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The complexity of the collection of measure distal transformations

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The Ohio State University, 1993

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**300 N. Zeeb Rd.
Ann Arbor, MI 48106**

**THE COMPLEXITY OF THE COLLECTION OF
MEASURE DISTAL TRANSFORMATIONS**

DISSERTATION

**Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University**

By

F. Beleznyay,

*** * * * ***

The Ohio State University

1993

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS	ii
VITA	iii
CHAPTER	PAGE
I Introduction	1
II Definitions and preliminaries	3
III Examples of measure distal systems	21
3.1 Notation	21
3.2 Systems on the finite dimensional tori	27
3.3 Systems on the infinite dimensional torus	34
IV The complexity of the collection of measure distal transformations	42
4.1 Lower bound	42
4.2 Upper bound	47
BIBLIOGRAPHY	56

CHAPTER I

Introduction

In [4], Halmos introduced three different topologies on the space of measure preserving transformations. He showed that the class of ergodic transformations is a dense G_δ set in what he called the neighborhood topology. This topology makes the space of measure preserving transformations to be a Polish (complete, separable, metrizable) space. In [5] he showed that the class of ergodic weakly mixing transformations was also a dense G_δ set in this space. In [10] Rokhlin showed that the collection of strongly mixing transformations is a meager set in this space, hence showing that there is a weakly mixing, not strongly mixing transformation. In this thesis I calculate the complexity of another class of transformations, the class of measure distal transformations.

In [3] Furstenberg gave an ergodic theoretic proof of Szemerédi's theorem. He used a structure theorem for ergodic transformations, which states that every ergodic transformation can be reached by transfinite induction from the trivial one point transformation by taking relatively compact and weakly mixing extensions and inverse limits. A transformation is called measure theoretically distal if only compact extensions are needed. There are several names for these. Zimmer in [11] where he gives a similar structure theorem, calls these transformations having generalized

discrete spectrum. He proves that an ergodic transformation has generalized discrete spectrum if and only if it has a separating sieve, the notion of Parry [9].

In [2] we computed the “complexity” of distal transformations of topological dynamics. This thesis contains similar results using the analogy between the distal and measure distal transformations. We show that the set of measure distal transformations forms a complete coanalytic set, hence it is not a Borel set, but universally measurable.

In Chapter 3 we give a “uniform” construction of measure distal transformations. Using this we show in Chapter 4 that the set of measure distal transformations is Π_1^1 -hard, i.e. every coanalytic set is reducible to it via a Borel function. In Chapter 4 we also give a Π_1^1 -definition of measure distality, which shows that the set of measure distal transformations is coanalytic. We use descriptive set theoretic methods. For reference on these see e.g. [8].

CHAPTER II

Definitions and preliminaries

In this chapter we introduce the necessary notation and preliminaries. We mostly follow the notation of chapter 5 and 6 of [3] and [4].

Definition 2.0.1 *The triple $(X, \mathcal{B}, \mu)$ is called a measure space if X is a set, $\mathcal{B}$ is a σ -algebra of subsets of X and μ is a σ -additive probability measure on $\mathcal{B}$.*

If $(X, \mathcal{B}, \mu)$ is a measure space, we can define another σ -algebra $\mathcal{B}^*$, consisting of equivalence classes of sets of $\mathcal{B}$ that differ by a set of measure 0. Since the measure μ naturally projects to this σ -algebra, we will use the same letter for the two measures. We can also define the Boolean operations on $\mathcal{B}^*$ by $[A] \cup [B] = [A \cup B]$, $[A] \cap [B] = [A \cap B]$ and $\neg[A] = [X \setminus A]$.

Definition 2.0.2 *The quadruple $\mathbf{X} = (X, \mathcal{B}, \mu, \Gamma)$ is called a measure preserving system, if $(X, \mathcal{B}, \mu)$ is a measure space and Γ is a group of measure preserving transformations of X .*

A measure preserving system $\mathbf{X} = (X, \mathcal{B}, \mu, \Gamma)$ is called separable if $\mathcal{B}^$ is generated by countably many elements and Γ is countable.*

We deal only with the case when $\Gamma = \mathbb{Z}$. In this case the group is generated by some $T : X \rightarrow X$ bijection, such that both T and T^{-1} are measurable and

measure preserving. We will use the notation $\mathbf{X} = (X, \mathcal{B}, \mu, T)$, indicating explicitly the generating element of the group.

Definition 2.0.3 (cf. Definition 5.1 of [3]) *A homomorphism $\alpha : (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{D}, \nu)$ between two measure spaces is given by an injection $\alpha^{-1} : \mathcal{D}^* \rightarrow \mathcal{B}^*$ satisfying*

- (i) $\alpha^{-1}([A] \cup [B]) = \alpha^{-1}([A]) \cup \alpha^{-1}([B])$, for all $A, B \in \mathcal{D}$;
- (ii) $\alpha^{-1}([A] \cap [B]) = \alpha^{-1}([A]) \cap \alpha^{-1}([B])$, for all $A, B \in \mathcal{D}$;
- (iii) $\alpha^{-1}(\neg[A]) = \neg\alpha^{-1}([A])$, for all $A \in \mathcal{D}$;
- (iv) $\mu(\alpha^{-1}([A])) = \nu([A])$, for all $A \in \mathcal{D}$.

Definition 2.0.4 *We say that the measure preserving system $(Y, \mathcal{D}, \nu, S)$ is a factor of $(X, \mathcal{B}, \mu, T)$ if there is a homomorphism α between the underlying measure spaces satisfying*

- (v) $\alpha^{-1}([S(A)]) = T(\alpha^{-1}([A]))$ and $\alpha^{-1}([S^{-1}(A)]) = T^{-1}(\alpha^{-1}([A]))$ for all $A \in \mathcal{D}$.

The systems are said to be equivalent, if this homomorphism is bijective.

Note that if $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $\mathcal{D}$ is a T -invariant σ -subalgebra of $\mathcal{B}$, then $(X, \mathcal{D}, \mu, T)$ is a factor of $(X, \mathcal{B}, \mu, T)$ by the identity map. Also if $(Y, \mathcal{D}, \nu, S)$ is a factor of $(X, \mathcal{B}, \mu, T)$ by α and $\mathcal{D}'$ is the α^{-1} -image of $\mathcal{D}$, then $(Y, \mathcal{D}, \nu, S)$ is equivalent to $(X, \mathcal{D}', \mu, T)$. This means that factors and invariant σ -subalgebras naturally correspond to each other.

Now we introduce a special kind of extension, the skew product of Anzai ([3]). Let $(Y, \mathcal{D}, \nu, S)$ be a measure preserving system, and $(W, \mathcal{E}, \theta)$ be a measure space.

Suppose also that for each $y \in Y$ we have a measurable ν -measure preserving map $\sigma(y)$ from W to itself such that the map $(y, w) \mapsto \sigma(y)w$ is measurable from $(Y \times W)$ to W . We define the skew product of Y with W to be the system $(X, \mathcal{B}, \mu, T)$, where $X = Y \times W$, $\mathcal{B} = \mathcal{D} \times \mathcal{E}$, $\mu = \nu \times \theta$ and $T(y, w) = (Sy, \sigma(y)w)$. Then X is a measure preserving system and by the projection on the first coordinate Y is a factor of X .

If $\alpha : (X, \mathcal{B}, \mu, T) \rightarrow (Y, \mathcal{D}, \nu, S)$ is a factor map then this induces an isometry $U_\alpha : L^2(Y) \rightarrow L^2(X)$ as follows. For any $f \in L^2(Y)$ define $B_t(f) = \{y : f(y) < t\}$. Let $C_t(f)$ be any set from $\alpha^{-1}([B_t(f)])$, and let us define $U_\alpha(f)(x) = \inf\{t : x \in C_t(f)\}$. This gives an isometric embedding of $L^2(Y)$ onto a closed subspace of $L^2(X)$. This closed subspace is also the space of functions measurable with respect to the α^{-1} -image of $\mathcal{D}^*$. Clearly X is equivalent to the factor Y if and only if U_α is onto.

We also need an inverse of this map. For any $f \in L^2(X)$ let Pf be the projection onto the image space of $L^2(Y)$. Let us define $E(f|Y) = U_\alpha^{-1}(Pf)$. The map $f \mapsto E(f|Y)$ is the conditional expectation map.

Proposition 2.0.5 (i) *If $f \in L^2(X)$ and g are bounded and measurable with respect to the σ -algebra $\alpha^{-1}(\mathcal{D}^*)$, then $E(gf|Y) = U_\alpha^{-1}(g)E(f|Y)$.*

(ii) $SE(f|Y) = E(Tf|Y)$.

Proof. See proposition 5.4 of [3]. □

Our goal is to study nonatomic, separable measure preserving systems and towers of these. We restrict our attention to systems on the infinite dimensional torus $T^{\mathbb{N}}$, with the σ -algebra of the Borel sets and the product of the Lebesgue measure. Since every non-atomic separable measure preserving system is equivalent to a system on

$\mathbb{T}^{\mathbb{N}}$, this is not a substantial restriction.

Let $\mathcal{O}$ be the algebra consisting of finite unions of sets of the form $\prod_{n \in \mathbb{N}} A_n$, where each A_n is a finite union of rational intervals (open, closed or half-open), and for all but finitely many n , $A_n = \mathbb{T}$. This is a countable algebra that generates the σ -algebra of Borel sets. Let $\{O_n : n \in \mathbb{N}\}$ be an enumeration of $\mathcal{O}$.

Definition 2.0.6 *Let's define an equivalence relation $\sim$ on the measure preserving transformations of $\mathbb{T}^{\mathbb{N}}$ by $T \sim S$ iff for all $n \in \mathbb{N}$, $\mu(T(O_n) \Delta S(O_n)) = 0$. Look at the space of these equivalence classes, and put the following metric on it:*

$$d([T], [S]) = \sum_{n=1}^{\infty} \frac{1}{2^n} [\mu(T(O_n) \Delta S(O_n)) + \mu(T^{-1}(O_n) \Delta S^{-1}(O_n))]. \quad (2.1)$$

We call this space the space of measure preserving transformations. We use the notation MPT to refer to this space.

Proposition 2.0.7 *The space of measure preserving transformations is a Polish space.*

Proof. See e.g. in [6]. □

Definition 2.0.8 *A measure preserving transformation is called ergodic if every invariant set has measure 1 or 0.*

Lemma 2.0.9 *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, and $\mathcal{A} \subset \mathcal{B}$ be an algebra generating $\mathcal{B}$. Then the transformation $T : X \rightarrow X$ is ergodic iff for every $B, C \in \mathcal{A}$ such that $\mu(B), \mu(C) > 0$, there is an m such that $\mu(T^m B \cap C) > \frac{1}{2} \mu(B) \mu(C)$.*

Proof. Ergodicity implies $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=0}^{N-1} \mu(T^m B \cap C) = \mu(B)\mu(C)$, hence for some $m \in \mathbb{N}$, $\mu(T^m B \cap C) > \frac{1}{2}\mu(B)\mu(C)$.

For the other direction suppose that T is not ergodic, i.e. there is an invariant $A \in \mathcal{B}$ such that $\delta < \mu(A) < 1 - \delta$ for some $\delta > 0$. Fix $\epsilon > 0$ such that $(\delta - \epsilon)^2 \geq 4\epsilon$, and choose $B, C \in \mathcal{A}$ such that $\mu(B \Delta A) < \epsilon$, and $\mu(C \Delta (X \setminus A)) < \epsilon$. Then for all $m \in \mathbb{N}$, $\mu(T^m A \Delta T^m B) < \epsilon$, and since $T^m A \cap (X \setminus A) = \emptyset$, $\mu(T^m B \cap C) < 2\epsilon$. On the other hand by assumption there is an $m \in \mathbb{N}$ such that $\mu(T^m B \cap C) > \frac{1}{2}\mu(B)\mu(C) > \frac{1}{2}(\delta - \epsilon)^2$, so $(\delta - \epsilon)^2 < 4\epsilon$, contradicting to the choice of ϵ . $\square$

Proposition 2.0.10 *The ergodic transformations form a dense G_δ set in the space of measure preserving transformations.*

Proof. The proof of this can be found i.e. [6]. We show here that it is a G_δ set to motivate the proof of Lemma 2.0.17. For $n, m \in \mathbb{N}$ let

$$D_{n,m} = \{[T] : \exists k, \mu(T^k O_n \cap O_m) > \frac{1}{2}\mu(O_n)\mu(O_m)\}. \quad (2.2)$$

The set $D_{n,m}$ is open for every $n, m \in \mathbb{N}$. Indeed, let k be a witness for $[T] \in D_{n,m}$, and let $\mu(T^k O_n \cap O_m) = \frac{1}{2}\mu(O_n)\mu(O_m) + \delta$ for some $\delta > 0$. Let $\mathcal{L} = \{O_l : l \in L\}$ be a finite list of elements of $\mathcal{O}$, such that for all $0 \leq j \leq k$ there are $l_1^j, \dots, l_j^j \in L$, such that $\mu(T^j O_n \Delta \bigcup_{i=1}^j O_{l_i^j}) < \delta/4k$. Let S be such that $\mu(SO_l \Delta TO_l) < \delta/2k|L|$ for all $l \in L$. Then for all $j < k$,

$$\mu(ST^j O_n \Delta T^{j+1} O_n) \leq \quad (2.3)$$

$$\mu(ST^j O_n \Delta S(\bigcup_{i=1}^j O_{l_i^j}) + \mu(S(\bigcup_{i=1}^j O_{l_i^j}) \Delta T(\bigcup_{i=1}^j O_{l_i^j})) + \mu(T(\bigcup_{i=1}^j O_{l_i^j}) \Delta T^{j+1} O_n) \leq \quad (2.4)$$

$$\delta/4k + \sum_{i=1}^{s_j} (SO_{O_i} \Delta TO_{O_i}) + \delta/4k \leq \delta/2k + |L|\delta/2k|L| = \delta/k. \quad (2.5)$$

Hence $\mu(S^{k-j}T^jO_n \Delta S^{k-j-1}T^{j+1}O_n) < \delta/k$. So $\mu(S^kO_n \Delta T^kO_n) < \mu(S^kO_n \Delta S^{k-1}TO_n) + \mu(S^{k-1}TO_n \Delta S^{k-2}T^2O_n) + \dots + \mu(ST^{k-1}O_n \Delta T^kO_n) \leq k\delta/k = \delta$, which implies that $\mu(S^kO_n \cap O_m) > \frac{1}{2}\mu(O_n)\mu(O_m) + \delta - \delta$, which witnesses that $[S] \in D_{n,m}$.

Finally since in this space $\mu(O_n) > 0$ for every $n \in \mathbb{N}$, according to Lemma 2.0.9 the set of ergodic transformations is the intersection of these $D_{n,m}$'s, which is a G_δ set. $\square$

In order to define the two main kind of extensions, we need to introduce the notion of disintegration of measures. However due to Theorem 5.15 of [3], we don't need this notion in it's most general form. We repeat this theorem here.

Theorem 2.0.11 *Let $\alpha : X \rightarrow Y$ be a homomorphism of separable measure preserving systems. There exist a system X' equivalent to X , a system Y' equivalent to Y and a measure space Z' such that X' is a skew product of Y' and Z' .* $\square$

Remark. From now on whenever we have a factor map, we assume that it is induced by a spatial map of the underlying spaces, which we denote by the same letter. Also, if X is a skew product of Y and some W , then we identify $U_\alpha(L^2(Y))$ with $L^2(Y)$. $\square$

Let $X = Y \times W$ be a skew product with $Tx = (Sy, \sigma(y)w)$. Let $\mu_y = \delta_y \times \theta$ be the measure on X , where δ_y is the point mass on y . Hence μ_y concentrates on the fiber above y and there it coincides with the measure on W .

Proposition 2.0.12 (i) $E(f|Y)(y) = \int f d\mu_y$ for $f \in L^2(X)$ and for a.e. $y \in Y$.

(ii) For every measurable $A \subset X$, $\mu(A) = \int \mu_y(A) d\nu(y)$.

(iii) $\int f d\mu = \int (\int f d\mu_y) d\nu(y)$ for every $f \in L^2(X)$.

Proof. This is an easy consequence of Theorem 5.8 of [3]. $\square$

Definition 2.0.13 Let X_1, X_2 be extensions of Y by ϕ_1 and ϕ_2 . The relative product $X_1 \times_Y X_2 = (X, \mathcal{B}, \mu, T)$ is defined as follows: $X = X_1 \times X_2$, $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, the transformation is defined pointwise: $T(x_1, x_2) = (T_1 x_1, T_2 x_2)$. Finally, for $A \in \mathcal{B}_1 \times \mathcal{B}_2$, $(\mu_1 \times_Y \mu_2)(A) = \int (\mu_{1,y} \times \mu_{2,y})(A) d\nu(y)$.

Proposition 2.0.14 (i) $X_1 \times_Y X_2$ is a measure preserving system, which is an extension of Y .

(ii) The measure on the relative product is characterized by

$$\int f_1(x_1) f_2(x_2) d(\mu_1 \times_Y \mu_2) = \int E(f_1|Y) E(f_2|Y) d\nu \quad (2.6)$$

for all $f_i \in L^2(X_i)$, $i = 1, 2$.

(iii) $(\mu_1 \times_Y \mu_2)_y = \mu_{1,y} \times \mu_{2,y}$.

Proof. See section 5.5 of [3]. $\square$

Lemma 2.0.15 If $(X_i, \mathcal{B}_i, \mu_i, T_i) = (Y, \mathcal{D}, \nu, S) \times (W_i, \mathcal{E}_i, \theta_i)$ are skew products for $i = 1, 2$ with $T_i(y, w) = (Sy, \sigma_i(y)w)$, then

$$(X_1 \times_Y X_2) \sim (Y \times W_1 \times W_2, \mathcal{D} \times \mathcal{E}_1 \times \mathcal{E}_2, \nu \times \theta_1 \times \theta_2, T) \quad (2.7)$$

where the action T is defined by $T(y, w_1, w_2) = (Sy, \sigma_1(y)w_1, \sigma_2(y)w_2)$.

Proof. This follows easily from 2.0.14. $\square$

Definition 2.0.16 An extension $\alpha : X \rightarrow Y$ is relatively ergodic if the only invariant functions in $L^2(X)$ are in $L^2(Y)$.

We will need a version of Lemma 2.0.9 for relatively ergodic extensions, i.e. we want to be able to decide whether an extension is relatively ergodic just by looking at the action on a countable family of sets.

Lemma 2.0.17 Let $(X, \mathcal{B}, \mu, T)$ be the skew product of $(Y, \mathcal{D}, \nu, S)$ and $(W, \mathcal{E}, \theta)$, and let π be the projection of X to Y . Let $\mathcal{A}$ be an algebra that generates $\mathcal{B}$ as a σ -algebra. Suppose that $\mathcal{A}$ is generated by sets of the form $B \times C$, where $B \in \mathcal{D}$ and $C \in \mathcal{E}$. Then this extension being relatively ergodic is equivalent to the following property:

(RE) Suppose that for $E, F \in \mathcal{A}$ such that $\pi(E) = \pi(F)$, and for $0 < \delta < 1/6$ there are $E_1, F_1 \in \mathcal{D}$ with the following properties:

$$(RE1) \nu(E_1), \nu(F_1) > (1 - \delta)\nu(\pi(F)),$$

$$(RE2) \text{ for every } w \in E_1, \theta(\{u \in E : \pi(u) = w\}) > \delta,$$

$$(RE3) \text{ for every } w \in F_1, \theta(\{u \in F : \pi(u) = w\}) > \delta, \text{ and}$$

$$(RE4) \text{ for all } m \in \mathbb{Z}, \nu(\pi(T^m(E)) \cap \pi(F)) > (1 - \delta)\nu(\pi(F)).$$

Then there is an $m \in \mathbb{N}$ such that $\mu(T^m(E) \cap (F)) > \frac{1}{2}\delta^2\nu(\pi(F))$.

Proof. First we show that relative ergodicity implies (RE). Relative ergodicity implies that for all $g, h \in L^2(X)$ such that $E(h|Y) = 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int E(gT^m(h)|Y) d\nu = 0. \quad (2.8)$$

(See e.g. [3]) On the other hand for every $f \in L^2(X)$, $E(f - E(f|Y)|Y) = 0$, so

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int E(gT^m(f) - gS^m(E(f|Y))|Y) d\nu = 0. \quad (2.9)$$

Using the properties of the conditional expectation,

$$\int E(gT^m(f) - gS^m(E(f|Y))|Y) d\nu = \quad (2.10)$$

$$\int E(gT^m(f)|Y) d\nu - \int E(g|Y) S^m(E(f|Y)) d\nu. \quad (2.11)$$

Hence

$$\limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int E(gT^m(f)|Y) d\nu = \quad (2.12)$$

$$\limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int E(g|Y) S^m(E(f|Y)) d\nu, \quad (2.13)$$

which is the same as

$$\limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int gT^m(f) d\mu = \quad (2.14)$$

$$\limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int E(g|Y) S^m(E(f|Y)) d\nu. \quad (2.15)$$

Now let $f = \chi_E$ and $g = \chi_F$ be the characteristic functions. For every $y \in Y$, $E(g|Y)(y)$ and $E(f|Y)(y)$ gives the θ -measure of the fibers of F and E above the point y . So $[E(g|Y)S^m(E(f|Y))](y) > \delta^2$ as long as $y \in F_1$ and $y \in S^{-m}E_1$. We now give a lower bound of the measure of $F_1 \cap S^{-m}E_1$. First notice that $F_1 \cap S^{-m}E_1$ contains every point of $\pi(F)$ which is not in any of $U = \pi(F) \setminus F_1$, $V = \pi(F) \setminus \pi(T^{-m}E)$ or $W = \pi(T^{-m}E) \setminus S^{-m}E_1$. Indeed,

$$F_1 \cap S^{-m}E_1 = F_1 \setminus (F_1 \setminus S^{-m}E_1) = \quad (2.16)$$

$$\{\pi(F) \setminus [\pi(F) \setminus F_1]\} \setminus [F_1 \setminus S^{-m}E_1] = \quad (2.17)$$

$$[\pi(F) \setminus U] \setminus [\pi(F) \setminus S^{-m}E_1] \supset \quad (2.18)$$

$$\{[\pi(F) \setminus U] \setminus [\pi(F) \setminus \pi(T^{-m}E)]\} \setminus [\pi(T^{-m}E) \setminus S^{-m}E_1] = \quad (2.19)$$

$$\{[\pi(F) \setminus U] \setminus V\} \setminus W. \quad (2.20)$$

But

$$\nu(U) = \nu(\pi(F) \setminus F_1) < \delta\nu(\pi(F)), \quad (2.21)$$

$$\nu(V) = \nu(\pi(F) \setminus \pi(T^{-m}(E))) < \delta\nu(\pi(F)), \text{ and} \quad (2.22)$$

$$\nu(W) = \nu(\pi(T^{-m}(E)) \setminus S^{-m}(E_1)) = \quad (2.23)$$

$$\nu(S^{-m}\pi(E) \setminus S^{-m}E_1) = \nu(\pi(E) \setminus E_1) < \delta\nu(\pi(E)) = \delta\nu(\pi(F)). \quad (2.24)$$

So $\mu(F_1 \cap S^{-m}E_1) \geq (1 - 3\delta)\nu(\pi(F)) > \frac{1}{2}\nu(\pi(F))$. Since $g = \chi_F$ and $f = \chi_E$ are non-negative functions, we can conclude that 2.15 is bigger than $\frac{1}{2}\delta^2\nu(\pi(F))$. Rewriting 2.14 we get that

$$\limsup \frac{1}{N+1} \sum_{m=0}^N \mu(T^m(E) \cap F) > \frac{1}{2}\delta^2\nu(\pi(F)). \quad (2.25)$$

Hence at least one of the terms should satisfy the inequality, which is what we wanted.

Now we show the other direction, i.e. that (RE) implies relative ergodicity. Suppose that there is a T -invariant set A such that for $A_1 = \pi(A)$, $\nu(A_1) > 0$. Also suppose that there is some $0 < \delta' < 1/3$ such that for every $y \in A_1$, $\delta' < \theta(\{x \in A : \pi(x) = y\}) < 1 - \delta'$. Such an A exist if $\mathbf{X}$ is not relatively ergodic over $\mathbf{Y}$. Indeed, if $\mathbf{X}$ is not relatively ergodic over $\mathbf{Y}$, then by definition there is an invariant A' such that for $A'_1 = \pi(A')$, $\nu(A'_1) > 0$ and for all $y \in A'_1$, $0 < \theta(\pi^{-1}(y)) < 1$. Then by Fubini's

theorem the function $f : A'_1 \rightarrow \mathbb{R}$ defined by $f(y) = \theta(\pi^{-1}(y))$ is almost everywhere defined and measurable. Hence there is a $0 < \delta' < 1/3$ such that the measurable $A_1 = \{y \in A'_1 : \delta' < \theta(\pi^{-1}(y)) < 1 - \delta'\}$ does not have zero measure. Clearly A_1 is also invariant, and $A = \pi^{-1}(A_1)$ works.

We show that in this case (RE) does not hold. Let $A' = (X \setminus A) \cap \pi^{-1}(A_1)$. Fix $\epsilon > 0$ and let $E, F \in \mathcal{A}$ be approximations of A and A' such that $\mu(E \Delta A) < \epsilon$, $\mu(F \Delta A') < \epsilon$, $\pi(F) = \pi(E)$ and $\nu(\pi(F) \Delta \pi(A')) = \nu(\pi(E) \Delta \pi(A)) < \epsilon$. (Without the condition $\pi(E) = \pi(F)$, these approximations exist because $\mathcal{A}$ is a dense algebra in $\mathcal{B}$. We can satisfy $\pi(E) = \pi(F)$, because we assumed that $\mathcal{A}$ is generated by sets from $\mathcal{D} \times \mathcal{E}$.) Then for all $m \in \mathbb{Z}$, $\mu(T^m(E) \cap F) \leq \mu(T^m E \Delta T^m A) + \mu(T^m A \cap A') + \mu(A' \Delta F) < 2\epsilon$. On the other hand let $\delta = \delta'/2$. If $0 < \epsilon < \delta$ is small enough to satisfy $\epsilon < \delta^2\nu(\pi(A)) - \delta^2\epsilon - \delta\epsilon$ and $2\epsilon < \delta(\nu(\pi(A')) - \epsilon)$, then E and F will satisfy the hypothesis of (RE) for δ . Indeed, let $E_1 = \{y : \theta\{x \in E : \pi(x) = y\} > \delta\}$ and $F_1 = \{y : \theta\{x \in F : \pi(x) = y\} > \delta\}$. If $\nu(E_1) \leq (1 - \delta)\nu(\pi(E))$, then $\nu((\pi(E) \setminus E_1) \cap \pi(A)) > \delta\nu(\pi(E)) - \epsilon$. For $y \in (\pi(E) \setminus E_1) \cap \pi(A)$, $\theta(\{x \in A : \pi(x) = y\}) > \delta' = 2\delta$, and $\theta(\{x \in E : \pi(x) = y\}) \leq \delta$. Hence $\mu(A \Delta E) \geq \delta(\delta\nu(\pi(E)) - \epsilon) \geq \delta(\delta(\nu(\pi(A)) - \epsilon) - \epsilon)$, which contradicts our choice of ϵ . To show that $\nu(F_1) > (1 - \delta)\nu(\pi(F))$ is similar. For (RE4) $\nu(\pi(T^m(E)) \cap \pi(F)) \geq \nu(\pi(T^m(A)) \cap \pi(F)) - \nu(\pi(T^m(E)) \Delta \pi(T^m(A))) = \nu(\pi(A') \cap \pi(F)) - \nu(\pi(E) \Delta \pi(A)) > \nu(\pi(F)) - 2\epsilon > (1 - \delta)\nu(\pi(F))$ by our choice of ϵ . The last inequality holds, because we chose $2\epsilon < \delta(\nu(\pi(A')) - \epsilon) < \delta\nu(\pi(F))$. So by the conclusion of (RE), there is an $m \in \mathbb{N}$ such that $\mu(T^m(E) \cap F) > \frac{1}{2}\nu(\pi(F))\delta^2$. We already showed that $2\epsilon > \mu(T^m(E) \cap F)$, hence $2\epsilon > \frac{1}{2}(\nu(\pi(A)) - \epsilon)\delta^2$. But

this cannot be true for arbitrarily small ϵ , which proves that (RE) implies relative ergodicity. $\square$

The following definitions are from chapter 6 of [3].

Definition 2.0.18 *Let X be an extension of Y .*

(i) *X is a weakly mixing extension of Y if $X \times_Y X$ is ergodic over Y .*

(ii) *Call $f \in L^2(X)$ almost periodic (AP) relative to Y if for every $\delta > 0$ there exist $g_1 \dots g_n \in L^2(X)$ such that for every $j \in \mathbb{Z}$ and for a.e. $y \in Y$*

$$\inf_{1 \leq k \leq n} \|T^j f - g_k\|_{L^2(\mu_y)} < \delta. \quad (2.26)$$

(iii) *Call $f \in L^2(X)$ compact relative to Y if it is a limit of relatively almost periodic functions.*

(iv) *X is a compact extension of Y if the set of relatively almost periodic functions are dense in $L^2(X)$. (i.e. every $f \in L^2(X)$ is compact relative to Y .)*

Remark. If we are talking about an extension $(X, \mathcal{B}, \mu, T)$ of $(X, \mathcal{D}, \mu, T)$, where $\mathcal{D}$ is a σ -subalgebra of $\mathcal{B}$, then we use the expressions almost periodic or compact relative to $\mathcal{D}$, instead of writing out the whole factor system.

Proposition 2.0.19 *Let $(X, \mathcal{B}, \mu, T)$ be an extension of $(X, \mathcal{D}, \mu, T)$. Then the collection of relatively compact functions is a closed, invariant subspace of $L^2(\mathcal{B})$. Moreover let $\mathcal{E}$ be the smallest σ -algebra such that every relatively compact function is measurable with respect to $\mathcal{E}$. Then $L^2(\mathcal{E})$ is the collection of all relatively compact functions.*

Proof. See the proof of Theorem 6.15 of [3]. $\square$

Definition 2.0.20 Let $\alpha : X \rightarrow Y$ be an factor map. Let $H \in L^2(X \times_Y X)$ and $\phi \in L^2(X)$. Then the convolution $H * \phi$ is defined by

$$H * \phi(x) = \int H(x, x') \phi(x') d\mu_{\alpha(x)}(x'). \quad (2.27)$$

Proposition 2.0.21

- (i) $H * \phi \in L^2(X)$
- (ii) $\|H * \phi\|_{\mu_Y} \leq \|H\|_{(\mu \times_Y \mu)_Y} \|\phi\|_{\mu_Y}$

Proof. See p.130-131 of [3]. $\square$

The following theorem gives some equivalent formulations of the compact extension. It is Theorem 6.13 of [3].

Theorem 2.0.22 Let $\alpha : X \rightarrow Y$ be an extension. The following are equivalent.

(C1) The functions $\{H * \phi\}$ span a dense subset of $L^2(X)$ as H ranges over bounded invariant functions in $L^2(X \times_Y X)$ and ϕ ranges over $L^2(X)$.

(C2) X is a compact extension of Y .

(C3) For every $f \in L^2(X)$ the following holds. For every $\epsilon, \delta > 0$ there exists a measurable set $B \subset Y$ with $\nu(B) > 1 - \epsilon$ and a finite set of functions $g_1 \dots g_k \in L^2(X)$ such that if we denote by f_B the function $f \chi_{\alpha^{-1}(B)}$, then for each $m \in \mathbb{Z}$, for a.e. $y \in Y$ $\min_{1 \leq j \leq k} \|T^m f_B - g_j\|_Y < \delta$.

(C4) For every $f \in L^2(X)$ the following holds. For every $\epsilon, \delta > 0$ there exists a finite set of functions $g_1 \dots g_k \in L^2(X)$ such that for each $m \in \mathbb{Z}$, $\min_{1 \leq j \leq k} \|T^m f - g_j\|_Y < \delta$ but for a set of $y \in Y$ of measure $< \epsilon$. $\square$

Definition 2.0.23 *Let V be a compact extension of Y such that both of them are factors of X . Then V is the maximal compact extension of Y in X , if no extension of V which is a factor of X is compact over Y .*

The following proposition states the existence of the maximal compact extension.

Proposition 2.0.24 *Let $Y = (Y, \mathcal{D}, \nu, S)$ be a factor of X . Let's define the following two subsets of $L^2(X)$:*

(M1) Let $\mathcal{H}$ be the closure of $\{H * \phi\}$ as H ranges over bounded invariant functions in $L^2(X \times_Y X)$ and ϕ ranges over $L^2(X)$.

(M2) Let $\mathcal{H}'$ be the closure of the relatively almost periodic functions. (i.e. the collection of relatively compact functions.)

Then $\mathcal{H} = \mathcal{H}'$. Moreover there is an invariant σ -algebra, $\mathcal{E}$, which is a sub- σ -algebra of $\mathcal{B}$, (and also can be seen as an extension of $\mathcal{D}$), such that both $\mathcal{H}$ and $\mathcal{H}'$ are the collection of the functions measurable with respect to $\mathcal{E}$.

Also $(X, \mathcal{E}, \mu, T)$ is the maximal compact extension of Y in X .

Proof. This is an immediate consequence of 2.0.19. □

Lemma 2.0.25 *Let Y be a factor of X and V be a compact extension of Y in X . (i.e. V is also a factor of X) Then V is the maximal compact extension of Y in X iff $X_1 = X \times_Y X$ is relatively ergodic over $V_1 = V \times_Y V$.*

Proof. The proof is based on characterization (M1) of the maximal compact extension. Let's denote the transformation on X_1 by T_1 . If X_1 is relatively ergodic

over V_1 , then if $H \in L^2(X_1)$ is T_1 -invariant, then $H \in L^2(V_1)$. So for any $\phi \in L^2(X)$, $H * \phi \in L^2(V)$. Since these are dense in the space of all compact functions, this implies that V is the maximal compact extension of Y in X .

For the other direction we need to introduce some notation. According to Theorem 2.0.11 we assume that there are measure spaces Z and W such that X is the skew product of V and W , and V is the skew product of Y and Z . Let $\{f_i : i \in \mathbb{N}\}$ be an orthonormal base of $L^2(Y)$, $\{g_i : i \in \mathbb{N}\}$ be an orthonormal base of $L^2(Z)$ and $\{h_i : i \in \mathbb{N}\}$ be an orthonormal base of $L^2(W)$. Then $\{f_i(y)g_j(z_1)g_k(z_2) : i, j, k \in \mathbb{N}\}$ is a base of $L^2(V_1)$, and $\{f_i(y)g_j(z_1)g_k(z_2)h_a(w_1)h_b(w_2) : i, j, k, a, b \in \mathbb{N}\}$ is a base of $L^2(X_1)$. Suppose now that $H \in L^2(X_1)$ is a T_1 -invariant function which is not in $L^2(V_1)$. Look at it's Fourier expansion

$$H(y, z_1, w_1, z_2, w_2) = \sum_{i,j,k,a,b} \alpha_{i,j,k,a,b} f_i(y)g_j(z_1)g_k(z_2)h_a(w_1)h_b(w_2). \quad (2.28)$$

Since $H \notin L^2(V_1)$, this means that $\alpha_{i_0,j_0,k_0,a_0,b_0}$ is not 0 for some a_0, b_0 such that $h_{a_0}(w_1)h_{b_0}(w_2)$ is not constant. Without loss of generality we can assume that h_{a_0} is not constant. Let $\phi(y, z_2, w_2) = g_{k_0}(z_2)h_{b_0}(w_2)$. Then

$$H * \phi = \sum_{i,j,k,a,b} \alpha_{i,j,k,a,b} \int f_i(y)g_j(z_1)g_k(z_2)h_a(w_1)h_b(w_2)g_{k_0}(z_2)h_{b_0}(w_2)dz_2dw_2. \quad (2.29)$$

But $\int g_k h_b(w_2)g_{k_0}(z_2)h_{b_0}(w_2)dz_2dw_2$ is 1 if $(k, b) = (k_0, b_0)$ and 0 otherwise, hence

$$H * \phi = \sum_{i,j,a} \alpha_{i,j,k_0,a,b_0} f_i(y)g_j(z_1)h_a(w_1) = \sum_a h_a(w_1) \sum_{i,j} \alpha_{i,j,k_0,a,b_0} f_i(y)g_j(z_1). \quad (2.30)$$

Let $G_a = \sum_{i,j} \alpha_{i,j,k_0,a,b_0} f_i(y)g_j(z_1)$. Usin the assumption that the $f_i(y)g_j(z_1)h_a(w_1)$'s are orthogonal to each other, $H * \phi$ can only be in $L^2(V)$ if all the G_a 's are zero

except possibly for the α' where $h_{\alpha'}$ is the constant function. But $\alpha_{i_0, j_0, k_0, n_0, b_0}$ is not 0 a.e., hence (again by orthogonality), G_{α_0} is not zero. This shows that $H * \phi \notin L^2(V)$. Hence V is not the maximal compact extension of Y in X . $\square$

Definition 2.0.26 *A system of measure preserving systems $\{X_\alpha : \alpha \leq \eta\}$ is called a compact tower of measure preserving systems for X_η , if*

- (i) X_0 is the trivial one-point system,
- (ii) every X_α is a factor of X_η , (let's denote the image of $L^2(X_\alpha)$ in $L^2(X_\eta)$ by $\mathcal{H}_\alpha$)
- (iii) for limit $\beta \leq \eta$, $\mathcal{H}_\beta$ is the closure in $\mathcal{H}_\eta$ of $\bigcup_{\alpha < \beta} \mathcal{H}_\alpha$,
- (iv) for every $\alpha < \eta$, $X_{\alpha+1}$ is a compact extension of X_α .

Furstenberg in [3] showed that every ergodic system is either the limit of a compact tower of ergodic systems, or a weak mixing extension of a limit of a compact tower.

Definition 2.0.27 *A system is called measure theoretically distal if it is the limit of a compact tower. The order of a measure distal system is the height of the shortest compact tower that reaches it from the trivial system by compact extensions.*

Theorem 2.0.28 *Let $X = (X, \mathcal{B}, \mu, T)$ be a separable measure distal system. Then the height of X is a countable ordinal.*

Proof. Let η be the order of X , and $\{X_\alpha : \alpha \leq \eta\}$ be a compact tower for $X = X_\eta$. Let $\mathcal{H}_\alpha$ be the image of $L^2(X_\alpha)$ in $L^2(X)$. We can assume that for $\beta < \gamma \leq \eta$, $\mathcal{H}_\beta \neq \mathcal{H}_\gamma$, otherwise $X_\beta \sim X_\gamma$, so we can shorten the compact tower, still reaching X

by compact extensions. Each $\mathcal{H}_\alpha$ is a closed subspace of $\mathcal{H}_\eta$. Hence $\{\mathcal{H}_\alpha : \alpha \leq \eta\}$ is a strictly increasing wellordered sequence of closed subspaces of $\mathcal{H}_\eta$. Since $\mathcal{H}_\eta$ is separable, the length of this sequence is countable, which shows the claim. $\square$

Lemma 2.0.29 *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, and $\mathcal{D} \subset \mathcal{E}$ be invariant σ -subalgebras of $\mathcal{B}$. Then if $f \in L^2(X)$ is compact over $\mathcal{D}$, then it is compact over $\mathcal{E}$.*

Proof. By theorem 2.0.11 there are $(X', \mathcal{B}', \mu', T') \sim (X, \mathcal{B}, \mu, T)$, $(Y', \mathcal{D}', \nu', S') \sim (X, \mathcal{D}, \mu, T)$ and $(Z', \mathcal{E}', \theta', U') \sim (X, \mathcal{E}, \mu, T)$ with the $\phi : X' \rightarrow Z'$ and $\psi : Z' \rightarrow Y'$ factor maps. Let $\mu'_z = \int \mu'_z d\theta'(z)$ and $\mu'_y = \int \mu'_y d\nu'(y)$ be the disintegrations of μ' over Z' and Y' and $\theta'_y = \int \theta'_y d\nu'(y)$ be the disintegrations of θ' over Y' . Fix $f \in L^2(X')$ that is compact over Y' . By (C4) of Theorem 2.0.22 it is enough to show, that for any $\epsilon, \delta > 0$ there are $g_1, \dots, g_k \in L^2(X')$ such that for all $m \in \mathbb{Z}$, $\min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_z} < \delta$ but for a set of $z \in Z'$ of θ' -measure $< \epsilon$. Since f is compact over Y' , we can choose $g_1, \dots, g_k$ such that that for all $m \in \mathbb{Z}$, $\min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_y} < \delta \epsilon^{1/2}/2^{1/2}$ but for a set of $y \in Y'$ of ν' -measure $< \epsilon/2$. We show that these g_i 's work. Indeed, fix $y \in Y'$ such that $\min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_y} < \delta \epsilon^{1/2}/2^{1/2}$, and choose j such that $\|T^m f - g_j\|_{\mu'_y} < \delta \epsilon^{1/2}/2^{1/2}$. Let $G_y = \{z \in Z' : \psi(z) = y \text{ and } \|T^m f - g_j\|_{\mu'_z} \geq \delta\}$. Then $\theta'_y(G_y) < \epsilon/2$, otherwise $\|T^m f - g_j\|_{\mu'_y}^2 = \int \|T^m f - g_j\|_{\mu'_z}^2 d\theta'_y(z) \geq \delta^2 \epsilon/2$. Let $G_1 = \cup\{\psi^{-1}(y) : \min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_y} \geq \delta \epsilon^{1/2}/2^{1/2}\}$, and $G_2 = \cup\{G_y : \min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_y} < \delta \epsilon^{1/2}/2^{1/2}\}$. Let $G = G_1 \cup G_2$. Then $\theta'(G_1) < \epsilon/2$ (by the choice of the g_i 's). Also $\theta'(G_2) = \int_{\psi[G_2]} \theta'_y(G_y) d\nu(y) < \epsilon/2$. So $\theta'(G) < \epsilon$, and if $z \notin G$, then $\min_{1 \leq j \leq k} \|T^m f - g_j\|_{\mu'_z} < \delta$, which is what we wanted. $\square$

Lemma 2.0.30 *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, $\mathcal{D} \subset \mathcal{D}'$, $\mathcal{E}$ be invariant σ -subalgebras of $\mathcal{B}$. Suppose that $(X, \mathcal{E}, \mu, T)$ is a compact extension of $(X, \mathcal{D}, \mu, T)$. Then if $\mathcal{E}'$ is the invariant σ -algebra generated by $\mathcal{E} \cup \mathcal{D}'$, then $(X, \mathcal{E}', \mu, T)$ is a compact extension of $(X, \mathcal{D}', \mu, T)$.*

Proof. Clearly every $\mathcal{D}'$ -measurable function is compact over $\mathcal{D}'$. On the other hand since every $\mathcal{E}$ -measurable function is compact over $\mathcal{D}$, by Lemma 2.0.29 they are compact over $\mathcal{D}'$. Since the collection of relatively compact functions is T -invariant and closed under multiplication, if it contains the $\mathcal{D}$ and $\mathcal{E}$ -measurable functions, it should contain the $\mathcal{E}'$ -measurable functions, which proves the claim. $\square$

Theorem 2.0.31 *Let $\{\mathbf{X}_\alpha = (X, \mathcal{B}_\alpha, \mu, T) : \alpha \leq \eta\}$ be a compact tower for the measure distal system $\mathbf{X} = \mathbf{X}_\eta = (X, \mathcal{B}, \mu, T)$. Suppose that for every $\alpha < \eta$, $\mathbf{X}_{\alpha+1}$ is the maximal compact extension of $\mathbf{X}_\alpha$ in $\mathbf{X}$. Then the order of $\mathbf{X}$ is η .*

Proof. Let $\{\mathbf{Y}_\alpha = (X, \mathcal{D}_\alpha, \mu, T) : \alpha \leq \theta\}$ be an other compact tower reaching $\mathbf{X}$. We have to show that $\eta \leq \theta$. For this it is enough to show that for every $\alpha \leq \eta$, $\alpha \leq \theta$, $\mathbf{X}_\alpha$ is a factor of $\mathbf{Y}_\alpha$. This is true for $\alpha = 0$. We proceed by induction. By the definition of the compact tower at limit stages, if α is limit, and $\mathbf{X}_\beta$ is a factor of $\mathbf{Y}_\beta$ for all $\beta < \alpha$, then $\mathbf{X}_\alpha$ is a factor of $\mathbf{Y}_\alpha$. Now suppose that $\mathbf{X}_\alpha$ is a factor of $\mathbf{Y}_\alpha$, i.e. $\mathcal{D}_\alpha \subset \mathcal{B}_\alpha$. Then every function that is compact over $\mathbf{Y}_\alpha$ is also compact over $\mathbf{X}_\alpha$. Hence they are in $L^2(\mathbf{X}_{\alpha+1})$, since $\mathbf{X}_{\alpha+1}$ is the maximal compact extension of $\mathbf{X}_\alpha$ in $\mathbf{X}$. Since every function in $L^2(\mathbf{Y}_{\alpha+1})$ is compact over $\mathbf{Y}_\alpha$, this shows that $L^2(\mathbf{Y}_{\alpha+1})$ is a subset of $L^2(\mathbf{X}_{\alpha+1})$, which is the same as $\mathbf{Y}_{\alpha+1}$ is a factor of $\mathbf{X}_{\alpha+1}$. $\square$

CHAPTER III

Examples of measure distal systems

3.1 Notation

In this chapter we build ergodic transformations on the infinite dimensional torus. The following are the notation used in this chapter.

Let's call a linear ordering I acceptable, if I is a countable linear ordering of $\mathbb{N}$, such that 0 is the smallest element, 1 is the largest element, and every $i \neq 1$ has a successor, denoted by i^+ . Let $<_I$ denote the ordering.

For a fixed acceptable ordering I , let $X = \prod_{j \in I} \mathbb{T}$. (Where $\mathbb{T}$ denotes the unit interval with 0 and 1 identified.)

We are looking at transformations $T : X \rightarrow X$ of the following form: $(T\vec{x})_0 = x_0 \oplus \alpha$ and $(T\vec{x})_j = x_j \oplus \sum_{i <_I j} t_j^i x_i$, where $\oplus$ denotes the mod 1 addition on $\mathbb{T}$. To get a well-defined transformation, we restrict our attention to the ones where for every fixed j , $\sum_{i <_I j} |t_j^i|$ is convergent. For well-ordered I 's this condition is also enough to guarantee that the transformation is measure preserving. However for I 's that are not well ordered we need further restrictions on the t_j^i 's to get a measure preserving transformation. We deal with this in Theorem 3.3.3.

Let's denote $(T^m \vec{x})_j$ by x_j^m . Then $x_j^m = x_j^{m-1} \oplus \sum_{i <_I j} t_j^i x_i^{m-1} = \dots = x_j \oplus$

$\sum_{i <_I j} t_j^i x_i^0 \oplus \dots \oplus \sum_{i <_I j} t_j^i x_i^{m-1}$. Note that here only the first term depends on x_j , i.e. we can write $x_j^m = x_j \oplus F_{\ell,m}(\{x_i : i <_I j\})$. We will also need to see how x_{j+}^m depends on x_j and x_{j+} , but only when t_{j+}^j is an integer. $x_{j+}^m = x_{j+} \oplus \sum_{k=0}^{m-1} \sum_{i <_{I+}} t_{j+}^i x_i^k = x_{j+} \oplus \sum_{k=0}^{m-1} t_{j+}^j x_j^k \oplus \sum_{k=0}^{m-1} \sum_{i <_{I+}} t_{j+}^i x_i^k$. The third part does not depend on x_j or x_{j+} . Since t_{j+}^j is an integer,

$$t_{j+}^j x_j^k = t_{j+}^j (x_j \oplus F_{\ell,k}(\{x_i : i <_I j\})) = t_{j+}^j x_j \oplus t_{j+}^j F_{\ell,k}(\{x_i : i <_I j\}), \quad (3.1)$$

hence for some $G_{\ell,m}$ that does not depend on x_j or x_{j+} ,

$$x_{j+}^m = x_{j+} \oplus m t_{j+}^j x_j \oplus G_{\ell,m}(\{x_i : i <_I j\}). \quad (3.2)$$

Let $\mathcal{A}$ be the countable family of finite unions of sets of the form $\prod_{i \in I} O_i$ where the O_i 's are rational intervals, and for all but finitely many $i \in I$, $O_i = \mathbb{T}$. Then $\mathcal{A}$ generates the σ -algebra of Borel sets. Let's say that S is a support of B if and only if for all $\vec{x}, \vec{y} \in X$, if $\vec{x}|_S = \vec{y}|_S$, then $\vec{x} \in B$ iff $\vec{y} \in B$. Clearly for every $B \in \mathcal{A}$ the support of B is finite. Let $\{(B_n, C_n) : n \in \mathbb{N}\}$ be an enumeration of pairs of elements of $\mathcal{A}$, such that the support of B_n or C_n is in $\{0, \dots, n\}$. Such an enumeration clearly exists. Note that we can fix this enumeration independently of I .

Let $Y_j = \prod_{k <_I j} \mathbb{T}$, $Z_j = \prod_{k \geq_I j} \mathbb{T}$, $X_j = X \times_{Y_j} X$, (so $X_j \simeq Y_j \times Z_j \times Z_j$), and $W_j = Y_j \times \mathbb{T} \times \mathbb{T}$ (here $\mathbb{T}$ corresponds to the j 'th coordinates). We will show that for every j , Y_{j+} is a compact extension of Y_j in X .

We will need a basis for the square-integrable functions of these spaces. Let us introduce the following notation. If A, B are topological spaces, $C = A \times B$ is the product space, $f : A \rightarrow \mathbb{C}$, $g : B \rightarrow \mathbb{C}$, then let $f \otimes g : C \rightarrow \mathbb{C}$ be the function $f(a)g(b)$.

Let $\mathcal{H}_j = \{e^{\vec{a}\vec{y}} : a_k \in 2\pi i\mathbb{Z}, \text{ and } a_k = 0 \text{ for all but finitely many } k\}$ be the usual basis of $L^2(Y_j)$, $\mathcal{G}_j = \{e^{\vec{b}\vec{z}} : b_k \in 2\pi i\mathbb{Z}, \text{ and } b_k = 0 \text{ for all but finitely many } k\}$ be a basis of $L^2(Z_j)$, $\mathcal{F} = \{f \otimes g : f \in \mathcal{H}_j, g \in \mathcal{G}_j\}$ be a basis of $L^2(X)$ and $\mathcal{D}_j = \{f \otimes g_1 \otimes g_2 : f \in \mathcal{H}_j, g_1, g_2 \in \mathcal{G}_j\}$ be a basis of $L^2(X_j)$.

Let $\pi_j : X \rightarrow Y_j$ and $\rho_j : X_j \rightarrow W_j$ be the canonical projections defined by the following formulas: For $i <_I j$, $(\pi_j(\vec{x}))_i = x_i$, $(\rho_j(\vec{y}, \vec{z}_1, \vec{z}_2))_i = y_i$, and the last two coordinates of $\rho_j(\vec{y}, \vec{z}_1, \vec{z}_2)$ are z_j^1 and z_j^2 .

Let $T_j^* : X_j \rightarrow X_j$ be the transformation on the relative product, induced by T . (cf. Lemma 2.0.15)

Let $\mathcal{A}_j$ be the countable family of finite unions of basic open sets of X_j . Let $\{(E_n, F_n) : n \in \mathbb{N}\}$ be an enumeration of pairs of these for all $j \in I$, such that $j < n$ and the support of E_n or F_n is in $\{0, \dots, n\}$. We also assume (for technical reasons) that every pair appears infinitely often in the enumeration.

For $S \subseteq I$ with $0 \in S$, let $X_S = \prod_{j \in S} T_j$, and $T_S : X_S \rightarrow X_S$ be the transformation induced by T the following way. For $j \in S$, $(T_S \vec{x})_j = x_j \oplus \sum_{i \in S, i <_I j} t_j^i x_i$. There is a corresponding $T_S' : X \rightarrow X$ which we get from T by replacing t_j^i by 0 if not both i and j are in S . We will only look at the action of T_S' on sets whose supports are subsets of S . On these sets there is a natural correspondence between the action of T_S' and T_S . More precisely for an open set B if the support of B is a subset of S , then with the projection $P_S : X \rightarrow X_S$, $T_S P_S B = P_S T_S' B$. So without confusion we will use T_S to denote both transformations. Define $Y_{S,j}$, $Z_{S,j}$, $X_{S,j}$, $W_{S,j}$ and $T_{S,j}^*$, $\pi_{S,j}$ and $\rho_{S,j}$ accordingly.

Lemma 3.1.1 *Let $Y = (Y, \mathcal{D}, \nu, S)$ be an arbitrary measure preserving system. Let X be the skew product of Y and the unit circle $\mathbb{T}$, where the action T on X is defined with some measurable $\sigma : Y \rightarrow \mathbb{T}$ by $T : (y, t) \mapsto (Sy, t \oplus \sigma(y))$.*

Then X is a compact extension of Y .

Remark. For the proof we use ideas from Section 6.2 of [3].

Proof. For $f \in L^2(\mathbb{T})$ let us use the notation f^α for the function $f(t \oplus \alpha)$. First note that for any $f \in L^2(\mathbb{T})$ the set $\{f^\alpha : \alpha \in \mathbb{T}\}$ is compact in $L^2(\mathbb{T})$.

Consider a function $f \in L^2(\mathbb{T})$ having the form $f(y, t) = \sum_{n=1}^N \psi_n(y) \phi_n(t)$ with $\psi_n \in L^\infty(Y)$ and $\phi_n \in L^2(\mathbb{T})$. Since these type of functions form a dense subset of $L^2(X)$, it is enough to show that these are almost periodic relative to Y . I.e. we have to show that for every $\delta > 0$ there are $g_1, \dots, g_M \in L^2(X)$ such that for almost every $y \in Y$ and for every $j \in \mathbb{N}$,

$$\min_{m \leq M} \|T^j f - g_m\|_{L^2(\mu_Y)} < \delta. \quad (3.3)$$

Fix $\delta > 0$ and choose K such that $K > \|\psi_n\|_\infty$ for all $n \leq N$. Choose also $h_1, \dots, h_M \in L^2(\mathbb{T})$ such that if for all $n \leq N$, $|\alpha_n| < K$, then for any $\beta_1, \dots, \beta_N$,

$$\min_{m \leq M} \left\| \sum_{n=1}^N \alpha_n \phi_n(t \oplus \beta_n) - h_m \right\|_{L^2(\mathbb{T})} < \delta. \quad (3.4)$$

Let $g_m(y, t) = h_m(t)$ and $H_j(y, t) = (T^j f)(y, t)$. Then for almost every $y \in Y$, $H_j(y, t) = \sum_{n=1}^N \alpha_n \phi_n(t \oplus \beta_n)$, for some $\beta_1, \dots, \beta_N$ and $|\alpha_n| < K$. By the choice of the h_m 's we get

$$\min_{m \leq M} \|H_j - g_m\|_{L^2(\mu_Y)} < \delta \quad (3.5)$$

for almost every $y \in Y$, which completes the proof. $\square$

This lemma proves that for every $j \in I$, Y_{j+} is a compact extension of Y_j . We will need the following characterization of the maximal compact extension.

Lemma 3.1.2 *Y_{j+} is the maximal compact extension of Y_j in X iff*

For every $f \in \mathcal{D}_j$, $\phi, \psi \in \mathcal{F}$, where $\phi(\vec{y}, \vec{z}^1) = e^{\vec{a}\vec{y} + \vec{b}\vec{z}^1}$ and $b_k \neq 0$ for some $k >_I j$,

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int_{Z_j} \int_{Z_j} \int_{Y_j} (T_j^*)^m(f(\vec{y}, \vec{z}^1, \vec{z}^2)) \overline{\phi(\vec{y}, \vec{z}^1)} \psi(\vec{y}, \vec{z}^2) d\vec{y} d\vec{z}^1 d\vec{z}^2 = 0 \quad (3.6)$$

Proof. Let $H = w \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N (T_j^*)^m(f)$. Then H is T_j^* -invariant. If Y_{j+} is the maximal compact extension of Y_j in X , then $H * \psi \in L^2(Y_{j+})$. Since by assumption $b_k \neq 0$ for some $k >_I j$, ϕ is orthogonal to $L^2(Y_{j+})$. This proves $\langle H * \psi, \phi \rangle = 0$ which is just a rewriting of 3.6.

For the other direction we have to show that if H is bounded and T_j^* -invariant and $\psi \in L^\infty(X)$ then $H * \psi \in L^2(Y_{j+})$. I.e. we have to show that for such H and ψ for any $\phi(\vec{y}, \vec{z}^1) = e^{\vec{a}\vec{y} + \vec{b}\vec{z}^1}$, where $b_k \neq 0$ for some $k >_I j$, $\langle H * \psi, \phi \rangle = 0$.

Let $K > \|H\|_\infty, \|\psi\|_\infty$ and fix any $\epsilon > 0$. Choose $\{f_l : l < L\} \subset \mathcal{D}_j$ and $\{\psi_l : l < L\} \subset \mathcal{F}$ so that for $F = \sum_{l < L} \alpha_l f_l$, $\|F - H\|_2 < \epsilon/K$, and for $G = \sum_{l < L} \beta_l \psi_l$, $\|G - \psi\|_2 < \epsilon/K$. For every $l_1, l_2 < L$, 3.6 implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int_{Z_j} \int_{Z_j} \int_{Y_j} (T_j^*)^m(f_{l_1}(\vec{y}, \vec{z}^1, \vec{z}^2)) \overline{\phi(\vec{y}, \vec{z}^1)} \psi_{l_2}(\vec{y}, \vec{z}^2) d\vec{y} d\vec{z}^1 d\vec{z}^2 = 0. \quad (3.7)$$

Hence

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \int_{Z_j} \int_{Z_j} \int_{Y_j} (T_j^*)^m(F(\vec{y}, \vec{z}^1, \vec{z}^2)) \overline{\phi(\vec{y}, \vec{z}^1)} G(\vec{y}, \vec{z}^2) d\vec{y} d\vec{z}^1 d\vec{z}^2 = 0, \quad (3.8)$$

i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \langle (T_j^*)^m(F) * \bar{\phi}, \bar{G} \rangle = 0. \quad (3.9)$$

But

$$\langle (T_j^*)^m(H) * \bar{\phi}, \bar{\psi} \rangle = \quad (3.10)$$

$$\langle (T_j^*)^m(H) * \bar{\phi}, \overline{\psi - G} \rangle + \langle (T_j^*)^m((H - F) * \bar{\phi}, \bar{G}) + \langle (T_j^*)^m(F) * \bar{\phi}, \bar{G} \rangle. \quad (3.11)$$

Since $(T_j^*)^m(H) = H$,

$$\langle (T_j^*)^m(H) * \bar{\phi}, \bar{\psi} \rangle = \langle H * \bar{\phi}, \bar{\psi} \rangle. \quad (3.12)$$

On the other hand, since $\|H\|_\infty < K$ and $\|\phi\|_\infty = 1$,

$$\langle (T_j^*)^m(H) * \bar{\phi}, \overline{\psi - G} \rangle \leq K \|\psi - G\|_2 \leq \epsilon. \quad (3.13)$$

Also

$$\langle (T_j^*)^m((H - F) * \bar{\phi}, \bar{G}) \leq \quad (3.14)$$

$$\|(T_j^*)^m((H - F) * \bar{\phi})\|_2 \|\bar{G}\|_2 \leq \|(T_j^*)^m((H - F))\|_2 \|G\|_2. \quad (3.15)$$

(Here we used that $|\phi(\tilde{y}, \tilde{z}^1)| = 1$ for all $(\tilde{y}, \tilde{z}^1)$.)

But $\|(T_j^*)^m((H - F))\|_2 = \|H - F\|_2 \leq \epsilon/K$, and $\|G\|_2 \leq \|\psi - G\|_2 + \|\psi\|_2 \leq \epsilon/K + K$. So

$$\langle (T_j^*)^m(H) * \bar{\phi}, \bar{\psi} \rangle \leq \epsilon + \frac{\epsilon^2}{K^2}. \quad (3.16)$$

So the limit of the averages as $0 \leq m < N$ of the first two terms of 3.11 is less than $2\epsilon + \epsilon^2/K^2$. We already have that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \langle (T_j^*)^m(F) * \bar{\phi}, \bar{G} \rangle = 0, \quad (3.17)$$

which gives

$$\langle H * \bar{\phi}, \bar{\psi} \rangle \leq \epsilon + \frac{\epsilon^2}{K^2}. \quad (3.18)$$

Since $\epsilon > 0$ was arbitrary, this implies that $\langle H * \bar{\phi}, \bar{\psi} \rangle = 0$, which is what we wanted.

□

3.2 Systems on the finite dimensional tori

In this section we give examples of ergodic systems on the finite dimensional tori, keeping in mind that we would like to put these together to ergodic systems on the infinite dimensional torus. I would like to thank Professor Benjamin Weiss for suggesting the approach we use to show the ergodicity of the systems.

Lemma 3.2.1 *Let $X = \prod_{i < n} \mathbb{T}$ be a finite product, and $T_{\vec{l}} : X \rightarrow X$ be the following transformation for $\vec{l} = \{l_j^i \neq 0 : i < j < n\}$: $(T_{\vec{l}} \vec{x})_0 = x_0 \oplus \alpha$ for any irrational α , $(T_{\vec{l}} \vec{x})_k = x_k \oplus \sum_{j < k} l_k^j x_j$ for $0 < k < n$. Let $X' = X \times \mathbb{T}$ and let $T' : X' \rightarrow X'$ be the transformation defined by $T'(\vec{x}, x_n) = (T_{\vec{l}}(\vec{x}), x_n \oplus \sum_{j < n} l_n^j x_j)$. Suppose that $T_{\vec{l}}$ is ergodic.*

Then T' is not ergodic if and only if there is a measurable nonzero $c_k : X \rightarrow \mathbb{C}$ which satisfies

$$c_k(x_0 \dots x_{n-1}) = c_k(T_{\vec{l}}(x_0 \dots x_{n-1})) e^{2\pi i k \sum_{j=0}^{n-1} l_n^j x_j} \quad (3.19)$$

for some $k \neq 0$.

Proof. First we show that nonergodicity implies the existence of a nonzero measurable c_k satisfying 3.19. Suppose for a particular $t_n^0, \dots, t_n^{n-1}$ the extended

transformation T' is not ergodic. We show that for the extended transformation coming from this $t_n^0, \dots, t_n^{n-1}$ there is a measurable c_k as in 3.19. Let $f(\vec{x}) = \sum_{k \in \mathbb{Z}} c_k(x_0, \dots, x_{n-1}) e^{2\pi i k x_n}$ be a nonconstant invariant function that witnesses the nonergodicity of the extended transformation. Because of the assumption that T is ergodic, c_k is almost everywhere different from 0 for some $k \neq 0$. Then using the fact that

$$k(x_n \oplus \sum_{j=0}^{n-1} t_n^j x_j) = kx_n \oplus k \sum_{j=0}^{n-1} t_n^j x_j, \quad (3.20)$$

and that in the exponent of e we can change expressions by an integer multiple of $2\pi i$, we get

$$T'(f)(\vec{x}) = \sum_{k \in \mathbb{Z}} c_k(T_{\vec{t}}(x_0, \dots, x_{n-1})) e^{2\pi i k \sum_{j=0}^{n-1} t_n^j x_j} e^{2\pi i k x_n}. \quad (3.21)$$

Hence the invariance of f and the uniqueness of the Fourier coefficients implies that for every $k \in \mathbb{Z}$,

$$c_k(x_0 \dots x_{n-1}) = c_k(T_{\vec{t}}(x_0 \dots x_{n-1})) e^{2\pi i k \sum_{j=0}^{n-1} t_n^j x_j}. \quad (3.22)$$

In particular this is true for the nonzero c_k .

Conversely if for some $k \neq 0$ there is a nonzero c_k satisfying 3.19, then

$$c_k(x_0, \dots, x_{n-1}) e^{2\pi i k x_n} \quad (3.23)$$

is a nontrivial T' -invariant function witnessing non-ergodicity. $\square$

Theorem 3.2.2 *Let $X = \prod_{i < n} \mathbb{T}$ be a finite product, and $T_{\vec{t}}: X \rightarrow X$ be the following transformation for $\vec{t} = \{t_j^i \neq 0 : i < j < n\}$: $(T_{\vec{t}}\vec{x})_0 = x_0 \oplus \alpha$ for some irrational α , $(T_{\vec{t}}\vec{x})_k = x_k \oplus \sum_{j < k} t_k^j x_j$ for $0 < k < n$.*

(i) If all the t_j^i 's are rational, then $T_{\vec{t}}$ is ergodic.

(ii) For almost every choice of the t_j^i 's for $0 < i < j \leq n$, $T_{\vec{t}}$ is ergodic.

Remark. For integer coefficients this is one of the standard examples of ergodic transformations.

Proof. By induction, it is enough to show that if $T_{\vec{t}}: X \rightarrow X$ is ergodic, then for almost every $(t_n^0, \dots, t_n^{n-1})$ and for every rational $(t_n^0, \dots, t_n^{n-1})$ the transformation

$$T' : (\vec{x}, x_n) \mapsto (T_{\vec{t}}\vec{x}, x_n \oplus \sum_{j < n} t_n^j x_j) \quad (3.24)$$

is also ergodic.

Suppose that for some $t_n^0, \dots, t_n^{n-1}$ the extended transformation T' is not ergodic. Then according to Lemma 3.2.1 there is $k \neq 0$ and a nonzero measurable c_k satisfying

$$c_k(x_0 \dots x_{n-1}) = c_k(T_{\vec{t}}(x_0 \dots x_{n-1})) e^{2\pi i k \sum_{j=0}^{n-1} t_n^j x_j}. \quad (3.25)$$

Hence if for a positive measure of $t_n^0, \dots, t_n^{n-1}$ the extended transformation is not ergodic, then there is a $k \neq 0$, such that for a positive measure of $t_n^0, \dots, t_n^{n-1}$ this is witnessed by 3.25 for this k . So fix this k , and let $B = \{(t_n^0, \dots, t_n^{n-1}) : \text{there is a measurable } c_k \neq 0 \text{ as in 3.25}\}$. Now suppose $(t^0, \dots, t^{n-1}) \in B$ is witnessed by $c_k \neq 0$ and $(s^0, \dots, s^{n-1}) \in B$ is witnessed by $d_k \neq 0$. Then

$$d_k c_k(x_0, \dots, x_{n-1}) = d_k c_k(T_{\vec{t}}(x_0, \dots, x_{n-1})) e^{2\pi i k \sum t_n^j x_j} e^{2\pi i k \sum s_n^j x_j} = \quad (3.26)$$

$$d_k c_k(T_{\vec{t}}(x_0, \dots, x_{n-1})) e^{2\pi i k \sum (t_n^j + s_n^j) x_j}. \quad (3.27)$$

This shows that $d_k c_k \neq 0$ is a witness for $(t^0 + s^0, \dots, t^{n-1} + s^{n-1}) \in B$. Similarly $c_k/d_k \neq 0$ is a witness for $(t^0 - s^0, \dots, t^{n-1} - s^{n-1}) \in B$. (Note that the ergodicity of

T_l implies that $|d_k|$ is constant, being invariant, hence if it is non-zero, then we can divide by it.) So B is closed under subtraction and addition. If $\mu(B) > 0$ then $B - B$ contains an open neighborhood of $\vec{0}$, and $B + B$ enlarges it. So if $\mu(B) > 0$ then $B = \mathbb{R}^n$. In particular any integer tuple is in B . Using similar argument, if there is a rational tuple in B then adding this to itself M times, where M is a common multiple of the denominators, we again get an integer tuple in B . So in both cases we found an integer tuple $(p_0, \dots, p_{n-1}) \in B$, such that for some non-zero measurable c_k , and for some $k \neq 0$

$$c_k(x_0 \dots x_{n-1}) = c_k(T_l(x_0 \dots x_{n-1}))e^{2\pi i k \sum_{j=0}^{n-1} p_j x_j}. \quad (3.28)$$

Writing $c_k(x_0 \dots x_{n-1}) = \sum_l d_{k,l}(x_0 \dots x_{n-2})e^{2\pi i l x_{n-1}}$, 3.28 becomes

$$\sum_l d_{k,l}(x_0 \dots x_{n-2})e^{2\pi i l x_{n-1}} = \quad (3.29)$$

$$\sum_l d_{k,l}(T_l(x_0 \dots x_{n-2}))e^{2\pi i l(x_{n-1} \oplus \sum_{j=0}^{n-2} l_{n-1}^j x_j)}e^{2\pi i k \sum_{j=0}^{n-1} p_j x_j}. \quad (3.30)$$

Using, that the p_j 's are integers, and that in the exponent of e we can change expressions by integer multiples of $2\pi i$, we can replace the mod 1 addition by ordinary one, this becomes

$$\sum_l d_{k,l}(x_0 \dots x_{n-2})e^{2\pi i l x_{n-1}} = \quad (3.31)$$

$$\sum_l d_{k,l}(T_l(x_0 \dots x_{n-2}))e^{2\pi i \sum_{j=0}^{n-2} s_j x_j}e^{2\pi i (l + p_{n-1}k)x_{n-1}}, \quad (3.32)$$

with $s_j = lt_{n-1}^j + kp_j$. So for every $l \in \mathbb{Z}$,

$$d_{k,l}(x_0 \dots x_{n-2}) = d_{k,l-p_{n-1}k}(T_l(x_0 \dots x_{n-2}))e^{2\pi i \sum_{j=0}^{n-2} s_j x_j}, \quad (3.33)$$

Hence for every $l \in \mathbb{Z}$, $\|d_{k,l}\|_2 = \|d_{k,l-p_{n-1}k}\|_2$, which is impossible, unless $\|d_{k,l}\|_2 = 0$ for every $l \in \mathbb{Z}$. This contradicts the assumption that $c_k \neq 0$, which completes the proof. $\square$

We also would like to control the maximal compact extensions. For this we have a similar claim.

Theorem 3.2.3 *Let $X = \prod_{i < n} \mathbb{T}$ be a finite product, and $T_{\vec{t}}: X \rightarrow X$ be the following transformation for $\vec{t} = \{t_j^i \neq 0 : i < j < n\}$: $(T_{\vec{t}}\vec{x})_0 = x_0 \oplus \alpha$ for some irrational α , $(T_{\vec{t}}\vec{x})_k = x_k \oplus \sum_{j < k} t_k^j x_j$ for $0 < k < n$. Let's assume that $T_{\vec{t}}$ is ergodic.*

(i) *If every t_j^i is rational, then for every $j < n - 1$, Y_{j+1} is the maximal compact extension of Y_j in X .*

(ii) *For almost every choice of the t_j^i 's for every $j < n - 1$, Y_{j+1} is the maximal compact extension of Y_j in X .*

Proof. Suppose that $X = \prod_{i < n} \mathbb{T}$, $T_{\vec{t}}: X \rightarrow X$ is ergodic and for every $j < n - 1$, Y_{j+1} is the maximal compact extension of Y_j in X . Let us extend $T_{\vec{t}}$ to $X \times \mathbb{T}$ by $(\vec{x}, y) \mapsto (T_{\vec{t}}(\vec{x}), y \oplus \sum_{i < n} t_n^i x_i)$ for some constants $(t_n^0, \dots, t_n^{n-1})$. By induction it is enough to show that for almost every $(t_n^0, \dots, t_n^{n-1})$ and for every rational $(t_n^0, \dots, t_n^{n-1})$, for every $j < n$, Y_{j+1} is the maximal compact extension of Y_j in $X \times \mathbb{T}$ for the extended transformation.

Suppose that for some $j < n - 1$, Y_{j+1} is not the maximal compact extension of Y_j in $X \times \mathbb{T}$. Let's use the notation $(y_0 \dots y_j, z_{j+1}^1 \dots z_n^1, z_{j+1}^2 \dots z_n^2)$ for the elements of $(X \times \mathbb{T}) \times_j (X \times \mathbb{T})$. According to Lemma 2.0.25 we get an invariant function H that depends on at least one of the variables $z_{j+2}^1 \dots z_n^1, z_{j+2}^2 \dots z_n^2$. Since in the induction

hypothesis we assumed that Y_{j+1} is the maximal compact extension of Y_j in X , H has to depend on either z_n^1 or z_n^2 , so we can write $H = \sum_{m,l \in \mathbb{Z}} d_{m,l} e^{2\pi i m z_n^1} e^{2\pi i l z_n^2}$, where $d_{m,l} \in L^2(X_j)$, and for some $(m, l) \neq (0, 0)$, $d_{m,l}$ is not constantly 0 a.e. Then by the invariance of H , for every $m, l \in \mathbb{Z}$,

$$d_{m,l} = (T_j^*)(d_{m,l}) e^{2\pi i m (\sum_{k \leq j} t_n^k v_k + \sum_{k > j} t_n^k z_k^1)} e^{2\pi i l (\sum_{k \leq j} t_n^k v_k + \sum_{k > j} t_n^k z_k^2)}. \quad (3.34)$$

(Here we again replace the mod 1 addition by the ordinary one as before.) In particular this has to be true for the non-zero $d_{m,l}$. Again, the converse is also true. A witness $d_{m,l}$ as in 3.34 implies that Y_{j+1} is not the maximal compact extension of Y_j in $X \times \mathbb{T}$.

Let B_{j,m,l,C_a} be the set of $(t_n^0, \dots, t_n^{n-1})$'s for which there is a measurable witness as in 3.34 which is non-zero on at least two thirds (in measure) of the basic open set C_a . If there is a witness as in 3.34 for a positive measure of the $(t_{n-1}^0, \dots, t_n^{n-1})$'s then there are j, m, l, C_a such that $\mu(B_{j,m,l,C_a}) > 0$. If $u_{m,l}$ witnesses that $(w_0, \dots, w_{n-1}) \in B_{j,m,l,C_a}$ and $v_{m,l}$ witnesses that $(s_0, \dots, s_{n-1}) \in B_{j,m,l,C_a}$, then define the following function: $d_{m,l} = u_{m,l}/v_{m,l}$ if $v_{m,l}$ is not zero, and 0 otherwise. This function is not zero where $u_{m,l}$ and $v_{m,l}$ are not zero, hence at least on one third of C_a . Also $d_{m,l}$ witnesses that for $(t_0, \dots, t_{n-1}) = (w_0 - s_0, \dots, w_{n-1} - s_{n-1})$, 3.34 is satisfied. So for any $(t_n^0, \dots, t_n^{n-1}) \in B_{j,m,l,C_a} - B_{j,m,l,C_a}$, there is a non-zero measurable $d_{m,l}$ as in 3.34. If $\mu(B_{j,m,l,C_a}) > 0$ then the difference set contains an open neighborhood of $\vec{0}$, hence there is a rational tuple in it. So from the assumption that the measure of $(t_n^0, \dots, t_n^{n-1})$'s for which there is a witness $d_{m,l}$ as in 3.34 is positive, we concluded that there is a rational such tuple $(q_0, \dots, q_{n-1})$. Let M be such that for every $i < n$,

$p_i = Mq_i$ is an integer. Then multiplying 3.34 M times we get

$$d_{m,l}^M = (T_j^*) (d_{m,l}^M) e^{2\pi i m (\sum_{k \leq j} p_k y_k + \sum_{k > j} p_k z_k^1)} e^{2\pi i l (\sum_{k \leq j} p_k y_k + \sum_{k > j} p_k z_k^2)}. \quad (3.35)$$

We show that this is impossible. We use the characterization in Lemma 3.1.2 of the maximal compact extension. Let S be the transformation on $(X \times \mathbb{T}) \times_j (X \times \mathbb{T})$ that corresponds to the choice $(t_n^0, \dots, t_n^{n-1}) = (p_0, \dots, p_{n-1})$. Let U denote the corresponding transformation on $L^2((X \times \mathbb{T}) \times_j (X \times \mathbb{T}))$. Let $F = d_{m,l}^M e^{2\pi i m z_n^1} e^{2\pi i l z_n^2}$. Then F is invariant. Without loss of generality we can assume that $m \neq 0$. Look at the Fourier expansion

$$F = \sum_{\vec{a}, \vec{b}, \vec{c}} \alpha_{\vec{a}, \vec{b}, \vec{c}} e^{\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2}. \quad (3.36)$$

Consider the functions

$$H_{\vec{a}, \vec{b}, \vec{c}} = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{t=0}^N U^t (e^{\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2}). \quad (3.37)$$

These are invariant functions. We claim that $H_{\vec{a}, \vec{b}, \vec{c}}$ is trivial if $b_n \neq 0$. Indeed, we'll show that for any $g = e^{\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2}$, $\langle H_{\vec{a}, \vec{b}, \vec{c}}, g \rangle = 0$.

We already saw when we introduced our special transformations in Section 3.1 that

$$U^t(\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2) = \Phi(\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2) \oplus b_{n-1} z_{n-1}^1 \oplus b_n p_{n-1} t z_{n-1}^1 \oplus b_n z_n^1 \quad (3.38)$$

for some function Φ that does not depend on z_{n-1}^1 or z_n^1 . Hence

$$\int U^t(e^{\vec{a} \vec{y} + \vec{b} \vec{z}^1 + \vec{c} \vec{z}^2}) g d\lambda = \int \int \int e^{(v_{n-1} + b_{n-1} + m p b_n) z_{n-1}^1} h(\vec{y}, \vec{z}^1, \vec{z}^2) d\vec{y} d\vec{z}^1 d\vec{z}^2, \quad (3.39)$$

where h does not depend on z_{n-1}^1 . (Again replacing $\oplus$ by the ordinary addition in the exponent of e .) So we can separate the z_{n-1}^1 variable to get

$$\int U^t(e^{\tilde{a}\tilde{y}+\tilde{b}\tilde{z}^1+\tilde{c}\tilde{z}^2})gd\lambda = \int e^{(v_{n-1}+b_{n-1}+tpb_n)z_{n-1}^1}dz_{n-1}^1 \int h d\lambda. \quad (3.40)$$

Since $b_n \neq 0$, the exponent of e can be 0 for at most one t . For the others the first integral is 0, so $\int U^t(e^{\tilde{a}\tilde{y}+\tilde{b}\tilde{z}^1+\tilde{c}\tilde{z}^2})gd\lambda = 0$ for all but at most one t . So $\langle H_{\tilde{a},\tilde{b},\tilde{c}}, g \rangle = 0$. Since this is true for every g , this shows that $H_{\tilde{a},\tilde{b},\tilde{c}} = 0$ if $b_n \neq 0$. But then

$$F = w \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{t=0}^N U^t F = \sum_{\tilde{a},\tilde{b},\tilde{c}} \alpha_{\tilde{a},\tilde{b},\tilde{c}} H_{\tilde{a},\tilde{b},\tilde{c}} \quad (3.41)$$

would not depend on z_n^1 which contradicts the assumption that $m \neq 0$. $\square$

Finally let's state the conclusion of these theorems that we will use to build the systems on the infinite torus.

Corollary 3.2.4 *Let S be a finite linear ordering of $\{0, \dots, k-1\}$ with 0 as the smallest element. Let $X = \Pi_{i \in S} \mathbb{T}$, and $T : X \rightarrow X$ be the transformation defined by $(T\tilde{x})_0 = x_0 \oplus \alpha$, for some irrational α , and $(T\tilde{x})_j = x_j \oplus \sum_{i <_S j} t_j^i x_i$ for rational non-zero t_j^i 's. Let S' be an extension of S on $\{0, \dots, k\}$, and let $0 < \epsilon_j^i$ be fixed for all $i <_{S'} j$ such that either $i = k$ or $j = k$.*

Then we can extend T to T' on $X' = \Pi_{i \in S'} \mathbb{T}$ which is defined by $(T'\tilde{x})_0 = x_0 \oplus \alpha$, and $(T'\tilde{x})_j = x_j \oplus \sum_{i <_{S'} j} t_j^i x_i$, where the new t_j^i 's are arbitrarily chosen rationals from the interval $(0, \epsilon_j^i)$, such that T' is ergodic, and for all $j < k$, Y'_{j+} is the maximal compact extension of Y'_j in X' . $\square$

3.3 Systems on the infinite dimensional torus

In this section we use the notation introduced in the first section without any further reference. Recall for example that $\mathcal{A}$ denotes the countable family of finite unions of basic open sets of X and $\mathcal{A}_j$ denotes the countable family of finite unions of basic open sets of X_j .

Lemma 3.3.1 *Let I be a linear ordering with first element, and $S \subset I$ be a finite subset containing this first element. Let $B \in \mathcal{A}$ be a set such that the support of B is a subset of S . Let $Q_S : X \rightarrow X$ be the transformation $\vec{x} \mapsto (x_0 \oplus \alpha, \dots, x_j \oplus \sum_{i <_I j} q_j^i x_i, \dots)$, where $q_j^i = 0$ if not both i and j are in S .*

Then for every fixed $m \in \mathbb{N}$ and $0 < \delta < \mu(B)$, there are $\epsilon_j^i > 0$ for all $i <_I j$, such that if $T : X \rightarrow X$ is such that $|t_j^i - q_j^i| < \epsilon_j^i$, then $\mu(Q_S^m B \Delta T^m B) < \delta$.

Proof. First look at the case $m = 1$. Let $C_1, \dots, C_n$ be disjoint basic open sets such that for $C = \bigcup_{i=1}^n C_i$, $\mu(C \Delta Q_S B) < \delta/8$. The projection of each C_i to the j 'th coordinate is an interval, let us denote the measure of it by $a_{i,j}$. Hence $\mu(C_i) = \prod_j a_{i,j}$. Choose $\delta_j > 0$ such that for all $i \leq n$, $\prod_j (1 + 2\delta_j/a_{i,j}) \leq 1 + \delta/16\mu(B)$. For any $C' \subset X$ let $C'_\delta = \{\vec{x} + \vec{y} : \vec{x} \in C', |y_i| < \delta_i \text{ for all } i \in I\}$. Then $\mu(C_{i,\delta}) \leq \prod_j (a_{i,j} + 2\delta_j) = \prod_j a_{i,j} (1 + 2\delta_j/a_{i,j}) = \mu(C_i) \prod_j (1 + 2\delta_j/a_{i,j}) < \mu(C_i) (1 + \delta/16\mu(B))$. Hence $\mu(C_{i,\delta} \setminus C_i) < \mu(C_i) (\delta/16\mu(B))$. Since $C_\delta \setminus C \subset \bigcup_{i=1}^n (C_{i,\delta} \setminus C_i)$, so $\mu(C_\delta \setminus C) \leq \mu(C) \delta/16\mu(B) \leq (\mu(B) + \delta/8) \delta/16\mu(B) = \delta/16 + \delta^2/128\mu(B) < \delta/8$. Hence $\mu(C_\delta \setminus Q_S B) \leq \mu(C_\delta \setminus C) + \mu(C \Delta Q_S B) < \delta/4$. Choose $\epsilon_j^i > 0$ so that for all j , $\sum_{i <_I j} |\epsilon_j^i| < \delta_j$. Let $|t_j^i - q_j^i| < \epsilon_j^i$. Then for all $\vec{x} \in X$ and $j \in I$, $|(Q_S \vec{x})_j - (T \vec{x})_j| < \delta_j$.

So for all $\vec{x}$ such that $Q_S \vec{x} \in C$, $T\vec{x} \in C_\delta$. Let $B_1 = B \cap Q_S^{-1}C$. Since Q_S is measure preserving, $\mu(B_1) \geq \mu(B) - \delta/8$. So $\mu(TB_1) \geq \mu(TB) - \delta/8$. But $TB_1 \subset C_\delta$, so $\mu(TB_1 \setminus Q_S B) \leq \mu(C_\delta \setminus Q_S B) \leq \delta/4$. Hence $\mu(TB \setminus Q_S B) \leq \delta/4 + \delta/8 < \delta/2$. So $\mu(TB \Delta Q_S B) < \delta$.

For $m > 1$ let's apply the previous case to $\delta' = \delta/m$ and $B, Q_S B, \dots, Q_S^{m-1} B$. We get that we can choose the ϵ_j^i 's so that for all $0 \leq k < m$, $\mu(Q_S^{k+1} B \Delta T Q_S^k B) < \delta'$. But $\mu(Q_S^{k+1} B \Delta T Q_S^k B) = \mu(T^l Q_S^{k+1} B \Delta T^{l+1} Q_S^k B)$ for any l , in particular for $l + k = m - 1$. Hence $\mu(Q_S^m B \Delta T Q_S^{m-1} B) < \delta'$, $\mu(T Q_S^{m-1} B \Delta T^2 Q_S^{m-2} B) < \delta', \dots, \mu(T^{m-1} Q_S B \Delta T^m B) < \delta'$, and putting these together $\mu(Q_S^m B \Delta T^m B) < m\delta' = \delta$. $\square$

This lemma with the equivalent formulation of ergodicity will enable us to build an ergodic flow by induction, each step specifying a finite approximation to it. We have a similar claim for transformations on relative products, to make sure that the factors and maximal compact extensions of the final flow are exactly what we want.

Lemma 3.3.2 *Let I be a linear ordering with first element, and $S \subset I$ be a finite subset containing this first element. Let $E \in \mathcal{A}_j$ be a set such that the support of E is a subset of S . Suppose also that $j \in S$. Let $Q_S : X \rightarrow X$ be the transformation $\vec{x} \mapsto (x_0 \oplus \alpha, \dots, x_j \oplus \sum_{i <_I j} q_j^i x_i, \dots)$, where $q_j^i = 0$ if not both i and j are in S .*

Then for every fixed $m \in \mathbb{N}$ and $\delta > 0$, there are $\epsilon_j^i > 0$ for all $i <_I j$, such that if $T : X \rightarrow X$ is such that $|t_j^i - q_j^i| < \epsilon_j^i$, then $\mu((Q_S^)^m(E) \Delta (T_j^*)^m(E)) < \delta$.*

Proof. Same as of the previous lemma. $\square$

We would like to get a “uniform” constructions of ergodic flows on $\prod_{i \in I} T$ for acceptable linear orderings I . We look at these linear orderings as orderings of the

natural numbers, i.e. there is the natural ordering of natural numbers of the elements of I other than the $<_I$ -ordering. By “uniformity” of the construction we mean, that if I and J are two orderings, such that for all $i, j < k$, $i <_I j$ iff $i <_J j$, then for all $i, j < k$ the coefficients t_j^i corresponding to the transformations are the same.

In Chapter 4 our goal will be to give a Borel map from the space of linear orderings to the space of measure preserving transformations which reduces the wellorderings to the measure distal transformations. We use the next theorem to construct this map. The uniformity of our construction is the reason why the reduction map in Theorem 4.1.5 is continuous on the Borel set of acceptable linear orderings.

Theorem 3.3.3 *For any acceptable I there is an ergodic $T : X \rightarrow X$ where every Y_{j+} is the maximal compact extension of Y_j in X .*

Proof. Recall that on $X = \prod_{i \in I} \mathbb{T}$ we are looking at transformations of the form $(T\vec{x})_0 = x_0 \oplus \alpha$ and $(T\vec{x})_j = x_j \oplus \sum_{i <_I j} t_j^i x_i$. We construct the transformation $T : X \rightarrow X$ inductively. To carry out the induction, at the k 'th step we also choose an $\epsilon_j^i(k)$ for each i, j , and $m_k, n_k, s_k, \delta_k, \eta_k, \nu_k, \kappa_k$. Let $S_k = \{0, \dots, k\}$, and $T_{S_k} : X_{S_k} \rightarrow X_{S_k}$ be the corresponding transformation. Recall that we already have an enumeration of the pairs of the basic open sets of $\mathcal{A}$, $\{(B_k, C_k) : k \in \mathbb{N}\}$ with the property that the support of B_k and C_k are in S_k . Also we have an enumeration of the pairs of the basic open sets of the $\mathcal{A}_j$'s, $\{(E_k, F_k) : k \in \mathbb{N}\}$ with the property that the support of E_k and F_k are in S_k . Also if j_k denotes the index for which $(E_k, F_k) \in \mathcal{A}_{j_k} \times \mathcal{A}_{j_k}$, then $j_k < k$. By induction we show that we can choose the t_j^i 's, $\epsilon_j^i(k)$'s and the other constants to satisfy the following conditions.

- (A) Every t_j^i is a non-zero rational,
- (B) If $i <_I k$ then $t_k^i < \epsilon_k^i(k-1)$,
- (C) If $k <_I i$ then $t_i^k < \epsilon_i^k(k-1)$,
- (D) m_k is the smallest m such that $\mu(T_{S_k}^m B_k \cap C_k) > \frac{1}{2}\mu(B_k)\mu(C_k)$, and $\mu(T_{S_k}^m B_k \cap C_k) - \frac{1}{2}\mu(B_k)\mu(C_k) = \delta_k$,
- (E) If $q_j^i = t_j^i$ for all $i, j \leq k$, $q_j^i < \epsilon_j^i(k)$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$, then $\mu(T_{S_k}^{m_k} B_k \Delta Q^{m_k} B_k) < \delta_k$,
- (F) η_k is the largest η such that there are $E'_k, F'_k \subset W_{S_k, j_k}$ such that
- (i) $\mu(E'_k), \mu(F'_k) \geq (1 - \eta)\mu(\rho_{S_k, j_k}(F_k))$,
 - (ii) for every $w \in E'_k$, $\mu(\rho_{S_k, j_k}^{-1}(w) \cap E_k) \geq \eta$, and
 - (iii) for every $w \in F'_k$, $\mu(\rho_{S_k, j_k}^{-1}(w) \cap F_k) \geq \eta$.
- (G) If there is an m such that $\mu(T_{S_k, j_k}^m(E_k) \cap F_k) > \frac{1}{2}\eta_k^2\mu(\rho_{S_k, j_k}(F_k))$, then n_k is the smallest such m , and ν_k is the difference.
- (H) If $q_j^i = t_j^i$ for all $i, j \leq k$ and $q_j^i < \epsilon_j^i(k)$ otherwise and we defined n_k in (G) and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$, then $\mu((T_{S_k, j_k}^{n_k}(E_k) \Delta (Q_{j_k}^{n_k}(E_k))) < \nu_k$.
- (I) If there is an m such that $\mu(\rho_{S_k, j_k}(T_{S_k, j_k}^m(E_k)) \cap \rho_{S_k, j_k}(F_k)) < (1 - \eta_k)\mu(\rho_{S_k, j_k}(F_k))$, then s_k is the smallest such m , and κ_k is the difference.
- (J) If $q_j^i = t_j^i$ for all $i, j \leq k$, $q_j^i < \epsilon_j^i(k)$ otherwise and we defined s_k in (I) and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$, then $\mu((T_{S_k, j_k}^{s_k}(E_k) \Delta (Q_{j_k}^{s_k}(E_k))) < \kappa_k$.

$$(K) \epsilon_j^i(k) \leq \epsilon_j^i(k-1)$$

(L) T_{S_k} is ergodic and for every $j <_I k$ if $j^+ <_I k$, then Y_{S_k, j^+} is the maximal compact extension of $Y_{S_k, j}$ in X_{S_k} .

(M) If $q_j^i = t_j^i$ for all $i, j \leq k$, $q_j^i < \epsilon_j^i(k)$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$, then $\mu(T_{S_k} B_k \Delta Q B_k) < 1/k$,

Now we show by induction that we can choose the t_j^i 's and $\epsilon_j^i(k)$'s and the other constants so that these conditions are satisfied. Let us start with the transformation $x_0 \mapsto x_0 \oplus \alpha$ for some irrational α .

Let's assume now that we have everything satisfied up to $k-1$. Since $\epsilon_j^i(k-1) > 0$, we can choose the positive rational t_k^i or t_i^k below $\epsilon_k^i(k-1)$ or $\epsilon_i^k(k-1)$. Hence (A), (B) and (C) can obviously be satisfied. Note also that (L) is a consequence of (A) using Corollary 3.2.4.

To choose an m_k as in (D), all we need is that there exist some m such that $\mu(T_{S_k}^m B_k \cap C_k) > \frac{1}{2} \mu(B_k) \mu(C_k)$. But this is a consequence of the ergodicity of T_{S_k} . Once we have m_k , we can certainly choose δ_k to satisfy the second part of (D).

According to Lemma 3.3.1 we can choose $(\epsilon_1)_j^i(k)$ so that if $q_j^i < (\epsilon_1)_j^i(k)$ for all i, j , then $\mu(T_{S_k}^{m_k} B_k \Delta Q^{m_k} B_k) < \delta_k$, and also $\mu(T_{S_k} B_k \Delta Q B_k) < 1/k$. Hence if $\epsilon_j^i(k) \leq (\epsilon_1)_j^i(k)$, then (E) and (M) are also satisfied.

(F) can be viewed as the definition of η_k .

Since we already know (L), Lemma 2.0.17 guarantees the existence of an m such that either $\mu(T_{S_k, j_k}^m(E_k) \cap F_k) > \frac{1}{2} \eta_k^2 \mu(\rho_{S_k, j_k}(F_k))$, or $\mu(\rho_{S_k, j_k}(T_{S_k, j_k}^m(E_k)) \cap \rho_{S_k, j_k}(F_k)) <$

$(1 - \eta_k)\mu(\rho_{S_k, j_k}(F_k))$. Indeed, the first inequality for some m is the conclusion of (RE). If that does not hold for any m , then (RE1-4) are contradicting each other. Since in (F) we assumed that η_k is such that (RE1-3) hold, this means that (RE4) should be false. But this is the same as for some m the second inequality holds. This shows that we can find n_k and ν_k to satisfy (G), or s_k and κ_k to satisfy (I).

According to Lemma 3.3.2 we can choose $(\epsilon_2)_j^i(k)$ so that if $q_j^i < (\epsilon_2)_j^i(k)$ for all i, j and if we defined n_k in (G), then $\mu((T_{S_k, j_k}^{n_k})^{n_k}(E_k) \Delta (T_{j_k}^{n_k})^{n_k}(E_k)) < \nu_k$. Also we can choose $(\epsilon_3)_j^i(k)$ so that if $q_j^i < (\epsilon_3)_j^i(k)$ for all i, j and if we defined s_k in (I), then $\mu((T_{S_k, j_k}^{s_k})^{s_k}(E_k) \Delta (T_{j_k}^{s_k})^{s_k}(E_k)) < \kappa_k$. Hence if $\epsilon_j^i(k)$ is not greater than $(\epsilon_2)_j^i(k)$ and $(\epsilon_3)_j^i(k)$, then (H) and (J) are satisfied.

According to these, $\epsilon_j^i(k) = \min\{(\epsilon_1)_j^i(k), (\epsilon_2)_j^i(k), (\epsilon_3)_j^i(k), \epsilon_j^i(k-1)\}$ will work, and also satisfies (K).

Hence we defined t_j^i for all $i < j$, so we can look at the transformation $T : X \rightarrow X$ defined by $T(\vec{x})_0 = x_0 \oplus \alpha$ and $T(\vec{x})_j = x_j \oplus \sum_{i < j} t_j^i x_i$.

We claim that this T satisfies the conclusion of the theorem.

First we show that it is measure preserving. It is enough to show that for every $B \in \mathcal{A}$, $\mu(TB) = \mu(B)$. For the k 's such that $B = B_k$, by (M) $\mu(T_{S_k}(B) \Delta TB) < 1/k$. Since T_{S_k} is measure preserving and $B = B_k$ for infinitely many k , this proves that $\mu(B) = \mu(TB)$.

Next we prove that it is ergodic. Indeed, let $(B, C) \in \mathcal{A} \times \mathcal{A}$. We have to show that there is an m such that $\mu(T^m B \cap C) > \frac{1}{2}\mu(B)\mu(C)$. But $(B, C) = (B_k, C_k)$ for some k . Then by (D) and (E), $\mu(T_{S_k}^{m_k} B \cap C) - \frac{1}{2}\mu(B)\mu(C) = \delta_k > 0$, $\mu(T_{S_k}^{m_k} B \Delta T^{m_k} B) < \delta_k$,

hence $\mu(T^{m_k} B \cap C) > \frac{1}{2}\mu(B)\mu(C)$.

To show that for every j , Y_{j+} is the maximal compact extension of Y_j in X , we have to show that for any $(E, F) \in \mathcal{A}_j \times \mathcal{A}_j$ whenever the hypothesis of (RE) in Lemma 2.0.17 are satisfied, then the conclusion also holds. We do this by showing that either the conclusion of (RE) holds, or (RE1-4) are contradicting to each other. The pair (E, F) is (E_k, F_k) for infinitely many k . Choose a k which is bigger than j and j^+ . Then in the k 'th step of our construction, if (i-iii) of (F) were satisfied for $T_{S_k j_k}$, then there are two cases. Either by (G) and (H) $\mu(T_{S_k j_k}^{m_k}(E_k) \cap F_k) - \frac{1}{2}\eta_k^2\mu(\rho_{S_k j_k}(F_k)) = \nu_k > 0$, $\mu(T_{S_k j_k}^{m_k}(E_k) \triangle T_{j_k}^{m_k}(E_k)) < \nu_k$, hence $\mu(T_{j_k}^{m_k}(E_k) \cap F_k) > \frac{1}{2}\eta_k^2\mu(\rho_{j_k}(F_k))$. Or if there is no m such that $\mu(T_{S_k j_k}^m(E_k) \cap F_k) > \frac{1}{2}\eta_k^2\mu(\rho_{S_k j_k}(F_k))$, then by (I) and (J), $\mu(\rho_{j_k}(T_{j_k}^{m_k}(E_k)) \cap \rho_{j_k}(F_k)) < (1 - \eta_k)\mu(\rho_{j_k}(F_k))$. Since this is true for the largest possible η_k with which (RE1-3) can be satisfied, (RE4) cannot hold with parameters satisfying (RE1-3). This completes the proof that (RE) holds for the transformation T . □

CHAPTER IV

The complexity of the collection of measure distal transformations

4.1 Lower bound

Theorem 4.1.1 *Suppose $(X, \mathcal{B}, \mu, T)$ is ergodic and suppose that $\{\mathcal{B}_\beta : \beta \leq \eta\}$ are sub- σ -algebras of $\mathcal{B}$ such that $\mathcal{B}_0$ is the trivial algebra, $\mathcal{B}_\eta = \mathcal{B}$, $(X, \mathcal{B}_{\beta+1}, \mu, T)$ is the maximal compact extension of $(X, \mathcal{B}_\beta, \mu, T)$ in $(X, \mathcal{B}, \mu, T)$, and for limit γ , $\mathcal{B}_\gamma$ is the σ -algebra generated by $\bigcup\{\mathcal{B}_\beta : \beta < \gamma\}$.*

Also let I be an acceptable linear ordering. Let $\{\mathcal{C}_i : i \in I\}$ be sub- σ -algebras of $\mathcal{B}$ such that $\mathcal{C}_0$ is the trivial algebra, $\mathcal{C}_1 = \mathcal{B}$, $(X, \mathcal{C}_{j+}, \mu, T)$ is the maximal compact extension of $(X, \mathcal{C}_j, \mu, T)$ in $(X, \mathcal{B}, \mu, T)$, and for limit j , $\mathcal{C}_j$ is the σ -algebra generated by $\bigcup\{\mathcal{C}_i : i <_I j\}$.

Then I is well-ordered in order-type $\eta + 1$, and if j_β denotes the β 'th element of I , then $\mathcal{B}_\beta = \mathcal{C}_{j_\beta}$.

Proof. Let J be the maximal well-ordered initial segment of I . Note that ω is an initial segment of I . Let θ be the order-type of J . Then since both the maximal compact extension and the inverse limit are unique, if we denote the β 'th element of J by j_β , then $\mathcal{B}_\beta = \mathcal{C}_{j_\beta}$. If $I \setminus J$ is nonempty, then it does not have a smallest element.

Indeed, if j were the smallest element of $I \setminus J$, then $J \cup \{j\}$ would be a wellordered initial segment of I , contradicting the maximality of J .

By transfinite induction on $\alpha \leq \eta$ we show that for every $i \in I \setminus J$, $\mathcal{B}_\alpha$ is a sub- σ -algebra of $\mathcal{C}_i$. This is clearly true for $\alpha < 0$. Also if it is true below a limit ordinal γ , then by the definition of the inverse limit, it is true for γ . Let us now suppose that it is true for β and let $\alpha = \beta + 1$. Fix $i \in I \setminus J$ and choose $j \in I \setminus J$ such that $j <_I i$ and $j^+ <_I i$. (Such a j exists since $I \setminus J$ has no minimal element.) Since $L^2(\mathcal{B}_\beta) \subseteq L^2(\mathcal{C}_j)$, every function of $L^2(\mathcal{B}_\alpha)$ is also compact over $(X, \mathcal{C}_j, \mu, T)$. Hence since $\mathcal{C}_{j^+}$ gives the maximal compact extension, $\mathcal{B}_\alpha \subseteq \mathcal{C}_{j^+} \subseteq \mathcal{C}_i$, which proves what we wanted.

But then $\mathcal{B} \subseteq \mathcal{C}_i$ for every $i \in I \setminus J$, which is a contradiction. $\square$

Theorem 4.1.2 *The transformation $T : X \rightarrow X$ constructed in Theorem 3.3.3 for the acceptable linear ordering I is measure distal iff I is a well-ordering.*

Proof. If I is a well-ordering, then the corresponding transformation is clearly measure distal. The converse is the consequence of Theorem 4.1.1. $\square$

Let us now introduce the coding of linear orderings of natural numbers in the Polish space $2^{\mathbb{N}^2}$. Every $f \in 2^{\mathbb{N}^2}$ codes a binary relation on $\mathbb{N}$: $R_f = \{\langle n, m \rangle : f(\langle n, m \rangle) = 1\}$.

Proposition 4.1.3 $LO = \{f : R_f \text{ is a linear ordering}\}$, and

$ALO = \{f : R_f \text{ is an acceptable linear ordering}\}$ are Borel sets in $2^{\mathbb{N}^2}$.

Proof. f codes a linear ordering iff for all $n \neq m$, ($f(\langle n, m \rangle) = 1$ iff $f(\langle m, n \rangle) = 0$), and for all n , $f(\langle n, n \rangle) = 1$, and for all n, m, k , ($f(\langle n, m \rangle) = 1$ and $f(\langle m, k \rangle) = 1$ implies $f(\langle n, k \rangle) = 1$). This gives a Borel definition of LO.

f codes an acceptable linear ordering iff it codes a linear ordering, and in addition for all n , ($f(\langle 0, n \rangle) = 1$ and $f(\langle n, 1 \rangle) = 1$), and for all $n \neq 1$ there exists $m \neq n$ such that $f(\langle n, m \rangle) = 1$ and for all k , $f(\langle n, k \rangle) = 1$ and $f(\langle k, m \rangle) = 1$ implies $k = n$ or $k = m$. This gives a Borel definition of ALO. $\square$

Proposition 4.1.4 *$WO = \{f : R_f \text{ is a well-ordering}\}$ is a complete coanalytic set.*

Proof. See e.g. [7] or [8]. $\square$

Theorem 4.1.5 *The set of measure distal transformations is a Π_1^1 -hard set.*

Proof. In Theorem 3.3.3 we proved that for every acceptable ordering I there is an ergodic transformation T on $X = \prod_{i \in I} \mathbb{T}$ such that for every $j \in I$, Y_{j+} is the maximal compact extension of Y_j in X . We also proved in Theorem 4.1.2 that this transformation is measure distal if and only if I is a wellordering. We use the construction of the proof of Theorem 3.3.3 to give a continuous function $F : ALO \rightarrow MPT$. This is enough to prove the claim. Indeed, since ALO is a Borel subset of $2^{\mathbb{N}^2}$, we can extend F to a Borel map $G : 2^{\mathbb{N}^2} \rightarrow MPT$ by assigning a fixed weakly mixing (and hence not measure distal) transformation to each $f \notin ALO$. According to Theorem 4.1.2 this G reduces the set of well-orderings to the set of measure distal transformations, which is what we need.

Let us look at the following tree. On the k 'th level of the tree we put the permutations of $\{0, \dots, k\}$. The ordering is defined by $\langle n_0, \dots, n_s \rangle \leq \langle m_0, \dots, m_t \rangle$ if and only if there is an increasing $f : (0, \dots, s) \rightarrow (0, \dots, t)$ such that for all $i \leq s$, $n_i = m_{f(i)}$.

We assign an ergodic transformation on a finite dimensional torus to each node of the tree. This transformation has the usual form: $(T_{\vec{s}}(\vec{x}))_0 = x_0 \oplus \alpha$, $(T_{\vec{s}}(\vec{x}))_j = x_j \oplus \sum t_j^i(\vec{s})x_i$, where the summation ranges over the i 's preceeding j in $\vec{s}$. By induction (similar to the one we used in the proof of Theorem 3.3.3) we can show that for every node $\vec{s}$ of the tree we can choose $t_j^i(\vec{s})$ if i preceeds j in $\vec{s}$ and $\epsilon_j^i(\vec{s})$ for all i, j to satisfy the following conditions

- (A) Every $t_j^i(\vec{s})$ is a non-zero rational,
- (B) If i preceeds j in $\vec{p}$ and $\vec{p} \leq \vec{s}$ then $t_j^i(\vec{s}) < \epsilon_j^i(\vec{p})$,
- (D) $m_{|\vec{s}|}$ is the smallest m such that $\mu(T_{\vec{s}}^m B_{|\vec{s}|} \cap C_{|\vec{s}|}) > \frac{1}{2}\mu(B_{|\vec{s}|})\mu(C_{|\vec{s}|})$, and $\mu(T_{\vec{s}}^m B_{|\vec{s}|} \cap C_{|\vec{s}|}) - \frac{1}{2}\mu(B_{|\vec{s}|})\mu(C_{|\vec{s}|}) = \delta_{|\vec{s}|}$,
- (E) If $q_j^i = t_j^i(\vec{s})$ for all $i, j \leq k$, $q_j^i < \epsilon_j^i(\vec{s})$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i < j} q_j^i x_i$, then $\mu(T_{\vec{s}}^{m_{|\vec{s}|}} B_{|\vec{s}|} \Delta Q^{m_{|\vec{s}|}} B_{|\vec{s}|}) < \delta_{|\vec{s}|}$,
- (F) $\eta_{|\vec{s}|}$ is the largest η such that there are $E'_{|\vec{s}|}, F'_{|\vec{s}|} \subset W_{S_{|\vec{s}|}, j_{|\vec{s}|}}$ such that
 - (i) $\mu(E'_{|\vec{s}|}), \mu(F'_{|\vec{s}|}) \geq (1 - \eta)\mu(\rho_{\vec{s}, j_{|\vec{s}|}}(F_{|\vec{s}|}))$,
 - (ii) for every $w \in E'_{|\vec{s}|}$, $\mu(\rho_{\vec{s}, j_{|\vec{s}|}}^{-1}(w) \cap E_{|\vec{s}|}) \geq \eta$, and
 - (iii) for every $w \in F'_{|\vec{s}|}$, $\mu(\rho_{\vec{s}, j_{|\vec{s}|}}^{-1}(w) \cap F_{|\vec{s}|}) \geq \eta$.
- (G) If there is an m such that $\mu(T_{\vec{s}, j_{|\vec{s}|}}^m(E_{|\vec{s}|}) \cap F_{|\vec{s}|}) > \frac{1}{2}\eta_{|\vec{s}|}^2\mu(\rho_{\vec{s}, j_{|\vec{s}|}}(F_{|\vec{s}|}))$, then $n_{|\vec{s}|}$ is the smallest such m , and $\nu_{|\vec{s}|}$ is the difference.
- (H) If $q_j^i = t_j^i(\vec{s})$ for all $i, j \leq k$, $q_j^i < \epsilon_j^i(\vec{s})$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i < j} q_j^i x_i$ and if we defined $n_{|\vec{s}|}$ in (G), then $\mu((T_{\vec{s}, j_{|\vec{s}|}}^*)^{n_{|\vec{s}|}}(E_{|\vec{s}|}) \Delta (Q_{j_{|\vec{s}|}}^*)^{n_{|\vec{s}|}}(E_{|\vec{s}|})) < \nu_{|\vec{s}|}$.

(I) If there is an m such that

$$\mu(\rho_{\vec{s},j|\eta}(T_{\vec{s},j|\eta}^m(E|\eta)) \cap \rho_{\vec{s},j|\eta}(F|\eta)) < (1 - \eta|\eta)\mu(\rho_{\vec{s},j|\eta}(F|\eta)), \quad (4.1)$$

then $s|\eta$ is the smallest such m , and $\kappa|\eta$ is the difference.

(J) If $q_j^i = t_j^i(\vec{s})$ for all $i, j \leq k$, $q_j^i < c_j^i(\vec{s})$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$ and if we defined $s|\eta$ in (I), then $\mu((T_{\vec{s},j|\eta}^*)^{s|\eta}(E|\eta) \Delta (Q_{j|\eta}^*)^{s|\eta}(E|\eta)) < \kappa|\eta$.

(K) If $\vec{p} \leq \vec{s}$ then $c_j^i(\vec{s}) \leq c_j^i(\vec{p})$

(L) $T_{\vec{s}}$ is ergodic and for every $j \in \vec{s}$ if we denote the successor of j in $\vec{s}$ by j^+ , then $Y_{\vec{s},j^+}$ is the maximal compact extension of $Y_{\vec{s},j}$ in $X_{\vec{s}}$.

(M) If $q_j^i = t_j^i(\vec{s})$ for all $i, j \leq k$, $q_j^i < c_j^i(\vec{s})$ otherwise and $Q : X \rightarrow X$ is the transformation defined by $Q(\vec{x})_0 = x_0 \oplus \alpha$ and $Q(\vec{x})_j = x_j \oplus \sum_{i <_I j} q_j^i x_i$, then for every $k \in \vec{s}$ $\mu(T_{\vec{s}}(B_k) \Delta T(B_k)) < 1/|\vec{s}|$.

(N) If $\vec{s} \leq \vec{p}$ and i preceeds j in $\vec{s}$ (and hence in $\vec{p}$, too) then $t_j^i(\vec{s}) = t_j^i(\vec{p})$.

The only difference here compared to the proof of Theorem 3.3.3 is that we added the extra conditions (M) and (N). Like (H) and (J), (M) is also a condition for an upper bound for the $c_j^i(\vec{s})$'s. Using Lemma 3.3.1 it can be satisfied similar to (H). (N) is trivially satisfied in the proof of Theorem 3.3.3, since we only choose the new t_j^i 's.

Given this tree and the assigned values of $t_j^i(\vec{s})$, let us now define the function $F : ALO \rightarrow MPT$. Any acceptable linear ordering I gives a branch through our tree, namely the collection of the $\vec{s}$'s, where i preceeds j in $\vec{s}$ if and only if $i <_I j$. For $i <_I j$ let us then define $t_j^i = t_j^i(\vec{s})$ for any $\vec{s}$ that contains both i and j . Because of (N) this is well defined. We thus have the transformation $T : X \rightarrow X$ defined as usual

by $T(\vec{x})_0 = x_0 \oplus \alpha$ and $T(\vec{x})_j = x_j \oplus \sum_{i < j} t_j^i x_i$. Hence we defined $F : ALO \rightarrow MPT$. All we need to show that this map is continuous on ALO. It is enough to show that for any $f \in 2^{\mathbb{N}^2} \cap ALO$, for any $\epsilon > 0$ and for any $k > 0$, there is an N such that for any $g \in 2^{\mathbb{N}^2} \cap ALO$ such that for every $n, m \leq N$, $f(\langle n, m \rangle) = g(\langle n, m \rangle)$, $\mu(T_f B_i \Delta T_g B_i) < \epsilon$ for every $i \leq k$.

Fix k and ϵ , and let N be bigger than k and $2/\epsilon$. We show that this N works. Let g be such that $f(\langle n, m \rangle) = g(\langle n, m \rangle)$ for all $n, m < N$. This means that for the orderings coded by f and g , $n <_f m$ if and only if $n <_g m$ for all $n, m < N$. Hence the branches corresponding to the orderings coded by f and g coincide until the N 'th level of our tree. By condition (M) of the construction, $\mu(T_f B_i \Delta T_{f,s} B_i) < 1/N < 1/2\epsilon$, $\mu(T_g B_i \Delta T_{g,s} B_i) < 1/N < 1/2\epsilon$ for every $i \leq k$. So $\mu(T_f B_i \Delta T_g B_i) < \epsilon$. This completes the proof. $\square$

4.2 Upper bound

In this section we give a Π_1^1 definition of measure distality, and hence show

Theorem 4.2.1 *The set of measure distal transformations form a complete coanalytic set in the space of measure preserving transformations.*

Lemma 4.2.2 *The system $(X, \mathcal{B}, \mu, T)$ is measure distal iff for every $\mathcal{D}$ invariant σ -subalgebra of $\mathcal{B}$, which is not equivalent to $\mathcal{B}$, there is a non- $\mathcal{D}$ -measurable relatively compact function over $\mathcal{D}$ in $L^2(\mathcal{B})$.*

Proof. If X is not measure distal, then there is a Y proper factor such that X is a weakly mixing extension of Y . Then $L^2(Y) \subset L^2(X)$. Let $\mathcal{D}$ be the smallest

σ -algebra that measures all functions of $L^2(Y)$. It is clearly invariant and a proper σ -subalgebra of $\mathcal{B}$. Since X is a weakly mixing extension of Y , there are no relatively compact functions, other than those in $L^2(Y)$, which are $\mathcal{D}$ -measurable.

For the other direction suppose that X is measure distal. Let $\{X_\xi : \xi \leq \eta\}$ be the compact tower reaching X . Let $\mathcal{B}_\xi \subset \mathcal{B}$ be the σ -algebra corresponding to X_ξ . Let ξ be the smallest ordinal such that $\mathcal{D} \not\supset \mathcal{B}_\xi$. Such exists, otherwise $\mathcal{B} = \mathcal{D}$. This ξ is a successor ordinal by the definition of the compact tower at limit stages. Let $\xi = \nu + 1$. Then since X_ξ is a compact extension of X_ν , every function in $L^2(\mathcal{B}_\xi)$ is compact over $\mathcal{B}_\nu$, and hence over $\mathcal{D}$. Since $\mathcal{D} \not\supset \mathcal{B}_\xi$, not every $f \in L^2(X_\xi)$ is $\mathcal{D}$ -measurable. This finishes the proof. $\square$

In the Lemma 4.2.4 we will need the following well-known fact

Proposition 4.2.3 *Let X be a measure space and $\mathcal{H} \subset L^2(X)$. Suppose that $\mathcal{H}$ is a closed subspace which is also closed under the operation $f \mapsto \min(f, 1)$. Also suppose that if $f, g \in \mathcal{H}$ are such that $fg \in L^2(X)$ then $fg \in \mathcal{H}$. Let $\mathcal{D}$ be the smallest σ -algebra that measures all functions from $\mathcal{H}$. Then $\mathcal{H} = L^2(\mathcal{D})$.*

Proof. $\mathcal{H} \subset L^2(\mathcal{D})$ is obvious. For the other direction let $f \in L^2(\mathcal{D})$. We need to show that $f \in \mathcal{H}$. For any rational p let $D_p = \{x : f(x) < p\}$. It is enough to show that the characteristic functions χ_{D_p} are in $\mathcal{H}$. (Indeed, we can approximate f by finite linear combinations of these, and we assumed that $\mathcal{H}$ is closed.) The set D_p is in the smallest σ -algebra that measures every function in $\mathcal{H}$. I.e. there are countably many sets $R_1, \dots, R_n, \dots$ such that $R_i = \{x : f_i < \alpha_i\}$ for some $f_i \in \mathcal{H}$ and $\alpha_i \in \mathbb{R}$, and we can get D_p in countably many steps from the R_n 's

using countable intersections, unions and taking complements. If $\chi_A, \chi_B \in \mathcal{H}$ then $\chi_{A \cap B} = \chi_A \chi_B \in \mathcal{H}$. If $\chi_{A_n} \in \mathcal{H}$ for all $n \in \mathbb{N}$ then $\chi_{\cap_{n \in \mathbb{N}} A_n} \in \mathcal{H}$ because $\mathcal{H}$ is closed. If $\chi_A \in \mathcal{H}$ and B is the complement of A then clearly $\chi_B \in \mathcal{H}$. From these if $\chi_{A_n} \in \mathcal{H}$ for all $n \in \mathbb{N}$ then $\chi_{\cup_{n \in \mathbb{N}} A_n} \in \mathcal{H}$. But using the closure of $\mathcal{H}$ under $f \mapsto \min(f, 1)$, and that $\mathcal{H}$ is a subspace, $\chi_{R_n} \in \mathcal{H}$ for all n . These imply that $\chi_{D_p} \in \mathcal{H}$, which completes the proof. $\square$

Lemma 4.2.4 *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Let $\mathcal{H} \subset L^2(X)$ be a closed proper invariant subspace, which is closed under the operation $f \mapsto \min(f, 1)$. Also suppose that if $f, g \in \mathcal{H}$ are such that $fg \in L^2(X)$ then $fg \in \mathcal{H}$. Let $\mathbf{F} = \{f_n : n \in \mathbb{N}\}$ be an enumeration of a dense set of $L^2(X)$, consisting of bounded functions. Then the following are equivalent.*

(1) *There is an $f \in L^2(X)$ which is not in $\mathcal{H}$, but relatively compact over $\mathcal{H}$.*

(2) *There is a bounded invariant $H \in L^2(X \times_{\mathcal{H}} X)$ and a bounded $\phi \in L^2(X)$ such that $H * \phi \notin \mathcal{H}$.*

(3) *There are bounded $f \in L^2(X \times_{\mathcal{H}} X)$ and $\phi \in L^2(X)$ such that*

$$(w \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N T^m f) * \phi \notin \mathcal{H} \quad (4.2)$$

(4) *There are $g_1, \dots, g_n, h_1, \dots, h_n, \phi \in \mathbf{F}$ and $a_1, \dots, a_n \in \mathbb{R}$, such that*

$$(w \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N T^m (\sum_{i=1}^n a_i g_i \otimes h_i)) * \phi \notin \mathcal{H} \quad (4.3)$$

(5) There are $g, h, \phi \in \mathbb{F}$, such that

$$(w \lim \frac{1}{N+1} \sum_{m=0}^N T^m(g \otimes h)) * \phi \notin \mathcal{H} \quad (4.4)$$

Proof. (5) $\rightarrow$ (4) $\rightarrow$ (3), and (2) $\rightarrow$ (1) are trivial.

(1) $\rightarrow$ (2): The functions of the form $H * \phi$ span the space of relatively compact functions as H ranges over the invariant $L^\infty(X \times_{\mathcal{H}} X)$ functions, and ϕ ranges over $L^\infty(X)$. So if (1) holds, then it cannot be true that all of these are in $\mathcal{H}$, which proves (2).

(2) $\leftrightarrow$ (3): This equivalence follows, since every invariant H is of the form $w \lim \frac{1}{N+1} \sum_{m=0}^N T^m f$, (e.g. with $f = H$), and every such weak limit is invariant.

(3) $\rightarrow$ (4): Let $f_N = \frac{1}{N+1} \sum_{m=0}^N T^m f$. Let $\delta > 0$ be the distance of $(w \lim f_N) * \phi$ and $\mathcal{H}$. Fix $\epsilon > 0$ so that $\epsilon(\|w \lim f_N\|_\infty + \|\phi\|_\infty + \epsilon) < \delta$. Let $g \in L^2(X \times_{\mathcal{H}} X)$ and $\psi \in L^2(X)$ be such that $\|f - g\|_2 < \epsilon$ and $\|\psi - \phi\|_\infty < \epsilon$. Let $g_N = \frac{1}{N+1} \sum_{m=0}^N T^m g$. Then for all N , $\|g_N - f_N\|_2 < \epsilon$. Hence $\|w \lim f_N - w \lim g_N\|_2 < \epsilon$. But then $\|(w \lim f_N) * \phi - (w \lim g_N) * \psi\|_2 \leq \|(w \lim f_N) * (\phi - \psi)\|_2 + \|(w \lim f_N - w \lim g_N) * \psi\|_2 < \epsilon(\|w \lim f_N\|_\infty + \|\psi\|_\infty) < \epsilon(\|w \lim f_N\|_\infty + \|\phi\|_\infty + \epsilon) < \delta$. This means that for any such g and ψ , $(w \lim g_N) * \psi \notin \mathcal{H}$. But the finite linear combinations of functions of the form $f_i \otimes f_j$ are dense in $L^2(X \times_{\mathcal{H}} X)$, this proves (4).

(4) $\rightarrow$ (5): This follows from the fact that $\mathcal{H}$ is closed under linear combinations, and that

$$\{w \lim \frac{1}{N+1} \sum_{m=0}^N T^m (\sum_{i=0}^n a_i g_i \otimes h_i)\} * \phi = \sum_{i=1}^n a_i [w \lim \frac{1}{N+1} \sum_{m=0}^N T^m (g_i \otimes h_i)] * \phi. \quad (4.5)$$

□

Combining these two lemmas, we get the following description of a measure distal transformation. Let $\mathbf{F} = \{f_i : i \in \mathbb{N}\}$ be an enumeration of a dense subset of $L^2(\mathbf{T})$, consisting of bounded functions.

Then T is measure distal iff for every $\mathcal{H}$ proper T -invariant closed subspace of $L^2(\mathbf{T})$, which is closed under multiplication and the operation $f \mapsto \min(f, 1)$, there are $g, h, f \in \mathbf{F}$ such that $w \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N T^m(g \otimes h) * f \notin \mathcal{H}$.

With the discrete topology on $\mathbf{F}$, $\mathbf{F} = \mathbf{F}^{\mathbb{N}^2}$ is a Polish space. Then $\mathbf{F} \times MPT$ is also a Polish space. Let's look at the following sets of this space.

MD1 = $\{(\vec{h}, T) \in \mathbf{F} \times MPT : \text{for all } i \in \mathbb{N}, \{h_{i,j} : j \in \mathbb{N}\} \text{ is a Cauchy sequence}\}$

Let $h_i = \lim_j h_{i,j}$.

MD2 = $\{(\vec{h}, T) \in \text{MD1} : \text{for all } i \in \mathbb{N} \text{ there is a } j \in \mathbb{N} \text{ such that } Th_i = h_j\}$

MD3 = $\{(\vec{h}, T) \in \text{MD1} : \text{for all } i \in \mathbb{N} \text{ there is a } j \in \mathbb{N} \text{ such that } \min(h_i, 1) = h_j\}$

MD4 = $\{(\vec{h}, T) \in \text{MD1} : \text{for all } i, j \in \mathbb{N} \text{ there is a } k \in \mathbb{N} \text{ such that } h_i h_j = h_k\}$

MD5 = $\{(\vec{h}, T) \in \text{MD1} : \text{there are } l, q \in \mathbb{N} \text{ such that for all } i \in \mathbb{N}, \|h_i - f_l\|_2 > 1/q\}$

MD6 = $\{(\vec{h}, T) \in \text{MD1} : \text{there are } i, j, k, p \in \mathbb{N} \text{ such that for all } l \in \mathbb{N},$

$\|(\omega \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N T^m(f_i \otimes f_j)) * f_k - h_l\|_2 > 1/p\}$

Every closed invariant $\mathcal{H}$ can be given by an invariant countable dense set. Every element of this dense set is the limit of a Cauchy-sequence of f_i 's from $\mathbf{F}$. Hence T is measure distal iff for every $\vec{h} \in \mathbf{F}$, if $(\vec{h}, T) \in \text{MD1} \cap \text{MD2} \cap \text{MD3} \cap \text{MD4} \cap \text{MD5}$, then $(\vec{h}, T) \in \text{MD6}$. To show that this describes a coanalytic set, it is enough to show that MD1-6 are Borel sets.

Let $MD1i = \{(\vec{h}, T) \in F \times \text{MPT} : \{h_{i,j} : j \in \mathbb{N}\} \text{ is a Cauchy sequence}\}$. Then $MD1 = \bigcap_{i \in \mathbb{N}} MD1i$, hence MD1 is Borel if all the MD1i's are Borel. $MD1i = \{(\vec{h}, T) : \forall n \exists m \forall j, k > m, \|h_{i,j} - h_{i,k}\|_2 < 1/n\} = \bigcap_n \bigcup_m \bigcap_{j,k > m} \{(\vec{h}, T) : \|h_{i,j} - h_{i,k}\|_2 < 1/n\}$. Since $\{(\vec{h}, T) : \|h_{i,j} - h_{i,k}\|_2 < 1/n\}$ is open, MD1i and hence MD1 is Borel.

$MD2 = \{(\vec{h}, T) \in MD1 : \forall i \exists j \forall n \exists m \forall k, l > m \|Th_{i,k} - h_{j,l}\|_2 < 1/n\}$. To show that this is a Borel set, suppose that $\|Th_{i,k} - h_{j,l}\|_2 < 1/n$, and let δ be the difference. Choose $\sum_{p=0}^N a_p \chi_{O_p} = h$ such that $\|h - h_{i,k}\|_2 < \delta/3$, where the O_p 's are basic open sets. Then clearly there is an $\epsilon > 0$ such that if $d(S, T) < \epsilon$, then $\|Sh - Th\|_2 < \delta/3$. Hence $\|Sh_{i,k} - h_{j,l}\|_2 \leq \|Sh_{i,k} - Sh\|_2 + \|Sh - Th\|_2 + \|Th - Th_{i,k}\|_2 + \|Th_{i,k} - h_{j,l}\|_2 < \delta + (1/n - \delta) = 1/n$. This shows that $MD2' = \{(\vec{h}, T) \in MD1 : \|Th_{i,k} - h_{j,l}\|_2 < 1/n\}$ is open. Indeed, if $(\vec{h}, T) \in MD2'$, then $U = \{(\vec{h}', S) : h'_{i,k} = h_{i,k}, h'_{j,l} = h_{j,l}, d(S, T) < \epsilon\}$ is an open neighborhood of $(\vec{h}, T)$ inside MD2'. Hence MD2 is Borel.

$MD3 = \{(\vec{h}, T) \in MD1 : \forall i \exists j \forall n \exists m \forall k, l > m, \|\min(h_{i,k}, 1) - h_{j,l}\|_2 < 1/n\}$. Since $\{(\vec{h}, T) : \|\min(h_{i,k}, 1) - h_{j,l}\|_2 < 1/n\}$ is open, MD3 is Borel.

$MD4 = \{(\vec{h}, T) \in MD1 : \forall i, j \exists k \forall n \exists m \forall a, b, c > m, \|h_{i,a}h_{j,b} - h_{k,c}\|_2 < 1/n\}$. Since $\{(\vec{h}, T) : \|h_{i,a}h_{j,b} - h_{k,c}\|_2 < 1/n\}$ is open, MD4 is Borel.

$MD5 = \{(\vec{h}, T) \in MD1 : \exists l, p \forall i \exists m \forall j > m, \|f_l - h_{i,j}\|_2 > 1/p\}$. Since $\{(\vec{h}, T) : \|f_l - h_{i,j}\|_2 > 1/p\}$ is open, MD5 is Borel.

For MD6 it is enough to show that for fixed i, j, k, l, p, q , $MD6' = \{(\vec{h}, T) \in MD1 : \|[\omega \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N T^m(f_i \otimes f_j)] * f_k - h_{l,q}\|_2^2 > 1/p^2\}$ is Borel. Let $G_m^T = T^m(f_i \otimes f_j)$, and $G^T = \omega \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N G_m^T$. Then

$$\|G^T * f_k - h_{l,q}\|_2^2 = \|G^T * f_k\|_2^2 - \langle G^T * f_k, h_{l,q} \rangle - \langle h_{l,q}, G^T * f \rangle + \|h_{l,q}\|_2^2. \quad (4.6)$$

Now look at

$$\langle G^T * f_k, h_{l,q} \rangle = \int (G^T * f_k)(x) \overline{h_{l,q}(x)} d\mu(x) = \quad (4.7)$$

$$\int [\int (w \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M G_m^T(x, x')) f_k(x') d\mu_{\alpha(x)}(x')] \overline{h_{l,q}(x)} d\mu(x) = \quad (4.8)$$

$$\int (w \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M G_m^T(x, x')) f_k(x') \overline{h_{l,q}(x)} d\mu_{\mathbf{T} \times_{\mathcal{H}} \mathbf{T}} = \quad (4.9)$$

$$\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M \int (G_m^T(x, x') f_k(x') \overline{h_{l,q}(x)} d\mu_{\mathbf{T} \times_{\mathcal{H}} \mathbf{T}}, \quad (4.10)$$

where $\mathcal{H}$ is the closed subspace of $L^2(\mathbf{T})$ which is spanned by the h_i 's. Similarly

$$\langle h_{l,q}, G^T * f_k \rangle = \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M \int h_{l,q}(x) \overline{G_m^T(x, x') f_k(x')} d\mu_{\mathbf{T} \times_{\mathcal{H}} \mathbf{T}}. \quad (4.11)$$

Also similarly

$$\langle G^T * f_k, G^T * f_k \rangle = \int (G^T * f_k)(x) \overline{G^T * f_k(x)} d\mu(x) = \quad (4.12)$$

$$\int [\int (w \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M G_m^T(x, x')) f_k(x') d\mu_{\alpha(x)}(x')] \overline{G^T * f_k(x)} d\mu(x) = \quad (4.13)$$

$$\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M \int (\int G_m^T(x, x') f_k(x') d\mu_{\alpha(x)}(x')) \overline{G^T * f_k(x)} d\mu(x) = \quad (4.14)$$

$$\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \int [\int G_m^T(x, x') f_k(x') d\mu_{\alpha(x)}(x')] \quad (4.15)$$

$$\overline{[\int G_n^T(x, x') f_k(x') d\mu_{\alpha(x)}(x')] d\mu(x)}. \quad (4.16)$$

Let $U_m^T = \int G_m^T(x, x') f_k(x') \overline{h_{l,q}(x)} d\mu_{\mathbf{T} \times_{\mathcal{H}} \mathbf{T}}$, and $W_m^T(x) = \int G_m^T(x, x') f_k(x') d\mu_{\alpha(x)}(x')$.

Let

$$P^T = [\lim_{M, N \rightarrow \infty} \frac{1}{M+1} \frac{1}{N+1} \sum_{m=0}^M \sum_{n=0}^N \int W_m^T(x) \overline{W_n^T(x)} d\mu(x)] + \quad (4.17)$$

$$\|h_{l,q}\|_2^2 - \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M (U_m^T + \overline{U_m^T}) \quad (4.18)$$

Then

$$\text{MD6}' = \{(\tilde{h}, T) \in \text{MD1} : P^T > 1/p^2\}. \quad (4.19)$$

Suppose that $(\tilde{h}, T) \in \text{MD6}'$. We'll show that there is an $\epsilon > 0$ such that

$$\{(\tilde{h}', S) : h'_{l,q} = h_{l,q}, d(S, T) < \epsilon\} \quad (4.20)$$

is an open neighborhood of $(\tilde{h}, T)$ in $\text{MD6}'$. This will show that $\text{MD6}'$ is open and hence MD6 is Borel. Let $\delta = P^T - 1/p^2$. From the definitions

$$U_m^T = \int G_m^T(x, x') f_k(x') \overline{h_{l,q}(x)} d\mu_{T \times T}(x, x'), \quad (4.21)$$

and

$$G_m^T(x, x') = T^m(f_i \otimes f_j) \quad (4.22)$$

it is clear that there is an $\epsilon_1 > 0$ such that if $d(S, T) < \epsilon_1$ then $\|G_m^T - G_m^S\|_1 < \delta/2 \|f_k h_{l,q}\|_\infty$. For such an S , $|U_m^T - U_m^S| < \delta/2$.

Also recall the definition $W_m^T(x) = \int G_m^T(x, x') f_k(x') d\mu_{\alpha(x)}(x')$, and look at

$$\int W_m^T(x) \overline{W_n^T(x)} - W_m^S(x) \overline{W_n^S(x)} d\mu(x) = \quad (4.23)$$

$$\int W_m^T(x) (\overline{W_n^T(x)} - \overline{W_n^S(x)}) d\mu(x) + \int \overline{W_n^S(x)} (W_m^T(x) - W_m^S(x)) d\mu(x). \quad (4.24)$$

If K is a bound for $\|f_i\|_\infty$, $\|f_j\|_\infty$ and $\|f_k\|_\infty$ then $\|G_m^T\|_\infty = \|G_m^S\|_\infty < K^2$. Also $\|W_m^T\|_\infty < K^3$ and $\|W_m^S\|_\infty < K^3$. But

$$W_m^T(x) - W_m^S(x) = \int (G_m^T(x, x') - G_m^S(x, x')) f_k(x') d\mu_{\alpha(x)}(x'). \quad (4.25)$$

There is an $\epsilon_2 > 0$ such that if $d(S, T) < \epsilon_2$ then $\|G_m^T - G_m^S\|_1 < \delta/2K^4$. Then for such an S ,

$$\|W_m^T(x) - W_m^S(x)\|_1 = \quad (4.26)$$

$$\int \int (G_m^T(x, x') - G_m^S(x, x')) f_k(x') d\mu_{\alpha(x)}(x') d\mu(x) < \delta/2K^3, \quad (4.27)$$

hence

$$\int \overline{W_n^S(x)} (W_m^T(x) - W_m^S(x)) d\mu(x) < \delta/2. \quad (4.28)$$

Similarly there is an $\epsilon_3 > 0$ such that if $d(S, T) < \epsilon_3$ then

$$\int W_m^T(x) (\overline{W_n^T(x)} - \overline{W_n^S(x)}) d\mu(x) < \delta/2. \quad (4.29)$$

Hence if $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$ and $d(S, T) < \epsilon$ then

$$\|[\lim_{M \rightarrow \infty} \frac{1}{M+1} \frac{1}{N+1} \sum_{m=0}^M \sum_{n=0}^N \int W_m^T(x) \overline{W_n^T(x)} d\mu(x)] + \|h_{l,q}\|_2^2 - \quad (4.30)$$

$$[\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M (U_m^T + \overline{U_m^T})] - \quad (4.31)$$

$$\|[\lim_{M \rightarrow \infty} \frac{1}{M+1} \frac{1}{N+1} \sum_{m=0}^M \sum_{n=0}^N \int W_m^S(x) \overline{W_n^S(x)} d\mu(x)] + \|h_{l,q}\|_2^2 - \quad (4.32)$$

$$[\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M (U_m^S + \overline{U_m^S})] \| < \quad (4.33)$$

$$[\lim_{M, N \rightarrow \infty} \frac{1}{M+1} \frac{1}{N+1} \sum_{m=0}^M \sum_{n=0}^N \delta + [\lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{m=0}^M \delta] = \delta. \quad (4.34)$$

This shows that if $(\vec{h}, T) \in \text{MD6}'$ then if S is in the ϵ -neighborhood of T and $h_{l,q}^T = h_{l,q}$ then $(\vec{h}', S) \in \text{MD6}'$, which is what we wanted.

Proof of Theorem 4.2.1. We just showed that the set of measure distal transformations is coanalytic, and since it is Π_1^1 -hard, it is a complete coanalytic set. $\square$

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