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**Implementation of variable structure control for sampled-data
systems**

Su, Wu-Chung, Ph.D.

The Ohio State University, 1994

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IMPLEMENTATION OF VARIABLE STRUCTURE CONTROL FOR SAMPLED-DATA SYSTEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Wu-Chung Su, B.S., M.S.

* * * * *

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To my parents, Yao-Tzung and Hsien-Fong Jiang Su.

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CHAPTER I

Introduction

1.1 Problem Statement

The theory of variable structure systems (VSS) has been developed to analyze systems where a discontinuous control is employed to steer the states towards and maintain them on a predesigned manifold, or switching surface. When the state trajectory is traveling on the surface, the system is said to be in sliding mode [1, 2, 3, 4, 5, 6, 7, 8, 9]. Such a trajectory describes a new type of motion different from any of the individual structures which result from various combination of the switching control laws. While in sliding mode, the system behavior is independent of the control. This also implies insensitivity to any system uncertainties lying in the control signal space.

During the past thirty years, numerous research efforts have been spent to explore the theory in greater depth and completeness. It has now become an established tool for solving linear, nonlinear, or time-varying control problems. Among the existing control techniques, sliding mode control is notable for its simplicity in controller design and robustness to uncertainties. Since the state trajectory is confined to a sliding manifold, the effective system order is reduced. Moreover, if the switching surface is chosen properly, the system in sliding mode can be linearized irrespective

of what type of system the original one may be. Up to the present, various applications can be found in robotics, power systems, electric machinery, aerospace, underwater vehicles, power electronics, and flexible structures [10, 11, 12, 13, 6].

A general mathematical formulation of VSS is given as the following:

$$\dot{x} = f(x, u, t), \quad (1.1)$$

where the state $x \in R^n$, the control $u \in R^m$, and the vector field $f \in R^n$. Let $s_i(x)$, $i = 1, 2, \dots, m$ be m scalar functions of the state x . Each equation $s_i(x) = 0$ represents a hyperplane in the n dimensional state space R^n . The s_i are called discontinuity surfaces since discontinuous control will be applied when the states travel on them. The sliding manifold is the intersection of these discontinuity surfaces

$$\mathcal{S} = \{x | s(x) = 0\}. \quad (1.2)$$

For the sliding manifold $\mathcal{S}$ to be well-defined, only the case when the intersection exists and contains the origin will be considered. The $m \times 1$ vector field $s(x)$ will be called the switching surface vector and each scalar function $s_i(x)$, a switching surface variable.

Each component in the control vector undergoes discontinuities on $\mathcal{S}$

$$u_i = \begin{cases} u_i^+, s_i > 0 \\ u_i^-, s_i < 0 \end{cases} \quad i = 1 \dots m, \quad (1.3)$$

where u_i^+ , u_i^- are continuous functions of x and t , chosen to render sliding mode on the intersection of the discontinuity surfaces $s_i = 0$. It can be observed that the VSS (1.1) with switching control (1.3) has totally 2^m possible structures selected according to the signs of s_i 's. With the switching on the discontinuity surfaces, new properties

of the dynamic system take place in the controlled motions which are not present in each of the individual systems. To describe the “sliding” motion mathematically, we adopt a formal definition of sliding mode by Drakunov and Utkin [4]:

Given a set M of points in the abstract space $R = \{t : -\infty < t < \infty\}$ with $t_0, t \in M, t_0 \leq t$, the state trajectory of the dynamic system (1.1) can be represented by a family of transformations

$$F(t, t_0, \cdot) : \mathcal{X} \rightarrow \mathcal{X}, \quad (1.4)$$

where $\mathcal{X}$ is a metric space in R^n ; F is a continuous function satisfying the semigroup condition $F(t, t_1, F(t_1, t_0, x_0)) = F(t, t_0, x_0)$ for every $t_0 \leq t_1 \leq t, x_0 \in \mathcal{X}$; and $F(t, t, x) = x$ for every $t \in M, x \in \mathcal{X}$. A point x in the state space $\mathcal{X}$ is said to be a sliding mode point at the moment $t \in M$ if for every $t_0 \in M, t_0 < t$, the transformation $F(t, t_0, \cdot)$ is not invertible at this point (that is, the equation $F(t, t_0, \xi) = x$ has more than one solution for ξ). A state trajectory $\{F(t, t_0, \cdot)\}_{t_0 \leq t}$ is said to be in sliding mode if each point in the trajectory is a sliding mode point. In other words, the families of state space transformations representing the closed loop system in sliding mode are semigroups rather than groups.

When the system (1.1) is in sliding mode, neither the right hand side $f(x, u, t)$ nor the Jacobian matrix $\nabla f = \frac{\partial f}{\partial x}$ is continuous. Therefore, according to the existence and uniqueness theorem for the solution of an n^{th} order ordinary differential equation [14] and classical, standard analysis, the solution to equation (1.1) is undecided. A way of continuation is necessary so that the system, when constrained on the sliding manifold, can be analyzed and the performance can be evaluated. The continuous

dynamic equations representing a VSS in sliding mode are called the equations of sliding mode. Although it is not always possible to find equations of sliding mode for general VSS, we will show later that a special class of nonlinear systems with discontinuous right hand side do have continuous representations.

There exist two methods for finding equations of sliding mode for the VSS. Filippov introduced a set of 2^m nonnegative parameters μ_j to formulate a continuous right hand side for the VSS

$$f^0 = \sum_{j=1}^{2^m} \mu_j f(x, u^j, t) \quad (1.5)$$

so that f^0 lies on the intersection of the surfaces geometrically tangential to the switching surface [15]. The parameters u_j are chosen to satisfy the conditions

$$\sum_{j=1}^{2^m} \mu_j = 1 \quad (1.6)$$

$$\nabla s \cdot f^0 = 0. \quad (1.7)$$

Such a formulation is not always possible. For example, the gradient vector ∇s has to lie in the range space spanned by the set of vector fields $\{f(x, u^j, t)\}_{j=1}^{2^m}$. Also, with the nonnegative constraint on μ_j 's, the angles between ∇s and each $f(x, u^j, t)$ should be within $\pm \frac{\pi}{2}$ for a solution of f_0 to exist.

The other formal technique to obtain continuation of the VSS's in sliding mode is called the "equivalent control method" proposed by Utkin [1]. The equivalent control, denoted u^{eq} , is defined by solving $\dot{s} = 0$, which renders

$$\nabla s \cdot f(x, u^{eq}, t) = 0. \quad (1.8)$$

The continuous vector field f^0 obtained from Filippov's method is determined by the 2^m parameters μ_j satisfying $(m + 1)$ linear algebraic equations in (1.6) and (1.7). Obviously, there is no unique solution (if the solution exists) to the set $\{\mu_j\}$, $j = 1, \dots, 2^m$ for $m > 1$. On the other hand, the equivalent control method in general does not guarantee an unambiguous solution to $f(x, u^{eq}, t)$ either. This is because the m algebraic equations (1.8) (to solve for the m components in u^{eq}) are nonlinear in u^{eq} . Nevertheless, if the nonlinear system is linear with respect to the control

$$\dot{x} = a(x, t) + b(x, t)u, \quad (1.9)$$

a unique solution for both methods can be obtained. For Filippov's method, substituting (1.6) and (1.9) into (1.7) leads to

$$f^0 = a(x, t) + b(x, t) \sum_{j=1}^{2^m} \mu_j u^j. \quad (1.10)$$

Since the $m \times 1$ vector $\sum_{j=1}^{2^m} \mu_j u^j$ is the parameter to be found, the m linear algebraic equations (1.7) can uniquely determine f^0 generically. In the same manner, substituting (1.9) into the equations for solving the equivalent control (1.8) yields m algebraic equations linear with respect to u^{eq}

$$\nabla s \cdot a(x, t) + \nabla s \cdot b(x, t)u^{eq} = 0. \quad (1.11)$$

The equivalent control u^{eq} can be solved for unambiguously provided the $m \times m$ matrix $\nabla s \cdot b(x, t)$ is nonsingular for all x and t . In fact, it was illustrated in [1] that Filippov's continuation method leads to identical result as the equivalent control method for nonlinear systems of the form (1.9).

For general nonlinear systems (1.1), however, the continued vector field f^0 does not coincide with $f(x, u^{eq}, t)$. Generally speaking, the sliding mode equations of motion cannot be obtained uniquely. As asserted in [1], no continuous control would result in a motion analogous to the one in sliding mode. Therefore, we will focus our study on the case when the system is linear with respect to the control (1.9) all through this dissertation.

With the control structure specified above, two fundamental problems arise in controller design. They are:

1. Choosing appropriate discontinuity surfaces $s_i = 0$ to meet a performance criterion.
2. Finding the switching control functions u_i^+ , u_i^- for each control variable to reach and maintain sliding mode on $\mathcal{S}$.

To deal with the first problem, there already exists a comprehensive theory for designing linear switching surface for linear dynamic systems [2, 16, 17, 5]. Several standard methods with specific steps will be addressed in Chapter II. In the case of nonlinear dynamics (1.9), methods of switching surfaces design are not as established. Thus far, the most common approach is to impose a nonlinear coordinate transformation on (1.9) such that the system can be put into the regular form [18] or into the Brunovsky canonical form [19]. Both methods have their merits and limitations. In Chapter II, we will introduce another approach from a Lyapunov perspective as a supplement to the existing methods. In the new approach, a switching surface determined by the control coefficient matrix and the associated Lyapunov function is able

to ensure asymptotic stability for the system in sliding mode. The proposed method may be used for systems with nonlinear dynamics and for linear systems with time delays. Although there is still no universal way of constructing the optimal sliding manifold for nonlinear systems so far, the first problem is essentially tractable for control engineers.

The second problem demonstrates most of the implementation difficulties. For a sliding mode to exist on $\mathcal{S}$, it is necessary that the algebraic equivalent control equation (1.8) has a solution for u^{eq} and that each component of the switching control satisfies the condition

$$\min(u_i^+, u_i^-) \leq u_i^{eq} \leq \max(u_i^+, u_i^-) \quad \forall i = 1, 2, \dots, m. \quad (1.12)$$

Also, to ensure existence of a region of sliding on the discontinuity surface, u_i^+ and u_i^- are chosen to satisfy the conditions

$$\lim_{s_i \rightarrow 0^+} \dot{s}_i < 0, \quad \lim_{s_i \rightarrow 0^-} \dot{s}_i > 0. \quad (1.13)$$

Equation (1.13) is useful only for mathematical analysis. Conventionally, one constructs a Lyapunov type of function (typically $V(s) = \frac{1}{2}s^T s$) for the sliding surface vector dynamics and chooses u_i^+, u_i^- in compliance with the necessary condition (1.12). There is no unique way of choosing the switching control law even though all possible solutions are actually equivalent to u^{eq} in the average sense. Once chosen, the switching control is implemented by either analog circuits or digital computers. However in both, several nonidealities such as delays and hysteresis of the system will degrade the performance [6, 20]. In the presence of switching nonidealities, the discontinuity

in the control variables may produce unmodeled fast dynamics causing undesirable system behavior referred to as the chattering phenomenon.

Chattering is not acceptable in many physical systems. For example, it causes rapid kicking in the actuator of a robotic manipulator and results in mechanical wear-down. In flexible structures, the high frequency modes in the plant usually not included in the mathematical model, are liable to be excited. Unexpected vibrations result if chattering is not reduced.

To cope with this phenomenon, it is believed that selection of a high switching frequency can ameliorate the problem. Nevertheless, this is only true for the state variables and not for the control inputs. In other words, the switching behavior of each control input u_i in between two functions, u_i^+ and u_i^- , is never removed no matter how fast it switches. In addition, the physical constraints of the system such as inherent time delays will not allow unlimited increase of switching frequency. Moreover, high switching frequency demands high sampling rate as well as fast processing capability of the digital processor, which also implies high cost for implementation [21].

Chattering reduction has been studied by many researchers [22, 23, 24, 25, 20, 26, 12] for continuous time implementation. Slotine and Sastry proposed considering a static boundary layer in the vicinity of each discontinuity plane ($s_i = 0$) as a transition region in the state space [22]. Instead of an abrupt change from control u_i^+ to u_i^- or vice versa, a continuous interpolation function is introduced to smooth the switching actions. Among them, the most commonly used one is a saturation function. Outside of the region, it behaves as a normal sliding mode controller that ensures convergence

of sliding motion into the boundary layer. When the state trajectory enters the boundary layer, the switching behavior of the control is smoothed and chattering is effectively reduced. Related works with similar ideas can be found in [23, 24].

Another method of removing chattering motion is to use unit-vector control [27], which gives discontinuity only at the intersection of discontinuity surfaces ($s = 0$) rather than on each individual discontinuity surface. To further smooth the control when the state is close to the sliding manifold, a small constant $\delta > 0$ is inserted to modify the control law such that the control does not change sign abruptly. This method has been implemented successfully by Dorling and Zinober in [16, 17]. No chattering was observed in the simulation results. In the paper coauthored by A. G. Bondarev, S. A. Bondarev, N. E. Kostyleva, and V. I. Utkin [25], an asymptotic state observer was built to maintain ideal sliding mode in spite of delays in the measurement. The inherent constraint on the control switching frequency is then eliminated and the controller can be implemented as near ideal as possible.

These methods, however, have compromised chattering reduction with robustness. Both the boundary layer and the modified unit-vector control approaches have produced a small region around the sliding manifold, in which the robustness property against uncertainties is lost although smooth control is acquired. The state observer, on the other hand, requires a precise knowledge of the original system, which excludes the robustness issues by definition. In [20], Young and Drakunov took a more general look at the problem and put forward an idea of prefiltering and postfiltering in the control loop. Based on this setting, the continuous interpolation function can

be viewed as a special case of the prefiltering blocks and the state observer pertains to the category of postfiltering blocks. Another example of such a filtration approach can be observed in [12, 28, 5], where Young and Özgüner adopted the idea of frequency shaping into the design of sliding surfaces for flexible structures. Low frequency content is discriminated from high by use of a sliding regime in the expanded state space. This can be accomplished using pole placement or quadratic regulator technique. The method was illustrated to be effective in suppressing high frequency behavior of the sliding system.

We have just elucidated the essence of continuous time variable structure control and the problems it encounters in implementation. In the following chapters, we will elaborate on chattering reduction from the discrete time sliding mode point of view. The idea of prefiltering and postfiltering will be exploited as a framework for solving output feedback sliding mode control problems. A comprehensive study on the implementation issues of variable structure control for sampled-data systems will cover various classes of systems. Applications to a robotic manipulator, a brushless DC motor, and a cantilever flexible beam will be presented as examples of the theoretical methods studied in this dissertation.

1.2 Variable Structure Control in Sampled-Data Systems

This dissertation mainly deals with the implementation issues of variable structure control for sampled-data systems (Figure 1). Since the theory of sliding mode has been originally developed from a continuous time prospective, a discrete time implementation for sampled-data systems encounters several problems due to limited

Figure 1: Block Diagram of a Sampled-Data Control System

sampling rate, sample/hold effect, and discretization errors. First of all, to attain ideal sliding mode requires a continuous processing of the plant state information and an infinite switching frequency in the control variables. The requirements contradict with the nature of a sampled-data system, where the states are sampled at discrete time instants and the controller switching frequency is the sampling rate. As a result, a direct translation of continuous time variable structure control design leads to the chattering phenomenon.

Although a considerable amount of work has been done analyzing discrete time sliding modes, very little was directly addressed to the sampling issues. In [29], Milosavljevic studied the oscillatory characteristic (quasisliding) in the neighborhood of the discontinuity surfaces due to discretization of continuous time signals. Existence conditions of quasisliding mode were derived as a discrete extension from the continuous time VSS theory. For control law design, Utkin and Drakunov proposed a definition of discrete time equivalent control that directs the states onto the sliding surface in one sampling period [30]. To remain on the surface, the associated con-

trol appears to be non-switching. Subsequently, the theoretical basis was furnished with a formal definition of sliding mode for discrete time systems in the context of contraction mapping [4].

Related works from different perspectives can be found in [31, 32, 33, 34, 35, 36]. Sarpturk *et al.* took a Lyapunov point of view for discrete time linear systems. It was asserted that the switching control had to be bounded in an open interval to guarantee convergence of sliding motion. This interval was later found to depend linearly on the distance of the state from the switching surface [32], which suggested a nonswitching control when discrete time sliding mode is attained. Sira-Ramirez imposed the geometrical concept to general nonlinear SISO systems along with the existence issues [34]. Parallel results with continuous time VSS's were obtained. To implement discrete time sliding mode control law in compliance with the existence conditions, Furuta proposed the idea of a sliding cone (sliding sector), where switching control takes place only when the states are out of the sector, while inside of the sector, the control law remains continuous [33]. Kaynak and Denker used an ARMA model to characterize the control-sliding surface relationship and yielded a non-switching type of control with a predictive-corrective scheme [35]. Bartolini *et al.* incorporated an adaptive control strategy to account for system uncertainties and designed control law in terms of a discrete time equivalent control [36].

This work has mainly extended the original idea of [30] to more general problems with detailed analysis of the sample/hold effects. Preliminary results were reported in [37, 38, 39, 40] and the full development is presented here. The control objective is to

Figure 2: Discrete time sliding mode in sampled-data systems

maintain the states on the switching surface at each sampling instant. Between samples, the states are allowed to deviate from the surface instead of being constantly and exactly on the switching surface, even the equivalent state travels within a boundary layer of that surface (see Figure 2). It will be demonstrated that the thickness of this boundary layer can be reduced significantly by proper consideration of the sampling phenomenon in the control design. Three classes of systems (linear, nonlinear, and stochastic systems) will be investigated individually. Robustness against both internal and external uncertainties will be considered. Furthermore, the chattering problem will be overcome.

1.3 Organization of Dissertation

The dissertation starts in Chapter I with a brief review of general continuous time VSS theory. The chapter continues with a discussion of the problems encountered in implementation and subsequently focuses on sample-data systems. A survey of contemporary publications on discrete time sliding mode control is conducted to demonstrate the need for this research. Chapter II presents a complementary result of

constructing discontinuity surfaces for nonlinear systems. The method is feasible for a broad class of nonlinearities, which the existing methods do not cover. It also suggests a new avenue for future development into the research of infinite dimensional systems.

Chapter III through Chapter V investigates three classes of finite dimensional systems one often encounters in control design problems. They are linear, nonlinear, and stochastic systems. Discrete time sliding mode control theory will be developed and applied for the sampled-data controller design. Various types of uncertainties (exogenous disturbances, system parameter variations, and control coefficient variations) are considered. The performance of each are derived in terms of the sampling period.

Chapter VI is a realization of the prefiltering and postfiltering idea presented in [20] for sampled-data systems. The main emphasis are placed on chattering reduction and output feedback sliding mode control problems. The proposed control system structure is able to remove chattering and release the high frequency switching and full state accessibility assumptions in variable structure control.

Chapter VII contains two applications to demonstrate the effectiveness of the proposed sampled sliding mode control methods. The first example is current and torque control of a brushless DC motor. The unique attribute of a brushless DC motor makes sliding mode control suitable and favorable to many control designers. The essence of controller design for brushless DC motors is in fact a set of switching logic for the three phase input voltages. Three algorithms for current and torque

control are developed based on the concept of sliding mode. A discrete time sliding mode approach is utilized for controller design to account for system uncertainties and sampling effects. The second example is the vibration control of a 5.6-meter-long flexible truss structure developed at The Ohio State University. A truncated model is employed to characterize the modes of interest. The experiment include system identification, state observer design, and discrete time sliding mode control.

Chapter VIII provides concluding remarks and discussion of future avenues of research regarding this dissertation. Finally, the appendix contains proofs of equations for Lemmas.

CHAPTER II

Constructing Discontinuity Surfaces - A New Approach

The characteristic feature of a variable structure system is that sliding mode occurs on a prescribed switching manifold. The manifold is the intersection of a set of discontinuity surfaces in the state space of a multiple state system, where each control input switches between two functions. These discontinuity surfaces are selected so that, while in sliding mode, the system performance satisfies the design objectives such as stability, performance index minimization, chattering reduction, enforcement of linear behavior, order reduction, $\dots$ *etc.*

2.1 Standard Approaches

We devote this section to outlining several existing, well-established methods of constructing sliding manifolds for linear and nonlinear systems. During the past decade, the problem of constructing discontinuity surfaces in VSS has been studied mainly on the basis established in [2]. A coordinate transformation is found to partition the states into a special form referred to as the canonical form. In the new coordinates, design of the switching surface is carried out for a reduced order system independent of the control variables. This idea has been extensively used in the literature

of VSS theory and has been exploited from different perspectives [2, 16, 5, 41, 18]. We call them standard approaches as a distinguishment from our new approach to be elaborated in the next section.

2.1.1 Linear Systems

The theory of designing linear switching surfaces for linear dynamics has been developed in great depth and completeness. Consider a linear system

$$\dot{x} = Ax + Bu, \quad (2.1)$$

where the state $x \in R^n$, the control $u \in R^m$, A and B are constant matrices with appropriate dimensions, and B is of full rank. In sliding mode, the state trajectory is constrained to a subspace defined by

$$s(x) = Cx = 0 \quad (2.2)$$

with C an $m \times n$ matrix such that CB is invertible. Assume there exists a nonsingular transformation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx$ leading to a canonical form for VSS design

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \quad (2.3)$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u, \quad (2.4)$$

where $x_1 \in R^{n-m}$, $x_2 \in R^m$, and (2.2) is transformed to $C_1x_1 + C_2x_2 = 0$. The dynamical behavior of sliding motion is determined by (2.2) and (2.3). By viewing x_2 as the control input to the constrained system (2.3), the problem of designing an appropriate sliding surface can be regarded as a reduced order state feedback design problem.

In the framework of the above formulation, Utkin and Young proposed three design procedures based on both pole placement and optimal control [2]. Dorling and Zinober proposed refinements to the pole placement method in [16, 17]. Recently, Young and Özgüner incorporated the optimal control approach with frequency shaping technique to design a dynamic switching surface that suppresses high frequency characteristics. Instead of being a static hyperplane in the original state space (x_1, x_2) , it appears as a linear operator on the states [5].

We summarize these methods below:

1. Pole placement

The dynamical behavior of the original system (2.1) in sliding mode can be determined by the reduced order system (2.3). It was proved in [2] that arbitrary pole placement in (2.3) can be achieved if the pair (A, B) is controllable. Under this assumption, one solves for the feedback gain F such that the closed loop system

$$\dot{x}_1 = (A_{11} - A_{12}F)x_1 \quad (2.5)$$

has the desired eigenvalue distribution. Consequently, the matrix C has the form

$$C = [C_1, C_2]T = C_2[F, I_m]T, \quad (2.6)$$

where C_2 is a nonsingular matrix to be specified for some other purposes.

2. Eigenstructure assignment

In addition to pole placement, the extra degree of freedom in C_2 can be utilized to partially assign the eigenvectors of the equivalent system

$$\dot{x} = (I_n - B(CB)^{-1}C)Ax. \quad (2.7)$$

Let $\{\lambda_i\}$, $i = 1, 2, \dots, n - m$ be the desired nonzero eigenvalues of (2.7) and $\{v_i\}$, $i = 1, 2, \dots, n - m$ be the corresponding eigenvectors. It is clear that v_i lies in the null space of C or

$$CV = 0, \quad V = [v_1, v_2, \dots, v_{n-m}]. \quad (2.8)$$

Thus the choice of the eigenstructure will affect the solution of C . Detailed analysis of this method can be found in [16].

3. Optimal sliding mode

Consider the problem of minimizing the quadratic performance index

$$I_1 = \int_{t_s}^{\infty} x^T Q x \, dt, \quad (2.9)$$

where $Q \geq 0$ and t_s is the time at which sliding mode begins. Minimizing I_1 subject to the dynamic equation (2.3) yields the optimal sliding surface with the form (2.6).

Alternatively, one can take the performance index

$$I_2 = \int_{t_s}^{\infty} (x^T Q x + (u^{eq})^T R u^{eq}) \, dt, \quad (2.10)$$

where $Q \geq 0$, $R > 0$, u^{eq} is the equivalent control defined in (1.8) and can be solved explicitly as

$$u^{eq} = -(CB)^{-1}CAx \quad (2.11)$$

assuming CB is invertible. The procedures to solve for the optimal sliding surface C are detailed in [2]. Similar work taking account of unmatched parameter variations can also be found in [27].

4. Frequency shaping

In stead of using a static hyperplane in the state space as the switching surface, this method incorporates compensator dynamics in sliding mode, which leads to a new class of switching surfaces having the interpretation of linear operators

$$\mathcal{L}(x, z) = Cx + C_0z = 0, \quad (2.12)$$

where z is the additional state vector realizing the compensator dynamics

$$\begin{aligned} \dot{z} &= Fz + Gx \\ \eta &= Hz + Dx. \end{aligned} \quad (2.13)$$

To determine the design parameters C , C_0 , F , G , H , D in such linear operators, one can use pole placement techniques to meet a given transient behavior specifications for the composite plant-compensator system in sliding mode. Another way is to follow the quadratic regulator design method to yield the optimal sliding mode in the augmented state space $\begin{bmatrix} x \\ z \end{bmatrix}$. In the latter case, a fre-

quency dependent quadratic cost function can be employed for supressing high frequency uncertainties

$$J_s = \int_{-\infty}^{+\infty} (x_1^*(jw)Q_{11}x_1(jw) + x_2^*(jw)Q_{22}(w)x_2(jw))dw, \quad (2.14)$$

where x_i^* is the complex conjugate transpose of x_i , $i = 1, 2$ and $Q_{11} \geq 0$, $Q_{22}(w) \geq 0$ for all frequencies w . $Q_{22}(w)$ is chosen to penalize the high frequency motion of the sliding system. Minimization of J_s will be able to reduce the chattering phenomenon. Furthermore, an appropriate choice of initial conditions for the additional dynamics $z(0)$ can eliminate the reaching phase [5, 28, 12].

2.1.2 Nonlinear Systems

Consider the nonlinear system of the form

$$\dot{x} = f(x, t) + g(x, t)u, \quad (2.15)$$

where $f(x, t) \in R^n$ and $g(x, t)$ is an $n \times m$ matrix. To extend the preceding techniques to the nonlinear system (2.15), two possible approaches have been suggested. The first approach is to impose a nonsingular coordinate transformation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \phi(x)$ to (2.15) yielding the regular canonical form [42, 43, 18]

$$\dot{x}_1 = f_1(x_1, x_2, t) \quad (2.16)$$

$$\dot{x}_2 = f_2(x_1, x_2, t) + g_2(x_1, x_2, t)u, \quad (2.17)$$

where $x_1 \in R^{n-m}$, $x_2 \in R^m$, and $\det[g_2(x, t)] \neq 0$. A switching surface can be found by designing a nonlinear feedback control law $x_2 = s(x_1)$ for the subsystem (2.16).

The other approach is to use feedback linearization technique to obtain a linear equivalence. Suppose the nonlinear system (2.15) is feedback linearizable. We apply a diffeomorphic transformation

$$z = \varphi(x), \quad (2.18)$$

so that the new state and control variables z and v satisfy the linear differential equation

$$\dot{z} = \hat{A}z + \hat{B}v \quad (2.19)$$

$$u = \Psi(x, v), \quad (2.20)$$

where $z \in R^n$ is the transformed coordinates, $v \in R^m$, $\hat{A}$ and $\hat{B}$ are constant matrices in Brunovsky canonical form [19]. With the linearized dynamic equations (2.19), a sliding surface can be easily found. This idea has been exploited by many researchers [44, 45, 41, 46, 47] with applications to vehicle maneuvers, stirred tank reactors, brushless dc motors, and robotic manipulators.

2.2 Lyapunov's Second Method

In this section, we take a different approach to the problem of constructing discontinuity surfaces. Motivated by the so called unit-vector control method [48, 49, 27], a switching surface that assures asymptotic stability is found by using Lyapunov's second method. This method does not require a partitioning of the system into the canonical form as in (2.3)-(2.4) or (2.16)-(2.17). It is especially useful for nonlinear systems and for linear systems with delays.

To apply Lyapunov's second method to asymptotically stabilize the dynamic systems (2.1) or (2.15) requires the following assumptions:

1. There exists a stabilizing feedback Kx for (2.1) or $k(x)$ for (2.15) such that the autonomous systems

$$\dot{x} = Ax + BKx \quad (2.21)$$

$$\dot{x} = f(x) + g(x)k(x) \quad (2.22)$$

are asymptotically stable.

2. A Lyapunov function $V(x)$ exists and satisfies $\dot{V}(x) \leq 0$.
3. $\dot{V}(x) \neq 0$ in the state trajectory except at the origin.

The proposed approach is to synthesize the Lyapunov function into the design of the switching surface to achieve the same performance as if the stabilizing feedback is employed. In other words, asymptotic stability is realized in sliding mode instead of being enforced separately through the control.

2.2.1 Linear systems

Consider the linear system given in (2.1). Assuming (A, B) is controllable, there exists a stabilizing feedback gain K such that $A + BK \triangleq A_s$ is asymptotically stable. It follows that there exists a positive definite matrix P which solves the Lyapunov equation

$$PA_s + A_s^T P = -Q; \quad Q > 0. \quad (2.23)$$

Choose the linear switching surface

$$Cx = DB^T Px = 0, \quad (2.24)$$

where D is an $m \times m$ nonsingular matrix. In fact, it can be regarded as a linear transformation for the equivalent surface $s^* = B^T Px$. A special choice of $D = (B^T PB)^{-1}$ will diagonalize CB leading to the sliding surface dynamics

$$\dot{s} = DB^T PAx + u. \quad (2.25)$$

The switching surface can be chosen more generally as (2.2), where C has a null space identical to that of $B^T P$. This enables the designer to assign appropriate eigenvectors for better transient response in sliding mode [16].

Rewrite the system (2.1) as

$$\dot{x} = A_s x + B\tilde{u}, \quad (2.26)$$

where $\tilde{u} = u - Kx$ and Kx is a fictitious feedback to system (2.1). We have the following lemma,

Lemma 2.1 *The system (2.26) with sliding mode on the surface (2.24) is asymptotically stable.*

Proof. Choose the Lyapunov function

$$V(x) = x^T Px > 0,$$

then

$$\dot{V}(x) = 2x^T PA_s x + 2x^T PB\tilde{u}.$$

On the switching surface (2.24), $x^T P B = 0$, we have

$$\dot{V}(x) = x^T (P A_s + A_s^T P) x = -x^T Q x < 0. \quad \square$$

On the switching surface (2.24), the effect of the control $\tilde{u}$ is excluded. The rate of energy decay is the same as the autonomous system $\dot{x} = A_s x$.

Lemma 2.2 *The equality*

$$x^T (P A + A^T P) x = -x^T Q x \quad (2.27)$$

holds on the surface (2.24).

Proof. Substituting $A_s = A + B K$ into (2.23) yields,

$$-x^T Q x = x^T (P A_s + A_s^T P) x = x^T (P A + A^T P) x + x^T P B K x + x^T K^T B^T P x.$$

On the surface (2.24), where $x^T P B = 0$ and $B^T P x = 0$, the equality (2.27) is satisfied. $\square$

Lemma 2.2 reveals that the stabilizing linear feedback $K x$ is realized on the switching surface instead of being applied explicitly in the control. It is well understood in VSS theory that, while in sliding mode, the system dynamics is insensitive to uncertainties in the range space of B [3]. Since any feedback through u will always lie in the range of B , the system performance in sliding mode with or without the control gain K will be the same. The following theorem is obtained directly from Lemmas 2.1 and 2.2.

Theorem 2.1 *The system (2.1) with sliding mode on the surface (2.24) is asymptotically stable.*

Example 2.1. Given the system (2.1) with

$$A = \begin{bmatrix} 0.2325 & -0.9285 & 0.0154 & 0.1222 \\ -0.7274 & 1.0116 & -0.0224 & 0.1576 \\ -1.6883 & 0.2214 & 0.6534 & 1.6278 \\ -0.5310 & -0.2603 & -0.0052 & 1.1025 \end{bmatrix},$$

$$B = \begin{bmatrix} 3.0 & 2.0 \\ 0 & 1.0 \\ 0.5 & -2.0 \\ 1.3 & 0 \end{bmatrix}.$$

The autonomous system A has eigenvalues $(-0.2, 0.5, 0.7, 1.0)$. Choose a stabilizing feedback gain K that places the poles at $(-0.2, -0.5, -0.7, -1.0)$. Solve the Lyapunov equation (2.23) with $Q = I$. Let $D = I$, it yields the switching surface

$$C = \begin{bmatrix} 42.5205 & -36.7318 & 16.6224 & -82.6258 \\ 3.7881 & -8.6858 & -9.0516 & 6.3276 \end{bmatrix}.$$

The resultant eigenvalues in sliding mode are located at $(-0.6120, -1.2204)$.

2.2.2 Nonlinear systems

A similar idea of constructing a switching surface using Lyapunov's second method can be used for the nonlinear system in (2.15). Suppose $W(x)$ is a Lyapunov function for the autonomous system (2.22). The nonlinear sliding surface can be chosen as

$$\sigma(x) = g^T(x) \left(\frac{\partial W(x)}{\partial x} \right)^T = 0 \quad (2.28)$$

where $\frac{\partial W}{\partial x} = (\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_n})$. The following lemma can be proven by direct substitution.

Lemma 2.3 *The equality*

$$\frac{\partial W(x)}{\partial x} f(x) = \frac{\partial W(x)}{\partial x} [f(x) + g(x)k(x)] \quad (2.29)$$

holds on the switching surface (2.28).

The above lemma implies that the autonomous systems

$$\begin{aligned} S_1 : \dot{x} &= f(x) \\ S_2 : \dot{x} &= f(x) + g(x)k(x) \end{aligned}$$

share the same Lyapunov function when the state is constrained on the surface (2.28).

In fact, any additive disturbances (known or unknown) through the control channel will not affect the system behavior in sliding mode.

Theorem 2.2 *The system (2.15) with sliding mode on the switching surface (2.28) is asymptotically stable.*

Example 2.2. Consider the second order nonlinear system [50]

$$f(x_1, x_2) = \begin{pmatrix} -2x_2^2 \\ -x_2^3 \end{pmatrix}, \quad g(x_1, x_2) = \begin{pmatrix} -1 \\ x_2 \end{pmatrix}.$$

A stabilizing feedback $k(x) = x_1$. The associated Lyapunov function is

$$W(x) = \frac{1}{2}(x_1^2 + 2x_2^2) > 0$$

so that $\dot{W}(x) = -x_1^2 - 2x_2^2 < 0$ for the system $\dot{x} = f(x) + g(x)k(x)$. A nonlinear switching surface can be found easily from (2.28).

$$\sigma(x) = -x_1 + 2x_2^2.$$

Since $\dot{\sigma} = (2x_2^2 + 4x_2^4) + (1 - 4x_2^2)u$, the system has relative degree 1 in the region $\Omega = \{(x_1, x_2) | x_2 \neq \pm \frac{1}{2}\}$. This guarantees the existence of the sliding regime in Ω [51].

To check the stability of the zero dynamics in sliding mode $\sigma(x) = 0$, we apply the method in [52] to convert the system into normal form. Define a new variable η satisfying $L_g \eta = 0$. One solution is $\eta = x_2 e^{x_1}$. The dynamic equation for η is stable

$$\dot{\eta} = -3x_2^2 \eta.$$

Therefore, the nonlinear system in sliding mode is asymptotically stable.

2.2.3 Delay Systems

Consider a linear system with delays in the state.

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + Bu(t), \quad (2.30)$$

where h is a fixed time delay. A memoryless stabilizing control law with Lyapunov function has been proposed in [53]. We will make use of the result to construct an appropriate sliding surface stabilizing the system.

Define

$$P = \int_0^T e^{-A_0 s} B B^T e^{-A_0^T s} ds, \quad T > 0. \quad (2.31)$$

We have the following theorem:

Theorem 2.3 *If (A_0, B) is controllable and the condition*

$$B B^T + e^{-A_0 T} B B^T e^{-A_0^T T} - Q - P A_1^T Q^{-1} A_1 P > 0 \quad (2.32)$$

holds for some $Q > 0$, then the system (2.30) with sliding mode on the surface

$$s(x) = B^T P^{-1} x = 0 \quad (2.33)$$

is asymptotically stable.

Proof. Introduce a coordinate transformation $y = P^{-1}x$ and rewrite the system (2.30) as

$$\dot{y}(t) = P^{-1}(A_0 P - B B^T)y(t) + P^{-1}A_1 P y(t - h) + P^{-1}B \tilde{u}(t), \quad (2.34)$$

where $\tilde{u}(t) = u(t) + B^T y(t)$. A Lyapunov functional for the new state variable y is chosen as

$$V(y) = y^T(t)Py(t) + \int_{t-h}^t y^T(s)Qy(s)ds > 0. \quad (2.35)$$

When the system is in sliding mode ($B^T y = 0$), using the equality

$$A_0 P + P A_0^T = B B^T - e^{-A_0 T} B B^T e^{-A_0^T T} \quad (2.36)$$

renders

$$\dot{V}(y) = - \begin{bmatrix} y(t) \\ y(t-h) \end{bmatrix}^T M \begin{bmatrix} y(t) \\ y(t-h) \end{bmatrix} \quad (2.37)$$

with

$$M = \begin{pmatrix} B B^T + e^{-A_0 T} B B^T e^{-A_0^T T} - Q & -A_1 P \\ -P A_1^T & Q \end{pmatrix}.$$

Since the condition (2.32) holds, the matrix M is positive definite [54]. This completes the proof. $\square$

The sliding surface (2.33) is also valid for systems with a continuously differentiable time delay $h = h(t)$ satisfying $h(t) \geq 0$ and $\dot{h}(t) < 1$ for all $t > 0$. In this case, the condition in Theorem 2.3 becomes

$$B B^T + e^{-A_0 T} B B^T e^{-A_0^T T} - Q - P A_1^T Q^{-1} A_1 P (1 - \dot{h}(t)) > 0. \quad (2.38)$$

The proof is similar. Simply take the Lyapunov function

$$V(y) = y^T(t)Py(t) + \int_{t-h(t)}^t y^T(s)Qy(s)ds > 0. \quad (2.39)$$

The corresponding M matrix becomes

$$M = \begin{pmatrix} B B^T + e^{-A_0 T} B B^T e^{-A_0^T T} - Q & -A_1 P \\ -P A_1^T & (1 - \dot{h}(t))Q \end{pmatrix},$$

which is positive definite provided $\dot{h}(t) < 1$ and the condition (2.38).

2.3 Generating Lyapunov Functions

Recall from the previous section that the dynamical behavior in sliding mode is determined by a reduced order system on the switching surface. None of the known and unknown portions lying in the range space of B will affect the system performance as long as sliding mode is maintained. This excludes an explicit realization of the stabilizing feedback since it simply plays the role of generating a Lyapunov function.

In linear systems, one can do without a stabilizing feedback and directly generate a Lyapunov function by solving the Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \quad (2.40)$$

with $R > 0$ and $Q \geq 0$.

In the canonical representation (2.3)-(2.4), the design of P can be independent of A_{21} and A_{22} since they both lie in the range space of B_2 . A possible choice is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \quad (2.41)$$

To find a Lyapunov function for any nonlinear systems is still an open research problem. For feedback linearizable systems, one can resort to differential geometric methods that formulate the system in Brunovsky canonical form through the diffeomorphic transformation (2.18). The new variables z and v satisfy the linear differential equation (2.19). One can solve for a Lyapunov function in the z coordinates.

In case of single input systems,

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (2.42)$$

Let the associated Lyapunov function be $w(z)$, then the switching surface is

$$s(z) = \frac{\partial w(z)}{\partial z_n}, \quad (2.43)$$

where z_n is the n^{th} component of the vector z .

For time delay systems as in (2.30), there is not much freedom in choosing the matrix P but the length of integration $\mathcal{T}$ in (2.31). P can be obtained by either solving the Lyapunov equation (2.36) or a direct evaluation of the integral in (2.31). To satisfy the condition (2.32), one would like to have the magnitude of A_1 as small as possible. This can be done by decomposing A_1 into the sum of two $n \times n$ matrices

$$A_1 = A_1^B + A_1^{B^\perp},$$

where $A_1^B \in \mathcal{N}[B^T]$ and $A_1^{B^\perp} \in \text{Im}[B]$. Since $A_1^{B^\perp}$ is in the control range space, it does not contribute any value in $\dot{V}(y)$ when sliding mode is maintained. Therefore, the condition (2.32) can be softened by replacing A_1 with A_1^B .

2.4 Conclusion

The Lyapunov approach contributes to the design of discontinuity surfaces a different perspective from that in [2]. This method does not have the advantage of order reduction; nevertheless, it suggests a new direction of solution which the framework

of [2] does not cover. As long as the system (2.1) or (2.15) has a stabilizing feedback with a known Lyapunov function, a sliding surface can be obtained immediately without the need of regular canonical form representation.

In the case of delay system (2.30), following the procedure of partitioning the states into the canonical form will lead to another delayed feedback control problem

$$\begin{aligned}\dot{x}_1 &= A_{11}^0 x_1(t) + A_{12}^0 x_2(t) + A_{11}^1 x_1(t-h) + A_{12}^1 x_2(t-h) \\ \dot{x}_2 &= A_{21}^0 x_1(t) + A_{22}^0 x_2(t) + A_{21}^1 x_1(t-h) + A_{22}^1 x_2(t-h) + B_2 u,\end{aligned}$$

which complicates the original problem even more. The proposed Lyapunov method in this chapter, however, leads to a feasible approach. If the delay system is stabilizable with a Lyapunov function, the associated switching surface can be found.

The versatility of Lyapunov's method provides a new avenue to deal with various problems such as nonlinear, time varying, delayed, infinite dimensional, $\dots$ *etc.* Investigation of these problems from the Lyapunov point of view as outlined above seems to be a promising approach.

CHAPTER III

Linear Systems

We come to the problem of sliding mode control law design for sampled-data systems. The need for research in discrete time sliding mode control is evident. A primary reason is that most control strategies nowadays are implemented in discrete time. In such systems the switching frequency in control variables is limited by T^{-1} , where T is the sampling period. Although high frequency switching is theoretically desirable from the robustness point of view, it is usually hard to achieve in practice due to physical constraints such as processor computational speed, A/D and D/A conversion delays, actuator (or process) bandwidth, $\dots$ *etc.* Even if it is possible, the unmodeled high frequency modes will likely be excited, which in turn deteriorates the performance. The use of a discontinuous control law (typically *sign* functions) in a sampled-data system will bring about chattering phenomenon in the vicinity of the sliding manifold $\mathcal{S}$, leading to a boundary layer with thickness $O(T)$.

In order to alleviate this problem, Utkin and Drakunov proposed to use discrete time equivalent control in the prescribed boundary layer, whose size is defined by the restriction applied to the control variables [30]. This approach results in the motion in $O(T^2)$ vicinity of the sliding manifold. The main difficulty arising in implementation of the proposed control law is that one needs to know the disturbances for calculating

the equivalent control. Lack of such information still leads to an $O(T)$ boundary layer, the same as with discontinuous control.

Sliding mode is originally a continuous time concept, and at first glance, it may not be clear that it will retain its properties in discrete time. In this chapter, we will analyze the sample/hold effect and show that perfect sliding mode generally can not exist in sampled-data systems. Since the system is discretized in time by the sample/hold processes, the control signal space is shrunk from $(L^2_{[0,T]})^m$ to R^m , m being the number of control inputs. Therefore, the controller will inherently be less capable than the continuous one [29].

This chapter deals with linear sampled-data systems. The control objective of the sliding mode control is to maintain the states on the switching surface at each sampling instant

$$s(x_k) = 0, \tag{3.1}$$

where x_k is the state value at the k^{th} sampling instant $t = kT$. If (3.1) is maintained for k greater than a positive integer k_s , we say the sampled-data system is in discrete time sliding mode [30]. Between samples, the states are allowed to deviate from the surface. We introduce a new control strategy based on the discrete time equivalent control concept proposed in [30, 4]. It will be demonstrated that the thickness of this boundary layer can be reduced to at least $O(T^2)$. Robustness against system uncertainties will be investigated. Furthermore, the notorious chattering problem will be addressed.

3.1 Problem Statement

Consider the dynamic system with a prescribed switching surface

$$\dot{x} = Ax + Bu + Df \quad (3.2)$$

$$\mathcal{S} = \{x | s(x) = Cx = 0\}, \quad (3.3)$$

where the state $x \in R^n$, the control $u \in R^m$, the disturbance vector $f \in R^l$, and the sliding surface vector $s \in R^m$; A , B , D are constant matrices of appropriate dimensions; and the $m \times n$ matrix C is chosen so that the system, when traveling on $\mathcal{S}$, will achieve the desired sliding mode dynamics [2]. We say the disturbance is completely “rejectable” if there exists a control u that can eliminate its effect instantaneously in the dynamic system. If the closed loop system behavior does not depend on $f(t)$, we say the controlled system is disturbance invariant. It is well known that the matching condition [3]

$$\text{rank}[B, D] = \text{rank}[B] \quad (3.4)$$

has to be satisfied for the system (3.2) to be disturbance invariant in sliding mode.

The dynamic equation for the sliding surface vector s is

$$\dot{s} = C(Ax + Df) + CBu. \quad (3.5)$$

The continuous time, variable structure approach leads to a discontinuous control law in each component of u that switches between two functions u_i^+ and u_i^- , $i = 1, \dots, m$, according to the sign of $s(x)$. With the assumption of infinite switching frequency,

the discontinuous control in sliding mode has an effect equivalent to a continuous one [1]. The continuous time “equivalent control” is defined by setting $\dot{s} = 0$ in (3.5)

$$u^{eq}(t) = -(CB)^{-1}C(Ax + Df). \quad (3.6)$$

Let the admissible control be bounded with an allowable range specified by $\mathcal{U}$. To ensure existence of sliding mode in $\mathcal{S}$ [1, 51], we impose the following assumptions:

Assumption 3.1

1. *The $m \times m$ matrix CB is nonsingular.*
2. *The equivalent control $u^{eq}(t) \in \mathcal{U}$.*

Applying a zero-order-hold to the control yields the discrete time representation for (3.2) and (3.5)

$$x_{k+1} = x_k + \int_{kT}^{(k+1)T} (Ax + Df)dt + BT u_k \quad (3.7)$$

$$s_{k+1} = s_k + \int_{kT}^{(k+1)T} C(Ax + Df)dt + CBT u_k. \quad (3.8)$$

To demonstrate that a direct implementation of the discontinuous control $u_i^\pm$, $i = 1, \dots, m$ to the sampled system (3.7) leads to unfavorable performance, let the right hand side of (3.5) with discontinuous control be bounded by a positive finite number M_s . It is observed in (3.8) that $\|s_{k+1}\| \leq M_s T$ in the vicinity of $\mathcal{S}$ ($s_k \approx 0$). Thus, instead of staying in the manifold $\mathcal{S}$, the state deviates from it with $O(T)$ error. Moreover, the switching control will force the state to penetrate the switching surface from one side to another and produce undesirable fast dynamics (chattering phenomenon), which deteriorates the system performance.

Figure 3: Direct implementation of $u^\pm$ in sampled-data systems

To alleviate chattering, a discrete time approach, which takes the sample/hold effects into account is necessary. We will show that proper consideration of the sampling phenomenon reduces the boundary layer to the size of $O(T^2)$. Chattering will be considerably reduced when the states travel in the boundary layer. Unfortunately, this is also an inherent limitation of the sampled controller. The controller, when constrained to remain constant between samples, will be less capable than the continuous one. As mentioned previously, a shrunk control signal space R^m no longer contains the disturbance space $(L^2_{[0,T]})^m$. Consequently, a complete disturbance rejection becomes out of the question for sampled controllers no matter what type of control techniques are employed [55, 56, 57, 58, 59].

Notwithstanding the limitation, perfect disturbance insensitivity is usually not necessary (even if possible) in practice. Robustness suffices for most control purposes. A controller is considered robust as long as it maintains certain properties of interest for the system. In the next section, we will show that the sampled-data controller

looses the disturbance invariance property but is still robust against bounded and smooth disturbances if discrete time sliding mode is achieved.

3.2 Discrete Time Sliding Mode

The discrete time representation of the dynamic equation (3.2) is obtained by applying u through a zero-order-hold:

$$x_{k+1} = \Phi x_k + \Gamma u_k + d_k, \quad (3.9)$$

where $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{A\lambda} d\lambda B$, $d_k = \int_0^T e^{A\lambda} Df((k+1)T - \lambda) d\lambda$. Note that the magnitude of Γ and d_k are both $O(T)$ if $f(t)$ is bounded.

The discrete time sliding mode control is to steer the states towards and maintain them on the surface $\mathcal{S}$ at each sampling instant

$$s_k = Cx_k = 0. \quad (3.10)$$

During the sampling interval $kT \leq t < (k+1)T$, the state will deviate from $\mathcal{S}$. The following Lemma provides bounds on the deviation from the sliding surface.

Lemma 3.1 *If the disturbance $f(t)$ in (3.2) is bounded and smooth in the intersample interval $kT \leq t \leq (k+1)T$, then $\begin{cases} s_k = 0 \\ s_{k+1} = 0 \end{cases}$ implies $s(t) = O(T^2)$ during that sampling period.*

Proof. With $s_k = 0$ and $s_{k+1} = 0$, the piecewise constant control law for $kT \leq t < (k+1)T$ can be solved from (3.8)

$$u_k = -(CBT)^{-1} \int_{kT}^{(k+1)T} C(Ax + Df) dt. \quad (3.11)$$

Let $0 \leq \tau \leq T$, the intersample sliding surface vector is evaluated as

$$s(kT + \tau) = \int_{kT}^{kT+\tau} C(Ax + Df)dt - \frac{\tau}{T} \int_{kT}^{(k+1)T} C(Ax + Df)dt.$$

Since both the state x and the disturbance f are smooth and bounded, the integrations can be approximated by Euler's method

$$\begin{aligned} \int_{kT}^{kT+\tau} C(Ax + Df)dt - \tau C(Ax(kT) + Df(kT)) &= e_1 = O(\tau^2) \\ \int_{kT}^{(k+1)T} C(Ax + Df)dt - TC(Ax(kT) + Df(kT)) &= e_2 = O(T^2), \end{aligned}$$

which leads to $s(kT + \tau) = e_1 - \frac{\tau}{T}e_2 = O(T^2)$. $\square$

The $O(T^2)$ deviation is inevitable for sampled-data systems. It is the ultimate performance a sampled controller (with zero-order-hold) can attain for continuous time plants. To acquire higher order intersample accuracy, one needs to resort to higher order holding mechanisms such as a first-order-hold, a second order hold, *etc*, which is not of interest in this research.

Note that if $s_k = O(T^2)$ and $s_{k+1} = O(T^2)$, the intersample value of the sliding surface vector $s(kT + \tau) = s_k + e_1 + \frac{\tau}{T}(s_{k+1} - s_k - e_2)$ is still $O(T^2)$. Although perfect sliding mode ($s(t) \equiv 0$ for $t \geq t_s$) is generally not possible in sampled-data systems, the $O(T^2)$ accuracy is sufficient for most applications.

Researchers have provided quite a few contribution in the design of sliding mode control laws for the discrete time system (3.9). In the work by Milosavljevic [29], a sliding condition similar to that of continuous time systems:

$$\lim_{s_k \rightarrow 0^+} \nabla s_k \leq 0, \quad \lim_{s_k \rightarrow 0^-} \nabla s_k \geq 0, \quad (3.12)$$

where $\nabla s_k = s_k - s_{k-1}$ denotes the backward difference of s_k was provided. An implicit condition for the validity of (3.12) is that the sampling period T be small

enough. To ensure convergence of s_k , Sarptürk *et al.* proposed

$$\begin{aligned} [{}^i s_{k+1} - {}^i s_k] \text{Sign}({}^i s_k) &< 0 \\ [{}^i s_{k+1} + {}^i s_k] \text{Sign}({}^i s_k) &\geq 0 \end{aligned} \quad (3.13)$$

on each hyperplane ${}^i s_k = 0$, $i = 1, 2, \dots, m$. The control law was then bounded by two limits [31]. Furuta [33] adopted the concept of equivalent control for continuous time sliding mode and proposed

$$s_{k+1} = s_k \quad (3.14)$$

as a basis of feedback gain design. All these methods, however, do not take the uncertainties into account in the control law design and therefore suffer the $O(T)$ error for sampled-data systems.

We define the discrete time equivalent control by solving $s_{k+1} = 0$ [30], which leads to

$$u_k^{eq} = -(C\Gamma)^{-1}C(\Phi x_k + d_k) \quad (3.15)$$

assuming $C\Gamma$ is invertible. Ideally, u_k^{eq} is a solution to the discrete time sliding mode control since it maintains the state on the sliding manifold at each sampling instant. In addition, it is not a switching type of control law; hence, no chattering phenomenon would occur if u_k^{eq} were to be employed.

However, unlike continuous time VSC, the matching condition (3.4) and the boundedness of disturbances are not sufficient for disturbance invariance in discrete time. Here by disturbance invariance in discrete time we mean that the disturbance d_k does not affect the next state value x_{k+1} in the discrete time system (3.9). The idea is similar but not exactly the same as the usual sense of disturbance invariance

in continuous time, where the effect of the continuous time disturbance $f(t)$ is completely rejected from the original plant (3.2). The difference not only lies in the time index $(t - k)$ but also in the control signal space depicted previously. If a sampled-data controller is not able to reject d_k , it certainly cannot reject $f(t)$ in continuous time. Likewise, perfect (continuous time) sliding mode implies discrete time sliding mode but the reverse argument is usually not true. The following Lemma indicates that the sample/hold effect not only destroys perfect sliding mode, it also invalidates the complete disturbance rejection property for the discrete time system (3.9).

Lemma 3.2 *The sampled-data system (3.9) is disturbance invariant on the discrete time sliding manifold (3.10) only if the continuous time matching condition (3.4) holds.*

Proof. Suppose the system is in discrete time sliding mode $s_k = 0$. Substituting (3.15) into (3.9) yields the sliding mode equation

$$x_{k+1} = (I - \Gamma(C\Gamma)^{-1}C)(\Phi x_k + \int_0^T e^{A\lambda} Df((k+1)T - \lambda)d\lambda) \quad (3.16)$$

The condition for disturbance invariance is

$$\int_0^T e^{A\lambda} Df((k+1)T - \lambda)d\lambda = \Gamma(C\Gamma)^{-1}C \int_0^T e^{A\lambda} Df((k+1)T - \lambda)d\lambda \quad (3.17)$$

for all possible disturbance functions $f(t)$. The necessary and sufficient condition for (3.17) is the existence of a constant $m \times 1$ vector g_k such that

$$\int_0^T e^{A\lambda} (Df((k+1)T - \lambda) - Bg_k)d\lambda = 0. \quad (3.18)$$

The proof for equivalence of (3.17) and (3.18) is in Appendix.

If the discrete time system (3.9) is disturbance invariant, then there exists a constant g_k such that (3.18) holds for all possible $f(t)$. In particular, if $f(t) = \bar{f}_k = \text{const.}$, for $kT \leq t < (k+1)T$ then $D\bar{f}_k = Bg_k$. This implies $\text{Rank}[B, D] = \text{Rank}[B]$. $\square$

Note that the reverse argument is not true. If the continuous time matching condition holds, it does not guarantee the existence of a constant g_k satisfying (3.18) because (3.18) is a system of n algebraic equations with m unknowns with $n > m$ in general.

3.3 Robustness Against Uncertainties

We consider three types of uncertainties that are important in affecting a dynamic system's behavior; namely, exogenous disturbances, system parameter variations, and control coefficient variations. Let the system be modeled as

$$\dot{x} = Ax + \Delta A(t)x + Bu + \Delta B(t)u + Df(t), \quad (3.19)$$

where $\Delta A(t)$ and $\Delta B(t)$ are time varying uncertainty matrices with appropriate dimensions. The continuous time matching conditions for $\Delta A(t)$ and $\Delta B(t)$ are:

$$\text{Rank}[\Delta A(t)\Theta, B] = \text{Rank}[B] \quad \forall t \quad (3.20)$$

$$\text{Rank}[\Delta B(t), B] = \text{Rank}[B] \quad \forall t, \quad (3.21)$$

where Θ is an $n \times (n - m)$ matrix with the columns forming a basis of the subspace $\text{null}(C)$ [3]. We apply the following assumptions:

Assumption 3.2

1. The uncertainties $f(t)$, $\Delta A(t)$, and $\Delta B(t)$ are bounded and smooth.

2. *The continuous time matching conditions (3.4), (3.20), and (3.21) hold in each sampling interval.*

The corresponding sampled-data representation of (3.19) is given in (3.9) with

$$d_k = \int_0^T e^{A\lambda} (Df((k+1)T-\lambda) + \Delta A((k+1)T-\lambda) + \Delta B((k+1)T-\lambda)u_k) d\lambda. \quad (3.22)$$

In this section, we will consider the design of discrete time sliding mode control against each individual uncertainty and then discuss the general case when they are all present simultaneously.

3.3.1 Exogenous Disturbances

Consider the case when only the exogenous disturbance $f(t)$ is present. The discrete time equivalent control u_k^{eq} in (3.15) is able to achieve $s_{k+1} = 0$. However, u_k^{eq} is generally not calculatable because it requires the value of d_k , which depends on the future values of the disturbance $f(t)$ from the present sampling instant kT to the next one $(k+1)T$. It is impossible to evaluate d_k exactly unless $f(t)$ is known. Nevertheless, with boundedness and smoothness assumptions imposed on $f(t)$, d_k can be predicted by its previous value d_{k-1} [10]

$$d_{k-1} = x_k - \Phi x_{k-1} - \Gamma u_{k-1}. \quad (3.23)$$

With Assumption 3.2, the error is

$$d_k - d_{k-1} = \int_0^T e^{A\lambda} D \int_{kT-\lambda}^{(k+1)T-\lambda} \dot{f}(\sigma) d\sigma d\lambda = O(T^2).$$

To approximate u_k^{eq} , define

$$u_k^{in} = -(C\Gamma)^{-1}C(\Phi x_k + d_{k-1}). \quad (3.24)$$

Taking the Taylor's series expansion of Φ yields the relationship $C\Phi x_k = s_k + (AT + \frac{T^2}{2!}A^2 \dots)x_k$. Since the magnitude of $C\Gamma$ and Cd_{k-1} are both $O(T)$, in order that u_k^{in} be admissible, the magnitude of $C\Phi x_k$ must also be $O(T)$. In other words, x_k must lie within an $O(T)$ boundary layer of $\mathcal{S}$ (so that $s_k = O(T)$) for u_k^{in} to be admissible. Let us adopt the following notation:

$$\mathcal{S}_T = \{x_k | u_k^{in} \in \mathcal{U}\} \text{ is the } O(T) \text{ boundary layer of } \mathcal{S}.$$

For the time being, we will study discrete time sliding mode control in the boundary layer $\mathcal{S}_T$. The case when $x_k \notin \mathcal{S}_T$ will be discussed in the latter section.

Theorem 3.1 *If the dynamic system (3.2) with exogenous disturbance $f(t)$ satisfies Assumption 3.2 and the state $x_k \in \mathcal{S}_T$, then the control u_k^{in} will lead to sliding motion such that $s_{k+1} = O(T^2)$. Furthermore, $s(t) = O(T^2)$ for $t \geq (k+1)T$.*

Proof. Since $x_k \in \mathcal{S}_T$, u_k^{in} is admissible by definition. Substituting u_k^{in} into s_{k+1} leads to

$$s_{k+1} = C(d_k - d_{k-1}) = O(T^2),$$

which maintains x_{k+1} in $\mathcal{S}_T$. By induction, $s_j = O(T^2)$ for $j \geq k+1$, which implies $s(t) = O(T^2)$ for $t \geq (k+1)T$ according to Lemma 3.1. $\square$

Additional switchings

Higher accuracy in sliding mode can be achieved by inserting additional switchings in the control during the sampling period. This requires a better estimation of d_k to

interpolate the intersample disturbance values. For bounded $f(t), \dot{f}(t), \dots, f^{q-1}(t)$, the disturbance d_k can be predicted by

$$\tilde{d}_k = \sum_{i=1}^{q-1} (-1)^{i-1} \binom{q-1}{i} d_{k-i} \quad (3.25)$$

with $O(T^q)$ error.

To insert additional control between samples, we divide each sampling period T into r subintervals. Let

$$u(t) = u_k^i, \quad (k + \frac{i}{r})T \leq t < (k + \frac{i+1}{r})T, \quad i = 0, \dots, r-1.$$

We come up with a discrete time system equation with additional degree of freedom in the control signal space.

$$x_{k+1} = \Phi x_k + \sum_{i=0}^{r-1} \Phi^{\frac{r-1-i}{r}} \Gamma_r u_k^i + d_k, \quad (3.26)$$

where $\Gamma_r = \int_0^{\frac{T}{r}} e^{A\lambda} d\lambda B$. The intersample disturbance d_k^i can be estimated by

$$\tilde{d}_k^i = \left(\sum_{j=0}^{r-1} \Phi^{\frac{j}{r}} \right)^{-1} \tilde{d}_k$$

with $O((\frac{T}{r})^2)$ error. Each intermediate state value x_k^i will be estimated by

$$\tilde{x}_k^{i+1} = \Phi_r^{\frac{1}{r}} \tilde{x}_k^i + \Gamma_r u_r^i + \tilde{d}_k^i \quad i = 0, \dots, r-2$$

with $\tilde{x}_k^0 = x_k$. Choose

$$\begin{aligned} u_k^i &= -(C\Gamma_r)^{-1} C \Phi_r^{\frac{1}{r}} \tilde{x}_k^i - (C\Gamma_r)^{-1} C \tilde{d}_k^i, \quad i = 0, \dots, r-2 \\ u_k^{r-1} &= -(C\Gamma_r)^{-1} C (\Phi x_k + \sum_{i=0}^{r-2} \Phi^{\frac{r-1-i}{r}} \Gamma_r u_k^{i-1} + \tilde{d}_k) \end{aligned}$$

to achieve

$$s_{k+1} = O(T^q), \quad s_k^i = O((\frac{T}{r})^2), \quad i = 1, \dots, r-1. \quad (3.27)$$

3.3.2 System Parameter Variations

Consider a dynamic system subject to parameter variations

$$\dot{x} = (A + \Delta A(t))x + Bu. \quad (3.28)$$

The unknown $\Delta A(t)x$ may be treated as a disturbance. The associated sampled-data representation is given in (3.9), where

$$d_k = \int_0^T e^{A\lambda} \Delta A((k+1)T - \lambda) x((k+1)T - \lambda) d\lambda. \quad (3.29)$$

With the smoothness assumption on $\Delta A(t)$, d_k can also be predicted by d_{k-1} in (3.23) with $O(T^2)$ error. Higher order accuracy as in (3.25) is not available in general because $\ddot{x}(kT)$ does not exist due to discontinuity in the control $u(t)$ at each sampling instant $t = kT$.

Theorem 3.2 *For a linear sampled-data system with parameter variations (3.28), the control u_k^n will lead to $s_{k+1} = O(T^2)$ if $\Delta A(t)$ satisfies Assumption 3.2 and $x_k \in \mathcal{S}_T$. Furthermore, the state will attenuate to zero asymptotically.*

Proof. Similar to Theorem 3.1, $s_{k+1} = O(T^2)$ and $s(t) = O(T^2)$ for $t \geq (k+1)T$. The state then enters the $O(T^2)$ boundary layer of $\mathcal{S}$. Since the disturbance satisfies the matching condition (3.20), the effect of $\Delta A(t)$ on the state is reduced to $O(T^2)$. This will induce an $O(T^2)$ perturbation on the zero dynamics of sliding mode. If one designs the surface with an adequate stability margin [27], $O(T^2)$ perturbation will still be within the tolerance of the performance criterion. The choice of sliding surface

that ensures system stability will lead to $x_j = O(T^2)$ asymptotically for some $j > k$. This implies $d_j \rightarrow O(T^3)$ according to (3.29). Therefore

$$s_j = C(d_j - d_{j-1}) \rightarrow O(T^3).$$

By induction, $x_j \rightarrow 0$ as $j \rightarrow \infty$. □

3.3.3 Control Coefficient Matrix Variations

Consider the case when there is an unknown variation $\Delta B(t)$ in the control coefficient matrix B .

$$\dot{x} = Ax + (B + \Delta B(t))u. \tag{3.30}$$

Systems with uncertainties in the control coefficient matrix have not been studied thoroughly in sliding mode control theory. The difficulty mainly lies in the fact that the induced disturbance is directly related to the control signal itself. In continuous time variable structure control, the control policy is largely determined by the sliding surface vector s . If the variation $\Delta B(t)$ is sufficiently small that the sign of each element in the control coefficient $B + \Delta B(t)$ coincides with that in B , the primary effect of $\Delta B(t)$ would only be to modulate the magnitude of the control signal. On the other hand, the magnitude of the control signal for continuous time VSC is immaterial as long as it is large enough to dominate the other uncertainties ($f(t)$ and $\Delta A(t)$). Therefore, most frequently, the design of continuous time sliding mode control simply ignores $\Delta B(t)$.

For sampled-data systems, however, the same policy can be adopted to steer the state only into the $O(T)$ boundary layer of the sliding manifold. Ignorance of $\Delta B(t)$

will lead to $O(T)$ error in sliding mode. If we desire a higher accuracy, the uncertainty needs to be taken into account in control design.

The system dynamic equation in discrete time is the same as (3.9), where

$$d_k = \Delta\Gamma_k u_k,$$

$$\Delta\Gamma_k = \int_0^T e^{A\lambda} \Delta B((k+1)T - \lambda) d\lambda.$$

As can be seen in (3.23), d_{k-1} contains the past control law u_{k-1} . Employing u_k^{in} induces a dynamic feedback in u_k through d_{k-1} . Since

$$s_{k+1} = C\Phi x_k + C(\Gamma + \Delta\Gamma_k)u_k, \quad (3.31)$$

the stability of s_k implies stability in x_k , which ensures stability in u_k if $C(\Gamma + \Delta\Gamma_k)$ is invertible.

Introduce the following notation:

$$e_{\Delta\Gamma_k} = \Delta\Gamma_k - \Delta\Gamma_{k-1}, \quad \Delta_k = C\Delta\Gamma_k(C\Gamma)^{-1}. \quad (3.32)$$

The quantity Δ_k can be regarded as a measure of variations in the control coefficient matrix with respect to the known one Γ . Based on Assumption 3.2, we have $e_{\Delta\Gamma_k} = O(T^2)$ and $\Delta_k = O(1)$.

Lemma 3.3 *For the linear sampled-data system with control coefficient matrix variation (3.30), if $\Delta B(t)$ satisfies Assumption 3.2 and the states x_{k-1} , x_k are in the boundary layer $\mathcal{S}_T$, then the control u_k^{in} will lead to*

$$s_{k+1} = -2\Delta_k s_k + \Delta_{k-1} s_{k-1} + \delta_k \quad (3.33)$$

with $\delta_k = O(T^2)$.

Proof. Applying u_k^{in} to the sampled-data system (3.9) yields

$$\begin{aligned} s_{k+1} &= C(d_k - d_{k-1}) = C(\Delta\Gamma_k u_k^{in} - \Delta\Gamma_{k-1} u_{k-1}^{in}) \\ &= -\Delta_k C(\Phi x_k + d_{k-1}) + \Delta_{k-1} C(\Phi x_{k-1} + d_{k-2}). \end{aligned}$$

Using the equality $C\Phi x_k = s_k + C(\Phi - I)x_k$ and the notation in (3.32), we obtain (3.33) with

$$\delta_k = -\Delta_k C(\Phi - I)(x_k - x_{k-1}) - Ce_{\Delta\Gamma_k}(C\Gamma)^{-1}C(\Phi - I)x_{k-1} - e_{\Delta\Gamma_k}(C\Gamma)^{-1}Cd_{k-2}.$$

With $\Phi - I = O(T)$, $x_k - x_{k-1} = O(T)$, $d_{k-2} = O(T)$, it follows $\delta_k = O(T^2)$. $\square$

Stability of s_k in (3.33) can be assured if Δ_k is small enough; that is, the variation of $\Delta B(t)$ is small with respect to B . This result justifies disregarding $\Delta B(t)$ in continuous time variable structure control design.

Theorem 3.3 *The state x_k attenuates to zero asymptotically if*

1. *The conditions in Lemma 3.3 hold.*
2. *Equation (3.33) is stable.*
3. *The $m \times m$ matrix $C(\Gamma + \Delta\Gamma_k)$ is nonsingular.*

Proof. Stability of (3.33) implies $s_k \rightarrow O(T^2)$, hence $x_k \rightarrow O(T^2)$.

With condition 3, $u_k \rightarrow O(T)$ according to (3.31), which leads to $d_k \rightarrow O(T^2)$ since $d_k = \Delta\Gamma_k u_k$.

Substituting $x_{j-1} = O(T^2)$, $x_j = O(T^2)$, and $d_j = O(T^2)$ yields $\delta_j = O(T^3)$ for some $j > k + 1$. Therefore, $s_j \rightarrow O(T^3)$.

By induction, $x_j \rightarrow 0$ as $j \rightarrow \infty$. $\square$

3.3.4 General Case

If all three types of uncertainties are present in the dynamic equation as in (3.19), the lumped disturbance d_k can be expressed as in (3.22). The approximate equivalent control u_k^{in} can still be employed to compensate the uncertainties simultaneously.

Theorem 3.4 *Given $x_k \in \mathcal{S}_T$, the control u_k^{in} in (3.24) will maintain $s(t) = O(T^2)$ for the system (3.19) if the conditions in Theorem 3.3 hold.*

Proof. Define

$$\begin{aligned} d_k^f &= \int_0^T D e^{A\lambda} f((k+1)T - \lambda) d\lambda \\ d_k^A &= \int_0^T e^{A\lambda} \Delta A((k+1)T - \lambda) x((k+1)T - \lambda) d\lambda \\ d_k^B &= \int_0^T e^{A\lambda} \Delta B((k+1)T - \lambda) d\lambda u_k. \end{aligned}$$

If the conditions in Theorem 3.3 hold, then (3.33) is stable and there exists $j \geq k$ such that

$$\begin{aligned} s_{j+1}^f &= C(d_j^f - d_{j-1}^f) = O(T^2) \\ s_{j+1}^A &= C(d_j^A - d_{j-1}^A) = O(T^2) \\ s_{j+1}^B &= C(d_j^B - d_{j-1}^B) = O(T^2) \end{aligned}$$

for sufficiently large j . It follows that

$$s_{j+1} = s_{j+1}^f + s_{j+1}^A + s_{j+1}^B = O(T^2).$$

According to Lemma 3.1, $s(t) = O(T^2)$ for $t \geq (j+1)T$. □

3.4 The Reaching Phase

We have thoroughly analyzed the condition when $x_k \in \mathcal{S}_T$, in which an approximation of the equivalent control (3.24) is employed. In the reaching phase, $x_k \notin \mathcal{S}_T$, the $(CT)^{-1}C\Phi x_k$ term in the expressions of (3.24) has the magnitude of order $O(\frac{1}{T})$ and u_k^{in} becomes inadmissible.

There are several alternatives to deal with the reaching phase. One can simply employ the continuous time variable structure control law and discretize it to yield

$$u_k = \begin{cases} -(C\Gamma)^{-1}C(\Phi x_k + d_{k-1}) & \text{when } x_k \in \mathcal{S}_T \\ -(C\Gamma)^{-1}C(\Phi x_k + d_{k-1} - s_k + K \operatorname{sgn}(s_k)) & \text{otherwise.} \end{cases} \quad (3.34)$$

The positive definite matrix K will determine the stepsize for the state to approach the boundary layer $\mathcal{S}_T$. The magnitude of K has to be chosen small enough not to overshoot $\mathcal{S}_T$.

In the scalar control case, a saturation function is feasible [36]. Let

$$u_k = \operatorname{Sat}(u_k^{\text{in}}, M) = \begin{cases} M \operatorname{sgn}(u_k^{\text{in}}) & \text{if } |u_k^{\text{in}}| > M \\ u_k^{\text{in}} & x_k \in \mathcal{S}_T \end{cases} \quad (3.35)$$

where $M > 0$ is a positive scalar within the allowable range of $\mathcal{U}$. Note that M determines the approaching speed of the state towards the boundary layer $\mathcal{S}_T$, it is usually desirable to select a large M to reduce the time span of the reaching phase. However, a large M will likely cause the state trajectory to overshoot the switching surface when the state is close to $\mathcal{S}_T$. To cope with this discrepancy, a self-adapting algorithm in adjusting M is proposed in [10]. Similar idea can also be found in [5], where additional dynamics in sliding mode is introduced to eliminate the reaching phase. We will discuss this issue in the latter chapter.

For linear multi-input systems, we can follow the “diagonalization method” [1] to decouple the m -dimensional dynamics of $s(x)$ into m independent first order differential equations and apply (3.35) for each control channel. Suppose the original sliding surface was $s^{\text{orig}} = C^{\text{orig}}x$, it can be diagonalized it by reconstructing an equivalent

sliding surface

$$\begin{aligned} s(x) &= Cx \\ C &= (C^{orig}B)^{-1}C^{orig}. \end{aligned} \quad (3.36)$$

The new sliding surface dynamic equation becomes

$$\dot{s} = u(t) + \sigma(x, u, t), \quad (3.37)$$

where $\sigma(x, u, t)$ contains all the known and unknown quantities of the system. Let

$$\sigma(x, u, t) = [\sigma^1, \sigma^2, \dots, \sigma^m]^T, u_k = [{}^1u_k, {}^2u_k, \dots, {}^mu_k]^T \text{ and } u_k^{in} = [{}^1u_k^{in}, {}^2u_k^{in}, \dots, {}^mu_k^{in}]^T.$$

In compliance with Assumption 3.1, the continuous time equivalent control for (3.37)

$u^{eq}(t) = -\sigma(x, u, t)$ needs to be admissible so that the saturation function ${}^iu_k = \text{Sat}({}^iu_k^{in}, M_i)$ with $M_i > |\sigma^i|$, $\forall i$, can be applied through each control channel.

3.5 Numerical Example

To illustrate the validity of the proposed method for discrete time sliding mode, we consider a similar example used in [33] under various kinds of uncertainties. The continuous time state equations are given as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \Delta Ax + \Delta Bu + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

The sliding surface is: $s = [1 \quad 1]x$. And the sampling period T is 0.1 second.

Exogenous disturbances

Consider the linear system with exogenous disturbance alone, that is, $\Delta A = 0$, $\Delta B = 0$ and

$$f(t) = 0.5 + 5 \sin(0.1t) - \cos(t).$$

Figure 4: Discrete sliding mode under exogenous disturbances

Figure 5: Chattering reduction and system performance

Obviously, the continuous time matching condition (3.4) is satisfied. The disturbance function $f(t)$ is smooth enough for arbitrary order of accuracy. In this example, we choose $q = 9$ as in (3.25).

Figure 6: One additional switching between samples

Figure 4 depicts the sliding variable $s(t)$ with initial condition $x^T(0) = [0.5 \ -1.0]$. For the first 8 samples(0.8 second), $s(kT) = O(T^2)$ because the prediction process with $O(T^9)$ error does not work until enough data of past d_i are collected. Two steps after that(1.0 second), we achieve the control goal.

Another exciting attribute we find in discrete time sliding mode is chattering reduction. As shown in Figure 5, there is no major chattering phenomenon in the control profile.

Better performance is obtained in Figure 6 when one additional switching in the control is inserted between consecutive samples.

Internal Uncertainties

Figure 7: Discrete sliding mode with ΔA variations

Figure 8: Discrete sliding mode with ΔB variations

In the presence of system parameter uncertainties alone, the controller is easier to implement. Suppose

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 2.0 \sin(0.05t) & -0.5 \cos(0.1t) \end{bmatrix}$$

and $f = 0$. The response is shown in Figure 7. After the first few samples, the effect of ΔA attenuates as the states approach zero.

A similar result is obtained in Figure 8 for control gain matrix variations.

$$\Delta B = \begin{bmatrix} 0 \\ -0.1 - 0.2 \cos(0.2t) \end{bmatrix}$$

The convergence rate is slower. The effect of ΔB is also attenuated to zero asymptotically.

If $f(t)$, ΔA , and ΔB are present simultaneously, no better accuracy than $O(T^2)$ is available. The only way to improve the performance is to increase the sampling rate.

3.6 Conclusion

In this chapter, we have investigated the sample/hold effect and taken it into account for discrete time sliding mode control design. We proved that the piecewise constant control law due to zero-order-hold has an inherent limitation in disturbance rejection. Even if the sampled-data system is kept exactly in the sliding manifold at each sampling instant, the intersample behavior still deviates from $\mathcal{S}$ by the amount $O(T^2)$. The error is inevitable since a similar “hold” does not take place in the disturbance channels. The sample/hold effect invalidates one’s attempt to achieve

complete disturbance invariance. Nevertheless, an $O(T^2)$ accuracy is sufficient in most applications; after all, there is no perfect sliding mode in reality.

A key assumption throughout the derivation is boundedness and smoothness of the disturbances. It allows one to approximate integrations and predict future unknowns by using past, discrete measurement. The induced error leads to an $O(T^2)$ boundary layer in the neighborhood of the sliding manifold.

For sampled implementation, the idea of a boundary layer enables one to smooth the control instead of instantaneous switchings. Meanwhile, it does not compromise chattering with robustness as many other smoothing functions do [20]. The proposed method is simple from an implementation point of view. It deals with both exogenous disturbances and internal uncertainties. Furthermore, the chattering problem associated with the continuous time variable structure control is removed with considerable robustness.

CHAPTER IV

Nonlinear Systems

In the previous chapter, we have developed a discrete time sliding mode control method applied to linear sampled-data systems to alleviate chattering. Here, we extend it to nonlinear systems. We assume the nonlinearities and uncertainties are bounded and smooth, in this case the past values of the states can be utilized to predict the effect. This approach allows us to keep the state in the $O(T^2)$ vicinity of the sliding manifold.

4.1 Problem Statement

Consider the nonlinear system

$$\dot{x} = a(x) + b(x)u + v(x, t) \quad (4.1)$$

with a prescribed sliding manifold

$$\mathcal{S} = \{x | s(x) = 0\}, \quad (4.2)$$

where $a(x)$ and $b(x)$ are known differentiable functions with dimensions $n \times 1$ and $n \times m$ respectively; $v(x, t)$ is an $n \times 1$ unknown vector-valued differentiable function.

The disturbance matching condition is

$$\text{Rank}[b(x), v(x, t)] = \text{Rank}[b(x)] \text{ when } x \in \mathcal{S}. \quad (4.3)$$

We say the disturbance $v(x, t)$ is matched if it satisfies the above matching condition. Let $G(x) = \frac{\partial}{\partial x}s(x)$ be the $m \times n$ Jacobian matrix of $s(x)$ so that the dynamics of the sliding surface vector can be expressed as

$$\dot{s}(x) = G(x)a(x) + G(x)b(x)u + G(x)v(x, t). \quad (4.4)$$

Solving $\dot{s} = 0$ for the continuous time equivalent control yields

$$u^{eq}(t) = -[G(x)b(x)]^{-1}G(x)[a(x) + v(x, t)]. \quad (4.5)$$

To design discrete time sliding mode control law for (4.1) requires a nonlinear sampled-data representation of that system. Conventionally, an accurate sampled-data system representation is obtained by solving the nonlinear differential equation with a constant control input $u(t) = u_k$ for $t \in [kT, (k+1)T]$ given the initial state value $x(kT) = x_k$. A major difficulty is that the solution to x_{k+1} (if the solution exists) is often difficult to obtain. The known nonlinearities in $a(x)$ and $b(x)$ have already been hard to deal with. The additional uncertainty part $v(x, t)$ makes it almost impossible to obtain an accurate discrete time representation.

Despite this problem, assuming the discrete time nonlinear representation corresponding to (4.1) can be found, it is often given as

$$x_{k+1} = \mathcal{F}(x_k, u_k), \quad (4.6)$$

where the mapping $\mathcal{F} : R^n \times R^m \rightarrow R^n$ contains the effect of the uncertainty $v(x, t)$ in some known fashion. To obtain the discrete time equivalent control, one needs to solve the imbedded nonlinear algebraic equation

$$s_{k+1} = s(\mathcal{F}(x_k, u_k^{eq})) = 0. \quad (4.7)$$

This equation incurs another difficulty that a unique solution to u_k^{eq} may not exist even if disturbances are not present in the original dynamics. We conclude that the above approach of obtaining discrete time version for (4.1) is not recommendable. Therefore, instead of coping with the cumbersome nonlinearities, we propose an easy, straight forward approach in following derivations.

Applying a zero-order-hold to the control yields the discrete time representation of (4.1) and (4.4)

$$x_{k+1} = x_k + \int_{kT}^{(k+1)T} a(x)dt + \int_{kT}^{(k+1)T} b(x)dt u_k + \int_{kT}^{(k+1)T} v(x, t)dt \quad (4.8)$$

$$s_{k+1} = s_k + \int_{kT}^{(k+1)T} G(x)a(x)dt + \int_{kT}^{(k+1)T} G(x)b(x)dt u_k + \int_{kT}^{(k+1)T} G(x)v(x, t)dt. \quad (4.9)$$

The discrete time sliding mode control is to steer the states towards and maintain them on the surface $\mathcal{S}$ at each sampling instant

$$s_k = s(x_k) = 0. \quad (4.10)$$

We introduce the notation:

$$\begin{aligned} F_k &= \int_{kT}^{(k+1)T} G(x)a(x)dt \\ B_k &= \int_{kT}^{(k+1)T} G(x)b(x)dt \\ h_k &= \int_{kT}^{(k+1)T} G(x)v(x, t)dt \end{aligned} \quad (4.11)$$

and rewrite (4.9) as

$$s_{k+1} = s_k + F_k + B_k u_k + h_k. \quad (4.12)$$

The quantities F_k , B_k , h_k usually can not be evaluated exactly since the integrations contain noncausal information about the state and the unknown disturbance.

However, they have the magnitude of $O(T)$ and can be approximated with $O(T^2)$ accuracy by using Euler's method if we impose some boundedness and smoothness assumptions to $G(x)$, $a(x)$, $b(x)$, and $v(x, t)$. This is especially important for sampled-data control law design to be discussed in the later sections. We do not pursue higher accuracy because the sampled controller will still result in an $O(T^2)$ error even if the discrete time system representation is exact.

With the notation (4.11), the discrete time equivalent control is obtained by solving $s_{k+1} = 0$, which renders

$$u_k^{eq} = -B_k^{-1}(s_k + F_k + h_k). \quad (4.13)$$

Denote the allowable range of admissible control by $\mathcal{U}$. For (4.5) and (4.13) to be valid for controller design, we apply the following assumptions:

Assumption 4.1

1. *The $m \times m$ matrix $G(x)b(x)$ is nonsingular for all x .*
2. *The $m \times m$ matrix B_k is nonsingular for all k .*
3. *The continuous time equivalent control $u^{eq}(t) \in \mathcal{U}$.*

It can be observed that u_k^{eq} will approach $u^{eq}(t)$ as $T \rightarrow 0$ when $s_k = 0$. Since the continuous time equivalent control $u^{eq}(t) \in \mathcal{U}$, the discrete time equivalent control u_k^{eq} will be admissible when the state is in the vicinity of $\mathcal{S}$. Note in (4.13) that $u_k^{eq} = O(\frac{1}{T})$ if x_k is far from $\mathcal{S}$ and hence is usually nonadmissible.

Since the control in the sampled data system will show jumps from the continuous time point of view, the formal concept of continuity is not compatible with the notion of sampled-data control. In order to make the analogy properly, let us introduce the following definition:

Definition 4.1 *The discrete time control law u_k is said to be equivalent to discontinuous if $\Delta u_k = O(1)$, continuous if $\Delta u_k = O(T)$, smooth if $\Delta^2 u_k = O(T^2)$, where Δ denotes the backward differencing operator $1 - z^{-1}$.*

To avoid a cumbersome description, the “continuity” of discrete time control to appear in the following text will refer to its equivalent meaning as defined above.

The following Lemma is the nonlinear version of Lemma 3.1.

Lemma 4.1 *If the functions $a(x)$, $b(x)$, and $v(x, t)$ in (4.1) are bounded and smooth in the intersample interval $kT \leq t \leq (k+1)T$, then $\begin{cases} s_k = 0 \\ s_{k+1} = 0 \end{cases}$ implies $s(t) = O(T^2)$ during that sampling period.*

Proof. With B_k being nonsingular by Assumption 4.1, the solution to u_k that maintains $s_k = 0$ and $s_{k+1} = 0$ is uniquely determined

$$u_k = -B_k^{-1}(F_k + h_k).$$

Let $0 \leq \tau \leq T$, the intersample sliding surface vector is

$$\begin{aligned} s(kT + \tau) = & \int_{kT}^{kT+\tau} G(x)a(x)dt \int_{kT}^{kT+\tau} G(x)b(x)dt B_k^{-1} F_k \\ & + \int_{kT}^{kT+\tau} G(x)v(x, t)dt - \int_{kT}^{kT+\tau} G(x)b(x)dt B_k^{-1} h_k. \end{aligned} \quad (4.14)$$

Since $a(x)$, $b(x)$, and $v(x, t)$ are smooth, the following approximations are legitimate:

$$\begin{aligned}
B_k - TG(x(kT))b(x(kT)) &= e_1 = O(T^2) \\
F_k - TG(x(kT))a(x(kT)) &= e_2 = O(T^2) \\
h_k - TG(x(kT))v(x(kT), T) &= e_3 = O(T^2) \\
\int_{kT}^{kT+\tau} G(x)b(x)dt - \tau G(x(kT))b(x(kT)) &= e_4 = O(\tau^2) \\
\int_{kT}^{kT+\tau} G(x)a(x)dt - \tau G(x(kT))a(x(kT)) &= e_5 = O(\tau^2) \\
\int_{kT}^{kT+\tau} G(x)v(x, t)dt - \tau G(x(kT))v(x(kT), T) &= e_6 = O(\tau^2).
\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14) yields

$$s(kT + \tau) = e_5 + e_6 - \frac{\tau}{T}(e_2 + e_3) - (e_4 + \frac{\tau}{T}e_1)B_k^{-1}(F_k + h_k).$$

Boundedness of $a(x)$, $b(x)$, and $v(x, t)$ implies $B_k = O(T)$, $F_k = O(T)$, and $h_k = O(T)$, which leads to $s(kT + \tau) = O(T^2)$. $\square$

Similar to the linear cases, this Lemma states that even the states reside on the sliding manifold at each sampling instant, the intersample state trajectory will deviate. Since an $O(T^2)$ error in sliding mode always exists, an exact discrete time system representation do not seem to be imperative for control design. An Euler's approximation suffices. This relieves the difficulties one encounters in discretizing a nonlinear system.

Another result parallel to Lemma 3.2 concerning the disturbance matching conditions for both the original (4.1) and the discretized (4.8) systems is presented below.

Lemma 4.2 *The sampled-data system (4.8) is disturbance invariant on the discrete time sliding manifold (4.10) only if the continuous time matching condition (4.3) holds.*

Proof. Suppose the discrete time system (4.8) is in sliding mode $s_k = 0$, one can substitute the discrete time equivalent control (4.13) into (4.8) to yield

$$x_{k+1} = x_k + \int_{kT}^{(k+1)T} a(x)dt + \int_{kT}^{(k+1)T} v(x, t)dt - \int_{kT}^{(k+1)T} b(x)dt B_k^{-1}(F_k + h_k). \quad (4.16)$$

The necessary and sufficient condition for the discrete time system (4.16) to be disturbance invariant is

$$\int_{kT}^{(k+1)T} b(x)dt B_k^{-1} h_k = \int_{kT}^{(k+1)T} v(x, t)dt,$$

which further implies that there exists a constant $m \times 1$ vector q_k satisfying

$$\int_{kT}^{(k+1)T} v(x, t)dt = \int_{kT}^{(k+1)T} b(x)dt q_k. \quad (4.17)$$

Therefore, $\text{Rank}[b(x), v(x, t)] = \text{Rank}[b(x)]$. $\square$

Note that the matching condition (4.3) is necessary but not sufficient to disturbance rejection when discrete time sliding mode is achieved. From the discrete time point of view, the disturbance vector $\int_{kT}^{(k+1)T} v(x, t)dt$ does not lie in the control coefficient matrix $\int_{kT}^{(k+1)T} b(x)dt$ in general. Suppose the condition (4.3) holds and there exists an $m \times 1$ differentiable vector field $q(x, t)$ such that

$$v(x, t) = b(x)q(x, t),$$

then the discrete time disturbance vector can be expressed as

$$\int_{kT}^{(k+1)T} b(x)q(x, t)dt = \int_{kT}^{(k+1)T} b(x)dt q(x(kT), kT) + \int_{kT}^{(k+1)T} b(x) \int_{kT}^{\tau} \dot{q}(x, t)dt d\tau. \quad (4.18)$$

The first term in the right hand side of equation (4.18) is rejectable by u_k , while the second term is general not. If furthermore, the time derivative $\dot{q}(x, t)$ is bounded,

the left over portion (out of the range space of $\int_{kT}^{(k+1)T} b(x)dt$) of the disturbance has magnitude $O(T^2)$. Thus, due to the zero-order-hold processes applied to the control signals, the matched disturbance $v(x, t)$ will have $O(T^2)$ “spillage”, not rejectable by the sampled control law. This agrees with the linear cases in Chapter III.

4.1.1 Matched Disturbances

Lemma 4.1 justifies the use of Euler’s approximation method in obtaining discrete time representation of nonlinear systems. Since an $O(T^2)$ deviation from the sliding surface always occurs even with a precise discrete time representation for the nonlinear system, an $O(T^2)$ error in discretization is satisfactory. We will make use of this idea in control design to maintain $O(T^2)$ accuracy in sliding mode. Approximate the quantities F_k , B_k , and h_k by the following expressions:

$$\begin{aligned}\bar{F}_k &= G(x_k)a(x_k)T \\ \bar{B}_k &= G(x_k)b(x_k)T \\ \bar{h}_k &= s_{k+1} - s_k - \bar{F}_k - \bar{B}_k.\end{aligned}\tag{4.19}$$

The legitimacy of the approximation (4.19) bases on the assumptions:

Assumption 4.2

1. *The functions $a(x)$, $b(x)$, $v(x, t)$, and $G(x)$ are bounded and smooth; $\frac{\partial}{\partial x}[G(x)a(x)]$, $\frac{\partial}{\partial x}[G(x)b(x)]$, and $\frac{\partial}{\partial x}[G(x)v(x, t)]$ are bounded.*
2. *There exist positive finite numbers m_g , M_g , M_f , and a nonnegative number m_f such that*

$$\begin{aligned}0 < m_g &\leq \|G(x)b(x)\| \leq M_g \\ 0 \leq m_f &\leq \|G(x)a(x)\| \leq M_f \text{ for all } x.\end{aligned}$$

3. The 2-norm of the admissible control u_k is bounded by a positive finite number

M_u such that $\|u\| \leq M_u$.

The approximation errors are defined as

$$\begin{aligned} e_{\bar{F}_k} &= F_k - \bar{F}_k, & \bar{e}_{\bar{F}_k} &= \bar{F}_k - \bar{F}_{k-1} \\ e_{\bar{B}_k} &= B_k - \bar{B}_k, & \bar{e}_{\bar{B}_k} &= \bar{B}_k - \bar{B}_{k-1} \\ e_{h_k} &= h_k - h_{k-1}, & e_{\bar{h}_k} &= h_k - \bar{h}_k. \end{aligned} \quad (4.20)$$

It can also be proved (see Appendix) that there exist finite positive numbers M_F , M_B , M_{h_1} , and M_{h_2} satisfying

$$\begin{aligned} \|e_{\bar{F}_k}\| &\leq M_F T^2, & \|\bar{e}_{\bar{F}_k}\| &\leq \frac{M_F}{2} T^2 \\ \|e_{\bar{B}_k}\| &\leq M_B T^2, & \|\bar{e}_{\bar{B}_k}\| &\leq \frac{M_B}{2} T^2 \\ \|e_{h_k}\| &\leq M_{h_1} T^2, & \|e_{\bar{h}_k}\| &\leq M_{h_2} T^2. \end{aligned} \quad (4.21)$$

With (4.19), we obtain an approximation for the equivalent control (4.13)

$$u_k^{in} = -\bar{B}_k^{-1}(s_k + \bar{F}_k + \bar{h}_{k-1}), \quad (4.22)$$

where $\bar{B}_k$ is invertible by Assumption 4.1. Similar to (3.24), x_k must lie in the boundary layer $\mathcal{S}_T$ for u_k^{in} to be admissible. We propose the following variable structure control law

$$u_k = \begin{cases} -\bar{B}_k^{-1}(s_k + \bar{F}_k + \bar{h}_{k-1}) = u_k^{in} & \text{when } x_k \in \mathcal{S}_T \\ -\bar{B}_k^{-1}(K \operatorname{sgn}(s_k) + \bar{F}_k + \bar{h}_{k-1}) = u_k^{out} & \text{otherwise.} \end{cases} \quad (4.23)$$

Note that u_k^{out} can always be made admissible by an appropriate choice of the positive definite matrix K since the second and the third terms $-\bar{B}_k^{-1}(\bar{F}_k + \bar{h}_{k-1})$ are close to the continuous time equivalent control evaluated at $t = kT$, which is admissible by assumption. If the state is out of the boundary layer $\mathcal{S}_T$, we apply the same control as continuous time design u_k^{out} . The sampling period T and the matrix K have to be chosen such that the next state approaches $\mathcal{S}_T$ without overshoot. This can be

referred to as the “reaching phase”. Chattering can occur in the reaching phase if there is partial sliding in some elements of the sliding surface vector s_k .

Once the state enters $\mathcal{S}_T$, the control u_k^{in} will further steer it into an $O(T^2)$ boundary layer and retain it there. Robustness against both external and internal uncertainties can then be acquired with sufficient accuracy. We will show that the proposed control in (4.23) will be continuous (see Definition 4.1) when $s_k = O(T^2)$.

Lemma 4.3 *The control u_k in (4.23) will lead to $s_{k+1} = O(T^2)$ if $x_k \in \mathcal{S}_T$.*

Proof. Since the state is in $\mathcal{S}_T$, the control u_k^{in} is admissible. Direct substitution of u_k^{in} into (4.12) yields

$$s_{k+1} = e_{\bar{F}_k} + e_{h_k} + e_{\bar{h}_{k-1}} - e_{\bar{B}_k} \bar{B}_k^{-1} (s_k + \bar{F}_k + \bar{h}_{k-1}).$$

With $\bar{B}_k = O(T)$, $s_k = O(T)$, $\bar{F}_k = O(T)$, $\bar{h}_k = O(T)$, $e_{\bar{F}_k} = O(T^2)$, $e_{\bar{B}_k} = O(T^2)$, $e_{h_k} = O(T^2)$, and $e_{\bar{h}_k} = O(T^2)$, it follows $s_{k+1} = O(T^2)$. $\square$

Lemma 4.4 *The control u_k in (4.23) is continuous if $s_{k-1} = O(T^2)$.*

Proof. According to Lemma 4.3, $s_{k-1} = O(T^2)$ implies $u_{k-1} = u_{k-1}^{in}$ and $s_k = O(T^2)$, which leads to $s_k = O(T^2)$ and $u_k = u_k^{in}$. It follows

$$\begin{aligned} \Delta u_k &= u_k^{in} - u_{k-1}^{in} \\ &= -\bar{B}_k^{-1} (s_k + \bar{e}_{\bar{F}_k} - \bar{e}_{\bar{B}_k} \bar{B}_{k-1}^{-1} \bar{F}_{k-1} + e_{h_{k-1}} - e_{\bar{h}_{k-1}} + e_{\bar{h}_{k-2}} - \bar{e}_{\bar{B}_k} \bar{B}_{k-1}^{-1} \bar{h}_{k-2}) \\ &\quad + \bar{B}_{k-1}^{-1} s_{k-1}. \end{aligned}$$

Applying the inequalities (4.21) with $s_{k-1} = O(T^2)$ and $s_k = O(T^2)$ yields $\Delta u_k = O(T)$. Therefore, the control is continuous as defined in Definition 4.1. $\square$

Theorem 4.1 *If the dynamic system (4.1) satisfies the matching condition (4.3), then the effect of the matched uncertainties $v(x, t)$ can be reduced to $O(T^2)$ if the variable structure control (4.23) is applied.*

Proof. When $x_k \notin \mathcal{S}_T$, u_k^{out} is employed. It will steer the state toward the boundary layer $\mathcal{S}_T$ with sliding surface equation

$$s_{k+1} - s_k = -K \operatorname{sgn}(s_k) + e_{\bar{F}_k} + e_{h_k} + e_{\bar{h}_{k-1}},$$

where the positive definite matrix K is chosen small enough so that $u_k^{out} \in \mathcal{U}$. and large enough to dominate the unknown $O(T^2)$ portion $e_{\bar{F}_k} + e_{h_k} + e_{\bar{h}_{k-1}}$ so that the state eventually enters the boundary layer $\mathcal{S}_T$.

Once the state enters $\mathcal{S}_T$ for some j , u_j^{in} will further maintain $s_{j+1} = O(T^2)$ according to Lemma 4.3. This implies $x_{j+1} \in \mathcal{S}_T$ and u_{k+1}^{in} is selected to render $s_{j+2} = O(T^2)$.

By induction, the discrete time states $x_j, x_{j+1}, \dots$ will stay in the $O(T^2)$ boundary layer, which guarantees $x(t) = O(T^2)$ for $t \geq jT$ (using Lemma 4.1). Thus the effect of the matched disturbance is reduced to $O(T^2)$. $\square$

4.1.2 Control Coefficient Variations

Consider the case when there is an unknown variation $\tilde{b}(x)$ in the control coefficient $b(x)$

$$\dot{x} = a(x) + b(x)u + \tilde{b}(x)u. \quad (4.24)$$

Following the same discretization procedure as indicated in (4.9), we have

$$s_{k+1} = s_k + F_k + B_k u_k + h_k, \quad (4.25)$$

where

$$\begin{aligned} h_k &= \tilde{B}_k u_k \\ \tilde{B}_k &= \int_{kT}^{(k+1)T} G(x) \tilde{b}(x) dt. \end{aligned}$$

The measurable portion of the disturbance is

$$\bar{h}_k = s_{k+1} - s_k - \bar{F}_k - \bar{B}_k u_k.$$

We would like to apply the variable structure control (4.23) for control coefficient variation problems so that the same control law can be used when both types uncertainties are present. The following assumptions are essential for our developments.

Assumption 4.3

1. *The unknown function $\tilde{b}(x)$ is bounded and smooth; $\frac{\partial}{\partial x}[G(x)\tilde{b}(x)]$ is bounded.*
2. *Variation $\tilde{b}(x)$ is matched: $\text{Rank}[b(x), \tilde{b}(x)] = \text{Rank}[b(x)]$ when $x \in \mathcal{S}$.*
3. *The $m \times m$ matrix $B_k + \tilde{B}_k$ is nonsingular for all k .*

It can be observed that the control (4.23) contains dynamic feedback through the disturbance prediction term $\bar{h}_{k-1}$. We rewrite equation (4.25) as

$$s_{k+1} = s_k + F_k + (B_k + \tilde{B}_k)u_k. \quad (4.26)$$

Suppose the matrix $(B_k + \tilde{B}_k)$ is nonsingular, according to the relationship in (4.26), the stability of u_k can be assured if the sliding surface vector s_k is stable. We shall prove that, with a small enough control coefficient matrix variation $\tilde{b}(x)$ and fast enough sampling rate, the control (4.23) will lead to a stable dynamic equation for s_k and that $s_k \rightarrow O(T^2)$ asymptotically.

Introduce the following notation:

$$\begin{aligned} e_{F_k} &= F_k - F_{k-1}, \quad e_{B_k} = B_k - B_{k-1}, \quad e_{\tilde{B}_k} = \tilde{B}_k - \tilde{B}_{k-1} \\ \Delta_k &= \tilde{B}_k B_k^{-1}, \quad \epsilon_{\Delta_k} = \Delta_k - \Delta_{k-1}. \end{aligned} \quad (4.27)$$

Based on Assumptions 4.3, the following results can be proved (see Appendix):

There exist finite positive numbers $M_{\tilde{B}}$ and M_{Δ} such that

$$\begin{aligned} \|e_{F_k}\| &\leq \frac{M_F}{4} T^2, \quad \|e_{B_k}\| \leq \frac{M_B}{4} T^2, \quad \|e_{\tilde{B}_k}\| \leq M_{\tilde{B}} T^2 \\ \|\epsilon_{\Delta_k}\| &\leq M_{\Delta} T, \quad \|I - B_k \tilde{B}_k^{-1}\| \leq \frac{M_B}{m_g} T. \end{aligned} \quad (4.28)$$

Direct substitution of equations (4.20), (4.27), and (4.22) into (4.12) leads to the following Lemma.

Lemma 4.5 *When the state is in the boundary layer $\mathcal{S}_T$, the following equalities are true:*

$$1. \quad s_{k+1} = h_k - h_{k-1} + \alpha_k^1(2s_k - s_{k-1}) + \alpha_k^2, \quad (4.29)$$

where $\alpha_k^1 = O(T)$, $\alpha_k^2 = O(T^2)$.

$$2. \quad h_k = -\Delta_k[(s_k + F_k + h_{k-1}) + \beta_k^1(2s_k - s_{k-1}) + \beta_k^2], \quad (4.30)$$

where $\beta_k^1 = O(T)$, $\beta_k^2 = O(T^2)$.

The quantities α_k^i , β_k^i , $i = 1, 2$ can be written explicitly:

$$\begin{aligned} \alpha_k^1 &= -2e_{\tilde{B}_k} \tilde{B}_k^{-1} \\ \alpha_k^2 &= -e_{\tilde{B}_k} \tilde{B}_k^{-1} F_k + e_{\tilde{F}_k} + e_{\tilde{B}_k} \tilde{B}_k^{-1} e_{\tilde{F}_k} + e_{\tilde{h}_{k-1}} + e_{\tilde{B}_k} \tilde{B}_k^{-1} (\tilde{F}_{k-1} + \tilde{B}_{k-1} u_{k-1}) \\ \beta_k^1 &= e_{\tilde{B}_k} \tilde{B}_k^{-1} \\ \beta_k^2 &= -e_{\tilde{F}_k} - e_{\tilde{h}_{k-1}} e_{\tilde{B}_k} \tilde{B}_k^{-1} (e_{\tilde{F}_k} - \tilde{B}_{k-1} u_{k-1}) \end{aligned} \quad (4.31)$$

The equalities (4.29) and (4.30) in Lemma 4.5 serve as preparation for the proof of the theorem stated below:

Theorem 4.2 *For the nonlinear system with control coefficient matrix variation (4.24), the variable structure control (4.23) will lead to $s_k \rightarrow O(T^2)$ asymptotically if*

1. *The state lie in the boundary layer $\mathcal{S}_T$.*
2. *The variation $\tilde{b}(x)$ satisfies Assumption 4.3.*
3. *The following dynamic equation for s_k is stable.*

$$s_{k+1} = a_k^0 s_k + a_k^1 s_{k-1} + a_k^2 s_{k-2} + \delta_k, \quad (4.32)$$

where

$$\begin{aligned} a_k^0 &= -2\Delta_k + O(T) \\ a_k^1 &= \Delta_{k-1} + O(T) \\ a_k^2 &= O(T) \\ \delta_k &= O(T^2). \end{aligned}$$

Proof. If $x_k \notin \mathcal{S}_T$, u_k^{out} will steer the state toward the boundary layer $\mathcal{S}_T$.

When $x_k \in \mathcal{S}_T$, substituting (4.30) into (4.29) with conditions 1 and 2, yields a third order dynamic equation for s_k (4.32). The coefficients and the perturbation matrices are

$$\begin{aligned} a_k^0 &= -2\Delta_k - 2\Delta_k \beta_k^1 + 2\alpha_k^1 \\ a_k^1 &= \Delta_{k-1} + 2\Delta_k \alpha_k^1 - \epsilon_{\Delta_k} + \Delta_k \beta_k^1 + 2\Delta_{k-1} \beta_k^1 - \alpha_k^1 \\ a_k^2 &= \epsilon_{\Delta_k} - \Delta_k \alpha_k^1 - \Delta_{k-1} \beta_k^1 \\ \delta_k &= -\Delta_k e_{F_k} - \epsilon_{\Delta_k} F_{k-1} + \Delta_k \alpha_k^2 + \epsilon_{\Delta_k} (F_{k-2} + B_{k-2} u_{k-2}) - \Delta_k \beta_k^2 + \Delta_{k-1} \beta_{k-1}^2 + \alpha_k^2 \end{aligned}$$

Condition 3 implies that $s_k \rightarrow \delta_k = O(T^2)$. $\square$

If the sampling rate is sufficiently high, the dynamic equation (4.32) is reduced to second order

$$s_{k+1} = -2\Delta_k s_k + \Delta_{k-1} s_{k-1}. \quad (4.33)$$

This concides with (3.33), the result for linear systems. Stability of s_k can be assured if Δ_k is small enough; that is, the variation of the control coefficient $\tilde{b}(x)$ is small with respect to $b(x)$. If the $O(T)$ terms in (4.32) are not negligible, the dynamic equation for s_k becomes third order with an $O(T^2)$ input. Stability of (4.32) implies that $s_k \rightarrow O(T^2)$ asymptotically.

We have studied sliding mode control for nonlinear sampled-data systems in the presence of matched disturbances $v(x, t)$ and control coefficient matrix variations $\tilde{b}(x)$ individually. Both cases share the same control law (4.23). In general, both uncertainties will appear in the original system

$$\dot{x} = a(x) + b(x)u + \tilde{b}(x)u + v(x, t). \quad (4.34)$$

To achieve $O(T^2)$ accuracy in sliding mode we still can use the variable structure control (4.23) provided the conditions in Theorems 4.1 and 4.2 are satisfied. The proof is similar to that in Theorem 3.4. We will demonstrate the effectiveness of the proposed method in the following example.

4.2 Numerical Example

A two-joint robotic manipulator with point mass m_i , link length r_i , and inertia with respect to each joint J_i , $i = 1, 2$ is shown in Figure 9. The dynamics are:

$$\begin{bmatrix} \alpha_{11}(q_2) & \alpha_{12}(q_2) \\ \alpha_{12}(q_2) & \alpha_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \beta_{12}(q_2)\dot{q}_1^2 + 2\beta_{12}(q_2)\dot{q}_1\dot{q}_2 \\ -\beta_{12}(q_2)\dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} \gamma_1(q_1, q_2)g \\ \gamma_2(q_1, q_2)g \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where q_1 and q_2 are the first and second joint angles respectively, g is the gravitational constant, and u_1 u_2 are the joint torques. The nonlinear elements α_{ij} , β_{12} , γ_j , $i, j = 1, 2$ and system parameters m_i , r_i , J_i , $i = 1, 2$ can be found in [6].

Figure 9: A two-joint manipulator

The control objective is to bring the robotic system from its initial configuration $[q_1(0), \dot{q}_1(0), q_2(0), \dot{q}_2(0)] = [-159.5^\circ, 0, -69^\circ, 0]$ to the desired configuration $[0, 0, 0, 0]$ by observing sliding motion on the switching surface:

$$\begin{aligned} s_1 &= 0.5q_1 + \dot{q}_1 = 0 \\ s_2 &= 0.4q_2 + \dot{q}_2 = 0. \end{aligned}$$

The amount of parameter variations considered in this example is:

$$\begin{array}{lll} \frac{\Delta m_1}{m_1} = 20\% & \frac{\Delta r_1}{r_1} = -10\% & \frac{\Delta J_1}{J_1} = -10\% \\ \frac{\Delta m_2}{m_2} = -8\% & \frac{\Delta r_2}{r_2} = 12.5\% & \frac{\Delta J_2}{J_2} = -4\%. \end{array}$$

Applying the sliding mode control law (4.22) with sampling period $T = 0.1$ second, the first joint angle has the response depicted in Figure 10. The manipulator achieves the desired configuration smoothly in spite of unknown system uncertainties. The

sampled-data control laws u_1 and u_2 exhibit no chattering in the steady state as shown in Figure 11.

Figure 10: Discrete time sliding mode in two-joint manipulator

Figure 11: Chattering reduction in discrete time sliding mode

CHAPTER V

Stochastic Systems

5.1 Problem Statement

This chapter is devoted to discrete time sliding mode control for continuous time linear systems with stochastic disturbances. The disturbance is assumed to satisfy an Ito type stochastic differential equation [60]

$$dx = Axdt + Budt + (D\xi dt + Rdw_1) \quad (5.1)$$

$$d\xi = W\xi dt + Qdw_2, \quad (5.2)$$

where $w_1(t)$, $w_2(t)$ are independent standard Wiener processes. Here $D\xi$ represents the colored part of the noise and the formal expression $R\frac{dw_1}{dt}$, the white part. Without loss of generality, the autocovariance matrices of the white noises are assumed to be identity.

We consider two possible state information channels: continuous measurement and discrete measurement. To compensate for the stochastic disturbances with a sampling controller, a mixed continuous-discrete type of filtering problem will be solved if continuous state information $x(t)$ is used. On the other hand, the use of discrete state information x_k will lead to a discrete time optimal filtering problem. In both cases, we apply the following assumption:

Assumption 1

1. W has asymptotically stable eigenvalues;
2. R is full rank;
3. The pair $(W, R^T D)$ is detectable;
4. The processes $Rw_1(t)$ and $Qw_2(t)$ are uncorrelated.

5.2 Continuous Measurement

Consider the discrete time version of (5.1)

$$x_{k+1} = \Phi x_k + \Gamma u_k + \eta_k + v_k, \quad (5.3)$$

where $\eta_k = \int_{kT}^{(k+1)T} e^{A((k+1)T-\lambda)} D \xi(\lambda) d\lambda$ is the colored part of the disturbance and $v_k = \int_{w_1(kT)}^{w_1((k+1)T)} e^{A((k+1)T-\lambda)} R dw_1(\lambda)$ is a white noise. Since the white noise can not be compensated, the control objective is to minimize the effect of the colored part η_k in the sampled-data system (5.3). Applying the control law with an estimation of η_k

$$u_k = -(C\Gamma)^{-1} C \Phi x_k - (C\Gamma)^{-1} C \hat{\eta}_k, \quad (5.4)$$

we obtain

$$s_{k+1} = C(\eta_k + v_k - \hat{\eta}_k). \quad (5.5)$$

For u_k (5.4) to be admissible, it still requires that the state lies in the $O(T)$ boundary layer $\mathcal{S}_T$. This boundary layer can be defined similarly as in the previous sections. Here, we only consider when $x_k \in \mathcal{S}_T$.

In the presence of stochastic disturbances, sliding mode is defined in terms of a probability measure. To minimize $E\|s_{k+1}\|^2$ given the observation of $x(t)$ up to the instant kT , the estimate $\hat{\eta}_k$ should be picked as a conditional expectation

$$\hat{\eta}_k = E(\eta_k | \mathcal{F}_{kT}), \quad (5.6)$$

where $\mathcal{F}_t$ denotes a σ -algebra generated by the process $x(t)$

$$\mathcal{F}_t = \sigma\{x(\tau) | 0 \leq \tau \leq t\}. \quad (5.7)$$

Since $w_1(t)$ is a martingale [60], the estimation η_k is obtained by taking a conditional expectation given the σ -algebra $\mathcal{F}_{kT}$

$$\hat{\eta}_k = \int_{kT}^{(k+1)T} e^{A((k+1)T-\lambda)} D d\lambda \hat{\xi}(kT) = \Xi \hat{\xi}_k, \quad (5.8)$$

where the matrix Ξ is of the form

$$\Xi = \int_0^T e^{A\lambda} D d\lambda, \quad (5.9)$$

and $\hat{\xi}$ is a conditional expectation

$$\hat{\xi}(t) = E(\xi(t) | \mathcal{F}_t). \quad (5.10)$$

Therefore, from (5.8) it suggests that in order to obtain the values of $\hat{\eta}_k$ we need a filter to calculate $\hat{\xi}_k$. The following theorem provides the form of such a filter.

Theorem 5.1 *If assumption 1 holds, the optimal estimate of ξ_k is*

$$\hat{\xi}_k = z_k + LR^T x_k, \quad (5.11)$$

where z_k satisfies

$$z_k = \Lambda z_{k-1} + \zeta_{k-1} + \Pi u_{k-1} \quad (5.12)$$

with Λ , Π and ζ_k defined by:

$$\begin{aligned}\Lambda &= e^{HT}, \\ \Pi &= \int_0^T e^{H\lambda} d\lambda M, \\ \zeta_{k-1} &= \int_{(k-1)T}^{kT} e^{H(kT-\lambda)} Nx(\lambda) d\lambda,\end{aligned}\tag{5.13}$$

where the matrices H , N , M , L are

$$\begin{aligned}H &= W - LR^T D, \\ N &= -LR^T A + WLR^T - LR^T DLR^T, \\ M &= -LR^T B, \\ L &= PD^T R(R^T R)^{-2}\end{aligned}\tag{5.14}$$

and P is the positive definite solution of the algebraic matrix Riccati equation

$$PW^T + WP - PD^T R(R^T R)^{-2} R^T DP + QQ^T = 0.\tag{5.15}$$

Proof. Multiplying the system equation (5.1) by R^T yields

$$R^T dx = R^T A x dt + R^T B u dt + R^T D \xi dt + R^T R dw_1.\tag{5.16}$$

With the σ -algebra $\mathcal{F}_t$, we obtain the innovation process $R^T D \xi dt + R^T R dw_1$ from (5.16) to be fed into the continuous time Kalman filter for $\xi(t)$

$$d\hat{\xi} = W \hat{\xi} dt + L(R^T D \xi dt + R^T R dw_1 - R^T D \hat{\xi} dt),\tag{5.17}$$

where the optimal feedback gain matrix $L = PD^T R(R^T R)^{-2}$ is determined by solving the algebraic Riccati equation (5.15) for P . With Assumption 1, the matrix $R^T R$ is invertible and an unique solution for P exists.

Substituting (5.16) into (5.17) renders

$$d\hat{\xi} = W \hat{\xi} dt + LR^T(dx - Ax dt - Bu dt - D \hat{\xi} dt).\tag{5.18}$$

Defining a new variable $z = \hat{\xi} - LR^T x$ and substituting z into (5.18) lead to the differential equation for z

$$\dot{z} = Hz + Nx + Mu, \quad (5.19)$$

which contains no stochastic processes. The constant matrices H , N , M are defined as in (5.14). Finally, discretization of the continuous time system (5.19) gives the difference equation (5.12). The optimal estimate for ξ_k is then obtained. $\square$

The filter (5.11) and (5.12) provides a way to calculate the conditional expectation of ξ_k given the information from the continuous time process $x(t)$. Substituting the estimate into the equation (5.8) we obtain an optimal $\hat{\eta}_k$ as:

$$\hat{\eta}_k = \Xi(z_k + LR^T x_k). \quad (5.20)$$

The quantity $\eta_k + v_k - \hat{\eta}_k$ from the right hand side of (5.5) is a discrete-time white noise process with variance of order $O(T)$. It means that the mean square deviation of s_{k+1} from zero is of order $O(T^{\frac{1}{2}})$.

5.3 Discrete Measurement

Suppose we only have access to the discrete state information, the σ -algebra generated by the discrete measurement $x_k, x_{k-1}, \dots$ is

$$\mathcal{F}_k = \sigma\{x_j | 0 \leq j \leq k\}. \quad (5.21)$$

The discrete time representations of (5.1) and (5.2) to be considered are

$$x_{k+1} = \Phi x_k + \Upsilon \xi_k + \Gamma u_k + v_k + \nu_k \quad (5.22)$$

$$\xi_{k+1} = \Psi \xi_k + q_k, \quad (5.23)$$

where $\nu_k = \int_0^T e^{A(T-\lambda)} D \int_0^\lambda e^{W(\lambda-\tau)} Q dw_2(kT+\tau) d\lambda$, $q_k = \int_{w_2(kT)}^{w_2((k+1)T)} e^{W((k+1)T-\lambda)} Q dw_2(\lambda)$

are discrete white noises generated from the process dw_2 ; the matrices Ψ and Υ are

$$\Psi = e^{WT}, \quad \Upsilon = e^{AT} \int_0^T e^{-A\lambda} D e^{W\lambda} d\lambda. \quad (5.24)$$

To minimize the effect caused by ξ_k , we apply the control law

$$u_k = -(C\Gamma)^{-1} C \Phi x_k - (C\Gamma)^{-1} C \Upsilon \hat{\xi}_k, \quad (5.25)$$

where $\hat{\xi}_k$ is the optimal estimation of ξ_k given $\mathcal{F}_k$

$$\hat{\xi}_k = E(\xi_k | \mathcal{F}_k). \quad (5.26)$$

From (5.22), the discrete measurements x_k and x_{k-1} yield the innovation process

$$y_{k-1} = x_k - \Phi x_{k-1} - \Gamma u_{k-1} = \Upsilon \xi_{k-1} + v_{k-1} + \nu_{k-1}. \quad (5.27)$$

Combining (5.27) with (5.23), one can construct a stationary Kalman filter with the process noise q_{k-1} in (5.23) correlated with the measurement noise $(v_{k-1} + \nu_{k-1})$ in (5.27) through dw_2 . To obtain a standard Kalman filter formulation with uncorrelated noises, we rewrite equation (5.23)

$$\xi_{k+1} = \bar{\Psi} \xi_k + \bar{H} \bar{R}^{-1} y_k + \bar{q}_k, \quad (5.28)$$

where $\bar{H}$ and $\bar{R}$ are the covariance matrices

$$\bar{H} = E(q_k(v_k + \nu_k)^*), \quad \bar{R} = E((v_k + \nu_k)(v_k + \nu_k)^*),$$

and $\bar{\Psi}$, $\bar{q}_k$ are defined as

$$\bar{\Psi} = \Psi - \bar{H} \bar{R}^{-1} \Upsilon, \quad \bar{q}_k = q_k - \bar{H} \bar{R}^{-1} (v_k + \nu_k).$$

With (5.28) and the innovation process y_k , an optimal estimation for ξ_k is acquired

$$\hat{\xi}_k = \bar{\Psi}\hat{\xi}_{k-1} + \bar{H}\bar{R}^{-1}y_{k-1} + \bar{L}(y_{k-1} - \Upsilon\hat{\xi}_{k-1}), \quad (5.29)$$

$$\bar{L} = \bar{P}\Upsilon^T(\Upsilon\bar{P}\Upsilon^T + \bar{R})^{-1}, \quad (5.30)$$

$$\bar{P} = \bar{\Psi}(\bar{P} - \bar{P}\Upsilon^T(\Upsilon\bar{P}\Upsilon^T + \bar{R})^{-1}\Upsilon\bar{P})\bar{\Psi}^T + \bar{Q}\bar{Q}^T, \quad (5.31)$$

where $\bar{Q}^T$ is the autocovariance matrix of the discrete white noise $\bar{q}_k$ in (5.28).

Applying the control law (5.25) yields

$$s_{k+1} = C(\Upsilon\xi_k + v_k + \nu_k - \Upsilon\hat{\xi}_k). \quad (5.32)$$

The quantity $\Upsilon(\xi_k - \hat{\xi}_k) + v_k + \nu_k$ from the right hand side of (5.32) is also a discrete time white noise with variance of order $O(T)$. Therefore, the mean square deviation of s_{k+1} from the sliding surface is $O(T^{\frac{1}{2}})$.

CHAPTER VI

Prefiltering and Postfiltering

6.1 Problem Statement

Full state accessibility has been assumed throughout the derivations in the previous chapters. In this chapter, we release this assumption and take a different perspective to deal with chattering phenomenon for sampled-data systems. In [20], Young and Drakunov summarized two main approaches to reduce chattering. The first one is the use of continuous approximation in the boundary layer of the sliding manifold [24, 26]. The second one is to utilize asymptotic state observers to localize the high frequency phenomenon [25]. These two approaches can then be unified into the framework of a pre-filtering and post-filtering processes. The system configuration is shown in Figure 12. We will analyze chattering reduction by exploiting the pre-filtering and post-filtering idea in discrete time for sampled data systems.

The original continuous time system is described by

$$\begin{aligned}\dot{x} &= Ax + Bu + Df \\ y &= Hx,\end{aligned}\tag{6.1}$$

where $x \in R^n$, $u \in R^m$ are state and control variables respectively; $f \in R^l$ corresponds to the lumped effect of both external and internal uncertainties; $y \in R^p$ is the measured output; and A , B , D , H are constant matrices of appropriate dimensions.

The desired sliding surface is

$$S = \{x | Cx = 0\}. \quad (6.2)$$

The use of a sliding mode control law for (6.1) allows one to achieve complete disturbance rejection if the matching condition holds [3].

$$\text{rank}[B] = \text{rank}[B, D] \quad (6.3)$$

Let T be the sampling period. Assuming u is applied through a zero-order-hold, the system (6.1) is discretized as

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k + d_k \\ y_k &= H x_k \end{aligned} \quad (6.4)$$

where $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{A\lambda} d\lambda B$, $d_k = \int_0^T e^{A\lambda} D f((k+1)T - \lambda) d\lambda$.

We impose the following assumptions:

1. (A, B, H) is complete.
2. B and Γ are full rank.
3. CB and $C\Gamma$ are invertible.
4. The continuous time equivalent control is admissible.

The pre-filter module assumes the main task of smoothing the control law u_k . Especially when the state lies in the $O(T^2)$ vicinity of the sliding surface, the pre-filter will eliminate discontinuity in the control law (in the sense of Definition 4.1), that is $\Delta u_k = O(T)$. The post-filter is a dynamic system which can accomplish multiple objectives. If only the measured output y_k is available, a state observer becomes

Figure 12: System Configuration

necessary. Under some special geometric conditions on the matrices (H, A, D) , the unmeasured states can be observed asymptotically irrespective of the disturbances. For the purpose of discrete time sliding mode in a sampled-data system, only $O(T^2)$ accuracy of state observation is required. A much milder condition on the triple (H, A, D) will suffice. If full state x_k is accessible for measurement, the post-filter will be used for constructing a dynamic sliding surface that reduces chattering by suppressing high frequency components and eliminating the reaching phase [5]. With the measured and/or estimated states, the switching function module is able to generate an appropriate signal to be fed into the pre-filter for chattering reduction.

6.2 Pre-filtering and Post-filtering

In this section, we implement discrete time sliding mode control in the pre-filtering and post-filtering setting to achieve chattering reduction.

6.2.1 Post-filtering

Most sliding control design requires full state accessibility for computing $s_k = Cx_k$. If only the measured output y_k is available, state observation becomes necessary. The design of an observer subject to disturbances is not as straight forward as the conventional Luenburger observer. In general, the matrices (H, A, D) should satisfy some special geometric conditions for the observer to be invariant of disturbances [61]. Since $O(T^2)$ error is inevitable for sampled data sliding mode control, we investigate a discrete time observer that allows $O(T^2)$ error for more general systems.

Applying the nonsingular linear transformation

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} H \\ P \end{bmatrix} x_k \quad (6.5)$$

yields,

$$\begin{aligned} y_{k+1} &= \Psi_{11}y_k + \Psi_{12}z_k + \Upsilon_1u_k + d_k^1 \\ z_{k+1} &= \Psi_{21}y_k + \Psi_{22}z_k + \Upsilon_2u_k + d_k^2. \end{aligned} \quad (6.6)$$

With

$$D_1 = H \int_0^T e^{A\lambda} D d\lambda, \quad D_2 = P \int_0^T e^{A\lambda} D d\lambda, \quad (6.7)$$

the disturbances d_k^1 and d_k^2 can be approximated by

$$d_k^1 = D_1 f_k + O(T^2), \quad d_k^2 = D_2 f_k + O(T^2). \quad (6.8)$$

We substitute (6.8) into the dynamic equation for y_k and define

$$\begin{aligned} m_{k-1} &= y_k - \Psi_{11}y_{k-1} - \Upsilon_1 u_{k-1} \\ &= \Psi_{12}z_{k-1} + D_1 f_{k-1} + O(T^2). \end{aligned} \quad (6.9)$$

Assuming $\dot{f}(t)$ exists and is bounded, so that $f_k = f_{k-1} + O(T)$, we define the augmented state $\xi_k = [z_k^T, f_k^T]^T$ and construct an observer for the augmented system.

$$\begin{aligned} \hat{\xi}_k = \begin{bmatrix} \Psi_{22} & D_2 \\ 0 & I \end{bmatrix} \hat{\xi}_{k-1} + \begin{bmatrix} \Psi_{12}y_{k-1} + \Upsilon_2 u_{k-1} \\ 0 \end{bmatrix} \\ + L_1(m_{k-1} - [\Psi_{12} \ D_1]\hat{\xi}_{k-1}) \end{aligned} \quad (6.10)$$

For the augmented system to be observable, it requires

$$\text{Rank} \begin{bmatrix} zI - \Psi_{22} & -D_2 \\ 0 & zI - I \\ \Psi_{12} & D_1 \end{bmatrix} = n - p + l \quad \forall z \quad (6.11)$$

A necessary condition is $p \geq l$. The number of measurements should be at least equal to the number of degrees of freedom in the disturbance domain. If it is observable, f_k can be estimated with $O(T)$ error. This implies that d_k can be estimated with $O(T^2)$ accuracy. Substituting the estimated d_k into the observer for the original discrete time system (6.4) yields the unmeasured portion of the state with $O(T^2)$ error.

$$\begin{aligned} \hat{x}_{k+1} &= \Phi \hat{x}_k + \Gamma u_k + L_2(y_k - H \hat{x}_k) + \hat{d}_k; \\ \hat{d}_k &= \int_0^T e^{A\lambda} D d\lambda \cdot \hat{f}_k. \end{aligned} \quad (6.12)$$

Thus, we have two layers of observers as shown in Figure 13. The first layer (6.10) estimates f_k with $O(T)$ accuracy. A similar approach can be found in [62] to yield the optimal estimate. The second layer further increases the accuracy to $O(T^2)$. With the observability assumption imposed on (6.12), L_2 can be found by standard Luenburger observer design.

Figure 13: Two layers of state observation

6.2.2 Switching Function

The post-filtering block provides an estimate for the state. One can use the information to calculate the approximate equivalent control

$$u_k^{in} = -(C\Gamma)^{-1}C\Phi x_k - (C\Gamma)^{-1}Cd_{k-1} \quad (6.13)$$

to be applied when the states lie in the boundary layer $\mathcal{S}_T$.

In the reaching phase, when $x_k \notin \mathcal{S}_T$, we switch the control law to

$$u_k^{out} = u_k^{in} + (C\Gamma)^{-1}[s_k - K \operatorname{sgn}(s_k)] \quad (6.14)$$

leading to

$$s_{k+1} = -K \operatorname{sgn} s_k + s_k, \quad (6.15)$$

where K is a positive definite matrix.

Thus, the control switches between u_k^{in} and u_k^{out} depending on whether $x_k \in \mathcal{S}_T$. The gain matrix K in (6.15) has to be small enough so that x_k enters the region $\mathcal{S}_T$ without overshooting. Once it enters, it stays in the region.

6.2.3 Pre-filtering

The control law in (6.13) can be related to its past value through d_{k-1} . We express d_{k-1} as a linear combination of x_k , x_{k-1} , and u_{k-1} . Equation (6.13) becomes

$$u_k^{in} = -(C\Gamma)^{-1}C(\Phi + I)x_k + (C\Gamma)^{-1}C\Phi x_{k-1} + u_{k-1}. \quad (6.16)$$

A similar expression holds for u_k^{out} . We obtain the following pre-filter:

$$u_k = \Pi u_{k-1} + v_k, \quad (6.17)$$

where $\Pi = I$ and v_k is the output signal from the switching function:

$$v_k = \begin{cases} -(C\Gamma)^{-1}C(\Phi + I)x_k + (C\Gamma)^{-1}C\Phi x_{k-1}, & x_k \in \mathcal{S}_T, \\ -(C\Gamma)^{-1}C\Phi(x_k - x_{k-1}) - K \operatorname{sgn}(s_k), & x_k \notin \mathcal{S}_T \end{cases} \quad (6.18)$$

The pre-filter not only maintains the state in the $O(T)$ boundary layer of $\mathcal{S}$ but also, as detailed in [37], it further shrinks the boundary layer to $O(T^2)$ thickness. The following theorem states that chattering in the control is removed when the state lies in the $O(T^2)$ boundary layer of $\mathcal{S}$.

Theorem 6.1 *The pre-filtered control u_k in (6.17) is continuous if the states lie in the $O(T^2)$ boundary layer of $\mathcal{S}$. Furthermore, u_k approaches the continuous time equivalent control as the sampling rate approaches infinity.*

$$u_k \xrightarrow{T \rightarrow 0} u^{eq}(t) = -(CB)^{-1}CAx(t) - (CB)^{-1}CDf(t) \quad (6.19)$$

Proof. Since the states lie in the $O(T^2)$ boundary layer of $\mathcal{S}$, $x_{k-1} \in \mathcal{S}_T$ and $u_{k-1} = u_{k-1}^{in}$. According to Theorem 3.1, $u_{k-1} = O(1)$ and $s_k = O(T^2)$.

From (6.17)-(6.18),

$$\Delta u_k = -(C\Gamma)^{-1}C(\Phi + I)x_k + (C\Gamma)^{-1}C\Phi x_{k-1}.$$

Replacing Φ with its Taylor series expansion yields

$$\Delta u_k = -(C\Gamma)^{-1}CAT(x_k - x_{k-1}) + O(T).$$

With $u_{k-1} = O(1)$, we have $\dot{x}(t) = O(1)$, $(k-1)T \leq t < kT$, hence, $x_k - x_{k-1} = O(T)$.

Therefore, $\Delta u_k = O(T)$.

As $T \rightarrow 0$, $s_k \rightarrow 0$, and

$$u_k^{in} \rightarrow -(C\Gamma)^{-1}C(e^{AT} - I)x_k - (C\Gamma)^{-1}C \int_0^T Df((k+1)T - \lambda)d\lambda.$$

Applying L'Hospital's rule gives

$$\begin{aligned} \lim_{T \rightarrow 0} u_k^{in} &= \lim_{T \rightarrow 0} \frac{\partial u_k^{in}}{\partial T} \\ &= -(CB)^{-1}CAx(t) - (CB)^{-1}CDf(t) \end{aligned}$$

□

6.3 Expanded State Space

In the previous section, chattering is reduced mainly by the pre-filtering and switching function blocks. The post-filter is only used for state estimation. Here, we further exploit the post-filtering block to assist chattering reduction.

As proved in Theorem 1, the control is continuous when the state lies in the $O(T^2)$ boundary layer of $\mathcal{S}$. It is still not continuous in the reaching phase. We adopt the idea of designing sliding mode in the expanded state space by Young and Özgüner to yield a new sliding surface. The new degree of freedom introduced in the expanded

state space enables one to choose the initial states so that the reaching phase of sliding mode is eliminated. We extend the idea to a more general point of view and apply it to sampled data systems.

Apply the coordinate transformation $\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = Mx_k$ to the sampled system (6.4) such that $M\Gamma = \begin{bmatrix} 0 \\ \Gamma_2 \end{bmatrix}$ and represent (6.4) by

$$\begin{aligned} x_{k+1}^1 &= \Phi_{11}x_k^1 + \Phi_{12}x_k^2 + \nu_k^1 \\ x_{k+1}^2 &= \Phi_{21}x_k^1 + \Phi_{22}x_k^2 + \Gamma_2 u_k + \nu_k^2 \end{aligned} \quad (6.20)$$

with Γ_2 being nonsingular and ν_k^1, ν_k^2 representing the disturbances in the new coordinate. Note that $\nu_k^1 = O(T^2)$ since the continuous disturbance coefficient matrix D is in the range space of B . We introduce a new augmented state η_k that satisfies

$$\eta_{k+1} = F\eta_k + G_1x_k^1 + G_2x_k^2, \quad (6.21)$$

where $\eta_k \in R^r$ with $r \geq m$ so that the new, expanded state space provides us the capability of placing the initial state in the new sliding manifold.

Rewrite the system equation in the expanded state space

$$x_{k+1}^e = \Phi_e x_k^e + \Gamma_e u_k + d_k^e, \quad x_k^e = [\eta_k^T, (x_k^1)^T, (x_k^2)^T]^T, \quad (6.22)$$

where

$$\Phi_e = \begin{bmatrix} F & G_1 & G_2 \\ 0 & \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{21} & \Phi_{22} \end{bmatrix}, \Gamma_e = \begin{bmatrix} 0 \\ 0 \\ \Gamma_2 \end{bmatrix}, d_k^e = \begin{bmatrix} 0 \\ \nu_k^1 \\ \nu_k^2 \end{bmatrix}.$$

By proper choice of the matrices F, G_1, G_2 , we are able to design an appropriate sliding manifold $S_e = \{x^e | C_e x^e = 0\}$. First of all, choose F to be stable to ensure minimum phase. The sliding surface can be selected either by pole placement

method [2] for desired performance or frequency shaping method [5] to attenuate the high frequency components in sliding mode. Before implementing the methods, the controllability of the augmented system (6.22) needs to be ensured. Let

$$\theta_k = H_0\eta_k + H_1x_k^1 + H_2x_k^2 \quad (6.23)$$

so that $\Theta(z) = H(z)X(z)$, where $\Theta(z)$ and $X(z)$ are the z-transform of θ_k and x_k respectively; $H(z)$ is a high frequency penalty transfer function. One can follow the frequency shaping technique depicted in [5] to solve for appropriate F , G_1 , G_2 , H_0 , H_1 , H_2 , and C_e . The frequency dependent cost function can be given in the general formulation:

$$\oint X^*(z)Q(z)X(z)dz, \quad (6.24)$$

where $Q(z)$ is the weighing function in corporate with $H(z)$. The initial condition of η should be assigned so that $C_e x_0^e = 0$. It renders

$$u_k = -(C_e\Gamma_e)^{-1}C_e(\Phi_e + I)x_k^e + (C_e\Gamma_e)^{-1}C_e\Phi_e x_{k-1}^e + u_{k-1}. \quad (6.25)$$

The reaching phase is eliminated by assigning the initial state to the new sliding surface. The control (6.25) is able to maintain the state in the $O(T^2)$ boundary layer. Thus chattering is eliminated.

6.4 Examples

We adopt an example similar to that in [33] for illustration.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} (u + f) \\ y &= [1 \ 0]x \end{aligned}$$

The disturbance satisfies the conditions in Theorem 3.1.

$$f(t) = (0.1 \sin 0.5t + 0.2)u(t) + 0.1 \sin t x(1) - 1.5 \cos 2t x(2) - 1.0 + \cos 0.5t.$$

The sampling period is 0.1 second. The admissible control range is $|u_k| < 10$.

6.4.1 Pre-filtering and Post-filtering

The sliding surface is chosen as: $s = [1 \ 1]x$. Assume only the output y_k is available for measurement. The post-filtering block in (6.10) and (6.12) are employed for estimating the unmeasured state. The accuracy is $O(T^2)$ as illustrated in Figure 14-(a). In Figure 14-(b), we show the control u_k and the sliding surface variable s_k . The control is discontinuous in the reaching phase and continuous after the state enters the $O(T^2)$ boundary layer of $\mathcal{S}$.

Figure 14: Pre-filtering and Post-filtering

6.4.2 Expanded State Space

Assume full state accessibility. The dynamic augmented state is chosen as

$$\theta_{k+1} = 0.5\theta_k + 0.2x_k^2 - x_k^2.$$

The sliding surface is found by pole placement method: $C_e = [0.2186 \quad -0.4639 \quad 1]$ that assigns the poles at $(0.85, 0.9)$ in sliding mode. The system performance is illustrated in Figure 15-(a). Figure 15-(b) shows the elimination of reaching phase. The state is in $\mathcal{S}_T$ at the beginning, where u_k is still discontinuous. The control becomes continuous afterwards.

Figure 15: Expanded State Space

6.5 Conclusion

This chapter deals with three major problems one often encounters in implementing sliding mode control. They are

1. high frequency switching;
2. full state accessibility;
3. chattering phenomenon.

By exploiting the pre-filtering and post-filtering feedback design, we are able to resolve these problems and, most importantly, sufficiently retain the robustness property of sliding mode.

One of the key assumptions throughout the derivation is boundedness and smoothness of the continuous time disturbance $f(t)$. This allows one to predict the future values by utilizing previous information. Disturbance prediction plays an important roll for chattering reduction. A similar idea can be found in [35], where an ARMA model was considered.

If the disturbance is due to parameter variations only, Furuta's sliding cone approach [33] has been shown to be effective without prediction. Recently, the idea was developed for self-tuning control that also relaxes the requirement of full state accessibility [63].

For inherent discrete time memoryless control systems, prediction is not available. The requirement of asymptotic stability about the zero state with complete disturbance rejection becomes too stringent. It was suggested by Coreless and Manela to achieve asymptotic stability about some bounded set which contains the zero state [64]. The same difficulty exists in discrete time sliding mode control. Without precise knowledge about the disturbances, sliding mode is hard to maintain. It remains an interesting research area that is worth exploring in the future.

CHAPTER VII

Applications

7.1 Brushless DC Motor Torque and Current Control

Figure 16: Brushless DC motor 3-phase, Y connected stator winding

A brushless DC motor has the same physical structure as a permanent-magnet AC synchronous machine. The permanent-magnet rotor provides a constant field against the armature winding on the stator, which is commutated electronically by a six-step inverter (see Figure 16) that switches according to the shaft position. With proper switching of the inverter, the need for slip rings in the rotor assembly is then eliminated. The rotor position information can be obtained from either Hall effect sensors or a shaft encoder mounted in the motor assembly. The Hall effect sensors divide each rotor revolution into six equal sectors (60 degrees each). This can only

reveal a coarse knowledge of the rotor position but is sometimes sufficient for switching logic design. The shaft encoder, on the other hand, can provide a higher resolution for the rotor angle which is especially useful for speed sensing and plays an important role in $(d - q)$ analysis for 3-phase motors.

With the above attributes, there are some inherent limitations in the brushless DC motor as well. Due to the dc direct current nature of the driver and the physical winding constraint, the stator can only provide six different magnetomotive forces (mmf), 60 degrees apart from each other as show in Figure (17). The control inputs (stator mmf's) being discrete values, a sliding mode control approach seems appropriate for most control purposes. As a matter of fact, there indeed exist several theoretical and experimental results that support the idea of sliding mode control for brushless DC motors [65, 11, 13, 21].

The dynamic equations in the orthogonal axes $(d - q)$ connected with the rotor can be written in the form

$$\begin{aligned} X_d \frac{di_d}{dt} &= -Ri_d + X_q i_q n + e_d \\ X_q \frac{di_q}{dt} &= -Ri_q - (X_d i_d + F)n + e_q, \end{aligned} \quad (7.1)$$

where i_d , i_q , e_d , e_q are components of stator currents and voltages in the direct and quadrature rotor axes respectively, X_d , X_q , R are active and passive resistances, F is the constant magnet flux coefficient, and n is the angular velocity of the rotor. The torque produced by the motor is given as

$$\tau = ((X_d - X_q)i_d + F)i_q. \quad (7.2)$$

Let θ be the angular rotor position. The $d - q$ control inputs are related to the 3-phase

Figure 17: Stator voltage vectors and rotor position sectors

input voltages by

$$\begin{aligned} e_d &= \frac{2}{3}(e_A \cos \theta + e_B \cos(\theta - \frac{2}{3}\pi) + e_C \cos(\theta + \frac{2}{3}\pi)) \\ e_q &= -\frac{2}{3}(e_A \sin \theta + e_B \sin(\theta - \frac{2}{3}\pi) + e_C \sin(\theta + \frac{2}{3}\pi)) \end{aligned} \quad (7.3)$$

with the inverse transformation [66]

$$\begin{aligned} e_A &= e_d \cos \theta - e_q \sin \theta \\ e_B &= e_d \cos(\theta - \frac{2}{3}\pi) - e_q \sin(\theta - \frac{2}{3}\pi) \\ e_C &= e_d \cos(\theta + \frac{2}{3}\pi) - e_q \sin(\theta + \frac{2}{3}\pi), \end{aligned} \quad (7.4)$$

where e_A , e_B , e_C are phase voltages and can take only two discrete values, $+V_{dc}$ and $-V_{dc}$. This results in the six possible composite control vectors depicted in Figure 17 (excluding two null vectors in a balanced 3-phase stator winding). As can be seen from the above relationships, the $d-q$ control inputs in (7.1) are also discrete valued for each rotor angle.

We consider current control as an important subject in the study of brushless DC

motors especially when one is interested in torque output control, energy loss, circuit protection, and any other current related issues. To control the currents, a general approach is to utilize equations (7.1) and design an appropriate controller to achieve the performance criterion. The control inputs e_d , e_q are functions of i_d , i_q , and n together with those system parameters R , X_d , X_q , and F . The computed values of e_d and e_q will be transformed into the 3-phase analog voltages ($\bar{e}_A$, $\bar{e}_B$, $\bar{e}_C$), which are then converted to the digital values (e_A , e_B , e_C) to be applied physically to the motor. Several papers have successfully verified the above scenario by using various control schemes [65, 11, 13, 21].

In this chapter, however, we have a different approach from the former methods. The switching logic of the three phase input voltages will be designed directly in discrete time. Several practical limitations of the system set up are taken into account such as sampling effects, limited computation speed, and system uncertainties. We will present three simple, fast algorithms that can lead to cheap, robust, and universal current controllers for brushless DC motors. The first algorithm takes a two-stage sliding mode approach. It computes the discrete time equivalent controls for each phase voltage and converts them to suitable DC values by implementing continuous time sliding mode. The second algorithm deals with physical phenomena and generates a proper switching sequence directly without the need of a conversion process. Finally, the third algorithm further assumes no knowledge about the system parameters. By using current feedback and rotor position information, a look up table is constructed to achieve the desired performance. All the three algorithms are robust

against parameter variations and external disturbances.

7.1.1 Controller Design

Figure 18: Constant torque output with constrained energy losses

The essence of designing controllers for brushless DC motors is in fact a set of switching logic for the three phase input voltages e_A , e_B , e_C . The switching logic depends on the states of the motor; namely i_d , i_q , θ , and possibly the angular velocity n . For current control, the performance criteria would be to reach some desired point (i_d^*, i_q^*) in the state space. This point can be determined by either mathematical optimization or physical considerations. Particularly, in the torque control problem, one may desire to achieve some torque value and at the same time minimize the energy loss. Thus (i_d^*, i_q^*) can be solved by

$$\min_{i_d, i_q} i_d^2 + i_q^2 = \sigma \quad \text{such that} \quad ((X_d - X_q)i_d + F)i_q = \tau_d, \quad (7.5)$$

where $\sigma = i_d^2 + i_q^2$ is the instantaneous energy loss, and τ_d is the desired torque output. Or in many cases, if we have a round, symmetric rotor; ie $X_d=X_q$, then it can be easily taken as

$$(i_d^*, i_q^*) = (0, \frac{\tau_d}{F}). \quad (7.6)$$

The minimum energy loss is defined as

$$\sigma_{min} = i_d^{*2} + i_q^{*2}. \quad (7.7)$$

We also impose an overcurrent constraint to the controller design, that is, during the transient process, σ is not to exceed a preset value σ_{max} .

We consider a brushless DC motor with system uncertainties:

$$\begin{aligned} \tilde{X}_d \frac{di_d}{dt} &= -\tilde{R}i_d + \tilde{X}_q i_q n + e_d + f_1(t) \\ \tilde{X}_q \frac{di_q}{dt} &= -\tilde{R}i_q - (\tilde{X}_d i_d + \tilde{F})n + e_q + f_2(t), \end{aligned} \quad (7.8)$$

where $\tilde{X}_d = X_d + \Delta X_d$, $\tilde{X}_q = X_q + \Delta X_q$, $\tilde{R} = R + \Delta R$, $\tilde{F} = F + \Delta F$; R , X_d , X_q , and F are the nominal system parameters; and $f_1(t)$, $f_2(t)$ are external disturbances. Three algorithms using discrete time sliding mode control methods are developed. The sliding surfaces are chosen to achieve the performance criterion depicted in Figure (18).

To obtain the desired torque output, a constant torque curve

$$s_0 = \tau - \tau_d = 0 \quad (7.9)$$

is a suitable switching surface. Since we have two degrees of freedom in control (7.1), one of them can be used for maintaining sliding mode on (7.9), the other can be used

to minimize the energy loss. In the case of current control, the switching surfaces are chosen as

$$\begin{aligned} s_1 &= i_d - i_d^* = 0 \\ s_2 &= i_q - i_q^* = 0. \end{aligned} \quad (7.10)$$

Algorithm 1. Two-stage sliding mode - (i_d, i_q) domain

Figure 19: Two-stage sliding mode block diagram

In Figure 19, the controller is composed of two independent sliding mode stages and an intermediate transformation block. The first sliding mode stage computes e_d^k and e_q^k to maintain sliding mode on the surfaces (7.10) at each sampling instant. The second stage generates appropriate switching logics by realizing continuous time sliding mode during the sampling period.

Let T be the sampling period. The discrete time representation of the system dynamics is obtained by applying a zero-order-hold through the control channels:

$$\begin{aligned} i_d^{k+1} &= i_d^k + (a_{11} + \Delta a_{11})i_d^k + (a_{12} + \Delta a_{12})i_q^k + (b_1 + \Delta b_1)e_d^k + d_1^k + w_1^k \\ i_q^{k+1} &= i_q^k + (a_{21} + \Delta a_{21})i_d^k + (a_{22} + \Delta a_{22})i_q^k - \frac{F}{X_q}n^kT + (b_2 + \Delta b_2)e_q^k + d_2^k + w_2^k, \end{aligned} \quad (7.11)$$

where

$$\begin{aligned} a_{11} &= -\frac{R}{X_d}T & a_{12} &= \frac{X_d}{X_d}n^kT & b_1 &= \frac{T}{X_d} & d_1^k &= \frac{1}{X_d} \int_{kT}^{(k+1)T} f_1(t)dt \\ a_{21}^k &= -\frac{X_d}{X_q}n^kT & a_{22} &= -\frac{R}{X_q}T & b_2 &= \frac{T}{X_q} & d_2^k &= \frac{1}{X_q} \int_{kT}^{(k+1)T} f_2(t)dt; \end{aligned}$$

Δa_{ij} and Δb_j , $i, j = 1, 2$ are due to parameter variations; d_1^k, d_2^k are $O(T)$ disturbances due to $f_1(t)$ and $f_2(t)$; w_1^k, w_2^k are $O(T^2)$ discretization errors resulted from Euler's method. We separate the known and unknown parts of (7.11) and arrive at

$$\begin{aligned} i_d^{k+1} &= i_d^k + a_{11}i_d^k + a_{12}i_q^k + b_1e_d^k + \epsilon_1^k \\ i_q^{k+1} &= i_q^k + a_{21}i_d^k + a_{22}i_q^k - \frac{F}{X_q}n^kT + b_2e_q^k + \epsilon_2^k, \end{aligned} \quad (7.12)$$

with

$$\begin{aligned} \epsilon_1^k &= \Delta a_{11}i_d^k + \Delta a_{12}i_q^k + \Delta b_1e_d^k + d_1^k + w_1^k \\ \epsilon_2^k &= \Delta a_{21}i_d^k + \Delta a_{22}i_q^k + \Delta b_2e_q^k + d_2^k + w_2^k. \end{aligned} \quad (7.13)$$

According to [30], the discrete time equivalent control (obtained by solving $s_j^{k+1} = 0$, $j = 1, 2$) is able to steer the states toward the sliding surfaces in one sampling period.

$$\begin{aligned} \{e_d^k\}_{eq} &= \frac{1}{b_1}(i_d^* - i_d^k - a_{11}i_d^k - a_{12}i_q^k - \epsilon_1^k) \\ \{e_q^k\}_{eq} &= \frac{1}{b_2}(i_q^* - i_q^k - a_{21}i_d^k - a_{22}i_q^k + \frac{F}{X_q}n^kT - \epsilon_2^k) \end{aligned} \quad (7.14)$$

Obviously, $\{e_d^k\}_{eq}$ and $\{e_q^k\}_{eq}$ are not accessible since they contain the unknown terms $\epsilon_1^k, \epsilon_2^k$. Nevertheless, the unknown terms can be approximated by their past values ϵ_1^{k-1} and ϵ_2^{k-1} , which can be calculated from (7.12). Therefore, we come up with an approximated equivalent control, which will result in sliding motion in a boundary layer of the switching surfaces (7.10) with thickness $O(T^2)$ [39].

$$\begin{aligned} e_d^k &= \frac{1}{b_1}(i_d^* - i_d^k - a_{11}i_d^k - a_{12}i_q^k - \epsilon_1^{k-1}) \\ e_q^k &= \frac{1}{b_2}(i_q^* - i_q^k - a_{21}i_d^k - a_{22}i_q^k + \frac{F}{X_q}n^kT - \epsilon_2^{k-1}). \end{aligned} \quad (7.15)$$

Lemma 7.1 *The control law (7.15) will lead to*

$$\begin{aligned} s_1^{k+1} &= -2\frac{\Delta b_1}{b_1}s_1^k + \frac{\Delta b_1}{b_1}s_1^{k-1} + \delta_1^k \\ s_2^{k+1} &= -2\frac{\Delta b_2}{b_2}s_2^k + \frac{\Delta b_2}{b_2}s_2^{k-1} + \delta_2^k \end{aligned} \quad (7.16)$$

with $\delta_j^k = O(T^2)$, $j = 1, 2$ if $f_1(t)$, $f_2(t)$, $\dot{f}_1(t)$, and $\dot{f}_2(t)$ are bounded.

Proof: Substituting (7.15) into (7.12) yields

$$s_j^{k+1} = \epsilon_j^k - \epsilon_j^{k-1}, \quad j = 1, 2. \quad (7.17)$$

For $j = 1$, equation (7.13) renders

$$\epsilon_1^k - \epsilon_1^{k-1} = \Delta b_1(e_d^k - e_d^{k-1}) + \nu_1^k, \quad (7.18)$$

with

$$\nu_1^k = \Delta a_{11}(i_d^k - i_d^{k-1}) + (\Delta a_{12}^k i_q^k - \Delta a_{12}^{k-1} i_q^{k-1}) + (d_1^k - d_1^{k-1}) + (w_1^k - w_1^{k-1}).$$

Since $f_1(t)$ and $\dot{f}_1(t)$ are bounded,

$$d_1^k - d_1^{k-1} = \int_0^T \int_{(k-1)T+\sigma}^{kT+\sigma} \dot{f}(t) dt d\sigma = O(T^2).$$

Also observe that $\Delta a_{11} = O(T)$, $\Delta a_{12}^k = O(T)$, $i_d^k - i_d^{k-1} = O(T)$, $i_q^k - i_q^{k-1} = O(T)$, $w_1^k - w_1^{k-1} = O(T^2)$, and $\Delta a_{12}^k - \Delta a_{12}^{k-1} = O(T^2)$. Therefore, $\nu_1^k = O(T^2)$.

With (7.15), we have

$$e_d^k - e_d^{k-1} = \frac{1}{b_1}(-s_1^k + s_1^{k-1}) - \frac{1}{b_1}(\epsilon_1^{k-1} - \epsilon_1^{k-2}) + \nu_2^k, \quad (7.19)$$

where $\nu_2^k = -\frac{1}{b_1}(a_{11}(i_d^k - i_d^{k-1}) + a_{12}^k i_q^k - a_{12}^{k-1} i_q^{k-1}) = O(T)$.

Combining (7.19), (7.18), and (7.17) renders equation (7.16), with

$$\delta_1^k = \Delta b_1 \nu_2^k + \nu_1^k = O(T^2).$$

Similar arguments holds for $j = 2$. $\square$

It can be observed that the dynamic equations (7.16) are stable if the variations Δb_j are small compared to b_j . In this particular case, the allowable range of variation in b_j is

$$-1 < \frac{\Delta b_j}{b_j} < \frac{1}{3}, \quad j = 1, 2. \quad (7.20)$$

With stability assumption on (7.16), the sliding surface variables s_j^k will approach $O(T^2)$ asymptotically. Therefore, the state trajectory converges to an $O(T^2)$ boundary layer of the switching surfaces (7.10).

To realize (e_d^k, e_q^k) , the computed values will be transformed back to the 3-phase voltages $(\bar{e}_A, \bar{e}_B, \bar{e}_C)$ by equations (7.4) (the intermediate block). Since $(\bar{e}_A, \bar{e}_B, \bar{e}_C)$ will generally not satisfy the DC constraint, it is necessary to convert them into the admissible values $\pm V_{dc}$. In [65], Utkin suggested that e_j be picked as

$$e_j = \text{sgn}(\bar{e}_j) * V_{dc}, \quad j = A, B, C. \quad (7.21)$$

This method has been shown to be effective for continuous time design [21]. The disadvantage is that (7.21) does not provide enough accuracy when $\bar{e}_j$'s are close to zero. This is especially important when one is dealing with small current control or with slow sampling. In either case, the chattering phenomenon will seriously degrade the system performance. A feasible way to increase accuracy is to use PWM (pulse-width-modulation), that effectively produces a fast switching digital signal by adjusting the on/off duty cycle

$$\text{duty cycle} = \begin{cases} 0.5(1 + \frac{\bar{e}_j}{V_{dc}}) & \text{if } V_{dc} > |\bar{e}_j| \\ 0.5(1 + \text{sgn}(\bar{e}_j)) & \text{if } V_{dc} \leq |\bar{e}_j|. \end{cases} \quad (7.22)$$

However, this method is subject to the base frequency and the linearity assumption of PWM.

Here we propose another way of converting $\bar{e}_j$ to e_j by building the following nonlinear circuitry

$$\dot{Z}_j = -V_{dc} \text{sgn}(Z_j) + \bar{e}_j, \quad j = A, B, C. \quad (7.23)$$

The circuitry can be realized by using analog power switches. It is shown in [1] that if $V_{dc} > |\bar{e}_j|$, the system (7.23) will reach sliding mode on the surface $Z_j = 0$ in finite time and we have

$$\bar{e}_j \equiv (V_{dc} \text{sgn}(Z_j))_{avg}, \quad j = A, B, C. \quad (7.24)$$

By letting

$$e_j = V_{dc} \text{sgn}(Z_j), \quad j = A, B, C, \quad (7.25)$$

we obtain the desired switchings for (e_A, e_B, e_C) , which is the actual voltage vector applied to the motor. If $V_{dc} \leq |\bar{e}_j|$, sliding mode is not attained and (7.25) will lead to $e_j = \text{sgn}(\bar{e}_j) * V_{dc}$, which is the same as (7.21) and (7.22).

Algorithm 2. Direct switching logic design - (τ, σ) domain

We can also attack the problem directly from the torque-energy point of view and work in the (τ, σ) state space. Our aim then would be to provide the desired torque value τ_d with minimum energy loss σ_{min} with the restriction that $\sigma \leq \sigma_{max}$ during the transient process. This is always a legitimate consideration to protect the circuit from an overcurrent.

The dynamic equations for τ and σ are

$$\begin{aligned}\dot{\tau} &= \frac{(X_d - X_q)i_q}{X_d}e_d + \frac{(X_d - X_q)i_d + F}{X_q}e_q + g_\tau + f_\tau \\ \dot{\sigma} &= \frac{i_d}{X_d}e_d + \frac{i_q}{X_q}e_q + g_\sigma + f_\sigma,\end{aligned}\tag{7.26}$$

where f_τ and f_σ collect all the system uncertainties; and

$$\begin{aligned}g_\tau &= \frac{(X_d - X_q)i_q}{X_d}(-Ri_d + X_q i_q n) + \frac{(X_d - X_q)i_d + F}{X_q}(-Ri_q - (X_d i_d + F)n) \\ g_\sigma &= \frac{i_d}{X_d}(-Ri_d + X_q i_q n) + \frac{i_q}{X_q}(-Ri_q - (X_d i_d + F)n).\end{aligned}\tag{7.27}$$

We ignore the unknown parts in (7.26) and substitute e_d, e_q by (7.3) to yield

$$\begin{aligned}\dot{\tau}^\circ &= b_{\tau A}e_A + b_{\tau B}e_B + b_{\tau C}e_C + g_\tau + \\ \dot{\sigma}^\circ &= b_{\sigma A}e_A + b_{\sigma B}e_B + b_{\sigma C}e_C + g_\sigma,\end{aligned}\tag{7.28}$$

where $b_{ij} = b_{ij}(i_d, i_q, \theta)$, $i = \tau, \sigma$, $j = A, B, C$ are functions of currents i_d, i_q , and rotor position θ .

$$\begin{aligned}b_{\tau A} &= \frac{2}{3}\left(\frac{(X_d - X_q)i_q}{X_d} \cos \theta - \frac{(X_d - X_q)i_d + F}{X_q} \sin \theta\right) \\ b_{\tau B} &= \frac{2}{3}\left(\frac{(X_d - X_q)i_q}{X_d} \cos\left(\theta - \frac{2}{3}\pi\right) - \frac{(X_d - X_q)i_d + F}{X_q} \sin\left(\theta - \frac{2}{3}\pi\right)\right) \\ b_{\tau C} &= \frac{2}{3}\left(\frac{(X_d - X_q)i_q}{X_d} \cos\left(\theta + \frac{2}{3}\pi\right) - \frac{(X_d - X_q)i_d + F}{X_q} \sin\left(\theta + \frac{2}{3}\pi\right)\right) \\ b_{\sigma A} &= \frac{2}{3}\left(\frac{i_d}{X_d} \cos \theta - \frac{i_q}{X_q} \sin \theta\right) \\ b_{\sigma B} &= \frac{2}{3}\left(\frac{i_d}{X_d} \cos\left(\theta - \frac{2}{3}\pi\right) - \frac{i_q}{X_q} \sin\left(\theta - \frac{2}{3}\pi\right)\right) \\ b_{\sigma C} &= \frac{2}{3}\left(\frac{i_d}{X_d} \cos\left(\theta + \frac{2}{3}\pi\right) - \frac{i_q}{X_q} \sin\left(\theta + \frac{2}{3}\pi\right)\right).\end{aligned}\tag{7.29}$$

Since there are only six possible values of the phase voltage vectors, $e = (e_A, e_B, e_C) \in E = \{e^1, e^2, \dots, e^6\}$, it is more convenient to describe the switching logic not analytically but a sequence of conditions. Let $e^* = e^*(\theta, n, i_d, i_q)$ denote the vector assigned as a control for the given values of θ, n, i_d, i_q .

Switching Logic

1. If $\sigma > \sigma_{max}$, consider the set E_1 of all possible $e \in E$ satisfying the condition

$\dot{\sigma}^\circ(e) < 0$ ($E_1 \neq \emptyset$). If there exists such $e \in E_1$ that $(\tau - \tau_d)\dot{\tau}^\circ < 0$, we assign

$e^* = e$, else, $e^* = e$ for any $e \in E_1$.

2. If $\sigma < \sigma_{max}$, we change the priority and consider the set E_2 of all $e \in E$ that condition $(\tau - \tau_d)\dot{\tau}^o < 0$ is satisfied ($E_2 \neq \emptyset$). If there exists such $e \in E_2$ that $\dot{\sigma}^o(e) < 0$, we assign $e^* = e$, else $e^* = e$ for any $e \in E_2$.

This algorithm works directly on the switching logic (e_A, e_B, e_C) and hence does not require the conversion steps. It ensures that the currents (i_d, i_q) do not trespass the allowable range and leads to sliding mode on the curve (7.9). Since only the signs of $\dot{\tau}$ and $\dot{\sigma}$ are significant in assigning the control vectors, a precise knowledge of the system dynamics is not necessary. The algorithm is effective as long as the uncertainties f_τ and f_σ are small enough not to affect the signs of $\dot{\tau}$ and $\dot{\sigma}$.

Algorithm 3. Table look-up

The above two algorithms assume a partial knowledge of the system parameters. Now we will exploit a different method that enables current control without the need for calculating (e_d, e_q) but simply makes use of Hall effect sensor information to design our switching logic.

Integrating the equations in (7.8) from $t = kT$ to $(k+1)T$, we observe that the incremental vector $(i_d^{k+1} - i_d^k, i_q^{k+1} - i_q^k)$ is a function of $i_d^k, i_q^k, n^k, e_d^k, e_q^k$, and the sampling time T .

$$\begin{aligned} i_d^{k+1} - i_d^k &= \frac{T}{\tilde{X}_d}(-\tilde{R}i_d^k + \tilde{X}_q i_q^k n + e_d^k) + O(T) \\ i_q^{k+1} - i_q^k &= \frac{T}{\tilde{X}_q}(-\tilde{R}i_q^k - (\tilde{X}_d i_d^k + \tilde{F})n + e_q^k) + O(T). \end{aligned} \tag{7.30}$$

To drive the current state i_d^k, i_q^k towards i_d^*, i_q^* , equations (7.30) suggest that we choose

e_d^k, e_q^k such that

$$\begin{aligned} \text{sgn}(e_d^k) &= \text{sgn}(i_d^* - i_q^k) = s_d \\ \text{sgn}(e_q^k) &= \text{sgn}(i_q^* - i_q^k) = s_q. \end{aligned} \quad (7.31)$$

This is appropriate from the point of view of sliding mode control. The existence condition of sliding mode on the surfaces (7.10) is

$$s_j \dot{s}_j < 0, \quad j = 1, 2, \quad (7.32)$$

which demands that

$$\begin{aligned} (-\tilde{R}i_d + \tilde{X}_q i_q n + e_d + f_1(t))(i_d - i_d^*) &< 0 \\ (-\tilde{R}i_q - (\tilde{X}_d i_d + \tilde{F})n + e_q + f_2(t))(i_q - i_q^*) &< 0. \end{aligned} \quad (7.33)$$

Since we have no information about $\tilde{R}$, $\tilde{X}_d$, $\tilde{X}_q$, $\tilde{F}$, and the external disturbances, a most feasible choice of e_d, e_q would be to satisfy (7.31). Note that only the sign of e_d and e_q are determined here. The calculation of the control laws is not needed.

Table 1: e_n v.s. (e_A, e_B, e_C)

stator mmf vector	(e_A, e_B, e_C)
e_1	$(+V_{dc}, -V_{dc}, -V_{dc})$
e_2	$(+V_{dc}, +V_{dc}, -V_{dc})$
e_3	$(-V_{dc}, +V_{dc}, -V_{dc})$
e_4	$(-V_{dc}, +V_{dc}, +V_{dc})$
e_5	$(-V_{dc}, -V_{dc}, +V_{dc})$
e_6	$(+V_{dc}, -V_{dc}, +V_{dc})$

Knowing the desired signs for e_d, e_q , the next step will be to determine which of the six stator mmf's $e_n, n = 1, \dots, 6$, are to be applied. As depicted in Figure 17, let θ_n be the associated angle for e_n with

$$\theta_n = \frac{(n-1)}{3}\pi. \quad (7.34)$$

For each e_n , the corresponding control variables e_d, e_q can be determined by projecting e_n onto the rotor direct axis and the quadrature axis. Given the rotor position θ , it can be seen that

$$\begin{aligned} \text{sgn}(e_d) &= \text{sgn}(\cos(\theta_n - \theta)) \\ \text{sgn}(e_q) &= \text{sgn}(\sin(\theta_n - \theta)). \end{aligned} \quad (7.35)$$

For each pair of the desired $\text{sgn}(e_d, e_q)$ and rotor angle θ , there exists either one or two e_n 's that satisfy (7.35). One can choose an appropriate e_n and obtain the corresponding switching logic (e_A, e_B, e_C) by looking up Table 1.

Another way of finding e_n is by using the Hall sensor information. As seen in Figure 17, for each sector(I,II,...,VI) the rotor resides in, we can always find one e_n that satisfies (7.35). Table 2 is then constructed.

Table 2: $\text{sgn}(e_d, e_q)$ as a function of e_n and rotor position

$\text{sgn}(e_d, e_q)$	rotor position sector number					
	I	II	III	IV	V	VI
e_1	(+, -)	X	(-, -)	(-, +)	X	(+, +)
e_2	(+, +)	(+, -)	X	(-, -)	(-, +)	X
e_3	X	(+, +)	(+, -)	X	(-, -)	(-, +)
e_4	(-, +)	X	(+, +)	(+, -)	X	(-, -)
e_5	(-, -)	(-, +)	X	(+, +)	(+, -)	X
e_6	X	(-, -)	(-, +)	X	(+, +)	(+, -)

where X denotes “don’t care” since the corresponding $\text{sgn}(e_d, e_q)$ is ambiguous.

Finally, with Tables 1 and 2, we come up with a single table of switching logic for (e_A, e_B, e_C) (Table 3).

Table 3: Current control switching logic look-up table

Hall sensor information sector number	(e_A, e_B, e_C)
I	$(s_d, s_q, -s_d) * V_{dc}$
II	$(-s_q, s_d, -s_d) * V_{dc}$
III	$(-s_d, s_d, s_q) * V_{dc}$
IV	$(-s_d, -s_q, s_d) * V_{dc}$
V	$(s_q, -s_d, s_d) * V_{dc}$
VI	$(s_d, -s_d, -s_q) * V_{dc}$

This method considers the physical structure of a brushless DC motor without the knowledge of system parameters. Hence, it is a robust against parameter variations. The stability bound is wider than that in (7.20).

7.1.2 Numerical Examples

We simulate the three algorithms to demonstrate the effectiveness of the proposed methods. The system parameters of the motor are taken as:

$$\begin{aligned}
 &\text{Phase winding resistance } R = 1.0\Omega \\
 &\text{d-axis inductance } X_d = 0.34H \\
 &\text{q-axis inductance } X_q = 0.11H \\
 &\text{Torque constant } F = 1.0NmA^{-1}
 \end{aligned}$$

The amount of parameter variations considered in this example is

$$\frac{\delta R}{R} = -25.0\% \quad \frac{\delta X_d}{X_d} = -17.2\% \quad \frac{\delta X_q}{X_q} = -54.5\% \quad \frac{\delta F}{F} = 25.0\%.$$

Also, additive disturbances through the control channels are present in the system dynamics

$$\begin{aligned}
 f_1(t) &= 0.4 + 0.3 \sin(31.4t) \\
 f_2(t) &= 0.5 - 0.1 \cos(31.4t).
 \end{aligned}$$

The other numerical data chosen for simulation are

Sampling time: $T_s = 0.001$ seconds

Applied DC voltage: $V_{dc} = 2.0$ volt

Desired torque output: $\tau_d = 1.0$ Nm

Optimal current values: $i_d^* = 0.2009$ A², $i_q^* = 0.9558$ A²

Maximal and minimal instant energy losses: $\sigma_{max} = 1.5$ A², $\sigma_{min} = 0.9540$ A²

Initial conditions: $[i_d(0), i_q(0), n(0), \theta(0)] = [0, 0, 0, 0]$.

Figure 20: Time response and phase diagram of i_d and i_q

In Algorithm 1, the control goal is to steer i_d, i_q to the desired values i_d^* and i_q^* . Both states achieve sliding mode in finite time (see Figure 20). Since the driving circuit is implemented by fast switching devices, less chattering is observed in the

Figure 21: Time response of τ and σ , phase trajectory of i_d and i_q

time response.

Algorithm 2 deals with τ and σ directly. As depicted in Figure 21, it reaches the sliding surface $\tau - \tau_d = 0$ first and then searches for the minimum σ . Due to system uncertainties, the states linger about the optimal point (i_d^*, i_q^*) .

Figure 22 is the system response obtained by using Algorithm 3. Sliding mode also occur on the surfaces (7.10) in finite time. The desired current values are attained in

Figure 22: Time response and phase trajectory of i_d and i_q

spite of the system uncertainties.

7.1.3 Conclusions

We have completed three different algorithms for brushless DC motor current and torque control. The same idea can be generalized to some other control applications, such as speed and position, where we need to include one or two more dynamic equations to fully describe the system. Different approaches as they take, the major

design philosophy stems from the theory of sliding mode.

One of the important concerns in sliding mode control is the switching frequency that essentially dominates the system performance. To increase the switching frequency as much as possible, it is desirable to reduce the switching logic computation time. Of the three algorithms proposed, the look-up table is notable for its simplicity and robustness against parameter variations. Since it is conservative in the choice of the 3-phase voltage vectors (Table 2), the system response is expected to be slower. This method is also more sensitive to external disturbances. Especially when the rotor position is close to the stator voltage vector angles θ_n , the projection of e_n to the quadrature axis will be small. If such e_n is chosen (from the look-up table), the associated control e_q may not be big enough to overcome the disturbance $f_2(t)$. Nevertheless, the effectiveness and ease for implementation make it worth considering.

The second algorithm is effective in torque control with overcurrent protection. It requires more computation than the others, yet has a faster torque response under normal operation ($\sigma < \sigma_{max}$). As seen in (7.5), the optimal point with minimum energy loss (i_d^* , i_q^*) is subject to the actual system parameters X_d , X_q , and F . Without a precise knowledge of these parameters, the optimal point can not be attained exactly.

In algorithm 1, the idea of two-stage sliding mode enables us to separate the discrete time control law design from the continuous time driving circuitry. The discrete time controller maintain sliding mode at each sampling instant. During the sampling interval, the analog driving circuitry will be able to provide a much higher chattering frequency than a digital computer can. Thus, robustness is attained in

the discrete time sliding mode stage, chattering is alleviated in the continuous time sliding mode stage. We believe this will be a promising direction for the future study of brushless DC motor as well as any other sliding mode control applications.

7.2 Flexible Structure Vibration Control

We apply the discrete time sliding mode control techniques to a flexible structure developed for Large Interconnected Vibration Experiment (LIVE) at the Ohio State University. The truss structure, constructed of hollow PVC tubes and Noryl nodes, is aligned on a vertically cantilevered configuration shown in Figure 23.

Figure 23: LIVE truss structure

The physical properties, control hardware setups, preliminary dynamic modelling and experimental results were published in [67] by Redmill and Özgüner. Extended works in modelling and sampled-data sliding mode control experiment are presented

here.

The dynamic equations of the vertically cantilevered truss can be modelled based on the Euler-Bernoulli beam formulation. The bending motion is discribed by a fourth order partical differential equation (PDE).

$$\rho A \ddot{Q} + EI Q'''' = 0 \quad (7.36)$$

where EI is the bending stiffness of the truss, A is the cross-sectional area and ρ is the density. The independent variable along the length of the truss is z , t is time, and $Q(z, t)$ is the flexural displacement.

The boundary conditions (BC) with one fixed end and one with an end mass M_L are given as

$$\begin{aligned} Q(0, t) &= 0 \\ Q'(0, t) &= 0 \\ -EI Q''(l, t) &= 0 \\ EI Q'''(l, t) - M_L \ddot{Q}(l, t) &= -u(t) \end{aligned} \quad (7.37)$$

To solve for $Q(z, t)$, one can follow the standard Fourier method and convert the PDE into infinite assumed modes which are driven parallely by the boundary control $u(t)$. Each mode can be expressed as a second order underdamped oscillation equation.

$$Q(z, t) = \sum_{i=1}^{\infty} q_i(t) \phi_i(z) \quad (7.38)$$

$$\ddot{q}_i + 2\xi_i w_i \dot{q}_i + w_i^2 q_i = k_i u(t), \quad i = 1, 2, \dots \quad (7.39)$$

where $\phi_i(z)$ is the spatially dependent mode shape coefficient, $q_i(t)$ is the time dependent modal state, w_i is the modal frequency, ξ_i is the damping ratio, and k_i is control gain for each mode.

Although $\phi_i(z)$, ξ_i , w_i , and k_i can be obtained by solving (7.36) in conjunction with (7.37), we do not follow this approach because the PDE and BC are subject to several idealistic assumptions, which may not be applicable to this system. Furthermore, even if our system matches all the assumptions, the boundary condition with an end mass will lead to nonorthogonal basis function set $\{\phi_i(z)\}_{i=1}^{\infty}$, which makes the analytical solution almost unsolvable. Therefore, instead of solving for the parameters analytically, we will identify them through experiment.

Control action is obtained using proof-mass actuators with neglected actuator dynamics [68]. Sensing of truss motion is accomplished through a piezoelectric accelerometer mounted on the structure to measure the relative velocity between the proof-mass actuator and the structure.

Combining the dynamic equation (7.39) with the actuation and sensing mechanism, we come up with the following I/O relationship:

$$y(t) = y_0(t) + \sum_{i=1}^{\infty} y_i(t) \quad (7.40)$$

$$\ddot{y}_i(t) + 2\xi_i w_i \dot{y}_i(t) + w_i^2 y_i(t) = k_i \dot{u}(t), i = 1, 2, \dots, \quad (7.41)$$

where $y(t)$ is the velocity measurement obtained from the output of the accelerometer; $y_0(t)$ is a low frequency (~ 0.1 Hz) measurement noise; $y_i(t)$, $i = 1, 2, \dots$ is the flexural velocity for each mode.

The main tasks of this experiment are:

1. Identify the system parameters w_i , ξ_i , k_i for each mode of interest (usually the first few modes)

2. Construct state observer for the modes of interest
3. Implement discrete time sliding mode control for vibration damping

7.2.1 Dynamic Modelling

The modal frequencies w_i of the truss are the prior parameters to be determined. In the preliminary work [67], the authors applied a white noise to the system and identify w_i 's from the spectrum of the output signal $y(t)$. The modal frequencies were found to be

$$\begin{aligned} f_1 &= 1.2 \text{ Hz} \\ f_2 &= 8.44 \text{ Hz} \\ f_3 &= 23.71 \text{ Hz} \\ &\vdots \end{aligned}$$

The result were later confirmed using a finite element method (FEM) package ANSYS 5.0. To identify ξ_i and k_i for mode i , we first apply a sinusoidal input to shake up the truss for a long period of time, then turn it off; i.e.

$$u(t) = \begin{cases} A_{amp} \sin wt & t \leq T_{ss} \\ 0 & t > T_{ss} \end{cases} \quad (7.42)$$

where T_{ss} has to be long enough for the system to reach steady state. Assume the modal frequencies w_i are known and the sinusoidal input has frequency $w = W_i$ for some mode i . For $t > T_{ss}$, the output $y_i(t)$ has an underdamped characteristic

$$y_i(t) = M_i e^{-\xi_i w_i t} \sin(w_i t + \delta_i). \quad (7.43)$$

By measuring the envelope of $y_i(t)$ and fitting it with an experimental decay function, the damping factor ξ_i can be found given w_i .

For $t \leq T_{ss}$, the steady state response is

$$y_i^{ss}(t) = \frac{k_i A_{amp}}{2\xi_i w_i} \sin w_i t \quad (7.44)$$

Given w_i , ξ_i and the amplitude of the input signal A_{amp} , the parameter k_i can be identified by measuring the amplitude of $y_i^{ss}(t)$. Before we proceed for state space realization, there are two legitimate concerns to be taken into account:

1. The response $y_i(t)$ is not accessible for measurement.
2. The given modal frequency may be inaccurate, i.e. $w \neq w_i$.

We will resolve these two questions together. For $t \rightarrow T_{ss}$, solving the steady state response for the sinusoidal input $u(t) = A_{amp} \sin wt$, we obtain

$$y_i^{ss}(t) = M_i^{ss} \sin(wt - \theta) \quad (7.45)$$

with

$$\theta = \arctan \frac{w^2 - w_i^2}{2\xi_i w w_i} \quad (7.46)$$

$$M_i^{ss} = \frac{k_i A_{amp} w}{\sqrt{w^4 + 2(2\xi_i^2 - 1)w^2 w_i^2 + w_i^4}} \quad (7.47)$$

The magnitude M_i^{ss} has a band pass profile centered at w_i with bandwidth $B_i = 4w_i \xi_i \sqrt{\xi_i^2 + 1}$. Since the modal frequencies of interest are well separated, the other modes with $w_j \neq w_i$ will not be too much excited by $u(t)$. (Note: this is not true for high frequency modes in which the neighbor modes can be within the range of the bandwidth $[w_i - B_i, w_i + B_i]$. Therefore, although the measurement $y(t)$ includes all modes, only the low frequency noise $y_0(t)$ and the excited mode $y_i(t)$ are significant

when $w = w_i$. If w is close to w_i , a phase shift θ will appear in $y_i^{ss}(t)$. To detect such phase shift, simply pass input $u(t)$ and the measurement $y(t)$ through a high pass filter $H(s)$ (off-line) to get rid of the low frequency disturbance and compare them in the time scale. Calibration can be made by adjusting w to synchronize both filtered signals. With the system parameters w_i , ξ_i , k_i , we obtain a state space representation for each mode of interest. One can apply the balanced realization technique for numerical stability [69]. In this work, a controllable canonical form [70] is used

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i u, \\ y_i &= C_i x_i,\end{aligned}\tag{7.48}$$

where $x_i = [q_i \dot{q}_i]^T$ is an 2×1 modal state vector, u is the scalar control, y_i is the bending velocity of mode i ,

$$A_i = \begin{bmatrix} 0 & 1 \\ -w_i^2 & -2\xi_i w_i \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, C_i = [0 \ c_i]\tag{7.49}$$

with

$$b_i c_i = k_i.\tag{7.50}$$

There are infinitely many possible choices for b_i and c_i satisfying (7.50). For simplicity, we choose

$$b_i = \sqrt{|k_i|}, \quad c_i = \text{sgn}(k_i) \sqrt{|k_i|}\tag{7.51}$$

7.2.2 State Observer

Full state accessibility is important in VSC design so that the sliding mode dynamics can be specified arbitrarily. With the state space model for the first N modes, we are able to construct a $2N$ dimensional state observer for these modes.

$$\dot{X} = AX + Bu \quad (7.52)$$

$$y = CX + y_0 + v(t) \quad (7.53)$$

where $X = (x_1^T, \dots, x_N^T)^T$ is a $2N \times 1$ state vector composed of the first N modes, $u \in R$ is the scalar control, $y \in R$ is the scalar sensor output, y_0 is the observation noise, and $v(t)$ collects all the higher frequency modes.

The constant matrices are:

$$\begin{aligned} A &= \text{Diag}(A_1, \dots, A_N), \\ B &= (B_1^T, \dots, B_N^T)^T, \\ C &= (C_1, \dots, C_N) \end{aligned} \quad (7.54)$$

Discretizing the system (7.53) yields

$$x_{k+1} = \Phi x_k + \Gamma u_k \quad (7.55)$$

$$y_k = y_k^0 + c x_k + v_k \quad (7.56)$$

The measurement y_k contains two disturbances y_k^0 and v_k . To reduce their effects, we propose three strategies in conjunction with standard linear Luenberger observer techniques.

Filtering

Since the frequency content of y_k^0 and v_k are beyond the frequency band of interest, a most intuitive way of suppressing them is to pass the measurement y_k through a filter $H(z)$, leading to a new measurement $p_k = H(z)y_k$, which preserves the useful information and rejects the disturbances. Introducing a dynamic filter induces phase shift. To correct the phase error, we include the filter dynamics into the observer. Let the state space representation of $H(z)$ be

$$\eta_{k+1} = a\eta_k + by_k \quad (7.57)$$

$$p_k = c\eta_k + dy_k \quad (7.58)$$

where η_k is the filter dynamics; a, b, c, d are constant matrices with appropriate dimensions such that

$$H(z) = c(zI - a)^{-1}b + d \quad (7.59)$$

Augmenting the additional state η_k into X_k yields the new state equations:

$$\begin{bmatrix} X_{k+1} \\ \eta_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ bC & a \end{bmatrix} \begin{bmatrix} X_k \\ \eta_k \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u_k \quad (7.60)$$

$$p_k = [dC \quad c] \begin{bmatrix} X_k \\ \eta_k \end{bmatrix}$$

Thus, the effects of the disturbances are suppressed and phase errors are corrected.

One can design observer for the augmented state $\begin{bmatrix} X_k \\ \eta_k \end{bmatrix}$.

Redundant Modes

The high frequency content in the measurement is the sum of higher modes:

$$v(t) = \sum_{i=N+1}^{\infty} y_i(t) \quad (7.61)$$

Although they are not to be controlled, we still can include them in the observer design to reduce errors. The reason is obvious. Inclusion of redundant modes $y_{N+1}(t), \dots, y_{N+L}(t)$ in the observer results in decreasing of higher frequency disturbance

$$\bar{v}(t) = \sum_{i=N+L+1}^{\infty} y_i(t) \quad (7.62)$$

in the measurement $y(t)$. Since $\bar{v}(t)$ contains less energy than $v(t)$, the observation error is reduced. This idea has been extensively used in finite state representation of infinite dimensional systems. Detailed analysis can be found in [71].

Observer pole assignment

The eigenvalues of a flexible structure are usually located far separated from each other (especially the first few modes). The system exhibits a multi-time scale nature, which makes the modal states difficult to track simultaneously. A possible way of solving this problem is to construct N full state observers operating in parallel. Each behaves as a band pass filter that amplifies one specific mode and suppresses the others. The poles are chosen to pass certain frequency range around that modal frequency.

7.2.3 Control Design

With the observed states, we are ready to design discrete time sliding mode control for vibration damping. Following the method of constructing discontinuity planes depicted in chapter II, we obtain the desired switching surface

$$s(x) = \Sigma X = 0. \quad (7.63)$$

Solving $s_{k+1} = 0$ for the discrete time equivalent control yields

$$u_k^{eq} = -(\Sigma\Gamma)^{-1}\Sigma\Phi X_k. \quad (7.64)$$

The variable structure control law to be applied is

$$u_k = Sat(u_k^{eq}, U) = \begin{cases} U \operatorname{sgn}(u_k^{eq}) & |u_k^{eq}| > U \\ u_k^{eq} & \text{otherwise,} \end{cases} \quad (7.65)$$

where $U > 0$ is a scalar constant containing the allowable range of control.

7.2.4 Experimental Results

This experiment involves dynamic modelling, state observation, and sliding mode vibration control. The physical properties of the truss are given in Table 4.

The data acquisition and control hardware setup are shown in Figure 24. For detailed description of each unit in the block diagram, please refer to [67] and [68]. The sampling frequency throughout this experiment is 100 Hz, which corresponds to sampling period $T = 0.01$ second. The experimental results presented here are for the first flextural mode only ($N = 1$). Further development for higher modes will be proceeded in the future research.

Table 4: Physical Properties of LIVE

Truss Element:0.35m Bar	78.4g
Truss Element:0.50m Bar	107.9g
Truss Element:Node	49.4g
Truss Element:Bar Outer Diameter	22mm
Truss Element:Bar Wall Thickness	2mm
Proof Mass Actuator: Total Mass	3.40kg
Proof Mass Actuator: Moving Mass	1.71kg
Panel Motor Assembly: Total Mass	2.78kg
Panel Mount Assembly: Total Mass	.95kg
Complete 16 Bay Truss: Total Length	5.6m
Complete 16 Bay Truss: Total Mass	20.61kg
Complete Truss + Actuators: Total Mass	31.5kg

Dynamic Modelling

Identification of the modal frequencies and damping coefficients for each mode has been accomplished in [67]. However, due to sensor/actuator aging and possible truss deformation, some of the system parameters deviate from its original values. We redo the modelling for the observer and controller design.

To excite mode 1, we apply the input signal

$$u(t) = \begin{cases} 0 & 0 \leq t < 2 \\ 0.3 \sin w_1 t & 2 \leq t < 12 \\ 0 & 12 \leq t < 20, \end{cases}$$

for open loop test where $w_1 = 2\pi f_1$ is the first modal frequency. We obtain the truss bending velocity information y_k from the accelerometer output. The vibration signal mixed with a low frequency noise is seen in Figure 25.

To diminish the effect, we pass y_k and u_k through a second order digital high pass

Figure 24: LIVE experimental hardware block diagram

Butterworth filter with cut-off frequency 0.05 Hz

$$H(z) = \frac{0.9780 - 1.9561Z^{-1} + 0.9780z^{-2}}{1.0000 - 1.9556z^{-1} + 0.9565z^{-2}} \quad (7.66)$$

The result is shown in Figure 26.

Analyzing the I/O relationship, we come up with the continuous time state space model for mode 1:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -56.8490 & -0.9143 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0.077161 \end{bmatrix}, \quad C_1 = [0 \quad -0.077161].$$

The corresponding discrete time version is

$$\Phi_1 = \begin{bmatrix} 0.99717 & 0.009945 \\ -0.56536 & 0.98808 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 3.8445 \times 10^{-6} \\ 7.6737 \times 10^{-4} \end{bmatrix}, \quad C_1 = [0 \quad -0.077161].$$

State observer

The high pass filter in 7.66 has the state space realization

$$\eta_{k+1} = \begin{bmatrix} 1.9556 & -0.9565 \\ 1 & 0 \end{bmatrix} \eta_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y_k \quad (7.67)$$

$$p_k = [-0.043446 \quad 0.042502] + 0.97803y_k$$

Figure 25: Noise corrupted measurement

Including the dynamics (7.67) into the discrete time state space model for mode 1 yields a fourth order system. The observer poles are chosen from the poles of a Butterworth low pass filter with cut-off frequency $f_c = 2$ Hz, slightly higher than the modal frequency $f_1 = 1.2$ Hz. Figure 27 depicts the effectiveness of the observer by comparing the measurement y_k and the corresponding observer output $\hat{y}_k$. To avoid visual disturbance due to the lower frequency measurement noise, both signals are filtered by $H(z)$.

Sliding mode control

To demonstrate the effectiveness of the proposed control method, we take the following senario:

Part 1: $0 \leq t < 2$ Settlement – $u_k = 0$.

Part 2: $2 \leq t < 12$ Shake up the truss – $u_k = 0.3 \sin(w_1 \cdot kT)$.

Figure 26: Dynamic modelling - I/O relationship

Part 3: $12 \leq t < 20$ Sliding mode control v.s. No control.

The sliding surface for the second order vibrating system is chosen as

$$s(x) = [c \ 1]x_1 = cq_i + \dot{q}_1 = 0.$$

Experiments are conducted for $c = 1$ and $c = 10$. The discrete time equivalent control for each are

$$c = 1 \quad : \quad u_k^{eq} = [5.5990 \times 10^2 \quad -1.2941 \times 10^3] x_1$$

$$c = 10 \quad : \quad u_k^{eq} = [-1.1673 \times 10^4 \quad -1.3496 \times 10^3] x_1$$

A saturation function in the form of (7.65) is employed to limit the applied control

$$u_k = \text{Sat}(u_k^{eq}, 0.1).$$

Figure 28 depicts the frequency content (FFT) of y_k in Part 3 for $c = 1$. Figure 29 is the time response of mode 1. Similar results for $c = 10$ are shown in Figures 30 and 31, in which better performance is obtained.

Figure 27: Observer including filter dynamics

Figure 28: Frequency content: Sliding mode control ($c=1$) v.s. No control

Figure 29: Time response: Sliding mode control ($c=1$) v.s. No control

Figure 30: Frequency content: Sliding mode control ($c=10$) v.s. No control

Figure 31: Time response: Sliding mode control ($c=10$) v.s. No control

CHAPTER VIII

Summary and Future Research

The VSS theory is, in nature, not suitable for sampled-data control systems. Several ideal assumptions of the VSS theory, such as infinite switching frequency and continuous processing of the state information, can not be maintained due to the sample/hold effects. The significance of this dissertation is, however, to reconcile the discrepancies between the theory and the physical limitations, to analyze and achieve the best possible performance, and to relieve the chattering problem encountered in implementation.

In Chapter II, we proposed a new approach of constructing discontinuity surfaces for linear, nonlinear, and delay systems as a supplement to the existing methods. This supplement, although covers a broad class of systems, is by no means a completion to the problem of sliding surface design. The use of Lyapunov's second method immediately induces a classical question of how to find an appropriate Lyapunov function for general nonlinear systems. This question also extends to infinite dimensional systems, which researchers are still investigating.

From Chapter III through Chapter V, we analyzed the sample/hold effects to find an avenue to smooth the control and hence to eliminate the chattering phenomenon. By using the proposed discrete time sliding mode control laws, the states can be

maintained in an $O(T^2)$ boundary layer of the sliding manifold $\mathcal{S}$ for deterministic systems and $O(T^{\frac{1}{2}})$ for stochastic systems. The results are summarized in Table 5.

Table 5: Sliding surface errors under different types of disturbances

<i>Linear Deterministic Systems</i> $\dot{x} = Ax + \Delta A(t)x + Bu + \Delta B(t)u + Df(t)$	
Exogenous disturbances $f(t)$	$s(t) = O(T^2), \quad t \geq k_s T$
System parameter variations $\Delta A(t)$	$s(t) \rightarrow 0$ as $t \rightarrow \infty$
Control coefficient matrix variations $\Delta B(t)$	$s(t) \rightarrow 0$ as $t \rightarrow \infty$
<i>Nonlinear Deterministic Systems</i> $\dot{x} = a(x) + b(x)u + \tilde{b}(x)u + v(x, t)$	
Matched disturbances $v(x, t)$	$s(t) = O(T^2), \quad t \geq k_s T$
Control coefficient matrix variations $\tilde{b}(x)$	$s(t) \rightarrow O(T^2)$ as $t \rightarrow \infty$
<i>Stochastic Systems</i> $dx = Axd\tau + Buds + (D\xi d\tau + Rdw_1)$ $d\xi = W\xi d\tau + Qdw_2$	
Continuous measurement $\mathcal{F}_t$	$(E\ s_k\ ^2)^{\frac{1}{2}} = O(T^{\frac{1}{2}}), \quad k \geq k_s$
Discrete measurement $\mathcal{F}_k$	$(E\ s_k\ ^2)^{\frac{1}{2}} = O(T^{\frac{1}{2}}), \quad k \geq k_s$

The idea of a boundary layer enables one to maintain quasisliding with continuous control instead of using instantaneous switchings. As to the error in sliding motion, it is proved in Lemmas 3.1 and 4.1 that the inaccuracy is inevitable as long as a zero-order-hold is applied to the control channels. With this inherent inaccuracy, the error due to approximation of disturbances using past information becomes acceptable since it does not increase the order of the total error.

The prefiltering and postfiltering settings in Chapter VI has not been exploited to its full extent yet. In our work, the postfiltering block contains a state estimator when output feedback problems are considered. Many contemporary researchers

have been working on the problem of output feedback sliding mode control by using different approaches. It would be instructional to incorporate these methods into our framework especially when additional dynamics are used. Recently, there are works employing adaptive control techniques to count for some classes of uncertainties and nonlinearities. This is another possible direction to enrich the idea of prefiltering and postfiltering.

Chapter VII included two applications as examples to our sampled-data sliding mode control techniques. The brushless DC motor torque and current control problem had been solved numerically. Experimental work is still necessary for the future development. Similar idea can be utilized in speed and position related control problems. The flexible truss structure vibration control experiment, on the other hand, had shown the effectiveness of the proposed method for the first flexural mode. Control of higher modes are to proceed using the same techniques outlined in Chapter VII.

Appendix A

Proof of Equations

I. Proof of equivalence of equations (3.17) and (3.18):

Equation (3.18) can be rewritten as

$$(A.1) \quad \int_0^T e^{A\lambda} Df((k+1)T - \lambda) d\lambda = \int_0^T e^{A\lambda} d\lambda B g_k = \Gamma g_k.$$

Sufficiency: If (3.17) holds, then (A.1) follows immediately with

$$g_k = (C\Gamma)^{-1} C \int_0^T e^{A\lambda} Df((k+1)T - \lambda) d\lambda.$$

Necessity: If (A.1) is true, we substitute $\int_0^T e^{A\lambda} Df((k+1)T - \lambda) d\lambda = \Gamma g_k$ into the right

hand side of (3.17) to yield

$$\Gamma(C\Gamma)^{-1} C \Gamma g_k = \Gamma g_k = \int_0^T e^{A\lambda} Df((k+1)T - \lambda) d\lambda.$$

□

II. Proof of (4.21) and (4.28):

1.

$$e_{\bar{F}_k} = \int_{kT}^{(k+1)T} G(x) a(x) d\tau - G(x_k) a(x_k) T$$

Let $e_{\bar{F}_k}(t) = \int_{kT}^{kT+t} G(x)a(x)d\tau - G(x_k)a(x_k)t$, $0 \leq t \leq T$, then

$$\ddot{e}_{\bar{F}_k}(t) = \frac{\partial}{\partial x}[G(x)a(x)][a(x) + b(x)u_k]|_{x=x(kT+t)}$$

With Assumption 4.1, $\exists M_F$ such that $|\ddot{e}_{\bar{F}_k}(t)| \leq \frac{M_F}{2}$. It yields,

$$|e_{\bar{F}_k}| \leq M_F T^2.$$

2.

$$\begin{aligned} \bar{e}_{\bar{F}_k} &= T \int_0^T \frac{\partial}{\partial \tau}[G(x(kT - T + \tau))a(x(kT - T + \tau))]d\tau \\ \rightarrow \|e_{\bar{F}_k}\| &\leq \frac{M_F}{2} T^2. \end{aligned}$$

3.

$$\begin{aligned} e_{F_k} &= \int_0^T \int_0^T \frac{\partial}{\partial \tau}[G(x(kT - T + \tau))a(x(kT - T + \tau))]d\tau \\ \rightarrow \|e_{\bar{F}_k}\| &\leq \frac{M_F}{4} T^2. \end{aligned}$$

4.

$$e_{\bar{h}_k} = -e_{\bar{F}_k} - e_{\bar{B}_k} u_k$$

With Assumption 4.2, u_k is bounded, therefore, $\exists M_{h_2}$ such that $\|e_{\bar{h}_k}\| \leq M_{h_2} T^2$.

5.

$$\begin{aligned} \epsilon_{\Delta_k} &= \tilde{B}_k B_k^{-1} - \tilde{B}_{k-1} B_{k-1}^{-1} = e_{\tilde{B}_k} B_k^{-1} - \tilde{B}_{k-1} B_{k-1}^{-1} e_{B_k} B_k^{-1}. \\ \rightarrow \|\epsilon_{\Delta_k}\| &\frac{M_{\tilde{B}}}{m_g} T + \frac{M_{\tilde{B}} M_B}{4m_g^2} T = M_{\Delta} T. \end{aligned}$$

6.

$$I - B_k \bar{B}_k^{-1} = -e_{\bar{B}_k} \bar{B}_k^{-1}$$

$$\rightarrow \|I - B_k \bar{B}_k^{-1}\| \leq \frac{M_B}{m_g} T.$$

7. similar proof for e_{B_k} , $e_{\bar{B}_k}$, $\bar{e}_{\bar{B}_k}$, and e_{h_k} .

□

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