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Strongly annular solutions to Mahler's functional equation

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The Ohio State University, 1993

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STRONGLY ANNULAR SOLUTIONS TO MAHLER'S FUNCTIONAL EQUATION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Richard Daquila, B.S., M.S.

* * * * *

The Ohio State University

1993

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A handwritten signature in cursive script that reads "Francis W. Carroll". The signature is written in black ink and is positioned above the word "Adviser".

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CHAPTER I

Definitions

1.1 Basic Definitions and Notation

The purpose of this chapter is to acquaint the reader with some definitions and notation that will be used throughout this paper. The reader will be expected to have an understanding of some of the elementary results of complex analysis such as: definition of an analytic function, series representation of analytic functions, and the maximum and minimum modulus theorems. All preliminary information and other results can be found in Conway [5]. $\mathbb{C}$ will stand for the set of complex numbers and $\mathbb{R}$ will stand for the set of real numbers.

Definition 1.1.1 *A Möbius transformation, τ , is given by two parameters θ and z_0 , where θ is a real number and z_0 is any complex number with modulus less than 1:*

$$\tau(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z} \quad (1.1)$$

All Möbius transformations are automorphisms of the unit disk; moreover, they map the boundary of the disk to the boundary of the disk.

Definition 1.1.2 *A Finite Blaschke Product, B , is a finite product of Möbius transformations.*

$B(z)$ will have only finitely many zeros interior to the disk and will also map the boundary of the disk to the boundary of the disk.

Definition 1.1.3 (Iteration) *Let G be a function defined on a domain D , and suppose that $G(D) \subset D$. Then G^0 is defined to be $G^0(z) = z$ the identity function and $G^n = G \circ G^{n-1}$ for $n = 1, 2, 3, \dots$*

We remark that since the notation G^n is being used for the n -th iterate, we shall use $\{G\}^n$ for the n -th power of G and $G^{(n)}$ for the n -th derivative.

Definition 1.1.4 *A function $g(z)$ holomorphic on the open unit disk is said to be an annular function if there is a sequence of Jordan curves $\{J_n\}$ so that:*

1. $0 \in \text{Interior of } J_0$.
2. $J_n \subset \text{Interior of } J_{n+1}$ for all n .
3. For every $\epsilon > 0$ there is a number N_ϵ so that $J_n \subset \{z : 1 - \epsilon < |z| < 1\}$ for all $n > N_\epsilon$
4. $\min_{z \in J_n} |g(z)| \rightarrow \infty$ as $n \rightarrow \infty$.

Definition 1.1.5 *A function $g(z)$ holomorphic on the open unit disk is said to be strongly annular if there is a strictly increasing sequence of numbers $\{r_n\}$ so that:*

1. $\lim_{n \rightarrow \infty} r_n = 1$.
2. $\min_{|z|=r_n} |g(z)| \rightarrow \infty$ as $n \rightarrow \infty$.

1.2 The Class BB

Definition 1.2.1 *A function $f(z)$ is said to be in the class BB if the following three conditions hold:*

1. $f(z)$ is holomorphic in the open unit disk.
2. $f(0) = 1$.
3. $\liminf_{r \rightarrow 1^-} \min_{|z|=r} |f(z)| > 1$.

Remark:

Functions in BB are “big near the boundary.” Our purpose in defining the class BB is to use these functions to form infinite products that are either annular or strongly annular.

In view of the Minimum Modulus Theorem, it is clear that all of the functions in BB have at least one zero inside the unit disk. Each function in BB will have only finitely many zero's interior to the unit disk. It is useful to consider what the graph of the minimum modulus function will look like for a function that is in BB .

Example 1.2.1 *The following gives an example of a function in the class BB . Some of the simplest are linear functions.*

$$f(z) = 1 + Az \quad |A| > 2 \quad (1.2)$$

Since $f(z)$ is continuous on the closed unit disk

$$1. \liminf_{r \rightarrow 1^-} \min_{|z|=r} |f(z)| = \min_{|z|=1} |f(z)|$$

2.

$$\min_{|z|=1} |f(z)| = \min_{|z|=1} |1 + Az| \quad (1.3)$$

$$\geq \min_{|z|=1} |1 - |A||z|| \quad (1.4)$$

$$= |A| - 1 > 1 \quad (1.5)$$

It follows that $f(z) = 1 + Az \in BB$ if $|A| > 2$.

Let $m(r) = \min_{|z|=r} |f(z)|$

Figure 1: Graph of minimum modulus for a typical BB function.

We want to consider the graph of $m(r)$. The function $m(r)$ is continuous, and its value at the origin is 1. It has a zero at $\frac{1}{|A|}$ and its value as r approaches 1 is $m(1)$

(see Figure 1).

We can choose a number $\epsilon > 0$ so that $1 + \epsilon < \liminf_{r \rightarrow 1^-} \min_{|z|=r} |f(z)|$.

This determines a number u_ϵ so that $r > u_\epsilon \Rightarrow m(r) > 1 + \epsilon$.

This also determines a number v_ϵ so that $0 < r < v_\epsilon \Rightarrow m(r) > \frac{1}{1+\epsilon}$.

1.3 Analytic Infinite Products

It is a standard exercise to show that the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent if the series $\sum_{n=1}^{\infty} |a_n|$ converges (see Stromberg [26]). Therefore, if a_n depends on the variable z (i.e. $a_n = a_n(z)$), and if $\sum_{n=1}^{\infty} |a_n(z)|$ is finite and converges uniformly on a domain D , then $\prod_{n=1}^{\infty} (1 + a_n(z))$ will also converge uniformly there. It is important to keep in mind that the almost uniform limit of analytic functions is an analytic function (see Saks and Zygmund, [24]).

The following lemmas show how certain infinite products are well defined analytic functions in the open unit disk.

Lemma 1.3.1 *Let $h(z)$ be holomorphic in the open unit disk and $h(0) = 0$. Let $\{m_k\}$ be a strictly increasing sequence of positive integers. Then the function*

$$g(z) = \prod_{n=1}^{\infty} (1 + h(z^{m_k})) \tag{1.6}$$

is a well defined analytic function on the open unit disk.

Proof:

Let $0 < r_0 < 1$. Then for $|z| \leq r_0$ we have $|h(z)| < K|z|$ with K a constant depending on r_0 . Hence, for $|z| \leq r_0$, we have $|h(z^{m_k})| < K|z|^{m_k}$, for $k = 1, 2, \dots$,

so that $\sum_{n=1}^{\infty} |h(z^{m_k})|$ is finite and uniformly convergent on $|z| \leq r_0$. Since $r_0 < 1$ is arbitrary, it follows that $\prod_{n=1}^{\infty} (1 + h(z^{m_k}))$ is an analytic function in the open unit disk. ■

Lemma 1.3.2 *Let $h(z)$ be holomorphic in the open unit disk and have a zero of order at least two at the origin, and further let $|h(z)| \leq 1$. Then $h(z)$ is a contraction of the disk and $|h^n(z)| \leq |z|^{2^n}$.*

Proof (by induction):

Case: $n = 1$

Note that the function $\frac{h(z)}{z^2}$ will be analytic in the open unit disk and bounded by 1, since for $0 < r < 1$,

$$\max_{|z| \leq r} \left| \frac{h(z)}{z^2} \right| = \max_{|z|=r} \frac{|h(z)|}{|z|^2} \quad (1.7)$$

$$\leq \max_{|z|=r} \frac{1}{r^2} = \frac{1}{r^2}. \quad (1.8)$$

Since $\frac{1}{r^2} \rightarrow 1$ as $r \rightarrow 1$, we have

$$\frac{|h(z)|}{|z|^2} \leq 1, \text{ or} \quad (1.9)$$

$$|h(z)| < |z|^2 \quad (1.10)$$

Let us assume that $|h^n(z)| \leq |z|^{2^n}$ for some n . Then

$$|h^{n+1}(z)| = |h^n(h(z))| \quad (1.11)$$

$$\leq |h(z)|^{2^n} \quad (1.12)$$

$$\leq (|z|^2)^{2^n} \quad (1.13)$$

$$\leq |z|^{2^{n+1}} \quad (1.14)$$

Therefore by induction we have $|h^n(z)| < |z|^{2^n}$ for all z in the open unit disk. ■

Lemma 1.3.3 *Let $F(z)$ be holomorphic in the open unit disk and let $F(0) = 1$. Let $h(z)$ be a holomorphic function in the open unit disk that has a zero of order at least two at the origin and $|h(z)| \leq 1$. Then*

$$\prod_{n=0}^{\infty} F(h^n(z)) \quad (1.15)$$

is a well defined analytic function in the open unit disk.

Proof:

Let $0 \leq r < 1$ so that for $|z| \leq r$ we have $|F(z) - 1| \leq K|z|$ where K depends on r . Then $\prod_{n=1}^{\infty} F(h^n(z))$ will be well defined and analytic for $|z| < r$ since

$$|F(h^n(z)) - 1| \leq K|h^n(z)| \quad (1.16)$$

$$\leq K|z|^{2^n} \quad (1.17)$$

$$\leq Kr^{2^n} \quad (1.18)$$

by applying Lemma 1.3.2. We also notice that

$$\sum_{n=1}^{\infty} Kr^{2^n} < \infty. \quad (1.19)$$

Hence the infinite product defined by:

$$\prod_{n=1}^{\infty} F(h^n(z)) \quad (1.20)$$

is an analytic function. ■

Corollary 1.3.1 *Let $F(z)$ be a holomorphic function on the open unit disk with $F(0) = 1$, and a zero inside the disk. Then for any positive integer $p > 1$ the function $g(z)$ defined by :*

$$g(z) = \prod_{n=0}^{\infty} F(z^{p^n}) \quad (1.21)$$

will have every point of $\{z : |z| = 1\}$ in the closure of $\{z : |z| < 1, g(z) = 0\}$.

Proof:

Applying Lemma 1.3.3 with $h(z) = z^p$, we obtain g analytic in $|z| < 1$. Let $\rho e^{i\theta}$ be a zero of $F(z)$. Then the points $\rho^{\frac{1}{p^n}} e^{\frac{i(\theta+2j\pi)}{p^n}}$ $j = 0, 1, \dots, p^n$ are roots of $F(z^{p^n})$. The modulus of this set of points goes to 1 as n goes to infinity, and the arguments only differ by $\frac{2\pi}{p^n}$, so that they are uniformly distributed about the circle $|z| = \rho^{\frac{1}{p^n}}$. So the closure of the set of all of the zeros of $g(z)$ will include the zeros of each factor $F(z^{p^n})$. Hence the closure of the set of zeros of $g(z)$ contains the unit circle. ■

A function satisfying the conditions of Corollary 1.3.1 can not be continued analytically beyond the unit disk, since such a continuation across an arc of $|z| = 1$ would vanish everywhere on the arc. Every function in Class BB satisfies the hypotheses of Corollary 1.3.1. Composing a function in BB with the iterates of z^p to form a infinite product results in a noncontinuable function on the open unit disk, with a set of zeros whose closure contains the unit circle.

1.4 Analytic and Algebraic Notions of Function Spaces

The set of functions holomorphic on a domain can be provided with a topology, the “topology of almost uniform convergence”, relative to which it is a complete metric space. We assume familiarity with this topology. Details may be found in many complex analysis texts. See, for instance, Saks and Zygmund [24].

Definition 1.4.1 *A sequence of functions $\{f_n\}$ is said to converge almost uniformly on a domain D to a function f if given K , a compact subset of D and $\epsilon > 0$, there is a $N_{K,\epsilon}$ so that for all $n > N_{K,\epsilon}$ we have $\max_{z \in K} |f_n(z) - f(z)| < \epsilon$.*

Definition 1.4.2 *A holomorphic function F is said to be universal on a domain D with respect to a group of automorphisms M of D if the set $\{F \circ \mu : \mu \in M\}$ is a dense subset of the set of holomorphic functions on D in the topology of almost uniform convergence.*

In other words a function is universal on a domain D with respect to a group M of transformations, if given a function f holomorphic on D there exists $\{\mu_i\} \subset M$ so that $F \circ \mu_i$ converges to f almost uniformly. To show that a function is universal it is enough to show that given any f holomorphic on D and $K \subset D$ with K compact, and for any $\epsilon > 0$, there is $\mu \in M$ so that

$$\max_{z \in K} |F \circ \mu(z) - f(z)| < \epsilon. \quad (1.22)$$

For the next set of definitions we will need some standardized notation. $\mathbb{C}[z]$ will denote the set of all polynomials with coefficients in $\mathbb{C}$. $\mathbb{C}[[z]]$ is the set of all rational

functions with coefficients in $\mathbb{C}$. $\mathbb{C}[z, y_0, \dots, y_m]$ is the set of all polynomials in the variables z and y_0 through y_m , for example, $zy_0^2 + y_0y_1$. $\mathbb{C}[[z]][y_0, \dots, y_n]$ will be the set of all polynomials in the variables y_0 through y_n that have coefficients in $\mathbb{C}[[z]]$, for example, $\frac{z^2+1}{z+2}y_3^2y_2y_0^3 + z^2y_1^2 + \frac{1}{1+z}y_0$.

Definition 1.4.3 *A function f is said to be differentially algebraic if there is a polynomial $P(z, y_0, y_1, \dots, y_n) \in \mathbb{C}[[z]][y_0, \dots, y_n]$ and $P(z, y_0, y_1, \dots, y_n) \not\equiv 0$ so that*

$$P(z, f(z), f^{(1)}(z), \dots, f^{(n)}(z)) \equiv 0. \quad (1.23)$$

Definition 1.4.4 *A function f is said to be transcendentially transcendental if it is not differentially algebraic.*

Studies of differentially algebraic and transcendentially transcendental functions have been published, see Kaplansky [14] and Moore [19]. Among the basic results are the facts that the sum, product, quotient, scalar multiple, inversion and derivative of differentially algebraic functions are differentially algebraic. A function that is transcendentially transcendental is a function whose derivatives form an algebraically independent set. A classical example of a transcendentially transcendental function is $\Gamma(z)$, the standard gamma function (see Rubel [23]).

CHAPTER II

Annular and Strongly Annular Products

2.1 Introduction

The purpose of this chapter is to study the solutions of Mahler's Functional Equation:

$$g(z) = q(z)g(z^p) \tag{2.1}$$

where p is an integer larger than one and $q(z)$ is a polynomial with $q(0) = 1$ (see Mahler [21]). The methods used will be similar to those used to obtain the known result for Schröder's Functional Equation:

$$sg(z) = g(B(z)) \tag{2.2}$$

in the case where $B(z)$ is a finite Blaschke product with $B(0) = 0$ and $s = B'(0) \neq 0$. In this case, nontrivial solutions of Equation 2.2 are annular (see Valiron [27] and Carroll [4] and Kuczma [16]). Notice in the Schröder equation the finite Blaschke product has a zero of order exactly one at $z = 0$, so that neither Equation 2.1 nor Equation 2.2 is a special case of the other.

Our principal results in this chapter are Theorem 2.2.1 and Theorem 2.3.1. The first says that if $q(z)$ (but not necessarily a polynomial) is in class BB and if g

satisfies Equation 2.1, then g is strongly annular. The second shows the annularity of the solutions G of an analogous equation, i.e., of $G(z) = f(z)G(B(z))$, where f is in class BB and B is a finite Blaschke product with a zero of at least order two at the origin.

Figure 2: Partitioning of the disk corresponding to partial products.

Theorem 2.2.1 and Theorem 2.3.1 are proved using similar techniques. We prove in some early lemmas that the iterates of a finite Blaschke product contract every point of the open unit disk toward the origin, making zero the limit point of the iterates. We then construct a Jordan curve “close” to the boundary so that the contraction to the origin is controlled. Using Example 1.2.1 and Lemma 2.2.1 we partition the open unit disk into four regions with the Jordan curve passing through each region as higher iterates of the curve are indexed (see Figure 2).

This suggests a non-standard method to partition an infinite product into four partial products. Each partial product has for its factors the value of a function in class BB when the image of the Jordan curve is in the corresponding region. In region 1 the factors are larger than $(1 + \epsilon)$. In region 2 the factors are uniformly bounded away from zero. In region 3 the factors are small, but not “so small” that the value of the factors in region 1 do not dominate the entire product. In region 4, the remaining factors are so close to 1 that their product can be estimated by a constant.

2.2 Strongly Annular Products

Lemma 2.2.1 *Let c be a fixed positive constant, let p be an integer greater than 1, and let $0 < r < 1$. Then there is a positive integer K so that :*

$$1 - cr^{p^k} \geq 1 - \frac{c}{p^k} > 0 \text{ for all } k > K \quad (2.3)$$

Proof:

We begin by choosing a positive integer N_0 so large that if $n > N_0$ we have $r^n < \frac{1}{n}$. Let K_1 be a positive integer which is larger than $\frac{\log N_0}{\log p}$. Then it follows that $p^k > N_0$ when $k > K_1$, hence $r^{p^k} < \frac{1}{p^k}$ when $k > K_1$. Choose K_2 so large that $c < p^k$ for $k > K_2$. Let $K = \max\{K_1, K_2\}$. Then for $k > K$ we have:

$$r^{p^k} < \frac{1}{p^k} \quad (2.4)$$

$$0 < 1 - \frac{c}{p^k} \leq 1 - cr^{p^k} \quad (2.5)$$

■

Theorem 2.2.1 Let $f(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^{\nu}$ be in class BB . Then for any fixed integer $p > 1$, the function $g(z) = \prod_{k=0}^{\infty} f(z^{p^k})$ is a strongly annular function.

Proof:

Applying Lemma 1.3.3 with $F(z) = f(z)$ and $h(z) = z^p$, we see that the function $g(z)$ defined above is analytic. Choose $\epsilon > 0$ so that $1 + \epsilon < \liminf_{r \rightarrow 1^-} \min_{|z|=r} |f(z)|$. Since $\liminf_{\rho \rightarrow 1^-} m(\rho) > 1$, the function $f(z)$ has only finitely many zeros in the disk (see Example 1.2.1). Suppose that the radii of these zeros are $\rho_1, \rho_2, \dots, \rho_s$.

Since the function $m(\rho) = \min_{|z|=\rho} |f(z)|$ is a continuous function of ρ , and since $m(0) = 1$ and considering the minimum modulus theorem, $m(\rho)$ is decreasing in a neighborhood of 0. There exist two constants u_{ϵ} and v_{ϵ} so that

$$\begin{aligned} 1 > r > u_{\epsilon} &\Rightarrow m(r) > 1 + \epsilon \\ v_{\epsilon} > r > 0 &\Rightarrow m(r) > \frac{1}{1 + \epsilon} \end{aligned}$$

(as in Example 1.2.1)

Choose r_0 so that:

1. $r_0 \neq \rho_j^{p^n}$ $1 \leq j \leq s$, n any integer
2. $r_0 > u_{\epsilon}$

Choose a positive integer d large enough that $r_0^{p^n} < v_{\epsilon}$ for all $n \geq d$. Choose K large enough so that if $k > K$ then $1 - v_{\epsilon}^{p^k} \sum_{j=1}^{\infty} |a_j| v_{\epsilon}^{j-1} > 1 - \frac{\sum_{j=1}^{\infty} |a_j| v_{\epsilon}^{j-1}}{p^k} > 0$. This can be done by Lemma 2.2.1.

We claim that if $n > K$, and if r_n is defined by $r_n = r_0^{p^{-2n}}$, then

$$\min_{|z|=r_n} |g(z)| \longrightarrow \infty \text{ as } n \rightarrow \infty \quad (2.6)$$

This will show that g is, indeed, a strongly annular function. We express g as a product of four subproducts. Recall that $z = r_n e^{i\theta} = e^{i\theta} r_0^{p^{-2n}}$, so that $|z|^{p^k} = r_0^{p^k - 2n}$

$$|g(z)| = \left| \prod_{k=0}^{\infty} f(z^{p^k}) \right| \quad (2.7)$$

$$= \left| \prod_{k=0}^{2n} f(z^{p^k}) \prod_{k=2n+1}^{2n+d} f(z^{p^k}) \prod_{k=2n+d+1}^{3n} f(z^{p^k}) \prod_{k=3n+1}^{\infty} f(z^{p^k}) \right| \quad (2.8)$$

$$= |P_1 \cdot P_2 \cdot P_3 \cdot P_4| \quad (2.9)$$

We will break down each of these four parts separately.

Estimate P_1 :

$$|P_1| = \prod_{k=0}^{2n} |f(r_0^{p^k - 2n} e^{i\theta p^k})| \quad (2.10)$$

Since

$$0 \leq k \leq 2n \quad (2.11)$$

$$-2n \leq k - 2n \leq 0 \quad (2.12)$$

$$\Rightarrow r_0^{p^k - 2n} > r_0 > u_\epsilon \quad (2.13)$$

$$\Rightarrow |f(r_0^{p^k - 2n} e^{i\theta p^k})| > 1 + \epsilon \quad \text{for } k = 0, \dots, 2n, \quad (2.14)$$

and therefore

$$\min_{|z|=r_n} |P_1| \geq \prod_{k=0}^{2n} (1 + \epsilon) = (1 + \epsilon)^{2n+1} \quad (2.15)$$

Estimate P_2 :

$$|P_2| = \prod_{k=2n+1}^{2n+d} |f(r_0^{p^{k-2n}} e^{i\theta p^k})| \quad (2.16)$$

We have $2n+1 \leq k \leq 2n+d$, so $1 \leq k-2n \leq d$.

Let $a = \min_{1 \leq j \leq d} m(r_0^{p^j})$. Note that $a \neq 0$, and its value is independent of n , since $r_0^{p^j} \neq \rho_m$, $1 \leq m \leq s$.

$$\min_{|z|=r_n} |P_2| \geq \prod_{k=2n+1}^{2n+d} |f(r_0^{p^{k-2n}} e^{i\theta p^k})| \geq \prod_{k=2n+1}^{2n+d} a = a^d \quad (2.17)$$

Estimate P_3 :

$$|P_3| = \prod_{k=2n+d+1}^{3n} |f(r_0^{p^{k-2n}} e^{i\theta p^k})| \quad (2.18)$$

Since $2n+d+1 \leq k \leq 3n$, and $d+1 \leq k-2n \leq n$, we have

$$\Rightarrow k-2n > d \quad (2.19)$$

$$\Rightarrow r_0^{p^{k-2n}} < v_\epsilon \quad (2.20)$$

$$\Rightarrow |f(r_0^{p^{k-2n}} e^{i\theta p^k})| > \frac{1}{1+\epsilon} \quad (2.21)$$

and therefore

$$\min_{|z|=r_n} |P_3| \geq \prod_{k=2n+d+1}^{3n} \frac{1}{1+\epsilon} = \left(\frac{1}{1+\epsilon} \right)^{n-d} \quad (2.22)$$

Estimate P_4 :

$$|P_4| = \prod_{k=3n+1}^{\infty} |f(r_0^{p^{k-2n}} e^{i\theta p^k})| \quad (2.23)$$

Since $k \geq 3n + 1$ for all n , we get $k - 2n \geq n + 1 \geq K$ and also $r_0^{p^{k-2n}} < v_\epsilon$.

We have

$$|f(r_0^{p^{k-2n}} e^{i\theta p^k})| \geq 1 - \sum_{j=1}^{\infty} |a_j| r_0^{j p^{k-2n}} \quad (2.24)$$

$$\geq 1 - r_0^{p^{k-2n}} \sum_{j=1}^{\infty} |a_j| r_0^{(j-1)p^{k-2n}} \quad (2.25)$$

$$\geq 1 - r_0^{p^{k-2n}} \sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1} \quad (2.26)$$

$$\geq 1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{p^{k-2n}} \quad (2.27)$$

$$\geq 1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{p^K} > 0 \quad (2.28)$$

$$\min_{|z|=r_n} |P_4| \geq \prod_{k=3n+1}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{p^{k-2n}} \right) \geq \prod_{k=K}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{p^k} \right) \quad (2.29)$$

Let $\omega = \prod_{k=K}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{p^k} \right)$. Then $\sum_{k=1}^{\infty} \frac{1}{p^k} < \infty$ implies $\omega \neq 0$. Therefore $|P_4| \geq \omega > 0$ on $|z| = r_n$.

Now it is important to consider that we have estimated the minimum modulus for $g(z)$ by estimating the minimum of each factor of the product.

Therefore on the circle $|z| = r_n$ we have

$$|g(z)| = |P_1 \cdot P_2 \cdot P_3 \cdot P_4| \quad (2.30)$$

$$\geq (1 + \epsilon)^{2n+1} a^d \frac{1}{(1 + \epsilon)^{n-d}} \omega \quad (2.31)$$

$$= a^d \omega (1 + \epsilon)^{n+d+1} \quad (2.32)$$

with a, d, ϵ, ω constants that do not depend on n .

Therefore $\lim_{n \rightarrow \infty} \min_{|z|=r_n} |g(z)| \geq \lim_{n \rightarrow \infty} \left(\frac{a}{1+\epsilon} \right)^d \omega (1 + \epsilon)^n = \infty$

Therefore g is a strongly annular function. ■

Corollary 2.2.1 *Let $f(z) \in BB$, let $\{a_\nu\}$ be a sequence of real numbers and let p be an integer greater than 1. Then the function g given by*

$$g(z) = \prod_{\nu=0}^{\infty} f(e^{ia_\nu} z^{p^\nu}) \quad (2.33)$$

is a strongly annular function.

Proof:

In the proof of Theorem 2.2.1 we define the same set of concentric circles and we notice that in Equations (2.15), (2.17), (2.22), (2.29) we can get the following equality.

$$\min_{|z|=r_n} |f(e^{i\theta} z^{p^\nu})| = \min_{|e^{i\theta} z^{p^\nu}|=r_n} |f(z^{p^\nu})| \quad (2.34)$$

$$= \min_{|z|=r_n} |f(z^{p^\nu})| \quad (2.35)$$

Since we estimated each factor of g , Equations (2.15), (2.17), (2.22), (2.29) hold for $g(z)$ on the circle of radius r_n , and there

$$|g(z)| \geq (1 + \epsilon)^{n+d+1} a^d \omega \quad (2.36)$$

which goes to infinity as n goes to infinity, so that $g(z)$ is a strongly annular function. ■

2.3 Annular Products

Lemma 2.3.1 *If $\tau(z) = e^{i\theta_0} \frac{z-z_0}{1-\bar{z}_0 z}$ is a Möbius transformation, then there is a ρ , $1 > \rho > 0$ and a positive integer k so that $1 \geq |z| > \rho \Rightarrow |z|^k \leq |\tau(z)|$.*

Proof:

Let $z = re^{i\theta}$. Let $|z_0| = \gamma$ and also let $r > \gamma > 0$. We further define the function $m(r)$ as $\min_{|z|=r} |\tau(z)|$. We begin by proving the following identity for $m(r)$ when $|z| > \gamma$.

$$m(r) = \min \left\{ \frac{r-\gamma}{1-\gamma r}, \frac{r+\gamma}{1+\gamma r} \right\} = \frac{r-\gamma}{1-r\gamma} \quad (2.37)$$

Let S be the segment through 0 and z_0 , hitting $|z| = r$ at the points $a = \frac{rz_0}{|z_0|}$ and $b = \frac{-rz_0}{|z_0|}$. The point $\frac{1}{\bar{z}_0}$ is on the extension of S , call it S^* . The image of S^* , $\tau(S^*)$ is a line or a circle, but in this case we know it must be a line because $\tau(\frac{1}{\bar{z}_0}) = \infty$.

The image of the circle $\{|z| = r\}$, $\tau(\{|z| = r\})$ is a circle since the pole of τ is outside the unit disk. The segment $\tau(S)$ is a diameter of a circle. Then $\tau(a)$ and $\tau(b)$ are diametrically opposed points on the circle $\tau(\{|z| = r\})$ and 0 lies on this diameter. From the geometry of the circle we can conclude that the maximum and minimum of the function τ are contained in the set $\{|\tau(a)|, |\tau(b)|\}$. We compute $|\tau(a)|$ and $|\tau(b)|$.

$$|\tau(a)| = \left| \frac{e^{i\theta_0} \left(\frac{rz_0}{|z_0|} - z_0 \right)}{\left(1 - \overline{z_0} \frac{rz_0}{|z_0|} \right)} \right| \quad (2.38)$$

$$= \frac{r - \gamma}{1 - \gamma r} \quad (2.39)$$

$$|\tau(b)| = \left| \frac{e^{i\theta_0} \left(\frac{-rz_0}{|z_0|} - z_0 \right)}{\left(1 - \overline{z_0} \frac{-rz_0}{|z_0|} \right)} \right| \quad (2.40)$$

$$= \frac{r + \gamma}{1 + \gamma r} \quad (2.41)$$

Consider the function $f(t)$, $-1 \leq t \leq 1$, given by:

$$f(t) = \frac{r + t\gamma}{1 + t\gamma r} \quad (2.42)$$

$$f'(t) = \frac{\gamma(1 + t\gamma r) - (r + t\gamma)\gamma r}{(1 + t\gamma r)^2} \quad (2.43)$$

$$= \frac{\gamma(1 - r^2)}{(1 + t\gamma r)^2} \quad (2.44)$$

Since the function $f'(t) > 0$ and $f(t)$ is continuous on the interval $-1 \leq t \leq 1$ we conclude $\min_{-1 \leq t \leq 1} f(t) = f(-1)$. We can apply equation (2.37) and conclude the following.

$$m(r) = \frac{r - \gamma}{1 - \gamma r} \quad (2.45)$$

$$m(r) - 1 = \frac{(1 + \gamma)(r - 1)}{(1 - \gamma r)} \quad (2.46)$$

$$m(r) = 1 + (1 + \gamma)(r - 1) \frac{1}{(1 - \gamma r)} \quad (2.47)$$

$$= 1 + (1 + \gamma)(r - 1) \frac{1}{(1 - \gamma - \gamma(r - 1))} \quad (2.48)$$

$$= 1 + \frac{(1 + \gamma)(r - 1)}{1 - \gamma} \frac{1}{1 - \frac{\gamma}{1 - \gamma}(r - 1)} \quad (2.49)$$

$$= 1 + \frac{1+\gamma}{1-\gamma}(r-1)\{1 + \frac{\gamma}{1-\gamma}(r-1) + \dots\} \quad (2.50)$$

$$= 1 + \frac{1+\gamma}{1-\gamma}(r-1) + O((r-1)^2) \quad \text{as } r \rightarrow 1 \quad (2.51)$$

We can also expand r^k as follows.

$$r^k = 1 + k(r-1) + O((r-1)^2) \quad \text{as } r \rightarrow 1 \quad (2.52)$$

Subtracting (2.51) from (2.52) we obtain the following.

$$m(r) - r^k = \left[\frac{1+\gamma}{1-\gamma} - k \right] (r-1) + O((r-1)^2) \quad \text{as } r \rightarrow 1 \quad (2.53)$$

$$= \left[k - \frac{1+\gamma}{1-\gamma} \right] (1-r) + O((r-1)^2) \quad \text{as } r \rightarrow 1 \quad (2.54)$$

We see $m(r) - r^k > 0$ provided that we choose $k > \frac{1+\gamma}{1-\gamma}$ and we also choose ρ close enough to 1 so that $r > \rho$ implies that $O((r-1)^2)$ is not significant compared to the rest of $m(r) - r^k$.

■

Remark. The previous result can also be obtained in the following manner. We use a similar equation which relates the modulus of the derivative to the modulus of the function.

Choose $\alpha > 2|\tau'(z)|$ for all z in the unit disk.

$$\alpha > (1+r)|\tau'(z)| \quad (2.55)$$

$$\alpha(1-r) > (1-r^2)|\tau'(z)| \quad (2.56)$$

$$-(1-r^2)|\tau'(z)| > \alpha(r-1) \quad (2.57)$$

$$1 - (1 - r^2)|\tau'(z)| > \alpha(r - 1) + 1 \quad (2.58)$$

Using the identity $|\tau(z)|^2 = 1 - (1 - |z|^2)|\tau'(z)|$ (see Garnett [8]), we obtain $|\tau(z)|^2 > \alpha(r - 1) + 1$.

The equation $y = \alpha(r - 1) + 1$ is a line passing through the point $(1, 1)$. If ρ is chosen close enough to 1 and k is large enough then $\alpha(r - 1) + 1 > r^{2k}$ for all $r > \rho$. Then $|\tau(z)|^2 > r^{2k}$ which gives $|\tau(z)| > |z|^k$.

Lemma 2.3.2 *Let $B(z)$ be a finite Blaschke Product. Then there exists a ρ and a positive integer k , $1 > \rho > 0$ so that $1 > |z| > \rho \Rightarrow |z|^{2k} \leq |B(z)|$.*

Proof:

$B(z) = \prod_{i=1}^p \tau_i(z)$ where $\tau_i(z)$ are Möbius Transformations. For each $\tau_i(z)$ there is a ρ_i and a k_i so that $1 > |z| > \rho_i \Rightarrow |z|^{k_i} \leq |\tau_i(z)|$ by Lemma 2.3.1. If $\rho = \max_{1 \leq i \leq p} \rho_i$ then we have $1 > |z| > \rho \Rightarrow |z|^{\sum_{i=1}^p k_i} \leq \prod_{i=1}^p |\tau_i(z)|$. Choose a positive integer k so that $2^k > \sum_{i=1}^p k_i$. Then $1 > |z| > \rho \Rightarrow |z|^{2k} \leq |z|^{\sum_{i=1}^p k_i} \leq |B(z)|$

■

Remark

If $B(z)$ is a finite Blaschke Product that has a zero of at least order 2 at the origin, then $|B(z)| \leq |z|^2$ for every z in the open unit disk. This is simply a variation of Schwarz's Lemma (see Lemma 1.3.2).

Lemma 2.3.3 *Suppose $B(z)$ is a finite Blaschke Product with a zero of at least order 2 at the origin. Then there exists ρ , $0 < \rho < 1$ and a positive integer k so that*

$1 > |z| > \rho^{2^{k-kn}} \Rightarrow |z|^{2^{kn}} \leq |B^n(z)| \leq |z|^{2^n}$ for all n . Moreover the second inequality is true for all z in the open unit disk.

Proof (by induction):

The case $n = 1$ determines the ρ and the k . This is simply the statement of Lemma 2.3.2.

Assume $|z| > \rho^{2^{k-kn}} \Rightarrow |z|^{2^{kn}} \leq |B^n(z)| \leq |z|^{2^n}$

We have

$$|z| > \rho^{2^{k-k(n+1)}} \Rightarrow |z| > \rho^{2^{-kn}} \quad (2.59)$$

$$\Rightarrow |z|^{2^{kn}} > \rho \quad (2.60)$$

$$\Rightarrow |B^n(z)| \geq \rho \quad (2.61)$$

$$\Rightarrow |B^n(z)|^{2^k} \leq |B^{n+1}(z)| \leq |B^n(z)|^2 \quad (2.62)$$

$$\Rightarrow (|z|^{2^{kn}})^{2^k} \leq |B^{n+1}(z)| \leq (|z|^{2^n})^2 \quad (2.63)$$

$$\Rightarrow |z|^{2^{k(n+1)}} \leq |B^{n+1}(z)| \leq |z|^{2^{n+1}} \quad (2.64)$$

■

Lemma 2.3.4 *Let $B(z)$ be a finite Blaschke Product with $B(0) = 0$. If ρ satisfies the conditions :*

$$1. |z| = \rho \Rightarrow B(z) \neq 0 \text{ and } B'(z) \neq 0$$

$$2. |B(w)| = \rho \Rightarrow B'(w) \neq 0$$

then one of the connected components of the set $\{w : |B(w)| = \rho\}$ is a Jordan curve with zero in its interior.

Proof:

Let $A = \{w : |B(w)| < \rho\}$. This is an open set containing zero. Let A_1 be the connected component of A containing zero. We assert that the boundary of A_1 is a Jordan curve whose image under $B(z)$ is the circle $|z| = \rho$.

Let z_0 be a element of the boundary of A_1 and D a sufficiently small open disk about z_0 so that the closure of D contains no zeros of $B(z)$. There exist points z_1 and z_2 with z_1, z_2 in the boundary of D so that $|B(z_1)| \leq |B(z)| \leq |B(z_2)|$ for all z in the closure of D . D contains points that are in A_1 and not in A_1 , so that there are points a_1 and a_2 in D for which $|B(a_1)| < \rho$ and $|B(a_2)| \geq \rho$, hence $|B(z_1)| < |B(a_1)| < \rho$ and $|B(z_2)| \geq |B(a_2)| \geq \rho$. As the diameter of the disk D gets smaller so does the difference between the maximum and the minimum of $B(z)$ on D so that $|B(z_0)| = \rho$.

Let w_0 be an element of the boundary of A_1 . Then $B(w_0) = z_0$, with $|z_0| = \rho$. Let $B_1(z)$ be the function defined in a neighborhood of z_0 with the property that $B(B_1(z)) = z$ and $B_1(z_0) = w_0$; this is possible since $B(z)$ has a local inverse for all z on $|z| = \rho$. Continue $B_1(z)$ analytically along $t \rightarrow \rho e^{i(t + \text{Arg } z_0)}$, $0 \leq t \leq 2\pi$ and let $w(t)$ be the value at t of the continuation, i.e. $w(t) = B_1(\rho e^{i(t + \text{Arg } z_0)})$ defines a curve. The curve is continuously differentiable. It remains to show that the mapping $t \rightarrow w(t)$ is one-to-one on $[0, 2\pi)$. Let t_2 be the smallest positive number so that there exists $t_1, 0 \leq t_1 < t_2$ with $B_1(\rho e^{i(t_1 + \text{Arg } z_0)}) = B_1(\rho e^{i(t_2 + \text{Arg } z_0)})$. There is such

a t_2 , otherwise the points $B_1(\rho e^{i(2j\pi + \text{Arg } z_0)})$, $j = 0, 1$ would be distinct. Moreover if $0 < t_1$ then the arcs $\{B_1(\rho e^{it}) : t_1 - \delta < t \leq t_1\}, \{B_1(\rho e^{it}) : t_1 \leq t < t_1 + \delta\}$ and $\{B_1(\rho e^{it}) : t_2 - \delta < t \leq t_2\}$ are distinct except for $B_1(\rho e^{it_1})$, and all are mapped by $B(z)$ into arcs bounded by ρe^{it_1} . This would contradict the fact that $B(z)$ has a local inverse at every point of its preimage (condition 2). Therefore $w(t)$ is a Jordan curve whose image is the boundary of A_1 , since A_1 cannot have a boundary which is disconnected because of the maximum modulus theorem. ■

It is important to realize that since a finite Blaschke Product is a rational function then the function and its derivative will only have finitely many zeros. There will only be finitely many numbers between 0 and 1 that do not satisfy the conditions of Lemma 2.3.4.

Theorem 2.3.1 *Suppose that $f(z)$ is in class BB and that $B(z)$ is a finite Blaschke Product with a zero of at least order 2 at the origin. Then the function $G(z) = \prod_{\nu=0}^{\infty} f(B^\nu(z))$ is an annular function.*

Proof:

Let $f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j$. Applying Lemma 1.3.3 with $F(z) = f(z)$ and $h(z) = B(z)$, we obtain $G(z)$ analytic in the open unit disk.

Choose u_ϵ and v_ϵ so that

$$1. \ 1 > |z| > u_\epsilon \Rightarrow |f(z)| > 1 + \epsilon$$

$$2. \ 0 < |z| < v_\epsilon \Rightarrow |f(z)| > \frac{1}{1+\epsilon}$$

Again, this is as in Example 1.2.1. We apply Lemma 2.3.3 to $B(z)$ and obtain a pair ρ, k for which

$$|z|^{2^{kn}} \leq |B^n(z)| \leq |z|^{2^n} \quad (2.65)$$

for all z with

$$\rho^{2^{k-kn}} < |z| < 1. \quad (2.66)$$

We now choose $r'_1 > \max\{\rho, u_\epsilon\}$.

Choose r'_2 so that $r'_1 < r'_2 < 1$.

Choose d a positive integer so that $(r'_2)^{2^d} < v_\epsilon$

Now consider the function $h(z) = \prod_{\nu=1}^d f(B^\nu(z))$

$h(z)$ is a well defined analytic function on the open unit disk and on the annulus $\{z : r'_1 < |z| < r'_2\}$.

Let $m_h(r) = \min_{|z|=r} |h(z)|$, and note that $m_h(r)$ is nonzero at some point in the interval (r'_1, r'_2) .

There is a positive real number α and a subinterval $(r_1, r_2) \subset (r'_1, r'_2)$ so that $m_h(r) > \alpha > 0$ on (r_1, r_2) .

Let $A = \{z : r_1 < |z| < r_2\}$. Let η_n be a number that satisfies the following three conditions :

1. $r_1 < \eta_n < r_2$
2. $|z_0| = \eta_n \Rightarrow B^{2n}(z_0) \neq 0$ and $\frac{d}{dz} B^{2n}(z_0) \neq 0$
3. $|B^{2n}(w)| = \eta_n \Rightarrow \frac{d}{dz} B^{2n}(w) \neq 0$

By Lemma 2.3.4 there exists a Jordan curve γ_n with zero in the curve's interior and for which $B^{2n}(\gamma_n)$ is a circle of radius η_n contained in A . Let $A'_n = \{z : r_1^{2^{-2n}} \leq |z| \leq r_2^{2^{-2kn}}\}$. By Lemma 1.3.2, if $|z| < r_1^{2^{-2n}}$ then $|B^{2n}(z)| \leq |z|^{2^{2n}} < r_1$ so that $z \notin \gamma_n$. By Lemma 2.3.3 if $|z| > r_2^{2^{-2kn}}$ then we have $r_2 < |z|^{2^{2kn}} \leq |B^{2n}(z)|$ so that again $z \notin \gamma_n$. It must be the case that the Jordan curve $\gamma_n \subseteq A'_n$. The Jordan curve γ_n will have the following three properties:

1. $\{0\} \in \text{interior of } \gamma_n$
2. $\gamma_n \subseteq A'_n$
3. $B^{2n}(\gamma_n) \subset A$

Choose a positive integer K large enough so that for $\nu > K$ we have the following inequality $1 - v_\epsilon^{2^\nu} \sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1} \geq 1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{2^\nu} > 0$ as in Lemma 2.2.1.

Now consider $z \in \gamma_n$ and $n > K + d$

$$|g(z)| \tag{2.67}$$

$$= \left| \prod_{\nu=0}^{\infty} f(B^\nu(z)) \right| \tag{2.68}$$

$$= \left| \prod_{\nu=0}^{2n} f(B^\nu(z)) \prod_{\nu=2n+1}^{2n+d} f(B^\nu(z)) \prod_{\nu=2n+d+1}^{3n} f(B^\nu(z)) \prod_{\nu=3n+1}^{\infty} f(B^\nu(z)) \right| \tag{2.69}$$

$$= |P_1 \cdot P_2 \cdot P_3 \cdot P_4| \tag{2.70}$$

We will consider these 4 products separately.

Estimate $|P_1|$:

$$P_1 = \prod_{\nu=0}^{2n} f(B^\nu(z)) \quad (2.71)$$

$B^\nu(z)$ is a contraction of the open unit disk in the sense that $|B^{\nu+1}(z)| \leq |B^\nu(z)|$ for all z in the open unit disk since :

$$|B^{\nu+1}(z)| = |B(B^\nu(z))| \quad (2.72)$$

$$\leq |B^\nu(z)|^2 \quad (2.73)$$

$$\leq |B^\nu(z)| \quad (2.74)$$

Hence for all $j \leq 2n$ we have $|B^j(z)| \geq |B^{2n}(z)| = \eta_n \geq r_1 \geq u_\epsilon$.

Therefore $|P_1| = \prod_{\nu=0}^{2n} |f(B^\nu(z))| \geq (1 + \epsilon)^{2n+1}$

Estimate $|P_2|$:

$$P_2 = \prod_{\nu=2n+1}^{2n+d} f(B^\nu(z)) \quad (2.75)$$

$$= \prod_{\nu=1}^d |f(B^\nu(B^{2n}(z)))| \quad (2.76)$$

$$= h(B^{2n}(z)) \quad (2.77)$$

$$z \in \gamma_n \Rightarrow B^{2n}(z) \in A \quad (2.78)$$

$$\Rightarrow \prod_{\nu=1}^d |f(B^\nu(B^{2n}(z)))| > \alpha > 0 \quad (2.79)$$

Therefore $|P_2| > \alpha$

Estimate $|P_3|$:

$$P_3 = \prod_{\nu=2n+d+1}^{3n} f(B^\nu(z)) \quad (2.80)$$

$$= \prod_{\nu=1}^{n-d} |f(B^\nu(B^d(B^{2n}(z))))| \quad (2.81)$$

By Lemma 1.3.2

$$z \in \gamma_n \Rightarrow B^{2n}(z) \in A \quad (2.82)$$

$$\Rightarrow |B^{2n}(z)| < r_2 \quad (2.83)$$

$$\Rightarrow |B^d(B^{2n}(z))| < |B^{2n}(z)|^{2^d} < r_2^{2^d} < v_\epsilon \quad (2.84)$$

If $|z| < 1$ then $|B^{\nu+1}(z)| < |B^\nu(z)|$ so that $|B^\nu(B^d(B^{2n}(z))))| < v_\epsilon$ for $\nu \geq 1$

$\Rightarrow |f(B^\nu(B^d(B^{2n}(z))))| > \frac{1}{1+\epsilon}$ for $\nu \geq 1$

$$|P_3| = \prod_{\nu=2n+d+1}^{3n} |f(B^\nu(z))| > \left(\frac{1}{1+\epsilon}\right)^{n-d} = \frac{1}{(1+\epsilon)^{n-d}} \quad (2.85)$$

Estimate $|P_4|$:

$$P_4 = \prod_{\nu=3n+1}^{\infty} f(B^\nu(z)) \quad (2.86)$$

From the above $\nu > 3n \Rightarrow |B^\nu(z)| < v_\epsilon$. Now the minimum modulus function for $f(z)$ is strictly decreasing for $|z| < v_\epsilon$. We know that $|B^\nu(z)| < |z|^{2^\nu}$, and we let $m = 2n + d$ so that $\nu - m > K + d > 0$.

$$|f(B^\nu(z))| \geq \min_{z \in \gamma_n} |f(B^{\nu-m}(B^d(B^{2n}(z))))| \quad (2.87)$$

$$\geq \min_{|z| \leq v_\epsilon} |f(B^{\nu-m}(z))| \quad (2.88)$$

$$\geq \min_{|z| \leq v_\epsilon} |f(z^{2^{\nu-m}})| \quad (2.89)$$

$$\geq \min_{|z|=v_\epsilon} |f(z^{2^{\nu-m}})| \quad (2.90)$$

If $f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j$, then we have

$$|f(z^{2^{\nu-m}})| \geq 1 - \sum_{j=1}^{\infty} |a_j| v_\epsilon^{j 2^{\nu-m}} \quad (2.91)$$

$$\geq 1 - v_\epsilon^{2^{\nu-m}} \sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1} \quad (2.92)$$

$$\geq 1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{2^{\nu-m}} \quad (2.93)$$

$$\Rightarrow |P_4| \geq \prod_{\nu=3n+1}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{2^{\nu-m}} \right) \geq \prod_{\nu=K}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{2^\nu} \right) \quad (2.94)$$

Define $\prod_{\nu=K}^{\infty} \left(1 - \frac{\sum_{j=1}^{\infty} |a_j| v_\epsilon^{j-1}}{2^\nu} \right) = \beta$

Therefore $|P_4| > \beta$

So we have

$$|g(z)| = |P_1 \cdot P_2 \cdot P_3 \cdot P_4| \quad (2.95)$$

$$\geq (1 + \epsilon)^{2n+1} \cdot \alpha \cdot \frac{1}{(1 + \epsilon)^{n-d}} \cdot \beta \quad (2.96)$$

$$= \alpha \beta (1 + \epsilon)^{n+d+1} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (2.97)$$

In the definition of annular we need that $\gamma_n \subset \text{interior of } \gamma_{n+1}$. In the construction

given above that is not guaranteed. Since $\gamma_n \subseteq A'_n$ the γ_n move toward the boundary, there exists a subsequence n_j so that $\gamma_{n_j} \subset \text{interior of } \gamma_{n_{j+1}}$. Hence we have, $\lim_{k \rightarrow \infty} \min_{z \in \gamma_{n_j}} |g(z)| = \infty$, therefore $g(z)$ is annular.

■

In the above proof we considered a sequence of Blaschke products $\{B^n(z)\}$ which is the sequence of iterates of $B(z)$. In order to form infinite products which are annular we need not restrict the sequence of Blaschke products to the iterates of a Blaschke product. The following corollary gives examples of other types of sequences of Blaschke products that may be composed with functions in BB to produce functions which are annular.

Corollary 2.3.1 *Let $f(z) \in BB$ and $B(z)$ a finite Blaschke product with a zero of at least order two at the origin. If $\tau(z)$ is a Möbius transformation, then*

$$g(z) = \prod_{\nu=0}^{\infty} f(B^{\nu}(\tau(z))) \quad (2.98)$$

is an annular function.

Proof:

Let $h(z) = \prod_{\nu=0}^{\infty} f(B^{\nu}(z))$. Then $h(z)$ is an annular function by Theorem 2.3.1. We also have

$$g(z) = h(\tau(z)) \quad (2.99)$$

Bonar showed that the composition of an annular function with a Möbius transformation results in an annular function (see Bonar, [1]). Hence $g(z)$ is an annular function.

2.4 Open Questions on Annular Products

In this section we will discuss some unresolved questions about creating infinite products which are annular or strongly annular. Bonar gives a method of constructing infinite products that are annular (see Bonar [1]). His examples show that if each factor of the product is “big near the boundary”, and for each factor there corresponds a circle on which the factor dominates the rest of the product, then the product will be strongly annular. This example shows how one can select a sequence of functions from BB of the form $1 + c_n z^{m_n}$ so that $\prod(1 + c_n z^{m_n})$ is a strongly annular product. It would be interesting to know if the class of functions BB is too small. In other words, are the conditions given necessary for a function which satisfies equation (2.1) to be annular?

Question 2.4.1 *Are the conditions given in Definition 1.2.1 both necessary and sufficient for the function given in Theorem 2.2.1 to be strongly annular?*

Question 2.4.2 *Are the conditions given in Definition 1.2.1 both necessary and sufficient for the function given in Theorem 2.3.1 to be annular?*

In order to answer the questions above it might be useful to have a better description of the class of functions BB . The first two conditions of Definition 1.2.1 can not be relaxed, or else the product would not necessarily be a holomorphic function. The third condition about the behavior of the function near the unit circle might have

a more useful description in terms of the measure of the set of points on which the function is larger than 1 near (or on) the unit circle. It might not be necessary for the measure of the set of points that are larger than 1 near the boundary to be 2π .

Question 2.4.3 *Can the third condition of Definition 1.2.1 be replaced by a condition involving the measure of the set of points on the boundary where the function is larger than 1?*

Can the functions in BB be described in terms of the classical factorization theorem valid for functions in, say, the Nevanlinna Class, into the product of three functions (see Hoffman, [12]). The first factor is a Blaschke Product giving the zeros, the second a singular function which is determined by a singular measure on the unit circle, and the third an outer function that determines the original function's behavior near the boundary of the disk. Not every function on the disk has such a factorization, for instance annular functions can not be factored in such a manner because the zeros of an annular function do not satisfy the Blaschke condition. Functions in BB cannot be annular since they have only finitely many zeros.

Question 2.4.4 *Do functions in BB have a factorization into a finite Blaschke Product, a singular function and an outer function?*

An affirmative answer to the question above then leads one to explore the following problem.

Question 2.4.5 *What would be a description in terms of the classical factorization theorem, of a function in the class BB into a Blaschke Product, a singular function,*

and an outer function?

One could also conjecture about how to construct annular or strongly annular infinite products by starting with a function in the class BB and asking what types of functions holomorphic on the disk it can be composed with to form a product that is annular or strongly annular. In light of Corollary 2.3.1 it is not necessary for the functions being composed to be iterates of a Blaschke Product, or even have a zero at the origin. The questions can be more precisely posed as follows:

Question 2.4.6 *Given a function $f(z)$ in BB and a sequence of positive integers $\{m_k\}$ going to infinity, what are necessary and sufficient conditions on the sequence $\{m_k\}$ so that $\Pi(f(z^{m_k}))$ is a strongly annular function?*

An answer would give conditions on how fast the sequence $\{m_k\}$ can go to infinity. We saw in Theorem 2.2.1 that if the sequence $\{m_k\}$ is $\{p^k\}$ the product is strongly annular, but this sequence goes to infinity quickly compared to the sequence $\{k^2\}$, so this leads one to ask the following:

Question 2.4.7 *If $f(z)$ is in BB , and if $k \geq 1$ is an integer, does it follow that $\prod_{n=0}^{\infty} (f(z^{n^k}))$ must be annular or strongly annular?*

It is important to keep in mind that the functions that are being composed can not go too quickly to the origin; for example, if we compose the iterates of the function $\frac{z}{2}$ with the BB function in Example 1.2.1 then $\Pi(1 + A\frac{z}{2^n})$ with $|A| > 2$ is bounded and hence not annular. Again keeping in mind Corollary 2.3.1 a question may be posed the following way:

Question 2.4.8 *Given a function $f(z)$ in BB and a sequence of finite Blaschke Products $\{B_k(z)\}$, what are necessary and sufficient conditions on the sequence $\{B_k(z)\}$ so that $\Pi(f(B_k(z)))$ is a strongly annular function?*

CHAPTER III

The Algebraic and Analytical Description of the Solution Space

3.1 Introduction

The purpose of this chapter is to give some information concerning the algebraic and analytical nature of the solutions of the functional equation

$$g(z) = f(z)g(B(z)) \tag{3.1}$$

In order for this equation to have a unique well-defined solution g , it is necessary to place restrictions on the functions $f(z)$ and $B(z)$. The function $f(z)$ will be taken to be holomorphic in the open unit disk with its value at the origin being one, while $B(z)$ is a finite Blaschke product with a zero of at least order two at the origin. Notice that the Mahler Equation (2.1) is a special case of this.

The following Lemma shows that the solution $g(z)$ to Equation 3.1 is unique up to a constant multiple.

Lemma 3.1.1 *Suppose that $g(z)$ is an analytic function for $|z| < 1$ and that $g(z)$ satisfies the functional equation:*

$$g(z) = f(z)g(B(z)) \tag{3.2}$$

with $f(0) = 1$ and B a finite Blaschke Product with a zero of at least order 2 at the origin.

Then

$$g(z) = g(0) \prod_{k=0}^{\infty} f(B^k(z)) \quad (3.3)$$

Proof:

Fix r in $0 < r < 1$ and let $z \in \{z : |z| < r\}$.

We wish to show that $g(z) = \prod_{k=0}^n f(B^k(z))g(B^{n+1}(z))$, for $n = 0, 1, \dots$

The proof is by induction. The case $n = 0$ is Equation 3.1.

Assume that for some $n \geq 0$, the following holds

$$g(z) = \prod_{k=0}^n f(B^k(z))g(B^{n+1}(z)). \quad (3.4)$$

Then applying Equation 3.1 with z replaced by $B^{n+1}(z)$ we obtain

$$g(z) = \prod_{k=0}^n f(B^k(z))f(B^{n+1}(z))g(B^{n+2}(z)) \quad (3.5)$$

$$= \prod_{k=0}^{n+1} f(B^k(z))g(B^{n+1}(z)) \quad (3.6)$$

Therefore $g(z) = \prod_{k=0}^n f(B^k(z))g(B^{n+1}(z))$

Since $|z| < 1$, the sequences $\prod_{k=0}^n f(B^k(z))$ and $g(B^{n+1}(z))$ are uniformly convergent in $\{z : |z| < r\}$.

Therefore

$$g(z) = \lim_{n \rightarrow \infty} \prod_{k=0}^n f(B^k(z))g(B^{n+1}(z)) \quad (3.7)$$

$$= g(0) \prod_{k=0}^{\infty} f(B^k(z)) \quad (3.8)$$

■

Recall that such products were considered in Chapter II. Multiplying an annular or strongly annular function by a nonzero constant does not change the annularity of the solution.

3.2 An Analytical Description

We shall give an analytical description of the solution space for Equation (3.1). Since $g(z)$ is determined by $g(0)$, the set of solutions of Equation (3.1) is equivalent to $\mathbb{C}$.[•] What we will be most concerned about is estimating a function with a solution of Equation (3.1) that is strongly annular. The general idea is that the solutions of Equation (3.1) are dense in the space of all holomorphic functions on the unit disk in the topology of almost uniform convergence. To show this we will need a version of Runge's Theorem as follows.

Theorem 3.2.1 (Runge's Theorem) *$\mathbb{C}[z]$ is dense in the set of all functions holomorphic on the open unit disk with the topology of almost uniform convergence. (See Leuking and Rubel [18])*

This just says that the polynomials are dense in the set of all functions holomorphic on the open unit disk with the topology of almost uniform convergence. To show another set of functions is dense it is enough to show that the new set is dense in $\mathbb{C}[z]$.

Lemma 3.2.1 *The set $\mathbb{C}[z] \cap \{cf(z) : c \in \mathbb{C}, c \neq 0, f(z) \in BB\}$ is dense in the set of all functions holomorphic in the open unit disk with the topology of almost uniform convergence.*

Proof:

Since polynomials are dense in this topology by Runge's Theorem it is enough to show that this set is dense in the set of polynomials.

Let K be a compact subset of the open unit disk, let $\epsilon > 0$ be given, and let $p(z) \in \mathbb{C}[z]$.

Let $p(z) = \sum_{j=0}^n a_j z^j$.

Choose $b_0 \neq 0$ so that $|a_0 - b_0| < \frac{\epsilon}{3}$.

Let $b_i = a_i$ for $i = 1, 2, \dots, n$.

Let $c_j = \frac{b_j}{b_0}$ for $j = 0, 1, 2, \dots, n$.

Choose a number $A \in \mathbb{C}$ so that $|A| - \sum_{j=0}^n |c_j| > 1$.

Choose a positive integer $m > n$ so that $|b_0| |A| |z|^m < \frac{\epsilon}{3}$ on K .

Let $b(z) = \sum_{j=0}^n b_j z^j$ and $c(z) = \sum_{j=0}^n c_j z^j + Az^m$.

Now

$$\min_{|z|=1} |c(z)| \geq |A| - \sum_{j=0}^n |c_j| > 1 \quad (3.9)$$

and $c_0 = 1$ so that $c(z) \in BB$. For z in K we have

$$|p(z) - b_0 c(z)| \leq |p(z) - b(z)| + |b(z) - b_0 c(z)| \quad (3.10)$$

$$= |a_0 - b_0| + |b_0| |A| |z|^m \quad (3.11)$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} < \epsilon \quad (3.12)$$

That is, for any $p(z) \in \mathbb{C}[z]$ and compact K a subset of the open unit disk and $\epsilon > 0$ we can find $b_0 \neq 0$ and $c(z) \in BB$ so that $|p(z) - b_0 c(z)| < \epsilon$, for $z \in K$.

This says that $\mathbb{C}[z] \cap \{cf(z) : c \in \mathbb{C}, c \neq 0, f(z) \in BB\}$ is dense in the set of all holomorphic functions on open unit disk with the topology of almost uniform convergence.

■

Theorem 3.2.2 *The set of strongly annular functions that satisfy equation (2.1) is dense in the set of all functions holomorphic on the open unit disk with the topology of almost uniform convergence.*

Proof:

It is enough to show that the set of polynomials can be estimated almost uniformly by such functions. Let $p(z) \in \mathbb{C}[z]$, K a compact subset of the open unit disk, and $\epsilon > 0$ be given. By Lemma 3.2.1 there is a polynomial $q(z) \in BB$ and a constant $c \neq 0$ so that $|p(z) - cq(z)| < \frac{\epsilon}{3}$ on K .

Define L_p as follows for positive integers p :

$$L_p(z) = \prod_{k=0}^{\infty} q(z^{p^k}) \quad (3.13)$$

It is important to notice that as $p \rightarrow \infty$ the sequence of functions $L_p(z^p)$ converge to the function $a(z) \equiv 1$, uniformly on K .

Choose $p > \max\{1, \deg q(z)\}$, and also choose p so large that for all $z \in K$ we have the following : $|cq(z)(1 - L_p(z^p))| < \frac{\epsilon}{3}$. Then for $z \in K$ we have

$$|p(z) - cL_p(z)| = |p(z) - cq(z)L_p(z^p)| \quad (3.14)$$

$$\leq |p(z) - cq(z)| + |cq(z)(1 - L_p(z^p))| \quad (3.15)$$

$$< \epsilon \quad (3.16)$$

Since $l(z)$ is in BB , $c \prod_{k=0}^{\infty} l(z^{p^k})$ is a strongly annular solution of Equation (2.1), by Theorem 2.2.1. Hence the strongly annular solutions of Equation (2.1) are dense in the set of all holomorphic functions on the open unit disk with the topology of almost uniform convergence. ■

Theorem 3.2.3 *Let M denote the set of all non-constant holomorphic functions g on the unit disk such that, for some integer p , $p \geq 2$, the meromorphic function $\frac{g(z)}{g(z^p)}$ is actually a polynomial. Then M is a first category set in the topology of almost uniform convergence of all functions holomorphic on the open unit disk.*

Proof:

For each fixed $p \geq 2$ and $n \geq 0$, let $M_{n,p}$ denote the subset of M for which the polynomial in question is of degree at most n . We shall show that each $M_{n,p}$ is closed in the set of all holomorphic functions on the open unit disk with the topology of almost uniform convergence, but contains no polynomial and hence is nowhere dense.

Let g be in the closure of $M_{n,p}$ and let $\{g_j(z)\} \subset M_{n,p}$ with $g_j(z) \rightarrow g$ almost uniformly.

$$\Rightarrow g_j(z^p) \rightarrow g(z^p) \text{ almost uniformly.}$$

$$\Rightarrow \frac{g_j(z)}{g_j(z^p)} \rightarrow \frac{g(z)}{g(z^p)}$$

But $\frac{g_j(z)}{g_j(z^p)}$ is a polynomial r_j of degree at most n . Then the almost uniform limit of $\frac{g_j(z)}{g_j(z^p)}$ is a polynomial of degree at most n . Hence $M_{n,p}$ is closed.

If $q(z)$ is a non-trivial polynomial than $\frac{q(z)}{q(z^p)}$ will have a pole so that $\frac{q(z)}{q(z^p)}$ is not in $M_{n,p}$ for any n and p . If $\mathcal{U}$ is a nonempty open set and $\mathcal{U} \subset M_{n,p}$ then there exists

a non-constant polynomial $q(z) \in \mathcal{U}$ by Theorem 3.2.1. But this is impossible since $q(z) \notin M_{n,p}$, so it must be the case that $\mathcal{U}$ is trivial.

Hence $M_{n,p}$ has no interior.

Therefore M is a countable union of closed nowhere dense sets.

Therefore M is of first category. ■

3.3 Interesting Examples

R.W.Howell [13] showed that strongly annular functions in the complete metric space of functions of the form $\sum \pm z^n$ formed a residual subset of that space. Using the results and methods of Chapter II, we show how to give an explicit construction for some such series.

Theorem 3.3.1 *Let g satisfy $g(0) = 1$, and*

$$g(z) = P(z)g(z^5) \tag{3.17}$$

where

$$P(z) = 1 - z + z^2 + z^3 + z^4. \tag{3.18}$$

Then g is strongly annular with coefficients ± 1 .

We proceed with the details.

Lemma 3.3.1 *If $P(z)$ is the polynomial of degree four given by (3.18) and if $\mu =$*

$$\min_{|z|=1} |P(z)|, \text{ then } \mu = \frac{\sqrt{11}}{2}$$

Proof:

$$P(e^{i\theta}) = e^{2i\theta}[e^{-2i\theta} - e^{-i\theta} + 1 + e^{i\theta} + e^{2i\theta}] \quad (3.19)$$

$$= e^{2i\theta}[2 \cos 2\theta + 1 + 2i \sin \theta] \quad (3.20)$$

$$|P(e^{i\theta})|^2 = (1 + 2 \cos 2\theta)^2 + 4 \sin^2 \theta \quad (3.21)$$

$$= (1 + 2 \cos^2 \theta - 2 \sin^2 \theta)^2 + 4 \sin^2 \theta \quad (3.22)$$

$$= (3 - 4 \sin^2 \theta)^2 + 4 \sin^2 \theta \quad (3.23)$$

$$= 16 \sin^4 \theta - 24 \sin^2 \theta + 4 \sin^2 \theta + 9 \quad (3.24)$$

$$= 16(\sin^4 \theta - \frac{20}{16} \sin^2 \theta) + 9 \quad (3.25)$$

$$= 16(\sin^4 \theta - \frac{5}{4} \sin^2 \theta + \frac{25}{64}) + 9 - \frac{25}{4} \quad (3.26)$$

$$= 16(\sin^2 \theta - \frac{5}{8})^2 + \frac{11}{4} \quad (3.27)$$

■

Corollary 3.3.1 *The polynomial (3.18) is in the class BB.*

Proof: (Theorem 3.3.1)

We define a sequence of polynomials f_n as follows:

$$f_1(z) = P(z) \quad (3.28)$$

$$f_{n+1}(z) = f_n(z^5)P(z), n = 1, 2, \dots \quad (3.29)$$

Then:

1. $f_n(z)$ is a polynomial of degree $5^n - 1$.

2. The coefficients of $f_n(z)$ are ± 1 .

Proof (by induction):

$$1. \deg(f_1(z)) = \deg(1 - z + z^2 + z^3 + z^4) = 5^1 - 1$$

If for some integer $n \geq 1$ we have $\deg f_n(z) = 5^n - 1$, then

$$\deg f_{n+1}(z) = \deg(f_n(z^5)P(z)) \quad (3.30)$$

$$= \deg f_n(z^5) + \deg P(z) \quad (3.31)$$

$$= 5 \deg f_n(z) + 4 \quad (3.32)$$

$$= 5(5^n - 1) + 4 \quad (3.33)$$

$$= 5^{n+1} - 5 + 4 \quad (3.34)$$

$$= 5^{n+1} - 1 \quad (3.35)$$

2. In $f_1(z) = 1 - z + z^2 + z^3 + z^4$, all the coefficients are ± 1 . Assume

$$f_n(z) = \sum_{\nu=0}^{5^n-1} \epsilon_\nu z^\nu \quad \text{with } \epsilon_\nu = \pm 1 \quad (3.36)$$

$$f_{n+1}(z) = f_n(z^5)P(z) \quad (3.37)$$

$$= \sum_{\nu=0}^{5^n-1} \epsilon_\nu z^{5\nu} P(z) \quad (3.38)$$

$$= \sum_{\nu=0}^{5^n-1} \epsilon_\nu z^{5\nu} - \epsilon_\nu z^{5\nu+1} + \epsilon_\nu z^{5\nu+2} + \epsilon_\nu z^{5\nu+3} + \epsilon_\nu z^{5\nu+4} \quad (3.39)$$

$$= \sum_{j=0}^{5^{n+1}-1} k_j z^j \quad \text{with } k_j = \epsilon_\nu = \pm 1. \quad (3.40)$$

The $f_k(z)$ are the partial products of the function

$$g(z) = \prod_{\nu=0}^{\infty} P(z^{5^\nu}) \quad (3.41)$$

which will therefore also have coefficients ± 1 . It follows from Lemma 3.3.1 and Theorem 2.2.1 that g is strongly annular.

■

The most common example of power series that represent strongly annular functions are those whose coefficients grow fast enough so that one term dominates on a particular circle (See Bonar [1]). In the power series in (3.41) there are no such dominant terms since all coefficients are ± 1 .

Example 3.3.1 *The following example illustrates the importance of the class BB . It gives an analog of (3.41) with P not in BB , and the product not annular.*

Let $Q(z) = 1 + z$, then we have

$$\prod_{\nu=0}^{\infty} Q(z^{2^\nu}) = \prod_{\nu=0}^{\infty} (1 + z^{2^\nu}) = \sum_{\nu=0}^{\infty} z^\nu = \frac{1}{1-z} \quad (3.42)$$

This function is bounded on any radial path not ending at 1, hence it is not annular.

This occurs despite the fact that $|Q(z)| > 1$ for $\frac{2}{3}$ of the unit circle. For a function to be sufficiently big near the boundary, the measure of the set of points near the boundary that have the modulus of the function greater than 1 would probably need to be at least 2π (see Question 2.4.3).

3.4 Algebraic Structure

While it is true that the products and sums of annular functions are not necessarily annular, we can restrict the set of annular functions so that they are closed under multiplication.

Let $f(z)$ and $h(z)$ be in BB . Then the following hold:

1. $f(z^p) \in BB$ for any positive integer p .
2. $f(z) \cdot h(z) \in BB$

Notice that conditions 1 and 2 of Definition 1.2.1 still hold so that we just need to show that the function stays bigger than 1 near the boundary. There would exist ρ_1 and ρ_2 with $0 < \rho_1, \rho_2 < 1$ so that if $|z| > \rho_1$ then $|f(z)| > \alpha > 1$ and if $|z| > \rho_2$ then $|g(z)| > \beta > 1$. If $|z| > \sqrt[p]{\rho_1}$ then $|z^p| > \rho_1$ so that $|f(z^p)| > \alpha > 1$ and hence $f(z^p)$ is in BB . For part 2 if $\rho = \max\{\rho_1, \rho_2\}$ then if $|z| > \rho$ then $|f(z)| |h(z)| > \alpha\beta > 1$, hence the product is in BB .

Now suppose that m, n are two positive integers with $m|n$, and further suppose that

$$g_1(z) = f(z)g_1(z^{p^n}) \quad (3.43)$$

$$g_2(z) = h(z)g_2(z^{p^m}) \quad (3.44)$$

Suppose that $n = lm$. Then $g_2(z) = \prod_{k=0}^{l-1} h(z^{p^{km}}) \cdot g_2(z^{p^{lm}})$ so that the product of the two functions is $g_1(z)g_2(z) = f(z) \prod_{k=0}^{l-1} h(z^{p^{km}})g_1(z^{p^n})g_2(z^{p^n})$

Notice that if $f(z)$ and $h(z)$ are in BB then the product of the strongly annular functions $g_1(z)$ and $g_2(z)$ is strongly annular. Their product also satisfies functional equation (3.1).

It is not difficult to show that an annular function cannot be algebraic. But if we recall that in Example 3.3.1 that if the functions that form the products do not satisfy condition 3 of Definition 1.2.1 then Equation (3.1) can have rational solutions.

So for the sake of completeness we will give some idea of what algebraic solutions will look like.

Lemma 3.4.1 *If $R(z) \in \mathbb{C}[[z]]$ with $R(0) = 1$ and $A \in \mathbb{C}[[z]]$ with the relation $(R(z))^k = \frac{A(z^p)}{A(z)}$ for $p \geq 2$ and $k > 1$ then A is a k^{th} power of a rational function.*

Proof:

Let $A(z) = c \prod_{\nu=1}^N (z - a_\nu)^{m_\nu}$, where $a_1, \dots, a_N$ are distinct, and m_ν are positive or negative integers. It is required to prove that every m_ν is a multiple of k , or that $\alpha_1, \dots, \alpha_N$ are all zero, where $m_\nu = \lfloor \frac{m_\nu}{k} \rfloor k + \alpha_\nu$ for $0 \leq \alpha_\nu \leq k - 1$.

Let

$$B(z) = c \prod_{\nu=1}^N (z - a_\nu)^{\lfloor \frac{m_\nu}{k} \rfloor k} \quad (3.45)$$

$$A_1(z) = \frac{A(z)}{B(z)} = \prod_{\nu=1}^N (z - a_\nu)^{\alpha_\nu} \quad (3.46)$$

$$R_1(z) = \prod_{\nu=1}^N \frac{(z - a_\nu)^{\lfloor \frac{m_\nu}{k} \rfloor k}}{(z^p - a_\nu)^{\lfloor \frac{m_\nu}{k} \rfloor k}} \quad (3.47)$$

Then

$$\frac{A_1(z^p)}{A_1(z)} = \frac{A(z^p)}{A(z)} \frac{B(z)}{B(z^p)} = [R(z)]^k [R_1(z)]^k \quad (3.48)$$

That is, we have

$$\frac{A_1(z^p)}{A_1(z)} = \left[Q(z) \right]^k \quad (3.49)$$

where Q is rational and A_1 is a rational polynomial given by (3.46), with $a_1, \dots, a_N$ distinct, and $0 \leq \alpha_\nu < k - 1$. Let s be the smallest value of ν such that $0 < \alpha_\nu$, so that

$$A_1(z) = \prod_{\nu=s}^N (z - a_\nu)^{\alpha_\nu} \quad \text{and,} \quad (3.50)$$

$$\frac{A_1(z^p)}{A_1(z)} = \frac{\prod_{\nu=s}^N (z^p - a_\nu)^{\alpha_\nu}}{\prod_{\nu=s}^N (z - a_\nu)^{\alpha_\nu}} \quad (3.51)$$

Let S be the set with the $p(N - s + 1)$ distinct elements

$$\eta_{s,1}, \dots, \eta_{s,p}, \eta_{s+1,1}, \dots, \eta_{N,p}$$

where $(z^p - a_\nu) = \prod_{\pi=1}^p (z - \eta_{\nu,\pi})$ $\nu = s, s+1, \dots, N$.

Considering (3.49) we see that unless $a_\nu = \eta_{s,1}$ for some ν , then $\eta_{s,1}$ is a zero of order α_s on the left side of (3.49), hence a zero of order α_s for the right side. But all zeros of the right side are zeros of order k at least. Since $0 < \alpha_s \leq k-1$, we have a contradiction.

On the other hand, suppose that $\eta_{s,1} = a_\nu$ for some ν . Then $\eta_{s,1}$ is a zero of order $\alpha_s - \alpha_\nu$ (or a pole of order $\alpha_\nu - \alpha_s$). Since $|\alpha_s - \alpha_\nu| \leq k-2$ and all poles and zeros have orders which are multiples of k , we must have $\alpha_\nu = \alpha_s$.

Since we are assuming $\alpha_s \geq 1$, the previous argument shows that we now must have $\eta_{s,\pi} = a_{\nu(\pi)}$ and $\alpha_{\nu(\pi)} = \alpha_s$ for $\pi = 1, \dots, p$. We continue in this way, arriving finally at the expression

$$\frac{A_1(z^p)}{A_1(z)} = \frac{\prod_{\nu \in S_1} (z^p - a_\nu)^{\alpha_\nu}}{\prod_{\nu \in S_2} (z - a_\nu)^{\alpha_\nu}} \quad (3.52)$$

where no a_ν ($\nu \in S_2$) is any $\eta_{\nu,\pi}$, ($\nu \in S_1$). Then $\frac{A_1(z^p)}{A_1(z)}$ has a pole at a_ν of order α_ν exactly. But the poles of Q^k have order multiples of k , so $\alpha_\nu = 0$, $\nu \in S_2$. Similarly, $\frac{A_1(z^p)}{A_1(z)}$ has a zero at each $\eta_{\nu,\pi} : \nu \in S_1$ of order α_ν , which must be a multiple of k , which then must be zero. Hence $\frac{A_1(z^p)}{A_1(z)} \equiv 1$. Then $A_1(z^p) = A_1(z)$, A_1 is a polynomial so $A_1 \equiv c$.

Theorem 3.4.1 *Let $f(z)$ be analytic at 0 and let $p \geq 2$. Suppose $f(z)$ satisfies the functional equation*

$$f(z^p) = R(z)f(z) \quad (3.53)$$

where $R(z)$ is rational and $R(0) = 1$. Then $f(z)$ is either rational or transcendental.

Proof:

Suppose that $f(z)$ is not rational and is algebraic. Then there is a polynomial $P(y) \in \mathbb{C}[[z]](y)$, P minimal with respect to the variable y for which $P(f(z)) \equiv 0$.

$$P(y) = y^k + \sum_{\nu=0}^{k-1} A_\nu(z)y^\nu \quad (3.54)$$

Without loss of generality we can assume the coefficient of y^k is 1 since each function $A_\nu(z)$ is rational.

$$(f(z))^k + \sum_{\nu=0}^{k-1} A_\nu(z)(f(z))^\nu \equiv 0 \quad (3.55)$$

$$(f(z^p))^k + \sum_{\nu=0}^{k-1} A_\nu(z^p)(f(z^p))^\nu \equiv 0 \quad (3.56)$$

$$(R(z))^k f(z)^k + \sum_{\nu=0}^{k-1} A_\nu(z^p)(R(z))^\nu (f(z))^\nu \equiv 0 \quad (3.57)$$

Multiplying Equation (3.55) by $(R(z))^k$ and subtracting from Equation (3.57) we get $\sum_{\nu=0}^{k-1} (f(z))^\nu \{A_\nu(z^p)(R(z))^\nu - A_\nu(z)(R(z))^k\} = 0$. By the minimality of the polynomial $P(y)$, each term must be identically zero so that $A_\nu(z^p)(R(z))^\nu = A_\nu(z)(R(z))^k$ for all ν . Let $\nu_0 < k$ be the largest ν so that $A_\nu(z) \not\equiv 0$. Then $(R(z))^{k-\nu_0} = \frac{A_{\nu_0}(z^p)}{A_{\nu_0}(z)}$

By Equation (3.53) we have $(f(z^p))^{k-\nu_0} = (R(z))^{k-\nu_0}(f(z))^{k-\nu_0}$, hence

$$\left(\frac{f(z^p)}{f(z)}\right)^{k-\nu_0} = \frac{A_{\nu_0}(z^p)}{A_{\nu_0}(z)} \quad (3.58)$$

Let the function $\mathcal{A}$ be any branch of $(A_{\nu_0}(z))^{\frac{1}{k-\nu_0}}$ analytic in a neighborhood of the origin. Let $C(z)$ be defined by $f(z) = C(z)\mathcal{A}(z)$.

Then Equation (3.58) gives $(C(z^p))^{k-\nu_0} = (C(z))^{k-\nu_0}$, so that $C(z) \equiv c$ a constant.

$$f(z) = c(A_{\nu_0}(z))^{\frac{1}{k-\nu_0}} \quad (3.59)$$

$$(f(z))^{k-\nu_0} = c^{k-\nu_0} A_{\nu_0}(z) \quad (3.60)$$

Unless $\nu_0 = 0$, this gives a contradiction since the degree of a minimum polynomial for $f(z)$ is k . It must be the case that $\nu_0 = 0$ and

$$P(y) = y^k + A_0(t) \quad (3.61)$$

$$(f(z))^k = -A_0(z) \quad (3.62)$$

$$(R(z))^k = \left(\frac{f(z^p)}{f(z)}\right)^k = \frac{A_0(z^p)}{A_0(z)} \quad (3.63)$$

But this contradicts Lemma 3.4.1 since it would say that $f(z)$ is rational. Hence it must be the case that $f(z)$ is transcendental.

■

3.5 Universal Functions

A connection between the concept of approximating analytic functions and the algebraic structure of the compositions of a set of functions on a given domain is represented by a universal function. Recall that a universal function is one whose composition with a group of automorphisms is dense in the almost uniform topology. Luecking and Rubel construct a function that is universal on the entire complex plane with the automorphism group being translation by a constant $z \mapsto z + c$ (see Luecking and Rubel [18]). In another paper on transcendently transcendental functions Rubel shows why such a function universal on the plane with respect to translations must also be transcendently transcendental. In an earlier paper Seidel and Walsh [25] showed how to construct a function universal on the open unit disk with the automorphism group being the set of all Möbius transformations. The following proof parallels the same idea as that of Rubel for the plane (see Rubel [23]).

Theorem 3.5.1 (Rubel) *Let $f(z)$ be a universal function on a domain D with respect to a set of transformations of D . Then $f(z)$ is transcendently transcendental.*

Proof:

It is known that if a function satisfied an algebraic differential equation then it would satisfy a autonomous algebraic differential equation. A differential equation $F(z, y^{(0)}, \dots, y^{(n)}) = 0$ is said to be autonomous if it does not involve z explicitly, i.e., if it is of the form $F(y^{(0)}, y^{(1)}, \dots, y^{(n)}) = 0$. Consider the autonomous polynomial $P(y_0, \dots, y_n) \in \mathbb{C}[y_0, \dots, y_n]$ and $P \not\equiv 0$ but with $P(f^{(0)}(z), \dots, f^{(n)}(z)) \equiv 0$.

Therefore we have

$$P(f^{(0)}(\tau(z)), \dots, f^{(n)}(\tau(z))) \equiv 0 \quad (3.64)$$

for every τ in the transformation set. Let $g(z)$ be any function holomorphic on the disk. Then there is a sequence $\{\tau_i(z)\}$ of transformations of D so that $f \circ \tau_i(z) \rightarrow g(z)$ almost uniformly. Since the sequence of derivatives of $\frac{d}{dz} f \circ \tau_i(z) \rightarrow g'(z)$ almost uniformly and the polynomial P is a continuous function of z , we have:

$$P(g^{(0)}(z), g^{(1)}(z), \dots, g^{(n)}(z)) \equiv 0 \quad (3.65)$$

This implies that every function satisfies the algebraic differential equation $P \equiv 0$, or is differentially algebraic on the domain D . This is a contradiction since there are entire functions that are transcendently transcendental. It is well known that $\Gamma(z)$ is transcendently transcendental (see Rubel [23]) so consider $\frac{1}{\Gamma(z)}$ restricted to the domain D and the contradiction is obvious. ■

To describe the set of universal functions in terms of their geometrical or analytical properties we want to be able to get an idea of the size of the set of universal functions in terms of category.

Lemma 3.5.1 *If $f_n \rightarrow f$ uniformly on a compact set K then $\max_{z \in K} |f_n| \rightarrow \max_{z \in K} |f|$*

Proof:

$$|f_n(z)| \leq |f_n(z) - f(z)| + |f(z)| \quad (3.66)$$

$$\leq \epsilon + \max_{z \in K} |f(z)| \quad \forall n > N_0 \quad (3.67)$$

$$\Rightarrow \max_{z \in K} |f_n(z)| \leq \max_{z \in K} |f(z)| \quad \forall n > N_0 \quad (3.68)$$

Conversely

$$|f(z)| \leq |f(z) - f_n(z)| + |f_n(z)| \quad (3.69)$$

$$\leq \epsilon + \max_{z \in K} |f_n(z)| \quad \forall n > N_0 \quad (3.70)$$

$$\Rightarrow \max_{z \in K} |f(z)| \leq \max_{z \in K} |f_n(z)| \quad \forall n > N_0 \quad (3.71)$$

Therefore $\lim_{n \rightarrow \infty} \max_{z \in K} |f_n(z)| = \max_{z \in K} |f(z)|$.

■

We introduce some notation for use in the next lemmas. If D is a domain, let M_D denote the group of its (analytic) automorphisms. In particular, let

$$\Delta = M_{\{z: |z| < 1\}} \quad (3.72)$$

Lemma 3.5.2 *The set of all polynomials with complex rational coefficients is dense in the set of all holomorphic functions on the open unit disk with the almost uniform topology. In other words, there exists $\{p_j\}$ with p_j a polynomial whose coefficients have rational real and imaginary parts, so that for any function g there is a subsequence $\{p_{m_k}\}$ such that $p_{m_k} \rightarrow g$ almost uniformly.*

Proof:

Using Theorem 3.2.1 we have for a function g on a compact subset K of the open unit disk $\max_{z \in K} |g(z) - p(z)| < \frac{\epsilon}{3}$.

Choose $p_j(z)$ with rational coefficients so that $|p_i(z) - p(z)| < \frac{\epsilon}{3}$

$$|g(z) - p_j(z)| \leq |g(z) - p(z)| + |p(z) - p_j(z)| \quad (3.73)$$

$$\leq \frac{2\epsilon}{3} \quad (3.74)$$

$$< \epsilon \quad \text{for } z \in K \quad (3.75)$$

The set of all polynomials with rational coefficients is dense and is also countable. There is a set $\{p_i\}$ countably dense, i.e. the set of all holomorphic functions on the open unit disk is a separable topological space. ■

Definition 3.5.1 Let $S(n, j, l) = \{g : \inf_{\tau \in \Delta} \max_{|z| \leq 1 - \frac{1}{l}} |g(\tau(z)) - p_j(z)| \geq \frac{1}{n}\}$ where n, j, l are all positive integers and $p_j(z)$ are as in Theorem 3.5.2.

Lemma 3.5.3 $S(n, j, l)$ is closed.

Proof

Let $\{g_m\} \in S(n, j, l)$ with $g_m \rightarrow g$ almost uniformly.

$$\Rightarrow \max_{|z| \leq 1 - \frac{1}{l}} |g_m(\tau(z)) - p_j(z)| \rightarrow \max_{|z| \leq 1 - \frac{1}{l}} |g(\tau(z)) - p_j(z)|$$

$$\Rightarrow \max_{|z| \leq 1 - \frac{1}{l}} |g(\tau(z)) - p_j(z)| \geq \frac{1}{n} \quad \tau \in \Delta$$

$$\Rightarrow g \in S(n, j, l)$$

Therefore $S(n, j, l)$ is closed. ■

Lemma 3.5.4 $\{g : g \text{ not universal} \} = \bigcup_{n,j,l} S(n,j,l)$

Proof

Suppose g is not universal.

$\Rightarrow \exists n_0, j_0, l_0$ so that

$$\forall \tau \in \Delta \quad |g(\tau(z)) - p_{j_0}(z)| > \frac{1}{n_0} \text{ for some } |z| < 1 - \frac{1}{l_0}$$

$$\Rightarrow g \in S(n_0, j_0, l_0)$$

$$\Rightarrow g \in \bigcup_{n,j,l} S(n,j,l)$$

$$\Rightarrow \{g : g \text{ not universal} \} \subset \bigcup_{j,n,l} S(n,j,l)$$

Conversely $g \in \bigcup_{j,n,l} S(n,j,l)$

$$\Rightarrow \exists n_0, j_0, l_0 \text{ so that } g \in S(n_0, j_0, l_0)$$

$$\Rightarrow \max_{|z| < 1 - \frac{1}{l_0}} |g(\tau(z)) - p_{j_0}(z)| \geq \frac{1}{n_0} \quad \forall \tau \in \Delta$$

$$\Rightarrow g \text{ is not universal}$$

$$\Rightarrow \{g : g \text{ is not universal} \} \supset \bigcup_{n,j,l} S(n,j,l)$$

Therefore $\{g : g \text{ is not universal} \} = \bigcup_{n,j,l} S(n,j,l)$

■

Lemma 3.5.5 *Each $S(n,j,l)$ has empty interior.*

Proof

Let $\mathcal{U}$ be an open set $\mathcal{U} \subset S(n, j, l)$

$$\begin{aligned}
 f \text{ universal} &\Rightarrow \{f \circ \tau : \tau \in \Delta\} \text{ is dense} \\
 &\Rightarrow f \circ \tau_0 \in \mathcal{U} \text{ if } \mathcal{U} \text{ is non empty for some } \tau_0 \\
 &\Rightarrow f \circ \tau_0 \text{ is not universal}
 \end{aligned}$$

But $f \circ \tau_0$ is universal since $\tau_0^{-1} \circ \tau$ is also an automorphism for any τ .

Therefore $S(n, j, l)$ has empty interior.

■

Theorem 3.5.2 *Universal functions on the open disk with respect to the full group of Möbius transformations form a residual set.*

Proof

By Lemma 3.5.4 $\{g : g \text{ not universal}\} = \bigcup_{n,j,l} S(n, j, l)$

By Lemma 3.5.3 each $S(n, j, l)$ is closed.

By Lemma 3.5.5 each $S(n, j, l)$ is nowhere dense.

$\Rightarrow \{g : g \text{ not universal}\}$ is a countable union of nowhere dense sets.

$\Rightarrow \{g : g \text{ universal}\}$ is a residual set.

■

Theorem 3.5.2 will have the same conclusion for any domain D and group of transformations M_D , if D can be approximated from within by compact sets. Theorem 3.5.2 tells us that most functions are universal in the sense of category. A result of

Bonar and Carroll [2] shows that most functions are annular in the sense of category. Hence there exist functions that are both universal and annular.

3.6 Open Questions Concerning the Set of Annular Functions

One source of motivation for research in this dissertation was a question raised by L.A. Rubel (private communication):

Question 3.6.1 *Can an annular function be differentially algebraic?*

There are a few things that one needs to keep in mind when trying to answer this very difficult question. In terms of category, most functions are transcendently transcendental. In the same sense, most functions are annular. One way of proceeding in order to answer the question in the affirmative is to build an annular function with “a lot” of algebraic structure and ask if that function satisfies an algebraic differential equation. The algebraic structure of the annular functions considered in Chapter II is the functional equation (3.1). We showed in Theorem 3.4.1 that the solutions of certain types of these products must be transcendental or rational (nonannular). It would be useful to know if a function that satisfies equation (3.1) must either be transcendently transcendental or rational. The question can be more formally stated as:

Question 3.6.2 *Let $g(z)$ be holomorphic on the open disk and satisfy equation (3.1) and let $g(z)$ satisfy a algebraic differential equation. Must $g(z)$ be rational?*

There are theorems listed in Dienes [7] that have the same conclusion, that is, if the power series expansion for a function holomorphic in the disk has only finitely many coefficients and is continuable, then the function must be rational. If a function g satisfies Equation (2.1) where the degree of the polynomial $q(z)$ is smaller than p , and the set of coefficients of $q(z)$ are a subset of a finite algebraically closed multiplicative set, then the function $g(z)$ will only have finitely many coefficients and have Hadamard gaps so $g(z)$ is non-continuable (see Hille [11]) and hence not rational.

CHAPTER IV

The Asymptotic Behavior of the Theta Function on the Disk

4.1 Introduction

Rubel [22] and [23] gives an amusing account of the role of the theta function in the study of algebraic differential equations. Namely, it provides an accessible example of an differentially algebraic function which nonetheless has $|z| < 1$ as its natural domain of existence. Hence the function itself, its derivatives or a polynomial in them, suggest themselves naturally as possible sources for annular but differentially algebraic functions. The existence of such a function would settle Question 3.6.1.

Theorem 4.3.2 shows that no such example can be constructed using a differential polynomial in theta. This result could have been obtained using diverse methods well known to various workers in this well developed area of research, had they been inclined to address the question.

$$\theta(z) = \sum_{n=1}^{\infty} z^{n^2} \tag{4.1}$$

This is what we will refer to as the Theta function. Theta is non-continuable in view of the Fabry Gap Theorem (see Hille [11]). This function is known to be

differentially algebraic. It will be demonstrated that no derivative or polynomial combination of derivatives can be an annular function. It is interesting to notice this theta function satisfies the following functional equation.

$$\theta(z) = 2\theta(z^4) - \theta(-z) , \quad (4.2)$$

which is somewhat similar to Equation 3.1. This fact suggests that $\theta(z)$ also might be annular. In fact, the function $2\theta(z) - 1$, the standard theta function, can be expressed by an equation of the form:

$$2\theta(z) - 1 = \prod_{n=1}^{\infty} (1 - z^{2n})(1 + z^{2n-1})^2 \quad (4.3)$$

It is interesting to notice that Equation 4.3 is similar to the annular products discussed in Section 2.2, except that the rate the exponent increases is $2n$ instead of 2^n . This gives even more evidence to suggest that the function $\theta(z)$ or one of its derivatives might be annular.

4.2 Unbounded Asymptotic Behavior

In this section we show how $\theta(z)$ and various forms of the derivatives of $\theta(z)$ go to infinity on radial paths ending at the boundary of the unit disk. We begin by proving the following approximation result for estimating infinite sums with integrals.

Lemma 4.2.1 *Let $f_{n,m}(r) = n^m r^{n^2}$ where $m \in \{1, 2, \dots\}$ and $0 < r < 1$. Then*

$$\left| \sum_{n=0}^{\infty} f_{n,m}(r) - \int_0^{\infty} f_{x,m}(r) dx \right| \leq \left(\frac{m}{-2e \log r} \right)^{\frac{m}{2}} \quad (4.4)$$

Proof:

Writing $f(x)$ for $f_{x,m}(r)$ (m, r fixed)

$$\frac{df}{dx} = mx^{m-1}r^{x^2} + 2x^{m+1}r^{x^2} \log r \quad (4.5)$$

$$= x^{m-1}r^{x^2}(m + 2x^2 \log r) \quad (4.6)$$

Recall the Euler Summation Formula (see Knopp, [15]):

$$g(0) + g(1) + \dots + g(n) = \int_0^n g(x) dx + \frac{1}{2}(g(0) + g(n)) + \int_0^n (x - \llbracket x \rrbracket - \frac{1}{2})g'(x) dx \quad (4.7)$$

Taking the limit as $n \rightarrow \infty$; and assuming all relevant limits exist, we obtain

$$\sum_{k=0}^{\infty} g(k) - \int_0^{\infty} g(x) dx = \int_0^{\infty} (x - \llbracket x \rrbracket - \frac{1}{2})g'(x) dx \quad (4.8)$$

$$\left| \sum_{k=0}^{\infty} g(k) - \int_0^{\infty} g(x) dx \right| \leq \frac{1}{2} \int_0^{\infty} |g'(x)| dx \quad (4.9)$$

Substituting for $f_{x,m}(r)$ we obtain

$$\left| \sum_{n=0}^{\infty} f_{n,m}(r) - \int_0^{\infty} f_{x,m}(r) dx \right| \quad (4.10)$$

$$\leq \frac{1}{2} \int_0^{\infty} \left| \frac{d}{dx} f_{x,m}(r) \right| dx \quad (4.11)$$

$$= \frac{1}{2} \int_0^{\sqrt{\frac{m}{-2 \log r}}} \frac{d}{dx} f_{x,m}(r) dx - \frac{1}{2} \int_{\sqrt{\frac{m}{-2 \log r}}}^{\infty} \frac{d}{dx} f_{x,m}(r) dx \quad (4.12)$$

$$= f_{\sqrt{\frac{m}{-2 \log r}}, m}(r) \quad (4.13)$$

$$= \left(\frac{m}{-2e \log r} \right)^{\frac{m}{2}} \quad (4.14)$$

Now we turn our attention to estimating the size of $\theta(z)$ and its derivatives near the boundary.

Theorem 4.2.1

$$\sum_{n=0}^{\infty} n^m r^{n^2} = \frac{C_m}{(-\log r)^{\frac{m}{2} + \frac{1}{2}}} + O\left(\frac{1}{(-\log r)^{\frac{m}{2}}}\right) \quad (4.15)$$

where $r \uparrow 1$ and

$$C_m = \frac{1}{2} \Gamma\left(\frac{m+1}{2}\right) \quad (4.16)$$

Proof:

Recall that the Γ -function is defined by

$$\Gamma(y) = \int_0^{\infty} t^{y-1} e^{-t} dt \quad (y > 0) \quad (4.17)$$

We consider the integral

$$\int_0^{\infty} x^m r^{x^2} dx \quad (4.18)$$

defined for $0 < r < 1$ and $m > 0$. The substitution $u = x^2$ in (4.18) gives

$$\frac{1}{2} \int_0^{\infty} u^{\frac{(m-1)}{2}} r^u du \quad (4.19)$$

The substitution $t = u \log(\frac{1}{r})$ in (4.19) yields

$$\frac{1}{2} \int_0^{\infty} \frac{t^{\frac{(m-1)}{2}} e^{-t}}{(-\log r)^{\frac{(m-1)}{2}} (-\log r)} dt = \frac{1}{2(-\log r)^{\frac{(m+1)}{2}}} \int_0^{\infty} t^{\frac{(m-1)}{2}} e^{-t} dt \quad (4.20)$$

$$= \frac{1}{2} \frac{1}{(-\log r)^{\frac{(m+1)}{2}}} \Gamma\left(\frac{m+1}{2}\right) \quad (4.21)$$

In summary, we have

$$\int_0^\infty x^m r^{x^2} dx = \frac{C_m}{(-\log r)^{\frac{(m-1)}{2}}} \quad (4.22)$$

where C_m is given by (4.16). We now apply this formula to the sum of the series.

$$\left| \frac{(-\log r)^{\frac{m}{2}+\frac{1}{2}}}{C_m} \sum_{n=0}^\infty n^m r^{n^2} \right| \quad (4.23)$$

$$= \left| \frac{(-\log r)^{\frac{m}{2}+\frac{1}{2}}}{C_m} \left[\sum_{n=0}^\infty n^m r^{n^2} - \int_0^\infty x^m r^{x^2} dx + \int_0^\infty x^m r^{x^2} dx \right] \right| \quad (4.24)$$

$$\leq \left| \frac{(-\log r)^{\frac{m}{2}+\frac{1}{2}}}{C_m} \right| \left| \left[\sum_{n=0}^\infty n^m r^{n^2} - \int_0^\infty x^m r^{x^2} dx \right] + \int_0^\infty x^m r^{x^2} dx \right| \quad (4.25)$$

Applying Lemma 4.2.1 to estimate the difference between the integral and the sum in expression 4.25, we obtain the following inequality.

$$\left| \frac{(-\log r)^{\frac{m}{2}+\frac{1}{2}}}{C_m} \sum_{n=0}^\infty n^m r^{n^2} \right| \leq 1 + O(\sqrt{-\log r}) \quad (4.26)$$

Expression (4.26) tends to 1 as $r \rightarrow 1$, so that asymptotically we have the following:

$$\sum_{n=0}^\infty n^m r^{n^2} = \frac{C_m}{(-\log r)^{\frac{m}{2}+\frac{1}{2}}} + O\left(\frac{1}{(-\log r)^{\frac{m}{2}}}\right) \quad (4.27)$$

■

Remark. The crucial identity (4.22) in the proof above could have been obtained using the following more self-contained approach.

First we obtain Formula 1:

$$\int_0^\infty x^m r^x dx = \frac{m!}{(-\log r)^{m+1}} \quad (4.28)$$

Proof by induction:

For $m = 0$:

$$\int_0^\infty r^x dx = \frac{r^x}{\log r} \Big|_0^\infty = \left(\frac{1}{-\log r} \right) \quad (4.29)$$

Assume

$$\int_0^\infty x^m r^x dx = \frac{m!}{(-\log r)^{m+1}} \quad (4.30)$$

Using integration by parts with $\begin{cases} u = x^{m+1} \\ dv = r^x dx \end{cases}$ $\begin{cases} du = (m+1)x^m dx \\ v = \frac{1}{\log r} r^x \end{cases}$, we obtain:

$$\int_0^\infty x^{m+1} r^x dx = \frac{x^{m+1} r^x}{\log r} \Big|_0^\infty - \frac{(m+1)}{\log r} \int_0^\infty x^m r^x dx \quad (4.31)$$

$$= \frac{(m+1)}{-\log r} \left(\frac{m!}{(-\log r)^{m+1}} \right) \quad (4.32)$$

$$= \frac{(m+1)!}{(-\log r)^{m+2}} \quad (4.33)$$

We wish to evaluate (4.28) when r^x is replaced by r^{x^2} . For m an odd positive integer (4.28) becomes Formula 2a:

Evaluate $\int_0^\infty x^{2n+1} r^{x^2} dx$

$$\int_0^\infty x^{2n+1} r^{x^2} dx = \frac{1}{2} \int_0^\infty u^n r^u du = \frac{1}{2} \frac{n!}{(-\log r)^{n+1}} \quad (4.34)$$

On the other hand when m in (4.28) is an even positive integer we obtain Formula 2b:

$$\int_0^\infty x^{2n} r^{x^2} dx = \frac{C_m}{(-\log r)^{\frac{m}{2} + \frac{1}{2}}} \quad (4.35)$$

Evaluate $\int_0^\infty x^{2n} t^{x^2} dx$ using $\left(\int_0^\infty x^{2n} t^{x^2} dx \right)^2$

$$= \left(\int_0^\infty x^{2n} t^{x^2} dx \right) \left(\int_0^\infty y^{2n} t^{y^2} dy \right) \quad (4.36)$$

$$= \int_0^\infty \int_0^\infty x^{2n} y^{2n} t^{x^2 + y^2} dx dy \quad (4.37)$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty r^{4n} \sin^{2n} \theta \cos^{2n} \theta t^{r^2} r dr d\theta \quad (4.38)$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{1}{2^{2n}} \sin^{2n} 2\theta r^{4n} t^{r^2} r dr d\theta \quad (4.39)$$

$$= \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\theta d\theta \int_0^\infty u^{2n} t^u du \quad (4.40)$$

$$= \frac{c_m(2n)!}{2^{2n+1}} \left(\frac{1}{-\log t} \right)^{2n+1} \quad (4.41)$$

The second factor can be evaluated by replacing n by $2n$ in Formula 2a, while evaluation of the first is a standard exercise. We have, in fact, from a table of integrals,

$$\int_0^{\frac{\pi}{2}} (\sin 2\theta)^n d\theta = \frac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{n}{2} + 1)} \quad (4.42)$$

The next two theorems deal with the unbounded asymptotic behavior of these types of functions. This result is based on a paper of Hardy and Littlewood [10] that gives the following result.

Lemma 4.2.2 (Hardy and Littlewood) *For all rational numbers $x = \frac{l}{s}$ where $\gcd(l, s) = 1$, and $S_n = \sum_{\nu=0}^n e^{\nu^2 \pi i x}$ then there exists a positive number D (depending on*

x) such that

$$S_n = (\pm 1 \pm i)Dn + O(1) \text{ if } s \equiv 0 \pmod{4} \quad (4.43)$$

$$S_n = \pm Dn + O(1) \text{ if } s \equiv 1 \pmod{4} \quad (4.44)$$

$$S_n = O(1) \text{ if } s \equiv 2 \pmod{4} \quad (4.45)$$

$$S_n = \pm iDn + O(1) \text{ if } s \equiv 3 \pmod{4} \quad (4.46)$$

$$(4.47)$$

as $n \rightarrow \infty$.

Corollary 4.2.1 For all rational numbers $x = \frac{l}{s}$ where $\gcd(l, s) = 1$, and $s \not\equiv 2 \pmod{4}$ and $S_n = \sum_{\nu=0}^n e^{\nu^2 \pi i x}$ then there exists a nonzero constant A (depending on x) such that

$$S_n = An + O(1) \quad (4.48)$$

Theorem 4.2.2 Let $A_{n,m} = \sum_{k=0}^n k^m e^{i\pi k^2 x}$ with $x = \frac{l}{s}$ and $s \not\equiv 2 \pmod{4}$. Then $A_{n,m} = \frac{A}{m+1} n^{m+1} + O(n^m)$ as $n \rightarrow \infty$, $A = A(x)$ as in Corollary 4.2.1

Proof:

$$\text{Let } S_n = \sum_{\nu=0}^n e^{\nu^2 \pi i x}$$

$$A_{n,m} \quad (4.49)$$

$$= \sum_{k=1}^n S_k \left(k^m - (k+1)^m \right) + S_n (n+1)^m \quad (4.50)$$

$$= \sum_{k=0}^n \left(Ak + O(1) \right) \left(-mk^{m-1} + O(k^{m-2}) \right) + \left(An + O(1) \right) (n+1)^m \quad (4.51)$$

$$= \sum_{k=0}^n -Amk^m + \sum_{k=1}^n O(k^{m-1}) + A(n)^{m+1} + O(n^m) \quad (4.52)$$

$$= -Am \sum_{k=0}^n k^m + \sum_{k=0}^n O(k^{m-1}) + A(n)^{m+1} + O(n^m) \quad (4.53)$$

$$= -Am \frac{1}{m+1} (n)^{m+1} + O(n^m) + An^{m+1} + O(n^m) \quad (4.54)$$

$$= \frac{A}{m+1} n^{m+1} + O(n^m) \quad (4.55)$$

■

Theorem 4.2.3 *The function $f_m(z) = \sum_{n=0}^{\infty} n^m z^{n^2}$ will have radial limit infinity on a dense set of points of the circle $|z| = 1$. More explicitly, we have*

$$\lim_{r \rightarrow 1-} \left| \sum_{n=0}^{\infty} n^m r^{n^2} e^{i\pi n^2 x} \right| = \infty \quad (4.56)$$

$x = \frac{l}{s}$ and $s \not\equiv 2 \pmod{4}$ and $m = 1, 2, \dots$

Proof:

We begin by using summation by parts on the $f_m(z)$. Since $0 < r < 1$ and the coefficients of r^{n^2} only grow at a polynomial rate, namely n^m , the new representation for the $f_m(z)$ is valid.

$$\left| f_m(r) \right| = \left| \sum_{n=0}^{\infty} n^m e^{i\pi n^2 x} r^{n^2} \right| \quad (4.57)$$

$$= \left| \sum_{n=0}^{\infty} A_{n,m} \left(r^{n^2} - r^{(n+1)^2} \right) \right| \quad (4.58)$$

$$= \left| \frac{(1-r)A}{r} \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \left(\frac{n^{m+1}}{m+1} + O(n^m) \right) r^{n^2+k+1} \right| \quad (4.59)$$

$$= |A| \left| \frac{1-r}{r} \right| \left| \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \left(\frac{(n+1)^{m+1}}{m+1} + O(n^m) \right) r^{n^2+k+1} \right| \quad (4.60)$$

By choosing N_0 large enough we get $\frac{(n+1)^{m+1}}{m+1} + O(n^m) \geq \frac{(n+1)^{m+1}}{m+2} > 0$, for $n \geq N_0$.

Continuing from Equation (4.60), we obtain:

$$= \frac{|A(1-r)|}{r} \left[\left(\sum_{n=N_0}^{\infty} \sum_{k=0}^{2n} \left(\frac{(n+1)^{m+1}}{m+1} + O(n^m) \right) r^{n^2+k+1} \right) - P(r) \right] \quad (4.61)$$

where P is a polynomial whose degree depends on N_0 . Applying what we know about N_0 to Equation (4.61) we again obtain:

$$\geq \frac{|A(1-r)|}{r} \left[\sum_{n=0}^{\infty} \frac{(n+1)^{m+1}}{m+2} r^{(n+1)^2} - \tilde{P}_1(r) \right] \quad (4.62)$$

$$= \frac{|A(1-r)|}{r} \left[\sum_{n=1}^{\infty} \frac{(n+1)^{m+1}}{m+2} r^{(n+1)^2} - \tilde{P}_2(r) \right] \quad (4.63)$$

$$\geq \frac{|A(1-r)|}{r} \left[\frac{C_{m+1}}{(m+2)(-\log r)^{\frac{m}{2}+1}} - O\left(\frac{1}{(-\log r)^{\frac{m}{2}+\frac{1}{2}}}\right) - \tilde{P}_3(r) \right] \quad (4.64)$$

$$= \frac{1-r}{(-\log r)^{\frac{m}{2}+1}} \frac{|A|}{r(m+2)} \left[C_{m+1} - O\left((- \log r)^{\frac{1}{2}}\right) - (-\log r)^{\frac{m}{2}+1} \tilde{P}_4(r) \right] \quad (4.65)$$

where the $\tilde{P}_j$ are polynomials.

For $m \geq 1$, $\frac{1-r}{(-\log r)^{\frac{m}{2}+1}} \rightarrow \infty$ as $r \rightarrow 1^-$, while

$$\frac{|A|}{r(m+2)} \left[C_{m+1} - O\left((- \log r)^{\frac{1}{2}}\right) - (-\log r)^{\frac{m}{2}+1} P(r) \right] \rightarrow \frac{|A|C_{m+1}}{(m+2)} \neq 0 \quad (4.66)$$

So $\lim_{r \rightarrow 1^-} |f_m(re^{i\pi x})| = \infty$, as claimed. ■

4.3 Bounded Asymptotic Behavior

Our results of this section depend heavily on the work of Hardy and Cartwright. Despite of the unbounded behavior of the theta function shown in Section 4.2 we find that there are points on the unit circle for which the radial limit exists.

Lemma 4.3.1 (Hardy [9]) *If we are given a strictly increasing sequence of positive numbers $\{\lambda_n\} \subset \mathbb{R}$ with $\lambda_n \rightarrow +\infty$ and if*

$$\sum_{n=1}^{\infty} a_n = S, \quad (4.67)$$

then, for every α , $0 < \alpha < \frac{\pi}{2}$, we have

$$\lim_{z \rightarrow 0} \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = S \quad (4.68)$$

uniformly in the angle $|y| \leq x \tan \alpha$, and $z = x + iy$.

Proof:(See Appendix A)

Our next lemma is due to Cartwright (see Hardy [9]); it gives bounds on the asymptotic behavior of the Fourier transform of e^{-x^k} .

Lemma 4.3.2 (Cartwright) *If $k > 0$, $\lambda = \frac{\pi}{2k}$, $x = re^{i\theta}$ and if the function $F(x)$ is defined by $\int_0^{\infty} e^{-t^k} \cos xt \, dt$ for positive x , then*

1. $F(x)$ is analytic if $|\theta| < \lambda$
2. $F(x) = O(1)$ is bounded for small x .
3. $F(x) = O(r^{-1-k})$ for large x .

Moreover 2) and 3) hold uniformly in any angle θ , $|\theta| \leq \lambda - \epsilon < \lambda$.

Proof:(See Appendix B)

The next theorem is due to Cartwright (see Hardy, [9]). It gives conditions on which the limit of a sum in an angle in the right half plane with vertex at the origin, will imply the limit of a related sum inside of a somewhat smaller angle that is contained in the original angle.

Theorem 4.3.1 (Cartwright) *If $p > 0$ and $q = kp$ with $k > 1$ and $\alpha_2 - \alpha_1 > \pi(1 - \frac{1}{k})$ and $\lim_{y \rightarrow 0} \sum_{n=1}^{\infty} a_n e^{-yn^p} = S$ where $y = re^{i\theta}$ and $-\frac{\pi}{2} < \alpha_1 < \theta < \alpha_2 < \frac{\pi}{2}$ then*

$$\lim_{y \rightarrow 0} \sum_{n=1}^{\infty} a_n e^{-yn^q} = S \quad (4.69)$$

inside any angle (β_1, β_2) where $-\frac{\pi}{2} + k(\frac{\pi}{2} + \alpha_1) < \beta_1 < \theta < \beta_2 < \frac{\pi}{2} - k(\frac{\pi}{2} - \alpha_2)$.

Proof:(See Appendix C)

Cartwright and Hardy's work will enable us to prove some important corollaries about functions of the form $\sum_{n=0}^{\infty} a_n z^{n^p}$ where p is an integer greater than 1. It is useful to notice that in Theorem 4.3.1 that we can limit ourselves to the case $y > 0$, so that making the substitution $r = e^{-y}$, we obtain

$$\lim_{y \rightarrow 0^+} \sum_{n=0}^{\infty} a_n e^{-yn^p} = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n r^{n^p} \quad (4.70)$$

Thus the function is defined on the interval $[0, 1)$ instead of the interval $[0, \infty)$, and therefore represents a function that is analytic in the open unit disk if $\sqrt[n]{|a_n|}$ is bounded by 1.

Corollary 4.3.1 *Let $\{a_n\} \subset \mathbb{C}$ be a sequence of numbers so that*

$$\lim_{\substack{z \rightarrow 1 \\ |z| \leq 1}} \sum_{n=0}^{\infty} a_n z^n = S \quad (4.71)$$

Then $\lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n r^{n^q} = S$ for $q = 2, 3, \dots$

Proof:

Let q be fixed. We substitute $z = e^{-y}$ in (4.71) and we note that the new limit will exist in the right half plane. We choose an interval (α_1, α_2) that satisfies the following:

$$-\frac{\pi}{2} < \alpha_1 < 0 < \alpha_2 < \frac{\pi}{2} \quad (4.72)$$

$$\alpha_2 - \alpha_1 > \pi(1 - \frac{1}{q}) \quad (4.73)$$

Taking $k = q$ in Theorem 4.3.1 we obtain an interval (β_1, β_2) with $0 \in (\beta_1, \beta_2)$ and we further notice

$$S = \lim_{\substack{z \rightarrow 1 \\ |z| \leq 1}} \sum_{n=0}^{\infty} a_n z^n \quad (4.74)$$

$$= \lim_{\substack{y \rightarrow 0 \\ \text{Arg } y \in (\alpha_1, \alpha_2)}} \sum_{n=0}^{\infty} a_n e^{-yn} \quad (4.75)$$

$$= \lim_{\substack{y \rightarrow 0 \\ \text{Arg } y \in (\beta_1, \beta_2)}} \sum_{n=0}^{\infty} a_n e^{-yn^q} \quad (4.76)$$

$$= \lim_{y \rightarrow 0^+} \sum_{n=0}^{\infty} a_n e^{-yn^q} \quad (4.77)$$

$$= \lim_{r \rightarrow 1^-} \sum_{n=1}^{\infty} a_n r^{n^q} \quad (4.78)$$

We wish to show how various types of functions have a limit as the boundary of the unit disk is approached along a radial path. We will use the boundary behavior of these functions to show that polynomial combinations of derivatives of the theta function have a limit as the boundary of the disk is approached along a radial path.

Corollary 4.3.2 *If $g_{m,p}(z) = \sum_{n=0}^{\infty} (-1)^n n^m z^{np}$, then $\lim_{r \rightarrow 1^-} g_{m,p}(r)$ exists for all non-negative integers m and positive integers p . Moreover the function represented by $g_{m,1}(z)$ in $|z| < 1$ is analytic in a neighborhood of 1.*

Proof (by induction on m):

For $m = 0$ and $p = 1$ we have

$$g_{0,1}(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \quad (4.79)$$

$$\lim_{z \rightarrow 1^-} g_{0,1}(z) = \frac{1}{2} \quad (4.80)$$

Hence by Corollary 4.3.1 we have

$$\lim_{r \rightarrow 1^-} g_{0,p}(z) = \frac{1}{2} \quad (4.81)$$

for all $p = 2, 3, 4, \dots$. Hence both assertions of the corollary are true for $m = 0$.

Let us assume that they are valid up to m . Then, in $|z| < 1$, we have

$$zg_{m,1}^{(1)}(z) = z \frac{d}{dz} \sum_{n=0}^{\infty} n^m (-z)^n \quad (4.82)$$

$$= -z \sum_{n=0}^{\infty} n^{m+1} (-z)^{n-1} \quad (4.83)$$

$$= \sum_{n=0}^{\infty} n^{m+1} (-z)^n \quad (4.84)$$

$$= g_{m+1,1}(z) \quad (4.85)$$

so that $g_{m+1,1}(z)$ represents a function analytic in a neighborhood of 1 and has a limit there.

Again by Corollary 4.3.1 we have

$$\lim_{r \rightarrow 1^-} g_{m+1,p}(r) = \lim_{z \rightarrow 1} g_{m+1,1}(z) \quad (4.86)$$

so by induction the assertion is true for all m and p . ■

The previous corollary says that $\lim_{r \rightarrow 1^-} g_{m,p}(r)$ exists for all non-negative m and p , so that this limit is going to exist for all scalar multiples and linear combinations of $g_{m,p}(r)$. The next corollary says how this can be generalized.

Corollary 4.3.3 *Let $h_p(z) = \sum_{n=0}^{\infty} q(n)(-z)^{n^p}$ where $q(x) \in \mathbb{C}[x]$. Then $\lim_{r \rightarrow 1^-} h_p(r)$ exists for $p = 1, 2, 3, \dots$*

Proof:

$$\text{Let } q(x) = \sum_{j=0}^m a_j x^j$$

$$h_p(z) = \sum_{n=0}^{\infty} q(n)(-z)^{n^p} \quad (4.87)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^m a_j n^j (-z)^{n^p} \quad (4.88)$$

$$= \sum_{j=0}^m a_j \sum_{n=0}^{\infty} n^j (-z)^{n^p} \quad (4.89)$$

$$= \sum_{j=0}^m a_j g_{j,p}(z), \quad (4.90)$$

where the $g_{j,p}(z)$ are defined in Corollary 4.3.2. Since the limit of a finite sum is the sum of the limits we have

$$\lim_{r \rightarrow 1^-} h_p(r) = \lim_{r \rightarrow 1^-} \sum_{j=0}^m a_j g_{j,p}(z) \quad (4.91)$$

■

The next corollaries show that certain functions cannot be combined in a specific manner to form annular functions. In particular, if a function is bounded along a radial path then it is not annular.

Corollary 4.3.4 *Let $q(z, y_0, \dots, y_n) \in \mathbb{C}[z, y_0, \dots, y_m]$, and let $p_i(x)$ be in $\mathbb{C}[x]$ and let m_i be positive integers for $i = 1, \dots, m$.*

If

$$f_i(z) = \sum_{n=0}^{\infty} p_i(n)(-z)^{n^{m_i}} \quad (4.92)$$

then

$$\lim_{r \rightarrow 1^-} q(r, f_0(r), f_1(r), \dots, f_m(r)) \quad (4.93)$$

exists.

Proof:

By Corollary 4.3.3

$$\lim_{r \rightarrow 1^-} f_i(r) = s_i \quad (4.94)$$

exists so by the continuity of the polynomial $q(z, y_0, \dots, y_n)$ we have

$$\lim_{r \rightarrow 1^-} q(r, f_0(r), \dots, f_m(r)) = q\left(\lim_{r \rightarrow 1^-} r, \lim_{r \rightarrow 1^-} f_0(r), \dots, \lim_{r \rightarrow 1^-} f_m(r)\right) \quad (4.95)$$

$$= q(1, s_0, \dots, s_m) \quad (4.96)$$

Hence the limit exists.

■

Theorem 4.3.2 *If $q(z, y_0, y_1, \dots, y_n) \in \mathbb{C}[z, y_0, y_1, \dots, y_m]$, then the function defined by $q(z, \theta^{(0)}(z), \dots, \theta^{(m)}(z))$ is not annular.*

Proof:

$$\theta^{(k)}(z) = \frac{1}{z^k} \sum_{n=1}^{\infty} p_k(n) z^{n^2} \quad (4.97)$$

where $p_k(x) \in \mathbb{C}[x]$ and $p_k(n) = 0$ for $n = 1, \dots, k$. So by Corollary 4.3.4 and the continuity of the functions $\frac{1}{z^k}$ at -1 we have

$$\lim_{z \rightarrow -1^+} q(z, \theta^{(0)}(z), \dots, \theta^{(m)}(z)) \quad (4.98)$$

exists so that $q(z, \theta^{(0)}(z), \dots, \theta^{(m)}(z))$ is bounded on $(-1, 0)$ function and is not annular. ■

The previous theorem says that no annular function is in the differential polynomial span of $\theta(z)$. There are bounded functions which have annular functions in their differential polynomial span. For example

$$\sum_{n=1}^{\infty} \frac{1}{7^n} z^{7^n} \quad (4.99)$$

has the function

$$\sum_{n=1}^{\infty} 7^n z^{7^n} \quad (4.100)$$

in its differential polynomial span which is a known annular function (see Bonar [1]).

Of course any function that is continuable beyond the disk is never going to have an annular function in its differential polynomial span.

4.4 Open Questions on Asymptotic Behavior

The purpose of this section is to use the collected results of Chapter IV in order to focus a strategy in answering Question 3.6.1. Notice that in Theorem 4.3.2 polynomial combinations of derivatives of the theta function do not yield a function which is annular, however, there are functions which are not annular, but whose differential polynomial span (with rational coefficients) includes functions which are annular. Consider the function $g(z)$ given in Theorem 3.3.1. The function defined by $(1-z)g(z)$ is not annular, because if z is positive and real $z = r$ then,

$$|(1-z)g(z)| = |(1-r) \sum_{n=0}^{\infty} \pm r^n| \quad (4.101)$$

$$\leq (1-r) \frac{1}{1-r} = 1 \quad (4.102)$$

Hence $(1-z)g(z)$ is bounded along a radial path and is not annular. There is a polynomial $P(z, y_0) \in \mathbb{C}[[z]][y_0]$, $P(z, y_0) = \frac{y_0}{(1-z)}$ so that $G(z) = P(z, (1-z)g(z))$ is strongly annular. This motivates the following question:

Question 4.4.1 *Does there exist $P(z, y_0, \dots, y_n) \in \mathbb{C}[[z]][y_0, \dots, y_n]$, $P \neq 0$ so that the function $P(z, \theta^{(0)}(z), \dots, \theta^{(n)}(z))$ is an annular function?*

Appendix A

Hardy's Proof for Summation Inside an Angle

The following is a proof of Lemma 4.3.1.

Proof:

Let $s_n = \sum_{j=1}^n a_j$. Without loss of generality suppose $S = 0$. We have

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = \sum_{n=1}^{\infty} s_n (e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) \quad (\text{A.1})$$

Now

$$\sum_{n=1}^{\infty} |e^{-\lambda_n z} - e^{-\lambda_{n+1} z}| = \sum_{n=1}^{\infty} \left| \int_{\lambda_n}^{\lambda_{n+1}} z e^{-tz} dt \right| \quad (\text{A.2})$$

$$\leq |z| \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} e^{-tx} dt \quad (\text{A.3})$$

$$= \frac{|z|}{x} \sum_{n=1}^{\infty} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) \quad (\text{A.4})$$

$$= \frac{|z|}{x} e^{-\lambda_1 x} \quad (\text{A.5})$$

$$\leq \sec \alpha e^{-\lambda_1 x} \quad (\text{A.6})$$

If we choose N_ϵ so that $|s_n| < \epsilon$ if $n \geq N_\epsilon$, then for $x \geq N$, we have

$$\left| \sum_{n=N}^{N'} (e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) s_n \right| < \epsilon \sec \alpha \quad (\text{A.7})$$

For $N' \geq N$, so that the sum is uniformly Cauchy, hence it is uniformly convergent.

Therefore the limit of the sum is the sum of the limits.

$$\lim_{z \rightarrow \infty} e^{-\lambda_n z} - e^{-\lambda_{n+1} z} = 0 \quad (\text{A.8})$$

Hence $f(z) \rightarrow 0$ inside the angle.

■

Appendix B

Cartwright's Proof on the Asymptotic Growth of the Fourier Transform of e^{-x^k}

The following is a proof for Lemma 4.3.2. Cartwright proves a stronger result in the original statement, but we have concentrated on the part of the proof that most suits our purposes.

Proof:

For positive values of x :

$$|e^{-t^k} \cos xt| < e^{-t^k} \quad (\text{B.1})$$

which is in $L_1(0, \infty)$ for all positive k . The function $F(x)$ may be seen to be analytic by applying Fubini's Theorem and Morera's Theorem.

$$F(x) = \frac{1}{x} \int_0^\infty e^{-\left(\frac{t}{x}\right)^k} \cos t \, dt \quad (\text{B.2})$$

We have

$$\left(\frac{t}{x}\right)^k = \frac{t^k}{r^k} e^{-ik\theta} \quad (\text{B.3})$$

$$= \frac{t^k}{r^k} \cos(k\theta) - i \frac{t^k}{r^k} \sin(k\theta) \quad (\text{B.4})$$

$$\left| e^{-\left(\frac{t}{x}\right)^k} \right| = e^{-\left(\frac{t^k}{r^k}\right) \cos k\theta} \quad (\text{B.5})$$

Since cosine is a decreasing function in $[0, \frac{\pi}{2}]$ we have

$$|e^{-\left(\frac{t}{x}\right)^k}| \leq e^{-\left(\frac{t}{r}\right)^k \cos k(\lambda - \epsilon)} \quad (\text{B.6})$$

Hence

$$|F(x)| \leq \frac{1}{r} \int_0^\infty e^{-\left(\frac{t}{r}\right)^k \cos k(\lambda - \epsilon)} dt \quad (\text{B.7})$$

So that 1) and 2) are true.

For positive x we have

$$xF(x) = x \int_0^\infty e^{-t^k} \cos xt \, dt \quad (\text{B.8})$$

$$= k \int_0^\infty e^{-t^k} t^{k-1} \sin xt \, dt \quad (\text{B.9})$$

$$= \frac{k}{2i} \int_0^\infty e^{-t^k} t^{k-1} e^{ixt} \, dt - \frac{k}{2i} \int_0^\infty e^{-t^k} t^{k-1} e^{-ixt} \, dt \quad (\text{B.10})$$

$$= xF_1(x) - xF_2(x). \quad (\text{B.11})$$

By doing another change of variable we get

$$xF_1(x) = \frac{k}{2ix^k} \int_0^{\infty e^{i\phi}} e^{-\left(\frac{t}{x}\right)^k} t^{k-1} e^{it} \, dt \quad (\text{B.12})$$

Note that the integral is independent of path along any ray inside a sufficiently small angle close to the positive real axis. The integral is taken along the line $t = \rho e^{i\phi}$ where ϕ is small and positive. The integral is an absolutely and uniformly convergent function of x in any angle θ of the form $|\theta| \leq \lambda - \phi - \eta < \lambda - \phi$ and is equal to $xF_1(x)$ throughout $|\theta| < \lambda - \phi$ we have

$$x^{k+1}F_1(x) \longrightarrow \frac{k}{2i} \int_0^{\infty e^{i\phi}} t^{k-1} e^{it} \, dt \quad (\text{B.13})$$

$$= \frac{k}{2i} \Gamma(k) e^{\frac{k\pi}{2}i} \quad (\text{B.14})$$

$$= \frac{\Gamma(k+1)}{2i} e^{\frac{k\pi}{2}i} \quad (\text{B.15})$$

when x tends to infinity uniformly in $|\theta| < \lambda - \phi - \eta$. Similarly we can use a path of integration below the real axis we see that

$$2ix^{k+1}F_2(x) \longrightarrow \Gamma(k+1)e^{-\frac{k\pi}{2}i}, \quad (\text{B.16})$$

uniformly in $|\theta| < \lambda - \phi - \eta$ and 3) follows.

■

Appendix C

Cartwright's Proof for Summation of Series with Larger Indices

The following is a proof of Theorem 4.3.1. This is the exact proof which Cartwright gave. She also gives an analogous result in the other direction. Interested readers may consult Hardy [9].

Proof:

We are given that $q = kp$ and $y > 0$. Let $Y = y^{\frac{1}{k}}n^p$; then $e^{-Y^k} = e^{-yn^q}$. If we recall the Fourier Inversion formula we get the following.

$$e^{-yn^q} = e^{-Y^k} \tag{C.1}$$

$$= \frac{2}{\pi} \int_0^\infty F(t) \cos Yt \, dt \tag{C.2}$$

$$= \frac{2}{\pi} \int_0^\infty F(t) \cos(y^{\frac{1}{k}}n^p t) \, dt \tag{C.3}$$

We recall that $\cos Yt = \frac{e^{iYt} + e^{-iYt}}{2}$ so that the above equation becomes:

$$\frac{1}{\pi} \int_0^\infty F(t) e^{iYt} \, dt + \frac{1}{\pi} \int_0^\infty F(t) e^{-iYt} \, dt \tag{C.4}$$

If $t = \rho e^{i\phi}$ and $0 \leq \phi_1 \leq \pi$ and $-\pi \leq \phi_2 \leq 0$ then the argument of $-iYt$ and the argument of iYt are in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $0 \leq \phi \leq \phi_1$ and $\phi_2 \leq \phi \leq 0$ respectively. If also

$$|\phi_1| < \lambda - \epsilon \text{ and } |\phi_2| < \lambda - \epsilon \tag{C.5}$$

then it follows from Lemma 4.3.2 and Cauchy's theorem that Equation C.4 can be replaced by:

$$e^{-yn^q} = \frac{1}{\pi} \int_{C_1} F(t) e^{iyt} dt + \frac{1}{\pi} \int_{C_2} F(t) e^{-iyt} dt \quad (\text{C.6})$$

Where C_1 and C_2 are the rays $\phi = \phi_1$ and $\phi = \phi_2$. It also follows that Equation C.6 is not only valid for $y > 0$ but it is also true if ϕ_1 and ϕ_2 satisfy Equation C.5 and

$$-\frac{\pi}{2} \leq \frac{\pi}{2} + \frac{\theta}{k} + \phi_1 \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq \frac{\pi}{2} + \frac{\theta}{k} + \phi_2 \leq \frac{\pi}{2} \quad (\text{C.7})$$

where $y = re^{i\theta}$ (instead of $0 \leq \phi_1 \leq \pi, -\pi \leq \phi_2 \leq 0$). So Equation C.7 will hold if

$$\alpha_1 \leq -\frac{\pi}{2} + \frac{\theta}{k} + \phi_1 \leq \alpha_2, \alpha_1 \leq \frac{\pi}{2} + \frac{\theta}{k} + \phi_2 \leq \alpha_2 \quad (\text{C.8})$$

and ϕ_1 and ϕ_2 satisfy Equation C.5.

Consider θ in the following interval:

$$-\frac{\pi}{2} + k\left(\frac{\pi}{2} + \alpha_1\right) < \theta < \frac{\pi}{2} + k\left(\frac{\pi}{2} + \alpha_2\right) \quad (\text{C.9})$$

in this interval we have

$$\alpha_1 + \frac{\pi}{2} + \frac{\theta}{k} < \frac{\pi}{2k} \text{ and } -\frac{\pi}{2k} < \alpha_2 + \frac{\pi}{2} - \frac{\theta}{k} \quad (\text{C.10})$$

so that if $\phi_1 \in \left(-\frac{\pi}{2k}, \frac{\pi}{2k}\right)$, then

$$\alpha_1 + \frac{\pi}{2} - \frac{\theta}{k} \leq \phi_1 \leq \alpha_2 + \frac{\pi}{2} - \frac{\theta}{k} \quad (\text{C.11})$$

$$\alpha_1 \leq -\frac{\pi}{2} + \frac{\theta}{k} + \phi_1 \leq \alpha_2. \quad (\text{C.12})$$

We can derive a similar equation for ϕ_2 .

So let

$$\beta_1 > -\frac{\pi}{2} + k\left(\frac{\pi}{2} + \alpha_1\right) \quad (\text{C.13})$$

$$\beta_2 < \frac{\pi}{2} - k\left(\frac{\pi}{2} - \alpha_2\right) \quad (\text{C.14})$$

$$\beta_1 < \beta_2. \quad (\text{C.15})$$

Suppose $\theta \in (\beta_1, \beta_2)$ and we choose ϕ_1 and ϕ_2 that satisfy Equation C.5 and Equation C.8, and we also let $\tau = e^{\frac{\alpha_1 + \alpha_2}{2}i}$ and $\delta > 0$. Then we have

$$\sum_{n=1}^{\infty} a_n e^{-yn^q - \delta\tau n^p} = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{C_1} a_n e^{-n^p z_1} F(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{C_2} a_n e^{-n^p z_2} F(t) dt \quad (\text{C.16})$$

$$= \chi_1 + \chi_2 \quad (\text{C.17})$$

where $z_1 = \delta\tau - iy^{\frac{1}{k}}t$ and $z_2 = \delta\tau + iy^{\frac{1}{k}}t$.

If $z_1 = \delta\tau + Z_1$ then the argument of Z_1 lies in (α_1, α_2) and z_1 is in the angle $A(\delta)$ whose vertex is at $\delta\tau$ and whose sides make angles of α_1 and α_2 with the positive x -axis. So if δ and y are fixed

$$\sum_{n=1}^{\infty} a_n e^{-n^p z} \quad (\text{C.18})$$

converges uniformly on the line whose argument is ϕ_1 and so

$$\int_{C_1} |F(t)| dt < \infty. \quad (\text{C.19})$$

By the dominated convergence theorem we can reverse the order of summation and integration and we get:

$$\chi_1 = \frac{1}{\pi} \int_{C_1} \sum_{n=1}^{\infty} a_n e^{-n^p z_1} F(t) dt \quad (\text{C.20})$$

and a similar conclusion for χ_2 .

If we let $f_p(z) = \sum_{n=1}^{\infty} a_n e^{-n^p z}$ then $f_p(z)$ is a uniformly continuous function on the positive real axis, and so by Lemma 4.3.1 we have,

$$f_q(y) = \sum_{n=1}^{\infty} a_n e^{-yn^q} \quad (\text{C.21})$$

$$= \lim_{\delta \rightarrow 0} \sum_{n=1}^{\infty} a_n e^{-yn^q - \delta zn^p} \quad (\text{C.22})$$

$$= \frac{1}{\pi} \int_{C_1} f_p(-iy^{\frac{1}{k}} t) F(t) dt + \frac{1}{\pi} \int_{C_2} f_p(iy^{\frac{1}{k}} t) F(t) dt \quad (\text{C.23})$$

Finally, $f_p(-iy^{\frac{1}{k}} t)$ and $f_p(iy^{\frac{1}{k}} t)$ go to S when $y \rightarrow 0$ in the angle $\beta_1 \leq \theta \leq \beta_2$

Since $\int_{C_1} |F(t)| dt < \infty$ and $\int_{C_2} |F(t)| dt < \infty$

For ϕ_1 and ϕ_2 satisfying Equation C.5 we get the following by applying the dominated convergence theorem:

$$\lim_{y \rightarrow 0} f_q(y) = \lim_{y \rightarrow 0} \frac{1}{\pi} \int_{C_1} f_p(-iy^{\frac{1}{k}} n^p) F(t) dt + \lim_{y \rightarrow 0} \frac{1}{\pi} \int_{C_2} f_p(iy^{\frac{1}{k}} n^p) F(t) dt \quad (\text{C.24})$$

$$= \frac{S}{\pi} \int_{C_1} F(t) dt + \frac{S}{\pi} \int_{C_2} F(t) dt \quad (\text{C.25})$$

$$= \frac{2S}{\pi} \int_0^{\infty} F(t) dt \quad (\text{C.26})$$

$$= S \quad (\text{C.27})$$

uniformly with $\theta \in (\beta_1, \beta_2)$.

■

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