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Comparing groups with umbrella orderings

Pan, Guohua, Ph.D.

The Ohio State University, 1994

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Comparing Groups with Umbrella Orderings

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the
Degree Doctor of Philosophy in the Graduate School of
The Ohio State University

by

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1994

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CHAPTER I

Introduction

Introduction

In this chapter we describe the background of the problem of interest, review the relevant literature, give the definition of an umbrella ordering and provide a formal statement of the intended research.

1.1 Background

In a one-factor experiment it is common that the response variable increases with an increase in the treatment level up to a point, then decreases with further increase in the treatment level. In the literature this up-then-down response pattern has interchangeably been called an umbrella ordering (see, for example, Mack and Wolfe, 1981) or a unimodal ordering (see, for example, Robertson, Wright and Dykstra, 1988). The treatment level corresponding to the turning point of the ordering is called the peak of the umbrella. Such umbrella or unimodal orderings can be observed with many physical and biological phenomena in a wide variety of scientific research areas. They represent additional information on the associated underlying physical relationship and this can be used to

improve the effectiveness of statistical analyses of sample data arising in these settings.

There has been considerable previous work on procedures designed to test homogeneity against umbrella ordering alternatives. Such test procedures specifically designed for umbrella ordering configurations are, of course, generally more powerful than corresponding procedures for the broader class of general alternatives. The primary questions of interest in this setting deal with whether or not the data are even supportive of the umbrella pattern and, if so, which treatment level(s) leads to the peak effect and how large is this effect.

However, in many practical settings we have two or more different underlying populations for which we are interested in responses to different levels of the treatment. Thus, population label could be viewed as a second factor (in addition to treatment level) affecting the subjects' responses. (One simple, but important, case of this type would be when the two populations correspond to a subject's sex.) Often in such situations it is not the basic appropriateness of the umbrella ordering for the various populations that is of primary interest (although that could certainly still be important), but rather the similarity or lack thereof of the several umbrella patterns across the populations, involving questions such as: Do they have the same peaks?, Are the magnitudes of the peak effects the same?, Do the effects rise and fall across treatment levels at the same rates for the various populations?, etc.

As an example, consider the effect of stress on the performance of a certain task. Due to biological differences, males and females have potentially different responses to various levels of stress. However, it is quite reasonable to postulate that the male and female populations do share a common up-then-down response pattern when subjected to increasing stress. The levels of stress leading to optimal performances might be the same

or different for males and females. Here the interest is primarily about whether the stress level leading to the greatest productivity is common to both genders, not specifically about the relative magnitudes of the peak performances.

In this current research, we are concerned primarily with the specific case of two populations, each of which is to be subjected to k ordered levels of a treatment. For each of the populations, the average responses to the various treatment levels are assumed to increase with an increase of the treatment level up to certain unknown magnitudes and then to decrease with further increases in the treatment level; that is, the two sets of average treatment responses follow an umbrella (unimodal) ordering. We are mainly interested in the following general hypothesis testing question:

Null Hypothesis: [the level that corresponds to the highest response for population 1 is the same as the level that corresponds to the highest response for population 2]

versus

Alternative Hypothesis: [the level that corresponds to the highest response for population 1 is different from the level that corresponds to the highest response for population 2].

1.2 Review of the Literature

As an important order restriction, umbrella ordering, also called unimodal ordering, has been previously studied by several authors in a variety of situations. Robertson (1978) considered the importance of unimodal orderings for multinomial parameters. Collings *et al* (1981) used a unimodal ordering for binomial data in their comparison of an isotonic test

and a regression test for mutagenicity. Mack and Wolfe (1981) considered nonparametric tests of homogeneity for both peak-known and peak-unknown umbrella alternatives. Likelihood ratio tests for homogeneity versus peak-known unimodal ordering alternatives in the normal means case were implicated in the work of Bartholomew (1961a, b). More recently, there has been considerable activity on testing homogeneity against umbrella alternatives in a one-way layout setting. Simpson and Margolin (1986, 1990) proposed recursive nonparametric tests for dose-response relationships subject to down turns at high doses. Hettmansperger and Norton (1987) proposed a class of rank tests against patterned alternatives. Shi (1988a) considered likelihood ratio tests and their corresponding rank analogues. Shi (1988b) also proposed an optimal rank test based on the work of Hettmansperger and Norton (1987). Chen and Wolfe (1990a) generalized Chacko's (1963) test to peak-known umbrella alternatives and proposed an adaptation of the Mack-Wolfe (1981) statistic to peak-unknown umbrella alternatives. Chen and Wolfe (1990b) considered a modification of the Mack-Wolfe test for a generalized Behrens-Fisher problem. Chen (1991) studied the optimality of the Mack-Wolfe tests and obtained the asymptotic null distribution of the Chen-Wolfe (1990a) statistic for the peak-unknown umbrella alternatives. Chen and Wolfe (1993) compared umbrella pattern treatment effects with a control in a one-way layout.

All of this previous work with umbrella or unimodal orderings has been concerned with the case where we have a single population with observations at various treatment levels. The only previous reference in the literature to the problems of similarities between two or more umbrella orderings appears in the Ph.D. dissertation of Chen (1989), where she considered the general question of whether two independent sets of one-way layout data have different umbrella patterned effects. Her discussion centered on assessing whether or not two umbrella patterns are parallel.

1.3 Definition of Umbrella Ordering

Suppose that X_{ij} is distributed with continuous cdf $F_i(x)$ for $i=1, \dots, k$ and $j=1, \dots, n_i$ and the observations are mutually independent. The following definition was given by Mack and Wolfe (1981).

Definition 1.3.1 $(F_1(x), \dots, F_k(x))$ is said to satisfy an **umbrella ordering with peak p** if $F_1(x) \geq \dots \geq F_{p-1}(x) \geq F_p(x) \leq F_{p+1}(x) \leq \dots \leq F_k(x)$ for all x and some $p \in \{1, 2, \dots, k\}$

Three special cases of the distributions $(F_1(x), \dots, F_k(x))$ are considered in this thesis:

Parametric:

Case(a): $(F_1(x), \dots, F_k(x))$ is $(N(\mu_1, \sigma_1^2), \dots, N(\mu_k, \sigma_k^2))$ with $\mu = (\mu_1, \dots, \mu_k)$ unknown and $\sigma^2 = (\sigma_1^2, \dots, \sigma_k^2)$ known.

Case(b): $(F_1(x), \dots, F_k(x))$ is $(N(\mu_1, a_1 \sigma^2), \dots, N(\mu_k, a_k \sigma^2))$ with $\mu = (\mu_1, \dots, \mu_k)$ and σ^2 unknown and $\mathbf{a} = (a_1, \dots, a_k)$ known positive constants. If $a_i=1, i=1, \dots, k$, this is the one-way layout with common unknown variance.

Nonparametric:

Case(c): $F_i(x) = F(x - \mu_i), i=1, \dots, k$, where $\mu = (\mu_1, \dots, \mu_k)$ is unknown and where $F(x)$ is an arbitrary continuous distribution function.

For these cases, it is more convenient to express the umbrella concept equivalently in terms of medians $\mu = (\mu_1, \dots, \mu_k)$ rather than $(F_1(x), \dots, F_k(x))$.

Definition 1.3.2 Let X_{ij} , $i=1, \dots, k$ and $j=1, \dots, n_i$, be $N = \sum_{i=1}^k n_i$ mutually independent random variables such that $X_{i1}, \dots, X_{in_i}$ is a random sample from a probability distribution with median μ_i , for $i=1, \dots, k$. The medians $(\mu_1, \dots, \mu_k)$ are said to satisfy an **umbrella (unimodal) ordering with peak p** if $\mu_1 \leq \mu_2 \leq \dots \leq \mu_p \geq \mu_{p+1} \geq \dots \geq \mu_k$ for some $p \in \{1, 2, \dots, k\}$. If all of the inequalities in this ordering are strict, then $(\mu_1, \dots, \mu_k)$ is called a **strict umbrella (unimodal) ordering**.

Experiments are sometimes designed so that the differences between adjacent population responses are greater than a preset value corresponding to a minimal difference of practical importance. Therefore we are also interested in the following more restrictive δ -umbrella ordering:

Definition 1.3.3 An umbrella ordering $(\mu_1, \dots, \mu_k)$ with peak p is called a **δ -umbrella ordering with peak p** if $|\mu_{i+1} - \mu_i| \geq \delta$, $i=1, \dots, k-1$, where δ is a positive number.

In a δ -umbrella ordering, the absolute differences between population responses at any two adjacent treatment levels are bounded below by a common positive number. There are several reasons to consider δ -umbrella orderings. First, when we design an experiment we usually choose treatment levels sufficiently separate to produce differences in responses of practically important magnitudes. Second, it does not make much sense to test whether two umbrella orderings have the same peaks if the differences between the peak responses and the second largest responses are not of practical importance. Third, we will show in Section 2.3 that δ -umbrella orderings provide support for using homogeneity configurations to obtain proper critical points for large samples.

1.4 Statement of the Basic Research Problem

Suppose that X_{ij} is distributed with continuous cdf $F_i(x)$ for $i=1, \dots, k$ and $j=1, \dots, n_i$ and the $N = \sum_{i=1}^k n_i$ observations are mutually independent. In addition, suppose that X_{ij}^* is distributed with continuous cdf $F_i^*(x^*)$ for $i=1, \dots, k$ and $j=1, \dots, n_i^*$ and the $N^* = \sum_{i=1}^k n_i^*$ observations are mutually independent, as well as independent of the N X_{ij} 's. Assume that $(F_1(x), \dots, F_k(x))$ and $(F_1^*(x^*), \dots, F_k^*(x^*))$ satisfy umbrella orderings with peaks p and p^* , respectively. The question we are interested in for this dissertation is whether these two umbrellas have the same peak. Formally, we want to test

$$Q_1: \quad H_0: p = p^* \quad \text{versus} \quad H_1: p \neq p^*. \quad (1.4.1)$$

The testing problem Q_1 is a special case of a more general class of testing problems. Let A be a nonempty proper subset of $B = \{(i, j): i, j=1, \dots, k\}$. A general class of testing problems is:

$$Q_g: \quad H_0: (p, p^*) \in A \quad \text{versus} \quad H_1: (p, p^*) \in B - A. \quad (1.4.2)$$

It is obvious that if $A = \{(p, p^*): p = p^* = 1, \dots, k\}$, then Q_g becomes Q_1 . Another special case with an explicit practical interpretation corresponds to

$A = \{(p, p^*): p \leq p^* = 1, \dots, k\}$. The associated Q_g corresponds to testing whether the first group obtains its peak no later than the second group. That is,

$$H_0: p \leq p^* \quad \text{versus} \quad H_1: p > p^*. \quad (1.4.3)$$

The main part of this dissertation focuses on strict umbrella orderings. However, all the results can be applied to general umbrella orderings with slight modifications. It is possible for an umbrella ordering to have multiple peaks. If two umbrella orderings

involved have at least one common peak, they will be considered to have the same peaks and included in the null hypothesis H_0 . Otherwise, they will be included in the alternative hypothesis H_1 . We will give a full description of the modifications of the problem and the testing procedures in Section 6.2. The problems and procedures discussed in this dissertation can also be applied to completely randomized complete block experiments, which will be discussed in Section 6.3.

CHAPTER II

Likelihood Ratio Tests for Normal Distributions with Known Variances

Introduction

In this chapter we develop the likelihood ratio testing procedure for testing Q_1 ((1.4.1)) when the underlying distributions are normal with known variances. In Section 2.1 we describe the likelihood ratio test statistic. In Section 2.2 we discuss the determination of exact critical points. In Section 2.3 we discuss the appropriate critical points for large samples from a practical point of view.

2.1 Likelihood Ratio Statistic

The likelihood ratio method is used extensively in order restricted inference (see Robertson *et al*, 1988). Likelihood ratio tests often have the advantage of a more uniform power function than other procedures such as contrast tests (see, for example, Robertson *et al*, 1988). For a rather large null hypothesis space such as H_0 , likelihood ratio tests usually maintain fairly high powers over the entire alternative hypothesis space H_1 even

though they are typically not locally most powerful at any specific alternative. In this section we develop the likelihood ratio procedure for testing Q_1 ((1.4.1)) when the underlying distributions are normal with known variances. The procedure proposed can be used when information about the variances is available from past experience or when the sample sizes are reasonably large so that the sample variances can be taken as reasonable approximations to the actual variances. The test statistic derived for known variances also suggests the form for a test statistic based on ranks, to be discussed in detail in Chapter 4.

The following is a detailed description of the assumptions under which the theory in this section is developed.

Let X_{ij} , $i=1, \dots, k$ and $j=1, \dots, n_i$, be a set of $N = \sum_{i=1}^k n_i$ mutually independent random variables such that $X_{ij} \sim N(\mu_i, \sigma_i^2)$, where the unknown mean vector $\mu = (\mu_1, \dots, \mu_k)$ satisfies a strict umbrella ordering with unknown peak p and the variances $\sigma^2 = (\sigma_1^2, \dots, \sigma_k^2)$ are known. In addition, let X_{ij}^* , $i=1, \dots, k$ and $j=1, \dots, n_i^*$, be a second set of $N^* = \sum_{i=1}^k n_i^*$ mutually independent random variables such that $X_{ij}^* \sim N(\mu_i^*, \sigma_i^{*2})$, where the unknown mean vector $\mu^* = (\mu_1^*, \dots, \mu_k^*)$ satisfies a strict umbrella ordering with unknown peak p^* and the variances $\sigma^{*2} = (\sigma_1^{*2}, \dots, \sigma_k^{*2})$ are known. Finally, we assume that the set of N X_{ij} 's is independent of the set of N^* X_{ij}^* 's.

Throughout this dissertation, we denote the set of all umbrella orderings with peak p by

$$\Phi_p = \{ (x_1, \dots, x_k) \in R^k : x_1 \leq x_2 \leq \dots \leq x_p \geq x_{p+1} \geq \dots \geq x_k \}. \quad (2.1.1)$$

Let $P_w(g|\phi_p)$ denote the projection of $g = (g_1, \dots, g_k) \in R^k$ onto ϕ_p with nonnegative weights $w = (w_1, \dots, w_k)$. That is, $P_w(g|\phi_p)$ is the solution to the problem of

$$\text{minimizing } \|g - x\|_w^2 = \sum_1^k (g_i - x_i)^2 w_i, \text{ subject to } (x_1, \dots, x_k) \in \phi_p. \quad (2.1.2)$$

The relevant theory and some examples involving $P_w(g|\phi_p)$ can be found in Chapter 1 of Robertson *et al.* (1988). Several algorithms to calculate $P_w(g|\phi_p)$, such as the *minimum violator algorithm* due to Thompson (1962), are described by Barlow, Bartholomew, Bremner and Brunk (1972). FORTRAN subroutines to calculate $P_w(g|\phi_p)$ are available from the author.

In this dissertation, we let

$$\bar{X} = (\bar{X}_1, \dots, \bar{X}_k) \text{ and } \bar{X}^* = (\bar{X}_1^*, \dots, \bar{X}_k^*) \quad (2.1.3)$$

be the sample means, where

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \text{ and } \bar{X}_i^* = \frac{1}{n_i^*} \sum_{j=1}^{n_i^*} X_{ij}^*, \text{ for } i=1, \dots, k. \quad (2.1.4)$$

Also let

$$n = (n_1, \dots, n_k) \text{ and } n^* = (n_1^*, \dots, n_k^*) \quad (2.1.5)$$

be the sample sizes. In this chapter, we set

$$w_n = \left(\frac{n_1}{\sigma_1^2}, \dots, \frac{n_k}{\sigma_k^2} \right) \text{ and } w_{n^*}^* = \left(\frac{n_1^*}{\sigma_1^{*2}}, \dots, \frac{n_k^*}{\sigma_k^{*2}} \right) \quad (2.1.6)$$

to represent the nonnegative weights.

It is well known that the sample means $\bar{\mathbf{X}} = (\bar{X}_1, \dots, \bar{X}_k)$ and $\bar{\mathbf{X}}^* = (\bar{X}_1^*, \dots, \bar{X}_k^*)$ are sufficient statistics for the parameters $\mu = (\mu_1, \dots, \mu_k)$ and $\mu^* = (\mu_1^*, \dots, \mu_k^*)$ when the variances $\sigma^2 = (\sigma_1^2, \dots, \sigma_k^2)$ and $\sigma^{*2} = (\sigma_1^{*2}, \dots, \sigma_k^{*2})$ are known. Therefore, we need only to consider the likelihood function based on $\bar{\mathbf{X}}$ and $\bar{\mathbf{X}}^*$, which is

$$L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) = c \exp\left(-\frac{1}{2}(\|\bar{\mathbf{x}} - \mu\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - \mu^*\|_{\mathbf{w}_n^*}^2)\right), \quad (2.1.7)$$

where c is the scaling value

$$c = (2\pi)^{-(N+N^*)/2} \left(\prod_{i=1}^k w_i \right)^{-1/2} \left(\prod_{i=1}^k w_i^* \right)^{-1/2}. \quad (2.1.8)$$

By the facts that

$$\phi_p \text{ is a closed convex cone} \quad (2.1.9)$$

and

$$P_w(\mathbf{g}|\phi_p) \text{ exists and is unique for any } k\text{-dimensional vector } \mathbf{g} \quad (2.1.10)$$

(see Section 1.3 of Robertson *et al*, 1988), we have that for any pair (p, p^*) such that $p, p^* \in \{1, \dots, k\}$,

$$\begin{aligned} & \max_{\mu \in \phi_p, \mu^* \in \phi_{p^*}} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \\ &= \max_{\mu \in \phi_p, \mu^* \in \phi_{p^*}} c \exp\left(-\frac{1}{2}(\|\bar{\mathbf{x}} - \mu\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - \mu^*\|_{\mathbf{w}_n^*}^2)\right) \\ &= c \exp\left(-\frac{1}{2} \min_{\mu \in \phi_p, \mu^* \in \phi_{p^*}} (\|\bar{\mathbf{x}} - \mu\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - \mu^*\|_{\mathbf{w}_n^*}^2)\right) \\ &= c \exp\left(-\frac{1}{2} \left(\min_{\mu \in \phi_p} (\|\bar{\mathbf{x}} - \mu\|_{\mathbf{w}_n}^2) + \min_{\mu^* \in \phi_{p^*}} (\|\bar{\mathbf{x}}^* - \mu^*\|_{\mathbf{w}_n^*}^2) \right)\right) \end{aligned}$$

$$\begin{aligned}
&= c \exp\left(-\frac{1}{2}(\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_p)\|_{\mathbf{w}_n^*}^2)\right) \\
&= L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p), P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_p)). \tag{2.1.11}
\end{aligned}$$

It follows from (2.1.11) that

$$\begin{aligned}
&\max_{H_0} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \\
&= \max_{\mu, \mu^* \in \varphi_p, p=1, \dots, k} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \\
&= \max_{p=1, \dots, k} \left(\max_{\mu, \mu^* \in \varphi_p} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \right) \\
&= \max_{p=1, \dots, k} (L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p), P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_p))) \\
&= c \exp\left(-\frac{1}{2} \min_{p=1, \dots, k} (\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_p)\|_{\mathbf{w}_n^*}^2)\right), \tag{2.1.12}
\end{aligned}$$

and

$$\begin{aligned}
&\max_{H_1} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \\
&= \max_{\mu \in \varphi_p, \mu^* \in \varphi_{p^*}, p \neq p^*} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \\
&= \max_{p \neq p^*} \left(\max_{\mu \in \varphi_p, \mu^* \in \varphi_{p^*}} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*) \right) \\
&= \max_{p \neq p^*} (L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p), P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_{p^*}))) \\
&= c \exp\left(-\frac{1}{2} \min_{p \neq p^*} (\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{x}}^*|\varphi_{p^*})\|_{\mathbf{w}_n^*}^2)\right). \tag{2.1.13}
\end{aligned}$$

By taking $-2 \ln$ of the likelihood ratio $\Lambda = \frac{\max_{H_0} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*)}{\max_{H_1} L_1(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*)}$, the likelihood ratio test

rejects H_0 for large values of

$$T_1 = \min_{p=1, \dots, k} \left\{ \|\bar{\mathbf{X}} - P_{\mathbf{w}_n}(\bar{\mathbf{X}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{X}}^* - P_{\mathbf{w}_{n^*}}(\bar{\mathbf{X}}^*|\varphi_{p^*})\|_{\mathbf{w}_{n^*}}^2 \right\} \\ - \min_{p \neq p^*} \left\{ \|\bar{\mathbf{X}} - P_{\mathbf{w}_n}(\bar{\mathbf{X}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{X}}^* - P_{\mathbf{w}_{n^*}}(\bar{\mathbf{X}}^*|\varphi_{p^*})\|_{\mathbf{w}_{n^*}}^2 \right\}. \quad (2.1.14)$$

For significance level α , $0 < \alpha < 1$, let $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ be the number such that

$$\max_{(\mu, \mu^*) \in H_0} P(T_1 \geq c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = \alpha. \quad (2.1.15)$$

Then we have established the following:

Theorem 2.1.1 The level α likelihood ratio test for Q_1 ((1.4.1)) rejects H_0 if and only if T_1 is greater than or equal to $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

The test statistic T_1 has a simple interpretation. Notice that $\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p)\|_{\mathbf{w}_n}^2$ and $\|\bar{\mathbf{x}}^* - P_{\mathbf{w}_{n^*}}(\bar{\mathbf{x}}^*|\varphi_{p^*})\|_{\mathbf{w}_{n^*}}^2$ are the squares of the "distances" from $\bar{\mathbf{x}}$ to φ_p and from $\bar{\mathbf{x}}^*$ to φ_{p^*} , respectively. Therefore,

$$\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{x}}^* - P_{\mathbf{w}_{n^*}}(\bar{\mathbf{x}}^*|\varphi_{p^*})\|_{\mathbf{w}_{n^*}}^2 \quad (2.1.16)$$

is the "total squared distances" from the point $(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*)$ to $\varphi_p \otimes \varphi_{p^*}$, where $\otimes$ denotes the Cartesian product. The statistic T_1 is hence the difference between the "total squared distance" from $(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*)$ to the null hypothesis space H_0 and the "total squared distance" from $(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*)$ to the alternative space H_1 . For any alternative $(\mu, \mu^*) \in H_1$ such that $\mu \in \varphi_p$ and $\mu^* \in \varphi_{p^*}$ for some $p \neq p^*$, the sample means $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^*$ tend to be in different cones φ_p and φ_{p^*} , or rather, the sample means are generally "closer" to the alternative

hypothesis space H_1 than to the null hypothesis space H_0 and the statistic T_1 tends to be positive. The farther away the alternative $(\mu, \mu^*) \in H_1$ is from the common margin of the alternative space H_1 and the null hypothesis space H_0 , the greater is the statistic T_1 . On the other hand, for any parameter configuration (μ, μ^*) in the null hypothesis space H_0 such that $\mu \in \phi_p$ and $\mu^* \in \phi_p$ for some p , the sample means $\bar{x}$ and $\bar{x}^*$ tend to be in the same cone ϕ_p , or rather, the sample means are generally "closer" to the null hypothesis space H_0 than to the alternative hypothesis space H_1 and the statistic T_1 tends to be negative. The farther away a parameter configuration $(\mu, \mu^*) \in H_0$ is from the common margin of the alternative space H_1 and the null hypothesis space H_0 , the smaller is the statistic T_1 . These observations suggest that T_1 is a reasonable statistic to distinguish between the null hypothesis space H_0 and the alternative hypothesis space H_1 .

The test statistic T_1 can be computed using the *minimum violator algorithm* due to Thompson (1962), which is also described in Barlow *et al* (1972). FORTRAN programs to compute T_1 are available from the author.

2.2 Critical Points Through Use of the Slippage Configuration

The critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ defined in (2.1.15) is difficult to determine exactly due to the vast size of the parameter region corresponding to the null hypothesis H_0 . The prime difficulty is to decide the parameter configuration in the null hypothesis space H_0 that yields the exact critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$. It is not possible to obtain an explicit closed form expression for the power function of the test due to the complexity of the test statistic T_1 .

However, simulation results clearly indicate the solution to the determination of $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for the following class of combinations of variances and sample sizes, as commonly assumed in applications:

$$\frac{\sigma_1^2}{n_1} = \dots = \frac{\sigma_k^2}{n_k} = \rho^2 \text{ and } \frac{\sigma_1^{*2}}{n_1^*} = \dots = \frac{\sigma_k^{*2}}{n_k^*} = \rho^{*2}. \quad (2.2.1)$$

A special case corresponds to equal variances and sample sizes separately within each group; that is,

$$\sigma_1^2 = \dots = \sigma_k^2 = \nu^2, \sigma_1^{*2} = \dots = \sigma_k^{*2} = \nu^{*2}, n_1 = \dots = n_k = m \text{ and } n_1^* = \dots = n_k^* = m^*. \quad (2.2.2)$$

Critical points for cases not included in (2.2.1) are discussed at the end of this section.

In studying the power of tests based on T_1 , it is more convenient to use differences between adjacent treatment levels, namely,

$$\delta_i = \mu_i - \mu_{i-1} \text{ and } \delta_i^* = \mu_i^* - \mu_{i-1}^*, \text{ for } i=2, \dots, k, \quad (2.2.3)$$

since these differences uniquely determine the power function. It is easy to see that

$$\mu_i = \mu_1 + \delta_2 + \dots + \delta_i \text{ and } \mu_i^* = \mu_1^* + \delta_2^* + \dots + \delta_i^* \text{ for } i=2, \dots, k. \quad (2.2.4)$$

The next theorem shows that a power study of the test based on T_1 can be simplified for cases (2.2.1). In the theorem, as throughout this dissertation, we let

$\mathbf{Z}_k = (Z_1, \dots, Z_k)$ and $\mathbf{Z}_k^* = (Z_1^*, \dots, Z_k^*)$, where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are iid

$N(0, 1)$ random variables. The k -dimensional unit vector is denoted by $\mathbf{1}_k = (\underbrace{1, \dots, 1}_k)$.

Theorem 2.2.1 If condition (2.2.1) is satisfied, then at any parameter configuration (μ, μ^*) , we have

$$\begin{aligned}
 T_i \stackrel{D}{=} \min_{p=1, \dots, k} \left\{ \|\mathbf{Z}_k + \vartheta_k\| - P_{1_k}((\mathbf{Z}_k + \vartheta_k)|\varphi_p)\|_{1_k}^2 \right. \\
 \left. + \|\mathbf{Z}_k^* + \vartheta_k^*\| - P_{1_k}((\mathbf{Z}_k^* + \vartheta_k^*)|\varphi_p)\|_{1_k}^2 \right\} \\
 - \min_{p \neq p^*} \left\{ \|\mathbf{Z}_k + \vartheta_k\| - P_{1_k}((\mathbf{Z}_k + \vartheta_k)|\varphi_p)\|_{1_k}^2 \right. \\
 \left. + \|\mathbf{Z}_k^* + \vartheta_k^*\| - P_{1_k}((\mathbf{Z}_k^* + \vartheta_k^*)|\varphi_{p^*})\|_{1_k}^2 \right\}, \quad (2.2.5)
 \end{aligned}$$

where

$$\vartheta_k = (0, \Delta_2, \Delta_2 + \Delta_3, \dots, \Delta_2 + \Delta_3 + \dots + \Delta_k) \quad (2.2.6)$$

and

$$\vartheta_k^* = (0, \Delta_2^*, \Delta_2^* + \Delta_3^*, \dots, \Delta_2^* + \Delta_3^* + \dots + \Delta_k^*), \quad (2.2.7)$$

with

$$\Delta_i = \frac{1}{\rho} \delta_i \text{ and } \Delta_i^* = \frac{1}{\rho^*} \delta_i^*, \quad i = 2, \dots, k. \quad (2.2.8)$$

Proof. Set

$$\mathbf{V}_n = \left(\sqrt{\frac{\sigma_1^2}{n_1}} Z_1, \dots, \sqrt{\frac{\sigma_k^2}{n_k}} Z_k \right) \quad (2.2.9)$$

and

$$\mathbf{V}_{n^*}^* = \left(\sqrt{\frac{\sigma_1^{*2}}{n_1^*}} Z_1^*, \dots, \sqrt{\frac{\sigma_k^{*2}}{n_k^*}} Z_k^* \right), \quad (2.2.10)$$

where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are iid $N(0, 1)$. It is obvious that

$$(\mathbf{V}_n + \mu, \mathbf{V}_n^* + \mu^*) \stackrel{D}{=} (\bar{\mathbf{X}}, \bar{\mathbf{X}}^*), \quad (2.2.11)$$

where here $\stackrel{D}{=}$ means that the vector on the left hand side has the same joint distribution as the vector on the right hand side. By (2.2.11) and (2.1.14), we have

$$\begin{aligned} T_1 &\stackrel{D}{=} \min_{p=1, \dots, k} \left\{ \|\mathbf{V}_n + \mu - \mathbf{P}_{\mathbf{w}_n}(\mathbf{V}_n + \mu | \varphi_p)\|_{\mathbf{w}_n}^2 + \|(\mathbf{V}_n^* + \mu^*) - \mathbf{P}_{\mathbf{w}_n^*}(\mathbf{V}_n^* + \mu^* | \varphi_p)\|_{\mathbf{w}_n^*}^2 \right\} \\ &\quad - \min_{p \neq p^*} \left\{ \|\mathbf{V}_n + \mu - \mathbf{P}_{\mathbf{w}_n}(\mathbf{V}_n + \mu | \varphi_p)\|_{\mathbf{w}_n}^2 + \|(\mathbf{V}_n^* + \mu^*) - \mathbf{P}_{\mathbf{w}_n^*}(\mathbf{V}_n^* + \mu^* | \varphi_{p^*})\|_{\mathbf{w}_n^*}^2 \right\}. \end{aligned} \quad (2.2.12)$$

From (2.2.1), (2.2.9), (2.2.10) and (2.2.12), it follows that

$$\begin{aligned} T_1 &\stackrel{D}{=} \min_{p=1, \dots, k} \left\{ \|\rho \mathbf{Z}_k + \mu - \mathbf{P}_{\mathbf{I}_k / \rho^2}(\rho \mathbf{Z}_k + \mu | \varphi_p)\|_{\mathbf{I}_k / \rho^2}^2 \right. \\ &\quad \left. + \|(\rho^* \mathbf{Z}_k^* + \mu^*) - \mathbf{P}_{\mathbf{I}_k / \rho^{*2}}(\rho^* \mathbf{Z}_k^* + \mu^* | \varphi_p)\|_{\mathbf{I}_k / \rho^{*2}}^2 \right\} \\ &\quad - \min_{p \neq p^*} \left\{ \|\rho \mathbf{Z}_k + \mu - \mathbf{P}_{\mathbf{I}_k / \rho^2}(\rho \mathbf{Z}_k + \mu | \varphi_p)\|_{\mathbf{I}_k / \rho^2}^2 \right. \\ &\quad \left. + \|(\rho^* \mathbf{Z}_k^* + \mu^*) - \mathbf{P}_{\mathbf{I}_k / \rho^{*2}}(\rho^* \mathbf{Z}_k^* + \mu^* | \varphi_{p^*})\|_{\mathbf{I}_k / \rho^{*2}}^2 \right\}. \end{aligned} \quad (2.2.13)$$

By the fact that

$$\mathbf{P}_{c\mathbf{w}}(\mathbf{g} | \varphi_p) = \mathbf{P}_{\mathbf{w}}(\mathbf{g} | \varphi_p), \text{ where } c \text{ is any positive constant,} \quad (2.2.14)$$

we have

$$\begin{aligned}
T_1 &= \min_{p=1, \dots, k}^D \left\{ \|\rho(\mathbf{Z}_k + \mu / \rho) - P_{1_k}(\rho(\mathbf{Z}_k + \mu / \rho) | \varphi_p)\|_{1_k/\rho^2}^2 \right. \\
&\quad \left. + \|\rho^*(\mathbf{Z}_k^* + \mu^* / \rho^*) - P_{1_k}(\rho^*(\mathbf{Z}_k^* + \mu^* / \rho^*) | \varphi_p)\|_{1_k/\rho^{*2}}^2 \right\} \\
&\quad - \min_{p \neq p^*} \left\{ \|\rho(\mathbf{Z}_k + \mu / \rho) - P_{1_k}(\rho(\mathbf{Z}_k + \mu / \rho) | \varphi_p)\|_{1_k/\rho^2}^2 \right. \\
&\quad \left. + \|\rho^*(\mathbf{Z}_k^* + \mu^* / \rho^*) - P_{1_k}(\rho^*(\mathbf{Z}_k^* + \mu^* / \rho^*) | \varphi_{p^*})\|_{1_k/\rho^{*2}}^2 \right\} \\
&= \min_{p=1, \dots, k}^D \left\{ \|\mathbf{Z}_k + \mu / \rho - P_{1_k}((\mathbf{Z}_k + \mu / \rho) | \varphi_p)\|_{1_k}^2 \right. \\
&\quad \left. + \|\mathbf{Z}_k^* + \mu^* / \rho^* - P_{1_k}((\mathbf{Z}_k^* + \mu^* / \rho^*) | \varphi_p)\|_{1_k}^2 \right\} \\
&\quad - \min_{p \neq p^*} \left\{ \|\mathbf{Z}_k + \mu / \rho - P_{1_k}((\mathbf{Z}_k + \mu / \rho) | \varphi_p)\|_{1_k}^2 \right. \\
&\quad \left. + \|\mathbf{Z}_k^* + \mu^* / \rho^* - P_{1_k}((\mathbf{Z}_k^* + \mu^* / \rho^*) | \varphi_{p^*})\|_{1_k}^2 \right\}. \quad (2.2.15)
\end{aligned}$$

Using the fact that

$$P_w(\mathbf{g} | \varphi_p) + \mathbf{c} = P_w(\mathbf{g} + \mathbf{c} | \varphi_p) \quad \text{for a constant vector } \mathbf{c} = (c, \dots, c), \quad (2.2.16)$$

the conclusion of Theorem 2.2.1 follows from (2.2.15) and (2.2.16) by letting $c = -\mu_1 / \rho$ and $c = -\mu_1^* / \rho^*$ in (2.2.16) for the first and second umbrella orderings, respectively.

Theorem 2.2.1 shows that the power function for a test based on T_1 has a common pattern for each of the configurations in the class (2.2.1). Therefore, in order to search for the least favorable configuration to determine $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$, it suffices to study the case with unit variance and a single observation for each treatment level; that is,

$$\sigma_1^2 = \dots = \sigma_k^2 = 1, \sigma_1^{*2} = \dots = \sigma_k^{*2} = 1, n_1 = \dots = n_k = 1 \text{ and } n_1^* = \dots = n_k^* = 1. \quad (2.2.17)$$

The distribution of the test statistic T_1 for this simple case is given by Theorem 2.2.1 with $\rho=1$. For distinction, we use upper case notations such as Δ , Δ_i and Δ_i^* for the special case (2.2.17) and lower case notations such as δ , δ_i and δ_i^* for any general case in (2.2.1).

The results of a simulation study (to be discussed in Chapter 5) suggest the following important observation:

Observation 2.2.1 The critical point $c_1(\alpha, \mathbf{l}_k, \mathbf{l}_k, \mathbf{l}_k)$ for case (2.1.17) can be obtained from the distribution of T_1 at the parameter configurations:

$$\Delta_2 = \dots = \Delta_{k-1} = 0, \Delta_k = +\infty \text{ and } \Delta_2^* = \dots = \Delta_k^* = 0. \quad (2.2.18)$$

Observation 2.2.1 combined with Theorem 2.2.1 solves the problem of finding the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for any general case in (2.2.1). The solution depends on the slippage configurations (SC):

$$\delta_2 = \dots = \delta_{k-1} = 0, \delta_k = +\infty \text{ and } \delta_2^* = \dots = \delta_k^* = 0. \quad (2.2.19)$$

It follows from (2.2.14) and (2.2.16) that the distribution of T_1 is the same at all of the slippage configurations (2.2.19). Therefore, for the purpose of critical point determination, we need only investigate the distribution of T_1 a single slippage configuration: $\mu_1 = \dots = \mu_{k-1} = \mu_1^* = \dots = \mu_k^* = 0$ and $\mu_k = +\infty$. In fact, this is the parameter configuration that we use in our simulation study. Let $s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ be the upper α percentile of the distribution of the test statistic T_1 at slippage configurations (2.2.19); that is, $s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is the number such that

$$P_{SC}(T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = \alpha. \quad (2.2.20)$$

Then we have the following important conclusion:

Corollary 2.2.1 For any case (2.2.1), the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ can be obtained from the distribution of T_1 at slippage configurations (2.2.19). In other words, we conclude that

$$c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) = s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}). \quad (2.2.21)$$

Furthermore, we have

$$s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) = s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k). \quad (2.2.22)$$

Proof. For the special case (2.2.17), Observation 2.2.1 suggests that at parameter configurations (2.2.18),

$$P(T_1 \geq s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)) = \max_{(\mu, \mu^*) \in H_0} P_{(\mu, \mu^*)}(T_1 \geq s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)) = \alpha. \quad (2.2.23)$$

It follows from Theorem 2.2.1 that for any general case (2.2.1), we have

$$\max_{(\mu, \mu^*) \in H_0} P_{(\mu, \mu^*)}(T_1 \geq s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)) = \alpha. \quad (2.2.24)$$

It also follows from Theorem 2.2.1 that the distribution of T_1 at parameter configurations (2.2.18) for case (2.2.17) is the same as that at parameter configurations (2.2.19) for any general case (2.2.1). Therefore, for any case (2.2.1), the equality in (2.2.23) still holds at slippage configurations (2.2.19). Comparing with the definition of $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$, we obtain (2.2.21). Equation (2.2.22) is an immediate result of Theorem 2.2.1.

Corollary 2.2.1 shows that only one table of $s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ values is needed to list the critical points for all the cases (2.2.1). The critical point $s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ can be obtained by simulation according to the distribution of T_1 at slippage configurations (2.2.19), as given in the following Theorem 2.2.2.

Theorem 2.2.2 Under condition (2.2.1), the distribution of T_1 at slippage configurations (2.2.19) is the same as that of the random variable

$$\|Z_k^* - P_{1_k}(Z_k^*|\varphi_k)\|_{1_k}^2 = \min_{p^*=1, \dots, k-1} \left\{ \|Z_k^* - P_{1_k}(Z_k^*|\varphi_{p^*})\|_{1_k}^2 \right\}, \quad (2.2.25)$$

where $Z_k^* = (Z_1^*, \dots, Z_k^*)$ with $Z_1^*, \dots, Z_k^*$ being iid $N(0, 1)$ random variables.

Proof. By Theorem 2.2.1, at slippage configurations (2.2.19),

$$\begin{aligned} T_1 &= \min_{p=1, \dots, k}^D \left\{ \| (Z_k + \vartheta_k) - P_{1_k}((Z_k + \vartheta_k)|\varphi_p) \|_{1_k}^2 + \| Z_k^* - P_{1_k}(Z_k^*|\varphi_p) \|_{1_k}^2 \right\} \\ &= \min_{p \neq p} \left\{ \| (Z_k + \vartheta_k) - P_{1_k}((Z_k + \vartheta_k)|\varphi_p) \|_{1_k}^2 + \| Z_k^* - P_{1_k}(Z_k^*|\varphi_p) \|_{1_k}^2 \right\}, \end{aligned} \quad (2.2.26)$$

where

$$\vartheta_k = (0, 0, \dots, +\infty). \quad (2.2.27)$$

By definition,

$$\begin{aligned} &\| (Z_k + \vartheta_k) - P_{1_k}((Z_k + \vartheta_k)|\varphi_p) \|_{1_k}^2 \\ &= +\infty, \end{aligned} \quad \text{for } p=1, \dots, k-1, \quad (2.2.28)$$

and

$$= \| Z_{k-1} - P_{1_{k-1}}(Z_{k-1}|\varphi_{k-1}^{k-1}) \|_{1_{k-1}}^2. \quad \text{for } p=k, \quad (2.2.29)$$

where $\varphi_{k-1}^{k-1} = \{(x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1} : x_1 \leq \dots \leq x_{k-1}\}$ is the set of umbrella orderings with peak $k-1$ in $(k-1)$ -dimensional Euclidean space. Therefore,

$$\begin{aligned}
& \min_{p=1, \dots, k} \left\{ \|(\mathbf{Z}_k + \mathbf{v}_k) - P_{\mathbf{I}_k}((\mathbf{Z}_k + \mathbf{v}_k)|\varphi_p)\|_{\mathbf{I}_k}^2 + \|\mathbf{Z}_k^* - P_{\mathbf{I}_k}(\mathbf{Z}_k^*|\varphi_p)\|_{\mathbf{I}_k}^2 \right\} \\
& = \|\mathbf{Z}_{k-1} - P_{\mathbf{I}_{k-1}}(\mathbf{Z}_{k-1}|\varphi_{k-1}^{k-1})\|_{\mathbf{I}_{k-1}}^2 + \|\mathbf{Z}_k^* - P_{\mathbf{I}_k}(\mathbf{Z}_k^*|\varphi_k)\|_{\mathbf{I}_k}^2
\end{aligned} \tag{2.2.30}$$

and

$$\begin{aligned}
& \min_{p \neq p^*} \left\{ \|(\mathbf{Z}_k + \mathbf{v}_k) - P_{\mathbf{I}_k}((\mathbf{Z}_k + \mathbf{v}_k)|\varphi_p)\|_{\mathbf{I}_k}^2 + \|\mathbf{Z}_k^* - P_{\mathbf{I}_k}(\mathbf{Z}_k^*|\varphi_p)\|_{\mathbf{I}_k}^2 \right\} \\
& = \min_{p=1, \dots, k-1} \left\{ \|\mathbf{Z}_{k-1} - P_{\mathbf{I}_{k-1}}(\mathbf{Z}_{k-1}|\varphi_{k-1}^{k-1})\|_{\mathbf{I}_{k-1}}^2 + \|\mathbf{Z}_k^* - P_{\mathbf{I}_k}(\mathbf{Z}_k^*|\varphi_p)\|_{\mathbf{I}_k}^2 \right\} \\
& = \|\mathbf{Z}_{k-1} - P_{\mathbf{I}_{k-1}}(\mathbf{Z}_{k-1}|\varphi_{k-1}^{k-1})\|_{\mathbf{I}_{k-1}}^2 + \min_{p=1, \dots, k-1} \left\{ \|\mathbf{Z}_k^* - P_{\mathbf{I}_k}(\mathbf{Z}_k^*|\varphi_p)\|_{\mathbf{I}_k}^2 \right\}.
\end{aligned} \tag{2.2.31}$$

Theorem 2.2.2 follows from (2.2.30) and (2.2.31).

In Table 1 we provide estimates of the critical point $c_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ for $k=3, 4, 5$, and 6 , obtained via Theorem 2.2.2 and simulation of the distribution of T_1 under the slippage configurations (2.2.19).

The use of slippage configurations (2.2.19) to obtain critical points $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for cases (2.2.1) is also supported by the fact that slippage configurations (2.2.19) are asymptotically the most difficult null hypothesis configurations for the likelihood ratio test based on T_1 to distinguish from the alternatives in H_1 .

In the following, we assume that $\{\mathbf{n}\}$ and $\{\mathbf{n}^*\}$ are two sequences such that

$$N = \sum_{i=1}^k n_i \rightarrow \infty \text{ and } \frac{n_i}{N} \rightarrow \lambda_i \in (0, 1), i=1, \dots, k, \text{ as } N \rightarrow \infty, \tag{2.2.32}$$

$$N^* = \sum_{i=1}^k n_i^* \rightarrow \infty \text{ and } \frac{n_i^*}{N^*} \rightarrow \lambda_i^* \in (0, 1), i=1, \dots, k, \text{ as } N^* \rightarrow \infty, \tag{2.2.33}$$

and

$$\frac{N}{N+N^*} \rightarrow \gamma \in (0, 1) \text{ as } N, N^* \rightarrow \infty. \quad (2.2.34)$$

Using $\overset{\Delta}{=}$ to mean 'defined as', we have

$$\frac{\mathbf{w}_n}{N} \rightarrow \left(\frac{\lambda_1}{\sigma_1^2}, \dots, \frac{\lambda_k}{\sigma_k^2} \right) \overset{\Delta}{=} \mathbf{w}_\infty, \text{ as } N \rightarrow \infty, \quad (2.2.35)$$

and

$$\frac{\mathbf{w}_n^*}{N^*} \rightarrow \left(\frac{\lambda_1^*}{\sigma_1^{*2}}, \dots, \frac{\lambda_k^*}{\sigma_k^{*2}} \right) \overset{\Delta}{=} \mathbf{w}_\infty^*, \text{ as } N^* \rightarrow \infty. \quad (2.2.36)$$

By (2.2.9) and (2.2.10), we have

$$\sqrt{N} \mathbf{V}_n \overset{D}{\rightarrow} \left(\sqrt{\frac{\sigma_1^2}{\lambda_1}} Z_1, \dots, \sqrt{\frac{\sigma_k^2}{\lambda_k}} Z_k \right) \overset{\Delta}{=} \mathbf{V}_\infty, \text{ as } N \rightarrow \infty, \quad (2.2.37)$$

and

$$\sqrt{N^*} \mathbf{V}_n^* \overset{D}{\rightarrow} \left(\sqrt{\frac{\sigma_1^{*2}}{\lambda_1^*}} Z_1^*, \dots, \sqrt{\frac{\sigma_k^{*2}}{\lambda_k^*}} Z_k^* \right) \overset{\Delta}{=} \mathbf{V}_\infty^*, \text{ as } N^* \rightarrow \infty. \quad (2.2.38)$$

The next theorem shows that the slippage configurations (2.2.19) are among the parameter configuration candidates that may yield the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

Theorem 2.2.3 For fixed μ and μ^* ,

$$\begin{aligned} 0 &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\ &< \lim_{N, N^* \rightarrow +\infty} P_{SC} (T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = \alpha \\ &< \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = 1. \end{aligned} \quad (2.2.39)$$

Proof. It follows from (2.2.9) and (2.2.10) that

$$\mathbf{V}_n \rightarrow \mathbf{0}_k \text{ and } \mathbf{V}_n^* \rightarrow \mathbf{0}_k \text{ almost surely as } N, N^* \rightarrow \infty. \quad (2.2.40)$$

By (2.2.12), (2.2.14), (2.2.32)-(2.2.34) and the continuity of the functions $P_w(\mathbf{g}|\varphi_p)$ and $\|\mathbf{g} - \mathbf{x}\|_w^2$, we have that

$$\begin{aligned} \frac{T_1}{N + N^*} & \stackrel{D}{=} \min_{p=1, \dots, k} \left\{ \frac{N}{N + N^*} \|\mathbf{V}_n + \mu\| - P_{w_n/N}(\mathbf{V}_n + \mu|\varphi_p)\|_{w_n/N}^2 \right. \\ & \quad \left. + \frac{N^*}{N + N^*} \|\mathbf{V}_n^* + \mu^*\| - P_{w_n^*/N^*}(\mathbf{V}_n^* + \mu^*|\varphi_p)\|_{w_n^*/N^*}^2 \right\} \\ & - \min_{p \neq p^*} \left\{ \frac{N}{N + N^*} \|\mathbf{V}_n + \mu\| - P_{w_n/N}(\mathbf{V}_n + \mu|\varphi_p)\|_{w_n/N}^2 \right. \\ & \quad \left. + \frac{N^*}{N + N^*} \|\mathbf{V}_n^* + \mu^*\| - P_{w_n^*/N^*}(\mathbf{V}_n^* + \mu^*|\varphi_{p^*})\|_{w_n^*/N^*}^2 \right\}, \end{aligned}$$

which, as $N, N^* \rightarrow +\infty$, converges almost surely to a constant

$$\begin{aligned} J_1 &= \min_{p=1, \dots, k} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1 - \gamma) \|\mu^* - P_{w_-^*}(\mu^*|\varphi_p)\|_{w_-^*}^2 \right\} \\ & - \min_{p \neq p^*} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1 - \gamma) \|\mu^* - P_{w_-^*}(\mu^*|\varphi_{p^*})\|_{w_-^*}^2 \right\}. \end{aligned} \quad (2.2.41)$$

By (2.2.16) and the facts that

$$P_w(\mathbf{c}|\varphi_p) = 0 \text{ for a constant vector } \mathbf{c} = (c, \dots, c) \quad (2.2.42)$$

(again see chapter 1 of Robertson *et al*, 1988), and that slippage configurations (2.2.19) satisfy

$$\begin{aligned} \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 &= +\infty \text{ for } p=1, \dots, k-1, \\ &= 0 \text{ for } p=k, \end{aligned} \quad (2.2.43)$$

we have that, under the slippage configurations (2.2.19),

$$\frac{T_1}{N + N^*} \xrightarrow{a.s} J_1 = 0, \text{ as } N, N^* \rightarrow \infty. \quad (2.2.44)$$

By the definition (2.2.20),

$$P_{sc}(T_1 / (N + N^*) \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) / (N + N^*)) = \alpha > 0. \quad (2.2.45)$$

This implies that

$$s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) / (N + N^*) \rightarrow 0 \text{ as } N, N^* \rightarrow \infty, \quad (2.2.46)$$

since otherwise there exists a subsequence $\{\mathbf{n}_s\}$ of $\{\mathbf{n}\}$ and a subsequence $\{\mathbf{n}_s^*\}$ of $\{\mathbf{n}^*\}$ such that

$$s_1(\alpha, \mathbf{n}_s, \mathbf{n}_s^*, \sigma^2, \sigma^{*2}) / (N + N^*) \rightarrow L \neq 0, \text{ as } N, N^* \rightarrow \infty.$$

In view of (2.2.44), the corresponding subsequence of $\frac{T_1}{N + N^*}$ will then have

$$\begin{aligned} P_{sc}(T_1 / (N + N^*) \geq s_1(\alpha, \mathbf{n}_s, \mathbf{n}_s^*, \sigma^2, \sigma^{*2}) / (N + N^*)) \\ \leq P_{sc}(T_1 / (N + N^*) > L/2) \rightarrow 0 \text{ as } N, N^* \rightarrow +\infty. \end{aligned} \quad (2.2.47)$$

This contradicts (2.2.45).

Now, we consider two cases. First, suppose that $(\mu, \mu^*) \in H_0$ so that $\mu_1 < \mu_2 < \dots < \mu_p > \mu_{p+1} > \dots > \mu_k$ and $\mu_1^* < \mu_2^* < \dots < \mu_p^* > \mu_{p+1}^* > \dots > \mu_k^*$ for some $p=1, \dots, k$.

Then, since

$$P_w(g|\varphi_p) = g \text{ for any } g \in \varphi_p \text{ and } P_w(g|\varphi_p) \neq g \text{ for any } g \notin \varphi_p, \quad (2.2.48)$$

we have that

$$\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 = 0 \text{ for any } g \in \varphi_p \quad (2.2.49)$$

and

$$\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 > 0 \text{ for any } g \notin \varphi_p. \quad (2.2.50)$$

Therefore,

$$\min_{p=1, \dots, k} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1-\gamma) \|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2 \right\} = 0 \quad (2.2.51)$$

and

$$\min_{p \neq p^*} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1-\gamma) \|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2 \right\} > 0. \quad (2.2.52)$$

It follows from (2.2.41) that, as $N, N^* \rightarrow \infty$,

$$\frac{T_1}{N + N^*} \xrightarrow{a.s} J_1 = - \min_{p \neq p^*} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1-\gamma) \|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2 \right\} < 0. \quad (2.2.53)$$

By Slutsky's Theorem, (2.2.46) and (2.2.53), we have

$$\begin{aligned} & \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_1 / (N + N^*) \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) / (N + N^*)) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_1 / (N + N^*) \geq 0) = 0. \end{aligned} \quad (2.2.54)$$

For the second case, let $(\mu, \mu^*) \in H_1$ so that $\mu_1 < \mu_2 < \dots < \mu_p > \mu_{p+1} > \dots > \mu_k$ and $\mu_1^* < \mu_2^* < \dots < \mu_{p^*}^* > \mu_{p^*+1}^* > \dots > \mu_k^*$ for some $p \neq p^* = 1, \dots, k$. By an approach similar to that for the case when $(\mu, \mu^*) \in H_0$, we find that

$$\min_{p=1, \dots, k} \left\{ \gamma \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 + (1-\gamma) \|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2 \right\} > 0 \quad (2.2.55)$$

and

$$\min_{p \neq p^*} \left\{ \gamma \|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2 + (1 - \gamma) \|\mu^* - P_{w_+}(\mu^* | \varphi_{p^*})\|_{w_+}^2 \right\} = 0. \quad (2.2.56)$$

Hence, for $(\mu, \mu^*) \in H_1$, we have that, as $N, N^* \rightarrow \infty$,

$$\frac{T_1}{N + N^*} \xrightarrow{a.s} J_1 = \min_{p=1, \dots, k} \left\{ \gamma \|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2 + (1 - \gamma) \|\mu^* - P_{w_+}(\mu^* | \varphi_{p^*})\|_{w_+}^2 \right\} > 0. \quad (2.2.57)$$

By Slutsky's Theorem, (2.2.46) and (2.2.57), we have

$$\begin{aligned} & \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_1 \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_1 / (N + N^*) \geq s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) / (N + N^*)) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_1 / (N + N^*) \geq 0) = 1. \end{aligned} \quad (2.2.58)$$

Theorem 2.2.3 follows by combining (2.2.54) and (2.2.58).

So far we have restricted our attention to cases (2.2.1). The determination of critical points for any general case not in (2.2.1) is more complicated. Simulation results indicate that the level α critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is the maximum of the upper α percentiles of the distributions of the test statistic T_1 under parameter configurations given by

$$\mu_1 = \dots = \mu_{i-1} = 0, \mu_i = +\infty, \mu_{i+1} = \dots = \mu_k = 0 \text{ and } \mu_1^* = \dots = \mu_k^* = 0, i=1, \dots, k \quad (2.2.59)$$

and

$$\mu_1 = \dots = \mu_k = 0 \text{ and } \mu_1^* = \dots = \mu_{j-1}^* = 0, \mu_j^* = +\infty, \mu_{j+1}^* = \dots = \mu_k^* = 0, j=1, \dots, k. \quad (2.2.60)$$

The distributions of T_1 under parameter configurations (2.2.59) and (2.2.60) are given in Theorem 2.2.4. It can be used to obtain simulation estimates of the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

Theorem 2.2.4 Under parameter configurations (2.2.59) with fixed i ,

$$T_1 \stackrel{D}{=} \|V_{\mathbf{n}}^* - P_{w_{\mathbf{n}}^*}(V_{\mathbf{n}}^* | \varphi_i)\|_{w_{\mathbf{n}}^*}^2 - \min_{p \neq i} \left\{ \|V_{\mathbf{n}}^* - P_{w_{\mathbf{n}}^*}(V_{\mathbf{n}}^* | \varphi_p)\|_{w_{\mathbf{n}}^*}^2 \right\} \quad (2.2.61)$$

and under parameter configurations (2.2.60) with fixed j ,

$$T_1 \stackrel{D}{=} \|V_{\mathbf{n}} - P_{w_{\mathbf{n}}}(V_{\mathbf{n}} | \varphi_j)\|_{w_{\mathbf{n}}}^2 - \min_{p \neq j} \left\{ \|V_{\mathbf{n}} - P_{w_{\mathbf{n}}}(V_{\mathbf{n}} | \varphi_p)\|_{w_{\mathbf{n}}}^2 \right\}, \quad (2.2.62)$$

where $w_{\mathbf{n}}, w_{\mathbf{n}}^*, V_{\mathbf{n}}, V_{\mathbf{n}}^*$ are defined in (2.1.6), (2.2.9) and (2.2.10).

Proof. The proof is similar to that of Theorem 2.2.2.

Simulation results indicate that the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is, in fact, the maximum of the upper α percentiles of the distributions of the test statistic T_1 under parameter configurations (2.2.59) with $i=1, k$ and (2.2.60) with $j=1, k$ (the symmetric transformations of the slippage configurations (2.2.19)) for most cases where the weights $w_{\mathbf{n}} = (\frac{n_1}{\sigma_1^2}, \dots, \frac{n_k}{\sigma_k^2})$ and $w_{\mathbf{n}}^* = (\frac{n_1^*}{\sigma_1^{*2}}, \dots, \frac{n_k^*}{\sigma_k^{*2}})$ are not extremely unbalanced.

However, simulation also suggests that this is not true for every situation. One example is the case when there are three treatment levels ($k=3$) with weights $w = (40, 20, 65)$ and $w^* = (1, 20, 1)$. Nevertheless, the maximum of the upper α percentiles of the distributions of the test statistic T_1 under parameter configurations (2.2.59) with $i=1, k$ and (2.2.60) with $j=1, k$ is very close to, and thus a good estimate of, the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for any situation.

A FORTRAN program that provides a simulation estimate of the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$, computes the test statistic T_1 and gives the P-value for any case of sample sizes and variances is available from the author. In the program, the user will need to input the number of iterations, or precision, desired for the simulation estimate of the critical point, the significance level of interest, the number of treatment levels k , the vectors of sample sizes, the vectors of sample means and the vectors of known variances. The output is the value of the test statistic, the simulated critical point and the P-value.

When the sample sizes are reasonably large (more than 30 observations at each treatment level), the test based on T_1 provides a good approximate test for the case with unknown variances. For large samples, we can view the sample variances as the unknown population variances and carry out the test based on T_1 as if the population variances are known.

2.3 Critical Points for Large Samples Through Use of the Homogeneity Configuration

Simulation result reveals that if $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is used as the critical point, then the test based on T_1 has serious biasedness. The critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is very conservative in the sense that a smaller critical point will be more appropriate for applications with large samples. To overcome this biasedness and conservativeness, we seek a class of parameter configurations that yield reasonable approximate critical points for applications with large sample sizes. We want these parameter configurations to be simple, intuitively reasonable and to provide practical approximate critical points.

One class of interesting parameter configurations are the homogeneity configurations (HC) for which

$$\mu_1 = \dots = \mu_k \text{ and } \mu_1^* = \dots = \mu_k^*. \quad (2.3.1)$$

It follows from (2.2.14) and (2.2.16) that the distribution of T_1 is the same at all the homogeneity configurations (2.3.1). Therefore, for the purpose of critical point determination, we need to investigate the distribution of T_1 at the single homogeneity configuration: $\mu_1 = \dots = \mu_k = \mu_1^* = \dots = \mu_k^* = 0$. Homogeneity configurations (2.3.1) are on the boundary of the null hypothesis space H_0 . Under such configurations, the statistic T_1 tends to be around zero. For any other parameter configuration (μ, μ^*) in the null hypothesis space H_0 , the statistic T_1 tends to be negative. This suggests that, with respect to the test statistic T_1 , homogeneity configurations (2.3.1) provide a reasonable division between the null hypothesis space H_0 and the alternative hypothesis space H_1 .

The parameter configuration under which the upper $100\alpha\%$ percentile of the distribution of T_1 is exactly $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is not the homogeneity configuration (2.3.1). However, we propose that the distribution of T_1 under homogeneity configurations may be used to obtain reasonable critical points for practical use with large samples. Let $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ be the number such that

$$P_{HC}(T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = \alpha. \quad (2.3.2)$$

The rationale that $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ provides a reasonable substitute for $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for large samples is based on the following four reasons (as well as empirical evidence from simulation studies, as discussed in Section 5.1):

(1). The homogeneity configurations (2.3.1) are asymptotically the most difficult null hypothesis configurations for the likelihood ratio test based on T_1 to distinguish from the alternatives in H_1 .

(2). The asymptotic distribution of T_1 under the homogeneity configurations (2.3.1) can be obtained.

(3). While the actual type I error is greater than the nominal level α when $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is used as the critical point, this type I error inflation results from parameter configurations in H_0 that are not very likely to occur in practice. (Later we provide an approximate relation between the level inflation and the region in H_0 causing the inflation.)

(4). The use of the actual critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ may lead to a serious bias problem (for details, see the results of a Monte Carlo simulation study in Chapter 5). This is because the parameter configuration yielding $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is very unusual in practice and is far away from the expected underlying parameter region in many applications. From a practical point of view, the use of $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ may thus make the testing procedure unduly conservative.

Before we discuss these statements in detail, let us point out that under the homogeneity configurations (2.3.1), the distribution of T_1 is the same for all the cases in (2.2.1).

Theorem 2.3.1 If condition (2.2.1) is satisfied, then under homogeneity configurations (2.3.1), we have

$$\begin{aligned}
T_1 = & \min_{p=1, \dots, k}^D \left\{ \|Z_k - P_{I_k}(Z_k | \varphi_p)\|_{I_k}^2 + \|Z_k^* - P_{I_k}(Z_k^* | \varphi_p)\|_{I_k}^2 \right\} \\
& - \min_{p \neq p^*} \left\{ \|Z_k - P_{I_k}(Z_k | \varphi_p)\|_{I_k}^2 + \|Z_k^* - P_{I_k}(Z_k^* | \varphi_{p^*})\|_{I_k}^2 \right\}
\end{aligned} \tag{2.3.3}$$

where $Z_k = (Z_1, \dots, Z_k)$ and $Z_k^* = (Z_1^*, \dots, Z_k^*)$ with $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ being iid $N(0, 1)$ random variables, and where $\mathbf{I}_k = \underbrace{(1, \dots, 1)}_k$ is the k -dimensional unit vector.

Proof. Theorem 2.3.1 follows from (2.2.15), (2.2.16) and (2.3.1).

Corollary 2.3.1 Under condition (2.2.1), we have

$$h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) = h_1(\alpha, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k). \tag{2.3.4}$$

Proof. By Theorem 2.3.1, the distribution of the test statistic T_1 under homogeneity configurations (2.3.1) is the same for all the cases in (2.2.1). Corollary 2.3.1 follows from the definition of $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

Therefore, only one set of $h_1(\alpha, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k)$ values is needed for all the different combinations of variances and sample sizes satisfying (2.2.1). In Table 2 we provide such simulation estimates of $h_1(\alpha, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k, \mathbf{I}_k)$ for $k=3, 4, 5$, and 6 .

A FORTRAN program that provides a simulation estimate of the critical point $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$, computes the test statistic T_1 and gives the P -value for any case of sample sizes and variances is available from the author. In the program, the user will need to input the number of iterations, or precision, desired for the simulation estimate of the critical point, the significance level of interest, the number of treatment levels k , the vectors of sample sizes, the vectors of sample means and the vectors of known variances. The output is the value of the test statistic, the simulated critical point and the P -value.

Since the procedure in this section is for large sample sizes, the test based on T_1 also provides a good approximate test for the cases with unknown variances. We can view the sample variances as the unknown population variances and carry out the test based on T_1 as if the population variances are known.

Now let us provide theoretical proofs of the statements made at the beginning of this section. The following theorems are for any general combination of sample sizes and variances. Again, we assume conditions (2.2.32)-(2.2.34). The next theorem shows that the homogeneity configurations (2.3.1) are asymptotically among the most difficult parameter configurations on the boundary of the null hypothesis space H_0 for the likelihood ratio test based on T_1 to distinguish from the alternatives in H_1 .

Theorem 2.3.2 For fixed μ and μ^* ,

$$\begin{aligned}
 0 &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{HC} (T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = \alpha \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) = 1.
 \end{aligned} \tag{2.3.5}$$

Proof. The proof follows the lines of the proof of Theorem 2.2.3 with $s_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ replaced by $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

The next theorem provides the limiting distribution of T_1 under homogeneity configurations (2.3.1).

Theorem 2.3.3 When $\mu_1 = \dots = \mu_k$ and $\mu_1^* = \dots = \mu_k^*$,

$$\begin{aligned} T_1 &\xrightarrow{D} \min_{p=1, \dots, k} \left\{ \|V_\infty - P_{w_\infty}(V_\infty | \phi_p)\|_{w_\infty}^2 + \|V_\infty^* - P_{w_\infty^*}(V_\infty^* | \phi_p)\|_{w_\infty^*}^2 \right\} \\ &\quad - \min_{p \neq p^*} \left\{ \|V_\infty - P_{w_\infty}(V_\infty | \phi_p)\|_{w_\infty}^2 + \|V_\infty^* - P_{w_\infty^*}(V_\infty^* | \phi_{p^*})\|_{w_\infty^*}^2 \right\} \\ &\stackrel{\Delta}{=} T_1(\lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}), \text{ as } N, N^* \rightarrow \infty, \end{aligned} \quad (2.3.6)$$

where V_∞ , V_∞^* , w_∞ and w_∞^* are defined in (2.2.35), (2.2.36), (2.2.37) and (2.2.38).

Proof. By (2.2.12), we have

$$\begin{aligned} T_1 &\stackrel{D}{=} \min_{p=1, \dots, k} \left\{ \|\sqrt{N}V_n - P_{w_n/N}(\sqrt{N}V_n | \phi_p)\|_{w_n/N}^2 + \|\sqrt{N^*}V_n^* - P_{w_n^*/N^*}(\sqrt{N^*}V_n^* | \phi_p)\|_{w_n^*/N^*}^2 \right\} \\ &\quad - \min_{p \neq p^*} \left\{ \|\sqrt{N}V_n - P_{w_n/N}(\sqrt{N}V_n | \phi_p)\|_{w_n/N}^2 + \|\sqrt{N^*}V_n^* - P_{w_n^*/N^*}(\sqrt{N^*}V_n^* | \phi_{p^*})\|_{w_n^*/N^*}^2 \right\}. \end{aligned} \quad (2.3.7)$$

By Wald's Theorem, (2.2.35), (2.2.36), (2.2.37), (2.2.38) and the continuity of the functions $P_w(g | \phi_p)$ and $\|g - x\|_w^2$, we obtain the conclusion of Theorem 2.3.3.

This limiting distribution of T_1 under homogeneity configurations (2.3.1) reduces the burden of producing a large number of tables of the critical points $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

When the sample sizes are sufficiently large, the critical point obtained from the distribution of $T_1(\lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2})$ provides a good approximation to $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$.

More specifically, let $h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2})$ be the constant such that

$$P(T_1(\lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}) \geq h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2})) = \alpha. \quad (2.3.8)$$

Then $h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2})$ will be a reasonable approximation for $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$

when the sample sizes $\mathbf{n}$ and $\mathbf{n}^*$ are large. We formally state this now in Corollary 2.3.2.

Corollary 2.3.2 Under conditions (2.2.32)-(2.2.34),

$$h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}) \rightarrow h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}), \text{ as } N, N^* \rightarrow \infty. \quad (2.3.9)$$

Proof. We will prove (2.3.9) by contradiction. Suppose that (2.3.9) is not true. Then there exists a subsequence $h_1(\alpha, \mathbf{n}_k, \mathbf{n}_k^*, \sigma^2, \sigma^{*2})$ such that

$$h_1(\alpha, \mathbf{n}_k, \mathbf{n}_k^*, \sigma^2, \sigma^{*2}) \rightarrow d \neq h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}), \text{ as } k, k^* \rightarrow \infty. \quad (2.3.10)$$

When the limit d is finite, it follows from Slutsky's Theorem and Theorem 2.3.3 that

$$\begin{aligned} &P(T_1 \geq h_1(\alpha, \mathbf{n}_k, \mathbf{n}_k^*, \sigma^2, \sigma^{*2})) \\ &\rightarrow P(T_1(\lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}) \geq d) \neq \alpha, \text{ as } k, k^* \rightarrow \infty, \end{aligned} \quad (2.3.11)$$

which contradicts (2.3.8). On the other hand, when the limit d is positive infinity, then there exist K and K^* such that when $k > K$ and $k^* > K^*$,

$$h_1(\alpha, \mathbf{n}_k, \mathbf{n}_k^*, \sigma^2, \sigma^{*2}) > h_1\left(\frac{\alpha}{2}, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}\right). \quad (2.3.12)$$

Therefore, when $k > K$ and $k^* > K^*$,

$$\begin{aligned} P(T_1 \geq h_1(\alpha, \mathbf{n}_k, \mathbf{n}_k^*, \sigma^2, \sigma^{*2})) &\leq P(T_1(\lambda, \lambda^*, \gamma) \geq h_1\left(\frac{\alpha}{2}, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}\right)) \\ &= \frac{\alpha}{2}, \end{aligned} \quad (2.3.13)$$

which also contradicts (2.3.8). A similar approach leads to a contradiction of (2.3.8) when d is negative infinity. Corollary 2.3.2 is thus established.

Theorem 2.3.2 establishes the homogeneity configurations (2.3.1) as reasonable division points between the null hypothesis space H_0 and the alternative hypothesis space

H_1 . Another reason to use homogeneity configurations (2.3.1) to obtain approximate critical points for large samples follows from a careful examination of the null hypothesis space H_0 . Since this null hypothesis space consists of a very large number of parameter configurations, some portions of it are quite unlikely to occur in applications. For example, the null hypothesis parameter configurations close to

$$\begin{cases} \mu_i = +\infty, \mu_1 = \dots = \mu_{i-1} = \mu_{i+1} = \dots = \mu_k \\ \mu_1^* = \dots = \mu_k^* \end{cases} \quad (2.3.14)$$

are not likely to be appropriate for most applications. On the other hand, some null hypothesis parameter configurations are quite commonly expected in applications. For example, null hypothesis parameter configurations close to the configuration

$$\begin{cases} \mu_1 = \mu + \delta, \dots, \mu_p = \mu + p\delta, \mu_{p+1} = \mu + (p-1)\delta, \dots, \mu_k = \mu + (2p-k)\delta \\ \mu_1^* = \mu^* + \delta^*, \dots, \mu_p^* = \mu^* + p\delta^*, \mu_{p+1}^* = \mu^* + (p-1)\delta^*, \dots, \mu_k^* = \mu^* + (2p-k)\delta^* \end{cases} \quad (2.3.15)$$

where δ and δ^* are positive numbers and $p \in \{1, \dots, k\}$, are very likely in practice.

Another common example in practice is the δ -umbrella ordering which we will discuss in detail later. Therefore if a critical point is determined by an unusual configuration such as in (2.3.14), it can be quite unrealistic. For the testing problem Q_1 ((1.4.1)) and the associated test statistic T_1 , we feel that a reasonable critical point should correspond to the overwhelming majority of parameter configurations that are expected in standard applications, even though, strictly speaking, we are risking an inflated type I error. In other words, we propose to ignore the portion of null parameter configurations rarely seen in applications and place more emphasis on those parameter configurations most expected in practice.

We have mentioned that the type I error is bigger than α when we use $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ instead of $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ as the critical point; that is,

$$\max_{(\mu, \mu^*) \in H_0} P(T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) > \alpha. \quad (2.3.16)$$

This type I error inflation from $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is caused by parameter configurations close to the margins of the null hypothesis space H_0 . However, this part of the null hypothesis space is typically not of most importance in practice. Experiments are usually designed so that the differences between adjacent population responses are greater than a preset value corresponding to a minimal difference of practical importance. Therefore we now turn our attention to that portion of the null hypothesis space satisfying $|\mu_{i+1} - \mu_i| \geq \delta$ and $|\mu_{i+1}^* - \mu_i^*| \geq \delta^*$, $i=1, \dots, k-1$, where δ and δ^* are positive numbers. Definition 1.3.3 formally labels such parameter configurations as δ -umbrella orderings. As we mentioned in Section 1.3, there are several reasons to consider δ -umbrella orderings. We denote the set of all δ -umbrella orderings with peak p by

$$\varphi(p, \delta) = \{ \mu = (\mu_1, \dots, \mu_k) : \mu \in \varphi_p \text{ and } |\mu_{i+1} - \mu_i| \geq \delta, i=1, \dots, k-1 \}. \quad (2.3.17)$$

The next theorem shows that if we restrict the null hypothesis space to δ -umbrella orderings, then the use of $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ as the critical point for the test T_1 does not lead to inflation of the type I error for large samples. Since the significance level α is usually chosen to be close to zero, we discuss only significance levels $0 < \alpha < 1$ such that $h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}) > 0$ in the following Theorems 2.3.4, 2.3.5 and 2.3.6.

Theorem 2.3.4 For any positive constants δ and δ^* ,

$$\max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \leq \alpha, \text{ as } N \text{ and } N^* \rightarrow +\infty. \quad (2.3.18)$$

Proof. Suppose that μ and μ^* are fixed and that

$$\mu \in \varphi(p, \delta) \text{ and } \mu^* \in \varphi(p, \delta^*) \text{ for a fixed } p \in \{1, \dots, k\}.$$

Notice that

$$(\bar{X} - \mu, \bar{X}^* - \mu^*)' \sim N(\mathbf{0}_{2k}, \Sigma_{2k}), \quad (2.3.19)$$

where $\mathbf{0}_{2k}$ is the $2k$ dimensional zero vector and

$$\Sigma_{2k} = \text{diag}\left(\frac{1}{w_1}, \dots, \frac{1}{w_k}, \frac{1}{w_1^*}, \dots, \frac{1}{w_k^*}\right) \quad (2.3.20)$$

is a $2k \times 2k$ matrix with diagonal elements $\frac{1}{w_1}, \dots, \frac{1}{w_k}, \frac{1}{w_1^*}, \dots, \frac{1}{w_k^*}$ and off-diagonal elements zero. Since

$$\frac{1}{w_i} \rightarrow 0 \text{ and } \frac{1}{w_i^*} \rightarrow 0, \text{ for } i=1, \dots, k, \text{ as } N, N^* \rightarrow +\infty, \quad (2.3.21)$$

we have that

$$(\bar{X} - \mu, \bar{X}^* - \mu^*)' \xrightarrow{P} \mathbf{0}_{2k}, \text{ as } N, N^* \rightarrow +\infty. \quad (2.3.22)$$

Now, let

$$A = \{|\bar{X}_1 - \mu_1| \leq \frac{\delta}{2}, \dots, |\bar{X}_k - \mu_k| \leq \frac{\delta}{2}, |\bar{X}_1^* - \mu_1^*| \leq \frac{\delta^*}{2}, \dots, |\bar{X}_k^* - \mu_k^*| \leq \frac{\delta^*}{2}\}. \quad (2.3.23)$$

Then for $\beta = 1 - \alpha$, there exists N_0 and N_0^* , independent of p , μ and μ^* , such that when $N > N_0$ and $N^* > N_0^*$, we have $P(A) > \beta$. Notice that by assumption, $\mu \in \varphi(p, \delta)$ and $\mu^* \in \varphi(p, \delta^*)$, which means

$$\mu_{i+1} - \mu_i \geq \delta \text{ and } \mu_{i+1}^* - \mu_i^* \geq \delta^*, \text{ for } i = 1, \dots, p-1, \quad (2.3.24)$$

and

$$\mu_{i+1} - \mu_i \leq \delta \text{ and } \mu_{i+1}^* - \mu_i^* \leq \delta^*, \text{ for } i = p, \dots, k-1. \quad (2.3.25)$$

Therefore, if $(\bar{X}, \bar{X}^*) \in A$, then

$$\begin{aligned}\bar{X}_{i+1} - \bar{X}_i &= (\bar{X}_{i+1} - \mu_{i+1}) - (\bar{X}_i - \mu_i) + (\mu_{i+1} - \mu_i) \\ &\geq -\frac{\delta}{2} - \frac{\delta}{2} + \delta = 0, \text{ for } i = 1, \dots, p-1,\end{aligned}\quad (2.3.26)$$

and

$$\begin{aligned}\bar{X}_{i+1} - \bar{X}_i &= (\bar{X}_{i+1} - \mu_{i+1}) - (\bar{X}_i - \mu_i) + (\mu_{i+1} - \mu_i) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} - \delta = 0, \text{ for } i = p, \dots, k-1.\end{aligned}\quad (2.3.27)$$

Similarly,

$$\bar{X}_{i+1}^* - \bar{X}_i^* \geq 0 \text{ for } i = 1, \dots, p-1 \quad (2.3.28)$$

and

$$\bar{X}_{i+1}^* - \bar{X}_i^* \leq 0 \text{ for } i = p, \dots, k-1. \quad (2.3.29)$$

So, for any $(\bar{X}, \bar{X}^*) \in A$, we have $\bar{X} \in \varphi_p$ and $\bar{X}^* \in \varphi_p$. From the definition of T_1 in (2.1.14), we have

$$T_1 = - \min_{p \neq p} \left\{ \|\bar{X} - P_w(\bar{X} | \varphi_p)\|_w^2 + \|\bar{X}^* - P_{w^*}(\bar{X}^* | \varphi_p^*)\|_{w^*}^2 \right\} \leq 0. \quad (2.3.30)$$

It follows from Corollary 2.3.1 and the assumption that $h_1(\alpha, \lambda, \lambda^*, \gamma, \sigma^2, \sigma^{*2}) > 0$ that there exist $M > 0$ and $M^* > 0$ such that when $n > M$ and $n^* > M^*$, we have $h_1(\alpha, n, n^*, \sigma^2, \sigma^{*2}) > 0$. Therefore, when $n > \max(N_0, M)$ and $n^* > \max(N_0^*, M^*)$, we have

$$\begin{aligned}
& \max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\
&= 1 - \max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(T_1 \leq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\
&\leq 1 - \max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(T_1 \leq 0) \\
&\leq 1 - \max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(A) \\
&\leq 1 - \beta = 1 - (1 - \alpha) = \alpha.
\end{aligned} \tag{2.3.31}$$

For fixed sample sizes $\mathbf{n}$ and $\mathbf{n}^*$, if we let δ and δ^* go to positive infinity, then the type I error over the subset of δ - and δ^* -umbrella orderings in the null hypothesis goes to zero. The following theorem is a formal statement of this fact.

Theorem 2.3.5 For fixed sample sizes $\mathbf{n}$ and $\mathbf{n}^*$,

$$\max_{\substack{\mu \in \varphi(p, \delta), \mu^* \in \varphi(p, \delta^*), \\ p=1, \dots, k}} P(T_1 \geq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \rightarrow 0, \text{ as } \delta \text{ and } \delta^* \rightarrow +\infty. \tag{2.3.32}$$

Proof. Let A be defined as in (2.3.23). Then

$$P(A) \rightarrow 1 \text{ as } \delta \text{ and } \delta^* \rightarrow +\infty. \tag{2.3.33}$$

Therefore, for any $0 < \beta < 1$, there exists Δ_0 and Δ_0^* , independent of p , μ and μ^* , such that when $\delta > \Delta_0$ and $\delta^* > \Delta_0^*$, we have $P(A) > \beta$. The conclusion of this theorem then follows by arguments similar to those in the proof of Theorem 2.3.4.

Theorems 2.3.4 and 2.3.5 show that for any fixed sample sizes $\mathbf{n}$ and $\mathbf{n}^*$, there exist $\delta(\mathbf{n}, \mathbf{n}^*)$ and $\delta^*(\mathbf{n}, \mathbf{n}^*)$ such that

$$\max_{\substack{\mu \in \Phi(p, \delta(n, n^*)), \mu^* \in \Phi(p, \delta^*(n, n^*)), \\ p=1, \dots, k}} P(T_1 \geq h_1(\alpha, n, n^*, \sigma^2, \sigma^{*2})) = \alpha. \quad (2.3.34)$$

We now investigate the relation between such a pair $(\delta(n, n^*), \delta^*(n, n^*))$ and (n, n^*) .

The next theorem provides one reasonable choice for $\delta(n, n^*)$ and $\delta^*(n, n^*)$. In

Theorem 2.3.6 we assume

$$\sigma_1^2 = \dots = \sigma_k^2 = v^2, \quad \sigma_1^{*2} = \dots = \sigma_k^{*2} = v^{*2}, \quad n_1 = \dots = n_k = s \text{ and } n_1^* = \dots = n_k^* = s^*. \quad (2.3.35)$$

Theorem 2.3.6 Let

$$\Delta(n, n^*) \geq \frac{2v}{\sqrt{s}} z\left(\frac{1}{2} - \frac{(1-\alpha)^{\frac{1}{2k}}}{2}\right) \quad (2.3.36)$$

and

$$\Delta^*(n, n^*) \geq \frac{2v^*}{\sqrt{s^*}} z\left(\frac{1}{2} - \frac{(1-\alpha)^{\frac{1}{2k}}}{2}\right), \quad (2.3.37)$$

where $z(\alpha)$ is the upper α percentile of the standard normal distribution. Then

$$\max_{\substack{\mu \in \Phi(p, \Delta(n, n^*)), \mu^* \in \Phi(p, \Delta^*(n, n^*)), \\ p=1, \dots, k}} P(T_1 \geq h_1(\alpha, n, n^*, \sigma^2, \sigma^{*2})) \leq \alpha. \quad (2.3.38)$$

Proof. Let $Z \sim N(0, 1)$. Since $(\bar{X} - \mu, \bar{X}^* - \mu^*)'$ has a $2k$ -variate normal distribution with mean vector $\mathbf{0}_{2k}$ and covariance matrix $\Sigma_{2k} = \text{diag}(\frac{1}{w_1}, \dots, \frac{1}{w_k}, \frac{1}{w_1^*}, \dots, \frac{1}{w_k^*})$, we

have

$$\begin{aligned} & P(|\bar{X}_i - \mu_i| \leq \frac{\Delta(n, n^*)}{2}) \\ &= P\left(\frac{\sqrt{s}}{v} |\bar{X}_i - \mu_i| \leq \frac{\sqrt{s}}{v} \frac{\Delta(n, n^*)}{2}\right) \end{aligned}$$

$$\begin{aligned}
&\geq P(|Z| \leq z(\frac{1}{2} - \frac{(1-\alpha)^{\frac{1}{2k}}}{2})) \\
&= (1-\alpha)^{\frac{1}{2k}}, \text{ for } i=1, \dots, k.
\end{aligned} \tag{2.3.39}$$

Similarly,

$$P(|\bar{X}_i^* - \mu_i^*| \leq \frac{\Delta^*(\mathbf{n}, \mathbf{n}^*)}{2}) \geq (1-\alpha)^{\frac{1}{2k}}, \text{ for } i=1, \dots, k. \tag{2.3.40}$$

Since $\bar{X}_1, \dots, \bar{X}_k, \bar{X}_1^*, \dots, \bar{X}_k^*$ are mutually independent, we have

$$P(A) = \prod_{i=1}^k P(|\bar{X}_i - \mu_i| \leq \frac{\Delta(\mathbf{n}, \mathbf{n}^*)}{2}) \prod_{j=1}^k P(|\bar{X}_j^* - \mu_j^*| \leq \frac{\Delta^*(\mathbf{n}, \mathbf{n}^*)}{2}) \geq 1 - \alpha, \tag{2.3.41}$$

where A is defined in (2.3.23). By the same reasoning as in the proof of Theorem 2.3.4, for any $(\bar{\mathbf{X}}, \bar{\mathbf{X}}^*) \in A$, we have

$$T_1 \leq 0 < h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2}). \tag{2.3.42}$$

Therefore

$$\begin{aligned}
&\max_{\substack{\mu \in \varphi(p, \Delta(\mathbf{n}, \mathbf{n}^*)), \mu^* \in \varphi(p, \Delta^*(\mathbf{n}, \mathbf{n}^*)) \\ p=1, \dots, k}} P(T_1 > h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\
&= 1 - \max_{\substack{\mu \in \varphi(p, \Delta(\mathbf{n}, \mathbf{n}^*)), \mu^* \in \varphi(p, \Delta^*(\mathbf{n}, \mathbf{n}^*)) \\ p=1, \dots, k}} P(T_1 \leq h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})) \\
&\leq 1 - P(A) \leq 1 - (1 - \alpha) = \alpha.
\end{aligned} \tag{2.3.43}$$

We note that Theorem 2.3.6 also holds under the slightly less restrictive condition that

$$\frac{n_1}{\sigma_1^2} = \dots = \frac{n_k}{\sigma_k^2} = \frac{s}{v^2} \text{ and } \frac{n_1^*}{\sigma_1^{*2}} = \dots = \frac{n_k^*}{\sigma_k^{*2}} = \frac{s^*}{v^{*2}}. \tag{2.3.44}$$

CHAPTER III

Likelihood Ratio Tests for Normal Distributions with Unknown Variances

Introduction

In this chapter we develop the likelihood ratio testing procedures for testing Q_1 ((1.4.1)) when the underlying distributions are normal with unknown variances. In Section 3.1 we describe two cases of assumptions about the variances: (1) the variances are unknown and possibly different for the two groups and (2) the variances are unknown but common for the two groups. In Section 3.2 we discuss the likelihood ratio test for case (1). In Section 3.3 we develop the corresponding likelihood ratio test for case (2).

3.1 Assumptions About Variances

We consider two cases of assumptions about the unknown variances.

Case (1) $\sigma_i^2 = a_i \tau^2$ and $\sigma_i^{*2} = a_i^* \tau^{*2}$, for $i = 1, \dots, k$, where the a_i 's and a_i^* 's are assumed to be known and both τ^2 and τ^{*2} are unknown.

In this case, the unknown variances τ^2 and τ^{*2} for the two groups can be different. This assumption is typically suitable when the two groups have structural differences in the responses measured. One example is the physical strength of male and female groups. When males and females perform a certain physical task, they usually have the same umbrella response pattern subject to increasing level of treatments, say stress, but the absolute magnitudes of their performances can be in totally different categories. Such a difference in the magnitudes of the responses between the two groups is often accompanied by a difference in the variances for the responses as well.

Case (2) $\sigma_i^2 = a_i \tau^2$ and $\sigma_i^{*2} = a_i^* \tau^2$, for $i = 1, \dots, k$, where the a_i 's and a_i^* 's are assumed to be known and τ^2 is unknown.

In this case, the unknown baseline parameter is the same (τ^2) for the two groups. This assumption is typically suitable when the two groups are of similar structure in the responses measured. One example of this type of responses could be the intelligence of male and female groups. When $a_i = a_i^* = 1$ for $i = 1, \dots, k$, we have the equal variances assumption commonly seen in analysis of variance models.

Throughout this chapter we set

$$\mathbf{w}_n = \left(\frac{n_1}{a_1}, \dots, \frac{n_k}{a_k} \right) \text{ and } \mathbf{w}_n^* = \left(\frac{n_1^*}{a_1^*}, \dots, \frac{n_k^*}{a_k^*} \right). \quad (3.1.1)$$

Note that this is different from the notation used in Section 2.1.

3.2 Likelihood Ratio Test for Case (1)

In this case the samples from the two umbrella orderings are independent and they do not have common parameters. Therefore, we can treat this case as the product of two one way layouts with umbrella orderings. The likelihood function is then

$$L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) = L(\bar{\mathbf{x}}, \mu, \tau^2) L^*(\bar{\mathbf{x}}^*, \mu^*, \tau^{*2}), \quad (3.2.1)$$

where

$$L(\bar{\mathbf{x}}, \mu, \tau^2) = (2\pi\tau^2)^{-N/2} \prod_{i=1}^k a_i^{-n_i/2} \exp\left[-\frac{1}{2\tau^2} \sum_{i=1}^k a_i^{-1} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)^2\right] \quad (3.2.2)$$

is the likelihood function for the sample from the first umbrella ordering and

$$L^*(\bar{\mathbf{x}}^*, \mu^*, \tau^{*2}) = (2\pi\tau^{*2})^{-N^*/2} \prod_{i=1}^k a_i^{*-n_i^*/2} \exp\left[-\frac{1}{2\tau^{*2}} \sum_{i=1}^k a_i^{*-1} \sum_{j=1}^{n_i^*} (x_{ij}^* - \mu_i^*)^2\right] \quad (3.2.3)$$

is the likelihood function for the sample from the second ordering. Using (2.2.9) in

Robertson *et al* (1988), we have that for any pair (i, j) , $i, j=1, \dots, k$,

$$\begin{aligned} & \max_{\mu \in \varphi_i, \mu^* \in \varphi_j} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) \\ &= \max_{\mu \in \varphi_i} L(\bar{\mathbf{x}}, \mu, \tau^2) \times \max_{\mu^* \in \varphi_j} L^*(\bar{\mathbf{x}}^*, \mu^*, \tau^{*2}) \\ &= (2\pi\hat{\tau}_{(i)}^2)^{-N/2} (2\pi\hat{\tau}_{(j)}^{*2})^{-N^*/2} e^{-(N+N^*)/2} \prod_{s=1}^k (a_s^{-n_s/2} a_s^{*-n_s^*/2}), \end{aligned} \quad (3.2.4)$$

where

$$\hat{\tau}_{(i)}^2 = \frac{1}{N} \left[\|\bar{\mathbf{x}} - P_{\mathbf{w}_n}(\bar{\mathbf{x}} | \varphi_i)\|_{\mathbf{w}_n}^2 + \sum_{s=1}^k \frac{1}{a_s} \sum_{t=1}^{n_s} (x_{st} - \bar{x}_s)^2 \right] \quad (3.2.5)$$

and

$$\hat{\tau}_{(j)}^{*2} = \frac{1}{N^*} \left[\|\bar{\mathbf{x}}^* - P_{\mathbf{w}_{n^*}^*}(\bar{\mathbf{x}}^* | \varphi_j)\|_{\mathbf{w}_{n^*}^*}^2 + \sum_{s=1}^k \frac{1}{a_s^*} \sum_{t=1}^{n_s^*} (x_{st}^* - \bar{x}_s^*)^2 \right]. \quad (3.2.6)$$

By (3.2.4), (3.2.5) and (3.2.6), we have

$$\begin{aligned} & -2 \ln \left(\frac{\max_{(\mu, \mu^*) \in H_0} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2})}{\max_{(\mu, \mu^*) \in H_1} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2})} \right) \\ &= -2 \left[\ln \left(\max_{(\mu, \mu^*) \in H_0} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) \right) - \ln \left(\max_{(\mu, \mu^*) \in H_1} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) \right) \right] \\ &= -2 \left[\max_{p=1, \dots, k} \left\{ \ln \left(\max_{\mu \in \varphi_p, \mu^* \in \varphi_p^*} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) \right) \right\} \right. \\ & \quad \left. - \max_{p \neq p^*} \left\{ \ln \left(\max_{\mu \in \varphi_p, \mu^* \in \varphi_{p^*}^*} L_2(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2, \tau^{*2}) \right) \right\} \right] \\ &= -2 \left[\max_{p=1, \dots, k} \left\{ \ln \left((2\pi \hat{\tau}_{(p)}^2)^{-N/2} (2\pi \hat{\tau}_{(p)}^{*2})^{-N^*/2} e^{-(N+N^*)/2} \prod_{s=1}^k (a_s^{-n_s/2} a_s^{*-n_s^*/2}) \right) \right\} \right. \\ & \quad \left. - \max_{p \neq p^*} \left\{ \ln \left((2\pi \hat{\tau}_{(p)}^2)^{-N/2} (2\pi \hat{\tau}_{(p^*)}^{*2})^{-N^*/2} e^{-(N+N^*)/2} \prod_{s=1}^k (a_s^{-n_s/2} a_s^{*-n_s^*/2}) \right) \right\} \right] \\ &= 2 \left[\max_{p=1, \dots, k} \left\{ \frac{N}{2} \ln(\hat{\tau}_{(p)}^2) + \frac{N^*}{2} \ln(\hat{\tau}_{(p)}^{*2}) \right\} \right. \\ & \quad \left. - \max_{p \neq p^*} \left\{ \frac{N}{2} \ln(\hat{\tau}_{(p)}^2) + \frac{N^*}{2} \ln(\hat{\tau}_{(p^*)}^{*2}) \right\} \right] \end{aligned}$$

$$\begin{aligned}
&= 2 \left[\max_{p=1, \dots, k} \left\{ \frac{N}{2} \ln \left(\frac{\hat{\tau}_{(p)}^2}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} \right) + \frac{N^*}{2} \ln \left(\frac{\hat{\tau}_{(p)}^{*2}}{\sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} \right) \right\} \right. \\
&\quad \left. - \max_{p \neq p^*} \left\{ \frac{N}{2} \ln \left(\frac{\hat{\tau}_{(p)}^2}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} \right) + \frac{N^*}{2} \ln \left(\frac{\hat{\tau}_{(p)}^{*2}}{\sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} \right) \right\} \right]. \quad (3.2.7)
\end{aligned}$$

Therefore, the likelihood ratio test rejects H_0 for large values of

$$\begin{aligned}
T_2 = \min_{p=1, \dots, k} &\left\{ N \ln \left[\frac{\|\bar{X} - P_{w_n}(\bar{X}|\varphi_p)\|_{w_n}^2}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} + 1 \right] + N^* \ln \left[\frac{\|\bar{X}^* - P_{w_n^*}(\bar{X}^*|\varphi_p)\|_{w_n^*}^2}{\sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right] \right\} \\
&- \min_{p \neq p^*} \left\{ N \ln \left[\frac{\|\bar{X} - P_{w_n}(\bar{X}|\varphi_p)\|_{w_n}^2}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} + 1 \right] + N^* \ln \left[\frac{\|\bar{X}^* - P_{w_n^*}(\bar{X}^*|\varphi_p)\|_{w_n^*}^2}{\sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right] \right\}. \quad (3.2.8)
\end{aligned}$$

For significance level α , $0 < \alpha < 1$, let $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ be the number such that

$$\max_{(\mu, \mu^*) \in H_0} P(T_2 \geq c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = \alpha. \quad (3.2.9)$$

Then, we have the following result:

Theorem 3.2.1 Under the assumptions of case (1), the level α likelihood ratio test for Q_1 ((1.4.1)) rejects H_0 if and only if T_2 is greater than or equal to $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$.

The test statistic T_2 can be computed using the *minimum violator algorithm* due to Thompson (1962), and described in Barlow *et al* (1972). FORTRAN programs to compute T_2 are available from the author.

The determination of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is difficult due to the large region occupied by the null hypothesis space H_0 . However, simulation results clearly indicate the appropriate solution for the following situations commonly assumed in practice:

$$\frac{a_1}{n_1} = \dots = \frac{a_k}{n_k} = \rho^2 \text{ and } \frac{a_1^*}{n_1^*} = \dots = \frac{a_k^*}{n_k^*} = \rho^{*2}. \quad (3.2.10)$$

Without loss of generality, we always assume $N \geq N^*$ for cases (3.2.10) in the following discussion. The determination of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for cases not in (3.2.10) is discussed at the end of this section.

Before we can describe the solution, we need to investigate the power function of the test based on T_2 . In this investigation, we use differences between adjacent treatment levels: $\delta_i = \mu_i - \mu_{i-1}$ and $\delta_i^* = \mu_i^* - \mu_{i-1}^*$, for $i=2, \dots, k$. Theorem 3.2.2 shows that the power study of the test based on T_2 can be simplified for cases (3.2.10). In the theorem, we let $\mathbf{Z} = (Z_1, \dots, Z_k)$ and $\mathbf{Z}^* = (Z_1^*, \dots, Z_k^*)$, where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are i.i.d. $N(0, 1)$ variables. Also let $Q_{(N-k)} \sim \chi_{(N-k)}^2$ and $Q_{(N^*-k)}^* \sim \chi_{(N^*-k)}^2$ where $\chi_{(v)}^2$ denotes a central chi-square variable with v degrees of freedom. Further assume that $Q_{(N-k)}$ is independent of $Q_{(N^*-k)}^*$ and that $Q_{(N-k)}$ and $Q_{(N^*-k)}^*$ are mutually independent of $\mathbf{Z}$ and $\mathbf{Z}^*$.

Theorem 3.2.2 If condition (3.2.10) is satisfied, then at parameter configuration (μ, μ^*) ,

$$\begin{aligned}
 T_2 \stackrel{D}{=} \min_{p=1, \dots, k} & \left\{ N \ln \left[\frac{\|(\mathbf{Z} + \vartheta) - P_{1_k}((\mathbf{Z} + \vartheta)|\varphi_p)\|_{1_k}^2}{Q_{(N-k)}} + 1 \right] \right. \\
 & \left. + N^* \ln \left[\frac{\|(\mathbf{Z}^* + \vartheta^*) - P_{1_k}((\mathbf{Z}^* + \vartheta^*)|\varphi_p)\|_{1_k}^2}{Q_{(N^*-k)}^*} + 1 \right] \right\} \\
 - \min_{p \neq p} & \left\{ N \ln \left[\frac{\|(\mathbf{Z} + \vartheta) - P_{1_k}((\mathbf{Z} + \vartheta)|\varphi_p)\|_{1_k}^2}{Q_{(N-k)}} + 1 \right] \right. \\
 & \left. + N^* \ln \left[\frac{\|(\mathbf{Z}^* + \vartheta^*) - P_{1_k}((\mathbf{Z}^* + \vartheta^*)|\varphi_p)\|_{1_k}^2}{Q_{(N^*-k)}^*} + 1 \right] \right\}. \quad (3.2.11)
 \end{aligned}$$

by setting

$$\vartheta = (0, \Delta_2, \Delta_2 + \Delta_3, \dots, \Delta_2 + \Delta_3 + \dots + \Delta_k) \quad (3.2.12)$$

and

$$\vartheta^* = (0, \Delta_2^*, \Delta_2^* + \Delta_3^*, \dots, \Delta_2^* + \Delta_3^* + \dots + \Delta_k^*), \quad (3.2.13)$$

with

$$\Delta_i = \frac{1}{\rho\tau} \delta_i \text{ and } \Delta_i^* = \frac{1}{\rho^*\tau^*} \delta_i^*, \quad i = 2, \dots, k. \quad (3.2.14)$$

Proof. We first describe the distribution of the test statistic T_2 for any general case. Let

$$\mathbf{U}_n \stackrel{\Delta}{=} \left(\sqrt{\frac{a_1}{n_1}} Z_1, \dots, \sqrt{\frac{a_k}{n_k}} Z_k \right) \quad (3.2.15)$$

be independent of

$$\mathbf{U}_n^* = (\sqrt{\frac{a_1^*}{n_1^*}} Z_1^*, \dots, \sqrt{\frac{a_k^*}{n_k^*}} Z_k^*), \quad (3.2.16)$$

where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are i.i.d. $N(0, 1)$. Let $Q_{(N-k)} \sim \chi_{(N-k)}^2$ and $Q_{(N^*-k)}^* \sim \chi_{(N^*-k)}^2$

be independent and furthermore mutually independent of $\mathbf{U}_n$ and $\mathbf{U}_n^*$. By the fact that

$$\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2, \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2, \bar{X} \text{ and } \bar{X}^*$$

are mutually independent and the facts that

$$\tau \mathbf{U}_n + \mu \stackrel{D}{=} \bar{X}, \tau^* \mathbf{U}_n^* + \mu^* \stackrel{D}{=} \bar{X}^*, \quad (3.2.17)$$

and

$$\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \sim \tau^2 \chi_{(N-k)}^2 \text{ and } \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2 \sim \tau^{*2} \chi_{(N^*-k)}^2, \quad (3.2.18)$$

it can be seen that at parameter configuration (μ, μ^*) ,

$$\begin{aligned}
T_2^D = \min_{p=1, \dots, k} & \left\{ N \ln \left[\frac{\|(\mathbf{U}_n + \frac{1}{\tau}\mu) - P_{w_n}((\mathbf{U}_n + \frac{1}{\tau}\mu)|\varphi_p)\|_{w_n}^2}{Q_{(N-k)}} + 1 \right] \right. \\
& \left. + N^* \ln \left[\frac{\|(\mathbf{U}_{n^*}^* + \frac{1}{\tau^*}\mu^*) - P_{w_{n^*}^*}((\mathbf{U}_{n^*}^* + \frac{1}{\tau^*}\mu^*)|\varphi_p)\|_{w_{n^*}^*}^2}{Q_{(N^*-k)}^*} + 1 \right] \right\} \\
& - \min_{p \neq p^*} \left\{ N \ln \left[\frac{\|(\mathbf{U}_n + \frac{1}{\tau}\mu) - P_{w_n}((\mathbf{U}_n + \frac{1}{\tau}\mu)|\varphi_p)\|_{w_n}^2}{Q_{(N-k)}} + 1 \right] \right. \\
& \left. + N^* \ln \left[\frac{\|(\mathbf{U}_{n^*}^* + \frac{1}{\tau^*}\mu^*) - P_{w_{n^*}^*}((\mathbf{U}_{n^*}^* + \frac{1}{\tau^*}\mu^*)|\varphi_{p^*})\|_{w_{n^*}^*}^2}{Q_{(N^*-k)}^*} + 1 \right] \right\}.
\end{aligned}
\tag{3.2.19}$$

Using an argument similar to that in the proof of Theorem 2.2.1, it can be seen that for cases (3.2.10), the distribution of T_2 at parameter configuration (μ, μ^*) is the same as that of

$$\begin{aligned}
& \min_{p=1, \dots, k} \left\{ N \ln \left[\frac{\|(\mathbf{Z} + \frac{1}{\rho\tau}\mu) - P_{I_k}((\mathbf{Z} + \frac{1}{\rho\tau}\mu)|\varphi_p)\|_{I_k}^2}{Q_{(N-k)}} + 1 \right] \right. \\
& \quad \left. + N^* \ln \left[\frac{\|(\mathbf{Z}^* + \frac{1}{\rho^*\tau^*}\mu^*) - P_{I_k}((\mathbf{Z}^* + \frac{1}{\rho^*\tau^*}\mu^*)|\varphi_p)\|_{I_k}^2}{Q_{(N^*-k)}} + 1 \right] \right\} \\
& - \min_{p \neq p^*} \left\{ N \ln \left[\frac{\|(\mathbf{Z} + \frac{1}{\rho\tau}\mu) - P_{I_k}((\mathbf{Z} + \frac{1}{\rho\tau}\mu)|\varphi_p)\|_{I_k}^2}{Q_{(N-k)}} + 1 \right] \right. \\
& \quad \left. + N^* \ln \left[\frac{\|(\mathbf{Z}^* + \frac{1}{\rho^*\tau^*}\mu^*) - P_{I_k}((\mathbf{Z}^* + \frac{1}{\rho^*\tau^*}\mu^*)|\varphi_{p^*})\|_{I_k}^2}{Q_{(N^*-k)}} + 1 \right] \right\}. \quad (3.2.20)
\end{aligned}$$

Theorem 3.2.2 follows from (2.2.16).

Notice that the distribution of the test statistic T_2 is the same for all those cases (3.2.10) with the same (N, N^*) . Therefore, within the sub-class of cases (3.2.10) with the same (N, N^*) , it suffices to study the power function for the case when

$$\rho = \rho^* = \tau = \tau^* = 1, \quad (3.2.21)$$

or rather,

$$\frac{a_1}{n_1} = \dots = \frac{a_k}{n_k} = \frac{a_1^*}{n_1^*} = \dots = \frac{a_k^*}{n_k^*} = 1 \text{ and } \tau = \tau^* = 1. \quad (3.2.22)$$

Theorem 3.2.2 gives the distribution of the test statistic T_2 for this simple case, which will be used later in conducting our simulation study. For distinction, we use upper case

notations such as Δ , Δ_i and Δ_i^* for the special cases (3.2.22) and lower case notations such as δ , δ_i and δ_i^* for any general case satisfying (3.2.10).

The results of a simulation study (to be discussed in Chapter 5) suggest the following important observation:

Observation 3.2.1 The critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for cases (3.2.22) can be obtained from the distribution of T_2 at parameter configurations where

$$\Delta_2 = \dots = \Delta_{k-1} = 0, \Delta_k = +\infty \text{ and } \Delta_2^* = \dots = \Delta_k^* = 0. \quad (3.2.23)$$

Observation 3.2.1 combined with Theorem 3.2.2 solves the problem of finding the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for any general case in (3.2.10). The solution depends on the distribution of the test statistic T_2 at slippage configurations (SC) where

$$\delta_2 = \dots = \delta_{k-1} = 0, \delta_k = +\infty \text{ and } \delta_2^* = \dots = \delta_k^* = 0. \quad (3.2.24)$$

It follows from (2.2.14) and (2.2.16) that the distribution of T_2 is the same at all of the slippage configurations (3.2.24). Therefore, for the purpose of critical point determination, we need only investigate the distribution of T_2 at a single slippage configuration: $\mu_1 = \dots = \mu_{k-1} = \mu_1^* = \dots = \mu_k^* = 0$ and $\mu_k = +\infty$. In fact, this is the parameter configuration that we use in our simulation study. Let $s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ be the upper α percentile of the distribution of the test statistic T_2 at slippage configurations (3.2.24); that is, $s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is the number such that

$$P_{\text{SC}}(T_2 \geq s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = \alpha. \quad (3.2.25)$$

Theorem 3.2.3 shows that $s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ depends only on N^* for cases (3.2.10). For convenience, we let $S_2(\alpha, N^*)$ denote the upper α percentile of the distribution of T_2 at

parameter configurations (3.2.23) for cases (3.2.22) with total sample size N^* from the second umbrella ordering. Then we have the following important result:

Corollary 3.2.1 For any case (3.2.10), the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ can be obtained from the distribution of T_2 at slippage configurations (3.2.24). In other words, we conclude that

$$c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*) = s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*). \quad (3.2.26)$$

Furthermore, we have

$$s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*) = S_2(\alpha, N^*). \quad (3.2.27)$$

Proof. The proof is similar to that of Corollary 2.2.1.

By Corollary 3.2.1, only one table of $S_2(\alpha, N^*)$ values is needed for the critical points for all the cases (3.2.10). The critical point $S_2(\alpha, N^*)$ can be obtained by simulation using the distribution of T_2 at slippage configurations (3.2.24), as given in the next theorem.

Theorem 3.2.3 If condition (3.2.10) is satisfied, then the distribution of T_2 at slippage configurations is the same as that of

$$N^* \left\{ \ln \left[\frac{\| \mathbf{Z}^* - \mathbf{P}_{\mathbf{I}_k}(\mathbf{Z}^* | \varphi_k) \|^2_{\mathbf{I}_k}}{Q_{(N^*-k)}^*} + 1 \right] - \min_{p^*=1, \dots, k-1} \ln \left[\frac{\| \mathbf{Z}^* - \mathbf{P}_{\mathbf{I}_k}(\mathbf{Z}^* | \varphi_{p^*}) \|^2_{\mathbf{I}_k}}{Q_{(N^*-k)}^*} + 1 \right] \right\}, \quad (3.2.28)$$

where $\mathbf{Z}^* = (Z_1^*, \dots, Z_k^*)$ with $Z_1^*, \dots, Z_k^*$ being i.i.d. $N(0, 1)$ variables are mutually independent of $Q_{(N^*-k)}^* \sim \chi^2_{(N^*-k)}$.

Proof. The result follows from an argument similar to that in the proof of Theorem 2.2.2.

In Table 3 we provide estimates of $S_2(\alpha, N^*)$ for selected significance levels α and sample sizes N^* with $N \geq N^*$.

The next theorem supports the conjecture that slippage configurations (3.2.24) yield the proper critical points by showing that such configurations are asymptotically the most difficult null hypothesis configurations for the likelihood ratio test based on T_2 to distinguish from the alternatives in H_1 .

Theorem 3.2.4 Under assumptions (2.2.32)-(2.2.34) in Section 2.2, we have, for fixed μ and μ^* , that

$$\begin{aligned}
 0 &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_2 \geq s_2(\alpha, n, n^*, a, a^*)) \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{SC} (T_2 \geq s_2(\alpha, n, n^*, a, a^*)) = \alpha \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_2 \geq s_2(\alpha, n, n^*, a, a^*)) = 1.
 \end{aligned} \tag{3.2.29}$$

Proof. It is easy to see from (3.2.8) that

$$\begin{aligned}
\frac{T_2}{N+N^*} = \min_{p=1, \dots, k} & \left\{ \frac{N}{N+N^*} \ln \left[\frac{\|\bar{\mathbf{X}} - P_{\mathbf{w}_n}(\bar{\mathbf{X}}|\phi_p)\|_{\mathbf{w}_n/N}^2}{\frac{1}{N} \sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} + 1 \right] \right. \\
& \left. + \frac{N^*}{N+N^*} \ln \left[\frac{\|\bar{\mathbf{X}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{X}}^*|\phi_p)\|_{\mathbf{w}_n^*/N^*}^2}{\frac{1}{N^*} \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right] \right\} \\
- \min_{p \neq p^*} & \left\{ \frac{N}{N+N^*} \ln \left[\frac{\|\bar{\mathbf{X}} - P_{\mathbf{w}_n}(\bar{\mathbf{X}}|\phi_p)\|_{\mathbf{w}_n/N}^2}{\frac{1}{N} \sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2} + 1 \right] \right. \\
& \left. + \frac{N^*}{N+N^*} \ln \left[\frac{\|\bar{\mathbf{X}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{X}}^*|\phi_{p^*})\|_{\mathbf{w}_n^*/N^*}^2}{\frac{1}{N^*} \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right] \right\}.
\end{aligned} \tag{3.2.30}$$

Notice that, as $N, N^* \rightarrow \infty$,

$$\frac{\mathbf{w}_n}{N} \rightarrow \left(\frac{\lambda_1}{a_1}, \dots, \frac{\lambda_k}{a_k} \right) \triangleq \mathbf{w}_\infty, \tag{3.2.31}$$

$$\frac{\mathbf{w}_n^*}{N^*} \rightarrow \left(\frac{\lambda_1^*}{a_1^*}, \dots, \frac{\lambda_k^*}{a_k^*} \right) \triangleq \mathbf{w}_\infty^*, \tag{3.2.32}$$

$$\bar{\mathbf{X}} \xrightarrow{P} \mu, \tag{3.2.33}$$

$$\bar{\mathbf{X}}^* \xrightarrow{P} \mu^*, \tag{3.2.34}$$

$$\frac{1}{N} \sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \xrightarrow{P} \tau^2 \tag{3.2.35}$$

and

$$\frac{1}{N^*} \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2 \xrightarrow{P} \tau^{*2}. \quad (3.2.36)$$

By the continuity of $P_w(g|\varphi_p)$ and $\|g - x\|_w^2$, we have that as $N, N^* \rightarrow \infty$, $\frac{T_2}{N + N^*}$

converges in probability to the constant

$$\begin{aligned} J_2 = & \min_{p=1, \dots, k} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\ & - \min_{p \neq p^*} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\}. \end{aligned} \quad (3.2.37)$$

Notice that slippage configurations (3.2.24) satisfy

$$\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 = +\infty, \quad \text{for } p=1, \dots, k-1 \quad (3.2.38)$$

and

$$\|\mu^* - P_{w_-}(\mu^*|\varphi_{p^*})\|_{w_-}^2 = 0. \quad (3.2.39)$$

Therefore, under slippage configurations (3.2.24),

$$J_2 = 0. \quad (3.2.40)$$

It follows from (3.2.25) and an argument similar to that in the proof of Theorem 2.2.3

that

$$\frac{s_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)}{N + N^*} \rightarrow 0, \text{ as } N, N^* \rightarrow \infty. \quad (3.2.41)$$

Now we consider two cases. First, suppose $(\mu, \mu^*) \in H_0$ so that there exists a $p_0 \in \{1, \dots, k\}$ such that

$$\mu_1 < \dots < \mu_{p_0-1} < \mu_{p_0} > \mu_{p_0+1} > \dots > \mu_k$$

and

$$\mu_1^* < \dots < \mu_{p_0-1}^* < \mu_{p_0}^* > \mu_{p_0+1}^* > \dots > \mu_k^*.$$

Hence

$$\|\mu - P_{w_-}(\mu|\varphi_{p_0})\|_{w_-}^2 = 0 \text{ and } \|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2 > 0 \text{ for } p \neq p_0 \quad (3.2.42)$$

and

$$\|\mu^* - P_{w_-}(\mu^*|\varphi_{p_0})\|_{w_-}^2 = 0 \text{ and } \|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2 > 0 \text{ for } p \neq p_0. \quad (3.2.43)$$

Therefore

$$\begin{aligned} & \min_{p=1, \dots, k} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu|\varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1-\gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^*|\varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\ &= \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu|\varphi_{p_0})\|_{w_-}^2}{\tau^2} + 1 \right] + (1-\gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^*|\varphi_{p_0})\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\ &= 0 \end{aligned} \quad (3.2.44)$$

and

$$\begin{aligned}
& \min_{p \neq p^*} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^* | \varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\
& = \min \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right], (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^* | \varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right], p \neq p_0, p^* \neq p_0 \right\} \\
& > 0.
\end{aligned} \tag{3.2.45}$$

By (3.2.37), we have that $J_2 < 0$ for any fixed $(\mu, \mu^*) \in H_0$.

For the second case, let $(\mu, \mu^*) \in H_1$ so that there exists $p_0 \neq p_0^* = 1, \dots, k$ such that

$$\mu_1 < \dots < \mu_{p_0-1} < \mu_{p_0} > \mu_{p_0+1} > \dots > \mu_k$$

and

$$\mu_1^* < \dots < \mu_{p_0^*-1}^* < \mu_{p_0^*}^* > \mu_{p_0^*+1}^* > \dots > \mu_k^*.$$

Hence

$$\|\mu - P_{w_-}(\mu | \varphi_{p_0})\|_{w_-}^2 = 0 \text{ and } \|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2 > 0 \text{ for } p \neq p_0 \tag{3.2.46}$$

and

$$\|\mu^* - P_{w_-}(\mu^* | \varphi_{p_0^*})\|_{w_-}^2 = 0 \text{ and } \|\mu^* - P_{w_-}(\mu^* | \varphi_p)\|_{w_-}^2 > 0 \text{ for } p \neq p_0^*. \tag{3.2.47}$$

Therefore

$$\min_{p=1, \dots, k} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^* | \varphi_p)\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} > 0 \tag{3.2.48}$$

and

$$\begin{aligned}
0 &\leq \min_{p \neq p^*} \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu | \varphi_p)\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^* | \varphi_{p^*})\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\
&= \left\{ \gamma \ln \left[\frac{\|\mu - P_{w_-}(\mu | \varphi_{p_0})\|_{w_-}^2}{\tau^2} + 1 \right] + (1 - \gamma) \ln \left[\frac{\|\mu^* - P_{w_-}(\mu^* | \varphi_{p_0})\|_{w_-}^2}{\tau^{*2}} + 1 \right] \right\} \\
&= 0.
\end{aligned} \tag{3.2.49}$$

Hence, by (3.2.37), we have that $J_2 > 0$ for any fixed $(\mu, \mu^*) \in H_1$. Theorem 3.2.4 then follows from an argument similar to that in the proof of Theorem 2.2.3.

The determination of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for any case not in (3.2.10) is more complicated. Simulation results indicate that the level α critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is the maximum of the upper α percentiles of the distributions of the test statistic T_2 under parameter configurations given by

$$\mu_1 = \dots = \mu_{i-1} = 0, \mu_i = +\infty, \mu_{i+1} = \dots = \mu_k = 0, \mu_1^* = \dots = \mu_k^* = 0, i=1, \dots, k, \tag{3.2.50}$$

and

$$\mu_1 = \dots = \mu_k = 0, \mu_1^* = \dots = \mu_{j-1}^* = 0, \mu_j^* = +\infty, \mu_{j+1}^* = \dots = \mu_k^* = 0, j=1, \dots, k. \tag{3.2.51}$$

The distributions of T_2 under parameter configurations (3.2.50) and (3.2.51) are given in Theorem 3.2.5. It can be used to obtain simulation estimates of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$.

Theorem 3.2.5 Under parameter configurations (3.2.50) with fixed i ,

$$T_2^D = N^* \ln \left[\frac{\|U_n^* - P_{w_n^*}(U_n^* | \varphi_i)\|_{w_n^*}^2}{Q_{(N^*-k)}^*} + 1 \right] - \min_{p \neq i} \left\{ N^* \ln \left[\frac{\|U_n^* - P_{w_n^*}(U_n^* | \varphi_p)\|_{w_n^*}^2}{Q_{(N^*-k)}^*} + 1 \right] \right\}, \quad (3.2.52)$$

and under parameter configurations (3.2.51) with fixed j ,

$$T_2^D = N \ln \left[\frac{\|U_n - P_{w_n}(U_n | \varphi_j)\|_{w_n}^2}{Q_{(N-k)}} + 1 \right] - \min_{p \neq j} \left\{ N \ln \left[\frac{\|U_n - P_{w_n}(U_n | \varphi_p)\|_{w_n}^2}{Q_{(N-k)}} + 1 \right] \right\}, \quad (3.2.53)$$

where w_n, w_n^*, U_n and U_n^* are defined in (3.1.1), (3.2.15) and (3.2.16) and where $Q_{(N-k)} \sim \chi_{(N-k)}^2$ and $Q_{(N^*-k)}^* \sim \chi_{(N^*-k)}^2$ are independent variables that are also mutually independent of U_n and U_n^* .

Proof. The proof is similar to that of Theorem 2.2.2.

Simulation results indicate that the critical point $c_2(\alpha, n, n^*, a, a^*)$ is in fact the maximum of the upper α percentiles of the distributions of the test statistic T_2 under parameter configurations (3.2.50) with $i=1, k$ and (3.2.51) with $j=1, k$ (the symmetric transformations of the slippage configurations (3.2.24) for most cases where the weights (3.1.1) are not extremely unbalanced. The maximum of the upper α percentiles of the distributions of the test statistic T_2 under parameter configurations (3.2.50) with $i=1, k$ and (3.2.51) with $j=1, k$ is very close to, and thus a good estimate of, the critical point $c_2(\alpha, n, n^*, a, a^*)$ for any case.

A FORTRAN program that provides a simulation estimate of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$, computes the test statistic T_1 and gives the P-value for any case of $\mathbf{n}, \mathbf{n}^*, \mathbf{a}$ and $\mathbf{a}^*$ is available from the author. In the program, the user will need to input the number of iterations, or precision, desired for the simulation estimate of the critical point, the significance level of interest, the number of treatment levels k , the vectors of sample sizes, the vectors of sample means, the vectors of sample variances and the vectors $\mathbf{a}$ and $\mathbf{a}^*$. The output is the value of the test statistic, the simulated critical point and the P-value.

3.3 Likelihood Ratio Test for Case (2)

In this case the likelihood function is given by (3.2.1), (3.2.2) and (3.2.3) with $\tau^{*2} = \tau^2$. That is,

$$L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2) = L(\bar{\mathbf{x}}, \mu, \tau^2) L^*(\bar{\mathbf{x}}^*, \mu^*, \tau^2), \quad (3.3.1)$$

where $L(\bar{\mathbf{x}}, \mu, \tau^2)$ and $L^*(\bar{\mathbf{x}}^*, \mu^*, \tau^2)$ are defined in (3.2.2) and (3.2.3), respectively. By an argument similar to that leading to (2.2.9), (2.2.10) and (2.2.11) in Robertson *et al* (1988), it can be seen that for any pair (i, j) , $i, j=1, \dots, k$,

$$\max_{\mu \in \Phi_i, \mu^* \in \Phi_j} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2) = (2\pi \hat{\tau}_{(i,j)}^2)^{-(N+N^*)/2} e^{-(N+N^*)/2} \prod_{i=1}^k (a_i^{-n_i/2} a_i^{*-n_i^*/2}), \quad (3.3.2)$$

where

$$\hat{\tau}_{(i,j)}^2 = \frac{1}{N+N^*} \left[\|\bar{\mathbf{x}} - \mathbf{P}_{\mathbf{w}_n}(\bar{\mathbf{x}}|\varphi_i)\|_{\mathbf{w}_n}^2 + \sum_{s=1}^k \frac{1}{a_s} \sum_{t=1}^{n_s} (x_{st} - \bar{x}_s)^2 + \right.$$

$$\left[\|\bar{\mathbf{x}}^* - P_{w_n^*}(\bar{\mathbf{x}}^* | \varphi_j)\|_{w_n^*}^2 + \sum_{s=1}^k \frac{1}{a_s^*} \sum_{t=1}^{n_i^*} (x_{st}^* - \bar{x}_s^*)^2 \right]. \quad (3.3.3)$$

Since

$$\begin{aligned} & -2 \ln \left(\frac{\max_{(\mu, \mu^*) \in H_0} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2)}{\max_{(\mu, \mu^*) \in H_1} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2)} \right) \\ &= (N + N^*) \ln \left(\frac{\max_{(\mu, \mu^*) \in H_0} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2)}{\max_{(\mu, \mu^*) \in H_1} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2)} \right)^{-2/(N+N^*)} \\ &= (N + N^*) \ln \left(\frac{\max_{p=1, \dots, k} \left\{ \max_{\mu \in \varphi_p, \mu^* \in \varphi_p} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2) \right\}}{\max_{p \neq p'} \left\{ \max_{\mu \in \varphi_p, \mu^* \in \varphi_{p'}} L_3(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*, \mu, \mu^*, \tau^2) \right\}} \right)^{-2/(N+N^*)} \\ &= (N + N^*) \ln \left(\frac{\min_{p=1, \dots, k} \hat{\tau}_{(p,p)}^2}{\min_{p \neq p'} \hat{\tau}_{(p,p')}^2} \right), \end{aligned} \quad (3.3.4)$$

the likelihood ratio test rejects H_0 for large values of

$$\begin{aligned} & (N + N^*) \left[\ln \left(\frac{\min_{p=1, \dots, k} \left\{ \|\bar{\mathbf{X}} - P_{w_n}(\bar{\mathbf{X}} | \varphi_p)\|_{w_n}^2 + \|\bar{\mathbf{X}}^* - P_{w_n^*}(\bar{\mathbf{X}}^* | \varphi_p)\|_{w_n^*}^2 \right\}}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 + \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right) \right. \\ & \quad \left. - \ln \left(\frac{\min_{p \neq p'} \left\{ \|\bar{\mathbf{X}} - P_{w_n}(\bar{\mathbf{X}} | \varphi_p)\|_{w_n}^2 + \|\bar{\mathbf{X}}^* - P_{w_n^*}(\bar{\mathbf{X}}^* | \varphi_{p'})\|_{w_n^*}^2 \right\}}{\sum_{i=1}^k \frac{1}{a_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 + \sum_{i=1}^k \frac{1}{a_i^*} \sum_{j=1}^{n_i^*} (X_{ij}^* - \bar{X}_i^*)^2} + 1 \right) \right]. \end{aligned} \quad (3.3.5)$$

For significance level α , $0 < \alpha < 1$, let $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ be the number such that

$$\max_{(\mu, \mu^*) \in H_0} P(T_3 \geq c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = \alpha. \quad (3.3.6)$$

Therefore, we have the following result.

Theorem 3.3.1 Under the assumptions in case (2), the level α likelihood ratio test for Q_1 ((1.4.1)) rejects H_0 if and only if T_3 is greater than or equal to $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$.

The test statistic T_3 can be computed using the *minimum violator algorithm* due to Thompson (1962) and described in Barlow *et al* (1972). FORTRAN programs to compute T_3 are available from the author.

The determination of the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is difficult due to the large region occupied by the null hypothesis space H_0 . However, simulation results clearly indicate the appropriate solution for the following situations commonly assumed in practice:

$$\frac{a_1}{n_1} = \dots = \frac{a_k}{n_k} = \rho^2 \text{ and } \frac{a_1^*}{n_1^*} = \dots = \frac{a_k^*}{n_k^*} = \rho^{*2}. \quad (3.3.7)$$

Without loss of generality, we always assume $N \geq N^*$ for cases (3.3.7) in the following discussion. The critical points for cases not in (3.3.7) are discussed at the end of this section.

The next theorem shows that the power study of the test based on T_2 can be simplified for cases (3.3.7). In the theorem, we let $\mathbf{Z} = (Z_1, \dots, Z_k)$ and $\mathbf{Z}^* = (Z_1^*, \dots, Z_k^*)$ where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are i.i.d. $N(0, 1)$. Also let $Q_{(N+N^*-2k)} \sim \chi_{(N+N^*-2k)}^2$, where $\chi_{(v)}^2$ denotes a central chi-square variable with v degrees of freedom. Assume that $Q_{(N+N^*-2k)}$ is independent of $\mathbf{Z}$ and $\mathbf{Z}^*$.

Theorem 3.3.2 If condition (3.3.7) is satisfied, then at parameter configuration (μ, μ^*) ,

$$T_3^D = (N + N^*) \times \left[\ln \left(\frac{\min_{p=1, \dots, k} \{ \|(\mathbf{Z} + \vartheta) - P_{1_k}((\mathbf{Z} + \vartheta)|\varphi_p)\|_{l_k}^2 + \|(\mathbf{Z}^* + \vartheta^*) - P_{1_k}((\mathbf{Z}^* + \vartheta^*)|\varphi_p)\|_{l_k}^2 \}}{Q_{(N+N^*-2k)}} + 1 \right) - \ln \left(\frac{\min_{p \neq p} \{ \|(\mathbf{Z} + \vartheta) - P_{1_k}((\mathbf{Z} + \vartheta)|\varphi_p)\|_{l_k}^2 + \|(\mathbf{Z}^* + \vartheta^*) - P_{1_k}((\mathbf{Z}^* + \vartheta^*)|\varphi_p)\|_{l_k}^2 \}}{Q_{(N+N^*-2k)}} + 1 \right) \right] \quad (3.3.8)$$

where

$$\vartheta = (0, \Delta_2, \Delta_2 + \Delta_3, \dots, \Delta_2 + \Delta_3 + \dots + \Delta_k) \quad (3.3.9)$$

and

$$\vartheta^* = (0, \Delta_2^*, \Delta_2^* + \Delta_3^*, \dots, \Delta_2^* + \Delta_3^* + \dots + \Delta_k^*), \quad (3.3.10)$$

with

$$\Delta_i = \frac{1}{\rho\tau} \delta_i \text{ and } \Delta_i^* = \frac{1}{\rho^*\tau} \delta_i^*, \quad i = 2, \dots, k \quad (3.3.11)$$

and

$$\delta_i = \mu_i - \mu_{i-1} \text{ and } \delta_i^* = \mu_i^* - \mu_{i-1}^*, \text{ for } i=2, \dots, k. \quad (3.3.12)$$

Proof. The proof is similar to that of Theorem 3.2.2.

Notice that the distribution of the statistic T_3 is the same for all the cases (3.3.1) with the same $d = N + N^*$. Therefore, within the sub-class of cases (3.3.7) with the same $d = N + N^*$, it suffices to study the power function for the case when

$$\rho = \rho^* = \tau = 1, \quad (3.3.13)$$

or rather,

$$\frac{a_1}{n_1} = \dots = \frac{a_k}{n_k} = \frac{a_1^*}{n_1^*} = \dots = \frac{a_k^*}{n_k^*} = 1 \text{ and } \tau = 1. \quad (3.3.14)$$

Theorem 3.3.2 give the distribution of the test statistic T_3 for this simple case, which will be used later to conduct our simulation study. For distinction, we use upper case notations such as Δ , Δ_i and Δ_i^* for the special cases (3.3.14) and lower case notations such as δ , δ_i and δ_i^* for any general case satisfying (3.3.7).

The results of a simulation study (to be discussed in Chapter 5) suggest the following important observation:

Observation 3.3.1 The critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for cases (3.3.14) can be obtained from the distribution of T_3 at parameter configurations where

$$\Delta_2 = \dots = \Delta_{k-1} = 0, \Delta_k = +\infty \text{ and } \Delta_2^* = \dots = \Delta_k^* = 0. \quad (3.3.15)$$

For convenience, we let $S_3(\alpha, d)$ denote the upper α percentile of the distribution of T_3 at parameter configurations (3.3.15) for cases (3.3.14) with total sample size $d = N + N^*$.

Observation 3.3.1 combined with Theorem 3.3.2 solves the problem of finding the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for any general case in (3.3.7). The answers depend on the distribution of the test statistic T_3 at slippage configurations (SC) where

$$\delta_2 = \dots = \delta_{k-1} = 0, \delta_k = +\infty \text{ and } \delta_2^* = \dots = \delta_k^* = 0. \quad (3.3.16)$$

It follows from (2.2.14) and (2.2.16) that the distribution of T_3 is the same at all of the slippage configurations (3.3.16). Therefore, for the purpose of critical point determination, we need only investigate the distribution of T_3 at a single slippage configuration: $\mu_1 = \dots = \mu_{k-1} = \mu_1^* = \dots = \mu_k^* = 0$ and $\mu_k = +\infty$. In fact, this is the parameter configuration that we use in our simulation study. Let $s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ be the upper α percentile of the distribution of the test statistic T_3 under slippage configurations (3.3.16); that is, $s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is the number such that

$$P_{sc}(T_3 \geq s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = \alpha. \quad (3.3.17)$$

Then we have the following important result:

Corollary 3.3.1 For any case (3.3.14), the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ can be obtained from the distribution of T_3 under slippage configurations (3.3.16). In other words, we conclude that

$$c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*) = s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*). \quad (3.3.18)$$

Furthermore, we have

$$s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*) = S_3(\alpha, N + N^*). \quad (3.3.19)$$

Therefore, only one table of $S_3(\alpha, d)$ values is needed for the critical points for all the cases (3.3.7). The critical point $S_3(\alpha, d)$ can be obtained by simulation using the distribution of T_3 under slippage configurations (3.3.16), as given in next theorem.

Theorem 3.3.3 If condition (3.3.7) is satisfied, then under slippage configurations (3.3.16),

$$T_3^D = (N + N^*) \left[\ln \left(\frac{\|Z - P_{i_{k-1}}(Z|\varphi_{k-1}^{k-1})\|_{i_{k-1}}^2 + \|Z^* - P_{i_k}(Z^*|\varphi_k)\|_{i_k}^2}{Q_{(N+N^*-2k)}} + 1 \right) \right. \\ \left. - \ln \left(\frac{\|Z - P_{i_{k-1}}(Z|\varphi_{k-1}^{k-1})\|_{i_{k-1}}^2 + \min_{p=1, \dots, k-1} \{ \|Z^* - P_{i_k}(Z^*|\varphi_p)\|_{i_k}^2 \}}{Q_{(N+N^*-2k)}} + 1 \right) \right], \quad (3.3.20)$$

where $Z = (Z_1, \dots, Z_{k-1})$ and $Z^* = (Z_1^*, \dots, Z_k^*)$, with $Z_1, \dots, Z_{k-1}, Z_1^*, \dots, Z_k^*$ being i.i.d. $N(0, 1)$ random variables, are independent of $Q_{(N+N^*-2k)} \sim \chi_{(N+N^*-2k)}^2$ and where φ_{k-1}^{k-1} is the set of umbrella orderings with peak $k-1$ in a $k-1$ sample problem.

Proof. The result follows from an argument similar to that in Theorem 2.2.2.

In Table 4 we provide simulation estimates of $S_3(\alpha, d)$ for selected significance levels α and total sample sizes $d = N + N^*$.

As in the previous two settings, it can be shown that the slippage configurations are asymptotically the most difficult null hypothesis configurations for the likelihood ratio test based on T_3 to distinguish from the alternatives in H_1 .

Theorem 3.3.4 Under assumptions (2.2.32)-(2.2.34) in Section 2.2, we have, for fixed μ and μ^* , that

$$\begin{aligned}
 0 &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_3 \geq s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{SC} (T_3 \geq s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = \alpha \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_3 \geq s_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)) = 1.
 \end{aligned} \tag{3.3.21}$$

Proof. The argument is similar to that used in the proof of Theorem 3.2.4.

The determination of the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for any case not in (3.3.14) is more complicated. Simulation results indicate that the level α critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is the maximum of the upper α percentiles of the distributions of the test statistic T_3 under parameter configurations given by

$$\mu_1 = \dots = \mu_{i-1} = 0, \mu_i = +\infty, \mu_{i+1} = \dots = \mu_k = 0, \mu_1^* = \dots = \mu_k^* = 0, i=1, \dots, k, \tag{3.3.22}$$

and

$$\mu_1 = \dots = \mu_k = 0, \mu_1^* = \dots = \mu_{j-1}^* = 0, \mu_j^* = +\infty, \mu_{j+1}^* = \dots = \mu_k^* = 0, j=1, \dots, k. \tag{3.3.23}$$

The distributions of T_3 under parameter configurations (3.3.22) and (3.3.23) are given in Theorem 3.3.5. It can be used to obtain simulation estimates of the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$. In Theorem 3.3.5, we use the following notations:

$$l_i = \left(\frac{n_1}{a_1}, \dots, \frac{n_{i-1}}{a_{i-1}} \right) \text{ and } u_i = \left(\frac{n_k}{a_k}, \dots, \frac{n_{i+1}}{a_{i+1}} \right), i=1, \dots, k, \tag{3.3.24}$$

$$l_j^* = \left(\frac{n_1^*}{a_1}, \dots, \frac{n_{j-1}^*}{a_{j-1}} \right) \text{ and } u_j^* = \left(\frac{n_k^*}{a_k}, \dots, \frac{n_{j+1}^*}{a_{j+1}} \right), j=1, \dots, k, \tag{3.3.25}$$

$$\mathbf{L}_i = (\sqrt{\frac{a_i}{n_i}} Z_1, \dots, \sqrt{\frac{a_{i-1}}{n_{i-1}}} Z_{i-1}) \text{ and } \mathbf{U}_i = (\sqrt{\frac{a_k}{n_k}} Z_k, \dots, \sqrt{\frac{a_{i+1}}{n_{i+1}}} Z_{i+1}), i=1, \dots, k, \quad (3.3.26)$$

and

$$\mathbf{L}_j^* = (\sqrt{\frac{a_j^*}{n_j^*}} Z_1^*, \dots, \sqrt{\frac{a_{j-1}^*}{n_{j-1}^*}} Z_{j-1}^*) \text{ and } \mathbf{U}_j^* = (\sqrt{\frac{a_k^*}{n_k^*}} Z_k^*, \dots, \sqrt{\frac{a_{j+1}^*}{n_{j+1}^*}} Z_{j+1}^*), j=1, \dots, k, \quad (3.3.27)$$

where $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$ are iid $N(0, 1)$ random variables. Let $\mathbf{w}_n, \mathbf{w}_n^*, \mathbf{U}_n$ and $\mathbf{U}_n^*$ be as defined in (3.1.1), (3.2.15) and (3.2.16). Assume that $Q_{(N+N^*-2k)} \sim \chi_{(N+N^*-2k)}^2$ is independent of $Z_1, \dots, Z_k, Z_1^*, \dots, Z_k^*$. Also let

$$\varphi_i^i = \{ (x_1, \dots, x_i) \in \mathbb{R}^i : x_1 \leq x_2 \leq \dots \leq x_i \}, i=1, \dots, k, \quad (3.3.28)$$

be the set of umbrella orderings with peak i in an i -dimensional Euclidean space.

Theorem 3.3.5 Under parameter configurations (3.3.22) with fixed i ,

$$\begin{aligned} T_3 &\stackrel{D}{=} (N + N^*) \times \\ &\left[\ln \left(\frac{\|\mathbf{L}_i - P_{\mathbf{L}_i}(\mathbf{L}_i | \varphi_{i-1}^{i-1})\|_{\mathbf{L}_i}^2 + \|\mathbf{U}_i - P_{\mathbf{U}_i}(\mathbf{U}_i | \varphi_{k-i}^{k-i})\|_{\mathbf{U}_i}^2 + \|\mathbf{U}^* - P_{\mathbf{w}_n^*}(\mathbf{U}^* | \varphi_i)\|_{\mathbf{w}_n^*}^2}{Q_{(N+N^*-2k)}} + 1 \right) \right. \\ &\quad \left. - \ln \left(\frac{\|\mathbf{L}_i - P_{\mathbf{L}_i}(\mathbf{L}_i | \varphi_{i-1}^{i-1})\|_{\mathbf{L}_i}^2 + \|\mathbf{U}_i - P_{\mathbf{U}_i}(\mathbf{U}_i | \varphi_{k-i}^{k-i})\|_{\mathbf{U}_i}^2 + \min_{p \neq i} \left\{ \|\mathbf{U}^* - P_{\mathbf{w}_n^*}(\mathbf{U}^* | \varphi_p)\|_{\mathbf{w}_n^*}^2 \right\}}{Q_{(N+N^*-2k)}} + 1 \right) \right]. \end{aligned} \quad (3.3.29)$$

Under parameter configurations (3.3.23) with fixed j ,

$$T_3^D = (N + N^*) \times$$

$$\begin{aligned} & \left[\ln \left(\frac{\|U - P_{w_n}(U|\varphi_j)\|_{w_n}^2 + \|L_j^* - P_{f_j^*}(L_j^*|\varphi_{j-1}^{j-1})\|_{f_j^*}^2 + \|U_j^* - P_{u_j^*}(U_j^*|\varphi_{k-j}^{k-j})\|_{u_j^*}^2}{Q_{(N+N^*-2k)}} + 1 \right) \right. \\ & \quad \left. - \ln \left(\frac{\min_{p \neq j} \{ \|U - P_{w_n}(U|\varphi_p)\|_{w_n}^2 \} + \|L_j^* - P_{f_j^*}(L_j^*|\varphi_{j-1}^{j-1})\|_{f_j^*}^2 + \|U_j^* - P_{u_j^*}(U_j^*|\varphi_{k-j}^{k-j})\|_{u_j^*}^2}{Q_{(N+N^*-2k)}} + 1 \right) \right]. \end{aligned}$$

(3.3.30)

Proof. By the symmetry of (3.3.29) and (3.3.30), it suffices to prove (3.3.29) for a fixed

i. According to (2.1.2), $P_w(g|\varphi_i)$ is the projection of $g = (g_1, \dots, g_k) \in R^k$ onto φ_i with nonnegative weights $w = (w_1, \dots, w_k)$. That is, $P_w(g|\varphi_i)$ is the solution to the problem of

$$\text{minimizing } \|g - x\|_w^2 = \sum_1^k (g_i - x_i)^2 w_i, \text{ subject to } (x_1, \dots, x_k) \in \varphi_i. \quad (3.3.31)$$

Suppose that for a fixed i, $g_i = +\infty$ and $g_s < +\infty$, $s=1, \dots, i-1, i+1, \dots, k$. Then it follows from Theorem 1.3.5 that $P_w(g|\varphi_i)_i = g_i = +\infty$. Therefore, the problem of minimizing

$\|g - x\|_w^2 = \sum_1^k (g_i - x_i)^2 w_i$ subject to $x = (x_1, \dots, x_k) \in \varphi_i$ is equivalent to the problems of

minimizing $\sum_1^{i-1} (g_i - x_i)^2 w_i$ subject to $(x_1, \dots, x_{i-1}) \in \varphi_{i-1}^{i-1}$ and minimizing

$\sum_{i+1}^k (g_i - x_i)^2 w_i$ subject to $(x_k, \dots, x_{i+1}) \in \varphi_{k-i}^{k-i}$. By (2.1.2), we have

$$P_w(g|\varphi_i) = (P_{(w_1, \dots, w_{i-1})}((g_1, \dots, g_{i-1})|\varphi_{i-1}^{i-1}), g_i, P_{(w_k, \dots, w_{i+1})}((g_k, \dots, g_{i+1})|\varphi_{k-i}^{k-i})). \quad (3.3.32)$$

Theorem 3.3.5 can be proved by using (3.3.32) and a technique similar to that used in the proof of Theorem 2.2.2.

Simulation results suggest that the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ is, in fact, the maximum of the upper α percentiles of the distributions of the test statistic T_3 under parameter configurations (3.3.22) with $i=1, k$ and (3.3.23) with $j=1, k$ (the symmetric transformations of the slippage configurations (3.3.16)) for most cases where the weights (3.1.1) are not extremely unbalanced. Again, the maximum of the upper α percentiles of the distributions of the test statistic T_3 under parameter configurations (3.3.22) with $i=1, k$ and (3.3.23) with $j=1, k$ is very close to, and thus provides a good estimate of, the critical point $c_3(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$ for any case.

A FORTRAN program that provides a simulation estimate of the critical point $c_2(\alpha, \mathbf{n}, \mathbf{n}^*, \mathbf{a}, \mathbf{a}^*)$, computes the test statistic T_1 and gives the P-value for any case of $\mathbf{n}, \mathbf{n}^*, \mathbf{a}$ and $\mathbf{a}^*$ is available from the author. In the program, the user will need to input the number of iterations, or precision, desired for the simulation estimate of the critical point, the significance level of interest, the number of treatment levels k , the vectors of sample sizes, the vectors of sample means, the vectors of sample variances and the vectors $\mathbf{a}$ and $\mathbf{a}^*$. The output is the value of the test statistic, the simulated critical point and the P-value.

CHAPTER IV

Nonparametric Distribution-Free Test Based on Ranks

Introduction

In this chapter we develop procedures for testing Q_1 (1.4.1) based on ranks. Simulation results show that the null distributions of the test statistic under slippage configurations and their symmetric transformations provide the proper critical points. The rank test has the advantage of being nonparametric distribution-free. The distributions and limiting distributions of the rank statistic under slippage configurations and their symmetric transformations are obtained. Large sample critical points for the rank test can be found from the critical points for T_1 (2.1.14) via simple transformations.

4.1 Test Statistic Based on Ranks

In this chapter, we let X_{ij} , $i=1, \dots, k$ and $j=1, \dots, n_i$, be a set of $N = \sum_{i=1}^k n_i$ mutually independent random variables such that $X_{ij} \sim F_i(x) = F(x - \mu_i)$, where the unknown mean

vector $\mu = (\mu_1, \dots, \mu_k)$ satisfies a strict umbrella ordering with unknown peak p . In addition, let X_{ij}^* , $i=1, \dots, k$ and $j=1, \dots, n_i^*$, be a second set of $N^* = \sum_{i=1}^k n_i^*$ mutually independent random variables such that $X_{ij}^* \sim F_i^*(x^*) = F^*(x^* - \mu^*)$, where the unknown mean vector $\mu^* = (\mu_1^*, \dots, \mu_k^*)$ satisfies a strict umbrella ordering with unknown peak p^* . Finally, we assume that the set of N X_{ij} 's is independent of the set of N^* X_{ij}^* 's.

Let R_{ij} be the rank of X_{ij} among the X_{ij} 's and R_{ij}^* be the rank of X_{ij}^* among the X_{ij}^* 's. Notice that if the original data are the ranks of X_{ij} and X_{ij}^* among all the X_{ij} 's and X_{ij}^* 's combined, we can easily get the rank of X_{ij} among the X_{ij} 's only and the rank of X_{ij}^* among the X_{ij}^* 's only.

Let $\bar{R}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} R_{ij}$ be the average rank of $X_{i1}, \dots, X_{in_i}$ among the X_{ij} 's and $\bar{R}_i^* = \frac{1}{n_i^*} \sum_{j=1}^{n_i^*} R_{ij}^*$ be the average rank of $X_{i1}^*, \dots, X_{in_i^*}^*$ among the X_{ij}^* 's.

In this chapter, we let

$$\mathbf{w}_n = (n_1, \dots, n_k) \text{ and } \mathbf{w}_n^* = (n_1^*, \dots, n_k^*). \quad (4.1.1)$$

By analogy to the test statistic T_1 (2.1.14), we propose the rank test which rejects H_0 for large values of

$$T_r = \frac{24}{N(N+1) + N^*(N^*+1)} \left[\min_{p=1, \dots, k} \left\{ \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{R}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{R}}^*|\varphi_p)\|_{\mathbf{w}_n^*}^2 \right\} \right. \\ \left. - \min_{p \neq p^*} \left\{ \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\varphi_p)\|_{\mathbf{w}_n}^2 + \|\bar{\mathbf{R}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{R}}^*|\varphi_{p^*})\|_{\mathbf{w}_n^*}^2 \right\} \right]. \quad (4.1.2)$$

For significance level α , $0 < \alpha < 1$, let $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ be the positive number such that

$$\max_{(\mu, \mu^*) \in H_0} P(T_r \geq c_r(\alpha, \mathbf{n}, \mathbf{n}^*)) = \alpha. \quad (4.1.3)$$

We have thus proposed the following:

Theorem 4.1.1 The level α rank test for Q_1 (1.4.1) rejects H_0 if and only if T_r is greater than or equal to $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$.

The test statistic T_r can be computed using the *minimum violator algorithm* due to Thompson (1962) and described in Barlow *et al* (1972). FORTRAN programs to compute T_r are available from the author.

4.2 Critical Points Through Use of the Slippage Configuration

The critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ defined in (4.1.3) depends on the sample sizes $\mathbf{n}$ and $\mathbf{n}^*$. Since the null hypothesis space H_0 is quite large, the null parameter configuration(s) yielding the critical point is difficult to find. Since the rank procedure is a nonparametric test, the δ -umbrella concept can not be applied. Therefore, even for large samples the distribution of T_r under homogeneity configurations can not be used to obtain proper critical points. Fortunately, a simulation study (summarized at the end of this section) suggests a solution to the problem of determining the critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$.

The easiest to describe are the solutions to the simple but important cases when the sample sizes are the same within either group; that is,

$$n_1 = \dots = n_k = n \text{ and } n_1^* = \dots = n_k^* = n^*. \quad (4.2.1)$$

Without loss of generality, we always assume $n \leq n^*$ in cases (4.2.1). For such cases, simulation shows that the critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ can be obtained from the distribution of T_r under the slippage configurations (SC), where

$$\mu_1 = \dots = \mu_{k-1} < +\infty, \mu_k = +\infty \text{ and } \mu_1^* = \dots = \mu_k^* < +\infty. \quad (4.2.2)$$

It is obvious that the distribution of T_r is the same for all the slippage configurations (4.2.2). Therefore, for the purpose of determining critical points, we need only consider the case when $\mu_1 = \dots = \mu_{k-1} = 0, \mu_k = +\infty$ and $\mu_1^* = \dots = \mu_k^* = 0$. Let $s_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ be the upper α percentile of the distribution of T_r under slippage configurations (4.2.2); that is, the number such that

$$P_{SC}(T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) = \alpha. \quad (4.2.3)$$

We have the following simulation study result, which will also be discussed in Chapter 5:

Observation 4.2.1 Under condition (4.2.1), we conjecture that

$$c_r(\alpha, \mathbf{n}, \mathbf{n}^*) = s_r(\alpha, \mathbf{n}, \mathbf{n}^*). \quad (4.2.4)$$

This observation leads to the nonparametric distribution-free property of the rank test T_r for cases (4.2.1). In fact, the nonparametric distribution-free property of the test statistic T_r under slippage configurations (4.2.2) holds for any general case not in (4.2.1).

Theorem 4.2.1 The test statistic T_r is nonparametric distribution-free under slippage configurations (4.2.2). Under such configurations, we have that

$$T_R^D = \frac{24}{N(N+1) + N^*(N^*+1)} \times \left[\min_{p=1, \dots, k} \left\{ \|\bar{\mathbf{R}}_{-k, c} - P_{w_n}(\bar{\mathbf{R}}_{-k, c} | \varphi_p)\|_{w_n}^2 + \|\bar{\mathbf{R}}^* - P_{w_n^*}(\bar{\mathbf{R}}^* | \varphi_p)\|_{w_n^*}^2 \right\} - \min_{p \neq p^*} \left\{ \|\bar{\mathbf{R}} - P_{w_n}(\bar{\mathbf{R}} | \varphi_p)\|_{w_n}^2 + \|\bar{\mathbf{R}}^* - P_{w_n^*}(\bar{\mathbf{R}}^* | \varphi_{p^*})\|_{w_n^*}^2 \right\} \right], \quad (4.2.5)$$

where $c = \sum_{i=1}^{k-1} n_i + \frac{n_k + 1}{2}$, $\bar{\mathbf{R}}_{-k} = \left(\frac{1}{n_1} \sum_{j=1}^{n_1} R_j, \frac{1}{n_2} \sum_{j=1}^{n_2} R_{n_1+j}, \dots, \frac{1}{n_{k-1}} \sum_{j=1}^{n_{k-1}} R_{n_1+\dots+n_{k-2}+j} \right)$, and

$\bar{\mathbf{R}}^* = \left(\frac{1}{n_1^*} \sum_{j=1}^{n_1^*} R_j^*, \frac{1}{n_2^*} \sum_{j=1}^{n_2^*} R_{n_1^*+j}^*, \dots, \frac{1}{n_k^*} \sum_{j=1}^{n_k^*} R_{n_1^*+\dots+n_{k-1}^*+j}^* \right)$, with the vectors

$\mathbf{R} = (R_1, \dots, R_{n_1+\dots+n_{k-1}})$ and $\mathbf{R}^* = (R_1^*, \dots, R_{N^*}^*)$ being independent and uniformly distributed over the set of all the permutations of $\{1, \dots, \sum_{i=1}^{k-1} n_i\}$ and $\{1, \dots, N^*\}$, respectively.

Proof. Under slippage configurations (4.2.2), the observations from level k of the first umbrella ordering are always infinite and therefore automatically take the ranks from $\sum_{i=1}^{k-1} n_i + 1$ to $\sum_{i=1}^k n_i$. On the other hand, the observations in samples 1 to $k-1$ from the first umbrella ordering will take ranks from 1 to $\sum_{i=1}^{k-1} n_i$. Therefore, we can view the first $k-1$ samples from the first umbrella ordering as $k-1$ independent samples from an imaginary $k-1$ sample problem. The corresponding ranks of the observations in the first $k-1$ samples in the original k samples can also be viewed as the ranks in the imaginary $k-1$ sample problem. Under slippage configurations, the distributions of the first $k-1$ samples in the first umbrella ordering are identical. Viewed as ranks from the imaginary $k-1$ sample problem with identical underlying distribution for each sample, these ranks are distribution-free and assume each permutation of the set $\{1, \dots, \sum_{i=1}^{k-1} n_i\}$ with equal

probability $1 / (\sum_{i=1}^{k-1} n_i)!$. It is also easy to see that the ranks of the observations from the second umbrella ordering are distribution-free under slippage configurations (4.2.2) and assume each permutation of the set $\{1, \dots, N^*\}$ with equal probability $1 / N^*!$. Notice that under slippage configurations (4.2.2), the average rank for the k th sample from the first umbrella ordering is a constant, namely,

$$\bar{R}_k = \sum_{i=1}^{k-1} n_i + \frac{n_k + 1}{2} = \sum_{i=1}^k n_i - \frac{n_k - 1}{2}, \quad (4.2.6)$$

and Theorem 4.2.1 is established.

The exact critical point $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ can be obtained by exhausting all the possible permutations of ranks according to Theorem 4.2.1. Approximations of the critical points $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ can be obtained through simulation using the distribution described in Theorem 4.2.1. Table 5 gives simulated $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ values for selected common sample sizes n and n^* . Later, in Section 4.3, we will show that for large samples the critical point $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ can be obtained from the critical point $s_l(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ (2.2.20) and (2.2.22) for case (2.2.17) via a simple transformation.

The determination of $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ is a bit more complex for a general case not in (4.2.1). Because the sample sizes are not the same for the k levels within an umbrella ordering, it is as plausible to investigate the following three classes of parameter configurations as the slippage configurations (4.2.2):

$$(1). \mu_1 = +\infty, \mu_2 = \dots = \mu_k = 0 \text{ and } \mu_1^* = \dots = \mu_k^* = 0, \quad (4.2.7)$$

$$(2). \mu_1 = \dots = \mu_k = 0 \text{ and } \mu_1^* = \dots = \mu_{k-1}^* = 0, \mu_k^* = +\infty, \quad (4.2.8)$$

$$(3). \mu_1 = \dots = \mu_k = 0 \text{ and } \mu_1^* = +\infty, \mu_2^* = \dots = \mu_k^* = 0. \quad (4.2.9)$$

Based on an extensive simulation study (see Chapter 5), we conclude that the critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ is indeed the maximum of the upper α percentiles of the distributions of the test statistic T_r under the four classes of parameter configurations (4.2.7), (4.2.8), (4.2.9) and slippage configurations (4.2.2) for any case that does not have extremely unbalanced sample sizes within a given umbrella ordering. We conjecture that the statement above is true even for cases with extremely unbalanced sample sizes. (The theorems to be developed in Section 4.3 will provide support for this conjecture.) We did not find any counterexample to this conjecture in our simulation study, but we were, of course, unable to exhaust all the possible combinations of sample sizes. While we are unable, due to the nature of simulation, to exclude with absolute certainty the possibility that the critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ may be obtained from the more extensive parameter configurations (3.3.22) and (3.3.23) for extremely unbalanced samples, simulation study of such cases gives us a strong impression that the maximum of the upper α percentiles of the distributions of the test statistic T_r under the parameter configurations (4.2.7), (4.2.8), (4.2.9) and (4.2.2) will be very close to, if not equal to, $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$. As we discussed in Section 2.3, the parameter configurations (4.2.7), (4.2.8), (4.2.9) and (4.2.2) are quite distant from the parameter configurations commonly encountered in applications. The critical point determined under just these configurations is already very conservative. The maximum of the upper α percentiles of the test statistic T_r under parameter configurations (4.2.7), (4.2.8), (4.2.9) and (4.2.2) is shown by simulation to be a proper level α critical point for any application in the sense that the significance level α dominates the value of the power function at virtually every null parameter configuration conceivable in practice. In other words, we can safely use the critical point determined from parameter configurations (4.2.7), (4.2.8), (4.2.9) and (4.2.2) even if these critical points might not be exactly equal to $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$. Therefore, we advocate the use of the maximum of the upper α percentiles of the distributions of the test statistic

T_r under the four classes of parameter configurations (4.2.7), (4.2.8), (4.2.9) and slippage configurations (4.2.2) as the critical point for the rank test based on T_r . Notice that, for cases (4.2.1), the distribution of the test statistic T_r is the same under all four classes of parameter configurations as a result of the symmetry of the classes. Therefore, for cases (4.2.1), slippage configurations (4.2.2) alone yield the critical points. For a general case, one of the four classes of parameter configurations will yield the proper critical points, depending on the combination of sample sizes. The distributions of the test statistic T_r under parameter configurations (4.2.7), (4.2.8) and (4.2.9) can be obtained easily from Theorem 4.2.1 (with slight modifications) by the symmetry among the four classes. Taking parameter configurations (4.2.7) as an example, the distribution of T_r can be obtained by revising the orders of $(\bar{R}_1, \dots, \bar{R}_k)$ and $(n_1, \dots, n_k)$ and then substituting into Theorem 4.2.1. These distributions can be used to obtain exact or approximate critical points $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ by exhausting all the possible permutations of ranks or by simulation, respectively. The limiting distributions of the test statistic T_r under parameter configurations (4.2.7), (4.2.8), (4.2.9) and slippage configurations (4.2.2) are described in Theorem 4.3.2, which can be used to obtain approximate large sample critical points.

A FORTRAN program that provides a simulation estimate of the critical point $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$, computes the test statistic T_r and gives the P-value for any case of sample sizes is available from the author. In the program, the user will need to input the number of iterations, or precision, desired for the simulation estimate of the critical point, the significance level of interest, the number of treatment levels k , the vectors of sample sizes and the vectors of sample average ranks. The output is the value of the test statistic, the simulated critical point and the P-value.

When the sample sizes are reasonably large (more than 20 observations at each treatment level), the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ for the test based on T_1 provides a good approximation to $c_r(\alpha, \mathbf{n}, \mathbf{n}^*)$. The details are given in Theorem 4.3.2, Corollary 4.3.1 and Corollary 4.3.2. A FORTRAN program for the calculation of the test statistic, the critical point and P-value for the test based on T_1 can also be used to calculate the approximate test statistic, the approximate critical point and the approximate P-value for the test based on T_r .

4.3 Asymptotic Properties of the Rank Test Statistic Under Slippage Configurations

The use of slippage configurations (4.2.2) and their symmetric transformations (4.2.7), (4.2.8) and (4.2.9) to obtain proper critical points has support from a theoretical study of the limiting properties of the rank test statistic T_r . In addition to the simulation evidence, we will now show theoretically that for any combination of the sample sizes $\mathbf{n}$ and $\mathbf{n}^*$,

1. The slippage configurations (4.2.2) are asymptotically the most difficult parameter configurations on the boundary of the null hypothesis space H_0 for the rank test based on T_r to distinguish from the alternatives in H_1 .
2. The asymptotic distribution of T_r under the slippage configurations (4.2.2) can be obtained.

In this section, we again assume that $\{\mathbf{n}\}$ and $\{\mathbf{n}^*\}$ are two sequences that satisfy conditions (2.2.32)-(2.2.34) in Section 2.2. In this section, we let

$$\mathbf{w}_\infty = (\lambda_1, \dots, \lambda_k) \text{ and } \mathbf{w}_\infty^* = (\lambda_1^*, \dots, \lambda_k^*). \quad (4.3.1)$$

The following lemmas are used in the proof of Theorem 4.3.1.

Lemma 4.3.1 Let X and X' be independent and have the same continuous distribution $F(x)$. Then, for any positive number $d > 0$,

$$P(X - X' < d) > \frac{1}{2}. \quad (4.3.2)$$

Proof. It is easy to see that

$$(X, X') \stackrel{D}{=} (X', X). \quad (4.3.3)$$

Applying Theorems 1.3.2 and 1.3.7 in Randles and Wolfe (1991), we have that

$$D = X - X' \text{ is distributed symmetrically about } 0. \quad (4.3.4)$$

Therefore,

$$P(X - X' \leq 0) = \frac{1}{2} \quad (4.3.5)$$

and

$$P(0 < X - X' < d) = \frac{1}{2} P(-d < X - X' < d). \quad (4.3.6)$$

To prove Lemma 4.3.1, it suffices to show that

$$\begin{aligned} \frac{1}{2} P(-d < X - X' < d) &= P(0 < X - X' < d) \\ &= P(X - X' < d) - P(X - X' \leq 0) > 0. \end{aligned} \quad (4.3.7)$$

Since $F(x)$ is a continuous cumulative distribution function, then for any $d > 0$ there exists an x_0 such that

$$P(x_0 < X < x_0 + d) > 0. \quad (4.3.8)$$

Since X' has the same distribution as X , we also have

$$P(x_0 < X' < x_0 + d) > 0. \quad (4.3.9)$$

Therefore, it follows from (4.3.8) and (4.3.9) that

$$\begin{aligned} P(-d < X - X' < d) &\geq P(x_0 < X < x_0 + d, x_0 < X' < x_0 + d) \\ &= P(x_0 < X < x_0 + d)P(x_0 < X' < x_0 + d) > 0, \end{aligned} \quad (4.3.10)$$

and the lemma is established.

The following Lemma 4.3.2 provides the fact that average ranks of observations from an umbrella ordering maintain an umbrella ordering. In Lemma 4.3.2, let X_{ij} , $i=1, \dots, k$ and $j=1, \dots, n_i$, be a set of $N = \sum_{i=1}^k n_i$ mutually independent random variables such that

$X_{ij} \sim F_i(x) = F(x - \mu_i)$. Also let R_{ij} be the rank of X_{ij} among the X_{ij} 's and let

$\bar{R}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} R_{ij}$ be the average rank of $X_{i1}, \dots, X_{in_i}$. Denote the vector of average ranks by

$\bar{\mathbf{R}} = (\bar{R}_1, \dots, \bar{R}_k)$. In addition, we assume that

$$\frac{n_i}{N} \rightarrow \lambda_i \in (0, 1), i=1, \dots, k, \text{ as } N \rightarrow \infty. \quad (4.3.11)$$

Lemma 4.3.2 If the location parameter $\mu = (\mu_1, \dots, \mu_k)$ satisfies

$$\mu_1 < \mu_2 < \dots < \mu_p > \mu_{p+1} > \dots > \mu_k \text{ for some } p \in \{1, 2, \dots, k\},$$

then there exists a vector $\eta = (\eta_1, \dots, \eta_k)$ with

$$\eta_1 < \eta_2 < \dots < \eta_p > \eta_{p+1} > \dots > \eta_k$$

such that

$$\frac{\bar{R}}{N} \xrightarrow{P} \eta \text{ as } N \rightarrow \infty. \quad (4.3.12)$$

Proof. Let $X_i \sim F_i(x) = F(x - \mu_i)$, $i=1, \dots, k$, be independent random variables. By the U-statistics theory, we have, for any $i \neq j=1, \dots, k$, that

$$\begin{aligned} U_{ij} &= \frac{1}{n_i n_j} \sum_{t=1}^{n_j} [\text{number of } X_{is} \text{'s such that } X_{is} < X_{jt}] \\ &= \frac{1}{n_i n_j} \sum_{s=1}^{n_i} \sum_{t=1}^{n_j} \Psi(X_{jt} - X_{is}) \\ &\xrightarrow{P} P(X_i < X_j), \end{aligned} \quad (4.3.13)$$

where $\Psi(t) = 1$, 0 as $t >, \leq 0$.

Now, fix $j \in \{1, \dots, k\}$ and group X_{is} , $i=1, \dots, k$, $i \neq j$, $s=1, \dots, n_i$, to form one sample ("sample one") and consider X_{jt} , $t=1, \dots, n_j$, as a second sample ("sample two"). Then the rank of X_{uv} among the combined "two samples" is still R_{uv} for $u=1, \dots, k$, $v=1, \dots, n_u$. By applying formula (2.3.9) in Randles and Wolfe (1991), we have, for a fixed $j=1, \dots, k$, that

$$\begin{aligned} &\sum_{\substack{i=1, \dots, k \\ i \neq j}} n_i n_j U_{ij} \\ &= \sum_{t=1}^{n_j} \left\{ \sum_{\substack{i=1, \dots, k \\ i \neq j}} [\text{number of } X_{is}, s=1, \dots, n_i, \text{ such that } X_{is} < X_{jt}] \right\} \end{aligned}$$

$$= \sum_{t=1}^{n_j} [\text{number of } X_{is}, i = 1, \dots, k, i \neq j, \text{ and } s = 1, \dots, n_i, \text{ such that } X_{is} < X_{jt}]$$

$$= \sum_{t=1}^{n_j} [\text{number of } X_{uv} \text{'s in "sample one" such that } X_{uv} < X_{jt}]$$

$$= \sum_{\substack{X_{uv} \text{ in} \\ \text{"sample one"}}} \sum_{\substack{X_{jt} \text{ in} \\ \text{"sample two"}}} \Psi(X_{jt} - X_{uv})$$

=Mann-Whitney statistic for the "two-sample" problem

$$\stackrel{\text{a.s.}}{=} \text{Wilcoxon statistic for the "two-sample" problem} - \frac{n_j(n_j + 1)}{2}$$

$$\stackrel{\text{a.s.}}{=} \sum_{t=1}^{n_j} R_{jt} - \frac{n_j(n_j + 1)}{2}. \quad (4.3.14)$$

Therefore,

$$\sum_{\substack{i=1, \dots, k \\ i \neq j}} n_i U_{ij} = \bar{R}_j - \frac{n_j + 1}{2}, \text{ for } j=1, \dots, k. \quad (4.3.15)$$

By (4.3.13) and (4.3.15), we have

$$\begin{aligned} \frac{\bar{R}_j}{N} &= \sum_{\substack{i=1, \dots, k \\ i \neq j}} \frac{n_i}{N} U_{ij} + \frac{n_j + 1}{2N} \\ &\xrightarrow{P} \sum_{\substack{i=1, \dots, k \\ i \neq j}} \lambda_i P(X_i < X_j) + \frac{\lambda_j}{2} \stackrel{\Delta}{=} \eta_j, \text{ for } j=1, \dots, k. \end{aligned} \quad (4.3.16)$$

Notice that, for any pair (i, j) such that $\mu_i < \mu_j$, we have

$$\begin{aligned} &P(X_t < X_j) - P(X_t < X_i) \\ &= P(X_t < X_i + (\mu_j - \mu_i)) - P(X_t < X_i) \end{aligned}$$

$$\begin{aligned}
&= P(X_t - X_i < \mu_j - \mu_i) - P(X_t - X_i < 0) \\
&\geq 0, \text{ for any } t=1, \dots, k.
\end{aligned} \tag{4.3.17}$$

By Lemma 4.3.1, we have, for any pair (i, j) such that $\mu_i < \mu_j$, that

$$P(X_i < X_j) - \frac{1}{2} = P(X_i - (X_j - (\mu_j - \mu_i)) < \mu_j - \mu_i) - \frac{1}{2} > 0. \tag{4.3.18}$$

Similarly,

$$\frac{1}{2} - P(X_j < X_i) > 0. \tag{4.3.19}$$

By (4.3.17), (4.3.18) and (4.3.19), we have, for any pair (i, j) such that $\mu_i < \mu_j$, that

$$\begin{aligned}
\eta_j - \eta_i = \sum_{\substack{t=1, \dots, k \\ t \neq i, j}} \lambda_t [P(X_t < X_j) - P(X_t < X_i)] + \lambda_j \left[\frac{1}{2} - P(X_j < X_i) \right] + \lambda_i \left[P(X_i < X_j) - \frac{1}{2} \right] > 0
\end{aligned} \tag{4.3.20}$$

The conclusion of Lemma 4.3.2 is then immediate.

The use of $s_r(\alpha, \mathbf{n}, \mathbf{n}^*)$ as the critical point for cases (4.2.1) is supported by the fact that the slippage configurations (4.2.2) are asymptotically the most difficult null hypothesis configurations for the rank test based on T_r to distinguish from the alternatives in H_1 . By the symmetry of the four classes of parameter configurations (4.2.7), (4.2.8), (4.2.9) and slippage configurations (4.2.2), it is also true that parameter configurations (4.2.7), (4.2.8), (4.2.9) are among the most difficult for T_r to distinguish from the alternatives in H_1 . The following Theorem 4.3.1 is for any combination of sample sizes, which supports the discussion of critical point determination in Section 4.2.

Theorem 4.3.1 For fixed μ and μ^* ,

$$\begin{aligned}
 0 &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{SC} (T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) = \alpha \\
 &< \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) = 1.
 \end{aligned} \tag{4.3.21}$$

Proof. It follows from Lemma 4.3.2 that there exist vectors $\eta = (\eta_1, \dots, \eta_k)$ and $\eta^* = (\eta_1^*, \dots, \eta_k^*)$ such that

$$\frac{\bar{\mathbf{R}}}{N} \xrightarrow{P} \eta \text{ and } \frac{\bar{\mathbf{R}}^*}{N^*} \xrightarrow{P} \eta^* \text{ as } N \text{ and } N^* \rightarrow \infty. \tag{4.3.22}$$

Therefore,

$$\begin{aligned}
 &\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \\
 &= 24 \min_{p=1, \dots, k} \left[\frac{N^3}{[N(N+1) + N^*(N^*+1)]^{3/2}} \left\| \frac{\bar{\mathbf{R}}}{N} - P_{\mathbf{w}_n/N} \left(\frac{\bar{\mathbf{R}}}{N} | \varphi_p \right) \right\|_{\mathbf{w}_n/N}^2 \right. \\
 &\quad \left. + \frac{N^{*3}}{[N(N+1) + N^*(N^*+1)]^{3/2}} \left\| \frac{\bar{\mathbf{R}}^*}{N^*} - P_{\mathbf{w}_{n^*}^*/N^*} \left(\frac{\bar{\mathbf{R}}^*}{N^*} | \varphi_p \right) \right\|_{\mathbf{w}_{n^*}^*/N^*}^2 \right] \\
 &- 24 \min_{p \neq p^*} \left[\frac{N^3}{[N(N+1) + N^*(N^*+1)]^{3/2}} \left\| \frac{\bar{\mathbf{R}}}{N} - P_{\mathbf{w}_n/N} \left(\frac{\bar{\mathbf{R}}}{N} | \varphi_p \right) \right\|_{\mathbf{w}_n/N}^2 \right. \\
 &\quad \left. + \frac{N^{*3}}{[N(N+1) + N^*(N^*+1)]^{3/2}} \left\| \frac{\bar{\mathbf{R}}^*}{N^*} - P_{\mathbf{w}_{n^*}^*/N^*} \left(\frac{\bar{\mathbf{R}}^*}{N^*} | \varphi_{p^*} \right) \right\|_{\mathbf{w}_{n^*}^*/N^*}^2 \right]
 \end{aligned}$$

$$\begin{aligned}
\stackrel{P}{\longrightarrow} J_r = & 24 \min_{p=1, \dots, k} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 \right. \\
& \left. + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta^* - P_{w_-}(\eta^*|\varphi_p)\|_{w_-}^2 \right] \\
& - 24 \min_{p \neq p^*} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 \right. \\
& \left. + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_{p^*})\|_{w_-}^2 \right].
\end{aligned}
\tag{4.3.23}$$

By the technique used in the proof of Theorem 4.2.1 of viewing the first $k-1$ samples from the first umbrella ordering as observations from an imaginary $k-1$ sample problem, and by (4.3.14) and well-known properties of the Wilcoxon rank sum statistic, we have,

$$E_{sc} \left(\frac{\bar{R}_i}{\sum_{i=1}^{k-1} n_i} \right) = \frac{\sum_{i=1}^{k-1} n_i + 1}{2 \sum_{i=1}^{k-1} n_i} \rightarrow \frac{1}{2}, \text{ as } \sum_{i=1}^{k-1} n_i \rightarrow +\infty, \text{ for } i=1, \dots, k-1, \tag{4.3.24}$$

and

$$\text{Var}_{sc} \left(\frac{\bar{R}_i}{\sum_{i=1}^{k-1} n_i} \right) = \frac{(\sum_{i=1}^{k-1} n_i - n_i)(\sum_{i=1}^{k-1} n_i + 1)}{12 n_i \left(\sum_{i=1}^{k-1} n_i \right)^2} \rightarrow 0, \text{ as } \sum_{i=1}^{k-1} n_i \rightarrow +\infty, \text{ for } i=1, \dots, k-1. \tag{4.3.25}$$

Therefore,

$$\begin{aligned}
E_{sc}\left(\frac{\bar{R}_i}{N}\right) &= \left(\frac{\sum_{i=1}^{k-1} n_i}{N}\right) E_{sc}\left(\frac{\bar{R}_i}{\sum_{i=1}^{k-1} n_i}\right) \\
&= \left(\frac{\sum_{i=1}^{k-1} n_i}{N}\right) \left(\frac{\sum_{i=1}^{k-1} n_i + 1}{2 \sum_{i=1}^{k-1} n_i}\right) \rightarrow \frac{1}{2} \sum_{i=1}^{k-1} \lambda_i \text{ as } N \rightarrow +\infty, \text{ for } i=1, \dots, k-1,
\end{aligned} \tag{4.3.26}$$

and, from (4.2.6),

$$E_{sc}\left(\frac{\bar{R}_k}{N}\right) = \frac{1}{N} \left(N - \frac{n_k - 1}{2}\right) \rightarrow 1 - \frac{\lambda_k}{2} \text{ as } N \rightarrow +\infty. \tag{4.3.27}$$

It is easy to see from (4.3.25) that

$$\text{Var}_{sc}\left(\frac{\bar{R}_i}{N}\right) = \left(\frac{\sum_{i=1}^{k-1} n_i}{N}\right)^2 \text{Var}_{sc}\left(\frac{\bar{R}_i}{\sum_{i=1}^{k-1} n_i}\right) \rightarrow 0, \text{ as } N \rightarrow +\infty, \text{ for } i=1, \dots, k-1, \tag{4.3.28}$$

and from (4.2.6) that

$$\text{Var}_{sc}\left(\frac{\bar{R}_k}{N}\right) = \frac{1}{N^2} \text{Var}_{sc}\left(N - \frac{n_k - 1}{2}\right) = 0. \tag{4.3.29}$$

Hence, under slippage configurations (4.2.2), we have from Chebychev's Inequality that

$$\frac{\bar{R}_i}{N} \xrightarrow{P} \eta_i = \frac{1}{2} \sum_{i=1}^{k-1} \lambda_i, \text{ for } i=1, \dots, k-1, \tag{4.3.30}$$

and

$$\frac{\bar{R}_k}{N} \xrightarrow{P} \eta_k = 1 - \frac{\lambda_k}{2}, \text{ as } N \rightarrow +\infty. \tag{4.3.31}$$

By a similar reason, we have that, under slippage configurations (4.2.2) ,

$$\frac{\bar{R}_i^*}{N^*} \xrightarrow{P} \eta_i^* = \frac{1}{2}, \text{ as } N^* \rightarrow +\infty, \text{ for } i=1, \dots, k. \quad (4.3.32)$$

Hence, under slippage configurations (4.2.2) ,

$$\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \xrightarrow{P} J_r = 0, \text{ as } N, N^* \rightarrow +\infty. \quad (4.3.33)$$

By (4.2.3) and an argument similar to that used in the proof of Theorem 2.2.3, we have that

$$\frac{s_r(\alpha, \mathbf{n}, \mathbf{n}^*)}{[N(N+1) + N^*(N^*+1)]^{1/2}} \rightarrow 0, \text{ as } N, N^* \rightarrow +\infty. \quad (4.3.34)$$

Now, we consider two cases. First, suppose that $(\mu, \mu^*) \in H_0$ so that

$$\mu_1 < \mu_2 < \dots < \mu_p > \mu_{p+1} > \dots > \mu_k$$

and

$$\mu_1^* < \mu_2^* < \dots < \mu_p^* > \mu_{p+1}^* > \dots > \mu_k^*.$$

By Lemma 4.3.2, we have that

$$\eta_1 < \eta_2 < \dots < \eta_p > \eta_{p+1} > \dots > \eta_k \quad (4.3.35)$$

and

$$\eta_1^* < \eta_2^* < \dots < \eta_p^* > \eta_{p+1}^* > \dots > \eta_k^*. \quad (4.3.36)$$

It follows from (2.2.48) that

$$24 \min_{p=1, \dots, k} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta^* - P_{w_-}(\eta^*|\varphi_p)\|_{w_-}^2 \right] = 0 \quad (4.3.37)$$

and

$$24 \min_{p \neq p^*} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta^* - P_{w_-}(\eta^*|\varphi_p)\|_{w_-}^2 \right] > 0. \quad (4.3.38)$$

It follows from (4.3.23) that, as $N, N^* \rightarrow \infty$,

$$\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \xrightarrow{P} J_r < 0. \quad (4.3.39)$$

By Slutsky's Theorem, (4.3.34) and (4.3.39), we have

$$\begin{aligned} & \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} (T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} \left(\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \geq \frac{s_r(\alpha, \mathbf{n}, \mathbf{n}^*)}{[N(N+1) + N^*(N^*+1)]^{1/2}} \right) \\ &= \lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_0} \left(\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \geq 0 \right) = 0. \end{aligned} \quad (4.3.40)$$

For the second case, let $(\mu, \mu^*) \in H_1$ so that for some $p \neq p^* = 1, \dots, k$, we have

$$\mu_1 < \mu_2 < \dots < \mu_p > \mu_{p+1} > \dots > \mu_k$$

and

$$\mu_1^* < \mu_2^* < \dots < \mu_p^* > \mu_{p+1}^* > \dots > \mu_k^*.$$

By Lemma 4.3.2, we have that

$$\eta_1 < \eta_2 < \dots < \eta_p > \eta_{p+1} > \dots > \eta_k \quad (4.3.41)$$

and

$$\eta_1^* < \eta_2^* < \dots < \eta_p^* > \eta_{p+1}^* > \dots > \eta_k^*. \quad (4.3.42)$$

By an approach similar to that used for the case when $(\mu, \mu^*) \in H_0$, we find that

$$24 \min_{p=1, \dots, k} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta^* - P_{w_-}(\eta^*|\varphi_p)\|_{w_-}^2 \right] > 0 \quad (4.3.43)$$

and

$$24 \min_{p \neq p^*} \left[\frac{\gamma^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta - P_{w_-}(\eta|\varphi_p)\|_{w_-}^2 + \frac{(1-\gamma)^3}{[\gamma^2 + (1-\gamma)^2]^{3/2}} \|\eta^* - P_{w_-}(\eta^*|\varphi_p)\|_{w_-}^2 \right] = 0 \quad (4.3.44)$$

It follows from (4.3.23) that, as $N, N^* \rightarrow \infty$,

$$\frac{T_r}{[N(N+1) + N^*(N^*+1)]^{1/2}} \xrightarrow{P} J_r > 0. \quad (4.3.45)$$

By Slutsky's Theorem, (4.3.34) and (4.3.45) and an argument similar to that leading to (2.2.58), we have

$$\lim_{N, N^* \rightarrow +\infty} P_{(\mu, \mu^*) \in H_1} (T_r \geq s_r(\alpha, \mathbf{n}, \mathbf{n}^*)) = 1. \quad (4.3.46)$$

Theorem 4.3.1 follows by combining (4.3.40) and (4.3.45).

Now we describe the limiting distribution of T_r under slippage configurations (4.2.2). (Theorem 4.3.2 can also be easily modified to obtain limiting distributions of T_r under parameter configurations (4.2.7), (4.2.8) and (4.2.9), using the symmetry of these parameter configurations with respect to slippage configurations (4.2.2). Take parameter configurations (4.2.7), for example. The distribution of T_r can be obtained by revising the orders of $(\bar{R}_1, \dots, \bar{R}_k)$ and $(n_1, \dots, n_k)$ and then substituting into Theorem 4.3.2.)

Theorem 4.3.2 Under slippage configurations (4.2.2), the rank test statistic T_r converges in distribution to

$$T_r(\lambda, \lambda^*, \gamma) \stackrel{\Delta}{=} \frac{2(1-\gamma)^2}{\gamma^2 + (1-\gamma)^2} \left\{ \|V_\infty^* - P_{w_\infty^*}(V_\infty^* | \varphi_k)\|_{w_\infty^*}^2 - \min_{p=1, \dots, k-1} \|V_\infty^* - P_{w_p^*}(V_\infty^* | \varphi_p)\|_{w_p^*}^2 \right\},$$

where V_∞^* and w_∞^* are defined in (4.3.50) and (4.3.1), respectively.

Proof. By a technique similar to that used in the proof of Theorem 4.2.1 (of viewing the first $k-1$ samples from the first umbrella ordering as samples from an imaginary $k-1$ sample problem) and the results in Kruskal (1952), we have that under the slippage configurations (4.2.2) (see also, for example, Hettmansperger, 1984),

$$\left(\frac{12}{\sum_{i=1}^{k-1} n_i} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) \xrightarrow{D} N \left(\mathbf{0}_{k-1}, \begin{bmatrix} \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_1} - 1 & -1 & . & . & . & -1 \\ -1 & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_2} - 1 & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_{k-1}} - 1 & . \\ -1 & -1 & . & . & . & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_{k-1}} - 1 \end{bmatrix} \right) \quad (4.3.47)$$

where $\bar{\mathbf{R}}_{-k} = (\bar{R}_1, \dots, \bar{R}_{k-1})$, and

$$\left(\frac{12}{N^*} \right)^{1/2} \left(\bar{\mathbf{R}}^* - \frac{N^* + 1}{2} \right) \xrightarrow{D} N \left(\mathbf{0}_k, \begin{bmatrix} \frac{1}{\lambda_1^*} - 1 & -1 & . & . & . & -1 \\ -1 & \frac{1}{\lambda_2^*} - 1 & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ -1 & -1 & . & . & \frac{1}{\lambda_k^*} - 1 & . \end{bmatrix} \right). \quad (4.3.48)$$

Define

$$\mathbf{V}_\infty = (V_1, \dots, V_{k-1}) \sim N_{\mathbf{0}_{k-1}}, \quad \begin{pmatrix} \left[\begin{array}{cccc} \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_2} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_{k-1}} \end{array} \right] \end{pmatrix} \quad (4.3.49)$$

and

$$\mathbf{V}_\infty^* = (V_1^*, \dots, V_k^*) \sim N_{\mathbf{0}_k}, \quad \begin{pmatrix} \left[\begin{array}{cccc} \frac{1}{\lambda_1^*} & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2^*} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \frac{1}{\lambda_k^*} \end{array} \right] \end{pmatrix}. \quad (4.3.50)$$

Also define

$$\hat{V} = \sum_{i=1}^{k-1} \left(\frac{\lambda_i}{\sum_{j=1}^{k-1} \lambda_j} \right) V_i \quad \text{and} \quad \hat{V}^* = \sum_{i=1}^k \lambda_i^* V_i^*. \quad (4.3.51)$$

Then

$$\mathbf{V}_\infty - \hat{\mathbf{V}} \sim N \left(\mathbf{0}_{k-1}, \begin{bmatrix} \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_1} - 1 & -1 & . & . & . & -1 \\ -1 & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_2} - 1 & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_{k-1}} - 1 & . \\ -1 & -1 & . & . & . & \frac{\sum_{i=1}^{k-1} \lambda_i}{\lambda_{k-1}} - 1 \end{bmatrix} \right) \quad (4.3.52)$$

and

$$\mathbf{V}_\infty^* - \hat{\mathbf{V}}^* \sim N \left(\mathbf{0}_k, \begin{bmatrix} \frac{1}{\lambda_1^*} - 1 & -1 & . & . & . & -1 \\ -1 & \frac{1}{\lambda_2^*} - 1 & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ -1 & -1 & . & . & . & \frac{1}{\lambda_k^*} - 1 \end{bmatrix} \right). \quad (4.3.53)$$

By (2.2.16), we have

$$\begin{aligned} & \frac{12}{N(N+1)} \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\varphi_p)\|_{\mathbf{w}_n}^2 \\ &= \frac{12}{N(N+1)} \left\| \bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} - P_{\mathbf{w}_n} \left(\left(\bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) |\varphi_p \right) \right\|_{\mathbf{w}_n}^2 \end{aligned}$$

$$\begin{aligned}
&= \left\| \left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) - P_{\mathbf{w}_n/N} \left(\left(\bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n/(N+1)}^2 \\
&= \left\| \left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) - P_{\mathbf{w}_n/N} \left(\left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n/(N+1)}^2 \\
&= \left\| \left(\frac{12}{N} \right)^{1/2} \left((\bar{\mathbf{R}}_{-k}, \bar{\mathbf{R}}_k) - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) - P_{\mathbf{w}_n/N} \left(\left(\frac{12}{N} \right)^{1/2} \left((\bar{\mathbf{R}}_{-k}, \bar{\mathbf{R}}_k) - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n/(N+1)}^2 \\
&= \left\| \left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2}, \frac{N}{2} \right) - P_{\mathbf{w}_n/N} \left(\left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2}, \frac{N}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n/(N+1)}^2 \\
&= \left\| \left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2}, (3N)^{1/2} \right) - \right. \\
&\quad \left. P_{\mathbf{w}_n/N} \left(\left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2}, (3N)^{1/2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n/(N+1)}^2. \tag{4.3.54}
\end{aligned}$$

Notice that

$$\begin{aligned}
\left(\frac{12}{N} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) &= \left(\frac{\sum_{i=1}^{k-1} n_i}{N} \right)^{1/2} \left(\frac{12}{\sum_{i=1}^{k-1} n_i} \right)^{1/2} \left(\bar{\mathbf{R}}_{-k} - \frac{\sum_{i=1}^{k-1} n_i + 1}{2} \right) \\
&\xrightarrow{D} \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} (V_\infty - \hat{V}), \text{ as } N \rightarrow +\infty \tag{4.3.55}
\end{aligned}$$

and

$$(3N)^{1/2} \rightarrow +\infty \text{ as } N \rightarrow +\infty. \tag{4.3.56}$$

Combining (4.3.54), (4.3.55) and (4.3.56), we have that, as $N \rightarrow +\infty$,

$$\begin{aligned}
 & \frac{12}{N(N+1)} \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\varphi_p)\|_{\mathbf{w}_n}^2 \\
 & \xrightarrow{D} \left\| \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} (\mathbf{V}_\infty - \hat{\mathbf{V}}), +\infty \right\rangle - P_{\mathbf{w}_\infty} \left(\left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} (\mathbf{V}_\infty - \hat{\mathbf{V}}), +\infty \right) | \varphi_p \rangle \right\|_{\mathbf{w}_\infty}^2 \\
 & = \infty, \quad \text{for } p=1, \dots, k-1,
 \end{aligned} \tag{4.3.57}$$

and

$$\begin{aligned}
 & = \left\| \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} (\mathbf{V}_\infty - \hat{\mathbf{V}}) - P_{\mathbf{w}_\infty^{k-1}} \left(\left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} (\mathbf{V}_\infty - \hat{\mathbf{V}}) | \varphi_{k-1}^{k-1} \right) \right\|_{\mathbf{w}_\infty^{k-1}}^2 \\
 & = \left\| \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty - P_{\mathbf{w}_\infty^{k-1}} \left(\left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty | \varphi_{k-1}^{k-1} \right) \right\|_{\mathbf{w}_\infty^{k-1}}^2, \quad \text{for } p=k,
 \end{aligned} \tag{4.3.58}$$

where $\mathbf{w}_\infty^{k-1} = (\lambda_1, \dots, \lambda_{k-1})$ and $\varphi_{k-1}^{k-1} = \{(x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1} : x_1 \leq \dots \leq x_{k-1}\}$ is the set of umbrella orderings with peak $k-1$ in $(k-1)$ -dimensional Euclidean space. Similarly,

$$\begin{aligned}
 & \frac{12}{N^*(N^*+1)} \|\bar{\mathbf{R}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{R}}^*|\varphi_p)\|_{\mathbf{w}_n^*}^2 \\
 & = \frac{12}{N^*(N^*+1)} \left\| \bar{\mathbf{R}}^* - \frac{N^*+1}{2} - P_{\mathbf{w}_n^*} \left(\left(\bar{\mathbf{R}}^* - \frac{N^*+1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n^*}^2 \\
 & = \left\| \left(\frac{12}{N^*} \right)^{1/2} \left(\bar{\mathbf{R}}^* - \frac{N^*+1}{2} \right) - P_{\mathbf{w}_n^*/N^*} \left(\left(\bar{\mathbf{R}}^* - \frac{N^*+1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n^*/(N^*+1)}^2 \\
 & = \left\| \left(\frac{12}{N^*} \right)^{1/2} \left(\bar{\mathbf{R}}^* - \frac{N^*+1}{2} \right) - P_{\mathbf{w}_n^*/N^*} \left(\left(\frac{12}{N^*} \right)^{1/2} \left(\bar{\mathbf{R}}^* - \frac{N^*+1}{2} \right) | \varphi_p \right) \right\|_{\mathbf{w}_n^*/(N^*+1)}^2 \\
 & \xrightarrow{D} \left\| (\mathbf{V}_\infty^* - \hat{\mathbf{V}}^*) - P_{\mathbf{w}_\infty^*} ((\mathbf{V}_\infty^* - \hat{\mathbf{V}}^*) | \varphi_p) \right\|_{\mathbf{w}_\infty^*}^2, \\
 & = \|\mathbf{V}_\infty^* - P_{\mathbf{w}_\infty^*}(\mathbf{V}_\infty^* | \varphi_p)\|_{\mathbf{w}_\infty^*}^2, \text{ as } N^* \rightarrow +\infty, \text{ for } p=1, \dots, k.
 \end{aligned} \tag{4.3.59}$$

Notice that

$$\begin{aligned}
 T_r = & \min_{p=1, \dots, k} \left[\frac{2N(N+1)}{N(N+1) + N^*(N^*+1)} \left(\frac{12}{N(N+1)} \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\phi_p)\|_{\mathbf{w}_n}^2 \right) \right. \\
 & \left. + \frac{2N^*(N^*+1)}{N(N+1) + N^*(N^*+1)} \left(\frac{12}{N^*(N^*+1)} \|\bar{\mathbf{R}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{R}}^*|\phi_p)\|_{\mathbf{w}_n^*}^2 \right) \right] \\
 & - \min_{p \neq p^*} \left[\frac{2N(N+1)}{N(N+1) + N^*(N^*+1)} \left(\frac{12}{N(N+1)} \|\bar{\mathbf{R}} - P_{\mathbf{w}_n}(\bar{\mathbf{R}}|\phi_p)\|_{\mathbf{w}_n}^2 \right) \right. \\
 & \left. + \frac{2N^*(N^*+1)}{N(N+1) + N^*(N^*+1)} \left(\frac{12}{N^*(N^*+1)} \|\bar{\mathbf{R}}^* - P_{\mathbf{w}_n^*}(\bar{\mathbf{R}}^*|\phi_p)\|_{\mathbf{w}_n^*}^2 \right) \right]. \quad (4.3.60)
 \end{aligned}$$

By Wald's Theorem, we have that, under slippage configurations (4.2.2), T_r converges in distribution to the following random variable:

$$\begin{aligned}
 & \left\{ \frac{2\gamma^2}{\gamma^2 + (1-\gamma)^2} \left\| \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty - P_{\mathbf{w}_\infty^{k-1}} \left(\left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty | \phi_{k-1}^{k-1} \right) \right\|_{\mathbf{w}_\infty^{k-1}}^2 \right. \\
 & \quad \left. + \frac{2(1-\gamma)^2}{\gamma^2 + (1-\gamma)^2} \left\| \mathbf{V}_\infty^* - P_{\mathbf{w}_\infty^*} (\mathbf{V}_\infty^* | \phi_k) \right\|_{\mathbf{w}_\infty^*}^2 \right\} \\
 & - \min_{p^*=1, \dots, k-1} \left\{ \frac{2\gamma^2}{\gamma^2 + (1-\gamma)^2} \left\| \left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty - P_{\mathbf{w}_\infty^{k-1}} \left(\left(\sum_{i=1}^{k-1} \lambda_i \right)^{1/2} \mathbf{V}_\infty | \phi_{k-1}^{k-1} \right) \right\|_{\mathbf{w}_\infty^{k-1}}^2 \right. \\
 & \quad \left. + \frac{2(1-\gamma)^2}{\gamma^2 + (1-\gamma)^2} \left\| \mathbf{V}_\infty^* - P_{\mathbf{w}_\infty^*} (\mathbf{V}_\infty^* | \phi_{p^*}) \right\|_{\mathbf{w}_\infty^*}^2 \right\}, \quad (4.3.61)
 \end{aligned}$$

which, after simple algebraic manipulation, is just $T_r(\lambda, \lambda^*, \gamma)$.

The limiting distribution of T_r under slippage configurations (4.2.2) can be further simplified for cases (4.2.1).

Corollary 4.3.1 For cases (4.2.1), the rank test statistic T_r under slippage configurations (4.2.2) converges in distribution to

$$T_r(\lambda, \lambda^*, \gamma) \stackrel{\Delta}{=} \frac{2(1-\gamma)^2}{\gamma^2 + (1-\gamma)^2} \left\{ \|Z^* - P_{I_k}(Z^* | \phi_k)\|_{I_k}^2 - \min_{p=1, \dots, k-1} \|Z^* - P_{I_k}(Z^* | \phi_p)\|_{I_k}^2 \right\}, \quad (4.3.62)$$

where $\gamma = \lim_{n, n^* \rightarrow \infty} \frac{n}{n + n^*}$ and $Z^* = (Z_1^*, \dots, Z_k^*)$ with $Z_1^*, \dots, Z_k^*$ being iid $N(0, 1)$ random variables.

Proof. For the cases (4.2.1),

$$\lambda_1 = \dots = \lambda_k = \lambda_1^* = \dots = \lambda_k^* = \frac{1}{k}. \quad (4.3.63)$$

Corollary 4.2.1 follows from (2.2.14).

Since the limiting distributions for cases (4.2.1) are only different by a constant scale, it suffices to consider a single special case. We choose the case when $n = n^*$ for which $\gamma = \frac{1}{2}$. Therefore, $T_r(\lambda, \lambda^*, \gamma)$ has the same distribution as that of the test statistic

T_1 (2.1.14) under slippage configurations (2.2.19), as given in Theorem 2.2.2.

Calculation of the critical point $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ for large samples is thus trivial, as illustrated in the next corollary.

Corollary 4.3.2 For large samples, the critical point $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ for cases (4.2.1) can be approximated from $s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ ((2.2.20)) by a simple scale change. That is,

$$s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k) = \frac{2(1-\gamma)^2}{\gamma^2 + (1-\gamma)^2} s_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k), \quad (4.3.64)$$

where

$$\gamma = \lim_{n, n^* \rightarrow \infty} \frac{n}{n + n^*}. \quad (4.3.65)$$

Simulation results show that this approximation is quite reasonable for $n \geq 20$.

CHAPTER V

Simulation Study

Introduction

In this chapter we obtain simulation estimates of the critical points for the various tests discussed in Chapters 2-4. Monte Carlo studies provide further support for the critical point determination methods we advocated in Chapters 2-4.

5.1 Critical Points for the Test for Normal Case with Known Variances

In this section we use the Monte Carlo method to estimate the critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ ((2.1.15)) for the test based on the statistic T_1 (2.1.14). Simulation results reveal that this test is seriously biased. The critical point $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ is conservative for many applications with large samples in the sense that a smaller critical point will be more appropriate. Support for $h_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ as a critical point comes from the fact that, with its use, values of the power function at null parameter configurations commonly expected in applications are, in fact, bounded by the nominal

significance level α .

Simulation studies of critical points are carried out with 100,000 iterations. Simulation studies of power performances are carried out with 10,000 iterations. Fortran subroutines from International Mathematics and Statistics Library (IMSL) are used. We use the Pool-Adjacent-Violator Algorithm (PAVA) and the Minimum Lower Sets Algorithm (see, for example, Robertson *et al*, 1988) to calculate the isotonic regression of a vector with respect to an umbrella ordering.

We begin with $k = 3$, for which it is convenient to express the umbrella ordering $\mu = (\mu_1, \mu_2, \mu_3)$ in terms of the differences between adjacent means: $\delta_i = \mu_i - \mu_{i-1}$ and $\delta_i^* = \mu_i^* - \mu_{i-1}^*$, $i=2, 3$. For example, an umbrella ordering with peak 2 is equivalent to the condition $\delta_2 > 0$ and $\delta_3 < 0$.

As far as the power performance is concerned, Theorem 2.2.1 shows that we need only study the case (2.1.17), where

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_1^{*2} = \sigma_2^{*2} = \sigma_3^{*2} = 1 \text{ and } n_1 = n_2 = n_3 = n_1^* = n_2^* = n_3^* = 1. \quad (5.1.1)$$

As in Section 2.2, we use upper case notations such as Δ_i and Δ for case (5.1.1).

Our simulation study indicates that the critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ can be obtained from the distribution of T_1 under slippage configurations:

$$\Delta_2 = 0, \Delta_3 = +\infty \text{ and } \Delta_2^* = \Delta_3^* = 0. \quad (5.1.2)$$

Table 1 provides the critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ for selected α . When there is no additional information about the shape of the parameter configuration, the critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ provides exact control of the type I error.

However, the critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ causes serious biasedness, as shown in Table 6. It is very conservative if the experiment is designed so that adjacent treatment effects are different by at least some prespecified minimum values of interest. In other words, $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ is unduly large if the umbrella orderings are in fact δ -umbrella orderings. We now investigate the advantages and disadvantages of using $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point for δ -umbrella ordered populations.

Table 2 lists $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ values for selected α . Table 6 gives the values of the power function at boundary parameter configurations with $\Delta_2 = 0$ and $\Delta_2^* = \Delta_3^* = 0$, using $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point. The last row in Table 6 can be considered as the power function at slippage configurations (5.1.2). It can be seen that the power function at parameter configurations $\Delta_2 = 0, \Delta_3 = 4, \Delta_2^* = \Delta_3^* = 0$ is close to that at parameter configurations (5.1.2). Because the power function is continuous around the common boundary of the null and alternative spaces H_0 and H_1 , we can see that the critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ causes serious biasedness in that area of the alternative hypothesis space H_1 where $0 < |\Delta_2|, |\Delta_3|, |\Delta_2^*|, |\Delta_3^*| < 1.5$.

Now let us look at the power function at null parameter configurations with δ -umbrella orderings. We recall from Theorem 2.2.1 that the power function for any general cases (2.2.1) can be calculated from that for case (2.2.17) by the transformations $\Delta_i = \frac{1}{\rho} \delta_i$ and $\Delta_i^* = \frac{1}{\rho^*} \delta_i^*$, $i = 2, \dots, k$. As far as power functions are concerned, δ -umbrella orderings in cases (2.2.1) are equivalent to Δ -umbrella orderings in case (2.2.17) with $\Delta = \min\left\{\frac{1}{\rho}, \frac{1}{\rho^*}\right\} \delta$. Without loss of generality, we assume $\rho = \rho^*$ in the following discussion. Let us use one example to illustrate this relationship. Suppose we sample from two umbrella orderings with

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_1^{*2} = \sigma_2^{*2} = \sigma_3^{*2} = 1 \text{ and } n_1 = n_2 = n_3 = n_1^* = n_2^* = n_3^* = 25.$$

If the experiment is designed so that $\delta=0.2$ is the minimum difference in effects between two adjacent treatment levels, then the power function study is need only be carried out for Δ -umbrella orderings in case (5.1.1) with $\Delta = \sqrt{\frac{25}{1}} 0.2 = 1$.

Table 7 provides support for the following observations:

(1) for Δ -umbrella orderings with $\Delta \geq 0.6$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.10$,

(2) for Δ -umbrella orderings with $\Delta \geq 0.8$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.05$,

and

(3) for Δ -umbrella orderings with $\Delta \geq 1.0$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.01$.

The corresponding recommendations regarding the choice of critical points for the general case (2.2.1) are:

(1) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.6 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.6 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, 2, 3$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point for $\alpha \geq 0.10$,

(2) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.8 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.8 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, 2, 3$, then it is preferable to use

$h_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point for $\alpha \geq 0.05$,

(3) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, 2, 3$, then it is preferable to use

$h_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point for $\alpha \geq 0.01$,

and

(4) if no other information is available, use $c_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point.

In fact, when the conditions in these recommendations are slightly violated, the power function over the null hypothesis space H_0 is still reasonably controlled by the corresponding significance level α . Table 8 suggests that

(5) for any $\alpha \geq .05$, if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, 2, 3$, then it is safe

to use $h_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point.

The simulation study improves on the conclusion in Theorem 2.3.6. An example may be helpful to illustrate the magnitude of Δ required to use $h_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point. Suppose the sample size for each treatment level is bigger than or equal to 25 and the significance level of interest is $\alpha=0.10$. As long as the unknown difference between two adjacent treatment levels can be believed to be roughly bigger than or equal to $0.6 \times 1/\sqrt{25} = 0.12$, then $h_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ is much preferable to $c_1(\alpha, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3, \mathbf{l}_3)$ as the critical point.

The null hypothesis space H_0 is a vast region of parameter configurations. For the absolute majority of null parameter configurations in H_0 , the values of the power function of the test based on T_1 using $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point are controlled at the corresponding significance level α . The values of the power function at some parameter configurations close to the boundary of the null hypothesis space H_0 can be three to ten times as large as the significance level α . These parameter configurations unlikely to occur in practice. However, the theoretical critical point $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ is determined at these boundary parameter configurations with rare patterns. Thus $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ can be considerably greater than $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ for the same nominal level α . We feel that if the sample sizes are reasonably large or the treatment levels are sparse enough to produce reasonable difference in effects between adjacent treatment levels, the use of $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point will yield conclusions closer to the truth.

The possibility of an inflated type I error due to the use of $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point is a price paid for the unbiasedness we obtain using $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$. Simulation suggests that the use of $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ achieves a reasonable compromise between the potential for an inflated type I error and the serious biasedness from the use of $c_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$. The test based on T_1 is usually carried out after separate tests of homogeneity against umbrella ordering (see, for example, Chen and Wolfe, 1991) have shown significant umbrella ordering patterns in both groups. This adds more support for the use of $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ as the critical point.

Now we study the case when $k = 4$. Theorem 2.2.1 shows that the discussion about general cases (2.2.1) can be reduced to the case (2.2.17), where

$$\sigma_1^2 = \dots = \sigma_4^2 = \sigma_1^{*2} = \dots = \sigma_4^{*2} = 1 \text{ and } n_1 = \dots = n_4 = n_1^* = \dots = n_4^* = 1. \quad (5.1.3)$$

The simulation study indicates that $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ is given by the distribution of T_1 under slippage configurations, where

$$\Delta_2 = \Delta_3 = 0, \Delta_4 = +\infty \text{ and } \Delta_2^* = \Delta_3^* = \Delta_4^* = 0. \quad (5.1.4)$$

Table 1 gives $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ values for selected significance levels α . When there is no other information than that the populations are strict umbrella orderings, $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ controls the type I error exactly.

However, $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ is unduly large if the populations are indeed δ -umbrella orderings. We consider $h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as an alternative. Table 2 gives $h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ values for selected significance levels α . Table 6 provides the values of the power function at boundary parameter configurations with $\Delta_2 = \Delta_3 = 0$ and $\Delta_2^* = \Delta_3^* = \Delta_4^* = 0$ using $h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point. The last row can be considered the power function at parameter configurations (5.1.4). It can be seen that the power function at parameter configurations $\Delta_2 = \Delta_3 = 0, \Delta_4 = 4$ and $\Delta_2^* = \Delta_3^* = \Delta_4^* = 0$ is close to that at slippage configurations (5.1.4). The values of the power function at some parameter configurations close to the boundary of the null hypothesis space H_0 can be three to ten times as large as the significant level α . While these parameter configurations are not likely to occur in practice, they determine the critical point $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$. Therefore, $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ can be considerably greater than $h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$. It can be seen that the critical point $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ cause serious biasedness in the area of the alternative hypothesis space H_1 with $0 < |\Delta_2|, |\Delta_3|, |\Delta_4|, |\Delta_2^*|, |\Delta_3^*|, |\Delta_4^*| < 1.5$.

In experiments where the treatment levels are chosen so that the effects have the δ -umbrella orderings, the simulation results presented in Table 7 yield the following observations:

(1) for Δ -umbrella orderings with $\Delta \geq 0.6$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.10$,

(2) for Δ -umbrella orderings with $\Delta \geq 0.7$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.05$,

and

(3) for Δ -umbrella orderings with $\Delta \geq 0.9$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.01$.

The corresponding recommendations regarding the choice of critical point for general cases (2.2.1) are:

(1) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.6 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.6 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 4$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point for $\alpha \geq 0.10$,

(2) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.7 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.7 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 4$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point for $\alpha \geq 0.05$,

(3) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.9 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.9 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 4$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point for $\alpha \geq 0.01$,

and

(4) if no other additional information is available, use $c_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point.

When the conditions in these recommendations are slightly violated, the power function over the null hypothesis space H_0 is still reasonably controlled by the corresponding α value. Simulations results presented in Table 8 indicate that the following conclusion also holds for $k = 4$:

(5) for any $\alpha \geq .05$, if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 4$, then it is safe to use $h_1(\alpha, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4, \mathbf{1}_4)$ as the critical point.

Now we study the case when $k=5$. Again, we need only discuss case (2.2.17), where

$$\sigma_1^2 = \dots = \sigma_5^2 = \sigma_1^{*2} = \dots = \sigma_5^{*2} = 1 \text{ and } n_1 = \dots = n_5 = n_1^* = \dots = n_5^* = 1. \quad (5.1.5)$$

The simulation study indicates that $c_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ is given by the distribution of T_1 under the slippage configurations

$$\Delta_2 = \Delta_3 = \Delta_4 = 0, \Delta_5 = +\infty \text{ and } \Delta_2^* = \Delta_3^* = \Delta_4^* = \Delta_5^* = 0. \quad (5.1.6)$$

Table 1 lists $c_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ values for selected significance levels α . Compared with $h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ in Table 2, $c_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ can be considerably bigger. Table 6 provides the values of the power function at boundary parameter configurations with $\Delta_2 = \Delta_3 = \Delta_4 = 0$ and $\Delta_2^* = \Delta_3^* = \Delta_4^* = \Delta_5^* = 0$ using $h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point. The last row can be considered the values of the power function at slippage configurations (5.1.6). The bias for $k = 5$ is more severe than it is for $k = 3$. The simulation results in Table 7 suggest:

(1) for Δ -umbrella orderings with $\Delta \geq 0.5$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.10$,

(2) for Δ -umbrella orderings with $\Delta \geq 0.6$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.05$,

and

(3).for Δ -umbrella orderings with $\Delta \geq 0.8$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.01$.

The corresponding recommendations regarding the choice of critical point for general cases (2.2.1) are:

(1) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 5$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point for $\alpha \geq 0.10$,

(2) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.6 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.6 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 5$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point for $\alpha \geq 0.05$,

(3) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.8 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.8 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 5$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point for $\alpha \geq 0.01$,

and

(4) if no other additional information is available, use $c_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point.

When the conditions in these recommendations are slightly violated, the power function over the null hypothesis space H_0 is still reasonably controlled by the corresponding α value. Table 8 indicates that the following conclusion also holds for $k=5$:

(5) for any $\alpha \geq .05$, if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 5$, then it is safe

to use $h_1(\alpha, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5, \mathbf{1}_5)$ as the critical point.

Finally we consider $k = 6$. It suffices to consider case (2.2.17), where

$$\sigma_1^2 = \dots = \sigma_6^2 = \sigma_1^{*2} = \dots = \sigma_6^{*2} = 1 \text{ and } n_1 = \dots = n_6 = n_1^* = \dots = n_6^* = 1. \quad (5.1.7)$$

The simulation study indicates that $c_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ is given by the distribution of T_1 under the following parameter configurations

$$\Delta_2 = \dots = \Delta_5 = 0, \Delta_6 = +\infty \text{ and } \Delta_2^* = \dots = \Delta_6^* = 0. \quad (5.1.8)$$

Table 1 and Table 2 list $c_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ and $h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ values for selected significance levels α . Table 6 gives the values of the power function at boundary parameter configurations with $\Delta_2 = \dots = \Delta_5 = 0$ and $\Delta_2^* = \dots = \Delta_6^* = 0$ using $h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point. The last row can be considered the power function at slippage configurations (5.1.8). The bias for $k = 6$ is severe. The simulation results in Table 7 suggest that if $h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ is used as the critical point, then

(1) for Δ -umbrella orderings with $\Delta \geq 0.4$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.10$,

(2) for Δ -umbrella orderings with $\Delta \geq 0.5$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.05$,

and

(3).for Δ -umbrella orderings with $\Delta \geq 0.6$, the power function over the null hypothesis space H_0 is controlled by α for $\alpha \geq 0.01$.

The corresponding recommendations regarding the choice of critical point for general cases (2.2.1) are:

(1) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.4 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.4 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 6$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point for $\alpha \geq 0.10$,

(2) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 6$, then it is preferable to use

$h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point for $\alpha \geq 0.05$,

(3) if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq 0.6 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 0.6 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 6$, then it is preferable to use $h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point for $\alpha \geq 0.01$,

and

(4) if no other additional information is available, use $c_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point.

When the conditions in these recommendations are slightly violated, the power function over the null hypothesis space H_0 is still reasonably controlled by the corresponding α value. Table 8 indicates that the following conclusion also holds for $k=6$:

(5) for any $\alpha \geq .05$, if the treatment effect differences between two adjacent levels for both groups are at least δ and if $\delta \geq .5 \frac{\sigma_i}{\sqrt{n_i}}$ and $\delta \geq 5 \frac{\sigma_i^*}{\sqrt{n_i^*}}$ for $i=1, \dots, 6$, then it is safe to use $h_1(\alpha, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6, \mathbf{1}_6)$ as the critical point.

5.2 Critical Points for the Tests for Normal Cases with Unknown Variances

Simulation studies of the critical points are carried out with 100,000 iterations while simulation studies of power performances are carried out with 10,000 iterations. Tables 2 and 3 give the simulation estimates of the critical point $S_2(\alpha, N^*)$ for the test based on T_2 (3.2.8) and the critical point $S_3(\alpha, d)$ for the test based on T_3 (3.3.4). For case (3.2.23), the power study suggests that the critical point $S_2(\alpha, N^*)$ for the test based on T_2 can be obtained from the distribution of the test statistic T_2 at slippage configurations

$$\Delta_2 = \dots = \Delta_{k-1} = 0, \Delta_k = +\infty \text{ and } \Delta_2^* = \dots = \Delta_k^* = 0. \quad (5.2.1)$$

For case (3.3.14), the critical point $S_3(\alpha, d)$ for the test based on T_3 can be obtained from the distribution of the test statistic T_3 at slippage configurations (5.2.1). The critical points in Tables 2 and 3 are precise to the first decimal place. We notice that both critical points $S_2(\alpha, N^*)$ and $S_3(\alpha, d)$ converge as the sample sizes increase. This fact can be deduced easily from the distributions of T_2 and T_3 given in Theorems 3.2.3 and 3.3.3, respectively, via a Taylor's expansion of the natural logarithm function and the fact that the mean of a chi-square distributed random variable is its degrees of freedom. It is worthwhile to point out the simulation estimates of the critical points for extremely small significance levels such as 0.001 are slightly erratic due to the few observations at the upper tail of the ordered simulated sample statistics. This observation is true for all the simulation studies in this dissertation.

5.3 Critical Points for the Test Based on Ranks

The test statistic T_r is given in (4.1.2) with critical points defined in (4.1.3). Simulation studies of these critical points are carried out with 100,000 iterations. Due to the use of the subroutine RANKS in IMSL, it may take hours to obtain critical points for one combination of sample sizes, especially when the sample sizes are quite large. Power studies are carried out with 10,000 iterations. Power studies indicate that a level α critical point for the rank test based on T_r is the maximum of the upper α percentiles of the distributions of the test statistic T_r under slippage configurations (4.2.2) and their transformations (4.2.7), (4.2.8) and (4.2.9). Since there are numerous combinations of

sample sizes n and n^* , we concentrate on the simplest and most important balanced experiments, where

$$n_1 = \dots = n_k = n \text{ and } n_1^* = \dots = n_k^* = n^*. \quad (5.3.1)$$

Table 5 gives simulation estimates of the critical points for the test based on T_r , according to Theorem 4.2.1. In order to illustrate the convergence rates of the small sample critical points to the limiting critical points given in Corollary 4.3.2, we group the entries in Table 5 with respect to the sample size ratio $\gamma = \frac{n}{n + n^*}$. A comparison between Table 1 and

Table 5 shows that Corollary 4.3.2 and Table 1 can be used to obtain fairly accurate large sample critical points for the rank test based on T_r when the smaller common sample size between the two umbrella orderings is above 20. Again, simulation estimates of the critical points are precise to the first decimal place. Critical points for cases with unbalanced sample sizes are not tabulated because of the tremendous number of possible cases.

Fortunately, FORTRAN programs to compute these critical points are available from the author. For an application, one can simply enter the sample sizes of the data into these FORTRAN programs and automatically obtain the desired critical points. To simplify tabulation of critical points for unbalanced experiments, we are interested in the robustness of the critical points. In other words, critical points for balanced experiments may provide estimates for unbalanced ones if the imbalance is not severe. Whether this is true and, if so, how to carry out the extension are topics for future research, as discussed in Chapter 6.

CHAPTER VII

Extensions and Future Research

Introduction

In this chapter we extend some of the results obtained in Chapter 2-5 to general umbrella orderings, discuss the applications to a completely randomized block design and outline some topics for future research.

6.1 Extensions

6.1.1 Alternatives to the Proposed Tests for Normal Cases with Unknown Variances

For the normal distribution setting with unknown variances discussed in Chapter 3, alternatives to test statistics T_2 (3.2.8) and T_3 (3.3.5) can be obtained from test statistic T_1 (2.2.14) by a technique similar to that used in Student's t test. For either T_2 or T_3 , first estimate the unknown population variances by the corresponding pooled sample variances and then carry out the test based on T_1 as if the sample variances were actually

the “known” population variances. When the sample sizes are large, $c_1(\alpha, \mathbf{n}, \mathbf{n}^*, \sigma^2, \sigma^{*2})$ in (2.1.15) provides approximate critical points for these modified procedures. Sample sizes can be considered large if the degrees for freedom of the pooled sample variances are at least 30.

6.1.2 Modifications for Umbrella Orderings

The problems and test procedures discussed thus far are all for strict umbrella orderings. However, they can be modified easily to deal with general umbrella orderings. Let P be the set of peaks for the first umbrella ordering and let P^* be the set of peaks for the second umbrella ordering. Then problem Q_1 (1.4.1) becomes

$$H_0: P \cap P^* \neq \emptyset \quad \text{versus} \quad H_1: P \cap P^* = \emptyset. \quad (6.1.2.1)$$

Let $A = \{P \otimes \{1, \dots, k\}\} \cap \{\{1, \dots, k\} \otimes P^*\}$ and $B = \{1, \dots, k\} \otimes \{1, \dots, k\}$, where $\otimes$ denotes the Cartesian product. Then, the likelihood ratio test and rank test statistics for problem (6.1.2.1) can be obtained easily by replacing $\min_{p=p}$ with $\min_{(p,p^*) \in A}$ and $\min_{p \neq p}$ with $\min_{(p,p^*) \in B-A}$ in the corresponding test statistics T_1 (2.2.14), T_2 (3.2.8), and T_2 (3.3.5), as developed for problem Q_1 (1.4.1) under strict umbrella orderings.

The critical points are the same as those for the corresponding tests for strict umbrella orderings.

6.1.3 Tests for a Randomized Complete Block Design

Two-way layout data arise in situations where a block effect is involved. A model for such a setting can be described as

$$X_{ij} = \mu_i + \beta_j + \varepsilon_{ij} \text{ for } i=1, \dots, k \text{ and } j=1, \dots, n,$$

where μ_i is the fixed treatment level i effect, β_j is the random effect of block j and ε_{ij} is the measurement error associated with the observation from treatment level i and block j . We assume that $\beta_j \sim N(0, \sigma_\beta^2)$ and $\varepsilon_{ij} \sim N(0, \sigma^2)$ for $i=1, \dots, k$ and $j=1, \dots, n$, and that all

$n(k+1)$ random variables are mutually independent. It is easy to see that the variables

$\bar{X}_{i.} = \frac{1}{n} \sum_{j=1}^n X_{ij} = \frac{1}{n} \sum_{j=1}^n (X_{ij} - \beta_j) \sim N\left(0, \frac{\sigma^2}{n}\right)$, $i=1, \dots, k$, are mutually independent. Let

$\bar{X}_{.j} = \frac{1}{k} \sum_{i=1}^k X_{ij}$, for $j=1, \dots, n$, and $\bar{X}_{..} = \frac{1}{kn} \sum_{i=1}^k \sum_{j=1}^n X_{ij}$. When divided by σ^2 , the error

sum of squares $\hat{\sigma}^2 = \sum_{i=1}^k \sum_{j=1}^n (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2$ has a chi-square distribution with $(k-$

$1)(n-1)$ degrees of freedom and it is independent of $\bar{X} = (\bar{X}_{1.}, \dots, \bar{X}_{k.})$.

Suppose there is a second two-way layout, with model

$$X_{ij}^* = \mu_i^* + \beta_j^* + \varepsilon_{ij}^* \text{ for } i=1, \dots, k \text{ and } j=1, \dots, n^*,$$

where $\beta_j^* \sim N(0, \sigma_{\beta^*}^2)$ and $\varepsilon_{ij}^* \sim N(0, \sigma^{*2})$ for $i=1, \dots, k$ and $j=1, \dots, n^*$, with

$\{\varepsilon_{ij}\}$ and $\{\beta_j^*\}$ being mutually independent. We also assume that $\{\varepsilon_{ij}\}$ and $\{\varepsilon_{ij}^*\}$ are

mutually independent. However, $\{\beta_i\}$ and $\{\beta_i^*\}$ can be correlated. Again, under this

model, $\bar{X}_{i.}^* = \frac{1}{n^*} \sum_{j=1}^{n^*} X_{ij}^*$ are mutually independent for $i=1, \dots, k$. Let $\bar{X}_{.j}^* = \frac{1}{k} \sum_{i=1}^k X_{ij}^*$, for

$j=1, \dots, n^*$, and $\bar{X}_{..}^* = \frac{1}{kn^*} \sum_{i=1}^k \sum_{j=1}^{n^*} X_{ij}^*$. When divided by σ^{*2} , the error sum of squares

$\hat{\sigma}^{*2} = \sum_{i=1}^k \sum_{j=1}^n (X_{ij}^* - \bar{X}_{i.}^* - \bar{X}_{.j}^* + \bar{X}_{..}^*)^2$ has a chi-squared distribution with $(k-1)(n^*-1)$ degrees of freedom and it is independent of $\bar{\mathbf{X}}^* = (\bar{X}_{1.}^*, \dots, \bar{X}_{k.}^*)$.

When both $\mu = (\mu_1, \dots, \mu_k)$ and $\mu^* = (\mu_1^*, \dots, \mu_k^*)$ have strict umbrella orderings, we want to test if their peaks are the same or not; that is,

$$H_0: p = p^* \quad \text{versus} \quad H_1: p \neq p^*.$$

Notice that the various statistics mentioned above have distributions similar to those discussed in Section 3.2. Therefore, a test procedure can be obtained easily by transforming the test statistic T_2 (3.2.8) in Section 3.2. We can take the test statistic for this two-way layout setting to be

$$T_b = \min_{p=1, \dots, k} \left\{ N \ln \left[\frac{n \|\bar{\mathbf{X}} - P_{1_k}(\bar{\mathbf{X}} | \varphi_p)\|_{1_k}^2}{\hat{\sigma}^2} + 1 \right] + N^* \ln \left[\frac{n^* \|\bar{\mathbf{X}}^* - P_{1_k}(\bar{\mathbf{X}}^* | \varphi_p)\|_{1_k}^2}{\hat{\sigma}^{*2}} + 1 \right] \right\} \\ - \min_{p \neq p^*} \left\{ N \ln \left[\frac{n \|\bar{\mathbf{X}} - P_{1_k}(\bar{\mathbf{X}} | \varphi_p)\|_{1_k}^2}{\hat{\sigma}^2} + 1 \right] + N^* \ln \left[\frac{n^* \|\bar{\mathbf{X}}^* - P_{1_k}(\bar{\mathbf{X}}^* | \varphi_{p^*})\|_{1_k}^2}{\hat{\sigma}^{*2}} + 1 \right] \right\},$$

where $N = kn - n + 1$ and $N^* = kn^* - n^* + 1$. The appropriate critical points are the same as the critical points $s_2(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k) = S_2(\alpha, N, N^*)$ ((3.2.25)) for the test statistic T_2 with $N = kn - n + 1$ and $N^* = kn^* - n^* + 1$.

6.2 Future Research

In this dissertation we derived the likelihood ratio test statistics under normality assumptions and found the associated critical points. For the normal case with known

variances, we discussed the advantages of using homogeneity configurations to determine critical points for large samples. We also derived a nonparametric distribution-free test procedure based on ranks.

A problem worth further investigation is that of the robustness of the obtained critical points. A positive answer would greatly reduce the burden of tabulation for the critical points for large samples. Consider the rank test based on T_r (4.1.2), for example. We would like to know if the limiting critical points are the same for all situations having the same $\lambda_1, \dots, \lambda_k, \lambda_1^*, \dots, \lambda_k^*$ and γ .

As we pointed out in Chapter 1, there is a large class of interesting testing problems regarding two umbrella ordered groups. The testing problem Q_1 (1.4.1) is a special case of a more general class of testing problems:

$$Q_g: \quad H_0: (p, p^*) \in A \quad \text{versus} \quad H_1: (p, p^*) \in B - A, \quad (6.2.1)$$

where A is a nonempty proper subset of $B = \{(i, j): i, j = 1, \dots, k\}$. If we take $A = \{(p, p^*): p \leq p^* = 1, \dots, k\}$, then the associated Q_g corresponds to testing whether the first group obtains its peak no later than the second group. That is,

$$H_0: p \leq p^* \quad \text{versus} \quad H_1: p > p^*. \quad (6.2.3)$$

The likelihood ratio test and rank test statistics for problem Q_g (6.2.2) can be obtained easily by replacing $\min_{p=p^*}$ with $\min_{(p, p^*) \in A}$ and $\min_{p \neq p^*}$ with $\min_{(p, p^*) \in B - A}$ in the corresponding test statistics for problem Q_1 (1.4.1). However, the determination of appropriate critical points is unknown and is a topic for future research.

The theory and techniques developed in this dissertation can be extended directly to the settings with more than two umbrella orderings. The test statistics would have forms

similar to those for the case with two umbrella orderings. However, the determination of critical points may be considerably more difficult.

Alternative methods for constructing test statistics other than the likelihood ratio method are widely available. The likelihood ratio tests and their corresponding nonparametric distribution-free counterparts based on ranks have the advantages of uniformly high powers over the entire (large) alternative spaces. However, the test statistics have complex forms and the critical points are difficult to obtain. An alternative class of tests with simpler test statistics are those based on contrasts. The idea behind the construction of a multiple contrast test is simple. We first select one or more "axes" in the "middle" of each convex cone of umbrella orderings with different peaks and define a measurement of the "angle" between any two k -dimensional vectors. The associated contrast test statistic is then the difference between the minimum of the sum of the angles between the two observed vectors and the axes in the same cones and the minimum of the sum of the angles between the two observed vectors and the axes in different cones. The choice of the axes is crucial to the simplicity of the associated multiple contrast test procedures.

Table 1

Critical points $c_1(\alpha, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k, \mathbf{1}_k)$ for the test based on the statistic T_1

k	α							
	.20	.15	.10	.05	.025	.01	.005	.001
3	1.71	2.20	2.93	4.20	5.53	7.28	8.51	11.75
4	2.46	3.05	3.90	5.31	6.71	8.64	10.10	13.27
5	3.11	3.76	4.65	6.15	7.66	9.61	11.19	14.23
6	3.61	4.28	5.20	6.77	8.34	10.33	11.88	14.99

Table 2

Approximate critical points $h_1(\alpha, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3, \mathbf{1}_3)$ for the test based on the statistic T_1 when sample sizes are large

k	α							
	.20	.15	.10	.05	.025	.01	.005	.001
3	0.57	0.76	1.05	1.56	2.10	2.84	3.43	4.79
4	0.88	1.07	1.39	1.98	2.56	3.35	3.95	5.33
5	1.08	1.33	1.70	2.32	2.98	3.82	4.47	5.89
6	1.33	1.61	2.01	2.68	3.36	4.26	4.93	6.42

Table 3Critical points $S_2(\alpha, N^*)$ for the test based on the statistic T_2

k=3		α						
N^*	.20	.15	.10	.05	.025	.01	.005	.001
4	6.87	8.88	11.92	17.26	22.70	29.99	35.54	48.38
5	4.30	5.57	7.37	10.60	14.01	18.43	21.94	29.39
6	3.45	4.46	5.89	8.38	11.05	14.51	17.31	23.35
7	3.00	3.90	5.19	7.45	9.78	12.96	15.26	20.90
8	2.73	3.57	4.74	6.76	8.90	11.76	13.87	18.70
9	2.53	3.29	4.39	6.33	8.26	11.03	12.92	17.62
10	2.47	3.20	4.25	6.04	7.92	10.51	12.33	16.71
11	2.35	3.04	4.06	5.82	7.63	10.11	11.82	15.78
12	2.26	2.93	3.92	5.64	7.42	9.76	11.54	15.83
13	2.21	2.87	3.81	5.52	7.19	9.45	11.13	14.86
14	2.16	2.80	3.75	5.40	7.02	9.26	10.96	15.32
15	2.14	2.78	3.68	5.30	6.93	9.23	10.85	14.80
18	2.06	2.68	3.59	5.11	6.73	8.91	10.59	14.59
21	2.01	2.62	3.47	5.00	6.57	8.55	10.10	13.45
24	1.92	2.51	3.33	4.79	6.28	8.34	9.85	13.66
27	1.90	2.48	3.30	4.73	6.24	8.21	9.77	13.30
30	1.87	2.43	3.26	4.67	6.09	7.98	9.63	12.89
36	1.84	2.39	3.18	4.58	6.00	7.92	9.43	13.02
42	1.83	2.36	3.14	4.51	5.89	7.78	9.15	12.69
48	1.79	2.33	3.12	4.48	5.87	7.71	9.25	12.78
54	1.76	2.29	3.07	4.47	5.86	7.68	9.15	12.23
60	1.78	2.32	3.12	4.47	5.82	7.54	8.93	12.40
72	1.78	2.32	3.08	4.43	5.80	7.66	8.96	12.08
84	1.76	2.28	3.05	4.36	5.72	7.51	8.85	12.22
96	1.74	2.27	3.01	4.36	5.66	7.40	8.76	12.18

Table 3 (continued)

k=4		α						
N*	.20	.15	.10	.05	.025	.01	.005	.001
5	8.53	10.63	13.82	19.51	25.88	29.99	34.33	56.27
6	6.01	7.41	9.46	12.98	16.59	21.82	25.58	34.64
7	4.96	6.13	7.78	10.62	13.53	17.39	20.41	27.61
8	4.40	5.43	6.87	9.40	11.98	15.39	17.88	23.55
9	4.04	5.00	6.33	8.66	10.98	14.08	16.50	22.01
10	3.81	4.69	5.95	8.13	10.40	13.34	15.59	20.56
11	3.63	4.45	5.66	7.77	9.82	12.60	14.59	19.27
12	3.46	4.28	5.44	7.47	9.40	11.82	13.99	18.87
13	3.42	4.20	5.33	7.24	9.20	11.77	13.65	17.91
14	3.27	4.02	5.11	7.03	8.89	11.48	13.33	17.92
15	3.22	3.97	5.05	6.93	8.66	11.15	13.10	17.38
16	3.15	3.89	4.95	6.79	8.55	11.03	12.98	16.91
20	3.01	3.70	4.68	6.36	8.02	10.26	11.97	15.48
24	2.90	3.58	4.54	6.20	7.82	9.92	11.50	15.07
28	2.84	3.50	4.43	6.02	7.68	9.90	11.62	15.50
32	2.80	3.46	4.37	5.95	7.55	9.72	11.38	15.18
40	2.71	3.34	4.23	5.76	7.26	9.27	10.68	14.21
48	2.67	3.31	4.18	5.76	7.24	9.25	10.77	14.52
56	2.61	3.24	4.15	5.63	7.09	9.04	10.61	14.20
80	2.58	3.19	4.05	5.55	7.02	8.98	10.45	14.13
96	2.56	3.17	4.01	5.47	6.93	8.82	10.47	13.93

Table 3 (continued)

k=5		α						
N*	.20	.15	.10	.05	.025	.01	.005	.001
6	9.06	11.06	13.98	19.32	25.29	33.97	41.30	58.27
7	7.02	8.45	10.48	14.08	17.80	22.97	27.22	37.68
8	6.04	7.25	9.02	12.09	15.26	19.36	22.46	30.26
9	5.45	6.55	8.08	10.79	13.49	17.09	19.84	26.55
10	5.10	6.13	7.59	10.04	12.52	15.76	18.18	24.45
11	4.79	5.78	7.17	9.46	11.72	14.77	17.24	22.68
12	4.57	5.49	6.79	9.00	11.20	14.12	16.41	21.58
13	4.42	5.31	6.59	8.72	10.80	13.75	16.03	21.01
14	4.29	5.15	6.36	8.46	10.49	13.19	15.25	20.31
15	4.19	5.03	6.20	8.19	10.14	12.73	14.66	19.22
16	4.10	4.94	6.12	8.11	10.04	12.55	14.46	19.04
17	4.03	4.84	6.00	7.94	9.87	12.42	14.20	18.44
18	3.94	4.75	5.87	7.77	9.62	12.00	13.89	18.00
19	3.89	4.69	5.80	7.63	9.46	11.88	13.84	18.44
20	3.85	4.65	5.72	7.58	9.44	11.81	13.75	17.70
25	3.66	4.41	5.46	7.20	9.00	11.22	12.92	16.98
30	3.53	4.25	5.26	7.01	8.76	11.05	12.74	16.48
35	3.49	4.21	5.21	6.87	8.57	10.78	12.37	15.93
40	3.42	4.12	5.09	6.78	8.45	10.73	12.31	16.00
60	3.31	4.00	4.93	6.54	8.17	10.25	11.71	15.54
100	3.25	3.89	4.80	6.40	7.97	9.94	11.53	14.70

Table 3 (continued)

k=6		α						
N*	.20	.15	.10	.05	.025	.01	.005	.001
7	9.22	10.96	13.56	18.42	23.76	31.94	39.35	59.13
8	7.72	9.14	11.16	14.63	18.29	23.33	27.65	37.38
9	6.80	8.06	9.84	12.83	15.89	20.17	23.19	31.11
10	6.19	7.35	8.93	11.68	14.50	18.15	21.09	27.12
11	5.81	6.88	8.37	10.95	13.48	16.90	19.45	25.97
12	5.55	6.56	7.98	10.37	12.68	15.87	18.19	23.89
13	5.35	6.34	7.73	10.02	12.35	15.53	18.00	23.57
14	5.12	6.06	7.38	9.60	11.77	14.71	16.84	21.49
15	5.03	5.95	7.23	9.44	11.58	14.45	16.70	21.89
16	4.91	5.82	7.05	9.19	11.25	14.03	16.02	20.75
17	4.78	5.65	6.87	8.91	10.90	13.56	15.64	20.20
18	4.68	5.55	6.74	8.81	10.81	13.50	15.35	20.02
19	4.61	5.45	6.64	8.64	10.62	13.19	15.21	19.98
20	4.58	5.45	6.62	8.59	10.60	13.21	15.08	19.35
21	4.53	5.36	6.51	8.44	10.40	12.94	14.73	19.20
22	4.45	5.26	6.38	8.27	10.14	12.58	14.28	18.53
23	4.43	5.24	6.38	8.25	10.09	12.51	14.32	18.40
24	4.33	5.15	6.29	8.13	10.01	12.32	14.03	17.81
30	4.19	4.99	6.08	7.88	9.67	11.97	13.62	17.90
36	4.11	4.85	5.92	7.69	9.42	11.71	13.59	17.46
42	4.01	4.76	5.78	7.52	9.19	11.46	13.19	16.80
48	3.96	4.70	5.70	7.46	9.17	11.32	12.93	16.92
60	3.88	4.59	5.59	7.28	8.95	11.05	12.63	16.29
90	3.80	4.52	5.52	7.15	8.73	10.90	12.54	15.77

Table 4Critical points $S_3(\alpha, d)$ for the test based on the statistic T_3

k=3		α							
d	.20	.15	.10	.05	.025	.01	.005	.001	
7	8.59	11.33	15.38	22.99	31.29	43.00	51.93	74.08	
8	5.59	7.24	9.75	14.19	18.81	25.29	30.29	42.88	
9	4.47	5.78	7.71	11.12	14.77	19.53	23.27	32.97	
10	3.85	4.96	6.60	9.52	12.49	16.50	19.73	27.11	
11	3.42	4.47	5.97	8.61	11.30	14.88	17.64	24.47	
12	3.17	4.11	5.50	7.85	10.28	13.66	16.11	22.12	
13	2.93	3.80	5.07	7.38	9.72	12.91	15.13	20.58	
14	2.84	3.67	4.88	6.98	9.20	12.22	14.56	20.12	
15	2.69	3.50	4.66	6.77	8.94	11.72	13.83	18.77	
16	2.58	3.33	4.42	6.38	8.40	11.13	13.22	18.18	
17	2.52	3.27	4.40	6.33	8.24	10.78	12.72	17.65	
18	2.46	3.18	4.21	6.07	8.03	10.62	12.57	17.21	
21	2.32	3.02	4.02	5.78	7.53	9.82	11.57	15.99	
24	2.20	2.83	3.80	5.46	7.17	9.46	11.06	14.94	
27	2.11	2.74	3.66	5.29	6.83	8.98	10.70	14.53	
30	2.08	2.71	3.59	5.17	6.80	9.00	10.67	14.24	
36	2.02	2.62	3.49	5.03	6.58	8.67	10.21	13.81	
42	1.96	2.54	3.39	4.84	6.32	8.36	9.86	13.63	
48	1.93	2.50	3.33	4.78	6.25	8.30	9.89	13.58	
60	1.89	2.45	3.26	4.68	6.15	8.18	9.65	12.81	
84	1.83	2.37	3.14	4.52	5.93	7.75	9.31	12.53	

Table 4 (continued)

k=4		α						
d	.20	.15	.10	.05	.025	.01	.005	.001
9	8.75	10.92	14.15	20.08	27.08	37.51	47.07	71.42
10	6.89	8.55	10.94	15.17	19.80	26.20	31.54	44.15
11	5.95	7.37	9.42	13.01	16.86	21.72	25.38	34.40
12	5.33	6.60	8.35	11.48	14.59	18.81	21.95	30.53
13	4.85	6.00	7.61	10.46	13.31	17.17	20.00	27.24
14	4.53	5.60	7.10	9.73	12.30	15.90	18.78	25.22
15	4.30	5.31	6.70	9.14	11.55	15.04	17.54	23.62
16	4.08	5.02	6.37	8.68	10.94	14.10	16.67	22.08
17	3.95	4.88	6.16	8.37	10.64	13.52	15.57	21.02
18	3.82	4.72	6.01	8.18	10.45	13.29	15.62	20.59
19	3.69	4.58	5.83	7.96	10.01	12.83	15.13	20.06
20	3.62	4.47	5.64	7.72	9.81	12.54	14.62	19.92
21	3.56	4.38	5.54	7.58	9.60	12.37	14.36	19.00
22	3.46	4.27	5.40	7.41	9.37	11.97	13.92	18.35
23	3.43	4.21	5.34	7.33	9.33	11.88	13.89	18.95
24	3.34	4.13	5.26	7.17	9.07	11.71	13.64	18.13
28	3.18	3.95	5.01	6.84	8.59	10.93	12.69	17.29
32	3.09	3.83	4.89	6.61	8.36	10.69	12.51	16.41
36	2.98	3.70	4.65	6.36	8.10	10.45	12.19	16.26
40	2.94	3.62	4.61	6.32	7.99	10.09	11.77	15.74
48	2.87	3.55	4.52	6.14	7.73	9.93	11.44	15.38
56	2.79	3.45	4.38	5.98	7.60	9.78	11.30	15.00
100	2.63	3.26	4.14	5.65	7.16	9.18	10.61	13.91

Table 4 (continued)

k=5		α						
d	.20	.15	.10	.05	.025	.01	.005	.001
11	8.80	10.64	13.36	18.14	23.18	30.72	37.12	54.94
12	7.55	9.16	11.44	15.30	19.25	24.86	28.92	40.14
13	6.77	8.15	10.11	13.52	16.88	21.64	25.38	33.39
14	6.23	7.51	9.30	12.38	15.51	19.81	23.13	31.03
15	5.86	7.06	8.77	11.63	14.49	18.57	21.50	28.22
16	5.49	6.61	8.20	10.90	13.69	17.33	20.00	26.65
17	5.34	6.40	7.92	10.46	12.98	16.53	19.14	25.79
18	5.07	6.11	7.53	9.99	12.42	15.77	18.23	23.66
19	4.93	5.92	7.33	9.70	12.08	15.24	17.48	22.98
20	4.82	5.78	7.12	9.45	11.73	14.75	16.98	22.41
21	4.65	5.60	6.94	9.17	11.44	14.43	16.77	21.62
22	4.56	5.50	6.79	8.99	11.27	14.17	16.12	21.34
23	4.46	5.38	6.64	8.77	10.88	13.67	15.75	20.94
24	4.38	5.25	6.49	8.60	10.71	13.43	15.58	20.31
25	4.30	5.20	6.43	8.48	10.60	13.35	15.46	20.00
30	4.05	4.88	6.04	8.03	10.07	12.68	14.74	19.13
35	3.91	4.68	5.76	7.62	9.45	11.82	13.73	18.36
40	3.76	4.53	5.61	7.44	9.27	11.58	13.27	17.15
45	3.72	4.46	5.53	7.33	9.12	11.44	13.12	17.41
50	3.62	4.35	5.39	7.14	8.87	11.13	12.76	16.51
60	3.53	4.25	5.24	6.92	8.59	10.87	12.47	16.34
70	3.45	4.16	5.13	6.79	8.44	10.59	12.22	16.23
80	3.40	4.09	5.06	6.67	8.33	10.41	12.09	15.94
100	3.32	4.00	4.97	6.59	8.16	10.28	11.73	15.48

Table 4 (continued)

k=6		α						
d	.20	.15	.10	.05	.025	.01	.005	.001
13	8.74	10.41	12.78	16.88	21.18	27.01	32.13	43.34
14	7.93	9.42	11.47	15.04	18.73	23.89	27.94	37.62
15	7.26	8.61	10.50	13.71	16.98	21.49	24.85	32.54
16	6.88	8.17	9.99	12.98	16.00	20.03	22.98	30.69
17	6.49	7.68	9.35	12.23	15.00	18.69	21.49	28.12
18	6.24	7.40	8.98	11.77	14.34	17.99	20.45	26.91
19	5.96	7.09	8.66	11.25	13.89	17.55	20.34	26.23
20	5.78	6.84	8.30	10.85	13.34	16.66	19.07	24.47
21	5.62	6.65	8.16	10.63	13.06	16.29	18.66	24.51
22	5.48	6.50	7.90	10.27	12.62	15.70	17.97	23.12
23	5.33	6.33	7.70	10.03	12.32	15.24	17.65	22.76
24	5.24	6.22	7.60	9.86	12.10	15.12	17.18	22.45
25	5.17	6.12	7.44	9.64	11.79	14.61	16.86	21.82
26	5.09	6.06	7.39	9.64	11.76	14.53	16.55	21.21
27	4.98	5.92	7.20	9.39	11.53	14.18	16.27	21.11
28	4.91	5.82	7.06	9.24	11.27	14.14	16.26	20.86
29	4.85	5.76	7.01	9.12	11.29	14.05	16.05	20.67
30	4.79	5.68	6.93	9.03	11.13	13.92	15.89	20.40
36	4.57	5.42	6.57	8.57	10.54	13.01	14.92	18.99
42	4.42	5.24	6.39	8.29	10.16	12.62	14.48	18.43
48	4.30	5.10	6.21	8.08	9.91	12.25	14.01	17.90
54	4.19	4.95	6.04	7.84	9.58	11.81	13.59	17.63
66	4.08	4.83	5.87	7.63	9.38	11.64	13.35	17.26
90	3.94	4.68	5.67	7.37	9.04	11.30	13.07	16.76

Table 5Critical points $s_r(\alpha, n\mathbf{1}_k, n^*\mathbf{1}_k)$ for the test based on the statistic T_r

k=3, $\gamma=1/2$		α			
n	n^*	.20	.10	.05	.01
3	3	1.80	2.76	3.76	5.40
6	6	1.73	2.99	4.22	6.74
9	9	1.72	2.97	4.22	6.95
12	12	1.71	2.94	4.19	6.94
15	15	1.71	2.93	4.20	7.14
18	18	1.73	2.95	4.19	7.10

k=3, $\gamma=3/7$		α			
n	n^*	.20	.10	.05	.01
3	4	2.06	3.07	3.76	4.10
6	8	2.22	3.75	5.10	7.41
9	12	2.19	3.75	5.35	8.47
12	16	2.20	3.78	5.40	9.05
15	20	2.18	3.78	5.39	9.19
18	24	2.19	3.79	5.39	9.14

Table 5 (continued)

k=3, $\gamma=3/8$		α			
n	n*	.20	.10	.05	.01
3	5	2.05	2.80	2.95	3.25
6	10	2.54	3.93	4.99	6.11
9	15	2.54	4.35	5.92	8.50
12	20	2.51	4.30	6.14	9.62
15	25	2.52	4.34	6.17	10.33
18	30	2.52	4.34	6.15	10.52

k=3, $\gamma=1/3$		α			
n	n*	.20	.10	.05	.01
3	6	1.81	2.25	2.25	2.33
6	12	2.60	3.64	4.5	4.65
9	18	2.71	4.41	5.57	7.04
12	24	2.74	4.72	6.35	9.11
15	30	2.73	4.68	6.65	9.78
18	36	2.70	4.69	6.73	10.68

k=4, $\gamma=1/2$		α			
n	n*	.20	.10	.05	.01
4	4	2.52	3.76	4.85	6.85
8	8	2.49	3.84	5.16	8.00
12	12	2.53	3.91	5.26	8.34
16	16	2.47	3.84	5.23	8.40
20	20	2.45	3.83	5.18	8.40

Table 5 (continued)k=4, $\gamma=4/9$ α

n	n*	.20	.10	.05	.01
4	5	2.93	4.25	5.23	6.66
8	10	3.07	4.71	6.30	9.53
12	15	3.05	4.74	6.36	10.04
16	20	3.04	4.74	6.43	10.19
20	25	2.99	4.70	6.39	10.28

k=4, $\gamma=2/5$ α

n	n*	.20	.10	.05	.01
4	6	3.10	4.18	4.98	5.45
8	12	3.46	5.27	6.84	9.66
12	18	3.44	5.38	7.28	11.07
16	24	3.42	5.36	7.26	11.53
20	30	3.46	5.41	7.33	11.80

k=4, $\gamma=4/11$ α

n	n*	.20	.10	.05	.01
4	7	2.97	3.81	4.25	4.55
8	14	3.71	5.42	6.76	8.68
12	21	3.74	5.84	7.68	11.02
16	28	3.74	5.85	7.91	12.00
20	35	3.72	5.83	7.94	12.44

Table 5 (continued)

k=4, $\gamma=1/3$		α			
n	n*	.20	.10	.05	.01
4	8	2.71	3.37	3.47	3.80
8	16	3.77	5.23	6.24	7.10
12	24	3.96	5.94	7.57	10.14
16	32	3.95	6.18	8.19	11.85
20	40	3.95	6.20	8.47	12.84

k=5, $\gamma=1/2$		α			
n	n*	.20	.10	.05	.01
5	5	3.14	4.54	5.79	8.24
10	10	3.11	4.60	6.01	9.16
15	15	3.13	4.64	6.10	9.29
20	20	3.10	4.59	6.05	9.34

k=5, $\gamma=5/11$		α			
n	n*	.20	.10	.05	.01
5	6	3.65	5.17	6.44	8.66
10	12	3.70	5.47	7.16	10.68
15	18	3.67	5.43	7.15	10.90
20	24	3.68	5.44	7.16	11.09

Table 5 (continued)

k=5, $\gamma=5/13$		α			
n	n*	.20	.10	.05	.01
5	8	4.00	5.26	6.09	6.72
10	16	4.48	6.51	8.31	11.57
15	24	4.48	6.64	8.73	12.97
20	32	4.47	6.65	8.72	13.57

k=5, $\gamma=5/14$		α			
n	n*	.20	.10	.05	.01
5	9	3.88	4.86	5.44	5.82
10	18	4.71	6.68	8.31	10.79
15	27	4.75	7.04	9.14	13.22
20	36	4.76	7.07	9.30	14.13

k=5, $\gamma=1/3$		α			
n	n*	.20	.10	.05	.01
5	7	3.90	5.37	6.50	7.85
10	14	4.16	6.13	7.97	11.55
15	21	4.11	6.11	8.05	12.25
20	28	4.10	6.07	8.06	12.38

Table 6

Value of the power function at boundary parameter configurations where $\Delta_2 = \dots = \Delta_{k-1} = 0$
and $\Delta_2^* = \dots = \Delta_k^* = 0$

k=3			
Δ_3	$\alpha=.10$	$\alpha=.05$	$\alpha=.01$
0.5	.105	.057	.014
1	.134	.076	.019
1.5	.167	.103	.030
2	.207	.133	.044
3	.257	.177	.073
4	.291	.215	.101
5	.288	.212	.104
6	.297	.217	.101
7	.299	.219	.107
8	.294	.218	.105
9	.2889	.2153	.105
10	.2981	.2181	.107
100	.2954	.2167	.105

k=4			
Δ_4	$\alpha=.10$	$\alpha=.05$	$\alpha=.01$
0.5	.108	.056	.001
1	.136	.072	.019
1.5	.187	.107	.034
2	.230	.147	.049
3	.293	.203	.086
4	.329	.237	.114
5	.345	.253	.125
6	.344	.256	.125
7	.346	.257	.128
8	.334	.253	.124
9	.347	.259	.131
10	.339	.252	.128
100	.345	.254	.129

Table 6 (continued)

k=5			
Δ_5	$\alpha=.10$	$\alpha=.05$	$\alpha=.01$
0.5	.110	.056	.012
1	.143	.078	.021
1.5	.186	.112	.032
2	.229	.148	.050
3	.323	.230	.093
4	.367	.275	.128
5	.373	.284	.138
6	.369	.281	.140
7	.368	.280	.141
8	.383	.292	.148
9	.371	.280	.146
10	.374	.283	.140
100	.371	.279	.143

k=6			
Δ_6	$\alpha=.10$	$\alpha=.05$	$\alpha=.01$
0.5	.117	.059	.013
1	.143	.078	.018
1.5	.185	.111	.032
2	.235	.153	.049
3	.324	.234	.100
4	.359	.271	.130
5	.375	.289	.150
6	.393	.298	.156
7	.389	.301	.156
8	.388	.300	.157
9	.387	.296	.152
10	.386	.293	.152
100	.388	.295	.153

Table 7

Value of the power function at parameter configurations where

$$\Delta_2 = \dots = \Delta_k = \Delta_2^* = \dots = \Delta_k^* = \Delta$$

k=3

Δ	$\alpha = .10$	Δ	$\alpha = .05$	Δ	$\alpha = .01$
0.6	.072	0.8	.028	1.0	.004
1.2	.072	1.6	.027	2.0	.005
1.8	.080	2.4	.037	3.0	.007
2.4	.087	3.2	.042	4.0	.009
3.0	.101	4.0	.045	5.0	.010
4.2	.107	5.6	.049	7.0	.011
5.4	.108	7.2	.048	9.0	.010
6.6	.112	8.8	.049	11.0	.011
7.8	.114	10.4	.043	13.0	.011
9.0	.116	12.0	.050	15.0	.010
10.2	.110	13.6	.047	17.0	.010
11.4	.108	15.2	.046	19.0	.011

k=4

Δ	$\alpha = .10$	Δ	$\alpha = .05$	Δ	$\alpha = .01$
0.6	.051	0.7	.023	0.9	.002
1.2	.061	1.4	.022	1.8	.004
1.8	.063	2.1	.027	2.7	.005
2.4	.076	2.8	.037	3.6	.006
3.0	.081	3.5	.040	4.5	.009
4.2	.089	4.9	.044	6.3	.008
5.4	.097	6.3	.047	8.1	.009
6.6	.095	7.7	.043	9.9	.010
7.8	.092	9.1	.048	11.7	.010
9.0	.091	10.5	.045	13.5	.009
10.2	.095	11.9	.044	15.3	.009
11.4	.094	13.3	.050	17.1	.010

Table 7 (continued)

k=5

Δ	$\alpha = .10$	Δ	$\alpha = .05$	Δ	$\alpha = .01$
0.5	.054	0.6	.022	0.8	.002
1.0	.049	1.2	.020	1.6	.002
1.5	.055	1.8	.026	2.4	.005
2.0	.066	2.4	.031	3.2	.007
2.5	.071	3.0	.036	4.0	.008
3.5	.087	4.2	.045	5.6	.008
4.5	.095	5.4	0.046	7.2	.009
5.5	.094	6.6	.047	8.8	.011
6.5	.102	7.8	.049	10.4	.010
7.5	.094	9.0	.045	12.0	.008
8.5	.094	10.2	.044	13.6	.008
9.5	.093	11.4	.045	15.2	.008

k=6

Δ	$\alpha = .10$	Δ	$\alpha = .05$	Δ	$\alpha = .01$
0.4	.053	0.5	.020	0.6	.003
0.8	.052	1.0	.017	1.2	.002
1.2	.052	1.5	.020	1.8	.003
1.6	.059	2.0	.026	2.4	.004
2.0	.068	2.5	.031	3.0	.007
2.8	.083	3.5	.037	4.2	.008
3.6	.089	4.5	.043	5.4	.010
4.4	.099	5.5	.044	6.6	.010
5.2	.096	6.5	.044	7.8	.010
6.0	.098	7.5	.042	9.0	.011
6.8	.101	8.5	.047	10.2	.012
7.6	.097	9.5	.044	11.4	.011

Table 8

Value of the power function at parameter configurations where

$\Delta_2 = \dots \Delta_k = \Delta_2^* = \dots = \Delta_k^* = \Delta$ is a multiple of 0.5.

k=3		
Δ	$\alpha = .10$	$\alpha = .05$
0.5	.079	.039
1.0	.079	.047
1.5	.087	.047
2.0	.099	.057
2.5	.109	.065
3.5	.125	.081
4.5	.129	.085
5.5	.131	.087
6.5	.131	.087
7.5	.132	.087
8.5	.131	.086
9.5	.130	.086

k=4		
Δ	$\alpha = .10$	$\alpha = .05$
0.5	.067	.032
1.0	.064	.032
1.5	.072	.036
2.0	.086	.048
2.5	.090	.050
3.5	.108	.067
4.5	.115	.072
5.5	.120	.076
6.5	.122	.079
7.5	.120	.079
8.5	.119	.074
9.5	.118	.076

Table 8 (continued)

k=5		
Δ	$\alpha = .05$	$\alpha = .01$
0.5	.025	.006
1.0	.025	.005
1.5	.031	.008
2.0	.037	.009
2.5	.039	.011
3.5	.053	.016
4.5	.058	.020
5.5	.060	.023
6.5	.064	.026
7.5	.059	.023
8.5	.059	.020
9.5	.060	.022

k=6		
Δ	$\alpha = .10$	$\alpha = .01$
0.5	.024	.004
1.0	.038	.003
1.5	.024	.004
2.0	.046	.007
2.5	.056	.007
3.5	.067	.012
4.5	.069	.013
5.5	.073	.014
6.5	.069	.014
7.5	.068	.015
8.5	.073	.017
9.5	.071	.015

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