airfoils, even though the freestream Mach number is only 0.2. From the numerical results, it is observed that boundary-layer flow on the downstream elements merging with the wake of the upstream element form a large separated flow region and small vortices on the surfaces behind the flaps. These flow separations play a dominate role in the generation of downforce. The formation of confluent boundary layers and flow separation actually depends on the size of the gap between the main airfoil and flap, flow conditions and the deflection angle of the second element. The c_p distribution of a two-dimensional multi-element airfoil shown in Figure 47 is different from the results of regular single element airfoils. The second element induces a high velocity around the trailing edge of the main airfoil. This allows an increase in downward momentum on the main element and reduction of the adverse pressure gradients near the trailing edge of the main element. This reduces flow recirculation zones and pressure on the

Figure 37: Geometry of a two-element racing car airfoil

lower surface of the trailing edge. The Gurney flap also brings downward turning flow over the trailing edge of the second element. Lower pressure behind the Gurney flap weakens the adverse pressure gradient and causes the confluent boundary layer to reattach over the entire length of the flap. The downforce can be increased by the Gurney flap. Due to grid coarseness around the Gurney flap and the leading edge of the second-element airfoil shown in Figure 39, the recirculation zones around the second-element airfoil and the flat tab are not clearly resolved, but it is seen that the most dominant physics are represented.

An optimized overlap gap size between two elements is investigated. The definition of the overlap gap size and the effect of an overlap size for negative Cl can be seen in Figure 41. The influence of the height of the gap between two elements on the production of downforce is not investigated in this study. Only an overlap position of the second element over the trailing edge of the main element is considered. In the numerical results, the flap position needs to be moved forward 5 percent of the chord length to achieve the maximum downforce under prescribed flow conditions. It is known that achievable maximum lift depends on the size of the gap and flight conditions. The optimized overlap may delay merging of the main element wake and flap boundary layers until the confluent boundary layer reaches the trailing edge of the flap. When the overlap size is smaller than the optimized size, wake interaction between the flap is not strong enough to keep the flap boundary layer attached. Flow separation cannot be avoided behind the flap and on the lower trailing edge surface of the main airfoil. The downforce significantly decreases with the reduction of overlap

Figure 38: Grid mesh 391x80 for the multi-element airfoil with a Gurney flap

Figure 39: Grid mesh distributions around the Gurney flap (the lower plot), Grid mesh distributions around the gap between the two-element airfoil (the upper plot)

Figure 40: Mach number contours for a two-dimensional racing car airfoil with a Gurney flap with Mach number=0.2

Figure 41: Effect of the overlap size on the lift coefficient of a two-dimensional racing car airfoil with a Gurney flap

Table 4: Lift Coefficient for Racing Car Wings

Wind Tunnel test(M=.166)	-2.16
2D computation(M=.166)	-2.24
2D computation(M=.2)	-2.38
uniform 3D computation (M=.2)	-2.35
small AoA non-uniform 3D computation(M=.2)	-2.41
large AoA non-uniform 3D computation(M=.2)	-.4

gap size. For larger overlap sizes, increased interaction between the wake of the main airfoil and flap leads to a reduction in the flap suction peak resulting in a loss in the downforce.

For the calculations of the three-dimensional cases, the surface grid is shown in Figures 42 and 43. The predicted lift coefficients for several cases are presented in Table IV. First, numerical validation shows that a two-dimensional computed lift coefficient of -2.24 for the Mach number 0.166 case compared with the lift coefficient of -2.16 obtained in a wind tunnel test. The accuracy of the current Navier-Stokes code for the prediction of downforce is satisfactory. The estimated lift coefficient of the three-dimensional uniform flow case is slightly lower than the two-dimensional case. From the Mach number contour plot, surface velocity profiles and the C_p distribution shown in Figures 44, 45 and 46, cross flow along spanwise direction is not observed in the case of a uniform flow over a three-dimensional wing with a Gurney flap. A small suction peak is observed in the surface C_p distribution near the wing tip. The pressure difference between the upper and lower surfaces is reduced on the wing tip,

which causes a slight downforce loss in the uniform flow case. Furthermore, a small amplitude variation of a non-uniform incoming flow from an angle of attack of 9° to 0° and a large amplitude variation of a non-uniform incoming flow from an angle of attack of 60° to 0° , assumed to be formed by the body of a racing car, are also considered in the three-dimensional calculations. A small downforce increase and weak three-dimensional effects are observed in the small amplitude non-uniform flow case from Table IV and Figures 48 and 49. The installation of a side fin maintains a maximum pressure difference between the upper and lower surfaces at the wing tip. In the spanwise loading distribution, shown in Figure 56, a small variation in angle of attack of the incoming flow and the installation of a side fin on the wing tip are shown to increase the downforce near the wing center and maintain the downforce near the wing tip. In the results for the large amplitude non-uniform flow case, a large recirculation region existing on the upper surface of the main wing center is seen in the Mach number contour plot depicted in Figure 51. The variation in the recirculation zone with the cross section surface near the wing tip and the side fin is shown in Figures 52 and 53. It is believed that a large downforce loss is caused by the large flow separation over the upper surface of the main wing. Strong cross flow effects can be found by the surface velocity profiles and C_p distributions described in Figures 54 and 55. In addition, the influence of the side fin or an end plate on maintaining downforce near the wing tip can be seen in Figure 56.

Figure 42: Surface grid meshes 392x80x10 for 3-D racing car airfoil with a Gurney flap

Figure 43: Surface grid meshes 391x80x10 for 3-D racing car airfoil with a Gurney flap and Side fin

Figure 44: Mach number contours of the three-dimensional uniform incoming flow case

Figure 45: Incoming velocity along spanwise direction of the three-dimensional uniform incoming flow case

Figure 46: Surface C_p distribution in spanwise direction of the three-dimensional uniform incoming flow case

Figure 47: Surface C_p distribution for the 2D and 3D racing car wings

Figure 48: Incoming velocity along spanwise direction of the three-dimensional small amplitude non-uniform incoming flow case

Figure 49: Surface C_p distribution in spanwise direction of the three-dimensional small amplitude non-uniform incoming flow case

Figure 50: Mach number contours of a racing car wing surface at 0.01 chord length height for the three-dimensional large amplitude non-uniform incoming flow case

Figure 51: Mach number contours of cross section plane for the three-dimensional large amplitude non-uniform incoming flow case

Figure 52: Mach number contours of cross section plane of the three-dimensional large amplitude non-uniform incoming flow case

Figure 53: Mach number contours of cross section plane for the three-dimensional large amplitude non-uniform incoming flow case

Figure 54: Surface U profiles of the three-dimensional large amplitude non-uniform incoming flow case

Figure 55: Surface C_p distribution in spanwise direction for the three-dimensional large amplitude non-uniform incoming flow case

Figure 56: Spanwise loading on the three-dimensional racing car wings, AoA is Angle of Attack

4.4.2 Oscillating Viscous Airfoils

Unsteady stall phenomena are the result of airfoils and wings oscillating in pitch and having a maximum angle of attack greater than the static stall angle. This delay of stall causes hysteresis in lift and moment coefficients which has challenged aerodynamicists for many years. This important viscous phenomena combines with unsteady effects in a variety of current aeronautical problems. The mechanism of dynamic stall appears on helicopter rotor blades, rapid maneuver aircraft and wind turbines, and even insects. This important aerodynamic characteristic was considered by Harris and Pruyn [57] for a helicopter rotor. It was observed that the extra lift on the helicopter rotor occurs when the blade moves opposite to the direction of flight. At the same time, Ham and Garelick [58] found that increased lift can be created by oscillating airfoils. In Ham's experiments [59], the hysteresis in lift and drag are believed to be caused as a result of vortex shedding on the surface of the airfoils. Carta [60] and Liiva and Davenport [61] showed that the dynamic pressure distribution is associated with the vortex passage. Dynamic stall was explored by McCroskey and Fisher [62] in an experimental investigation of a model rotor. The dynamic effects were verified to be a result of a vortex-dominated flowfield. More information was presented in the experiments [63, 64, 65, 66] of McCroskey et al. and McAlister and Carr's research [67, 68]. In this work, dynamic stall behavior of various airfoils were studied over a wide range of mean angles, amplitude of oscillation, reduced frequency, Reynolds numbers and Mach numbers.

In the numerical simulation, the linearized unsteady airfoil theory developed by

Theodorsen [69] and von Karman and Sears [70] has been widely used for the analysis of small amplitude oscillations. Metha [71] first used the two-dimensional incompressible Navier-Stokes equations to solve the flowfield around an oscillating NACA 0012 airfoil. Ono et al [72] studied the same problem by using the discrete vortex approximation method. Considering compressibility and high Reynolds number effects, Tassa and Sankar [73] used the compressible Navier-Stokes equations for a numerical model of dynamic stall. The deep dynamic stall process associated with a NACA 0012 airfoil oscillating in pitch at low reduced frequency and high Reynolds number, was analyzed by Ono [74] and Visbal [75, 76] by using an implicit Beam-Warming code. Rumsey [77] applied an upwind-biased implicit approximate factorization algorithm for self-excited and forced unsteady separated flows to the same problem. Shida et al [78] used the block pentadiagonal matrix inversion scheme and considered grid refinement effects for dynamic stall of NACA 0012 airfoils. Visbal [79, 80] extended his study into the structure of vortex breakdown on a pitching wing. He has conducted a detailed description of the 3-D instantaneous structure of the flow field for the first time using critical point theory. So far, the excessive numerical diffusion still cannot be avoided in the current numerical simulation. Further accuracy requirement and reduced cost in computer time are big challenges for present researchers.

In the oscillating airfoil flow simulation, the movement of a NACA 0012 airfoil is assumed in a sinusoidal oscillation. In the experiments of McCroskey et al. [65], the viscous-inviscid interaction on oscillating NACA 0012 airfoils is described by four regimes. First, if the important parameter θ_{max} is 13° , there is almost no separa-

tion throughout the oscillating cycle. This regime is interpreted as weak interaction between viscous and inviscid regions. When the θ_{max} increased to 14° , the flow separation occurs in a small fraction of the oscillation cycle. The hysteresis loops of unsteady pressure distributions and airloads are distorted in this stall onset regime. This stall-onset condition represents the limiting case of the increased lift without significant influence in the pitching moment. In these two regimes, viscous layers remain relatively thin. However, the challenge is greater if the magnitude of viscous-inviscid interaction increases rapidly with the increase of θ_{max} . For the NACA 0012 airfoil at Mach number of 0.3, the extent of flow separation increases with greater reduced frequency, but this separation is similar to boundary layer disturbances originating at the upper leading edge and propagating downstream. There is neither complete flow separation nor reversed flow over the entire airfoil in this stall-onset region.

Furthermore, with a slight additional increase in θ_{max} to 15° , the viscous-inviscid interaction and unsteady effects on the pressure distributions and airloads become strong. In this so-called light stall regime, the duration and extent of flow separation experience a major increase. The hysteresis losses in lift and moment are first observed in this regime. Further increase in angle of attack leads to the deep stall regime, and the flow separation zone appears over half of the oscillating cycle. The unsteady effects are characterized by the shedding of a large vortex-like disturbance. The distorted hysteresis loop of pressure and airloads is seen during the entire sinusoidal oscillating cycle.

One light and one deep dynamic stall case in the experiments of McCroskey et

al. are used for validating the calculations in the following sections. The boundary conditions for viscous flow over the dynamic airfoil are similar to those for the stationary airfoil. The density is determined from the point next to the body surface. The tangential and normal velocities of the moving surface are equal to the surface grid velocities ($u=x_r$, $v=y_r$). The pressure is determined from solving the normal momentum equation. Uniform flow conditions are specified at the farfield flow boundaries. Downstream outer boundary conditions are extrapolated from the inner points near the outlet boundaries. The instantaneous locations of the points on the inner boundary (airfoil surface) are given by the prescribed surface motion. The entire mesh is moved with the airfoil surface at each time step of the calculations. The minimum spacing is specified to the chord length of 0.00005 in η direction for 249x64 O type grid.

In the light stall case, the parameters chosen in the calculations are Mach number of 0.3, reduced frequency of 0.2, angle of attack of $10+5 \sin(\omega t)$ with an oscillation mode pitching at a chord of 0.25. The unsteady calculations are started from steady solutions at the mean pitching angle of attack. Figures 57 and 58 show the Mach number contours and pressure coefficient distributions of the light dynamic stall process obtained by computing two cycles of sinusoidal oscillation. In the numerical results, the small trailing edge vortices are captured in the entire sinusoidal oscillation. The distinguishing feature of the light stall is the scale of the interaction. The vertical extent of the viscous zone tends to remain within the thickness of the boundary layers. Only small vortices behind the trailing edge can be observed in the numerical results.

However, from unsteady lift and moment coefficient distributions shown in Figure 59, the entire event of light dynamic stall can be seen clearly. The pitching airfoil passes the static-stall angle during the upstroke without the lift loss that usually occurs during the static stall. The unsteady lift is increased until the pitching motion reaches an angle of attack of 15° . The lift and moment stall occur near an angle of attack of 15° at the beginning of the downstroke. The increasing negative moment is stopped near the static stall angle of attack during the downstroke. Stall is completed when the pitching down reaches an angle of attack of 10° . The reattachment of the boundary layer occurs at an angle of attack of approximately 10° during downward pitching. The downstroke is stopped and upstroke restarts at an angle of attack of 5° , and the lift and moment coefficients are returned to unstalled values. Numerical prediction of lift coefficient distribution is satisfactory. However, improvements are needed in the computation of the moment coefficient distribution. The resolution of viscous-inviscid interaction in the boundary layer might not be satisfactory as a result of excessive numerical dissipation. Grid refinement is needed for further studies.

In the deep stall case, the parameters chosen for the calculations are Mach number of 0.3, reduced frequency of 0.25, angle of attack of $15 + 10 \sin(\omega t)$ with an oscillation mode pitching at a chord of 0.25. The unsteady calculations are started from steady solutions at the mean pitching angle of attack as defined in equation (4.1). Figures 60, 61 and 62 show the Mach number contours and pressure coefficient distributions of the deep dynamic stall process obtained by computing two cycles of sinusoidal oscillation. In the numerical results, an obvious separation zone is seen over the

entire airfoil surface near the maximum angle of attack in both the upstroke and downstroke directions. A large vortex is formed at the leading edge and then convects along the airfoil surface, shedding into the wake during nose-down pitching motion. Figure 63 shows the development of unsteady C_l and C_m versus angle of attack, and the corresponding boundary-layer behavior for a dynamic stall on the NACA 0012 airfoil is observed. The static stall angle is exceeded at the beginning of nose-up pitching with increased lift. The discernible occurrence of small vortices appears on the trailing edge near an angle of attack of 15° . The unsteady lift is increased until the pitching motion reaches an angle of attack of around 25° . The reversal flow around the trailing edge propagated up along the surface to form leading edge flow separation. At this stage, the boundary layer no longer remains thin and attached and a strong separated flow forms. At the start of the downstroke, the lift stall begins and the vortex moves down along the surface. At the same time, a strong negative pitching moment is induced. This phenomenon is called a dynamic stall. As the angle of attack decreases, the vortex moves into the downstream wake and fully separated flow is developed. From the unsteady lift distributions, dynamic stall seems to be completed around an angle of attack of 13° and the boundary-layer reattaches for a short period during the pitching motion near the minimum angle of attack. As the angle of attack reaches its minimum, the lift restarts from its minimum and returns to the unstalled values. In the numerical results, a similar tendency of lift coefficient distribution is observed. However, improvements are needed in the computation of the moment coefficient distribution. The resolution of vortex shedding is not fine

enough due to excessive numerical dissipation. In this calculation, the scale of the computed vortices is of insufficient resolution to predict real physics. This could decrease the time for the vortex shed from the leading edge to reach the trailing edge. This causes the computed maximum lift coefficient to be smaller than that from experimental data as well as discrepancies in the moment coefficient prediction. It is seen that the accuracy of the current MUSCL type upwind code is not good for the deep dynamic stall case. The unsteady behavior could be damped out by using the algebraic turbulence model of Baldwin-Lomax. In this deep dynamic stall case, separation occurs from the leading edge of airfoils. The transitional nature of boundary layers over airfoil surfaces is not evaluated in the Baldwin-Lomax model. Grid refinement and incorporation of suitable turbulence models are suggested for future studies of dynamic stall problems.

Figure 57: Mach number contours in an oscillation for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=10+5 \sin(\omega t)$, $k=0.2$, AoA is Angle of Attack

Figure 58: Surface pressure coefficient distribution for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=10+5 \sin(\omega t)$, $k=0.2$, AoA is Angle of Attack

Figure 59: Lift and moment coefficient hysteresis vs angle of attack.

Figure 60: Mach number contours in an first half oscillating for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack

Figure 61: Mach number contours in an second half oscillating for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack

Figure 62: Surface pressure coefficient distribution for $M_\infty = 0.3$, $Re=4 \times 10^6$, $\theta=15+10 \sin(\omega t)$, $k=0.25$, AoA is Angle of Attack

Figure 63: Lift and moment coefficient hysteresis vs angle of attack.

CHAPTER V

Conclusions and Suggestions

5.1 Conclusions

A systematic investigation has been performed on the influence of limiters in MUSCL type upwind schemes with flux-vector splitting. Several observations and conclusions are drawn from this study. The results show that a limiter is an important factor in the stability and accuracy of MUSCL type upwind schemes.

In NVD analysis and numerical comparisons on the shock tube and high-speed blunt body flow problems, it is seen that MUSCL type spatial differencing without limiters may produce nonphysical solutions and the addition of limiters in MUSCL methods reduces numerical oscillations around discontinuities. The van Albada limiter satisfies accuracy and stability criteria of NVD analysis. It is shown that the van Albada limiter enhances resolution of discontinuities and achieves stable convergence with less diffusive error. The superbee limiter is less accurate in some monotonic regions of a normalized variable diagram, resulting in convergence problems in steady-state problems. The resolution of discontinuities in these calculations is reduced due to the first-order time accuracy. The addition of the minmod limiter smears out discontinuities due to excessive numerical dissipation effects. It is noted that a lim-

iter with less diffusive error like the van Albada limiter achieves good resolution of discontinuities and stable convergence.

An important property for limiters is that the second-order TVD condition is contained in monotonicity criteria of Spekreijse for multiple dimensional MUSCL type finite volume methods. It is noted that a limiter obeys the second-order accurate TVD condition which means it also contains monotonicity in multi dimensions. Modifications to existing limiters have been developed in which the averaging function like that in the van Albada limiter is used in the smooth regions of the solution. First-order upwind differencing is used in oscillating regions. This modified van Albada limiter function improves the convergence rate of the current MUSCL type upwind schemes without sacrificing high-order accuracy. A new formulation of β limiters with a parameter, S , is obtained by reformulating the averaging function of the van Leer limiter and the β limiters of Sweby. In numerical tests, using the shock tube and blunt body flow problems, applications of the new β limiters demonstrate a significant reduction in the diffusive errors produced by the minmod type limiters in resolving discontinuities. These tests also show that the new limiters provide significant improvement over the divergence observed with the van Leer limiter. By investigating the second-order TVD condition, NVD analysis and numerical experiments, the use of the parameter S , where S is 1.2, the new β limiter is shown to have the good accuracy and stability performance.

Numerical validation of MUSCL methods with new limiters shows good agreement with experimental data or previously validated computations on steady inviscid

transonic and viscous subsonic airfoil problems. In the inviscid oscillating transonic airfoil problem, numerical estimates of lift and moment coefficients as compared with experimental data are satisfactory.

In numerical applications, generation of the maximum downforce is predicted for a racing car wing. It is seen that installation of a Gurney flap and a side fin enhances the production of downforce. Three-dimensional effects are not obvious in the case of an uniform incoming flow over a three-dimensional two-element racing car wing with Gurney flap. The estimated downforce of three-dimensional uniform flow case is slightly lower than downforce in the two-dimensional case. In the non-uniform three-dimensional flow cases, it is shown that a small downforce increase is enhanced by the installation of a side fin on the wing tip and utilizing a small amplitude variation incoming flow assumption, however, a large downforce loss is observed in the case of the large amplitude incoming flow even with a side fin mounted on the wing tip. Computational results also show that the second element needs to be moved forward 5 percent chord length to achieve the maximum downforce.

In the dynamic stall problems of an oscillating subsonic airfoil, the corresponding boundary-layer behavior for dynamic stall is observed in the current numerical simulation. The static stall angle is exceeded during the upstroke with increased lift. The unsteady lift is increased until the pitching motion reaches near the maximum angle of attack. In the upstroke, the reversal flow around the trailing edge propagates up along the surface to form leading-edge flow separation. At this stage, the boundary layer is no longer thin and attached and a strong separated flow forms. At the early

downstroke, the lift stall begins and a vortex moves down along the surface. At that time, a strong negative pitching moment is induced. Dynamic stall is completed and the boundary-layer reattaches for a short period in the downstroke near the minimum angle of attack. As the angle of attack reaches its minimum, the lift restarts from its minimum and returns to the unstalled values. Numerical estimates of unsteady lift coefficient distributions is satisfactory. However, the prediction of moment coefficient distributions needs further improvement.

5.2 Suggestions for future works

In the calculations of compressible low-speed viscous flow problems, inefficiency is a disadvantage of the current explicit time-accurate schemes. The difficulty can be seen by comparing the largest and the smallest eigenvalues of the flux Jacobian:

$$\frac{\lambda_{max}}{\lambda_{min}} = \frac{u + a}{u} = \frac{M + 1}{M} \quad (5.1)$$

This ratio (5.1) approaches ∞ if the Mach number is near 0. Since the eigenvalues are the speeds of waves carrying information, the flow information of smallest eigenvalues can be ignored in the time steps of explicit methods which are limited by the fastest eigenvalues for small Mach number flow. In the incompressible flow region, the number of time steps needed to reach steady state will tend to infinity. Numerical inaccuracy cannot be avoided in the explicit time-accurate calculations of low-speed high lift systems whose flow regions vary from incompressible flow to transonic flow. An implicit time-accurate scheme with fewer stability requirements is suggested to

modify the current explicit upwind codes for low-speed high-lift flow problems.

The use of Lomax-Baldwin turbulence model in the current upwind codes is the first step to develop the capability of turbulence phenomena simulations. The evaluation of accurate and efficient turbulence models is necessary for future studies of high lift system problems.

Appendix A

Scaling of the Governing Equations

The normalization of a governing equation can be done by replacing the dimensional variables with the properly scaled nondimensional variables.

The choice of scaling parameters for (2.1) to (2.8), the variables are be scaled as

$$\begin{aligned}
 \bar{x} &= \frac{x}{L} & \bar{y} &= \frac{y}{L} & \bar{t} &= \frac{a_\infty t}{L} \\
 \bar{\rho} &= \frac{\rho}{\rho_\infty} & \bar{u} &= \frac{u}{a_\infty} & \bar{v} &= \frac{v}{a_\infty} \\
 \bar{T} &= \frac{T}{T_\infty} & \bar{e} &= \frac{e}{\rho_\infty a_\infty^2} & \bar{p} &= \frac{p}{\rho_\infty a_\infty^2} \\
 \bar{\tau} &= \frac{\tau}{\rho_\infty a_\infty^2} & \bar{\mu} &= \frac{\mu}{\mu_\infty} & \bar{Re} &= \frac{\rho_\infty L a_\infty}{\mu_\infty}
 \end{aligned} \tag{A.1}$$

First, each term in the mass conservation equation in (2.1) can be reformulated as

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} &= \frac{\rho_\infty}{L/a_\infty} \frac{\partial(\rho/\rho_\infty)}{\partial(a_\infty t/L)} = \frac{\rho_\infty a_\infty}{L} \frac{\partial \bar{\rho}}{\partial \bar{t}} \\
 \frac{\partial \rho u}{\partial x} &= \frac{\rho_\infty}{L/a_\infty} \frac{\partial(\rho/\rho_\infty)(u/a_\infty)}{\partial(x/L)} = \frac{\rho_\infty a_\infty}{L} \frac{\partial \bar{\rho} \bar{u}}{\partial \bar{x}}
 \end{aligned}$$

In the same manner,

$$\frac{\partial \rho v}{\partial x} = \frac{\rho_\infty}{L/a_\infty} \frac{\partial(\rho/\rho_\infty)(v/a_\infty)}{\partial(x/L)} = \frac{\rho_\infty a_\infty}{L} \frac{\partial \bar{\rho} v}{\partial \bar{x}}$$

Combining them into

$$\frac{\rho_\infty a_\infty}{L} \left[\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho} u}{\partial \bar{x}} + \frac{\partial \bar{\rho} v}{\partial \bar{y}} \right] = 0.$$

Dropping the common factor outside the bracket, the mass conservation equation becomes

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho} u}{\partial \bar{x}} + \frac{\partial \bar{\rho} v}{\partial \bar{y}} = 0. \quad (\text{A.2})$$

The momentum equations can be scaled as

$$\frac{\partial \rho u}{\partial t} = \frac{\rho_\infty}{L/a_\infty^2} \frac{\partial(\rho/\rho_\infty)(u/a_\infty)}{\partial(a_\infty t/L)} = \frac{\rho_\infty a_\infty^2}{L} \frac{\partial \bar{\rho} u}{\partial \bar{t}}$$

$$\frac{\partial}{\partial x}(\rho u^2 + p - \tau_{xx}) = \frac{\rho_\infty a_\infty^2}{L} \frac{\partial}{\partial \bar{x}}(\bar{\rho} \bar{u}^2 + \bar{p} - \tau_{xx}^-)$$

$$\frac{\partial}{\partial y}(\rho uv - \tau_{xy}) = \frac{\rho_\infty a_\infty^2}{L} \frac{\partial}{\partial \bar{y}}(\bar{\rho} \bar{u} v + \bar{p} - \tau_{xy}^-)$$

Combining them into

$$\frac{\rho_\infty a_\infty^2}{L} \left[\frac{\partial \bar{\rho} u}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{\rho} \bar{u}^2 + \bar{p} - \tau_{xx}^-) + \frac{\partial}{\partial \bar{y}}(\bar{\rho} \bar{u} v + \bar{p} - \tau_{xy}^-) \right] = 0.$$

Hence the momentum equations become

$$\frac{\partial \bar{\rho} u}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{\rho} \bar{u}^2 + \bar{p} - \tau_{xx}^-) + \frac{\partial}{\partial \bar{y}}(\bar{\rho} \bar{u} v + \bar{p} - \tau_{xy}^-) = 0. \quad (\text{A.3})$$

likewise,

$$\frac{\partial \bar{\rho} v}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{\rho} \bar{u} v + \bar{p} - \tau_{xy}^-) + \frac{\partial}{\partial \bar{y}}(\bar{\rho} \bar{v}^2 + \bar{p} - \tau_{yy}^-) = 0. \quad (\text{A.4})$$

For the energy conservation equation, the nondimensional process is as

$$\begin{aligned} \frac{\partial e}{\partial t} &= \frac{\rho_\infty}{L/a_\infty^3} \frac{\partial(e/(\rho_\infty a_\infty^2))(u/a_\infty)}{\partial(a_\infty t/L)} \\ &= \frac{\rho_\infty a_\infty^3}{L} \frac{\partial \bar{e}}{\partial \bar{t}} \\ \frac{\partial(\rho u(e+p))}{\partial x} &= \frac{\rho_\infty}{L/a_\infty^3} \frac{\partial(\rho/\rho_\infty)(e/(\rho_\infty a_\infty^2) + p/(\rho_\infty a_\infty^2))(u/a_\infty)}{\partial(x/L)} \\ &= \frac{\rho_\infty a_\infty^3}{L} \frac{\partial \bar{\rho} \bar{u}(\bar{e} + \bar{p})}{\partial \bar{x}} \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{\partial(\rho v(e+p))}{\partial y} &= \frac{\rho_\infty}{L/a_\infty^3} \frac{\partial(\rho/\rho_\infty)(e/(\rho_\infty a_\infty^2) + p/(\rho_\infty a_\infty^2))(v/a_\infty)}{\partial(y/L)} \\ &= \frac{\rho_\infty a_\infty^3}{L} \frac{\partial \bar{\rho} \bar{v}(\bar{e} + \bar{p})}{\partial \bar{y}} \end{aligned}$$

The stress terms can be scaled as

$$\rho_\infty a_\infty^2 \frac{\tau_{xx}}{\rho_\infty a_\infty^2} = \frac{\mu_\infty a_\infty}{L} ((\lambda + 2\mu) \frac{\partial(u/a_\infty)}{\partial(x/L)} + \lambda \frac{\partial(v/a_\infty)}{\partial(y/L)})$$

$$\rho_\infty a_\infty^2 \tau_{xx}^- = \frac{\mu_\infty a_\infty}{L} ((\lambda + 2\mu) \frac{\partial \bar{u}}{\partial \bar{x}} + \lambda \frac{\partial \bar{v}}{\partial \bar{y}})$$

$$\tau_{xx}^- = \frac{\mu_\infty}{a_\infty \rho_\infty L} ((\lambda + 2\mu) \frac{\partial \bar{u}}{\partial \bar{x}} + \lambda \frac{\partial \bar{v}}{\partial \bar{y}})$$

Assuming $Re^{-1} = \frac{\mu_\infty}{a_\infty \rho_\infty L}$, the nondimensional shear stress terms become

$$\tau_{xx}^- = \frac{1}{Re} ((\lambda + 2\mu) \frac{\partial \bar{u}}{\partial \bar{x}} + \lambda \frac{\partial \bar{v}}{\partial \bar{y}})$$

likewise,

$$\tau_{yy}^- = \frac{1}{Re} ((\lambda + 2\mu) \frac{\partial \bar{v}}{\partial \bar{y}} + \lambda \frac{\partial \bar{u}}{\partial \bar{x}})$$

$$\tau_{xy}^- = \frac{1}{Re} \bar{\mu} (\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}})$$

For the derivatives of stress terms in the energy conservation equation,

$$\frac{\partial R}{\partial x} = \frac{\rho_\infty}{L/a_\infty^3} \frac{\partial}{\partial \bar{x}} (\bar{u} \tau_{xx}^- + \bar{v} \tau_{xy}^- + \alpha \frac{1}{Re} \frac{\partial \bar{a}^2}{\partial \bar{x}})$$

$$\frac{\partial S}{\partial y} = \frac{\rho_\infty}{L/a_\infty^3} \frac{\partial}{\partial \bar{y}} (\bar{u} \tau_{xy}^- + \bar{v} \tau_{yy}^- + \alpha \frac{1}{Re} \frac{\partial \bar{a}^2}{\partial \bar{y}})$$

Combining all terms, the energy conservation equation becomes

$$\frac{\rho_\infty a_\infty^3}{L} (\frac{\partial \bar{e}}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u} (\bar{e} + \bar{p})}{\partial \bar{x}} + \frac{\partial \bar{\rho} \bar{v} (\bar{e} + \bar{p})}{\partial \bar{y}} + \frac{\partial \bar{R}}{\partial \bar{x}} + \frac{\partial \bar{S}}{\partial \bar{y}}) = 0$$

Hence the energy conservation equation becomes

$$\begin{aligned} & \frac{\partial \bar{e}}{\partial \bar{t}} + \frac{\partial \bar{\rho} \bar{u} (\bar{e} + \bar{p})}{\partial \bar{x}} + \frac{\partial \bar{\rho} \bar{v} (\bar{e} + \bar{p})}{\partial \bar{y}} + \\ & \frac{\partial}{\partial \bar{x}} (\bar{u} \tau_{xx}^- + \bar{v} \tau_{xy}^- + \alpha \frac{1}{Re} \frac{\partial \bar{a}^2}{\partial \bar{x}}) + \frac{\partial}{\partial \bar{y}} (\bar{u} \tau_{xy}^- + \bar{v} \tau_{yy}^- + \alpha \frac{1}{Re} \frac{\partial \bar{a}^2}{\partial \bar{y}}) = 0 \end{aligned} \quad (A.5)$$

For the equation of state,

$$\rho_{\infty} a_{\infty}^2 \frac{e}{\rho_{\infty} a_{\infty}^2} = \rho_{\infty} a_{\infty}^2 \frac{p/(\rho_{\infty} a_{\infty}^2)}{\gamma - 1} + \frac{1}{2}(\rho/\rho_{\infty})[(u/a_{\infty})^2 + (v/a_{\infty})^2]$$

the nondimensional equation of state is

$$\bar{e} = \frac{\bar{p}}{\gamma - 1} + \frac{1}{2}\bar{\rho}(\bar{u}^2 + \bar{v}^2) \quad (\text{A.6})$$

Appendix B

Stability Analysis for MUSCL Methods

Consider the linear wave equation in one dimension, which is a model for the Euler equations.

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (\text{B.1})$$

where x is the spatial coordinate, u is velocity, t is time and a is the constant wave speed and is non-negative.

The higher-order MUSCL flux splitting schemes are made TVD by defining the numerical flux of the semi-discretized schemes as

$$f_{i+1/2} = f(u_{i+1/2}^L, u_{i+1/2}^R) \quad (\text{B.2})$$

Such as

$$f_{i+1/2} = f^+(u_{i+1/2}^L) + f^-(u_{i+1/2}^R) \quad (\text{B.3})$$

For the linear wave equation $f=au$, the numerical flux can be reduced to

$$f_{i+1/2} = a^+ u_{i+1/2}^L + a^- u_{i+1/2}^R \quad (\text{B.4})$$

and the explicit Euler method is selected as time integration leading to

$$u_i^{n+1} = u_i^n - \lambda [a_{i+1/2}^+(u_{i+1/2}^L - u_{i-1/2}^L) + a_{i-1/2}^-(u_{i+1/2}^R - u_{i-1/2}^R)] \quad (\text{B.5})$$

where λ is the mesh ratio $\frac{\Delta t}{\Delta x}$.

Using MUSCL type interpolation to evaluate $u_{i+1/2}^L$

$$u_{i+1/2}^L = u_i + \frac{1}{4} [(1 - \eta)\Psi(R^L)(u_i - u_{i-1}) + (1 + \eta)\Psi(1/R^L)(u_{i+1} - u_i)] \quad (\text{B.6})$$

with

$$R^L = \frac{u_{i+1} - u_i}{u_i - u_{i-1}} \quad (\text{B.7})$$

The left interface value can be modified as

$$u_{i+1/2}^L = u_i + \frac{1}{2}\Phi^L(u_i - u_{i-1}) \quad (\text{B.8})$$

with

$$\Phi^L = \frac{1}{2} [(1 - \eta)\Psi(R^L) + (1 + \eta)R^L\Psi(1/R^L)] \quad (\text{B.9})$$

Following the same procedure as the above, the right interface value can be

$$u_{i+1/2}^R = u_{i+1} - \frac{1}{2}\Phi^R(u_{i+2} - u_{i+1}) \quad (\text{B.10})$$

with

$$\Phi^R = \frac{1}{2}[(1 - \eta)\Psi(R^R) + (1 + \eta)R^R\Psi(1/R^R)] \quad (\text{B.11})$$

Any of the limiter Ψ can be selected in the above formulation. It is noticed that Ψ is identical to Φ in fully upwind scheme i.e. $\eta = -1$.

Substitution of (B.8) and (B.10) into (B.5) give

$$\begin{aligned} u_i^{n+1} = & u_i^n - \lambda a_{i-1/2}^+ [u_i^n + \frac{1}{2}\Phi^L(u_i^n - u_{i-1}^n) \\ & - u_{i-1}^n - \frac{1}{2}\Phi^L(u_{i-1}^n - u_{i-2}^n)] \\ & - \lambda a_{i+1/2}^- [u_{i+1}^n - \frac{1}{2}\Phi^R(u_{i+2}^n - u_{i+1}^n) \\ & - u_i^n + \frac{1}{2}\Phi^R(u_{i+1}^n - u_i^n)] \end{aligned} \quad (\text{B.12})$$

rearranging (B.12) as

$$\begin{aligned} u_i^{n+1} = & u_i^n - \lambda a_{i-1/2}^+ [u_i^n - u_{i-1}^n \\ & + \frac{1}{2}\Phi^L(u_i^n - u_{i-1}^n) - \frac{1}{2}\Phi^L(u_{i-1}^n - u_{i-2}^n)] \\ & - \lambda a_{i+1/2}^- [u_{i+1}^n - u_i^n \\ & - \frac{1}{2}\Phi^R(u_{i+2}^n - u_{i+1}^n) + \frac{1}{2}\Phi^R(u_{i+1}^n - u_i^n)] \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} u_i^{n+1} = & u_i^n - \lambda a_{i-1/2}^+ [1 + \frac{1}{2}\Phi^L \\ & - \frac{1}{2} \frac{\Phi^L}{R_{i-1/2}^L}] (u_i^n - u_{i-1}^n) \end{aligned}$$

$$\begin{aligned}
& -\lambda a_{i+1/2}^- [1 + \frac{1}{2} \Phi^R \\
& - \frac{1}{2} \frac{\Phi^R}{R_{i+1/2}^R}] (u_{i+1}^n - u_i^n)
\end{aligned} \tag{B.14}$$

Using TVD conditions (3.8) for (B.14), the TVD conditions for MUSCL methods become

$$1 + \frac{1}{2} \Phi^L - \frac{1}{2} \frac{\Phi^L}{R_{i-1/2}^L} \leq 0 \tag{B.15}$$

$$1 + \frac{1}{2} \Phi^R - \frac{1}{2} \frac{\Phi^R}{R_{i+1/2}^R} \leq 0 \tag{B.16}$$

These two TVD conditions lead a generalized formulation as

$$\frac{\Phi(R)}{R} - \Phi(W) \leq 2 \tag{B.17}$$

where R and W can be any value.

In the TVD schemes, $\Phi=0$ for $R \leq 0$ is used to turn off the limiter functions to avoid oscillating behaviors when extremum is encountered. With this assumption, the additional constraint is

$$\Phi(R) \geq 0 \tag{B.18}$$

for any R.

Hence, the final TVD sufficient condition for MUSCL methods becomes

$$0 \leq (\Phi(R)/R, \Phi(R)) \leq 2 \tag{B.19}$$

In MUSCL type interpolation, Ψ is identical to Φ not only in fully upwind scheme i.e. $\eta = -1$ but also for any limiter Φ has a symmetric property as

$$\frac{\Phi(R)}{R} = \Phi\left(\frac{1}{R}\right) \quad (\text{B.20})$$

Substituting (B.20) into (B.9) and (B.11) leads $\Psi(R)=\Phi(R)$ for any η . In this study, all limiters have this property which ensures the backward and forward facing gradients treated in the same manner. We can obtain

$$0 \leq (\Psi(R)/R, \Psi(R)) \leq 2 \quad (\text{B.21})$$

In addition, Sweby proposes that any second order scheme relying on the points $(Q_{i-2}, Q_{i-1}, Q_i, Q_{i+1}, Q_{i+2})$ must use a weighted average of the second order centered Lax-Wendroff scheme and the second order upwind Warming and Beam scheme [36], i.e. in terms of limiters,

$$\Psi(R) = (1 - \delta(R))\Psi(R)^{LW} + \delta(R)\Psi(R)^{WB} \quad (\text{B.22})$$

where $0 \leq \delta(R) \leq 1$. Since

$$\Psi(R)^{LW} = 1, \Psi(R)^{WB} = R \quad (\text{B.23})$$

(B.22) becomes

$$\Psi(R) = 1 + \delta(R)(R - 1) \quad (\text{B.24})$$

Hence, the second order TVD condition (B.24) is obtained and also shown in 20(b).
By means of the definition R such as (B.7). All the current limiters can be expressed as

Minmod limiter:

$$\Psi(R) = \text{Max}(0, \text{Min}(R, 1)) \quad (\text{B.25})$$

Superbee limiter:

$$\Psi(R) = \text{Max}(0, \text{Min}(2R, 1), \text{Min}(R, 2)) \quad (\text{B.26})$$

van Albada's limiter:

$$\Psi(R) = \frac{R^2 + R}{R^2 + 1} \quad (\text{B.27})$$

Modified van Albada's limiter:

$$\Psi(R) = \frac{1/2(|R| + R)(1 + R)}{R^2 + 1} \quad (\text{B.28})$$

New β limiters:

$$\Psi(R) = \text{Min}(\Psi^{\text{vl}}, \text{Max}(0, (\text{Min}(SR, 1), \text{Min}(R, S)))) \quad (\text{B.29})$$

with

$$\Psi^{\text{vl}}(R) = \frac{|R| + R}{|R| + 1}$$

where $1 \leq S \leq 2$.

All the above limiters can be shown in Figures 3 and 22. It is seen that the curve of any limiter lies in the boundness of the second-order TVD conditions for $R \geq 0$ and the upper bound of each limiter can be achieved.

- Ψ of minmod limiter ≤ 1
- Ψ of superbee limiter ≤ 2
- Ψ of van Albada limiter ≤ 1.25
- Ψ of modified van Albada limiter ≤ 1.25
- Ψ of new β limiter $\leq S$

where $1 \leq S \leq 2$.

The defined upper bound for each limiter is important to the following derivation of CFL-like conditions for each limiter. Using TVD conditions (3.13) and (B.19) for (B.14), The additional TVD condition for MUSCL methods is

$$\begin{aligned}
 \lambda(C_{i+1/2}^+ - C_{i+1/2}^-) &= \lambda a_{i+1/2}^+ [1 + \frac{1}{2} \Phi^L - \frac{1}{2} \frac{\Phi^L}{R_{i+1/2}^L}] \\
 &\quad - \lambda a_{i+1/2}^- [1 + \frac{1}{2} \Phi^R - \frac{1}{2} \frac{\Phi^R}{R_{i+1/2}^R}] \\
 &= \lambda [|a|_{i+1/2} + a_{i+1/2}^+ \frac{1}{2} (\Phi^L - \frac{\Phi^L}{R_{i+1/2}^L}) \\
 &\quad - a_{i+1/2}^- \frac{1}{2} (\Phi^R - \frac{\Phi^R}{R_{i+1/2}^R})] \\
 &= \lambda (|a|_{i+1/2} + a_{i+1/2}^+ \frac{1}{2} \delta - a_{i+1/2}^- \frac{1}{2} \delta)
 \end{aligned}$$

$$\begin{aligned}
&= \lambda |a|_{i+1/2} (1 + \frac{1}{2}\delta) \\
&\leq 1
\end{aligned} \tag{B.30}$$

where δ is the upper bound of limiters and $0 \leq \delta \leq 2$ is obtained from (B.17).

The stability requirement of explicit Euler type MUSCL methods under TVD conditions is

$$\begin{aligned}
CFL &= \lambda |a| \\
&\leq \frac{2}{2 + \delta}
\end{aligned} \tag{B.31}$$

Hence CFL requirement for each limiters is described as

- CFL of minmod limiter $\leq 2/3$ for $\delta=1$
- CFL of superbee limiter $\leq 1/3$ for $\delta=2$
- CFL of van Albada limiter ≤ 0.61 for $\delta=1.25$
- CFL of modified van Albada limiter ≤ 0.61 for $\delta=1.25$
- CFL of new β limiter $\leq \frac{2}{2+\delta}$ for $0 \leq \delta \leq 2$

BIBLIOGRAPHY

- [1] P. D. Lax. "Weak Solutions of Nonlinear Hyperbolic Equations and their Numerical Computations". *Commun. Pure Appl. Math.*, 7:159–193, 1954.
- [2] P. D. Lax and B. Wendroff. "Difference Schemes for Hyperbolic Equation with High Order of Accuracy". *Commun. Pure Appl. Math.*, 117:381–398, 1964.
- [3] P. D. Lax. "*Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*", SIAM Publications, 1973.
- [4] B. van Leer. "Towards the Ultimate Conservative Difference Scheme. II: Monotonicity and Conservation Combined in a Second Order Scheme,". *Journal of Computational Physics*, 14:361–370, 1974.
- [5] A. Harten. "High Resolution Schemes for Hyperbolic Conservative Laws ,". *Journal of Computational Physics*, 49:357–393, 1983.
- [6] H. C. Yee. "Upwind and Symmetric Shock Capturing Schemes,". NASA, TM-89464, 1987.
- [7] A. Harten. "On a Class of High Resolution Total Variation Stable Finite Difference Schemes,". *SIAM J. Num. Anal.*, 21:1–23, 1984.
- [8] H. C. Yee. "A Class of High-Resolution Explicit and Implicit Shock-Capturing Methods,". Technical report, NASA TM-101088, 1989.
- [9] H. C. Yee, G. H. Klopfer, and J. L. Montagne. "High-Resolution Shock-Capturing Schemes for Inviscid and Viscous Hypersonic Flows,". *Journal of Computational Physics*, 88:31–61, 1990.
- [10] B. van Leer. "Upwind-Difference Methods for Aerodynamic Problems Governed by the Euler Equations,". *Lecture in Applied Mathematics*, 22:327–336, 1985.
- [11] P. K. Sweby. "High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws,". *Siam J. Numer Anal*, 21(7):995–1011, 1984.
- [12] P. K. Sweby. "High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws,". *Lectures in Applied Mathematics*, 22:289–309, 1985.
- [13] S. Spekreijse. "Multigrid Solution of Monotone Second-Order Discretization of Hyperbolic Conservation Laws,". *Mathematics and Computations*, 49(179):135–155, 1987.

- [14] P. L. Roe. "Approximate Riemann Solver, Parameter Vectors and Difference Schemes,". *Journal of Computational Physics*, 43:357-372, 1981.
- [15] S. Osher and S. Chakravathy. "High Resolution Schemes and the Entropy,". *ICASE Report NASA CR*, 1772218, 1983.
- [16] J. L. Steger and R. F. Warming. "Flux Vector Splitting of the Inviscid Gasdynamics Equations with Application to finite Difference Methods,". *Journal of Computational Physics*, 40:263-293, 1981.
- [17] B. van Leer. "Flux-Vector Splitting for the Euler Equations,". *Lecture notes in Physics*, 170:507-512, 1982.
- [18] J. L. Anderson, W. K. Thomas and B. van Leer. "Comparison of finite Volume Flux vector Splittings for the Euler Equations,". *AIAA Journal*, 24:1453-1460, 1986.
- [19] D. Kroll, N. Gaitonde and M. Aftosmis. "A Systematic Comparative Study of Several High Resolution Schemes for Complex Problems in High Speed Flows,". *AIAA Paper*, 91-0603, 1991.
- [20] J. L. Roe P. L. van Leer, B. Thomas and R. W. Newsome. "A Comparison of Numerical Flux Formulas for the Euler and Navier-Stokes Equations,". *AIAA Paper*, 87-1104, 1987.
- [21] E. von Lavante. "Accuracy of Upwind Schemes Applied to Navier-Stokes Equations,". *AIAA Journal*, 28:1312-1314, 1990.
- [22] D. Gaitonde. "High-Speed Viscous Flows Past Blunt Bodies and Compression Corners with Flux-Split Methods,". *WL-TR*, 92-3018, March 1992.
- [23] D. Hanel, R. Schwane, , and G. Seider. "On the Accuracy of Upwind schemes for the Solution of the Navier-Stokes Equations ,". *AIAA Paper*, 87-1105CP, 1987.
- [24] D. Hanel and R. Schwane. "An implicit Flux-Vector Splitting Scheme for the Computati on of Viscous Hypersonic Flows ,". *AIAA Paper*, 89-0274, 1989.
- [25] B. P. Leonard. "Simple High Accuracy Resolution Program for Convective Modelling of Discontinuities,". *Int. J. Numer. Methods Fluids*, 8:1291-1318, 1988.
- [26] W. Jameson, A. Schmidt and E. Turkel. "Numerical Solution Schemes of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time Stepping schemes,". *AIAA Paper*, 81-11259, 1981.
- [27] B. van Leer. "Towards the Ultimate Conservative Difference Scheme. V: A Second Order Sequel to Godunov's Method,". *Journal of Computational Physics*, 32:101-136, 1979.
- [28] S. K. Godunov. "A Finite-Difference Method for the Numerical Computation of Discontinuous Solutions of the Equations of Fluid Dynamics,". *Mat. Sb*, 47:271-306, 1959.

- [29] B. van Albada, G. D. van Leer and Jr. Roberts. "A Comparative Study of Computational Methods in Cosmic Gas Dynamics,". *Astron. Astrophys.*, 108:76-84, 1982.
- [30] B. S. Baldwin and H. Lomax. "Thin Layer Approximation and Algebraic Model for Separated Turbulent Flows,". *AIAA Paper*, 78-257, 1978.
- [31] T. Cebeci and A. M. O. Smith. "*Analysis of Turbulent Boundary Layers*". Academic Press, 1974.
- [32] A. Jameson and P. D. Lax. "Conditions for the construction of multi-point total variation diminishing difference schemes". Technical report, University of Princeton MAE Report 1650, 1984.
- [33] C. W. Shu and S. Osher. "Efficient Implementation of Essentially Non-oscillatory Shock Capturing Schemes,". *Journal of Computational Physics*, 77:439-471, 1988.
- [34] C. W. Shu and S. Osher. "Efficient Implementation of Essentially Non-oscillatory Shock Capturing Schemes II,". *Journal of Computational Physics*, 83:32-78, 1989.
- [35] H. W. Liepmann and A. Roshko. "*Elements of Gasdynamics*,". Addison-Wesley, 1956.
- [36] R. F. Warming and R. M. Beam. "Upwind Second Order Difference Schemes and Applications in Aerodynamics,". *AIAA Journal*, 14:1241-1249, 1976.
- [37] T. H. Pulliam and J. Steger. "Recent Improvements in Efficient, Accuracy and Convergence for Implicit Approximate Factorization Algorithms,". *AIAA Paper*, 85-0360, 1985.
- [38] R. Magnus and Yoshihara. "Unsteady Transonic Flows over an Airfoil,". *AIAA Journal*, 13:1722-1628, 1975.
- [39] A. Lerat and J. Sides. "Numerical Simulation of Unsteady Transonic Flows Using the Euler Equations in Integral Form,". *Israel J. of Tech.*, 17:302-310, 1979.
- [40] J. L. Steger. "Implicit Finite-difference Simulation of Flow about arbitrary Two-dimensional Geometries,". *AIAA Journal*, 17:679-686, 1978.
- [41] J. L. Steger and H. E. Bailey. "Calculation of Transonic Aileron Buzz,". *Journal of Computational Physics*, 40:263-293, 1981.
- [42] W. J. Chyu, S. S. Davis, and K. S. Chang. "Calculation of Unsteady Transonic Flow over an Airfoil,". *AIAA Journal*, 19:684-690, 1979.
- [43] W. J. Chyu and S. S. Davis. "Numerical Studies of Unsteady Transonic Flow over an Oscillating Airfoil,". *AGARD*, CP-374, 1985.

- [44] D. M. Belk. "Unsteady Three-Dimensional Euler Equations Solutions on Dynamic Blocked Grids". PhD dissertation, Mississippi State University, 1986.
- [45] W. K. Anderson, J. L. Thomas, and C. L. Rumsey. "Extension and Applications of Flux-Vector Splitting to Unsteady Calculations on Dynamic Meshes,". *AIAA Paper*, 87-1152, 1987.
- [46] J. T. Batina. "Unsteady Euler Airfoil Solutions Using Unstructured Dynamic Meshes,". *AIAA Journal*, 28:1381-1388, 1990.
- [47] B. S. Stratford. "Flow in the Laminar Boundary Layer near Separation,". Technical report, British ARC Report and Memo 3002, 1954.
- [48] B. S. Stratford. "The Prediction of Separation of the Turbulent Boundary Layer,". *Journal of Fluid Mechanics*, 5:1-16, 1959.
- [49] A. M. O. Smith. "High Lift Aerodynamics,". *Journal of Aircraft*, 12:234-244, 1975.
- [50] R. H. Leibeck and Smith A. M. O. "A Class of Aorfoils Designed for High Maximum Lift in Incompressible Flow". Technical report, Douglas Aircraft Co. Report MDC-J1097/01, 1971.
- [51] R. H. Leibeck. "Design of Subsonic Airfoils for High Lift,". *Journal of Aircraft*, 15:547-561, 1978.
- [52] D. M Schuster and Brickelbaw. "Numerical Computations of Viscous Flowfields about Multiple Component Airfoils,". *AIAA Paper*, 85-0167, 1985.
- [53] E. Shima. "Numerical Analysis of Multiple Element High-Lift Devices by Navier-Stokes Equations Using Implicit TVD Finite-Volume Method,". *AIAA Paper*, 88-2574, 1988.
- [54] R. Chow and K. Chu. " Navier-Stokes Solution for High Lift Multielement Airfoil System with Flap Separation,". *AIAA Paper*, 91-1623, 1991.
- [55] K. J. Renze and K. Chu. "A Comparative Study of Turbulence Models for Overset Grids,". *AIAA Paper*, 92-0437, 1992.
- [56] S. E. Rogers. "Progress in High-lift Aerodynamic Calculations,". *AIAA Paper*, 93-0194, 1993.
- [57] F. D. Harris and R. R. Pruyn. " Blade Stall-Half Fact, Half Ficition,". *Journal of the American Helicopter Society*, 13(2):27-48, Apr 1968.
- [58] N. D. Ham and M. S. Garelick. " Dynamic Stall Considerations in Helicopter Rotors,". *Journal of the American Helicopter Society*, 13(2):49-55, Apr 1968.
- [59] N. D. Ham. " Aerodynamic loading on a Two-Dimensional Airfoil during Dynamic Stall,". *AIAA Journal*, 10:1927-1934, Oct 1968.

- [60] F. O. Carta. Experimental investigation of the unsteady aerodynamic characteristic of a naca 0012 airfoil,. Technical report, United Aircraft Corp., 1960.
- [61] J. Liiva and F. J. Davenport. "Dynamic Stall of Airfoil Sections for High-Speed Rotors ,". *Journal of the American Helicopter Society*, 14(2):26-33, Apr 1969.
- [62] W. J. McCroskey and R. K. Fisher. "Detailed Aerodynamic Measurements on a Model Rotor in the Blade Stall Regime,". *Journal of the American Helicopter Society*, 17(1):20-30, Jan 1972.
- [63] W. J. McCroskey, K. W. McAlister, and L. W. Carr. "Dynamic Stall Experiments on Oscillating Airfoils ,". *AIAA Journal*, 14(1):57-63, Jan 1976.
- [64] W. J. McCroskey. "The Phenomenon of Dynamic Stall,". Technical report, NASA TM-81264, 1981.
- [65] W. J. McCroskey, K. W. McAlister, L. W. Carr, and S. L. Pucci. "An Experimental Study of Dynamic Stall On Advanced Airfoil Sections ,. Technical report, NASA TM-84245, 1982.
- [66] W. J. McCroskey. "Unsteady Airfoils ,". *Ann. Rev. Fluid Mech*, 14:285-311, 1982.
- [67] K. W. McAlister, L. W. Carr, S. L. Pucci, and W. J. McCroskey. "Dynamic Stall Experiments on the NACA 0012 Airfoil,". Technical report, NASA TP-1100, 1978.
- [68] L. W. Carr. "Dynamic Stall Progress in Analysis and Prediction ,". *AIAA Paper*, 85-1796, 1985.
- [69] T. Theodorsen. "General Theory of Aerodynamic Instability and the Mechanism of Flutter,". Technical report, NACA Rept. 496, 1935.
- [70] T. von Karman and W. R. Sears. "Airfoil Theory for Nonuniform Motion,". *Journal of the Aeronautical Sciences*, 5(10), 1938.
- [71] U. B. Metha. "Dynamic Stall of an Oscillating NACA 0012 Airfoil,". Technical report, AGARD Paper 23, 1977.
- [72] K. Ono, K. Kuwahara and Oshima. "Numerical Analysis of Dynamic Phenomena of an Oscillating Airfoil by the Discrete Vortex Approximation,". Technical report, Proc. 7th ICNMFD STOC, 1981.
- [73] Y. Tassa and NL Sankar. "Dynamic Stall of NACA 0012 Airfoil ,". *AIAA Paper*, 81-1289, 1981.
- [74] K. Ono. "Numerical Study of Dynamic Stall Process of a NACA 0012 Airfoil,". *AIAA Paper*, 85-0128, 1985.
- [75] M. R. Visbal. "Evaluation of An Implicit Navier-Stokes Solver for Some Unsteady Separated Flows,". *AIAA Paper*, 86-1053, 1986.

- [76] M. R. Visbal. "Effect of Compressibility on Dynamic Stall of a Pitching Airfoil ,". *AIAA Paper*, 88-0132, 1988.
- [77] C. L. Rumsey and W. K. Anderson. "Some Numerical and Physical Aspects of Unsteady Navier -Stokes Computations over Airfoils Using Dynamic Meshes,". *AIAA Paper*, 88-0329, 1988.
- [78] K. Shida, Y. Ono and H. Takami. "Computation of Dynamic Stall of a NACA 0012 Airfoil ,". *AIAA Journal*, 25:408-413, 1987.
- [79] M. R. Visbal and J. S. Shang. "Investigation of the Flow Structure Around a Rapidly Pitching Airfoil ,". *AIAA Journal*, 30:911-922, 1992.
- [80] M. R. Visbal. "Structure of Vortex Breakdown on a Pitching Delta Airfoil ,". *AIAA Paper*, 94-0434, 1993.