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BLOCKS AND SOURCE ALGEBRAS
FOR THE DOUBLE COVERS OF THE SYMMETRIC GROUPS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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CHAPTER I

INTRODUCTION

A representation of a finite group G is a homomorphism $\rho : G \rightarrow GL(n, k)$ for some field k and integer $n > 1$. Equivalently, ρ may be thought of as a homomorphism of the group algebra $k[G]$ into the ring of all $n \times n$ matrices over k . The study of such representations and their associated characters was initiated by F.G Frobenius in 1896.

The group algebra of a finite group G over an algebraically closed field k is semi-simple if the characteristic of the field does not divide the order of G . In 1902 L. E. Dickson showed that in this case the representation theory of G over k is equivalent to its representation theory over the field of complex numbers. If, however, the characteristic of k does divide the order of G , the group algebra has a much more complicated structure. Block theory, the study of the indecomposable two-sided ideals of such group algebras (called block algebras) was first developed by Richard Brauer during the course of his efforts to understand the arithmetic properties of ordinary group characters by decomposing them modulo a rational prime. Among other things he established that if k has characteristic p and B is a

block algebra of the group algebra kG , then there is associated to B a unique (upto G -conjugacy) p -subgroup of D of G called the defect group of B . In particular, there is a close relationship between B and a unique block algebra b of $k[N_G(D)]$.

One of the problems of block theory is to describe the relationship between the defect group of a block and the other invariants of the block. Peter Donovan made the following conjecture in this context for a fixed p -group P .

CONJECTURE 1.1. (*Donovan [A]*). *Up to Morita equivalence there are only a finite number of block algebras of $k[G]$ with a fixed defect group P , where G ranges over all finite groups.*

An account of Morita's theory may be found in [CR]. Briefly, a Morita equivalence between two algebras means that their module categories are equivalent. There are many instances of Morita equivalence among blocks algebras of group algebras especially in the theory of the blocks of p -solvable groups. Also, the Clifford theory of blocks may be viewed as a series of Morita equivalences (see [B]).

The Morita type of a block algebra determines many of its numerical invariants like the entries of its Cartan matrix, the number and defects of irreducible modules belonging to it, etc. Hence Donovan's conjecture would in particular imply that, for any group P , all the above invariants for blocks having P as a defect group

are bounded by a function of $|P|$.

In [Sc] Joanna Scopes proved that Donovan's conjecture holds for the block algebras of the symmetric groups. She used the theory of Specht modules which provide a bridge between the character theory (both ordinary and modular) of the symmetric groups and the theory of partitions to find a criterion which forces two block algebras of $k[S_n]$ and $k[S_m]$ respectively, where S_n and S_m are respectively the symmetric groups on n and m letters, with a common defect group P to be Morita equivalent. Donovan's conjecture followed by showing that the block algebras with defect group P could be divided into finitely many families such that for any two elements b_1 and b_2 in the same family, there was a chain of blocks starting at b_1 and ending at b_2 with consecutive elements of the chain satisfying her criterion.

One of the new concepts introduced in representation theory in recent years is that of a source algebra for a block and is due to Lluís Puig (see Chapter II, §2). The source algebra of a block is Morita equivalent to the block algebra itself and also incorporates much other useful information about the block, such as the vertices and sources of its indecomposable modules, generalised decomposition numbers, its Brauer category etc. Puig stated the following conjecture at the 1982 conference on representations of finite groups held in Oberwolfach:

CONJECTURE 1.2. (*Puig*). *There are only finitely many isomorphism classes of interior P -algebras which are source algebras for blocks of finite groups with a fixed defect group P .*

In [P2], Puig showed that the Morita equivalence demonstrated by Scopes between two blocks b and c of the symmetric groups S_n and S_m respectively that satisfy her criterion may be obtained through an interior P -algebra isomorphism of the corresponding source algebras.

Before stating his precise result we need to introduce the following notation to which we will adhere throughout this thesis. The definitions of blocks, source algebras, and other related terms that are used in the statements of the subsequent results may be found in Chapter II.

Notation

Let $(K, \mathcal{O}, k)$ be a p -modular system, i.e. $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal $J(\mathcal{O})$, $k = \mathcal{O}/J(\mathcal{O})$ is its residue field with characteristic p and K is its field of quotients with characteristic zero. Suppose further that k is algebraically closed. A proof of the existence of such systems may be found in [NT, Chapter 1, §13].

THEOREM 1.3. (Puig). *Let b and c be as above and let $m < n$. There exists an isomorphism*

$$c(OS_nb)c \simeq OS_mc \otimes_{\mathcal{O}} \text{End}_{\mathcal{O}}(OS_{n-m}) \quad (1.1)$$

as interior $S_m \times S_{n-m}$ algebras.

In his proof of the above theorem, Puig makes use of the connections between the character theory of the symmetric groups and partitions only at the level of ordinary characters. In particular, no explicit information regarding the decomposition matrices of these groups is used and in fact, after establishing a suitable correspondence between the ordinary characters of the concerned blocks, the proof does not invoke the specific properties of the symmetric groups to a very large extent. Thus, as remarked by him in [P2,section 1.11], it seems that this method of proof may be applied to blocks of other families of finite groups where the ordinary characters may be parametrized in a way similar to that of the symmetric groups.

Issai Schur initiated the study of projective representations of finite groups (i.e homomorphisms of G into the projective general linear group $\text{PGL}(n,k)$) in the first decade of the century and showed that the projective representations of any finite group G were equivalent to the linear representations of certain covering groups of G called the representation groups of G . Besides developing the general theory, he

studied intensively the case of the projective representations of the symmetric and alternating groups. For $n > 3$, the groups $\tilde{S}_n$ and $\hat{S}_n$ (see (3.1) and (3.26)) are the only representation groups of S_n and they are non-isomorphic except when $n = 6$. As these are central extensions of S_n by groups of order 2, they are also known as the double covers of S_n . In [S], Schur completely enumerated the ordinary irreducible characters of these groups via a class of symmetric functions now known as the Q-functions and also gave degree formulae and some explicit character values. In a series of papers published in the 1960's A. O. Morris addressed many questions regarding the construction and manipulation of these characters by using the concept of shifted tableaux (see [M1, M2]). Since then, the projective representations of the symmetric and alternating groups have been of considerable interest to both combinatorialists and group theorists.

In this thesis we investigate the existence of equivalences of the type (1.1) in the blocks of the families of groups $\tilde{S}_n$ and $\hat{S}_n$ which would lead to the verification of Conjectures 1.1 and 1.2 for these families. We shall concentrate only on the groups $\tilde{S}_n$. However, as is explained in Remark 3.30 the character theory of $\hat{S}_n$ can be described completely in terms of the character theory of $\tilde{S}_n$ and all the results stated in this work for the family of groups $\{\tilde{S}_n\}$ may be proved for the family $\{\hat{S}_n\}$ in exactly the same fashion. We establish the following notation before stating our main result:

Notation

Let p be an odd prime and let P be a non-identity finite p group. By a faithful block of $\tilde{S}_n$ (see Chapter III, §3) we mean a block of $\tilde{S}_n$ that does not contain any ordinary irreducible characters that are “lifts” of characters of S_n . Let

$$\mathcal{B}_P = \{b : b \text{ a faithful block of } \tilde{S}_n, n \in \mathbb{Z}^+, P \text{ is a defect group of } b\},$$

$$\mathcal{B}_P^* = \{b : b \text{ a block of } \tilde{S}_n, n \in \mathbb{Z}^+, P \text{ is a defect group of } b\}.$$

For $b \in \mathcal{B}_P^*$, put $|b| = n$ if b is a block of $\tilde{S}_n$.

THEOREM 1.4. *There exists an integer N_P with the following property: For any b in $\mathcal{B}_P$ with $n = |b| > N_P$ there exists c in $\mathcal{B}_P$ with $m = |c| < |b|$ and an idempotent I of $(\mathcal{O}\tilde{S}_n b)^{\tilde{S}_m}$ such that*

$$I\mathcal{O}\tilde{S}_n b I \simeq \mathcal{O}\tilde{S}_m c \otimes_{\mathcal{O}} \text{Mat}_L(\mathcal{O}) \quad (1.2)$$

as interior $\tilde{S}_m$ -algebras. In particular, there is an interior P algebra isomorphism between a source algebra for b and a source algebra for c .

As is explained in Chapter III, §3, when p is an odd prime, faithful characters and non-faithful characters of $\tilde{S}_n$ always belong to different p blocks. Moreover, if a block b of $\tilde{S}_n$ contains non-faithful characters of $\tilde{S}_n$, then b is isomorphic to a block of S_n via the extension of θ to the group algebra of $\tilde{S}_n$ over $\mathcal{O}$ where θ is the

surjection of $\tilde{S}_n$ onto S_n described in (3.2) and then the isomorphism described by Puig in (1.1) for two blocks b and c of S_n and S_m respectively immediately lifts via θ to an isomorphism for the corresponding blocks of $\tilde{S}_n$ and $\tilde{S}_m$. Then Scopes' criterion (cf. [P2, eq. 1.71]) insures the existence of an integer N'_P such that the Theorem (1.4) holds when we consider instead the set of non-faithful blocks of the family $\{\tilde{S}_n\}$ with defect group P (in fact we may choose c such that $I = bc$). Hence, from Theorem (1.4) and [P2, Theorem 1.7], we get

COROLLARY 1.5. *Theorem 1.4 is true if we replace B_P by B_P^* .*

Consequently, we get

COROLLARY 1.6. *Puig's conjecture holds for the p -blocks of the groups $\tilde{S}_n$, for every odd prime p .*

COROLLARY 1.7. *Donovan's conjecture holds for the p -blocks of the groups $\tilde{S}_n$, for every odd prime p .*

The following is an outline of what is contained in the subsequent chapters of this work. Chapter II is divided into three sections. Section 1 will state some general results about algebras over a complete discrete valuation ring, versions of the idempotent lifting lemma, the Krull-Schmidt theorem and some results on matrix algebras. Section 2 deals with algebras which have a group action. We give definitions of the Brauer homomorphisms, pointed groups, defect pointed groups,

interior group algebras and the source algebras. We also state and prove Lemma 2.24 which will be needed in Chapter III. In Section 3 we apply the definitions of the previous section to the case of a group algebra, $\mathcal{O}G$, of a finite group G and state some general results on the distribution of ordinary irreducible characters of G into p -blocks.

Chapter III is devoted to the double covers of the symmetric groups. In Section 1, we give the presentation of $\tilde{S}_n$ via generators and relations, define the groups $\tilde{A}_n$ and state some facts about partitions and conjugacy classes in $\tilde{S}_n$. In Section 2, we state Schur's theorems parametrizing the irreducible characters of $\tilde{S}_n$ and $\tilde{A}_n$ by the strict partitions of n , which gives also the values of characters on conjugacy classes not corresponding to the partitions of n with only odd parts. We also introduce the concept of r -bars of partitions which allow us to state the analogue of the Murnaghan-Nakayama formula given by A. O. Morris to calculate inductively the remaining character values and to determine the restriction of characters to a subgroup $\tilde{S}_m$ of $\tilde{S}_n$. Section 3 deals with the work of J.F Humphreys, Meir-Tappe, Joern Olsson, and Marc Cabanes in describing the distribution of irreducible characters into p -blocks via the p -bar cores of the corresponding partitions, the description of the defect groups as Sylow p -subgroups of $\tilde{S}_{pw}$ where w is the weight of the block and a description of the faithful blocks of $\tilde{A}_n$ in terms of the blocks of $\tilde{S}_n$.

Having laid out the basic framework in which our problem is set, we give a proof of Theorem 1.4 in Chapters IV and V. We start with a 1 – 1 correspondence between the set of faithful blocks of $\tilde{S}_n$ of a fixed weight (and hence with a fixed defect group) with the set of strict partitions which are their own p -bar cores. In section 1 of Chapter IV, we define the Scopes maps on $\mathbb{Z}/p\mathbb{Z}$, and on the set of strict partitions, prove some of their properties and find a criterion which forces a suitable correspondence between strict partitions having a fixed weight w and p -bar core μ and strict partitions of weight w and p -bar core $\tilde{S}_{c_x}(\mu)$. This leads to an estimate for N_P , and a method of choosing the block c of the statement of Theorem (1.4) given a block b in $\mathcal{B}_P$ for which $|b| > N_P$. These results along with those of Chapter III are used to obtain a rank comparison between certain subalgebras of $\mathcal{O}\tilde{S}_{|b|}$ and $\mathcal{O}\tilde{S}_{|c|}$ in Section 2.

Chapter V shows the existence of the isomorphism (1.2) by following the approach used by Puig in [P2,section 4] and we end by proving Corollaries 1.6 and 1.7.

The proof of Theorem 1.4 is modeled entirely on the methods used by Scopes and Puig for the symmetric groups but there are some differences in the two cases. Firstly, a strict partition of n labels either a single faithful irreducible character of $\tilde{S}_n$ or a pair of associate faithful irreducible characters of $\tilde{S}_n$ depending on its parity (see Theorem 3.6). So, given a strict partition μ which is its own p -

bar core and an x such that condition (4.23) is satisfied, if μ and $\widetilde{Sc}_x(\mu)$ do not have the same parity, we are unable to get a suitable correspondence between the irreducible characters belonging to the blocks b_μ and $b_{\widetilde{Sc}_x(\mu)}$ respectively. This difficulty is circumvented at present by using Proposition 4.12 and taking for c instead the block corresponding to a partition obtained by successively applying the Scopes maps to μ finitely many times. However, if $b_{\widetilde{Sc}_x(\mu)}$ is considered instead as a block of $\tilde{A}_{|\widetilde{Sc}_x(\mu)|}$ there does exist a “nice” correspondence at the level of ordinary characters and it is possible that the source algebras of b_μ , considered as a block for $\tilde{S}_{|b_\mu|}$, and of $b_{\widetilde{Sc}_x(\mu)}$, considered as a block of $\tilde{A}_{|\widetilde{Sc}_x(\mu)|}$, are isomorphic.

Also, if $G = S_n$ and if the p -group P is a defect group for a block b of G , then $N_G(P)$ induces inner automorphisms on $C_G(P)/Z(P)$. On the other hand, if $G = \tilde{S}_n$, $N_G(P)$ induces an outer automorphism of order two on $C_G(P)/Z(P)$ and this causes the group $\tilde{S}_m$ to have two points on the algebra A (see Chapter V) whereas in the corresponding situation for the symmetric groups, there is only a single point of S_m on $c\mathcal{O}S_nbc$.

CHAPTER II

ALGEBRAS

As mentioned in the introduction, throughout this thesis, $(K, \mathcal{O}, k)$ will denote a p -modular system and we will work under the assumption that k is algebraically closed although this is not strictly needed for some of the results of the next sections.

§1. Miscellaneous results on $\mathcal{O}$ algebras

In this section, A will denote an associative $\mathcal{O}$ algebra, with unit element, finitely generated as $\mathcal{O}$ module and either $\mathcal{O}$ -free or such that $J(\mathcal{O})A = 0$, i.e such that A is also a k algebra.

Let A^* denote the group of invertible elements of $\mathcal{O}$ and $J(A)$ the Jacobson radical of A , i.e $J(A)$ is the intersection of all maximal ideals of A .

DEFINITION 2.1. Two primitive idempotents e and f of A are called *associate* if there exist elements a in eAf and b in fAe such that $ab = e$ and $ba = f$.

We now state a special case of the well known idempotent lifting lemma:

LEMMA 2.2. *An idempotent e of A is primitive in A iff $e + J(\mathcal{O})A$ is primitive*

in $A/J(\mathcal{O})A$ and two primitive idempotents e and f are associate in A if $e+J(\mathcal{O})A$ and $f+J(\mathcal{O})A$ are associate in $A/J(\mathcal{O})A$.

PROOF. See [NT, Theorem 14.2]. ■

PROPOSITION 2.3. *Two idempotents e and f are associate in A if and only if they are conjugate in A .*

PROOF. Chapter 2 of [K] provides a proof for the case $J(\mathcal{O})A = 0$. However, it is easy to check that Lemma 2.2 allows us to adapt the proof for the general case. The same remark applies to the proofs of 2.5, 2.6 and 2.10 below. ■

DEFINITION 2.4. A *point* of A is a conjugacy class of primitive idempotents of A and we let $\mathcal{P}(A)$ be the set of points of A .

We now state a generalized version of the idempotent lifting lemma:

THEOREM 2.5. *Let I be a two sided ideal of A , and let α be a point of A . Then either $\alpha \subset I$ or $\alpha + I \in \mathcal{P}(A/I)$ and the map $\alpha \rightarrow \alpha + I$ is a bijection from the set of $\alpha \in \mathcal{P}(A)$ such that $\alpha \not\subset I$ onto $\mathcal{P}(A/I)$.*

PROOF. See [K, 4.5] ■

THEOREM 2.6. *Let*

$$\sum_{\alpha \in \mathcal{P}(A)} \sum_{i=1}^{i=k_\alpha} e_i^\alpha = 1_A = \sum_{\beta \in \mathcal{P}(A)} \sum_{j=1}^{j=l_\beta} f_j^\beta \quad (2.1)$$

be two orthogonal primitive idempotent decompositions of the identity, where $e_i^\alpha \in \alpha$ and $f_j^\beta \in \beta$ for $\alpha, \beta \in \mathcal{P}(A), i = 1, \dots, k_\alpha, j = 1, \dots, l_\beta$. Then, $k_\alpha = l_\alpha$ for $\alpha \in \mathcal{P}(A)$, and there is a unit u in A such that $ue_i^\alpha u^{-1} = f_i^\alpha$ and $i = 1, \dots, k_\alpha$.

REMARK 2.7. Since a primitive idempotent decomposition

$$1_A = \sum_{\alpha \in \mathcal{P}(A)} \sum_{i=1}^{i=k_\alpha} e_i^\alpha \quad (2.2)$$

of the identity in A corresponds to a decomposition

$$A_A = \sum_{\alpha \in \mathcal{P}(A)} \sum_{i=1}^{i=k_\alpha} e_i^\alpha A \quad (2.3)$$

of the right A -module A into indecomposable summands with $e_i^\alpha A \simeq e_j^{\alpha'} A$ iff $\alpha = \alpha'$ (cf. [NT, Theorems 4.2 and 4.3]), the above theorem is just a version of the Krull-Schmidt theorem.

DEFINITION 2.8. (i) The *multiplicity* of any $\alpha \in \mathcal{P}(A)$, denoted m_α is defined to be the number of elements of α that appear in any primitive idempotent decomposition of the identity. By the theorem above, m_α is well defined. Let e be an idempotent of A . We will write

$$e = m_\alpha \alpha \quad (2.4)$$

if a decomposition of e into primitive idempotents contains elements only from α and there are m_α of these. Naturally, in that case no elements of α appear in any decomposition of $1 - e$ into primitive idempotents.

(ii) We define the m -rank of a , denoted $r_m(A)$ by

$$r_m(A) := \sum_{\alpha \in \mathcal{P}(A)} (m_\alpha)^2. \quad (2.5)$$

We now state a version of the Wedderburn-Malcev theorem.

THEOREM 2.9. *Let A be $\mathcal{O}$ free. Then*

(i) There is a unitary subalgebra T of A such that

$$T \simeq \prod_{\alpha \in \mathcal{P}(A)} \text{Mat}_{m_\alpha}(\mathcal{O}) \quad (2.6)$$

and $A = T \oplus J(A)$. Here, as usual, $\text{Mat}_n(\mathcal{O})$ denotes the $n \times n$ matrix algebra over $\mathcal{O}$.

(ii) If T' is a subalgebra of A isomorphic to a direct product of matrix algebras over $\mathcal{O}$, there exists $a \in A^$ such that $T' \subset aTa^{-1}$.*

PROOF. This is contained in [P1, Proposition 2.3 and Corollary 2.4]. ■

We will also need

LEMMA 2.10. *(Rosenberg) Let e be a primitive idempotent of the $\mathcal{O}$ -algebra A and let $I_1, \dots, I_n$ be ideals of A such that $e \in I_1 + \dots + I_n$. Then $e \in I_j$ for some $j \in \{1, \dots, n\}$.*

PROOF. See [K, 5.1] ■

Let B be a subalgebra of A . We denote by $C_A(B)$ the subalgebra of all the elements of A that commute with every element of B . In Chapter V, we will need the following result:

PROPOSITION 2.11. *Let T be a subalgebra of A containing 1_A . Suppose that T is isomorphic to the full $n \times n$ matrix algebra over $\mathcal{O}$ or k . Let j be a primitive idempotent of T . Then,*

$$C_A(T) \simeq jAj \text{ via } c \rightarrow cj = jc \quad (2.7)$$

and

$$T \otimes C_A(T) \simeq A \text{ via } t \otimes c \rightarrow tc. \quad (2.8)$$

PROOF. $\mathcal{P}(T)$ has just one element, of multiplicity n . Let j be any primitive idempotent in T . Then there is a subset V of T^* such that

$$T = \sum_{v, v' \in V} \mathcal{O} \cdot (v^{-1} j v') \text{ and } j^v j^{v'} = 0 \text{ unless } v = v'. \quad (2.9)$$

In particular,

$$A = \sum_{v, v' \in V} j^v A j^{v'}. \quad (2.10)$$

Consider the map

$$f : T \otimes_{\mathcal{O}} jAj \rightarrow A$$

defined by

$$f\left(\sum_{v,v' \in V} (v^{-1}jv') \otimes a_{v,v'}\right) = \sum_{v,v' \in V} v^{-1}av'. \quad (2.11)$$

It is easy to check using (2.9) and (2.10) that f is an isomorphism. Moreover, $f(T \otimes j) = T$ and $C_{T \otimes_{\mathcal{O}} jAj}(T \otimes j) = 1_A \otimes jAj$, i.e, $f(1_A \otimes jAj) = C_A(T)$ and if $c \in C_A(T)$, then $f^{-1}(c)$ is $1 \otimes jc$. Hence $c \rightarrow jc$ is an isomorphism of $C_A(T)$ onto jAj proving (2.7).

Now if

$$g : T \otimes_{\mathcal{O}} C_A(T) \rightarrow T \otimes_{\mathcal{O}} jAj$$

is the map given by

$$g\left(\sum_{v,v' \in V} (v^{-1}jv') \otimes c_{v,v'}\right) = \sum_{v,v' \in V} (v^{-1}jv') \otimes jc_{v,v'}j, \quad (2.12)$$

then $f \circ g$ is the isomorphism required in (2.8). ■

COROLLARY 2.12. *Let A and T be as in the proposition. Let $1 = e_1 + e_2$ be a decomposition of the identity in T . Set $B = e_1Te_1 \oplus e_2Te_2$. Then,*

$$C_A(B) = e_1C_A(T) \oplus e_2C_A(T) \simeq C_A(T) \oplus C_A(T). \quad (2.13)$$

PROOF. By (2.8), we have

$$C_A(B) \simeq C_{T \otimes_{\mathcal{O}} C_A(T)}(B \otimes_{\mathcal{O}} \mathcal{O}) = C_T(B) \otimes_{\mathcal{O}} C_A(T) = (\mathcal{O}e_1 \oplus \mathcal{O}e_2) \otimes_{\mathcal{O}} C_A(T) \quad (2.14)$$

which proves the corollary. ■

LEMMA 2.13. *Let A be an $\mathcal{O}$ -free $\mathcal{O}$ -algebra of finite rank. Let $1 = e_1 + e_2$ be an idempotent decomposition of the identity. Further, let B_1 be a unitary subalgebra of $e_1 A e_1$ and let B_2 be a unitary subalgebra of $e_2 A e_2$. If $B = B_1 \oplus B_2$, then,*

$$C_A(B) = C_{e_1 A e_1}(B_1) \oplus C_{e_2 A e_2}(B_2). \quad (2.15)$$

PROOF. As e_1 and e_2 are orthogonal, $C_A(B) \supset C_{e_1 A e_1}(B_1) \oplus C_{e_2 A e_2}(B_2)$ and

$$A = e_1 A e_1 \oplus e_1 A e_2 \oplus e_2 A e_1 \oplus e_2 A e_2 \quad (2.16)$$

as $\mathcal{O}$ -modules.

Let $a \in e_i A e_j \cap C_A(B)$ where $i, j \in \{1, 2\}, i \neq j$. Then $a = e_i a = a e_i = 0$.

Hence, $C_A(B) \subset e_1 A e_1 \oplus e_2 A e_2$, proving the lemma. ■

§2. Groups acting on algebras

Let A be as in Section 1 and let G be a finite group. Suppose that A is also a G -algebra, i.e there is a homomorphism from G into $\text{Aut}(A)$, the group of automorphisms of A . For $a \in A, g \in G$, we will denote by ${}^g a$ the image of a under the action of g .

Let K and H be subgroups of G , with $K \subset H$. We denote by A^H the subalgebra of elements of A that are fixed under the action of H . Clearly, A^H is invariant under the action of $N_G(H)$ and thus A^H is naturally an $N_G(H)$ -algebra.

DEFINITION 2.14. The *relative trace map from K to H* is the map:

$$\mathrm{Tr}_K^H : A^K \rightarrow A^H \text{ given by } a \rightarrow \sum_{hK \in H/K} {}^h a.$$

Then

$$A_K^H := \mathrm{Tr}_K^H(A^K)$$

is an ideal in A^H and hence so is

$$A_{<H}^H := \sum_{K \subsetneq H} A_K^H.$$

Let

$$A(H) := A^H / (A_{<H}^H + J(\mathcal{O})A^H).$$

Note that $A(H)$ is annihilated by $J(\mathcal{O})$ and it is thus a k -algebra. Also, the ideal

$A_{<H}^H + J(\mathcal{O})A^H$ is invariant under $N_G(H)$ and A^H is left fixed by H . Hence $A(H)$

is a natural $N_G(H)/H$ algebra, via

$${}^n H(a + A_{<H}^H + J(\mathcal{O})A^H) = {}^n a + A_{<H}^H + J(\mathcal{O})A^H, \text{ for } n \in N_G(H), a \in A^H. \quad (2.17)$$

DEFINITION 2.15. The *Brauer homomorphism A with respect to H* is the natural map:

$$\mathrm{Br}_H : A^H \rightarrow A(H).$$

By equation (2.17) Br_H is a homomorphism of $N_G(H)$ algebras.

Let P be a Sylow p -subgroup of H and let $a \in A^H$. It is immediate from the definition of the relative trace map that

$$\text{Tr}_P^H(a) = [H : P]a. \quad (2.18)$$

As $[H : P]$ is a unit in $\mathcal{O}$, we get that $a \in A_P^H$ and hence that $A^H = A_P^H \subset A_{<H}^H$ if P is proper in H . So,

$$A(H) \neq 0 \Rightarrow H \text{ is a } p \text{ group.} \quad (2.19)$$

DEFINITION 2.16. Let G be a finite group and A a G -algebra.

(i) A *pointed group on A* is a pair $(H, \beta) =: H_\beta$ where H is a subgroup of G and β is a point of A^H . We also refer to β as a point of H on A . Let H_β and K_γ be two pointed groups on A . We say that K_γ is contained in H_β and write $K_\gamma \leq H_\beta$ if $K \subset H$ and for any i in β , there exists a j in γ such that $ji = j = ij$.

(ii) The group G acts on the set of pointed groups on A via

$${}^g(H_\beta) = (gHg^{-1}) {}_g\beta. \quad (2.20)$$

The *normalizer of H_β in G* , denoted $N_G(H_\beta)$ is defined as

$$N_G(H_\beta) := \{g \in G : {}^g(H_\beta) = H_\beta\}.$$

Clearly,

$$H \subset N_G(H_\beta) \subset N_G(H).$$

By the structure theorems for $\mathcal{O}$ -algebras, there is a unique maximal ideal M_β of A^H not containing β . For $g \in G$, ${}^g(M_\beta) = M_{{}^g\beta}$ and thus M_β is left invariant by $N_G(H_\beta)$. Thus, if we set

$$A(H_\beta) := A^H/M_\beta,$$

then $A(H_\beta)$ is naturally an $N_G(H_\beta)/H$ -algebra over k .

DEFINITION 2.17. We let

$$S_\beta : A^H \rightarrow A(H_\beta), \quad a \mapsto a + M_\beta$$

be the natural surjection. It is clear that S_β is a homomorphism of $N_G(H_\beta)$ algebras.

DEFINITION 2.18. A pointed group H_β on A is called *local* if $\text{Br}_H(i) \neq 0$ for some (and hence any) $i \in \beta$. and in that case β is also called a local point of H on A . Note that by (2.19), if H_β is a local pointed group, then H is a p -group.

Let us now state a theorem of Puig completing a result of Green :

THEOREM 2.19. *Let H_β be a pointed group on A .*

(i) *If P is a minimal subgroup of H such that $\beta \subset A_P^H$, there exists a local point γ of P on A such that $P_\gamma \subset H_\beta$ and $\beta \subset \text{Tr}_P^H(A^P \gamma A^P)$.*

(ii) *A local pointed group P_γ contained in H_β is a maximal one if and only if $\beta \subset A_P^H$.*

(iii) The group H acts transitively via conjugation on the set of maximal local pointed groups P_γ such that $P_\gamma \subset H_\beta$.

PROOF. See [P1, Theorem 1.2]. ■

DEFINITION 2.20. The maximal local pointed groups P_γ such that $P_\gamma \subset H_\beta$ are called the *defect pointed groups* of H_β .

DEFINITION 2.21. Let G be a finite group. An *interior G -algebra* is a pair (A, ϕ) , where A is an $\mathcal{O}$ -algebra, and ϕ is a homomorphism from G into the group of units of A . If H is a subgroup of G , then $(A, \phi|_H)$ is an interior H -algebra.

Any interior G -algebra (A, ϕ) gives A a left $\mathcal{O}G$ -module structure via $ga := (\phi(g))a$. Similarly setting $ag := a(\phi(g))$ gives it a right $\mathcal{O}G$ -module structure. In this way A becomes an $\mathcal{O}G$ -bimodule. This bimodule satisfies the following associativity relation:

$$(ag)b = a(bg), \quad (a, b \in A, g \in \mathcal{O}G). \quad (2.21)$$

We will therefore omit parenthesis and write agb instead. For $g \in G$, we have $\phi(g) = g1_A = 1_Ag$. Thus ϕ is uniquely determined by the left $\mathcal{O}G$ -module structure of A . We will usually omit ϕ and define the interior G -algebra structure on A by defining the induced left $\mathcal{O}G$ -module structure instead.

DEFINITION 2.22. Let A and B be two interior G -algebras. An algebra homomorphism $\Gamma : A \rightarrow B$ is said to be an *interior G -algebra homomorphism* if

$$\Gamma(g.a) = g\Gamma(a) \text{ and } \Gamma(ag) = \Gamma(a)g \text{ for all } a \in A, g \in G. \quad (2.22)$$

An interior G -algebra is naturally an G -algebra, via ${}^ga = gag^{-1}$ and whenever we are looking at an $\mathcal{O}$ -algebra as both an interior G algebra and a G -algebra the latter structure will be the natural one inherited from the former.

Then (2.22) is easily seen to be equivalent to the condition

$$\Gamma(1_A) \in B^G \text{ and } \Gamma(g1_A) = g\Gamma(1_A) \text{ for any } g \in G. \quad (2.23)$$

DEFINITION 2.23. Let A be an interior G -algebra. An interior G -subalgebra of A is a subalgebra B (not necessarily containing 1_A) of A which is also an $\mathcal{O}G$ -subbimodule of A and in this case B itself becomes an interior G -algebra. Further, if I is an ideal of A , then A/I inherits a natural interior G -algebra structure, via $g \rightarrow g + I$.

Again, let H be a subgroup of G . Clearly, A^H contains the image of $C_G(H)$ in A and thus A^H is an interior $C_G(H)$ algebra. This allows us to consider $A(H)$ and $A(H_\beta)$ as interior $C_G(H)$ -algebras over k for $\beta \in \mathcal{P}(A^H)$ and as such Br_H and S_β are homomorphisms of interior $C_G(H)$ -algebras.

In Chapter V, Section 3 we will need the following lemma:

LEMMA 2.24. *Let H and H' be two groups with $H' \trianglelefteq H$. Let A be an interior H algebra. Suppose β is a point of H on A such that $\beta \in A_{H'}^H$. Then if δ_1 and δ_2 are two points of H' on A such that $H_\beta \supset H'_{\delta_1}, H'_{\delta_2}$, then δ_1 and δ_2 are H conjugate, i.e. ${}^h\delta_1 = \delta_2$ for some h in H .*

PROOF. It suffices to show that for some $j \in \delta_1$, there exists an $h \in H$ such that ${}^hj \in \delta_2$. Let $i \in \beta$ and set $B = iAi$. Then B is an interior H algebra (as i commutes with H) and $B^H = iA^Hi$. Since i is primitive in A^H , we get that $i = 1_B$ is the unique idempotent in B^H .

H acts on $\mathcal{P}(B^{H'})$ since H' is normal in H .

CLAIM. *All points of H' on B are H -conjugate.*

The lemma follows immediately from the claim:

As $B^{H'} = iA^{H'}i : \mathcal{P}(B^{H'}) = \{\alpha \in \mathcal{P}(A^{H'}), \alpha \cap B \neq \emptyset\}$. Also, $\delta_1 \cap B$ and $\delta_2 \cap B$ are both non-empty by the hypothesis of the lemma. By the claim there exists $h \in H$ such that ${}^h(\delta_1 \cap B) = \delta_2 \cap B$ and we are done.

PROOF OF CLAIM:.. We have $\beta \in A_{H'}^H$. So there exists $a \in A^{H'}$ such that $i = \text{Tr}_{H'}^H(a)$. So

$$i = i^3 = i \text{Tr}_{H'}^H(a) i = \text{Tr}_{H'}^H(ia i) \in B_{H'}^H. \quad (2.24)$$

Let $Cl_{\mathcal{P}(B^{H'})}(H)$ denote the set of orbits of the H -action on $\mathcal{P}(B^{H'})$. For $\Omega \in Cl_{\mathcal{P}(B^{H'})}(H)$, let $I_\Omega = \sum_{\gamma \in \Omega} B^{H'} \gamma B^{H'}$ where by $B^{H'} \gamma B^{H'}$ we mean the

ideal generated in $B^{H'}$ by any element of γ . We have

$$h(I_\Omega) \subset I_\Omega \text{ for } h \in H. \quad (2.25)$$

Also,

$$B^{H'} = \sum_{\gamma \in \mathcal{P}(B^{H'})} B^{H'} \gamma B^{H'} = \sum_{\Omega \in Cl_{\mathcal{P}(B^{H'})}(H)} I_\Omega. \quad (2.26)$$

Thus

$$\text{Tr}_{H'}^H(B) = \sum_{\Omega \in Cl_{\mathcal{P}(B^{H'})}(H)} \text{Tr}_{H'}^H(I_\Omega). \quad (2.27)$$

The idempotent i is primitive in B^H , and so by Rosenberg's Lemma (Lemma 2.10) and equations (2.24), (2.25) and (2.27),

$$1_B = i \in \text{Tr}_{H'}^H(I_{\Omega_0}) \subset I_{\Omega_0} \text{ for some } \Omega \in Cl_{\mathcal{P}(B^{H'})}(H). \quad (2.28)$$

In other words, $B^{H'} = I_{\Omega_0}$. Let

$$\pi : B^{H'} \rightarrow B^{H'} / J(B^{H'})$$

be the usual surjection. Denote by $\pi(\gamma)$, the point of $B^{H'} / J(B^{H'})$ containing $\pi(j)$ where j is any idempotent in γ . By the idempotent lifting lemma (Theorem 2.5), this notation makes sense. Then,

$$B^{H'} / J(B^{H'}) = \pi(I_{\Omega_0}) = \sum_{\gamma \in \Omega_0} (B^{H'} / J(B^{H'})) \pi(\gamma) (B^{H'} / J(B^{H'})). \quad (2.29)$$

$B^{H'}/J(B^{H'})$ is a direct sum of matrix algebras over k and it is easy to see that the ideal generated by any primitive idempotent of such an algebra is a simple summand of it and hence the extreme terms of the above equation yield:

$$\mathcal{P}(B^{H'}/J(B^{H'})) = \{\pi(\gamma) : \gamma \in \Omega_0\}. \quad (2.30)$$

The claim now follows from the idempotent lifting lemma, thus completing the proof of lemma 2.24

■

DEFINITION 2.25. Let A be an interior G -algebra and let α be a point of G on A . Suppose that P_γ is a defect pointed subgroup of G_α . A *source algebra* of A_α is an algebra of the type iAi , where $i \in \gamma$. As $i \in A^P$, the map $p \rightarrow ip = pi$ is a homomorphism from P into $(iAi)^*$ and thus a source algebra is an interior P -algebra. Note that a source algebra depends both on the choice of the local point α of P on A and on the choice of $i \in \alpha$. However, if i' is another element of α , then there exists a unit u in A^P , such that $i' = ui$ and the map

$$f : iAi \rightarrow i'Ai' \text{ defined by } f(a) \rightarrow u_a$$

is clearly an isomorphism of interior P -algebras such that $f(pi) = pi'$ for $p \in P$. In other words f is an interior P -algebra isomorphism. This means that up to isomorphism as interior P -algebras, source algebras are independent of the choice of $i \in \alpha$ for a fixed α .

So the number of interior P -algebra isomorphism classes of source algebras for A_α is at most equal to the number of local points of P on A which by Theorem (2.19) is

$$[N_G(P) : N_G(P_\gamma)] \leq |\text{Aut}(P)|. \quad (2.31)$$

§3. Group algebras

In this section we will let $A = \mathcal{O}G$, the group algebra over $\mathcal{O}$ of the finite group G . Then A is of course an interior G -algebra (via $g \rightarrow g$). Let

$$\pi : \mathcal{O}G \rightarrow kG, \quad \sum_{g \in G} a_g g \rightarrow \sum_{g \in G} (a_g + J(\mathcal{O}))g$$

be the natural map. We will denote by $\bar{a}$ the image under π of any element a of $\mathcal{O}G$.

Let H be a subgroup of G . H acts on the group G via conjugation, i.e via ${}^h g = hgh^{-1}$. We call the orbits of this action the H -conjugacy classes of G and denote the set of all orbits by $Cl_G(H)$. It is easy to see that the set $\{C^+ : C \in Cl_G(H)\}$ is a basis for the algebra A^H . Here, for any subset X of G , $X^+ := \sum_{x \in X} x \in \mathcal{O}G$.

Now suppose that L and H are subgroups of G with $L \subset H$ and let $g \in D \in Cl_G(L)$. Then

$$D^+ = \sum_{d \in D} d = \sum_{l \in L/C_L(g)} {}^l g = \text{Tr}_{C_L(g)}^L(g). \quad (2.32)$$

Hence,

$$\begin{aligned}
\mathrm{Tr}_L^H(D) &= \mathrm{Tr}_L^H(\mathrm{Tr}_{C_L(g)}^L(g)) = \mathrm{Tr}_{C_L(g)}^H(g) = \mathrm{Tr}_{C_H(g)}^H(\mathrm{Tr}_{C_L(g)}^{C_H(g)}(g)) \\
&= \mathrm{Tr}_{C_H(g)}^H([C_H(g) : C_L(g)]g) = [C_H(g) : C_L(g)]\mathrm{Tr}_{C_H(g)}^H(g) \\
&= [C_H(g) : C_L(g)]C^+.
\end{aligned} \tag{2.33}$$

where C denotes the H -conjugacy class of G containing g .

It is clear from (2.33) that the elements $C^+ + J(\mathcal{O})A^H$, $C \in Cl_G(H)$ such that p does not divide $[C_H(g) : C_L(g)]$ form a k -basis for $(A_L^H + J(\mathcal{O})A^H)/J(\mathcal{O})A^H$.

Now suppose that Q is a p -subgroup of H . Applying the above remarks to the case where $H = Q$, we get that the elements $C^+ + J(\mathcal{O})A^H$, $C \in Cl_G(H)$, such that $C_Q(g)$ is proper in Q for some $g \in C$ form a basis for $(A_{<Q}^Q + J(\mathcal{O})A^H)/J(\mathcal{O})A^H$.

As

$$A^Q = \mathcal{O}C_G(Q) \oplus \sum_{\substack{C \in Cl_G(H), \\ C_Q(g) \subsetneq Q}} \mathcal{O}C^+, \tag{2.34}$$

we have that

$$\mathrm{Ker}(\mathrm{Br}_Q) = J(\mathcal{O})C_G(Q) \oplus \sum_{\substack{C \in Cl_G(H), \\ C_Q(g) \subsetneq Q}} \mathcal{O}C^+. \tag{2.35}$$

Hence

$$A(Q) \simeq kC_G(Q) \text{ via } \mathrm{Br}_Q(a) \rightarrow \bar{a} \text{ for } a \in \mathcal{O}C_G(Q). \tag{2.36}$$

The composite of Br_Q with the above isomorphism is a map from A^Q onto $kC_G(Q)$

which sends $\sum_{g \in G} a_g g$ to $\sum_{g \in C_G(P)} \bar{a}_g g$ and from now on we shall identify Br_Q

with this composite map.

DEFINITION 2.25. (i) The algebra $(\mathcal{O}G)^G$ is just the center $Z(\mathcal{O}G)$ of $\mathcal{O}G$ and hence any point of G on $A = \mathcal{O}G$ consists of a single primitive idempotent of $Z(A)$. Any such idempotent is called a *block* of A . Let b be a block of G . We write G_b for the pointed group G_α where $\alpha = \{b\}$. The block algebra of b is the algebra $\mathcal{O}Gb$ which is an interior G -algebra via $g \rightarrow gb$.

(ii) A *defect group* of b is a group P such that P_γ is a defect pointed subgroup of G_b .

Now suppose that K is a splitting field for G , and let χ be an irreducible character of G over K and C a conjugacy class in G . Let

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x \quad (2.37)$$

be the central idempotent of KG associated with χ and let the function

$$\omega_\chi : Z(KG) \rightarrow K \text{ defined by } \omega_\chi(C^+) = \frac{|G|\chi(x)}{|C_G(x)|\chi(1)} \quad (2.38)$$

be the central character of KG associated with χ , where C^+ is the sum of all elements in C and $x \in C$ (cf. [NT, Chapter 3, Theorems 2.22, 2.23]).

Let b be a block of G , $\text{Irr}(b)$ be the set of irreducible characters of G that belong to b . Then,

PROPOSITION 2.26.

$$b = \sum_{\chi \in \text{Irr}(b)} e_{\chi} \quad (2.39)$$

and if χ' is another irreducible representation of G over K , then χ and χ' belong to the same block of G if and only if the following holds for all p' conjugacy classes of G :

$$w_{\chi}(C^+) \equiv w_{\chi'}(C^+) \pmod{J(\mathcal{O})}. \quad (2.40)$$

PROOF. A proof may be found in [NT] Chapter3, Theorem 6.24. ■

REMARK 2.27. Note that if K is such that it contains all character values for the group G , the idempotents e_{χ} may still be defined over K . In other words, we do not need the condition that K is a splitting field for G to make these definitions, just that it contains all the ordinary character values of G .

CHAPTER III

THE DOUBLE COVERS OF THE SYMMETRIC GROUPS

In this chapter, we describe how various properties of the double covers of the symmetric groups may be understood in terms of partitions of integers. We shall introduce new notation as needed in each section. As usual $\mathbb{Z}$ will denote the set of integers and $\mathbb{N}$ will denote the set of natural numbers.

§1. The groups $\tilde{S}_n$ and $\tilde{A}_n$

DEFINITION 3.1. Let $n \in \mathbb{N}$, $n > 1$. (i) The group $\tilde{S}_n$ is defined via the following presentation:

$$\begin{aligned}\tilde{S}_n &= \langle z, t_1, \dots, t_{n-1} \rangle \\ z^2 &= 1 \\ zt_i &= t_i z \\ t_i^2 &= z \\ t_i t_j &= z t_j t_i \text{ if } |i - j| \geq 2 \\ (t_i t_{i+1})^3 &= z.\end{aligned}\tag{3.1}$$

We also define $\tilde{S}_0 := \tilde{S}_1 := \langle z \rangle$, where z has order 2. The Schur multiplier of S_n for $n > 3$ is of order two and the group $\tilde{S}_n$ is a representation group for S_n . Let $U = \{1, \dots, n\}$ and let S_n be considered the group of permutations of U . The map

$$\theta : \tilde{S}_n \rightarrow S_n \text{ with } \theta(t_i) = (i, i+1) \quad (3.2)$$

is surjective and has kernel $\langle z \rangle$.

(ii) The inverse image $\theta^{-1}(A_n)$ of the alternating group of S_n is a normal subgroup of index 2 in $\tilde{S}_n$ and is denoted by $\tilde{A}_n$.

We will also use the notation $S(U)$ for S_n , $\tilde{S}(U)$ for $\tilde{S}_n$ and $\tilde{A}(U)$ for $\tilde{A}_n$. If $0 \leq m \leq n$, and V is a subset of U of cardinality m , we denote by $S(V)$ its symmetric group considered as the subgroup of $S(U)$ fixing each element of U/V and by $A(V)$ its alternating group. Then $\tilde{S}(V) =: \theta^{-1}(S(V))$ is clearly isomorphic to $\tilde{S}_m$ and in this way we may regard the latter group as a subgroup of $\tilde{S}_n$. Note that the choice of V is unimportant as $\tilde{S}(V)$ and $\tilde{S}(V')$ are conjugate in $\tilde{S}(U)$ if V and V' are of the same cardinality. Also, we have $\tilde{A}(V) := \theta^{-1}(A(V)) \simeq \tilde{A}_m \triangleleft \tilde{S}(V) \leq \tilde{S}(U)$.

The following fact which we will need later is quite easy to prove:

LEMMA 3.2. *Let V and W be disjoint subsets of U and let $v \in \tilde{S}(V)$, $w \in \tilde{S}(W)$. Then, v and w commute if either v or w is in $\tilde{A}(U)$. Otherwise, $vw = zwv$.*

PROOF. [HH, Cor. 3.5] ■

A partition, λ of a positive integer n is a sequence of positive integers: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$ whose sum is n . Each λ_i is called a part of λ . We formally allow $\{\phi\}$ to be a partition of 0. We will need the following notation pertaining to partitions:

Notation

For $n \in \mathbb{N} \cup \{0\}$, let

$\mathcal{P}(n) :=$ the set of partitions of n ,

$\mathcal{P}^o(n) :=$ the set of partitions of n with no even parts ,

$\mathcal{D}(n) :=$ the set of strict partitions of n i.e. partitions with no repeated parts ,

$\mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}(n)$.

Let λ be a partition. Define

$|\lambda| :=$ the sum of the parts of λ ,

$n(\lambda) :=$ the number of parts of λ .

The parity $\epsilon(\lambda)$ of λ is defined as

$$\epsilon(\lambda) := \begin{cases} 0 & \text{if } |\lambda| - n(\lambda) \text{ is an even integer} \\ 1 & \text{otherwise .} \end{cases}$$

We say that λ is an *even* partition if $\epsilon(\lambda) = 0$ and λ is called an *odd* partition otherwise. We let

$\mathcal{D}^+(n) :=$ the set of even, strict partitions of n ,

$\mathcal{D}^-(n) :=$ the set of odd, strict partitions of n .

The cycle type of a permutation x in S_n is the partition of n determined by the multi-set of cycle lengths in the decomposition of x into disjoint cycles. It is easy to see that the conjugacy class of x is determined by its cycle type, i.e, x' in S_n is conjugate to x iff they have the same cycle type.

For y and y' in $\tilde{S}_n$, clearly, $\theta(y)$ and $\theta(y')$ are conjugate in S_n if y and y' are conjugate in $\tilde{S}_n$. The following theorem describes when the inverse image $\theta^{-1}(C)$ of a conjugacy class C in S_n is a single conjugacy class in $\tilde{S}_n$:

THEOREM 3.3. *Let C be a conjugacy class in S_n . Then its inverse image, $\theta^{-1}(C)$, is itself a conjugacy class in $\tilde{S}_n$ iff there is an element y of $\theta^{-1}(C)$ such that zy is conjugate to y . Otherwise, $\theta^{-1}(C)$ will be a union of two conjugacy classes in $\tilde{S}_n$. Further, the case that $\theta^{-1}(C)$ splits into two conjugacy classes in $\tilde{S}_n$ occurs precisely when the cycle type of elements of C is in $\mathcal{P}^o(n)$ or in $\mathcal{D}^-(n)$.*

PROOF. [HH, Theorems 3.6 and 3.8] ■

Recall that the signature map $\text{sgn} : S_n \rightarrow \{ \pm 1 \}$ is the natural map from S_n onto the cyclic group of order 2 with kernel A_n which is given by $\text{sgn}(x) \rightarrow (-1)^{\epsilon(\lambda)}$, where λ is the cycle type of x , for any x in S_n . Let y be an element of $\tilde{S}_n$. By the cycle type and the signature of y we will mean respectively the cycle type and the

signature of $\theta(y)$.

§2. The irreducible characters of $\tilde{S}_n$

Let χ be an ordinary irreducible character of $\tilde{S}_n$ and let $T : \tilde{S}_n \rightarrow \text{GL}(V)$ be a representation affording χ .

DEFINITION 3.4. The map $T^a : \tilde{S}_n \rightarrow \text{GL}(V)$ given by $T^a(y) = \text{sgn}(y)T(y)$ for y in $\tilde{S}_n$ again defines an irreducible representation of $\tilde{S}_n$ and is called the *associate* of T . Its character, denoted by χ^a , is called the *associate character* of χ . In case χ^a is equal to χ , χ is called a *self-associate character* of $\tilde{S}_n$. Of course, $(\chi^a)^a$ is always χ as $(\text{sgn}(y))^2$ is 1 for all y in $\tilde{S}_n$.

Let χ and T be as above. By Schur's lemma, $T(z)$ is $\pm Id$, where Id is the identity element of $\text{GL}(V)$.

DEFINITION 3.5. We call χ a *faithful* character of $\tilde{S}_n$ if $T(z) = -Id$ and non-faithful otherwise and will adopt the same nomenclature for the irreducible characters of $\tilde{A}_n$. For a non-faithful character, χ of $\tilde{S}_n$, we define the map $T' : \tilde{S}_n \rightarrow \text{GL}(V)$ by $T'(x) = T(y)$ where $\theta(y) = x$. Clearly, T' is well defined and defines an irreducible representation of S_n with character χ' such that $\chi' \circ \theta = \chi$. Conversely, any irreducible character χ' of S_n may be lifted to a non faithful character, $\chi' \circ \theta$, of $\tilde{S}_n$.

For the remainder of this work, unless stated otherwise, by an irreducible char-

acter of $\tilde{S}_n$ (respectively $\tilde{A}_n$) we will always mean a faithful irreducible character of $\tilde{S}_n$ (respectively $\tilde{A}_n$). We now state parts of Schur's main results on the characters of $\tilde{S}_n$, namely a complete enumeration of these via the set of strict partitions of n , the nature of their restrictions to $\tilde{A}_n$ and the determination of some character values:

THEOREM 3.6 (SCHUR). *Every element λ of $\mathcal{D}(n)$ indexes an irreducible faithful representation χ_λ of $\tilde{S}_n$ such that:*

(i) *If $\epsilon(\lambda) = 0$, then $\chi_\lambda = \chi_\lambda^a$ and $\chi(y)$ is non-zero only if the cycle type of y is in $\mathcal{P}^o(n)$.*

(ii) *If $\epsilon(\lambda) = 1$, then $\chi_\lambda \neq \chi_\lambda^a$ and $\chi(y)$ is non-zero only if the cycle type of y is either λ or is in $\mathcal{P}^o(n)$. In the former case,*

$$\chi_\lambda(y) = \begin{matrix} + \\ - \end{matrix} i^{\frac{(n-l+1)}{2}} \frac{(\lambda_1 \lambda_2 \dots \lambda_l)^{\frac{1}{2}}}{2}, \quad (3.3)$$

where λ is the partition $(\lambda_1 \lambda_2 \dots \lambda_l)$.

(iv) *When $\epsilon(\lambda) = 1$, the character χ_λ restricts to an irreducible character of $\tilde{A}_n$ and this character is also the restriction of χ_λ^a to $\tilde{A}_n$. If $\epsilon(\lambda) = 0$, and $n > 1$, the restriction of χ_λ to $\tilde{A}_n$ is a sum of two distinct (conjugate) irreducibles. Further, the restrictions as described above give a complete irredundant list of the irreducible representations for $\tilde{A}_n$.*

PROOF. See [HH, Theorems 8.6 and 8.7] ■

Let χ be a character of $\tilde{S}_n$. Then $\chi(y) = -\chi(zy)$ for any y in $\tilde{S}_n$. In particular, $\chi(y)$ is zero if y and zy are in the same conjugacy class. By Theorem 3.3, therefore, χ vanishes off conjugacy classes whose cycle type is not in $\mathcal{P}^o(n) \cup \mathcal{D}^-(n)$. This fact is contained in the statement of Theorem 3.6 which also determines (up to sign) the values of χ on classes whose elements have cycle type in $\mathcal{D}^-(n)$. To completely determine χ it remains only to specify its values on elements with cycle type in $\mathcal{P}^o(n)$. A.O.Morris has given a method which reduces the calculation of such character values to the calculation of character values for characters of $\tilde{S}_m$ where $m < n$. This is analogous to the Murnaghan-Nakayama formula for calculating the character values of S_n .

Before stating Morris' theorem, we need to introduce the concept of r -bar removal from partitions.

DEFINITION 3.7. Let r be an odd positive integer and let $\lambda \in \mathcal{D}(n)$ be given by $\lambda = (\lambda_1, \lambda_2 \dots \lambda_l)$ where $\lambda_i > \lambda_{i+1} > 0$ for i between 1 and $l-1$. We define:

- (i) a subset $I_+ \cup I_0 \cup I_- = I(\lambda, r)$ of integers between 1 and l ; and
- (ii) for each $i \in I(\lambda, r)$, a strict partition $\lambda^{(i,r)}$ in $\mathcal{D}(n-r)$. Let

$$I_+ := \{i : \lambda_j > \lambda_i - r > \lambda_{j+1} \text{ for some } j \leq l, \text{ taking } \lambda_{l+1} = 0\}.$$

If $i \in I_+$, then we define $\lambda^{(i,r)}$ to be the partition obtained from λ by removing the part λ_i and inserting the integer $\lambda_i - r$ between λ_j and λ_{j+1} . Let

$$I_0 := \{i : \lambda_i = r\}.$$

Clearly, I_0 contains at most one element. If $i \in I_0$, then we define $\lambda^{(i,r)}$ to be the partition obtained from λ by removing the part λ_i . Let

$$I_- := \{i : \lambda_i + \lambda_j = r \text{ for some } j \leq l, \text{ with } i < j \leq l\}.$$

If $i \in I_-$, then we define $\lambda^{(i,r)}$ to be the partition obtained from λ by removing the parts λ_i and λ_j .

If $i \in I(\lambda, r)$ the partition $\lambda^{(i,r)}$ is said to be obtained from λ by the removal of an r -bar.

DEFINITION 3.8.

The character $\chi_{(n)}$, indexed by the partition (n) of n is also known as the *basic faithful character of $\tilde{S}_n$*

DEFINITION 3.9. For $1 \leq a \leq b \leq 2m+2$, define

$$t(a, b) := t_a t_{a+1} \dots t_{b-1} \in \tilde{S}_{2m+2},$$

so that $t(a, a) = 1$ and $t(a, b) \in \tilde{S}_{2m-1}$ for $b < 2m$ (see Definition 3.1). The projection of $t(a, b)$ in S_n is the cycle $(a \ a+1 \dots b)$ of length $b - a + 1$.

PROPOSITION 3.10. *Let $n = 2m + 2$ and let $1 \leq a_1 < b_1 < a_2 < \dots < b_l \leq 2m + 2$ where $l \geq 1$, and where $(b_i - a_i)$ is even for all i , $1 \leq i \leq l$.*

Then

$$\chi_{(n-1)} = \chi_{(n)}|_{\tilde{S}_{n-1}} \text{ and } \chi_{(n)}(t(a_1, b_1) \dots t(a_l, b_l)) = 2^{m - \sum \frac{(b_i - a_i)}{2}}. \quad (3.4)$$

PROOF. For the first part of (3.4) see [HH, Propositions 6.1 and Theorem 6.2].

The second part is contained in Proposition 6.6 of [HH]. ■

Any element π of $\mathcal{P}^o(n)$ is the cycle type of $t(a_1, b_1) \dots t(a_l, b_l)$ for some sequence $1 \leq a_1 < b_1 < a_2 < \dots < b_l \leq n$ of the type in the proposition above. Also, there are two conjugacy classes in $\tilde{S}_n$ corresponding to any element of $\mathcal{P}^o(n)$, and the value of an irreducible character χ on any one of these is the negative of its value on the other. So Proposition 3.10 gives, in particular that the basic faithful character does not vanish on any element of $\tilde{S}_n$ whose cycle type is in $\mathcal{P}^o(n)$.

We adopt the following convention :

For any partition π in $\mathcal{P}^o(n)$, and for any character χ of $\tilde{S}_n$, $\chi(\pi)$ will denote $\chi(y)$ for any y in $\tilde{S}_n$ which has cycle type π and for which $\chi_{(n)}(y)$ is positive.

Now consider $\tilde{S}_{2m} = \langle t_1, \dots, t_{2m-1} \rangle \subset \tilde{S}_{2m+2}$ and let $y \in \tilde{S}_{2m}$ have the same cycle type as the element $t(a_1, b_1) \dots t(a_l, b_l)$ for some sequence $1 \leq a_1 < b_1 < a_2 < \dots < b_l \leq 2m$ with $b_i - a_i$ even for all i . By Proposition 3.10, applied to

the cases $n = 2m$ and $n = 2m + 2$, $\chi_{(2m)}(y)$ will be positive iff y is conjugate to $t(a_1, b_1) \dots t(a_l, b_l)$ in $\tilde{S}_{2m}$ iff y is conjugate to $t(a_1, b_1) \dots t(a_l, b_l)$ in $\tilde{S}_{2m+2}$ iff $\chi_{(2m+2)}(y)$ is positive. Again, by Proposition 3.10, $\chi_{(2m+1)}(y) = \chi_{(2m+2)}(y)$ for y in $\tilde{S}_{2m+1}$. Hence, we get the following corollary:

COROLLARY 3.11. *Let $n > 1$ and let y be in $\tilde{S}_{n-1}$, with cycle type in $\mathcal{P}^0(n-1)$. Then $\chi_{(n-1)}(y)$ is positive iff $\chi_{(n)}(y)$ is positive.*

We are now ready to state Morris' theorem :

THEOREM 3.12. *Let r be a positive odd integer and $\pi \in \mathcal{P}^o(n)$ be a partition containing r at least once. Define $\pi' \in \mathcal{P}^o(n-r)$ by removing a copy of r from π . Then for any λ in $\mathcal{D}(n)$*

$$\chi_\lambda(\pi) = \sum_{i \in I(\lambda, r)} n_i \chi_{\lambda(i, r)}(\pi'), \quad (3.5)$$

where

$$n_i = \begin{cases} (-1)^{j-i} 2^{1-\epsilon(\lambda)} & \text{if } i \in I_+ \\ (-1)^{l-i} & \text{if } i \in I_0 \\ (-1)^{j-i+\lambda_i} 2^{1-\epsilon(\lambda)} & \text{if } i \in I_- \end{cases} \quad (3.6)$$

Here $l = n(\lambda)$ and the integer j is that occurring in the definitions of $I_\pm$.

PROOF. See [HH, Theorem 10.1]. ■

Let us consider the above theorem for the case $r = 1$. First of all, notice that by Corollary 3.11, if π and π' are as in the statement of the theorem and if $y \in \tilde{S}_{n-1}$ has cycle type π' , then we may replace both π and π' in (3.5) by y .

From definition 3.7, it follows immediately that when $r = 1$,

$$I_- = \phi. \quad (3.7)$$

We also get that if 1 is a part of λ , $I_0 = \{l\}$ and I_0 is empty otherwise. Further, if 1 is a part of λ , then $\epsilon(\lambda^{(l,1)}) = \epsilon(\lambda)$, the number of parts, $n(\lambda^{(l,1)})$ of $\lambda^{(l,1)}$ is $n(\lambda) - 1$ and $|\lambda^{(l,1)}| = |\lambda| - 1$. Then from (3.6) we have

$$n_l = 2^{\frac{|\lambda| - |\lambda^{(l,1)}| - (n(\lambda) - n(\lambda^{(l,1)})) - (\epsilon(\lambda) - \epsilon(\lambda^{(l,1)}))}{2}} \quad (3.8)$$

for $l \in I_0$.

Similarly, it is easy to see that $I_+ = \{i : \lambda_i - \lambda_{i+1} > 1\}$ and for $i \in I_+$, the corresponding j is just i itself, $n_i = 2^{1-\epsilon(\lambda)}$ and $\epsilon(\lambda(i, 1)) \neq \epsilon(\lambda)$. So, for i in $I_+(\lambda, 1)$, we again may rewrite equation (3.6) as

$$n_i = 2^{\frac{|\lambda| - |\lambda^{(i,1)}| - (n(\lambda) - n(\lambda^{(i,1)})) - (\epsilon(\lambda) - \epsilon(\lambda^{(i,1)}))}{2}} \quad (3.9)$$

for $i \in I_+$.

So, by equations (3.7), (3.8) and (3.9), we have that Morris's theorem for the case $r = 1$ is

PROPOSITION 3.13.

$$\chi_\lambda(y) = \sum_{i \in I_+ \cup I_0} 2^{\frac{|\lambda| - |\lambda^{(i,1)}| - (n(\lambda) - n(\lambda^{(i,1)})) - (\epsilon(\lambda) - \epsilon(\lambda^{(i,1)}))}{2}} \chi_{\lambda(i,r)}(y) \quad (3.10).$$

for any y in $\tilde{S}_{n-1}$ with cycle type in $\mathcal{P}^o(n-1)$.

As in Section 1, we set $U = \{1, \dots, n\}$. We choose a subset V of U of cardinality m and use the notation $\tilde{S}(U)$ for $\tilde{S}_n$ and $\tilde{S}(V)$ for $\tilde{S}_m$ to specify the containment $\tilde{S}_m \subset \tilde{S}_n$.

LEMMA 3.14. *Let $V \subset U$ and suppose that $|V| - |U| \geq 2$. Let χ be a character of $\tilde{S}(U)$. Then, $\chi|_{\tilde{S}(V)}$ vanishes outside $\tilde{A}(V)$ and*

$$\langle \chi|_{\tilde{S}(V)}, \phi \rangle = \langle \chi^a|_{\tilde{S}(V)}, \phi \rangle = \langle \chi|_{\tilde{S}(V)}, \phi^a \rangle. \quad (3.11)$$

for any character ϕ of $\tilde{S}(V)$. Here $\langle, \rangle$ denotes the usual inner product on group characters.

PROOF. Let χ be an irreducible character of $\tilde{S}(U)$ and ϕ be an irreducible character of $\tilde{S}(V)$.

Recall that $\tilde{S}(V)$ is naturally contained in $\tilde{S}(U)$ and that $\theta : \tilde{S}(U) \rightarrow S(U)$ is the projection map. As $|U| - |V| \geq 2$, for $g \in \tilde{S}(V)$, the cycle type of $\theta(g)$ will contain at least two cycles of length one. In particular, the cycle type of $\theta(g)$ corresponds to a partition of $|U|$ with repeated parts and so $\chi(g) = 0$, unless $g \in \tilde{A}(U)$. So

$$\langle \chi|_{\tilde{S}(V)}, \phi \rangle = \frac{1}{|\tilde{S}(V)|} \sum_{g \in \tilde{A}(V)} \chi(g) \phi(g^{-1}). \quad (3.12)$$

As $\chi(g) = \chi^a(g)$ and $\phi(g) = \phi^a(g)$ for $g \in \tilde{A}(V)$, the lemma is proved. $\blacksquare$

DEFINITION 3.15. Let $l \in \mathbb{N} \cup \{0\}$. (i) We denote by $\mathcal{M}^l(\lambda)$ the set of all sequences,

$$\Lambda : \lambda = \Lambda^0 > \dots > \Lambda^l$$

of elements of $\mathcal{D}$, where for each i , $0 \leq i \leq l$, Λ^{i+1} is the partition obtained by removing a 1-bar from Λ^i , i.e $\Lambda^{i+1} = \Lambda^{i(j,1)}$ for some j in $I(\Lambda^i, 1)$.

(ii) For $\lambda \in \mathcal{D}(n)$ and $\lambda' \in \mathcal{D}(m)$ we let

$$\mathcal{M}^l(\lambda, \lambda') := \{\Lambda \in \mathcal{M}^l(\lambda) : \Lambda^l = \lambda'\}.$$

DEFINITION 3.16. For $\lambda \in \mathcal{D}(n)$ and $\lambda' \in \mathcal{D}(m)$ set

$$n(\lambda, \lambda') := \frac{(|\lambda| - |\lambda'| - (n(\lambda) - n(\lambda')) - (\epsilon(\lambda) - \epsilon(\lambda')))}{2}.$$

Note that $n(\lambda, \lambda')$ is an integer as $|\lambda| - n(\lambda) - \epsilon(\lambda) \equiv 0 \pmod{2}$ and likewise for $|\lambda'| - n(\lambda') - \epsilon(\lambda')$.

By iterating the equation (3.10), and using the definitions above we get that for y in $\tilde{S}(V)$ with cycle type in $\mathcal{P}(m)$ and $\lambda \in \mathcal{D}(n)$, $l = n - m \geq 2$,

$$\begin{aligned} \chi_\lambda(y) &= \sum_{\Lambda \in \mathcal{M}^l(\lambda)} \prod_{0 \leq i \leq l-1} 2^{\frac{|\Lambda^i| - |\Lambda^{i+1}| - (n(\Lambda^i) - n(\Lambda^{i+1})) - (\epsilon(\Lambda^i) - \epsilon(\Lambda^{i+1}))}{2}} \chi_{\Lambda^l}(y) \\ &= \sum_{\Lambda \in \mathcal{M}^l(\lambda)} 2^{\frac{|\Lambda^0| - |\Lambda^l| - (n(\Lambda^0) - n(\Lambda^l)) - (\epsilon(\Lambda^0) - \epsilon(\Lambda^l))}{2}} \chi_{\Lambda^l}(y) \\ &= \sum_{\lambda' \in \mathcal{D}(m)} |\mathcal{M}^l(\lambda, \lambda')| 2^{n(\lambda, \lambda')} \chi_{\lambda'}(y). \end{aligned} \tag{3.13}$$

PROPOSITION 3.17. Let $|U| = n$ and $|V| = m$, $\lambda \in \mathcal{D}(n)$ and $\lambda' \in \mathcal{D}(m)$. If $l = n - m \geq 2$, then

$$\langle \chi_\lambda |_{\tilde{S}(V)}, \chi_{\lambda'} \rangle = \begin{cases} 2^{n(\lambda, \lambda')} |\mathcal{M}^l(\lambda, \lambda')| & \text{if } \epsilon(\lambda') = 0 \\ \frac{1}{2} 2^{n(\lambda, \lambda')} |\mathcal{M}^l(\lambda, \lambda')| & \text{if } \epsilon(\lambda') = 1. \end{cases} \quad (3.14)$$

PROOF. First notice that equation (3.13) holds for any y in $\tilde{A}(V)$ as by Theorem 3.6 (ii) both sides are zero if $y \in \tilde{A}(V)$ and the cycle type of y is not in $\mathcal{P}^o(m)$. Hence for any y in $\tilde{A}(V)$,

$$\begin{aligned} \chi_\lambda(y) &= \sum_{\lambda' \in \mathcal{D}(m)} |\mathcal{M}^l(\lambda, \lambda')| 2^{n(\lambda, \lambda')} \chi_{\lambda'}(y) \\ &= \sum_{\substack{\lambda' \in \mathcal{D}(m), \\ \epsilon(\lambda')=0}} 2^{n(\lambda, \lambda')} |\mathcal{M}^l(\lambda, \lambda')| \chi_{\lambda'}(y) + \frac{1}{2} \sum_{\substack{\lambda' \in \mathcal{D}(m), \\ \epsilon(\lambda')=1}} 2^{n(\lambda, \lambda')} |\mathcal{M}^l(\lambda, \lambda')| (\chi_{\lambda'} + \chi_{\lambda'}^a)(y). \end{aligned} \quad (3.15)$$

As by Lemma 3.14,

$$\chi_\lambda(y) = 0 = (\chi_{\lambda'} + \chi_{\lambda'}^a)(y) \quad (3.16),$$

for y in $\tilde{S}(V) - \tilde{A}(V)$, λ in $\mathcal{D}(n)$ and λ' in $\mathcal{D}(m)$, we get an equality between the extreme terms of the above equation for all y in $\tilde{S}(V)$, i.e

$$\chi_\lambda|_{\tilde{S}(V)} = \sum_{\substack{\lambda' \in \mathcal{D}(m), \\ \epsilon(\lambda')=0}} 2^{n(\lambda, \lambda')} \chi_{\lambda'} + \frac{1}{2} \sum_{\substack{\lambda' \in \mathcal{D}(m), \\ \epsilon(\lambda')=1}} 2^{n(\lambda, \lambda')} (\chi_{\lambda'} + \chi_{\lambda'}^a) \quad (3.17),$$

from which the proposition is immediate. ■

§3. The blocks of $\tilde{S}_n$

Applying the criterion (2.40) in Proposition 2.26, to the case $G = \tilde{S}_n$ where p is odd and C is the class of the central element z , we get that a faithful character of $\tilde{S}_n$ is never in the same p block as a non-faithful character when p is odd and the same holds for characters of $\tilde{A}_n$. Hence we have the following definition:

DEFINITION 3.18. The blocks containing faithful characters of $\tilde{S}_n$ (respectively $\tilde{A}_n$) are called *faithful blocks* of $\tilde{S}_n$ (respectively $\tilde{A}_n$), the others are called *non-faithful blocks*.

If b is a non-faithful block of $\tilde{S}_n$, then it is clear from (3.2) that if $\hat{\theta} : \mathcal{O}\tilde{S}_n \rightarrow \mathcal{O}S_n$ is the natural extension of θ to the corresponding group algebras, $\hat{\theta}(b)$ is a block of S_n , $\text{Irr}(\hat{\theta}(b)) = \{\chi \circ \theta^{-1} : \chi \in \text{Irr}(b)\}$ and $b \rightarrow \hat{\theta}(b)$ is a 1-1 correspondence between non-faithful blocks of $\tilde{S}_n$ and blocks of S_n .

$\mathcal{O}S_n\hat{\theta}(b)$ has an interior $\tilde{S}_n$ algebra structure via $x \rightarrow \theta(x)\hat{\theta}(b)$ and the restriction of $\hat{\theta}$ to $\mathcal{O}\tilde{S}_nb$ induces an interior $\tilde{S}_n$ -algebra isomorphism between $\mathcal{O}\tilde{S}_nb$ and $\mathcal{O}\tilde{S}_n\theta(b)$. If P is a defect group of b , then P is a defect group of $\mathcal{O}S_n\hat{\theta}(b)$ considered as an interior $\tilde{S}_n$ -algebra, and hence $\theta(P) (\simeq P)$ is a defect group of $\mathcal{O}S_n\hat{\theta}(b)$ considered as an interior S_n algebra. In particular, two non-faithful blocks b and c of $\tilde{S}_n$ and $\tilde{S}_m$ respectively with $m \leq n$, have isomorphic defect groups if and only if $\theta(b)$ and $\theta(c)$ have isomorphic defect groups. Moreover, an interior S_n algebra

isomorphism between $\hat{\theta}(c)\mathcal{O}S_n\hat{\theta}(b)\hat{\theta}(c)$ and $\mathcal{O}S_n\hat{\theta}(c) \otimes_{\mathcal{O}} \text{Mat}_L(\mathcal{O})$, will lift to an interior $\tilde{S}_m$ algebra homomorphism between $c\mathcal{O}\tilde{S}_nbc$ and $\mathcal{O}\tilde{S}_nc \otimes_{\mathcal{O}} \text{Mat}_L(\mathcal{O})$.

As the theory of the non-faithful blocks of $\tilde{S}_n$ reduces to that of the blocks of S_n , from now on by a block of $\tilde{S}_n$, we shall always mean a faithful block of $\tilde{S}_n$.

DEFINITION 3.19. Let $\lambda \in \mathcal{D}$. An element μ of $\mathcal{D}$ is said to be a p -bar core of λ if there is a chain of elements of $\mathcal{D}$: $\lambda = \lambda^0, \lambda^1, \dots, \lambda^s = \mu$ such that for $0 \leq i \leq s-1$, λ^{i+1} is obtained from λ^i by the removal of a p -bar and $I(\mu, p) = \phi$. Here note that μ may be $\{\phi\}$.

Every partition λ in $\mathcal{D}$ has a unique p -bar core(cf. [C, Proposition 6]) which we shall denote by $c_p(\lambda)$.

DEFINITION 3.20. The weight of the partition λ , denoted $w(\lambda)$, is defined by

$$w(\lambda) := \frac{1}{p}(|\lambda| - |c_p(\lambda)|).$$

It is clear from the definition of the p -core that $w(\lambda)$ is an integer. Let

$\bar{\mathcal{D}} :=$ the set of strict partitions which are their own p -bar cores.

For p an odd prime, the question of distribution of faithful characters into p blocks is related to p -bar cores via the following :

THEOREM 3.21 (J.F. HUMPHREYS, CABANES, OLSSON).

Let $\lambda \in \mathcal{D}(|U|)$. Then

(i) If $\lambda \neq c_p(\lambda)$, i.e if λ is not its own p -bar core, χ_λ and χ_λ^a belong to the same p block and for any element ζ of $\mathcal{D}(|U|)$, χ_ζ is in the same p block as χ_λ iff $c_p(\zeta) = c_p(\lambda)$.

(ii) If $\lambda = c_p(\lambda)$, then χ_λ and χ_λ^a lie in blocks of defect zero. In particular, if $e(\lambda) = 1$, then $\chi_\lambda \neq \chi_\lambda^a$, and χ_λ and χ_λ^a belong to different blocks.

PROOF. See [C, Theorems A and B]. ■

The defect groups of $\tilde{S}_n$ are described in the following theorem:

THEOREM 3.22. *Let b be a faithful block of $\tilde{S}_n$ and λ be in $\mathcal{D}(n)$. If χ_λ (or χ_λ^a) $\in b$, then a defect group of b is a Sylow p -subgroup of $\tilde{S}_{pw(\lambda)}$.*

PROOF. See [C, Theorems A and B] ■

By Theorem 3.21, we may denote by $c_p(b)$ and by $w(b)$ the common core and common weight respectively of all the partitions λ of $\mathcal{D}(|U|)$, such that χ_λ (or χ_λ^a) belong to the block b . We will refer to $c_p(b)$ and $w(b)$ as the core and weight respectively of b .

Now, let b be as in the above theorem and let P be a defect group of b . Let U^P be the subset of U on which $\theta(P)$ acts trivially. Then, by [C, Theorem 12 and Corollary 26], $|U| - |U|^P$ is $pw(b)$ and P is a Sylow p -subgroup of $\tilde{S}(U - U^P)$.

By [C, Remark 18]

$$C_{\tilde{S}(U)}(P) = \tilde{S}(U^P) \times Z(P). \quad (3.18)$$

Also, it is easy to see that

$$N_{\tilde{S}(U)}(P) = \tilde{S}(U^P) \cdot N_{\tilde{S}(U-U^P)}(P). \quad (3.19)$$

As $|U| - |U^P| = pw(b)$, $c_p(b)$ is a strict partition of $|U^P|$ and by part (ii) of Theorem 3.21, $\chi_{c_p(b)}$ and $\chi_{c_p(b)}^a$ both belong to blocks of defect zero of $\tilde{S}(U^P)$. Let these blocks be denoted by $e_{c_p(b)}$ and $e_{c_p(b)}^a$ respectively. Here of course, if $c_p(b)$ is an even partition of $|U^P|$, then $\chi_{c_p(b)} = \chi_{c_p(b)}^a$ and $e_{c_p(b)} = e_{c_p(b)}^a$.

DEFINITION 3.23. We let

$$e_{c_p(b)}^* := \begin{cases} e_{c_p(b)} & \text{if } e_{c_p(b)} = e_{c_p(b)}^a \\ e_{c_p(b)} + e_{c_p(b)}^a & \text{otherwise} \end{cases}.$$

Recall the Brauer homomorphism $\text{Br}_P : (\mathcal{O}\tilde{S}(U))^P \rightarrow kC_{\tilde{S}(U)}(P)$ of $\mathcal{O}\tilde{S}(U)$ with respect to P . (Chapter II, §3)

From (3.18) we have that the image of the Brauer homomorphism,

$$(\mathcal{O}\tilde{S}(U))(P) \simeq kC_{\tilde{S}(U)}(P) \simeq k\tilde{S}(U^P) \otimes_k kZ(P) \quad (3.20)$$

and we shall identify $(\mathcal{O}\tilde{S}(U))(P)$ with $k\tilde{S}(U^P) \otimes_k kZ(P)$.

As P is a defect group of b , the image of $\text{Br}_P(b)$ is a non-zero central idempotent of $k\tilde{S}(U^P) \otimes_k kZ(P)$. But $Z(k\tilde{S}(U^P) \otimes_k kZ(P)) = Z(k\tilde{S}(U^P)) \otimes_k kZ(P)$ as $Z(P)$

is abelian. Also, $kZ(P)$ is a local ring and so $\text{Br}_P(b)$ is of the form $e \otimes 1$ where e is a central idempotent of $(k\tilde{S}(U^P))$.

PROPOSITION 3.24. *If $\lambda \neq c_p(\lambda)$, then*

$$\text{Br}_P(b) = \overline{e_{c_p(b)}^*} \otimes 1, \quad (3.21)$$

where $\overline{e_{c_p(b)}^*}$ is the image of $e_{c_p(b)}^*$ in $k\tilde{S}(U^P)$.

PROOF. [C, Theorem A and Corollary 26] ■

As is seen in Theorem 3.21, the relationship of blocks to p -bar cores for blocks of non zero defect differs from that in the case of blocks of defect zero. Hence, we make the following definitions:

DEFINITION 3.25. Let

$$\mathcal{B} := \{b : b \text{ a block of } \tilde{S}_n, n \geq 0, b \text{ is of non-zero defect}\},$$

$$\mathcal{E} := \{e : e \text{ a block of } \tilde{S}_n, n \geq 0, e \text{ is of defect zero}\}.$$

By Theorem 3.22 we have that the defect group of a block b of $\tilde{S}(U)$ is non-trivial if and only if $w(b)$ is positive.

Hence, $b \rightarrow (c_p(b), w(b))$ is a map from $\mathcal{B}$ into $\overline{\mathcal{D}} \times N$, i.e into the set of ordered pairs (μ, w) where μ is a strict partition with no p -bars and w is a positive integer.

By part (i) of Theorem 3.21, this map is 1-1. We will show that it is also surjective:

Let (μ, w) be any element of $\overline{\mathcal{D}} \times N$. As μ has no p -bars the partition $\lambda = \mu \cup \{pw\}$ of $|\mu| + pw$ is also strict and $c_p(\lambda) = \mu$. Let b be the block of $\tilde{S}_{|\mu|+pw}$ containing λ . Then, clearly, $c_p(b) = c_p(\lambda) = \mu$ and $w(b) = w(\lambda) = w$. Hence we have proved

PROPOSITION 3.26. *The map $b \rightarrow (c_p(b), w(b))$ is a 1-1 correspondence between $\mathcal{B}$ and the set $\overline{\mathcal{D}} \times N$.*

DEFINITION 3.27. Let

$$\mathcal{B}_w := \{b : b \text{ a block of } \tilde{S}_n, n \geq 0, w(b) = w\},$$

We have

$$\mathcal{B} = \bigcup_{w > 0} \mathcal{B}_w. \quad (3.22)$$

By Theorem 3.24, for a non-identity p -group P , the set of blocks of $\tilde{S}_n$, ($n \geq 0$) which have P as a defect group is non-empty only if P is a Sylow- p subgroup of $\tilde{S}_{pw}$ for some positive integer w in which case, this set is just $\mathcal{B}_w$.

We next discuss the faithful blocks of $\tilde{A}_n$.

LEMMA 3.28. *Let b be a faithful block of $\tilde{S}_n$ of positive defect. Then b covers a unique block of $\tilde{A}_n$.*

PROOF. [O, 5.1] ■

This gives easily the following proposition characterizing the faithful blocks of $\tilde{A}_n$:

PROPOSITION 3.29. *Let $n > 1$. Then*

(i) *If $b \in \mathcal{B}$, then b is a block of $\tilde{A}_n$ and b has the same defect groups when regarded as a block of $\tilde{S}_n$ and of $\tilde{A}_n$.*

(ii) *If $e \in \mathcal{E}$, and $e = e^a$, then e is a non-primitive central idempotent of $\mathcal{O}\tilde{A}_n$ and $e = e' + e'^c$ is a primitive decomposition of e in $\mathbb{Z}(\mathcal{O}\tilde{A}_n)$, where $e'^c = e'^g$ for any g in $\tilde{S}_n - \tilde{A}_n$. If $e \neq e^a$, then $e + e^a = e^*$ is a block of $\tilde{A}_n$.*

(iii) *The set $\{b : b \in \mathcal{B}\} \cup \{e', e'^c : e \in \mathcal{E}, e = e^a\} \cup \{e^* : e \in \mathcal{E}, e \neq e^a\}$ is a complete set of the faithful blocks of $\tilde{A}_n$.*

PROOF. Let b be a block of $\tilde{S}_n$ of non-zero defect and $\lambda \in \mathcal{D}(n)$ with $\chi_\lambda \in b$. Then by Theorem 3.21 χ_λ^a is also in b . Recall from (2.37) that e_{χ_λ} and $e_{\chi_\lambda^a}$ are the central idempotents of $K\tilde{S}_n$ associated with χ_λ and χ_λ^a respectively. From the definition of the associate character it is clear that χ_λ and χ_λ^a agree on $\tilde{A}_n$ and their values off $\tilde{A}_n$ are the negatives of each other. So

$$e_{\chi_\lambda} + e_{\chi_\lambda^a} = \frac{\chi_\lambda(1)}{|\tilde{A}_n|} \sum_{x \in \tilde{A}_n} \chi_\lambda(x^{-1})x. \quad (3.23)$$

Also both χ_λ and χ_λ^a belong to b and hence it is clear from (3.23) and (2.39) that $b \in \mathcal{O}\tilde{A}_n$. But then by Lemma 3.28 above, b is also a block of $\tilde{A}_n$. Also, as any

p -group of $\tilde{S}_n$ is contained in $\tilde{A}_n$, b has the same defect groups when regarded as a block of $\tilde{S}_n$ or of $\tilde{A}_n$ [NT, Theorem 5.16 , (ii)].

Now let e be a block of $\tilde{S}_n$ of zero defect. Then e is e_μ (or e_μ^a) for some $\mu \in \mathcal{D}(n)$, $c_p(\mu) = \mu$ and χ_μ (or χ_μ^a) is the unique ordinary irreducible character in e .

If $\epsilon(\mu) = 0$, $\chi_\mu|_{\tilde{A}_n}$ is the sum of two irreducible characters say χ'_μ and χ'^c_μ . Each of these must belong to a block of defect zero (cf. [NT, Lemma 5.7 and Theorem 5.16]), and if e' and e'^c are the blocks of $\tilde{A}_n$ containing χ'_μ and χ'^c_μ respectively, then e' and e'^c are the only blocks of $\tilde{A}_n$ covered by b . As χ_μ is zero off $\tilde{A}_n$, $e \in \mathcal{O}\tilde{A}_n$ and hence, we get that

$$e = e' + e'^c. \quad (3.24)$$

The equality $e'^g = e'^c$ holds as $\chi'^g_\mu = \chi'^c_\mu$ for g in $\tilde{S}_n - \tilde{A}_n$.

If $\epsilon(\mu) = 1$, $\chi_\mu|_{\tilde{A}_n} = (\chi^a_\mu|_{\tilde{A}_n})$ is an irreducible character say χ'_μ of $\tilde{A}_n$. Hence the block e^* of $\tilde{A}_n$ containing χ'_μ is of zero defect, both e and e^a cover it and further, it is the unique block of $\tilde{A}_n$ covered by e and e^a [NT, Lemma 5.7 and Theorems 5.9, 5.16]. In other words,

$$e + e^a = e^*. \quad (3.25)$$

The last part of the proposition follows from parts (i) and (ii) since by Theorem 3.6 (iv) every faithful character of $\tilde{A}_n$ belongs to a block of $\tilde{A}_n$ of the type mentioned in the statement of (iii). ■

REMARK 3.30. Let $n \in \{0\} \cup \mathbb{N}$. The group defined by

$$\begin{aligned}
 \hat{S}_n &= \langle z, t_1, \dots, t_{n-1} \rangle \\
 z^2 &= 1 \\
 zt_i &= t_i z \\
 t_i^2 &= 1 \\
 t_i t_j &= z t_j t_i \text{ if } |i - j| \geq 2 \\
 (t_i t_{i+1})^3 &= z,
 \end{aligned} \tag{3.26}$$

when $n > 1$ and by $\hat{S}_0 = \hat{S}_1 = \langle z \rangle$ is also a representation group of S_n for $n > 3$.

Moreover, $\tilde{S}_n$ and $\hat{S}_n$ are non-isomorphic except when $n = 6$. It is easy to check that there is a 1-1 correspondence

$$\chi \rightarrow \hat{\chi} \text{ via } \hat{\chi}(s_j) = i\chi(t_j) \tag{3.27}$$

between the faithful irreducible characters of $\tilde{S}_n$, and those of $\hat{S}_n$. Hence the character theory of these groups may be described in exactly the same fashion as done here for the groups $\tilde{S}_n$.

CHAPTER IV

CORRESPONDENCE OF CHARACTERS

Let P be as in the statement of Theorem 1.4, a non-identity p -group. By the remarks following Definition 3.27 we may assume that P is a Sylow- p subgroup of $\tilde{S}_{pw}$, where w is a positive integer. Then the set $\mathcal{B}_P$ of the statement of Theorem 1.4 is the set $\mathcal{B}_w$ of definition 3.27.

Clearly, the map $b \rightarrow (c_p(b), w(b))$ defined in Proposition 3.26 when restricted to $\mathcal{B}_w$ will give rise to a 1-1 correspondence

$$b \rightarrow c_p(b)$$

between elements of $\mathcal{B}_w$ and $\overline{\mathcal{D}}$. Let the reverse correspondence be given by

$$\rho \rightarrow b_\rho$$

To prove Theorem 1.4, therefore, we will find an integer N_w such that for any μ in $\overline{\mathcal{D}}$ with $|\mu| > N_w$, there exists a ν in $\overline{\mathcal{D}}$, $|\nu| < |\mu|$, and such that b_μ and b_ν are related by an isomorphism of the type (1.2). Then Theorem 1.4 will hold for $N_P := N_w + pw$.

§1. The Scopes maps

Notation

We retain the notation of Chapter III. Further, let

$$X := \{0, \dots, p-1\}.$$

Recall the notion of the p -bar core and weight of a strict partition introduced in Definitions 3.19 and 3.20.

One way to visualize the p -bar core of a strict partition, λ , is to think of the parts of λ as beads on an abacus. The abacus has p columns indexed by the set $\{0\} \cup X$. The runner on the extreme left is numbered 0 and the rest are numbered in increasing order from left to right. The rows of the abacus are numbered in increasing order by the set $\{0\} \cup \mathbb{N}$ from the 0th row down. For $x \in \{0\} \cup X$, a bead will be placed on the a th row of the x th column of the abacus if and only if $x + pa$ is a part of the partition λ . Clearly, every part of λ will correspond to a unique bead on the abacus in this way, and as λ has no repeated parts, no two parts will correspond to the same bead. More formally, we make the following definition:

DEFINITION 4.1. (i) Let $\lambda \in \mathcal{D}$. For $x \in \mathbb{Z}$ the p -set of λ through x , denoted by $\lambda(x)$ is defined by

$$\lambda(x) := \{a : x + pa \text{ is a part of } \lambda\}.$$

$|\lambda(x)|$ will denote the cardinality of the set $\lambda(x)$. As $\{0\} \cup X$ is a transversal for $\mathbb{Z}/p\mathbb{Z}$, the strict partition λ is completely determined by the sets $\lambda(x)$, $x \in \{0\} \cup X$. So for any $x \in \{0\} \cup X$, there will be beads on the a th row of the x th column iff $a \in \lambda(x)$.

(ii) For $x \in X$, let

$$m_x(\lambda) := |\lambda(x)| - |\lambda(p - x)|.$$

As is described in [HH, Appendix 10], the removal of a p -bar from λ is registered on the abacus by making one of the following types of moves:

(Type 1) Slide a bead up one position on the same column.

(Type 2) Remove a bead on the zeroth row of the zeroth column.

(Type 3) For $x \in X$, remove the beads on the zeroth row of the x th and $p - x$ th columns.

The p -bar core is attained by the following process. First move all the beads as far up as possible on the same column and then perform all possible moves of type (2) and (3). Repeat this procedure until no further moves are possible. Here note that once a bead reaches the zeroth row of the zeroth column, it is no longer a part of the partition as it represents the number 0. The positions of the beads at the end of this process represents the p -bar core of λ . In other words, we have

PROPOSITION 4.2. *Let $\lambda \in \mathcal{D}$. Then there is only one p -bar core $c_p(\lambda)$ of λ and it is given by:*

- (i) $c_p(\lambda)(0) = \phi$.
- (ii) For $x \in X$ such that $m_x(\lambda) \leq 0$, $c_p(\lambda)(x) = \phi$.
- (iii) For $x \in X$ such that $m_x(\lambda) > 0$, $c_p(\lambda)(x) = \{0, \dots, m_x(\lambda) - 1\}$.

PROOF. See [C], Proposition 6. ■

DEFINITION 4.3. Let $\lambda \in \mathcal{D}$. For $x \in X \cup \{0\}$, define

$$W_x(\lambda) := \sum_{a \in \lambda(x)} (x + pa) - \sum_{a' \in c_p(\lambda)(x)} (x + pa')$$

Clearly, the weight $w(\lambda)$ (Definition 3.20) is given by

$$w(\lambda) = \frac{1}{p} \sum_{x \in \{0\} \cup X} W_x(\lambda). \quad (4.1)$$

The weight of the partition λ may also be understood as being the number of moves of type (1) and (3) that are performed on the abacus to get to the p -bar core of λ .

Now suppose that $\lambda(x) = \{a_i : 0 \leq i \leq |\lambda(x)| - 1\}$ is an indexing of the elements of $\lambda(x)$ in increasing order, i.e $a_i < a_{i+1}$ for $0 \leq i < |\lambda(x)| - 1$.

By Proposition 4.2, $c_p(\lambda)(x) = \{0, \dots, |c_p(\lambda)(x)| - 1\}$, and $|c_p(\lambda)(x)| \leq |\lambda(x)|$. Hence, we clearly have

$$W_x(\lambda) = (|\lambda(x)| - |c_p(\lambda)(x)|)x + p \left(\sum_{i=0}^{|c_p(\lambda)(x)|-1} (a_i - i) + \sum_{i=|c_p(\lambda)(x)|}^{|\lambda(x)|-1} a_i \right). \quad (4.2)$$

LEMMA 4.4. *Let w be a nonnegative integer, and let λ in $\mathcal{D}$ be such that $w(\lambda) = w$. Then,*

(i)

$$\lambda(x) \subset \{0, \dots, |\lambda(x)| + w - 1\} \text{ and } |\lambda(x)| \leq |c_p(\lambda)(x)| + w \quad (4.3)$$

for $x \in X$.

(ii) *If $|\lambda(x)| > w + 1$ for $x \in X$, then*

$$|\lambda(x)| = |c_p(\lambda)(x)|, \quad \{0, \dots, |\lambda(x)| - w - 1\} \subset \lambda(x) \text{ and } \lambda(p - x) = \phi. \quad (4.4)$$

(iii)

$$\lambda(0) \subset \{1 \dots w\}. \quad (4.5)$$

REMARKS. The Lemma states how the weight and p -bar core of a strict partition control its representation on the abacus. For example, suppose the first part of (4.3) does not hold. This would mean that between the 0th row and the lowest row on which there is bead on the x th column, there are at least $w + 1$ rows on which there are no beads. But then clearly, at least $w + 1$ moves of type (1) can be made on the x th column. But this is impossible as λ has weight w . For the second part of (4.3), note that $|c_p(\lambda)(x)|$ is the number of beads left on the x th column of the abacus at the end of the process described before Proposition

4.2. Thus $|\lambda(x)| > |c_p(\lambda(x))| + w$ would imply that at least $w + 1$ moves of type (3) may be made on the abacus, which again contradicts the weight of λ .

Now for $x \in X$, $|c_p(\lambda(x))| < |\lambda(x)|$ if and only if a move of type (3) is made on the x th and $p - x$ th columns and this can happen if and only if there is a bead on both these columns. Also, once a move of type (3) is made on the x th column, the remaining beads on the x th column can all be moved up one position. This proves the first and third parts of (4.4). Again, suppose that there is no bead on the a th row of the x th column. Then all the beads on the x th column below the a th row can be moved up at least one position and thus there can be at most w beads below the a th row on the x th column but this is just the second assertion of (4.4).

(4.5) is immediate as all beads on the 0th runner are removed in going to the p -bar core. Let us now present these observations more rigorously.

PROOF. For $x \in X$, (4.2) gives in particular,

$$W_x(\lambda) \geq p \left(\sum_{i=0}^{i=|c_p(\lambda)(x)|-1} (a_i - i) + \sum_{i=|c_p(\lambda)(x)|}^{i=|\lambda(x)|-1} a_i \right), \quad (4.6)$$

which may be further weakened to yield the following inequalities:

$$\frac{1}{p} W_x \geq a_{|\lambda(x)|-1} - |\lambda(x)| - 1, \quad (4.7)$$

$$\frac{1}{p} W_x \geq |\lambda(x)| - |c_p(\lambda)(x)|. \quad (4.8)$$

If $|c_p(\lambda)(x)| < |\lambda(x)|$, (4.3) also yields

$$\frac{1}{p}W_x \geq a_{|\lambda(x)|-1} \geq |\lambda(x)| - 1. \quad (4.9)$$

Let i_0 be the least i for which $a_i > i$. Then $a_i - i \geq 1$ for all $i \geq i_0$ and

hence,

$$\frac{1}{p}W_x \geq \sum_{i \geq i_0}^{|\lambda(x)|-1} (a_i - i) \geq |\lambda(x)| - i_0. \quad (4.10)$$

Also,

$$w \geq \frac{1}{p}W_x. \quad (4.11)$$

So, (4.7) and (4.11) imply the first part of (i). The second follows from (4.8) and (4.11). (4.9) and (4.11) imply the first part of (ii) and the second part of (ii) follows from (4.10) and (4.11). We have

$$|\lambda(x)| - |\lambda(p-x)| = |c_p(\lambda)(x)| - |c_p(\lambda)(p-x)|. \quad (4.12)$$

If $|\lambda(x)| > w + 1$, then as has been shown above, $|c_p(\lambda)(x)| = |\lambda(x)|$. But then $|c_p(\lambda)(p-x)| = 0$ and the last part of (ii) follows from (4.12).

■

DEFINITION 4.5. For each x in X we define an involution of the set $\{0\} \cup X$ called the *Scopes involution at x* and denoted Sc_x which in the usual cycle notation is defined as

$$Sc_x := \begin{cases} (1, p-1) & \text{if } x = 1 \\ (\frac{p-1}{2}, \frac{p+1}{2}) & \text{if } x = \frac{p+1}{2} \\ (x-1, x)(p-x, p-x+1) & \text{otherwise.} \end{cases}$$

It is immediate from the above definitions that $Sc_x = Sc_{p-x+1}$, when $x \neq 1$, and $Sc_x(0) = 0$, $Sc_x(p - z) = p - Sc_x(z)$ for any $x, z \in X$.

DEFINITION 4.6. We define another set of maps also indexed by the elements of X and denoted $\widetilde{Sc}_x$ on the elements of $\mathcal{D}$ (the set of strict partitions) via

$$\widetilde{Sc}_x(\lambda)(z) := \lambda(Sc_x(z)) \text{ if } x \neq 1 \text{ or } z \neq p - 1$$

and

$$\widetilde{Sc}_1(\lambda)(p - 1) := \{a - 1 : a \in \lambda(1), a \neq 0\}.$$

These maps are called the Scopes maps on the set $\mathcal{D}$.

Again, the Scopes maps on partitions may be registered as the following moves on the abacus:

For $x \in X$, $x \neq 1$, $\widetilde{Sc}_x(\lambda)$ is obtained from λ by exchanging the positions of the beads on the x th and $x - 1$ st columns and doing the same for the beads on the $(p - x)$ th and $(p - x + 1)$ st columns.

The abacus representation $\widetilde{Sc}_1(\lambda)$ may be described in the following way:

- Put a bead on the a th row of the first column iff there is a bead on the a th row of the $p - 1$ st column in the abacus representation for λ .

- Put a bead on the a th row of the $p - 1$ st column iff there is a bead on the $a + 1$ st row of the first column in the representation for λ .
- For all other columns, put a bead on the a th row of a column if there is a bead on the a th row of the same column in the abacus representation for λ .

Note that if there is a bead on the 0th row of the first column in the abacus representation for λ , then the number of beads on the $p - 1$ st column in the abacus representation for $\widetilde{S}_{c_1}(\lambda)$ is one less than the number of beads on the first column in the representation for λ .

It is clear from definition that $\widetilde{S}_{c_x} : \mathcal{D} \rightarrow \mathcal{D}$ is an involution if $x \neq 1$. Now suppose that $x = 1$. We consider the map

$$\widetilde{S}_{c_1} : \mathcal{D} \rightarrow \mathcal{D}.$$

This is clearly onto as given any χ in $\mathcal{D}$, consider the partition $\text{inv}(\chi)$ given by

$$\text{inv}(\chi)(y) = \begin{cases} \chi(1) & \text{if } y = p - 1 \\ \{a + 1 : a \in \chi(p - 1)\} \cup \{0\} & \text{if } y = 1 \\ \chi(y) & \text{otherwise .} \end{cases}$$

It is clear that 1 is a part of $\text{inv}(\chi)$. Let $\text{inv}(\chi)'$ be the partition obtained from $\text{inv}(\chi)$ by deleting the part 1 from it , i.e if $\text{inv}(\chi)$ is $(a_1, \dots, a_{l-1}, 1)$, then $\text{inv}(\chi)'$ is the partition $(a_1, \dots, a_{l-1})$. Then it is easy to see that the image of both $\text{inv}(\chi)$ and $\text{inv}(\chi)'$ under $\widetilde{S}_{c_1}$ is χ and that these are the only such partitions. So, if $\mathcal{D}'$ is defined to be the set of all strict partitions λ for which $0 \in \lambda(1)$, then the above

remarks show that

$$\widetilde{Sc}_1|_{\mathcal{D}'} : \mathcal{D}' \rightarrow \mathcal{D}$$

is a bijection.

LEMMA 4.7. *Let $\lambda \in \mathcal{D}$, $x \in X$. If $x \neq 1$, then $w(\widetilde{Sc}_x(\lambda)) = w(\lambda)$ and $\widetilde{Sc}_x(c_p(\lambda)) = c_p(\widetilde{Sc}_x(\lambda))$.*

Moreover, if λ is such that $0 \in \lambda(1)$ and $\lambda(p-1) = \phi$, then the same is true for $\widetilde{Sc}_1(\lambda)$.

REMARKS. The Lemma is quite easy for the case $x \neq 1$ if one thinks of the representations of λ and $\widetilde{Sc}_x(\lambda)$ on an abacus. For any $y \in X$, there is a unique y' in X such that the configuration of beads on the pair of columns y and $p-y$ in the representation for $\widetilde{Sc}_x(\lambda)$ is the configuration on the pair of columns y' and $p-y'$ in the representation for λ . Hence the p -bar core of $\widetilde{Sc}_x(\lambda)$ may be attained by making the same type and same number of moves on the pair of columns y and $p-y$ as one would make on the pair of columns y' and $p-y'$ when going from the abacus representation for λ to a representation for $c_p(\lambda)$. In particular, the configuration of beads on the columns y and $p-y$ in the representation for $c_p(\widetilde{Sc}_x(\lambda))$ is the configuration of beads on the columns y' and $p-y'$ in the representation for $c_p(\lambda)$. As the configuration of beads on the 0th row is the same in the representation for λ and $\widetilde{Sc}_x(\lambda)$, the assertion of the Lemma becomes clear

in the case $x \neq 1$.

The case $x = 1$ is a bit more complicated and is best explained formally in the language of p -sets. We shall do this below and will also give a formal proof for the case $x \neq 1$.

PROOF. We assume first that x is not 1 or $y \neq 1, p-1$. Then, by Definition 4.6,

$$|\widetilde{Sc}_x(\lambda)(y)| = |\lambda(Sc_x(y))| \quad (4.13)$$

In particular,

$$\begin{aligned} m_y(\widetilde{Sc}_x(\lambda)) &= |\widetilde{Sc}_x(\lambda)(y)| - |\widetilde{Sc}_x(\lambda)(p-y)| \\ &= |\lambda(Sc_x(y))| - |\lambda(Sc_x(p-y))| \\ &= |\lambda(Sc_x(y))| - |\lambda(p-Sc_x(y))| \\ &= m_{Sc_x(y)}(\lambda) \end{aligned} \quad (4.14).$$

and so,

$$\begin{aligned} |c_p((\widetilde{Sc}_x(\lambda))(y))| &= \max(0, m_y(\widetilde{Sc}_x(\lambda))) \\ &= \max(0, m_{Sc_x(y)}(\lambda)) \\ &= |c_p(\lambda)(Sc_x(y))| \\ &= |\widetilde{Sc}_x(c_p(\lambda))(y)|. \end{aligned} \quad (4.15)$$

Then, by the definition of the Scopes maps and (4.15), we also get that for $x \neq 1$ or $y \neq 1$,

$$W_y(\widetilde{Sc_x}(\lambda)) = W_{Sc_x(y)}(\lambda). \quad (4.16)$$

The equations (4.11), (4.15) and (4.16) prove the lemma for $x \neq 1$.

To prove the lemma when $x = 1$, we will show that under the conditions $0 \in \lambda(x)$ and $\lambda(p - x) = \phi$, there is equality between the extreme terms of (4.15) and that (4.16) holds for $x = 1$ and $y = 1, p - 1$.

First of all notice these conditions yield

$$|c_p(\lambda)(1)| = |\lambda(1)| \neq 0 \text{ and } |c_p(\lambda)(p - 1)| = |\lambda(p - 1)| = 0. \quad (4.17)$$

In particular, $c_p(\lambda) \in D'$ and hence

$$|\widetilde{Sc_1}(c_p(\lambda))(p - 1)| = |c_p(\lambda)(1)| = |\lambda(1)| - 1 \text{ and } |\widetilde{Sc_1}(\lambda)(1)| = |c_p(\lambda)(p - 1)| = 0. \quad (4.18)$$

This is precisely equation (4.15) for the cases $x = 1$ and $y = 1$ and $y = p - 1$

By (4.17), $|\lambda(1)| = |c_p(\lambda(1))|$ and hence (4.2) will give

$$W_1(\lambda) = p \left(\sum_{i=0}^{|\lambda(x)|-1} (a_i - i) \right) \quad (4.19)$$

where we write the set $\lambda(1)$ as $\{a_i : 0 \leq i \leq |\lambda(1)| - 1 : a_i < a_{i+1}\}$. As $0 \in \lambda(1)$,

$a_0 = 0$ and the above equation may be rewritten as

$$W_1(\lambda) = p \left(\sum_{i=1}^{i=|\lambda(x)|-1} (a_i - 1 - (i - 1)) \right). \quad (4.20)$$

But by definition $\widetilde{S}_{c_1}(\lambda)(p-1) = \{a_i - 1 : a_i \in \lambda(1), i \geq 1\}$. Also, $(c_p(\widetilde{S}_{c_1}(\lambda)))(p-1) = \{i : 0 \leq i \leq |\lambda(1)| - 1\}$ by (4.17).

Thus the right side of (4.20) is just $W_{p-1}(\widetilde{S}_{c_1}(\lambda))$, i.e.

$$W_1(\lambda) = W_{p-1}(\widetilde{S}_{c_1}(\lambda)) \quad (4.21)$$

Clearly,

$$W_{p-1}(\lambda) = 0 = W_1(\widetilde{S}_{c_1}(\lambda)) \quad (4.22)$$

The result for the the case $x = 1$ follows from (4.15), (4.16), (4.18) and (4.21) completing the proof of the Proposition. ■

We have

PROPOSITION 4.8. *Let w be a non negative integer. Let $\mu \in \bar{\mathcal{D}}$ and suppose that $x \in X$ is such that*

$$|\mu(x)| - |\mu(x-1)| > 3(w+1) \quad (4.23)$$

Then, $\widetilde{S}_{c_x}(\mu) \in \bar{\mathcal{D}}$, and $|\mu| - |\widetilde{S}_{c_x}(\mu)| > 3(w+1)$.

REMARKS. The first assertion of Proposition 4.8 is basically the assertion of Lemma 4.7.

For the second part, note that if a bead on the a th row of the y th column in the abacus representation for λ is moved one position left to the a th row of the $(y - 1)$ st column, the resulting partition has magnitude one less than λ . Here by moving a bead on the 0th column and a th row one position to the left on the abacus we will mean that the bead moves to the $a - 1$ st row on the $(p - 1)$ st column. Conversely, if a bead on the a th row of the y th column in the abacus representation of λ is moved one position right to the a th row of the $y + 1$ st column, the resulting partition has magnitude one more than λ .

For $x \neq 1$, $\widetilde{Sc}_x(\mu)$ may be thought of as obtained from λ by moving the beads on the x th column one position to the left, those on the $(x - 1)$ st position one column to the right and by moving the beads on the $(p - x + 1)$ th column one position to the left and those on the $(p - x)$ st position one column to the right. Now equation (4.23) ensures that there are more beads on the x th column than on the $(x - 1)$ st column. Since $\mu \in D$, Proposition 4.2 gives that there are no beads on the $p - x$ th column. Thus, there will be more beads moving one position to the left than there will be moving one position to the right which will yield the second assertion of Proposition 4.8.

When $x = 1$, (4.23) ensures that there are no beads on the $p - 1$ st column. Hence, $\widetilde{Sc}_1(\mu)$ may be thought of as being obtained from λ by moving all beads, except the bead on the 0th row, on the first column in the representation of λ to a

one higher row on the $p - 1$ st row. In other words these beads move two positions to the left. The bead on the 0th row of the first column is moved one position to the left, which is equivalent to removing it from the abacus. No beads are moved to the right and the lemma follows. We explain this in greater detail below.

PROOF. Let $\mu' = \widetilde{Sc}_x(\mu)$ and $l = n - m$.

The condition (4.23) and the fact that $\mu \in \bar{\mathcal{D}}$ imply

$$0 \in \mu(x), |\mu(x)| > 3(w + 1) > w + 1 \text{ and } |\mu(p - x)| = \phi. \quad (4.24)$$

So, by definition of the Scopes maps,

$$|\mu'(Sc_x(x))| > 3(w + 1) - 1 > (w + 1) \text{ and } |\mu'(Sc_x(p - x))| = \phi. \quad (4.25)$$

Now if $\lambda \in \mathcal{D}(n)$ has p -bar core μ , then $|\lambda(x)| \geq |\mu(x)| > w + 1$ and hence by Lemma 4.4 and (4.24)

$$0 \in \lambda(x), |\lambda(x)| = |\mu(x)| \text{ and } |\lambda(p - x)| = \phi. \quad (4.26)$$

Similarly, if $\chi \in \mathcal{D}(m)$ has p -bar core μ' , then by Lemma 4.4 and (4.25),

$$|\chi(Sc_x(x))| = |\mu'(Sc_x(x))| \text{ and } |\chi(Sc_x(p - x))| = \phi. \quad (4.27)$$

The equations (4.24) and (4.26) show in particular that the conditions of Lemma 4.7 are satisfied by μ and λ as above and hence $w(\mu') = w(\mu) = 0$ and $Sc_x(\lambda)$ is a partition of m with core μ' .

Using the equation (4.24) and the definitions of the Scopes maps, it is easy to see that

$$|\mu| - |\mu'| = \begin{cases} 2|\mu(1)| - 1 & \text{if } x = 1 \\ |\mu(\frac{p+1}{2})| & \text{if } x = \frac{p+1}{2} \\ |\mu(x)| - |\mu(x-1)| + |\mu(p-x+1)| & \text{otherwise} \end{cases} \quad (4.28)$$

Then by (4.23), $|\mu| - |\mu'| \geq 3(w+1) > 0$, proving Proposition 4.8. $\blacksquare$

PROPOSITION 4.9. *Keep the hypothesis of the the previous Proposition.*

Then the map $\widetilde{Sc}_x$ establishes a 1-1 correspondence between strict partitions λ of $n := |\mu| + pw$ with core μ and partitions λ' of $m := |\widetilde{Sc}_x(\mu)| + pw$ with p -bar core $\widetilde{Sc}_x(\mu)$.

PROOF. Let $\lambda \in \mathcal{D}(n)$ with $c_p(\lambda) = \mu$. Put $\lambda' = \widetilde{Sc}_x(\lambda)$ and $\mu' = \widetilde{Sc}_x(\mu)$.

If $x \neq 1$, the map $\widetilde{Sc}_x : \mathcal{D} \rightarrow \mathcal{D}$ is clearly an involution which preserves p -bar cores and weights by Lemma 4.7 and so $\lambda \rightarrow \lambda'$ is a 1-1 correspondence of partitions of n with p -bar core μ and partitions of m with p -bar core μ' .

Now suppose that $x = 1$. Let

$$T := \{\lambda : \lambda \in \mathcal{D}(n), c_p(\lambda) = \mu\}$$

and

$$S := \{\chi : \chi \in \mathcal{D}(m), c_p(\chi) = \mu'\}.$$

Then by equation (4.26), $T \subset \mathcal{D}'$, hence $\widetilde{Sc}_1|_T$ is injective. Further, λ satisfies the hypothesis of Lemma 4.7 and thus $\widetilde{Sc}_1(T) \subset S$.

Now let χ be an element of S . Recall the definition of $\text{inv}(\chi)$ from the remarks following Definition (4.6). By equation (4.27), $\text{inv}(\chi)$ will satisfy $0 \in \text{inv}(\chi)(1)$ and $\text{inv}(\chi)(p-1) = \phi$. By Lemma 4.7, as $\chi = \widetilde{Sc}_1(\text{inv}(\chi))$, $w(\chi) = w(\text{inv}(\chi))$ and $\widetilde{Sc}_1(c_p(\text{inv}(\chi))) = \mu'$. In other words, $c_p(\text{inv}(\chi))$ is either $\text{inv}(c_p(\chi)) = \mu$ or $\text{inv}(c_p(\chi))'$. But $\text{inv}(c_p(\chi))'$ is clearly not its own p -bar core and hence $c_p(\text{inv}(\chi))$ is $\text{inv}(c_p(\chi))$, i.e. $\text{inv}(\chi) \in T$. This shows the 1-1 correspondence between the sets T and S via $\widetilde{Sc}_1$, thus completing the proof. $\blacksquare$

Recall the sets $\mathcal{M}^l(\lambda, \lambda')$ from Definition 3.15 (ii). We have

PROPOSITION 4.10. *Keep the hypothesis of Proposition 4.8. Let λ be a partition of $n := |\mu| + pw$ with core μ . If χ is a partition of $m := |\widetilde{Sc}_x(\mu)| + pw$ with p -bar core $\widetilde{Sc}_x(\mu)$, then $\mathcal{M}^{n-m}(\lambda, \chi) \neq \emptyset$ iff $\chi = \widetilde{Sc}_x(\lambda)$ in which case:*

(i)

$$|\mathcal{M}^{n-m}(\lambda, \chi)| = |\mathcal{M}^{n-m}(\mu, \widetilde{Sc}_x(\mu))| = \begin{cases} (n-m)! & \text{if } x \neq 1 \\ \frac{(n-m)!}{2^{\frac{n-m-1}{2}}} & \text{if } x = 1 \end{cases} \quad (4.29)$$

(ii)

$$n(\lambda) - n(\chi) = n(\mu) - n(\widetilde{Sc}_x(\mu)) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \quad (4.30)$$

(iii)

$$\epsilon(\lambda) - \epsilon(\chi) \equiv \epsilon(\mu) - \epsilon(\widetilde{Sc}_x(\mu)) \pmod{2} \quad (4.31)$$

REMARKS. Again, before we begin the proof, we explain the results of Proposition 4.10 in terms of the abacus representations for strict partitions. Let

λ be a strict partition of n with p -bar core μ and let χ be a strict partition of m with p -bar core $\widetilde{S}_{c_x}(\mu)$. Also, let $l = n - m$. We suppose that $\mathcal{M}^l(\lambda, \chi) \neq \emptyset$ and let $\Lambda : \lambda = \Lambda^0 > \dots > \Lambda^l = \chi$ be an element of $\mathcal{M}^l(\lambda, \chi)$. For $1 \leq i \leq l - 1$, Λ^{i+1} is the partition obtained by removing a 1 bar from Λ^i (cf. Definition 3.7). In other words, the abacus representation for Λ^{i+1} is obtained from the representation for Λ^i by moving a bead in the abacus representation of Λ^i one position to the left. As Λ^{i+1} is also a strict partition, there must not already be a bead one position to the left of the bead being moved. We use the constraints placed by (4.23) and Lemma 4.4 on the abacus representations of λ and χ to show that when $x \neq 1$, the number of beads on the $(x - 1)$ st column in the representation for χ is exactly equal to the number of beads on the x th column in the representation for λ and that there are no beads on the $(p - x + 1)$ st column in the representation for χ . We will also show that if there is a bead on a row of the $x - 1$ st column in the representation for λ , then by (4.23) and by Lemma 4.4 there will be a bead on the corresponding row in the x th column. Further, we will show that there are no beads on the $(p - x)$ th column in the representation for λ . From this it will follow that the only way to get from λ to χ by moving beads successively to the left l times is to move all the beads on the x th column in the representation for λ for which there is no bead on the corresponding row in the $x - 1$ st column one position to the left and to do the same for the all beads on the $p - x + 1$ st column,

which will show that χ must be $\widetilde{Sc}_x(\lambda)$. We will also show that these beads may be moved in any order, which will give (4.29) for the case $x \neq 1$. Then (4.30) and (4.31) will follow from the definitions of the Scopes maps.

When $x = 1$, Lemma 4.4 will give that there are no beads on the first column in the representation for χ , and the number of beads on $(p - 1)$ st column in the representation for χ is exactly one less than the number of beads on the first column in the representation for λ . Lemma 4.4 and (4.23) will also give that there are no beads on the $(p - 1)$ st column in the representation for λ and that if there is a bead on a row on the 0th column in the representation for λ , there is a bead on the corresponding row of the first column in the representation for λ .

From these observations, we will show that the only way we can get to the representation for χ from the representation for λ is by making the following moves in an admissible order:

- The beads on the 0th column in the representation for λ are moved one position to the left.
- A bead on the first column in the representation for λ is moved one position to the left if there is a bead on the corresponding row of the 0th column in the representation for λ .
- A bead on the first column and not on the first row in the representation for λ is moved two positions to the left if there is no bead on the corresponding

row of the 0th column.

- The bead on the 0th position of the first column is moved one position to the left, i.e it is removed from the abacus.

It will follow that χ must be $\widetilde{Sc}_1(\lambda)$. Also, there will be certain constraints on the order in which these movements are made which will result in (4.29).

PROOF. First, notice that part (iii) follows from part (ii) as $|\lambda| - |\chi| = |\mu| - |\mu'|$. Let λ , χ and Λ be as in the statement of the Proposition. Let us put $\lambda' := \widetilde{Sc}_x(\lambda)$, $\mu' := \widetilde{Sc}_x(\mu)$ and $l := n - m$.

Λ^{i+1} is the partition obtained by removing a 1 bar from Λ^i (cf. Definition 3.7), but this is easily seen to be equivalent to replacing one of the parts say b , of Λ^i by $b - 1$, where b is such that $b - 1$ is not a part of Λ^i . Here note that if b is 1, then Λ^{i+1} is simply $\Lambda^i / \{1\}$. There is a unique t in $\{0\} \cup X$ and a unique non-negative integer a such that $b \equiv t + pa$. Then a is in $\Lambda^i(t) / \lambda^i(t - 1)$ and we may describe Λ^{i+1} as follows:

$$\Lambda^{i+1}(t) = \Lambda^i(t) / \{a\}, \quad (4.32)$$

$$\Lambda^{i+1}(t - 1) = \Lambda^i(t) \cup \{a\}, \quad (4.34)$$

$$\Lambda^{i+1}(y) = \Lambda^i(y), \text{ for } y \neq t, t - 1. \quad (4.35)$$

We then write,

$$\Lambda^{i+1} = \Lambda^i(t, a). \quad (4.36)$$

Note that, if $t = 0$, the above expression gives us the p -set through -1 , but

$$\Lambda^{i+1}(p-1) = \{a-1 : a \in \Lambda^{i+1}(-1)\}$$

At this point, we consider 2 cases.

Case 1 : $x \neq 1$.

Then the equation (4.27) yields

$$|\chi(x-1)| = |\mu'(x-1)| = |\mu(x)|. \quad (4.37)$$

$$\chi(p-x+1) = \phi. \quad (4.38)$$

Let

$$|\lambda(x-1)| = |\mu(x-1)| + t. \quad (4.39)$$

Then, by Proposition 4.2,

$$|\lambda(p-x+1)| = |\mu(p-x+1)| + t, \quad (4.40)$$

where $w \geq t \geq 0$ by Lemma 4.4.

We have

$$|\mu(x)| - |\mu(x-1)| - t \geq 3(w+1) - w > 2w+1 \quad (4.41)$$

So, by (4.26), (4.39) and (4.41), we have that $|\lambda(x)| - |\lambda(x-1)| > 2w+1$. But by Lemma 4.4 (i) and (ii), $\lambda(x) \supset \{0 \dots |\lambda(x)| - w - 1\}$, $\{0 \dots |\lambda(x-1)| + w\} \supset \lambda(x-1)$.

So, in fact,

$$\lambda(x-1) \subset \lambda(x). \quad (4.42)$$

Subcase 1: $x \neq \frac{p+1}{2}$.

By equations (4.37) and (4.39) there is a subset A of $\lambda(x-1) - \lambda(x-1)$ of cardinality $|\mu(x)| - |\mu(x-1)| - t$ and a subset $I = \{i_a : a \in A\}$ of $\{0 \dots l-1\}$ such that $\Lambda^{i_a+1} = \Lambda^{i_a}(x, a)$ for a in A (cf. equations 4.32-4.36).

By equations (4.38) and (4.40), there is a subset $J = \{j_b : b \in \lambda(p-x+1)\}$ of $\{0 \dots l-1\}$, of cardinality $|\mu(p-x+1)| + t$, such that for b in $\lambda(p-x+1)$, $\Lambda^{j_b+1} = \Lambda^{j_b}(p-x+1, b)$.

Clearly, I and J are disjoint subsets of $\{0 \dots l-1\}$ and $|I| + |J| = |\mu(x)| - |\mu(x-1)| + |\mu(p-x+1)|$. But as was checked in (4.9), this is just $|\mu| - |\mu'| = l$ and so $I \cup J = \{0 \dots l-1\}$.

Let $a \in A$. Suppose that a is not in $\lambda(x)$. By definition, $a \in \Lambda^{i_a}(x)$, so there is an r in $\{0 \dots l-1\}$ such that $\Lambda^{r+1} = \Lambda^r(x+1, a)$ which means that x is $p-x+1$, a contradiction. Hence, $A \subset \lambda(x) - \lambda(x-1)$. But by equation (4.42), $\lambda(x-1) \subset \lambda(x)$ and as $|A| = |\mu(x)| - |\mu(x-1)| - t = |\lambda(x)| - |\lambda(x-1)|$, by equations (4.37) and (4.39) and as $A \cap \lambda(x-1) = \emptyset$, we get that $A = \lambda(x) / \lambda(x-1)$.

Now let

$$C := \{(x, a) : a \in A\} \cup \{(p-x+1, a) : a \in \lambda(p-x+1)\}.$$

We have shown above that there is a bijection $\psi : \{0 \dots l-1\} \rightarrow C$, such that for any r in $\{0 \dots l-1\}$, $\Lambda^{r+1} = \Lambda^r(y, a)$, where $\psi(r) = (y, a)$. Conversely, let $\phi : \{0 \dots l-1\} \rightarrow C$ be any bijection. For r in $\{0 \dots l-1\}$, let $\Lambda_\phi^0 = \lambda$. Also, let $\Lambda_\phi^{r+1} = \Lambda_\phi^r(y, a)$, where $(y, a) = \phi(r)$.

We claim that Λ_ϕ^{r+1} is well defined, i.e, we need to check that if $\phi(r) = (y, r)$, then $y \in \Lambda^r(y)$ and $y \notin \Lambda^r(y-1)$. Suppose first that $\phi(r) = (x, a)$. If a is not in $\Lambda_\phi^r(x)$, then for some s in $\{0 \dots l-1\}$, $s < r$, $\phi(s)$ must be $(x+1, a)$ but this is impossible as $x+1 \neq p-x+1$. Again, $a \notin \Lambda_\phi^r(x-1)$, so if a is in $\Lambda_\phi^r(x-1)$, then there is an s in $\{0 \dots l-1\}$, $s < r$, for which $\phi(s)$ is (x, a) but this contradicts the fact that ϕ is a bijection. So, $a \in \Lambda_\phi^r(x) - \Lambda_\phi^r(x-1)$. Similarly, we can show that if $\phi(r) = (p-x+1, a)$, then $a \in \Lambda_\phi^r(p-x+1) - \Lambda_\phi^r(p-x)$.

Clearly, $\Lambda_\phi^l(y) = \lambda(y) = \lambda'(y)$ for $y \in \{0\} \cup X$, $y \notin \{x, x-1, p-x+1, p-x\}$. Also, it is easy to see that $\Lambda_\phi^l(x) = \lambda(x) - A = \lambda(x-1)$, $\Lambda_\phi^l(p-x+1) = \phi = \lambda(p-x) = \lambda'(p-x)$ and $\Lambda_\phi^l(p-x) = \lambda(p-x+1) = \lambda'(p-x)$. In other words, $\Lambda_\phi^l = \lambda'$ and thus the sequence $\Lambda_\phi : \lambda = \Lambda_\phi^0, \dots, \Lambda_\phi^l$ is an element of $\mathcal{M}^l(\lambda, \lambda')$.

Let ϕ' be another bijection between $\{0 \dots l-1\}$ and C . We will show that $\Lambda_\phi \neq \Lambda_{\phi'}$. Let r be the least integer for which $\phi(r) \neq \phi'(r)$. Then, $\Lambda_\phi^r = \Lambda_{\phi'}^r$ and consequently, $\Lambda_\phi^{r+1} \neq \Lambda_{\phi'}^{r+1}$.

As $\Lambda = \Lambda_\phi$, for some bijection ϕ between $\{0 \dots l-1\}$ and C , we get from the above remarks that $\chi = \lambda'$ and again as any element of $\mathcal{M}^l(\lambda, \lambda')$ is Λ_ϕ for

some ϕ , we get that

$$|\mathcal{M}^l(\lambda, \lambda')| = |C|! = l!. \quad (4.43)$$

The equations (4.37)-(4.42) hold with λ and χ replaced by μ and μ' respectively, and repeating the above treatment, we get that

$$|\mathcal{M}^l(\mu, \mu')| = |C|! = l!, \quad (4.44)$$

proving (4.29) when $x \neq 1, \frac{p+1}{2}$.

(4.30) follows immediately from the definition of the Scopes maps.

We now consider

Subcase 2: $x = \frac{p+1}{2}$.

As $x = p - x + 1$, we have $\chi(x) = \phi$ by (4.38) but then there must be a subset

$$I := \{i_a : a \in \lambda(\frac{p+1}{2})\}$$

of $\{0 \dots l-1\}$ of cardinality $|\lambda(\frac{p+1}{2})|$ such that $\Lambda^{i_a+1} = \Lambda^{i_a}(x, a)$ for a in $\lambda(\frac{p+1}{2})$.

As $|\Lambda(\frac{p+1}{2})| = l$, $I = \{0 \dots l-1\}$, and then it follows as in the previous case that $\chi = \lambda'$ and that $\mathcal{M}^l(\lambda, \lambda')$ is in one to one correspondence with the bijections between $\{0 \dots l-1\}$ and $\lambda(\frac{p+1}{2})$, i.e. $|\mathcal{M}^l(\lambda, \lambda')| = l!$. The proposition now follows as in the previous case.

It remains only to prove

Case 3: $x = 1$.

Then equation (4.27) gives

$$|\chi(p-1)| = |\mu'(p-1)| = |\mu(1)| - 1. \quad (4.45)$$

$$\chi(1) = \phi. \quad (4.46)$$

By using condition (4.23) and Lemma (4.4) with $x = 1$, we can show that

$$\lambda(0) \subset \lambda(1). \quad (4.47)$$

By equations (4.26) and (4.46), there is a subset $I = \{i_a : a \in \lambda(1)\}$ of $\{0 \dots l-1\}$ of cardinality $|\mu(1)| = |\lambda(1)|$ such that $\Lambda^{i_a+1} = \Lambda^{i_a}(1, a)$ for $a \in \lambda(1)$. In particular, as $0 \in \lambda(1)$, i_0 is an element of I and $\Lambda^{i_0+1} = \Lambda^{i_0}/\{1\}$.

Again, as $|\lambda(-1)| = |\lambda(p-1)| = 0$ and $|\chi(-1)| = |\chi(p-1)| = |\mu(1)| - 1$, there is a subset $J = \{j_b : b \in \chi(-1)\}$ of $\{0 \dots l-1\}$, such that $\Lambda^{j_b+1} = \Lambda^{j_b}(0, b)$, for b in $\chi(-1)$.

By definition, $I \cap J = \phi$ and $|I| + |J| = 2|\mu(1)| - 1 = l$, and so $I \cup J = \{0 \dots l-1\}$.

CLAIM.

$$\chi(-1) = \lambda(1) - \{0\}. \quad (4.48)$$

PROOF OF CLAIM. Let $b \in \chi(-1)$. If b belongs to $\lambda(0)$, then it belongs to $\lambda(1)$ by equation (4.47). If $b \notin \lambda(0)$, then for some r in $\{0 \dots l-1\}$, $r < j_b$,

$\Lambda^{r+1} = \Lambda^r(1, b)$. This r then must belong to I and consequently b must belong to $\lambda(1)$. So $\chi(-1) \subset \lambda(1)$ and as 0 can not belong to $\chi(-1)$, in fact $\chi(-1) \subset \lambda(1)/\{0\}$. As the two sets have the same cardinality, the claim is proved.

Now let a be a non-zero element of $\lambda(1)$. Suppose first $a \in \lambda(0) \cup \lambda(1)$. Then, as a cannot be an element of $\Lambda^{i_a}(0)$, there is an r in $\{0 \dots l-1\}$, with $r < i_a$ and such that $\Lambda^{r+1} = \Lambda^r(0, a)$. But then, r is in J and is in fact j_a which shows that $i_a > j_a$. On the other hand, if a is not in $\lambda(0)$, then the fact that a is in $\Lambda^{j_a+1}(0)$ means that there is an s in $\{0 \dots l-1\}$, $s < j_a$, such that $\Lambda^{s+1} = \Lambda^s(1, a)$ and as s must then be i_a , we get that $i_a < j_a$.

Now let

$$C := \{(1, 0)\} \cup \{(1, a) : a \in \lambda(1) - \{0\}\} \cup \{(0, a) : a \in \lambda(1) - \{0\}\}.$$

We have shown above that the sequence Λ determines a bijection $\psi : \{0 \dots l-1\} \rightarrow C$ via $\Lambda^{r+1} = \Lambda^r(y, a)$ where $\psi(r) = (y, r)$. Further, ψ satisfies the following property:

If r_1 and r_2 in $\{0 \dots l-1\}$ are such that $\psi(r_1) = (0, a)$ and $\psi(r_2) = (1, a)$ for some a in $\lambda(1) - \{0\}$, then

$$r_1 < r_2 \text{ iff } a \in \lambda(0). \quad (4.49)$$

Conversely, for any bijection ϕ between $\{0 \dots l-1\}$ and C which satisfies the above property, it is easy to see that the sequence Λ_ϕ of strict partitions

defined by $\Lambda_\phi^0 = \lambda$, and $\lambda_\phi^{r+1} = \Lambda_\phi^r(y, r)$ where $\phi(r) = (y, r)$ is well defined. For this, all we need to check is that if $\phi(r) = (y_r, a_r)$, then $a_r \in \Lambda^r(y_r)$ and $a_r \notin \Lambda^r(y_r - 1)$. Suppose first that $r = \phi^{-1}(1, a)$, i.e $y_r = 1$ and $a_r = a$ for some $a \in \lambda(1) - \{0\}$, and let $s = \phi^{-1}(0, a)$. Suppose, if possible, that $a \in \Lambda_\phi^r(0)$. If $a \in \lambda(0)$, then $s < r$, and as $a \notin \Lambda_\phi^{s+1}(0)$ there must be some t , $s < t < r$, such that $\Lambda_\phi^{t+1} = \Lambda_\phi^t(1, a)$, which would imply that $\phi(t) = (1, a) = \phi(s)$, hence $t = r$ but this contradicts the choice of s . If $a \notin \lambda(0)$, there must be some t , $t < r$, such that $\Lambda_\phi^{t+1} = \Lambda_\phi^t(1, a)$, which would imply that $\phi(t) = (1, a) = \phi(s)$, hence $t = s$. But $t < r < s$. Similarly, we can show that $a \in \Lambda^r(1)$.

The cases $y_r = 0$ and $y_r = 1, a_r = 0$ can be handled using very similar arguments.

It is easy to see that

$$\Lambda_\phi^l(-1) = \Lambda_\phi^0(-1) \cup \lambda(1) - \{0\} = \lambda(1) - \{0\}. \quad (4.50)$$

This is the same thing as saying that

$$\Lambda_\phi^l(p-1) = \{a-1 : a \in \lambda(0) - \{0\}\}. \quad (4.51)$$

Also,

$$\Lambda_\phi^l(1) = \phi. \quad (4.52)$$

and

$$\Lambda_\phi^l(y) = \lambda(y) \quad (4.53)$$

for all other y in X . Using the same sort of arguments as were used to prove that Λ_ϕ is a well defined sequence, one can easily show that

$$\Lambda_\phi^l(0) = \lambda(0) \quad (4.54)$$

which, along with the equations (4.51), (4.53) and (4.53) shows that $\Lambda_\phi = \lambda'$.

Also, clearly, if $\phi \neq \phi'$, then $\Lambda_\phi \neq \Lambda_{\phi'}$. Then as $\Lambda = \Lambda_\phi$ for some ϕ , we simultaneously get that $\chi = \lambda'$ and that the elements of $\mathcal{M}^l(\lambda, \lambda')$ are in one to one correspondence with bijective maps $\phi \rightarrow C$, which satisfy the condition (4.49). But the number of such maps is

$$\frac{|C|!}{2^{\frac{|C|-1}{2}}} = \frac{l!}{2^{\frac{l-1}{2}}} \quad (4.55)$$

A similar equation may be obtained for μ and μ' .

As μ and λ are in $\mathcal{D}'$, (4.29) will hold for the case $x = 1$. Now (4.30) is clear from the definition of the Scopes maps. This completes the proof of Proposition 4.10. ■

DEFINITION 4.11. Let w be a non-negative integer, and let μ be in $\bar{\mathcal{D}}$. Let μ' be another element of $\bar{\mathcal{D}}$. We will write $\mu' \sim \mu$ if there exists a chain of elements of $\bar{\mathcal{D}}$, starting at μ and ending at μ' such that if ν and η are two elements of this chain, with η the immediate predecessor of ν , then for some x in X , $\nu = \widetilde{Sc}_x(\eta)$ where $|\eta(x)| - |\eta(x-1)| > 3(w+1)$. Note that $\sim$ is a transitive relation on $\bar{\mathcal{D}}$.

PROPOSITION 4.12. *Let $\mu \in \bar{\mathcal{D}}$, such that $n(\mu) > 3p^2(w+1) + 3(p-1)$. Then there exists μ' in $\bar{\mathcal{D}}$ such that $\mu' \sim \mu$ and $\epsilon(\mu') = \epsilon(\mu)$ and $|\mu| > |\mu'|$.*

REMARKS. Let $\tau \in \mathcal{D}$ and suppose that $x \in X$ is such that $|\tau(x)| - |\tau(x-1)| > 3(w+1)$. It is easy to see from (4.28) and (4.34) that if $x = 1$, then $\widetilde{Sc}_x(\tau)$ has the same parity as τ and when $x \neq 1$, the parity of $\widetilde{Sc}_x(\tau)$ depends on the number of beads on the x th, $x-1$ st and $p-x+1$ st columns in the representation for τ on an abacus. The critical point of the proof below is that the number of beads on the representation for $\widetilde{Sc}_1(\tau)$ is one less than the number of beads in the representation for τ . We will define a chain of elements in $\mathcal{D}$ starting at μ , as in Definition 4.11, and this chain will periodically have successive elements η and ν such that $\nu = \widetilde{Sc}_1(\eta)$. This will ensure that the number of beads in the representations of the elements of this chain keeps decreasing. We will use this fact to isolate an element in the chain which has the same parity as μ .

PROOF. First, let χ be any element of $\bar{\mathcal{D}}$ with $n(\chi) > 3p^2(w+1)$. We will construct another element $\tilde{\chi}$ of $\bar{\mathcal{D}}$ in the following way:

As $n(\chi) = \sum_{x \in X} |\chi(x)|$, there is some x in X such that $|\chi(x)| > 3p(w+1) > 3x(w+1)$. Let x_χ be the least such x . Then for any $x < x_\chi$, certainly, $|\chi(x_\chi)| - |\chi(x)| > 3(w+1)$. In particular, $|\chi(x_\chi)| - |\chi(x_\chi-1)| > 3(w+1)$ and hence $\widetilde{Sc}_{x_\chi}(\chi) \sim \chi$.

If $x_\chi > 1$, then $\widetilde{Sc}_{x_\chi}(\chi)(y) = \chi(Sc_{x_\chi}(y))$ for all y in $\{0\} \cup X$. In particular, $\widetilde{Sc}_{x_\chi}(\chi)(x_\chi - 1) = \chi(x_\chi)$, and $\widetilde{Sc}_{x_\chi}(\chi)(y) = \chi(y')$ where y' is $y, y - 1$ or $y + 1$. In any case $y' \leq x_\chi - 1$ for $1 \leq y < x_\chi - 1$, which means that

$$|\widetilde{Sc}_{x_\chi}(\chi)(x_\chi - 1)| = |\chi(x_\chi)| > |\chi(y')| + 3(w + 1) = |\widetilde{Sc}_{x_\chi}(y)| + 3(w + 1) \quad (4.56)$$

for all y in $\{0\} \cup X$ such that $y < x_\chi - 1$. In particular, $\widetilde{Sc}_{x_\chi - 1} \circ \widetilde{Sc}_{x_\chi}(\chi) \sim \widetilde{Sc}_{x_\chi}(\chi) \sim \chi$. As $\sim$ is a transitive relation, $\widetilde{Sc}_{x_\chi - 1} \circ \widetilde{Sc}_{x_\chi}(\chi) \sim \chi$.

Applying the above argument successively to $x_\chi, x_\chi - 1, \dots, 2$, we get that if

$$\tilde{\chi} := \widetilde{Sc}_1 \circ \dots \circ \widetilde{Sc}_{x_\chi - 1} \circ \widetilde{Sc}_{x_\chi}(\chi),$$

then $\tilde{\chi} \sim \chi$. Here, if $x_\chi = 1$, then $\tilde{\chi} = \widetilde{Sc}_1(\chi)$.

Let $x \in X$ and let

$$\sigma := Sc_1 \circ \dots \circ Sc_{x-1} \circ Sc_x.$$

By Definition 4.5, using the usual cycle notation for permutations, it is easy to see that for any x in X ,

$$\sigma = \begin{cases} (1 \dots x - 1, x, p - 1, \dots, p - x + 1, p - x) & \text{if } 1 \leq x < \frac{p+1}{2} \\ (1, 2, \dots, p - x)(p - 1, p - 2, \dots, x) & \text{if } x \geq \frac{p+1}{2}. \end{cases} \quad (4.57)$$

Applying this to $x = x_\chi$, it becomes clear from the definitions of $\tilde{\chi}$ that $\tilde{\chi}(p - 1) = \{a - 1 : a \in \chi(x_\chi), a \neq 0\}$ and that $\tilde{\chi}(y) = \chi(\sigma^{-1}(y))$ for all y in

$\{0\} \cup X$ different from $p - 1$. In other words, as $0 \in \chi(x_\chi)$,

$$|\tilde{\chi}(\sigma(x_\chi))| = |\chi(x_\chi)| - 1 \text{ and for all other } y \text{ in } \{0\} \cup X, |\tilde{\chi}(y)| = |\chi(\sigma^{-1}(y))|. \quad (4.58)$$

Again, using x_χ for x in the equation (4.57), we get that $\sigma^{-1}(y) \leq y < x_\chi - 1$ for $1 < y < x_\chi$. Hence, by definition of x_χ ,

$$|\tilde{\chi}(y)| = |\chi(\sigma^{-1}(y))| \leq 3\sigma^{-1}(y)(w+1) \leq 3y(w+1) \quad (4.59)$$

for all y in X , $1 < y < x_\chi$.

The inequality between the extreme terms of (4.59) holds trivially for $y = 1$ since $\chi(x_\chi) \neq \phi$ and $\chi \in \bar{\mathcal{D}}$ imply that $\tilde{\chi}(1) = \chi(p - x_\chi) = \phi$.

If $1 < x_\chi < \frac{p+1}{2}$, then $\sigma^{-1}(x_\chi) = x_\chi - 1$ and again,

$$|\tilde{\chi}(x_\chi)| = |\chi(x_\chi - 1)| < 3x_\chi(w+1). \quad (4.60)$$

The above remarks show that either $\tilde{\chi}(x) \leq 3x(w+1)$ for all $x \in X$ or, denoting by $x_{\tilde{\chi}}$ the least $x \in X$ such that $\tilde{\chi}(x) \leq 3x(w+1)$,

$$x_{\tilde{\chi}} \geq x_\chi \text{ and if } x_\chi < \frac{p+1}{2}, \text{ then } x_{\tilde{\chi}} > x_\chi. \quad (4.61)$$

Now (4.58) gives, in particular,

$$n(\tilde{\chi}) = n(\chi) - 1. \quad (4.62)$$

As $n(\mu) > 3p^2(w+1) + 3(p-1)$, we use the above construction inductively to get a sequence:

$$\mu = \mu_1 \sim \mu_2 \sim \dots \sim \mu_{3(p-1)+1} \quad (4.63)$$

of elements of $\bar{D}$ where $\mu_{i+1} = \tilde{\mu}_i$.

Note that by equation (4.62), $n(\mu_{i+1}) = n(\mu_i) - 1 \geq n(\mu) - 3(p-1) > 3p^2(w+1)$, which shows that $x_{\mu_{i+1}}$ exists for $1 \leq i \leq 3(p-1)$ and the above sequence is therefore well defined.

Let $x_i = x_{\mu_i}$ and let σ_i be the permutation $Sc_1 \circ \dots \circ Sc_{x_{i-1}} \circ Sc_{x_i}$ of $\{0\} \cup X$.

As $\mu_{i+1} = \tilde{\mu}_i$, equation (4.58) gives $|\mu_{i+1}(\sigma_i(x_i))| = |\mu_i(x_i)| - 1$ and for all other y in $\{0\} \cup X$, $|\mu_{i+1}(\sigma_i(y))| = |\mu_i(y)|$. So, if y is not one of $\sigma_l(x_l)$, $\sigma_l \circ \sigma_{l-1}(x_{l-1}), \dots, \sigma_l \circ \sigma_{l-1} \circ \dots \circ \sigma_i(x_i)$, then

$$\mu_{l+1}(y) = \mu_i(\sigma_i^{-1} \circ \dots \circ \sigma_{l-1}^{-1} \circ \sigma_l^{-1}(y)). \quad (4.64)$$

CLAIM. *There exist r and s , such that $\frac{p+1}{2} \leq r < s \leq 3(p-1)$ and such that $|\mu_s(x_s)| = |\mu_r(x_r)| - 1$.*

PROOF. For $\frac{p+1}{2} \leq i \leq 3(p-1)$, consider $y_i = \sigma_{3(p-1)} \circ \dots \circ \sigma_i(x_i)$. As each of these numbers belongs to X , and $3(p-1) - (p-1) > p-1 = |X|$, they cannot all be different. Choose r and s such that $y_r = y_s$, $r < s$ and $s - r$ as small as possible.

If $r = s - 1$, then $x_s = \sigma_{s-1}(x_{s-1})$ and then by equation (4.58), $|\mu_s(x_s)| = |\mu_{s-1}(x_{s-1})| - 1$ and the claim is proved.

Otherwise, $x_s = \sigma_{s-1} \circ \dots \circ \sigma_r(x_r)$ and $\sigma_{t-1} \circ \dots \circ \sigma_r(x_r) \neq x_t$ for any t strictly in between r and s , as this would contradict the minimality of $s - r$. So, by equation (4.64) with $i = r + 1, l = s - 1$, $|\mu_s(x_s)| = |\mu_{r+1}(\sigma_r(x_r))| = |\mu_r(x_r)| - 1$, thus proving the claim.

By equation (4.61), $x_{\mu_{i+1}} \geq x_{\mu_i}$, with strict inequality if $x_{\mu_i} < \frac{p+1}{2}$ and so, in particular, for all $i \geq \frac{p-1}{2}$,

$$x_{\mu_{i+1}} \geq \frac{p+1}{2}. \quad (4.65)$$

For $i \geq \frac{p+1}{2}$, let

$$\rho_i := \begin{cases} \mu_i & \text{if } x_{\mu_i} = \frac{p+1}{2} \\ \widetilde{Sc}_{\frac{p+1}{2}+1} \circ \dots \circ \widetilde{Sc}_{x_{\mu_i}}(\mu_i) & \text{otherwise.} \end{cases}$$

Also, let

$$\tau_i := \widetilde{Sc}_{\frac{p+1}{2}}(\rho_i).$$

In either case, $\mu_{i+1} = \tilde{\mu}_i = \widetilde{Sc}_1 \circ \dots \circ \widetilde{Sc}_{\frac{p-1}{2}}(\tau_i)$, i.e, ρ_i and τ_i are intermediate partitions in the sequence of partitions that are used to construct μ_{i+1} from μ_i and therefore, it is easy to see that $\tau_i \sim \rho_i \sim \mu_i \sim \mu$.

Also, it is easy to check that

$$\rho_i\left(\frac{p+1}{2}\right) = \mu_i(x_{\mu_i}) \quad (4.66)$$

and

$$\rho_i\left(\frac{p-1}{2}\right) = \mu_i(p - x_{\mu_i}) = \phi. \quad (4.67)$$

So

$$\epsilon(\rho_i) - \epsilon(\tau_i) \equiv |\rho_i| - |\tau_i| - n(\rho_i) + n(\tau_i) \equiv |\mu_i(x_{\mu_i})| \pmod{2}. \quad (4.68)$$

In particular, for r and s as in the claim, $\epsilon(\rho_r) - \epsilon(\tau_r) \neq \epsilon(\rho_s) - \epsilon(\tau_s)$.

But then $\epsilon(\mu)$ must be one of $\tau_r, \rho_r, \tau_s, \rho_s$. As μ is not equal to any of $\tau_r, \rho_r, \tau_s, \rho_s$, $|\mu'| < |\mu|$ by Proposition 4.8, which proves the proposition. $\blacksquare$

§2. A rank comparison

Notation

Set

$$a_w := 3p^2(w+1) + 3(p-1),$$

the integer that shows up in the statement of Proposition 4.12. Also, set

$$N_w := a_w(p-1) + p\left(\frac{a_w(a_w+1)}{2}\right).$$

Let $\mu \in \overline{\mathcal{D}}$ with $|\mu| > N_w$. Then it is easy to see that $n(\mu) > a_w$, i.e., μ satisfies the hypothesis of Proposition 4.12 and hence there exists $\nu \in \overline{\mathcal{D}}$ such that $\nu \sim \mu$, $|\nu| < |\mu|$ and $\epsilon(\nu) = \epsilon(\mu)$. In other words, there exists a sequence :

$$\mu =: \nu_0, \nu_1, \dots, \nu_t, \nu_{t+1} =: \nu$$

in $\bar{\mathcal{D}}$ such that for $0 \leq i \leq t$, $\nu_{i+1} = \widetilde{Sc_{x_i}}(\nu_i)$ where $|\nu_i(x_i)| - |\nu_i(x_i - 1)| > 3(w+1)$.

Also, replacing if necessary ν by ν_{i+1} , for some i less than t , we may assume that ν and the sequence above are chosen such that t is the smallest i for which $\epsilon(\nu_{i+1}) = \epsilon(\mu)$. Here, note that t may equal 0.

Recall the correspondence $(\rho \rightarrow b_\rho)$ between the set $\bar{\mathcal{D}}$ and $\mathcal{B}_w$ established in the beginning of the chapter.

By Proposition (4.12), $|b_\nu| < |b_\mu|$. To prove Theorem (1.4), it therefore suffices to exhibit an isomorphism of the type (1.2) for the blocks b_μ and b_ν and then to show that their source algebras are isomorphic as interior P -algebras. This section and all of Chapter 5 will be devoted to producing such an isomorphism.

For the remainder of this chapter, we will adopt the following notation:

Notation

Set

$$b_i := b_{\nu_i} \text{ for } 0 \leq i \leq t+1,$$

$$\sigma := b_1 \dots b_t.$$

Here $\sigma = 1$ if $t = 0$.

Then $|b_i| = |\nu_i| + pw$ and by Proposition 4.8 we have that $|b_0| > |b_1| \dots > |b_t| > |b_{t+1}|$. So we may choose a sequence of sets:

$$Y_0 \supset Y_1 \supset \dots \supset Y_t \supset Y_{t+1} \tag{4.70}$$

which gives the sequence of natural inclusions of groups :

$$\tilde{S}(Y_0) \supset \tilde{S}(Y_1) \supset \dots \supset \tilde{S}(Y_t) \supset \tilde{S}(Y_{t+1}), \quad (4.71)$$

and of algebras:

$$\mathcal{O}\tilde{S}(Y_0) \supset \mathcal{O}\tilde{S}(Y_1) \supset \dots \supset \mathcal{O}\tilde{S}(Y_t) \supset \mathcal{O}\tilde{S}(Y_{t+1}). \quad (4.72)$$

such that $|Y_i| = |\nu_i| + pw$, and b_i is a block of $\tilde{S}(Y_i)$. As b_i is central in $\mathcal{O}\tilde{S}(Y_{i+1})$ and $\mathcal{O}\tilde{S}(Y_{i+1})$ is contained in $\tilde{S}(Y_i)$, we get that b_i and b_{i+1} commute for all i , $0 \leq i \leq t$, and hence σ is an idempotent of $\mathcal{O}\tilde{S}(Y_0)$.

P is a defect group for all the b_i 's and in particular for b_{t+1} . By Theorem (3.22), we may choose a subset W of Y_{t+1} of cardinality pw such that P is a Sylow p -subgroup of $\tilde{S}_{pw}$ and hence if we set $Y_i^P = Y_i - W$ for $0 \leq i \leq t+1$, we get another sequence of inclusions of sets :

$$Y_0^P \supset Y_1^P \supset \dots \supset Y_t^P \supset Y_{t+1}^P \quad (4.73)$$

and of groups and algebras:

$$\tilde{S}(Y_0^P) \supset \tilde{S}(Y_1^P) \supset \dots \supset \tilde{S}(Y_t^P) \supset \tilde{S}(Y_{t+1}^P), \quad (4.74)$$

$$\mathcal{O}\tilde{S}(Y_0^P) \supset \mathcal{O}\tilde{S}(Y_1^P) \supset \dots \supset \mathcal{O}\tilde{S}(Y_t^P) \supset \mathcal{O}\tilde{S}(Y_{t+1}^P) \quad (4.75)$$

where Y_i^P is the subset of Y_i which is left fixed under the action of $\theta(P)$. Here θ is the surjection of $\tilde{S}(U)$ on $S(U)$ described in (3.2). We have

$$W = Y_0 - Y_0^P = Y_1 - Y_1^P = \dots = Y_t - Y_t^P = Y_{t+1} - Y_{t+1}^P. \quad (4.76)$$

Further, we have that $c_p(b_i) = \nu_i$, and so we also get a sequence (see Definition 3.23)):

$$e_{\nu_0}^*, e_{\nu_1}^*, \dots, e_{\nu_t}^*, e_{\nu_{t+1}}^*$$

where $e_{\nu_i}^*$ is a central idempotent of $\mathcal{O}\tilde{S}(Y_i^P)$. Again $e_{\nu_i}^*$ and $e_{\nu_{i+1}}^*$ commute in $\mathcal{O}\tilde{S}(Y_i^P)$. Thus, if we set

$$\tau = e_{\nu_1}^* \dots e_{\nu_t}^*,$$

then τ is an idempotent of $\mathcal{O}\tilde{S}(Y_0^P)$.

PROPOSITION 4.13. *There exists an integer L such that :*

(i)

$$\text{rank}_{\mathcal{O}}(b_{t+1}\sigma\mathcal{O}\tilde{S}(Y_0)b_0\sigma b_{t+1}) = L^2 \text{rank}_{\mathcal{O}}(\mathcal{O}\tilde{S}(Y_{t+1})b_{t+1}) \quad (4.77)$$

(ii)

$$\text{rank}_{\mathcal{O}}(e_{\nu_{t+1}}^* \tau \mathcal{O}\tilde{S}(Y_0^P) e_{\nu_0} \tau e_{\nu_{t+1}}^*) = L^2 \text{rank}_{\mathcal{O}}(\mathcal{O}\tilde{S}(Y_{t+1}^P) e_{\nu_{t+1}}) \quad (4.78)$$

(iii) *If $\epsilon(\nu_0) = 1$, then*

$$\text{rank}_{\mathcal{O}}(e_{\nu_{t+1}} \tau \mathcal{O}\tilde{S}(Y_0^P) e_{\nu_0} \tau e_{\nu_{t+1}}) = \frac{L^2}{4} \text{rank}_{\mathcal{O}}(\mathcal{O}\tilde{S}(Y_{t+1}^P) e_{\nu_{t+1}}) \quad (4.79)$$

PROOF. Let F be a finite extension of K containing all the $|Y_0|$ th roots of unity, i.e, F contains splitting fields of all $\tilde{S}(Y_i)$, $0 \leq i \leq t+1$.

All the $\mathcal{O}$ -algebras in the statement of the proposition are $\mathcal{O}$ -free and hence it suffices to compare the F -dimensions of the algebras obtained by extending their rings of scalars by F .

F is a splitting field for $\tilde{S}(Y_0)$. Hence,

$$F\tilde{S}(Y_0)b_0 \simeq \prod_{\text{Irr}(b_0)} \text{Mat}_{n_\chi}(F) \quad (4.80)$$

where $\text{Irr}(b_0)$ is the set of ordinary representations of $\tilde{S}(Y_0)b_0$ which belong to the block b_0 and $n_\chi = \chi(1)$.

As the identity element of each matrix algebra in the decomposition of $F\tilde{S}(Y_0)b_0$ above is a central idempotent of that algebra, we have that

$$b_{t+1}\sigma F\tilde{S}(Y_0)b_0\sigma b_{t+1} \simeq \prod_{\text{Irr}(b_0)} \text{Mat}_{m_\chi}(F) \quad (4.81)$$

where $m_\chi = \chi(\sigma b_{t+1})$.

By Theorem (3.21)(i),

$$b_{t+1}\sigma F\tilde{S}(Y_0)b_0\sigma b_{t+1} \simeq \prod_{\substack{\lambda \in \mathcal{D}(Y_0) \\ c_p(\lambda) = \nu_0 \\ \epsilon(\lambda) = 0}} \text{Mat}_{m_{\chi_\lambda}}(F) \prod_{\substack{\lambda \in \mathcal{D}(Y_0) \\ c_p(\lambda) = \nu_0 \\ \epsilon(\lambda) = 1}} \text{Mat}_{m_{\chi_\lambda}}(F) \prod_{\substack{\lambda \in \mathcal{D}(Y_0) \\ c_p(\lambda) = \nu_0 \\ \epsilon(\lambda) = 1}} \text{Mat}_{m_{\chi_\lambda^a}}(F). \quad (4.82)$$

Let $\lambda \in \mathcal{D}(Y_0)$ with $c_p(\lambda) = \nu_0$.

For any y in $F\tilde{A}(Y_1)$, by Lemma 3.14 and Propositions 3.17 and 4.8,

$$\begin{aligned}
\chi_\lambda(b_1 y) &= \sum_{\phi \in \text{Irr}(b_1)} \langle \chi_\lambda |_{\tilde{S}(Y_1)}, \phi \rangle \phi(y) \\
&= \sum_{\substack{\lambda' \in \mathcal{D}(Y_1) \\ c_p(\lambda') = \nu_1 \\ \epsilon(\lambda') = 0}} \langle \chi_\lambda |_{\tilde{S}(Y_1)}, \chi_{\lambda'} \rangle \chi_{\lambda'}(y) \\
&\quad + \sum_{\substack{\lambda' \in \mathcal{D}(Y_1) \\ c_p(\lambda') = \nu_1 \\ \epsilon(\lambda') = 1}} \langle \chi_\lambda |_{\tilde{S}(Y_1)}, \chi_{\lambda'} \rangle \chi_{\lambda'}(y) + \langle \chi_\lambda |_{\tilde{S}(Y_1)}, \chi_{\lambda'}^a \rangle \chi_{\lambda'}^a(y) \\
&= \sum_{\lambda' \in \mathcal{D}(Y_1), c_p(\lambda') = \nu_1} 2^{n(\lambda, \lambda')} |\mathcal{M}^{|Y_0| - |Y_1|}(\lambda, \lambda')| \chi_{\lambda'}(y). \tag{4.83}
\end{aligned}$$

Let $\lambda_0 = \lambda$ and define λ_i , $0 < i \leq t$, by $\lambda_{i+1} = Sc_{x_i}(\lambda_i)$. Now $\nu_1 = \widetilde{Sc}_{x_0}(\nu_0)$ where $|\nu_0(x_0)| - |\nu_0(x_0 - 1)| > 3(w + 1)$, and so by Proposition 4.10, the above equation with $y = b_2 \dots b_{t+1}$, yields

$$\chi_\lambda(\sigma b_{t+1}) = 2^{n(\lambda_0, \lambda_1)} |\mathcal{M}^{|Y_0| - |Y_1|}(\nu_0, \nu_1)| \chi_{\lambda_1}(b_2 \dots b_{t+1}). \tag{4.84}$$

Applying the above argument successively to

$$\chi_{\lambda_1}(b_2 \dots b_t b_{t+1}), \chi_{\lambda_2}(b_3 \dots b_t b_{t+1}), \dots, \chi_{\lambda_i}(b_{i+1} \dots b_t b_{t+1}), \dots, \chi_{\lambda_{t+1}}(1),$$

we have

$$\chi_\lambda(\sigma b_{t+1}) = \prod_{0 \leq i \leq t} 2^{n(\lambda_i, \lambda_{i+1})} |\mathcal{M}^{|Y_i| - |Y_{i+1}|}(\nu_i, \nu_{i+1})| \chi_{\lambda_{t+1}}(1). \tag{4.85}$$

If $t = 0$, then $\epsilon(\nu_0) = \epsilon(\nu_1)$ and so by Proposition 4.10 (iii), $\epsilon(\lambda_0) = \epsilon(\lambda_1)$. Also, $|\lambda_0| - |\lambda_1| = |\nu_0| - |\nu_1|$, and by Proposition 4.10 (ii) $n(\lambda_0) - n(\lambda_1) = n(\nu_0) - n(\nu_1)$. So, $n(\lambda_0, \lambda_1) = n(\nu_0, \nu_1)$. ■

If $t > 0$, again by Proposition 4.10, we have that $|\lambda_i| - |\lambda_{i+1}| = |\nu_i| - |\nu_{i+1}|$, and $n(\lambda_i) - n(\lambda_{i+1}) = n(\nu_i) - n(\nu_{i+1})$ for all i between 0 and t .

By choice of the ν_i 's and Proposition 4.10,

$$\epsilon(\nu_0) \neq \epsilon(\nu_1) = \epsilon(\nu_t) \neq \epsilon(\nu_{t+1}) \text{ and } \epsilon(\lambda_0) \neq \epsilon(\lambda_1) = \epsilon(\lambda_t) \neq \epsilon(\lambda_{t+1}). \quad (4.86)$$

Thus,

$$\epsilon(\nu_0) - \epsilon(\nu_1) + \epsilon(\nu_t) - \epsilon(\nu_{t+1}) = 0 = \epsilon(\lambda_0) - \epsilon(\lambda_1) + \epsilon(\lambda_t) - \epsilon(\lambda_{t+1}). \quad (4.87)$$

Also, for $1 < i < t$,

$$\epsilon(\nu_i) - \epsilon(\nu_{i+1}) = 0 = \epsilon(\lambda_i) - \epsilon(\lambda_{i+1}). \quad (4.88)$$

From the above equations, we get

$$n(\lambda_0, \lambda_1) + n(\lambda_t, \lambda_{t+1}) = n(\nu_0, \nu_1) + n(\nu_t, \nu_{t+1}) \quad (4.89)$$

and

$$n(\lambda_i, \lambda_{i+1}) = n(\nu_i, \nu_{i+1}) \quad (4.90)$$

for $1 < i < t$.

Hence,

$$\chi_\lambda(\sigma b_{t+1}) = \prod_{0 \leq i \leq t} 2^{n(\nu_i, \nu_{i+1})} \mathcal{M}^{|Y_i| - |Y_{i+1}|}(\nu_i, \nu_{i+1}) \chi_{\lambda_1}. \quad (4.91)$$

for every λ in $\mathcal{D}(|Y_0|)$ with $c_p(\lambda) = \nu_0$

By Proposition 4.9 and the choice of ν_i 's, $\lambda \rightarrow \lambda_{t+1}$ is a 1-1 correspondence between strict partitions of $|Y_0|$ whose p -bar core is ν_0 and strict partitions of $|Y_{t+1}|$ whose p -bar core is ν_{t+1} and this correspondence preserves parity.

Consequently, if

$$L := \prod_{0 \leq i \leq t} 2^{n(\nu_i, \nu_{i+1})} |\mathcal{M}^{|Y_i| - |Y_{i+1}|}(\nu_i, \nu_{i+1})|,$$

then from (4.82) and (4.91)

$$\dim_F(b_{t+1} \sigma F \tilde{S}(Y_0) b_0 \sigma b_{t+1}) = L^2 \sum_{\substack{\lambda_{t+1} \in \mathcal{D}(Y_{t+1}) \\ c_p(\lambda_{t+1}) = \nu_0 \\ e(\lambda_{t+1}) = 0}} (\chi_{\lambda_{t+1}}(1))^2 + 2 \sum_{\substack{\lambda_{t+1} \in \mathcal{D}(Y_{t+1}) \\ c_p(\lambda_{t+1}) = \nu_0 \\ e(\lambda) = 1}} (\chi_{\lambda_{t+1}}(1))^2 \quad (4.92)$$

But the right side of the above equation is precisely $L^2 \dim_F(F \tilde{S}(Y_{t+1}) b_{t+1})$

and this proves part (i) of the proposition.

We have that

$$e_{\nu_{t+1}}^* \tau F \tilde{S}(Y_0^P) e_{\nu_0} \tau e_{\nu_{t+1}}^* = \text{Mat}_n(F), \quad (4.93)$$

where $n = \chi_{\nu_0}(\tau e_{\nu_{t+1}}^*)$ and again by Lemma 3.14 and Propositions 3.17, 4.9 and 4.10,

$$\begin{aligned}
\chi_{\nu_0}(\tau e_{\nu_{t+1}}^*) &= \sum_{\phi \in \text{Irr}(\tilde{S}(Y_1^P))} \langle \chi_{\nu_0} |_{\tilde{S}(Y_1^P)}, \phi \rangle \phi(e_{\nu_1}^* e_{\nu_2}^* \dots e_{\nu_{t+1}}^*) \\
&= \sum_{\phi \in \text{Irr}(e_{\nu_1}) \cup \text{Irr}(e_{\nu_1}^a)} \langle \chi_{\nu_0} |_{\tilde{S}(Y_1^P)}, \phi \rangle \phi(e_{\nu_1}^* e_{\nu_2}^* \dots e_{\nu_{t+1}}^*) \\
&= \langle \chi_{\nu_0} |_{\tilde{S}(Y_1^P)}, \chi_{\nu_1}^* \rangle \chi_{\nu_1}(e_{\nu_2}^* \dots e_{\nu_{t+1}}^*) \\
&= 2^{n(\nu_0, \nu_1)} |\mathcal{M}^{Y_0^P - |Y_1^P|}(\nu_0, \nu_1)| \chi_{\nu_1}(e_{\nu_2}^* \dots e_{\nu_{t+1}}^*). \tag{4.94}
\end{aligned}$$

Here

$$\chi_{\nu_1}^* := \begin{cases} \chi_{\nu_1} & \text{if } \epsilon(\nu) = 0 \\ \chi_{\nu_1} + \chi_{\nu_1}^a & \text{otherwise.} \end{cases}$$

Note that we are using again the fact that $\chi_{\nu_1}(e_{\nu_2}^* \dots e_{\nu_{t+1}}^*) = \chi_{\nu_1}^a(e_{\nu_2}^* \dots e_{\nu_{t+1}}^*)$ as $|\tilde{S}(U_1)^P| - |\tilde{S}(U_2)^P| > 2$ and $e_{\nu_2}^* \dots e_{\nu_{t+1}}^*$ is in $F\tilde{A}(Y_2^P)$.

Iterating this process, we get

$$\begin{aligned}
\chi_{\nu_0}(\tau e_{\nu_{t+1}}^*) &= \prod_{0 \leq i \leq t} 2^{n(\nu_i, \nu_{i+1})} |\mathcal{M}^{Y_i^P - |Y_{i+1}^P|}(\nu_i, \nu_{i+1})| \text{rank}_F \chi_{\nu_{i+1}}^* \\
&= \prod_{0 \leq i \leq t} 2^{n(\nu_i, \nu_{i+1})} |\mathcal{M}^{Y_i^P - |Y_{i+1}^P|}(\nu_i, \nu_{i+1})| \text{rank}_F \chi_{\nu_{i+1}}. \tag{4.95}
\end{aligned}$$

which proves part (ii) of the proposition.

For part (iii), note that $e_{\nu_{t+1}}^a = u^{-1} e_{\nu_{t+1}} u$ for any u in $\tilde{S}(Y_t - Y_{t+1}) - \tilde{A}(Y_t - Y_{t+1})$. As u belongs to $\tilde{S}(Y_t)$, it commutes with $e_{\nu_0}, \dots, e_{\nu_t}$ and hence $\tau e_{\nu_0} u$ is a unit in $\tau \tilde{\mathcal{O}}\tilde{S}(Y_0^P) e_{\nu_0} \tau$ such that $e_{\nu_{t+1}}^a \tau e_{\nu_0} = (\tau e_{\nu_0} u^{-1}) e_{\nu_{t+1}} (u \tau e_{\nu_0})$. The result follows. ■

CHAPTER V

ISOMORPHISM OF ALGEBRAS

In this chapter, we produce an isomorphism of the type (1.2) for the blocks b_μ and b_ν of §2 , Chapter 4. For the sake of convenience we make the following changes in the notation introduced in that section :

Notation

We set

$$b := b_\mu,$$

$$c := b_\nu,$$

$$e := e_\mu,$$

$$f := e_\nu,$$

$$U := Y_0,$$

$$V := Y_{t+1},$$

$$G := \tilde{S}(U),$$

$$H := \tilde{S}(V),$$

$$G_0 := \tilde{S}(U^P),$$

$$H_0 := \tilde{S}(V^P),$$

$$D(H) := \tilde{A}(V),$$

$$D(H_0) := \tilde{A}(V^P).$$

Recall from Chapter IV, §2, that $\sigma = b_1 \dots b_t$ and $\tau = e_1^* \dots e_t^*$. The following will be our approach for proving the required isomorphism:

Let

$$A := c\sigma OGb\sigma c,$$

considered as an interior H algebra. It is clear from (3.20) and (3.21) that

$$\text{Br}_P(A) = \bar{f}^* \bar{\tau} k G_0 \bar{e}^* \bar{\tau} \bar{f}^* \otimes_k kZ(P). \quad (5.1)$$

We will find a suitable point, say ι , of H on A such that $H_\iota \supset P_\gamma$, where γ is a local point of P on A . Note that it is possible to so choose ι and γ as P is a defect group of b and because $\text{Br}_P(b\sigma c) \neq 0$ by (5.1) and by Proposition 4.13(ii). We then consider the interior H algebra IAI where I will be an idempotent of A^H such that if i is a primitive idempotent of A^H for which $iI = i$, then i is in ι and further, there is no element of ι in any primitive idempotent decomposition of $1 - I$ in A^H . In other words, using the notation introduced in (2.4), $I = m_\iota \iota$. Here, 1 is the identity of A .

By Theorem 2.4, we may find a subalgebra T of $IA^H I$, such that T is a product of matrix algebras over O and T is maximal such. As ι is the only point of H on IAI , T has actually one simple factor, i.e, T is isomorphic to the $m_\iota \times m_\iota$ matrix

algebra over $\mathcal{O}$. Let $C := C_{IAI}(T)$. T is a unitary $\mathcal{O}$ simple subalgebra of IAI .

Hence, by Proposition (2.11),

$$C \otimes_{\mathcal{O}} T \simeq IAI, \text{ via } s \otimes t \mapsto ct. \quad (5.2)$$

and

$$C \simeq iIAIi = iAi, \text{ via } c \mapsto ici \quad (5.3)$$

where i is any primitive idempotent of T .

Note that T is contained in $(IAI)^H$. So, C contains the image of H in IAI and we give $C \otimes_{\mathcal{O}} T$ an interior H algebra structure via $h \mapsto hI \otimes I = h1_C \otimes 1_T$. iAi also has a natural interior H algebra structure as i commutes with H , and it is clear that as such, both isomorphisms shown above are interior H algebra isomorphisms. In particular, passing to the subalgebras fixed by H in the isomorphism (5.2), we get

$$IA^HI = (IAI)^H \simeq (C \otimes_{\mathcal{O}} T)^H = C^H \otimes_{\mathcal{O}} T \simeq C^H \otimes_{\mathcal{O}} \text{Mat}_{m_i}(\mathcal{O}), \quad (5.4)$$

which gives

$$m_i^2 = r_m(IA^HI) = r_m(C^H \otimes_{\mathcal{O}} \text{Mat}_{m_i}(\mathcal{O})) \geq m_i^2. \quad (5.5)$$

Here recall that for any $\mathcal{O}$ -algebra B , $r_m(B)$ denotes the m -rank(B) (cf. Definition 2.8(ii)). This shows that $1 \otimes_{\mathcal{O}} i$ is primitive even in $C \otimes_{\mathcal{O}} T$, for otherwise strict

inequality would hold on the extreme right of the sequence displayed above. Thus $i \cdot 1 = i$ is in ι .

Now the interior H -algebra structure of iAi induces a homomorphism $\mathcal{O}Hc \rightarrow iAi$, where clearly $c \mapsto b\sigma ci = i$, making the homomorphism unitary. Moreover, P stabilizes an $\mathcal{O}$ -basis of $\mathcal{O}G$ (via the diagonal action of P) and iAi is a P module summand of $\mathcal{O}G$. Thus by Green's theorem, P stabilizes an $\mathcal{O}$ -basis of iAi . We have $\text{Br}_P(i) \neq 0$ as i is in ι and $H_\iota \supset P_\gamma$, and thus $iAi(P) = \text{Br}_P(i)kC_G(P)\text{Br}_P(i)$ is a non-zero projective $kZ(P)$ module via the left multiplication action of $Z(P)$. P , H and iAi satisfy therefore the hypothesis of [P2], Prop. 3.8, and hence the map $\mathcal{O}Hc \rightarrow iAi$ is a direct inclusion of $\mathcal{O}H$ bimodules and in particular, of $\mathcal{O}$ modules.

Using (5.4), we consider the sequence of interior H -algebra homomorphisms:

$$\mathcal{O}Hc \otimes_{\mathcal{O}} \text{Mat}_{m_\iota}(\mathcal{O}) \rightarrow iAi \otimes_{\mathcal{O}} \text{Mat}_{m_\iota}(\mathcal{O}) \simeq IAI \quad (5.6)$$

By the above argument, the composed map between the two extreme terms in (5.6) is a direct inclusion of $\mathcal{O}$ -modules. If we could show that

$$\text{rank}_{\mathcal{O}}(\mathcal{O}Hc) \cdot m_\iota^2 \geq \text{rank}_{\mathcal{O}}(IAI), \quad (5.7)$$

then (5.6) would give

$$\mathcal{O}Hc \otimes_{\mathcal{O}} \text{Mat}_{m_\iota}(\mathcal{O}) \simeq IAI \quad (5.8)$$

as interior H -algebras. But $IAI = IOGbI$ and thus (5.8) would be the isomorphism required in (1.2).

Let us note at this point that once we have that $OHc \otimes_{\mathcal{O}} Mat_{m_i}(\mathcal{O}) \simeq IAI$ as interior H algebras, the isomorphism of source algebras of b and c as interior P algebras is immediate:

Let i be a primitive idempotent of $(OHc)^P$ which belongs to a local point of P on OHc and let j be any primitive idempotent of $Mat_{m_i}(\mathcal{O})$. Further, let t be the image of $i \otimes j$ in IAI under the above isomorphism. Then clearly from (5.8),

$$iOHc \simeq (i \otimes j)(OHc \otimes_{\mathcal{O}} Mat_{m_i}(\mathcal{O}))(i \otimes j) \simeq tIAIt = tOGkt \quad (5.9)$$

as interior P algebras since P is a subgroup of H . Also, t must belong to a local point of P on IAI and hence to a local point of OGb . As P is maximal among groups having a local point on OGb , the two extreme terms of (5.9) are source algebras for c and b respectively.

We now find a suitable γ , ι and show that (5.7) will hold.

PROPOSITION 5.1. *Fix a local point γ of P on A . Then,*

(i) *There are two points ι and ζ , of H on A such that $P_\gamma \leq H_\iota, H_\zeta$ and these are the only two such points of H on A .*

(ii) *There is exactly one point δ , of $D(H)$ on A such that $P_\gamma \in D(H)_\delta$.*

(iii) $m_\iota + m_\zeta = m_\delta$.

(iv) For any $i \in \iota$ and $j \in \zeta$, i and j remain primitive in $A^{D(H)}$, i.e. ι and ζ are contained in δ , and, in particular, there is an a in $A^{D(H)}$ such that $a^{-1}ia = j$, i.e. ι and ζ are conjugate in $A^{D(H)}$.

COROLLARY 5.2. For i and j as in (iv) above,

$$\begin{aligned} \text{rank}_O(iAi) &= \text{rank}_O(iAj) \\ &= \text{rank}_O(jAi) \\ &= \text{rank}_O(jAj). \end{aligned} \tag{5.10}$$

PROOF. This is clear. ■

CLAIM. Part (iv) of the proposition follows from parts (i) (ii) and (iii).

PROOF OF CLAIM. $D(H)$ is normal in H , so H acts on $A^{D(H)}$ via conjugation and hence it acts on the points of $A^{D(H)}$.

We will first show that $N_H(D(H)_\delta) = H$. As $[H : D(H)] = 2$, it suffices to show that for some h in $H - D(H)$, $h^{-1}\delta h = \delta$. Pick $h \in H_0 - D(H)$ and let $d \in \delta$. As $D(H)_\delta \supset P_\gamma$, there is a $d_1 \in \gamma$ such that $d_1 d = d_1$.

Then $h^{-1}d_1 d h = h^{-1}d_1 h$, i.e. $(h^{-1}d_1 h)(h^{-1}dh) = h^{-1}d_1 h$.

As h is in H_0 , h centralizes P and thus $h^{-1}\gamma h = \gamma$. The above equation therefore gives that $h^{-1}dh \in \delta$, as by part (ii) of the Proposition δ is the only point of $A^{D(H)}$ for which $P_\gamma \leq D(H)_\delta$.

Now, since $H_\iota \supset P_\gamma$, we must have $H_\iota \supset D(H)_\delta$. Let $i \in \iota$. Then, any decomposition of i into a sum of primitive idempotents in $A^{D(H)}$ must contain an element of δ , which, by the above argument, is a full conjugacy class of points of $A^{D(H)}$ under the action of H . Hence, by Lemma 2.24, any such decomposition contains idempotents only from δ .

By exactly the same argument, we get that δ is the only point of $A^{D(H)}$ such that $H_\zeta \supset D(H)_\delta$. In particular, we get that

$$m_\delta \geq m_\iota + m_\zeta. \quad (5.11)$$

with strict inequality if idempotents in either ι or ζ are imprimitive in $A^{D(H)}$.

On the other hand, Part (iii) of the Proposition says that

$$m_\iota + m_\zeta = m_\delta. \quad (5.12)$$

So, ι and ζ remain primitive in $A^{D(H)}$, i.e. i, j belong to δ for i in ι and j in ζ and the claim follows. ■

PROOF OF PROPOSITION 5.1. To prove the proposition, we consider two cases.

Case 1: $\epsilon(\mu) = \epsilon(\nu) = 0$, i.e. $e^* = e$ and $f = f^*$

Part (ii) of Proposition 4.13 gives

$$e\tau f \neq 0 \text{ and } \text{rank } \mathcal{O}(f\tau\mathcal{O}\tilde{S}(U^P)e\tau f) = L^2 \text{ rank } \mathcal{O}(\mathcal{O}\tilde{S}(V^P f) \quad (5.13)$$

The natural inclusion of V^P in U^P induces the sequence of maps

$$\mathcal{O}H_0f \rightarrow f\tau\mathcal{O}G_0e\tau f \rightarrow f\mathcal{O}G_0fe.$$

The maps above are interior $\mathcal{O}H_0$ -algebra homomorphisms since τe commutes with H_0 . As $bc \neq 0$, the composed map above is not the zero map. Then, as $\mathcal{O}H_0f$ is a matrix algebra, the above map is actually an injection. From now on we identify $\mathcal{O}H_0f$ with its image in $f\tau\mathcal{O}G_0e\tau f$. Let $E := C_{f\tau\mathcal{O}G_0e\tau f}(\mathcal{O}H_0f)$. By Proposition 2.11

$$f\tau\mathcal{O}G_0e\tau f \cong \mathcal{O}H_0f \otimes E. \quad (5.14)$$

As $\mathcal{O}G_0e$ is a matrix algebra, so is $f\tau\mathcal{O}G_0e\tau f$, and hence, from (5.13) and (5.14), we get that E is isomorphic to the full $L \times L$ matrix algebra over $\mathcal{O}$.

Let γ denote the unique local point of P on $\mathcal{O}Gb$. As $\text{Br}_P(b\sigma c) = e\tau f$, and $e\tau f \neq 0$, $\gamma \cap A$ is non empty and so there is a unique local point of P on A , which, without ambiguity, we may call γ also. For the same reason, there is a point β of H on A such that $H_\beta \supset P_\gamma$. Now, (3.21) in this case gives,

$$\begin{aligned} \text{Br}_P(c\sigma\mathcal{O}Gb\sigma c) &= \text{Br}_P(b\sigma c)(kG_0 \otimes kZ(P))\text{Br}_P(b\sigma c) \\ &= \bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f} \otimes kZ(P). \end{aligned} \quad (5.15)$$

From (5.15), it is clear that

$$A(P_\gamma) \simeq \bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f} \quad (5.16)$$

as interior H_0 algebras and this isomorphism is canonical. Since P is a defect group of c , we have $c \in (\mathcal{O}Hc)_P^H$. Let $x \in (\mathcal{O}Hc)^P$ be such that $c = \text{Tr}_P^H(x)$. Then, $b\sigma c = \text{Tr}_P^H(b\sigma x)$, as $b\sigma$ commutes with H . Also, $b\sigma x \in A^P$; hence we get that $b\sigma c$ is in A_P^H . A_P^H is an ideal of A^H and it contains the identity element of the same, namely $b\sigma c$, hence, A_P^H is actually equal to A^H .

Let

$$S_\gamma : A^P \rightarrow A(P_\gamma)$$

be the homomorphism given in Definition 2.17.

By [P1 Proposition 1.3], $S_\gamma(A_P^H) = (A(P_\gamma))_1^N$, where $N = N_H(P_\gamma)/P$. But since $bc \in A_P^H$, $S_\gamma(bc) \in (A(P_\gamma))_1^N$ and as S_γ is a unitary homomorphism, we get that $(A(P_\gamma))_1^N = A(P_\gamma)^N$, and thus the restriction of S_γ to A^H still gives a surjective map:

$$\text{Res}S_\gamma : A^H \rightarrow (A(P_\gamma))^N.$$

Suppose β is a point of H on A . Then, $S_\gamma(\beta) \neq 0$ iff $H_\beta \supset P_\gamma$. Hence, by a consequence of the lifting theorem for idempotents, $\text{Res}S_\gamma$ establishes a 1-1 correspondence between those points β of H on A for which H_β contains P_γ and the points of $(A(P_\gamma))^N$. Further, if β is such a point of H on A , then

$$m_{\text{Res}S_\gamma(\beta)} = m_\beta. \quad (5.17)$$

As γ is the only local point of P on A , $N_G(P_\gamma) = N_G(P)$ and hence

$N_H(P_\gamma) = N_H(P) = \tilde{S}(V^P).N_{\tilde{S}(W)}(P)$. Let $\bar{E}$ denote the canonical image of E in $\bar{f}k\tilde{S}(U^P)\bar{e}\bar{f}$. Then, $\bar{E} = (\bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f})^{H_0}$. From (3.19), we get:

$$\begin{aligned} (A(P_\gamma))^N &\simeq (\bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f})^{H_0.N_{\tilde{S}(W)}(P)} \\ &= (\bar{E})^{N_{\tilde{S}(W)}(P)}. \end{aligned} \quad (5.18)$$

$N_{\tilde{S}(W)}(P)$ contains $N_{\tilde{A}(W)}(P)$ as a subgroup of index 2. Let $t \in N_{\tilde{S}(W)}(P) - N_{\tilde{A}(W)}(P)$. As W and U^P are disjoint sets, $\tilde{A}(W)$ commutes with $\tilde{S}(U^P)$, and so we get that

$$A(P_\gamma)^N = \bar{E}^{<t>}. \quad (5.19)$$

CLAIM. $\bar{E}^{<t>} \neq \bar{E}$.

PROOF. Suppose, if possible, that $\bar{E}^{<t>} = \bar{E}$. Then from (5.18)

$$A(P_\gamma)^N = \bar{E}. \quad (5.20)$$

Thus there is exactly one point ι of H on A for which $P_\gamma \subset H_\iota$ and $m_\iota = L$. Then for $I = m_\iota \iota$,

$$\text{rank}_{\mathcal{O}}(IAI) \leq \text{rank}_{\mathcal{O}}(A) = L^2 \text{rank}_{\mathcal{O}}(\mathcal{O}Hc) = m_\iota^2 \text{rank}_{\mathcal{O}}(\mathcal{O}Hc) \quad (5.21)$$

and (5.8) holds, i.e $\mathcal{O}Hc \otimes_{\mathcal{O}} \text{Mat}_L(\mathcal{O}) \simeq A$.

But by comparing $\mathcal{O}$ -ranks, we in fact get that $IAI = A$, and so

$$\mathcal{O}Hc \otimes_{\mathcal{O}} \text{Mat}_L(\mathcal{O}) \simeq IAI \quad (5.22)$$

as interior H -algebras. Tensoring over K gives

$$KHc \otimes_K \text{Mat}_L(K) \simeq cKGbc \quad (5.23)$$

as interior H -algebras.

This means that for every ordinary irreducible character χ_H in c , there is an irreducible character of G , say χ_G , in b , such that the decomposition of χ_G , considered as an H character, into a direct sum of irreducible H characters contains L copies of χ_H and χ_H is the only character of H in c that appears in this decomposition. But for a non self associate χ_H , this is impossible, for if the restriction of any character of G contains copies of χ_H , then it contains an equal number of copies of χ_H^e and vice versa. On the other hand, every block of non-zero defect contains a non-self associate character [O, Section 5]. This proves the claim, i.e. t does not act trivially on $\bar{E}$.

We have that t^2 is in $\tilde{A}(W)$, hence t acts as an involutory automorphism on $\bar{E}$. As the action of t on $\bar{f}kG_0\bar{e}\bar{f}$ is induced by the conjugation of G_0 by t , it is clear that t fixes all elements of $\bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f}$ of the type $q\bar{e}\bar{\tau}\bar{f}$, where $q \in k$, that is, t fixes the center of $\bar{f}\bar{\tau}k\tilde{S}(U^P)\bar{e}\bar{\tau}\bar{f}$, and hence it fixes the center of $\bar{E}$. Then, by the Noether-Skolem theorem, t induces an inner automorphism of $\bar{E}$. Let it be the inner automorphism induced by $s \in \bar{E}^*$. Let $\psi : \bar{E}^* \rightarrow \text{Aut}(\bar{E}^*)$ be the natural

homomorphism, where $\psi(x)$ is the automorphism of $\bar{E}^*$ induced by conjugation by x , so that $\text{Ker}\psi$ is $Z(\bar{E}^*)$. Then as $\psi(s)^2$ is the identity of $\text{Aut}(\bar{E})$, $s^2 = \lambda \cdot I$, where I is the identity of $\bar{E}$ and λ is in k . The minimal polynomial of s is therefore $X^2 - \lambda$, which splits in $k[X]$ into two distinct linear factors, as k is algebraically closed and of odd characteristic, and s is not in the center of $\bar{E}$. Then, we can find y in $\bar{E}^*$ such that ysy^{-1} is the diagonal matrix $z = \text{diag}(\nu \dots \nu, -\nu \dots -\nu)$, where $\nu = \sqrt{\lambda}$. Clearly, $\bar{E}^{<t>} = \bar{E}^{<s>} = y^{-1}\bar{E}^zy$. So, there is an idempotent decomposition $I = \bar{i} + \bar{j}$ of I in $\bar{E}$ such that $\bar{E}^{<t>} = \bar{i}\bar{E}\bar{i} \oplus \bar{j}\bar{E}\bar{j}$. From (5.19),

$$A(P_\gamma)^N = \bar{i}\bar{E}\bar{i} \oplus \bar{j}\bar{E}\bar{j}. \quad (5.24)$$

We have already established a 1-1 correspondence between points of $A(P_\gamma)$ and points β of H on A for which $H_\beta \supset P_\gamma$. As $\bar{i}\bar{E}\bar{i}$ and $\bar{j}\bar{E}\bar{j}$ are matrix algebras, it is clear from (5.24) that there are exactly two such points of H on A , proving part (i) of the proposition. Let these be ι and ζ respectively. Also, from (5.17) and (5.24),

$$m_\iota + m_\zeta = \sqrt{\dim_k(\bar{i}\bar{E}\bar{i})} + \sqrt{\dim_k(\bar{j}\bar{E}\bar{j})} = \sqrt{\dim_k(\bar{E})} = L. \quad (5.25)$$

Since c is a block of $D(H)$ with defect group P , there is an element u in $\mathcal{O}D(H)$ such that $c = \text{Tr}_P^{D(H)}(u)$. In particular, $b\sigma c = \text{Tr}_P^{D(H)}(ub\sigma)$, as $b\sigma$ commutes with H . Clearly, $cubc \in A^P$, and so, we get that $bc \in A_P^{D(H)}$ and thus $A_P^{D(H)} = A^{D(H)}$. Again, we can find a point δ of $D(H)$ on A such that

$D(H)_\delta \supset P_\gamma$ and by [P1, Prop. 1.3], we get a surjective map:

$$\text{res } S_\gamma : A^{D(H)} \rightarrow A(P_\gamma)^{D(N)}$$

and this map establishes a 1-1 correspondence between points δ of $A^{D(H)}$ such that $D(H)_\delta \supset P_\gamma$ and points of $A(P_\gamma)^{D(N)}$ where $D(N) := N_{D(H)}(P_\gamma) = N_{D(H)}(P)$.

We now examine $A(P_\gamma)^{D(N)}$. As P is contained in $D(H)$, and $D(H)$ is normal in H of index two, $[N : D(N)] = 2$ and for the same reason $[N_{\tilde{S}(W)}(P) : N_{\tilde{A}(W)}(P)] = 2$.

Clearly, $D(N)$ contains $D(H_0) \cdot N_{\tilde{S}(W)}(P)$. Also, $[N : D(H_0) \cdot N_{\tilde{S}(W)}(P)] = 4$. Choose r_1 in $H_0 - D(H_0)$, r_2 in $N_{\tilde{S}(W)}(P) - \tilde{A}(W)$. Let $r = r_1 r_2$. Then r is in $D(H)$ but not in $D(H_0) \cdot N_{\tilde{A}(W)}(P)$. We get easily that

$$D(N) = \langle D(H_0) \cdot N_{\tilde{A}(W)}(P), r \rangle. \quad (5.26)$$

By Proposition 3.29, $\bar{f} = \bar{f}' + \bar{f}'^c$ where f' and f'^c are blocks of $D(H_0)$ and $\bar{f}'^c = g^{-1} \bar{f}' g$ for any g in $H_0 - D(H_0)$.

For g as above, we have

$$\bar{f} k H_0 \bar{f} = \bar{f} k D(H_0) \oplus \bar{f} k D(H_0) g \quad (5.27)$$

as vector spaces over k . Right multiplication by g induces a vector space isomorphism between $\bar{f} k D(H_0) \bar{f}$ and $\bar{f} k D(H_0) f g$. Hence,

$$\dim_k(\bar{f} k D(H_0) \bar{f}) = \frac{1}{2} \dim_k(\bar{f} k H_0 \bar{f}). \quad (5.28)$$

Also,

$$\begin{aligned}
 \bar{f}kH_0\bar{f} &= (\bar{f}' + \bar{f}'^c)kH_0(\bar{f}' + \bar{f}'^c) \\
 &= \bar{f}'kH_0\bar{f}' \oplus \bar{f}'kH_0\bar{f}'^c \oplus \\
 &\quad \bar{f}'^ckH_0\bar{f}' \oplus \bar{f}'^ckH_0\bar{f}'^c.
 \end{aligned} \tag{5.29}$$

As the four summands on the right hand side of the above are isomorphic as vector spaces over k , we easily get from above that

$$\begin{aligned}
 \dim_k(\bar{f}'kH_0\bar{f}') &= \dim_k(\bar{f}'^ckH_0\bar{f}'^c) \\
 &= \frac{1}{4}\dim_k(\bar{f}kH_0\bar{f}).
 \end{aligned} \tag{5.30}$$

But,

$$\bar{f}kD(H_0)\bar{f} = \bar{f}'kD(H_0)\bar{f}' \oplus \bar{f}'^ckH_0'\bar{f}'^c \tag{5.31}$$

as $\bar{f}'$ and $\bar{f}'^c$ are central idempotents of $k\tilde{A}(V^P)$.

In particular, $\bar{f}kD(H_0)\bar{f} \subset \bar{f}'kH_0\bar{f}' \oplus \bar{f}'^ckH_0\bar{f}'^c$ and then, by (5.28), (5.30) and (5.31), we get that

$$\bar{f}'kD(H_0)\bar{f}' = \bar{f}'kH_0\bar{f}' \tag{5.32}$$

$$\bar{f}'^ckH_0'\bar{f}'^c = \bar{f}'^ckH_0\bar{f}'^c. \tag{5.33}$$

Then the image of $\bar{f}kD(H_0)\bar{f}$ in $\bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f}$ via the natural inclusion $\bar{f}kH_0\bar{f} \hookrightarrow \bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f}$, as defined earlier, is equal to the image of $\bar{f}'kH_0\bar{f}' \oplus \bar{f}'^ckH_0\bar{f}'^c$.

Let $\bar{E}' := C_{\bar{f}\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f}}(\bar{f}\bar{\tau}kD(H_0)\bar{e}\bar{\tau}\bar{f})$.

By Corollary 2.12,

$$\bar{E}' = \bar{f}' E \oplus \bar{f}'^c E. \quad (5.34)$$

From (5.28),

$$A(P_\gamma)^{D(N)} = \bar{E}'^{<r>} \quad (5.35)$$

as G_0 is centralized by $\tilde{A}(W)$.

Choose $w \in \bar{E}$. Then, $\bar{f}' w \in \bar{E}$ and

$$\begin{aligned} r^{-1} \bar{f}' w r &= r_2^{-1} r_1^{-1} \bar{f}' w r_1 r_2 \\ &= r_2^{-1} \bar{f}'^c r_1^{-1} w r_1 r_2 \\ &= \bar{f}'^c r_2^{-1} r_1^{-1} w r_1 r_2 \\ &= \bar{f}'^c r^{-1} w r \in \bar{f}'^c \bar{E} \end{aligned} \quad (5.36)$$

as $\bar{f}'^c \in kD(H_0)$ and hence commutes with r_2 .

Thus, $r^{-1} \bar{f}' \bar{E} r = \bar{f}'^c \bar{E}$ and, similarly, $r^{-1} \bar{f}'^c \bar{E} r = \bar{f}' \bar{E}$.

So, from (5.34) and (5.35) and Corollary 2.12, $A(P_\gamma)^{D(N)} \simeq \bar{f}' \bar{E} \simeq \bar{f}'^c \bar{E} \simeq \bar{E}$.

Thus, there is exactly one point, say δ , of $D(H)$ on A for which $D(H)_\delta \supset P_\gamma$, proving (ii) of Proposition 5.1. Further, $m_\delta = L$. Part (iii) then follows from (5.27). This completes the proof of Proposition 5.1 for the case $\epsilon(\mu) = 0$.

We now consider

Case 2: $\epsilon(\mu) = \epsilon(\nu) = 1$.

Let us rename e and e^a as e_1 and e_2 respectively, and f and f^a as f_1 and f_2 respectively.

Then Proposition 4.13 (ii) and (iii) will give

$$\text{Br}_P(b\sigma c) \neq 0 \text{ and } \text{rank}_{\mathcal{O}}(f_j\tau\mathcal{O}G_0e_j\tau f_j) = \frac{L^2}{4}\text{rank}_{\mathcal{O}}(\mathcal{O}H_0f_j) \quad (5.37)$$

for any i and j in $\{1,2\}$.

We consider the homomorphisms

$$\mathcal{O}H_0f_1 \rightarrow f_1\tau\mathcal{O}G_0e_1\tau f_1 \text{ and } \mathcal{O}H_0f_2 \rightarrow f_2\tau\mathcal{O}G_0e_1\tau f_2$$

induced by the natural inclusion of H_0 in G_0 . As before these homomorphisms are injective and if we let

$$E_1 := C_{f_1\tau\mathcal{O}G_0e_1\tau f_1}(\mathcal{O}H_0f_1) \text{ and } E_2 := C_{f_2\tau\mathcal{O}G_0e_1\tau f_2}(\mathcal{O}H_0f_2),$$

then

$$f_1\tau\mathcal{O}G_0e_1\tau f_1 \simeq \text{Cal}\mathcal{O}H_0f_1 \otimes_{\mathcal{O}} E_1 \text{ and } f_2\tau\mathcal{O}G_0e_1\tau f_2 \simeq \mathcal{O}H_0f_2 \otimes_{\mathcal{O}} E_2. \quad (5.38)$$

Furthermore,

$$E_1 \simeq E_2 \simeq \text{Mat}_{\frac{L}{2}}(\mathcal{O}) \quad (5.39).$$

Recall from Proposition 3.29 that $f^* = f_1 + f_2$ is a block of $D(H_0)$.

We have that $f^*\tau\mathcal{O}G_0e_1\tau f^*$ is a matrix algebra over $\mathcal{O}$ as is $\mathcal{O}G_0e_1$. Moreover, $\mathcal{O}H_0f^* = \mathcal{O}H_0f_1 \oplus \mathcal{O}H_0f_2$ as f_1 and f_2 are central in $\mathcal{O}H_0$. Therefore the sum of the above maps induces an injective map of $\mathcal{O}H_0f^* \rightarrow f^*\tau\mathcal{O}G_0e_1\tau f^*$.

If we set

$$E := C_{f^*\tau\mathcal{O}G_0e_1\tau f^*}(\mathcal{O}H_0f^*),$$

then by Lemma 2.11

$$E = E_1 \oplus E_2. \quad (5.40)$$

We have

$$\begin{aligned} \mathrm{Br}_P(A^P) &= \bar{f}^*\bar{\tau}kG_0\bar{e}\bar{\tau}\bar{f}^* \otimes_k kZ(P) \\ &= \bar{f}^*\bar{\tau}kG_0\bar{e}_1\bar{\tau}\bar{f}^* \otimes_k kZ(P) \oplus \bar{f}^*\bar{\tau}kG_0\bar{e}_2\bar{\tau}\bar{f}^* \otimes_k kZ(P). \end{aligned} \quad (5.41)$$

as e_1 and e_2 are central in G_0 .

As $\bar{f}^*\bar{\tau}kG_0\bar{e}_1\bar{\tau}\bar{f}^*$ and $\bar{f}^*\bar{\tau}kG_0\bar{e}_2\bar{\tau}\bar{f}^*$ are non-zero matrix algebras and $kZ(P)$ is a local ring, we get that P has two local points on A . We may call the two P -local points of A γ_1 and γ_2 .

Without loss of generality,

$$A(P_{\gamma_1}) \simeq \bar{f}^*\bar{\tau}kG_0\bar{e}_1\bar{\tau}\bar{f}^* \text{ and } A(P_{\gamma_2}) \simeq \bar{f}^*\bar{\tau}kG_0\bar{e}_2\bar{\tau}\bar{f}^*. \quad (5.42)$$

and as in the earlier case these isomorphisms are canonical.

We can find a point β of H on A such that $H_\beta \supset P_{\gamma_1}$. Then P_{γ_1} is a maximal local pointed subgroup of H_β .

By the same arguments as in Case 1, we get that $A_P^H = A^H$ and then [P1, Proposition 1.3] gives a surjective homomorphism:

$$\text{Res} S_{\gamma_1} : A^H \rightarrow (A(P_{\gamma_1}))^{N_1}.$$

where $N_1 = N_H(P_{\gamma_1})$, and as before this establishes a 1-1 correspondence between points β of H on A for which $H_\beta \supset P_\gamma$ and points of $(A(P_{\gamma_1}))^{N_1}$. Now $N_H(P) = H_0 \cdot N_{\tilde{S}(W)}(P)$. As there are two local points of P on A and P is a defect group for A , $[N_H(P) : N_1] = 2$. Moreover, H_0 is contained in N_1 as it centralizes P . Let $x \in \tilde{A}(W) \cap N_H(P)$. Then x acts on $\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*$ and $\bar{e}_1$ is fixed under this action. Hence x fixes γ_1 in its action on A^P . In other words, $N_{\tilde{A}(W)}(P)$ is contained in N_1 and thus

$$N_1 = H_0 \cdot N_{\tilde{A}(W)}(P). \quad (5.43)$$

Then (5.39) gives that

$$\begin{aligned} (A(P_{\gamma_1}))^{N_1} &\simeq (\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*)^{H_0 \cdot N_{\tilde{A}(W)}(P)} \\ &= (\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*)^{H_0} \\ &= \bar{E} \end{aligned} \quad (5.44)$$

as $\tilde{A}(W)$ commutes with G_0 .

From (5.39) and (5.40), $\bar{E} = \bar{E}_1 \oplus \bar{E}_2$ and both of these are matrix algebras over k of dimension $\frac{L^2}{4}$. Thus there are two points, say ι and ζ , of H on A for which P_{γ_1} is contained in H_ι and H_ζ and these are the only such points of H on A . Taking γ_1 for γ , part (i) of Proposition 5.1 follows. Also,

$$m_\iota = \frac{L}{2} \text{ and } m_\zeta = \frac{L}{2}. \quad (5.45)$$

Let $N'_1 = N_{D(H)}(P_{\gamma_1})$. It follows easily from the description of N_1 given in (5.43) that

$$N'_1 = D(H_0) \cdot N_{\tilde{A}(W)}(P). \quad (5.46)$$

Hence from (5.42),

$$\begin{aligned} (A(P_{\gamma_1}))^{N'_1} &\simeq (\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*)^{D(H_0) \cdot N_{\tilde{A}(W)}(P)} \\ &\simeq (\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*)^{D(H_0)} \end{aligned} \quad (5.47)$$

as $\tilde{A}(W)$ commutes with $\bar{f}^* \bar{\tau} k G_0 \bar{e}_1 \bar{\tau} \bar{f}^*$. By Proposition 3.29, $f^* = f_1 + f_2$ is a block of $D(H_0)$ of defect zero. Further, it is easy to see that $\mathcal{O}H_0 f^* = \mathcal{O}D(H_0)f^* \oplus \mathcal{O}(H_0 - D(H_0))f^*$ as $\mathcal{O}$ -modules. As left multiplication by any element of $H_0 - D(H_0)$ induces an $\mathcal{O}$ -module isomorphism between $\mathcal{O}D(H_0)f^*$ and

$\mathcal{O}(H_0 - D(H_0))f^*$, using Proposition 4.13, we have

$$\begin{aligned}
 \text{rank}_{\mathcal{O}}(\mathcal{O}D(H_0)f^*) &= \frac{1}{2}\text{rank}_{\mathcal{O}}(\mathcal{O}H_0f^*) \\
 &= \text{rank}_{\mathcal{O}}(\mathcal{O}H_0f_1) \\
 &= \frac{4}{L^2}\text{rank}_{\mathcal{O}}(f_1\tau\mathcal{O}G_0e_1\tau f_1) \\
 &= \frac{1}{L^2}\text{rank}_{\mathcal{O}}(f^*\tau\mathcal{O}G_0e_1\tau f^*). \tag{5.48}
 \end{aligned}$$

Since $\mathcal{O}D(H_0)f^*$ and $f^*\tau\mathcal{O}G_0e_1\tau f^*$ are both matrix algebras, the non zero algebra homomorphism, $\mathcal{O}D(H_0)f \rightarrow f^*\tau\mathcal{O}G_0e_1\tau f^*$, induced by the inclusion of $D(H_0)$ in G_0 is injective and we identify $\mathcal{O}D(H_0)$ with its image in $f^*\tau\mathcal{O}G_0e_1\tau f^*$.

Let $E' = C_{f^*\tau\mathcal{O}G_0e_1\tau f^*}(\mathcal{O}D(H_0)f^*)$. It follows from Proposition 2.11 that E' is the full matrix algebra over $\mathcal{O}$ of rank $\frac{1}{4}L^2$.

By (5.47),

$$(A(P_\gamma))^{N'_1} \simeq \bar{E}'. \tag{5.49}$$

Again, by [P1, Prop.1.3], the restriction of S_{γ_1} to $A^{D(H)}$ gives a 1-1 correspondence between points of $D(H)$ on A containing P_{γ_1} and points of $\bar{E}'$. Thus, there is exactly one point say δ of $D(H)$ on A for which $D(H)_\delta \supset P_{\gamma_1}$ and it is of multiplicity L which proves part (ii) of the proposition. Part (iii) follows from (5.45).

This completes the proof of Proposition 5.1.

■

We now show that for ζ, ι as in the Proposition 5.1, (5.7) is immediate. Indeed, choose I as stipulated. Let $1 = I + Z + R$ be an idempotent decomposition of the identity in A^H such that $Z = m_\zeta \zeta$. Note that here R may be zero.

By Proposition 4.23(ii) and Corollary 5.2 we clearly have

$$\text{rank}_{\mathcal{O}}(IAI) = \frac{m_\iota^2}{L^2} \text{rank}_{\mathcal{O}}((I + Z)A(I + Z)) \leq \frac{m_\iota^2}{L^2} \text{rank}_{\mathcal{O}}(A) = m_\iota^2 \text{rank}_{\mathcal{O}} \mathcal{O}Hc, \quad (5.50)$$

but this is exactly (5.7). As explained at the beginning of this chapter, this completes the proof of Theorem 1.4 and hence of Corollary 1.5. This in turn yields Corollaries 1.6 and 1.7 as we now indicate.

PROOF OF COROLLARY 1.6. By iterating Theorem 1.2, we immediately get that a source algebra for any $b \in \mathcal{B}_P$ is isomorphic to a source algebra for a block $c \in \mathcal{B}_P$ where $|c| \leq N_P$. The corollary follows by (2.31). ■

PROOF OF COROLLARY 1.7. This is immediate from Corollary 1.6 and the fact that a block algebra is Morita equivalent to its source algebra (cf. [P1, Cor.3.5]). ■

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