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**Electromagnetic scattering by an inhomogeneous penetrable
material with an embedded resistive sheet**

Gilmore, Sean William, Ph.D.

The Ohio State University, 1992

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Ann Arbor, MI 48106**

ELECTROMAGNETIC SCATTERING BY AN
INHOMOGENEOUS PENETRABLE MATERIAL WITH AN
EMBEDDED RESISTIVE SHEET

A Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Sean William Gilmore, B.S.E.E., M.S.E.E.

* * * * *

The Ohio State University

1992

Dissertation Committee:

Professor Edward H. Newman

Professor Prabhakar H. Pathak

Professor Roger C. Rudduck

Approved by:

Adviser

Department of Electrical
Engineering

DEDICATION

To my mother and the memory of my father.

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VITA

1982 – 1986	Earth Terminal Antenna Department, COMSAT Laboratories, Clarksburg, Maryland
June 1986	B.S. Electrical Engineering Magna Cum Laude with Distinction, Virginia Polytechnic Institute and State University, Blacksburg, Virginia
1986 – 1992	The ElectroScience Laboratory, The Ohio State University, Columbus, Ohio
June 1988	M.S. Electrical Engineering, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Electrical Engineering

Electromagnetics	Robert G. Kouyoumjian, Prabhakar H. Pathak
Antennas	Roger C. Rudduck
Communications	Frederick D. Garber, Dean T. Davis
Applied Mathematics	Ulrich H. Gerlach, David W. Dean

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CHAPTER I

INTRODUCTION

The scattering from penetrable bodies has been the focus of much research in recent years, especially in the areas of hyperthermia and the effect of electromagnetic waves on living organs [1, 2, 3, 4, 5, 6]. Since exact solutions exist for only the most simple of canonic shapes, techniques to solve these problems usually involve computer intensive numerical methods. All of these methods require the modeling of a system by the appropriate mathematical equations and boundary conditions followed by a discretization of this system. The discrete system is then solved by direct matrix methods or iterative techniques.

Our concerns lie with developing a numerical approach to predict the electromagnetic scattering from three-dimensional material bodies. The materials of interest involve isotropic dielectrics and ferrites that may include some loss (complex permittivities and permeabilities) as well as resistive sheets. These sheets may be finitely or infinitely conducting and fully or partially embedded in a material volume.

The geometry of the scatterer will have some bearing on choosing the numerical approach. We will primarily be concerned with a general inhomogeneous material that has some volume but is very thin. Specifically, we intend to determine the scattered fields from material based propeller blades. Much work has been done previously with this type of scatterer for the perfectly conducting case [7, 8, 9, 10]. Figure 1 illustrates a typical blade geometry. The technique chosen must be robust

enough to handle this type of geometry, but must also be amenable to efficient implementation on a large main frame or super computer.

1.1 Available Numerical Methods

A number of different techniques are available for the solution of the material body problem, depending upon the problem formulation. Here, we will consider three formulations and discuss their relative merits with respect to the scattering geometries of interest. These three techniques are as follows: the volume differential equation formulation, the surface integral equation formulation, and the volume integral equation formulation.

Figure 1: Propeller blade geometry

1.1.1 Volume Differential Equation Formulation

The Volume Differential Equation (VDE) formulation relies on the differential, or point form, of Maxwell's equations and leads to what is commonly called a Finite Element Method (FEM) approach. In general, the volume is discretized, or meshed, and Maxwell's equations are satisfied either in the so-called "weak sense" using the method of weighted residuals [1, 5], or by a variational technique [11, 12, 13].

Because it begins with the differential form of Maxwell's equations, the FEM approach leads to a sparse matrix to invert. With appropriate nodal numbering, this sparse matrix can be banded about the main diagonal [14, 15] and efficient methods exist to solve the resulting set of simultaneous equations. Also, since the matrix is banded, less computer storage is required than for a comparable full matrix. However, this set of equations must be solved according to the given boundary conditions on the differential equations governing the volume.

Boundary Conditions

The FEM approach solves the set of linear differential equations according to a complementary set of boundary conditions (BC's). Thus, this technique has been used with extreme effectiveness in closed region problems such as waveguides and cavities where the BC's are well defined (Dirichlet, Neumann, or mixed) and are easy to apply. Unfortunately for the open region problem, the radiation condition at infinity is well defined but not easy to apply. For integral equation formulations, the radiation condition is naturally enforced by the Green's function kernel in the integral. For the differential equation formulation, the point form of Maxwell's equations are used and the radiation condition must be enforced explicitly. Two approaches are commonly used: global techniques and local techniques.

Global Boundary Conditions: Global techniques require formulation of an integral equation on the closed surface of the volume which then enforces the radiation condition everywhere outside the volume, resulting in a hybrid FEM solution [2, 13, 16]. Unfortunately, when the exterior region is coupled to the interior region the sparse banded matrix is modified if not completely destroyed. Thus, many of the advantages to the FEM approach are lost.

Techniques do exist to decouple the surface integral from the interior, thereby retaining the sparse banded matrix [5, 6, 17, 18, 19]. Unfortunately, these techniques are even more difficult to implement, especially for three-dimensional problems. In either case, if the body is very thin the solution becomes dominated by the surface integral equation, coupled or decoupled. Therefore, the projected savings with the hybrid FEM approach are not realized.

Local Boundary Conditions: Local techniques involve the use of absorbing boundary conditions (ABC's) [20, 21, 22] that are applied locally at each exterior node. These ABC's are applied in the same way that other BC's are applied, thus no implementation cost is incurred. However, the ABC's can only approximate the radiation condition.

To apply the ABC's, the volume must be meshed up to a boundary some distance away from the actual scatterer [23] (this distance is itself the subject of current research). Therefore, although complexity of the formulation has been saved over the hybrid approach, the number of unknowns may have significantly increased using ABC's. A good estimate is that the mesh must be extended out in all directions some percentage of the largest dimension of the scatterer. In the case of a thin body, a large number of unknowns are generated in an attempt to compensate for

the radiation condition. Thus, we see that the FEM approach with ABC's is also not a good approach for our problem.

Other problems exist in the FEM approach that also must be considered (e.g., spurious modes and ill-conditioned matrices). Coupled with the problems in applying the radiation condition, an integral approach seems more suited to our needs.

1.1.2 Surface Integral Equation Formulation

One Surface Integral Equation (SIE) formulation employs Schelkunoff's surface equivalence theorem to replace a *homogeneous* volume with equivalent surface currents [24]. These currents are then used to calculate the scattered fields. Because the volume is required to be homogeneous, the material characteristics of the scatterer are somewhat restricted. This technique can however be applied to a volume made up of homogeneous sub-regions such as a layered material. Another SIE formulation may be used to construct an integral formulation for the scattering from an infinitely thin sheet by directly employing the boundary conditions. A surface integral formulation may also be used to approximate thin dielectric shells. These SIE formulations lead to what is commonly referred to as a Method of Moments (MoM) approach. The unknown currents are expanded in terms of a set of known basis functions, and the integral equations are enforced by testing at certain points or regions on the surface or surfaces.

Using an SIE approach for a two layer volume, eight scalar unknowns must be calculated: two scalar components for the unknown electric current $\vec{J}$ and two scalar components for the unknown magnetic current $\vec{M}$ on each of two surfaces. If the volume is very thin, the SIE approach may have more unknowns than a corresponding volume integral equation approach, as demonstrated in the next section.

Also, numerical problems may exist in a SIE formulation if the surfaces are in close proximity (e.g., closely spaced layers).

1.1.3 Volume Integral Equation Formulation

The Volume Integral Equation (VIE) formulation uses the volume equivalence theorem to replace the *inhomogeneous* volume scatterer with volume equivalent currents [24]. An integral equation is formulated to calculate the unknown currents. Fields interior and exterior to the volume can then be found readily.

In the VIE formulation, three orthogonal components of both the unknown electric current $\vec{J}$ and the unknown magnetic current $\vec{M}$ must be calculated, leading to six scalar equations over the volume. However, if the volume is thin, the VIE approach may have less unknowns than a comparable formulation for a homogeneous scatterer using the SIE approach. Consider, for example, the cases illustrated in Table 1.

Table 1 outlines the number of unknowns required for the electromagnetic scattering from a three-dimensional thin material rectangular slab of dimensions 25 cm by 10 cm by 0.5 cm, which are on the order of the dimensions for a typical propeller fan blade (see Figure 1). For a homogeneous slab of relative permittivity ϵ_r and relative permeability μ_r , the SIE formulation requires four unknowns per square cell. For the VIE formulation, six unknowns per cubic cell are required. A modest sampling of $\lambda_o/10$ in both cases is assumed (where λ_o is the free-space wavelength).

As illustrated in Table 1, the SIE approach has more unknowns than the VIE approach at the lower frequencies for this thin geometry. Also, if another material were embedded to form a two-layer geometry, the number of unknowns would increase substantially for the SIE formulation. The VIE formulation would not require more unknowns to account for material inhomogeneities provided these inhomogeneities

Table 1: Comparison of MoM formulations for a thin volume.

frequency (GHz)	surface area (λ_o^2)	volume (λ_o^3)	SIE unknowns	VIE unknowns
1 ($\lambda_o = 30.00$ cm)	0.5944	0.0046	~ 240	~ 28
2 ($\lambda_o = 15.00$ cm)	2.3778	0.0370	~ 950	~ 220
10 ($\lambda_o = 3.00$ cm)	59.4444	4.6296	$\sim 24,000$	$\sim 28,000$
18 ($\lambda_o = 1.67$ cm)	192.6000	27.0000	$\sim 77,000$	$\sim 162,000$

were not smaller than a cubic cell. Thus, for this thin material geometry a VIE approach is preferable to a SIE approach at lower frequencies.

1.2 Method of Investigation

We will employ a Volume Integral Equation formulation to calculate the scattered fields from a thin material body. Since we wish to embed a resistive sheet in the material, a Surface Integral Equation formulation will be added for the sheet. In addition, we shall use this SIE formulation to approximate thin dielectric materials as infinitely thin sheet impedances. The geometry will be discretized and a Method of Moments approach will then be used to numerically solve the linear system of equations.

Because we wish to study scatterers with complex shapes, tetrahedral modeling of the inhomogeneous volume and triangular modeling of the surface resistive

boundary sheet will be used. These types of elements will provide a better geometric model than simple rectangular brick meshing. However, tetrahedral and triangular cells require more sophisticated basis functions which support the unknown currents or fields in the MoM formulation. We have chosen to use special basis functions developed by Schaubert, Wilton, Glisson, and Rao [25, 26]. These will be discussed at some length.

We will begin our analysis of the scattering from material bodies with a discussion of the plane wave scattering at normal incidence from planar materials of infinite extent in Chapter II. This simple analysis will provide a motivation for the development of a more general numerical solution for the scattering from complex material bodies. Chapter III details the formulation of the integral equations used to mathematically model the scattering material. Chapter IV then presents the discretization of the system of equations and expansion of the unknown currents in terms of tetrahedral and triangular basis functions. The subsequent transformation of the set of equations into matrix form via the Method of Moments is discussed in Chapter V. Mesh parameters involving the number of unknowns, sampling, and discretization are then briefly considered in Chapter VI. Results calculated by a computer program based on the MoM approach are presented in Chapters VII, VIII, and IX for the material coated resistive sphere, the grounded slab, and the material fan blade, respectively. Comparisons are made to both calculated canonical problems as well as measurements. Conclusions and recommendations are presented in Chapter X. The dissertation is followed by several appendices which discuss various topics associated with this study including a new eigenfunction solution for the electromagnetic scattering from a material coated resistive sphere.

CHAPTER II

MOTIVATION: FRESNEL SPECULAR ANALYSIS

As a motivation for studying the dielectric/ferrite material with an embedded resistive sheet, we find it to our advantage to study the specular scattering from various planar geometries of infinite extent. These configurations will give us some indication of the effect of the resistive sheet on the specular reflection from a penetrable material. The following analysis focuses upon the normal incidence backscatter fields.

2.1 Resistive Sheet

First consider a principal plane analysis of scattering from an infinite resistive sheet as illustrated in Figure 2. The sheet is lying in the x - z plane with a constant surface resistance R (ohms/square). A plane wave traveling in the $-\hat{y}$ direction with electric field polarized in the $\hat{x}$ direction is incident normal to the sheet.

The incident fields on the resistive sheet can be written ($e^{j\omega t}$ is assumed and suppressed)

$$\vec{E}^i = \hat{x}e^{jk_oy}, \quad (2.1)$$

$$\vec{H}^i = \hat{z}\frac{e^{jk_oy}}{Z_o}, \quad (2.2)$$

where k_o is the free space wave number and Z_o is the intrinsic impedance of free space. The reflected fields can be expressed as

$$\vec{E}^r = \hat{x}\Gamma e^{-jk_oy}, \quad (2.3)$$

Figure 2: Plane wave incidence normal to a resistive sheet of infinite extent.

$$\vec{H}^r = -\hat{z} \frac{\Gamma e^{-jk_o y}}{Z_o}, \quad (2.4)$$

where Γ is the *reflection coefficient* associated with the resistive boundary. Likewise, the transmitted field can be written as

$$\vec{E}^T = \hat{x} T e^{jk_o y}, \quad (2.5)$$

$$\vec{H}^T = \hat{z} \frac{T e^{jk_o y}}{Z_o}, \quad (2.6)$$

for a *transmission coefficient* T .

The boundary conditions on the resistive sheet are as follows [27]:

The tangential electric field is continuous across the sheet,

$$\hat{n} \times (\vec{E}^+ - \vec{E}^-) = \vec{0}, \quad (2.7)$$

and the tangential magnetic field is discontinuous across the sheet,

$$\hat{n} \times \hat{n} \times \vec{E} = -R \hat{n} \times (\vec{H}^+ - \vec{H}^-), \quad (2.8)$$

where the $+/-$ signs indicate either side of the thin resistive boundary, and $\hat{n}$ is the surface normal on the $+$ side of the boundary. See Appendix A for a more complete discussion of the resistive boundary condition.

For the geometry illustrated above, the normal is given by $\hat{n} = \hat{y}$. On the boundary $y = 0$,

$$\vec{E}^+ = (\vec{E}^i + \vec{E}^r)|_{y=0} \quad (2.9)$$

$$= \hat{z}(1 + \Gamma), \quad (2.10)$$

$$\vec{E}^- = (\vec{E}^T)|_{y=0} \quad (2.11)$$

$$= \hat{z}T. \quad (2.12)$$

From (2.7),

$$\hat{x} \times (\vec{E}^+ - \vec{E}^-) = (\hat{y} \times \hat{x}) [(1 + \Gamma) - T], \quad (2.13)$$

$$= \vec{0}. \quad (2.14)$$

That is,

$$T = 1 + \Gamma, \quad (2.15)$$

which is the expected relation. Also, on the boundary $y = 0^+$

$$\vec{H}^+ = (\vec{H}^i + \vec{H}^r)|_{y=0} \quad (2.16)$$

$$= \hat{z} \frac{(1 - \Gamma)}{Z_o}, \quad (2.17)$$

$$\vec{H}^- = (\vec{H}^T)|_{y=0} \quad (2.18)$$

$$= \hat{z} \frac{T}{Z_o}, \quad (2.19)$$

such that from (2.8)

$$-\hat{x}(1 + \Gamma) = -\hat{x} \frac{R}{Z_o} [(1 - \Gamma) - T]. \quad (2.20)$$

Substituting (2.15) into the above equation and solving for Γ , the reflection coefficient for the resistive sheet may be written as

$$\Gamma = -\frac{Z_o}{Z_o + 2R}, \quad (2.21)$$

and again using (2.15), the transmission coefficient for the resistive sheet may be written as

$$T = \frac{2R}{Z_o + 2R}. \quad (2.22)$$

Note that from (2.21) and (2.22), in the limit as $R \rightarrow 0$ (the sheet becomes a perfect electric conductor)

$$\Gamma = -1, \quad (2.23)$$

$$T = 0, \quad (2.24)$$

and as $R \rightarrow \infty$ (the sheet disappears)

$$\Gamma = 0, \quad (2.25)$$

$$T = 1, \quad (2.26)$$

Figure 3: Plane wave incidence normal to an infinite dielectric slab with embedded resistive sheet.

which are the expected results.

2.2 Dielectric Slab with Embedded Resistive Sheet

We now wish to use the results obtained in the previous section to analyze the backscatter from a dielectric slab of infinite extent with an embedded resistive sheet at normal incidence. Consider the geometry illustrated in Figure 3. A plane wave is incident normal to a dielectric slab of relative permittivity ϵ_r and relative permeability $\mu_r = 1$. The slab has a width of $2d$ and characteristic impedance Z_1 with a resistive sheet R embedded half way between.

At the $Z_1 - R - Z_1$ interface, the *local* reflection and transmission coefficients

$$r = -\frac{Z_1}{Z_1 + 2R}, \quad (2.27)$$

$$\tau = \frac{2R}{Z_1 + 2R}, \quad (2.28)$$

may be written using the results of the previous section. At the Z_o - Z_1 interface, define an appropriate set of s parameters:

$$s_{11} = \text{reflection in } Z_o, \quad (2.29)$$

$$s_{21} = \text{transmission from } Z_o \text{ to } Z_1, \quad (2.30)$$

$$s_{12} = \text{transmission from } Z_1 \text{ to } Z_o, \quad (2.31)$$

$$s_{22} = \text{reflection in } Z_1. \quad (2.32)$$

Local incident and transmitted *scalar* fields at the two types of material interfaces may also be defined as illustrated in Figure 3:

$$A^+ = \text{field in } Z_1 \text{ incident on } Z_o, \quad (2.33)$$

$$A^- = \text{field in } Z_1 \text{ transmitted from } Z_o, \quad (2.34)$$

$$B^+ = \text{field in } Z_1 \text{ incident on } R, \quad (2.35)$$

$$B^- = \text{field in } Z_1 \text{ transmitted through } R. \quad (2.36)$$

Assuming plane wave incidence normal to the slab, the *global* reflection and transmission coefficients may be written by inspection as

$$\Gamma = s_{11}E^i + s_{12}A^+, \quad (2.37)$$

$$T = s_{12}PB^-, \quad (2.38)$$

where E^i is the magnitude of the incident plane wave and P is a propagation factor

$$P = e^{-jk_1d}, \quad (2.39)$$

for a wave number k_1 in the dielectric. Also by inspection,

$$A^+ = rP^2A^- + P\tau B^+, \quad (2.40)$$

$$A^- = s_{21}E^i + s_{22}A^+, \quad (2.41)$$

$$B^+ = s_{22}P^2B^-, \quad (2.42)$$

$$B^- = \tau PA^- + rB^+. \quad (2.43)$$

Substituting (2.42) into (2.43),

$$B^- = \tau PA^- + rP^2s_{22}B^- \quad (2.44)$$

$$= \frac{\tau PA^-}{1 - rP^2s_{22}}. \quad (2.45)$$

Further, substituting (2.45) back into (2.42) leaves

$$B^+ = s_{22}P^2 \frac{\tau PA^-}{1 - rP^2s_{22}}, \quad (2.46)$$

which can then be substituted into (2.40) such that

$$A^+ = rP^2A^- + P\tau \frac{\tau s_{22}P^3}{1 - rP^2s_{22}}A^- \quad (2.47)$$

$$= MA^-, \quad (2.48)$$

with the definition

$$M \equiv rP^2 + \frac{\tau^2 s_{22}P^4}{1 - rP^2s_{22}}. \quad (2.49)$$

From (2.41) and (2.48),

$$A^- = s_{21}E^i + s_{22}MA^-. \quad (2.50)$$

In other words,

$$A^- = \frac{s_{22}E^i}{1 - s_{22}M}, \quad (2.51)$$

and from (2.48)

$$A^+ = Ms_{21}E^i + s_{22}M^2A^-. \quad (2.52)$$

Then, substituting into (2.37) the reflection coefficient for normal incidence to a dielectric slab with embedded resistive sheet for $E^i = 1$ may be written as

$$\Gamma = s_{11} + \frac{s_{12}s_{21}M}{1 - s_{22}M}. \quad (2.53)$$

To find the corresponding transmission coefficient, substitute (2.46) into (2.43) and write

$$B^- = \tau P A^- + \frac{rs_{22}\tau P^3 A^-}{1 - rP^2 s_{22}} \quad (2.54)$$

$$= N A^-, \quad (2.55)$$

with

$$N \equiv \tau P + \frac{rs_{22}\tau P^3}{1 - rP^2 s_{22}}. \quad (2.56)$$

By substituting (2.51) into (2.55)

$$B^- = \frac{s_{21}E^i}{1 - s_{22}M}N, \quad (2.57)$$

and from (2.41) with $E^i = 1$

$$T = \frac{s_{12}Ps_{21}}{1 - s_{22}M}N. \quad (2.58)$$

2.3 Parametric Study

The above analysis may now be used to study the effects of various parameters on the level of backscatter for normal incidence from an infinite material sheet. Plots of Γ and T versus frequency for varying configurations are presented. A value of zero dB for the reflection coefficient indicates perfect reflection ($\Gamma = 1$) from a perfect electric conductor (PEC). A value of zero dB for the transmission coefficient indicates perfect transmission ($T = 1$) through free space.

Figures 4 and 5 illustrate the reflection and transmission coefficients for an infinite dielectric slab across the frequency band from 2 to 18 GHz. The total thickness of the slab is 0.25 in. and the relative permittivity is varied for three cases: $\epsilon_r = 3, 6, 9$. As expected, reflection increases for increasing relative permittivity.

The nulls in the reflection coefficient curves are formed by the cancellation of the reflected fields from the front and rear faces of the dielectric slab. The reflection coefficient from the front face is 180° out of phase with the reflection coefficient from the back face. Since the reflected field in the material travels a total of one wavelength or 360° of phase at these frequencies, the net phase shift in the $\lambda/2$ thick slab is zero and the reflected fields from the two faces cancel.

The reflection and transmission coefficients for an infinite resistive sheet (of vanishing thickness) are illustrated in Figures 6 and 7, respectively. The resistance is varied for $R = 50, 100, 200$, and $400 \Omega/\text{sq}$. Again, as expected, the backscatter level from the resistive sheet decreases as the resistance gets larger. Also note that the reflection and transmission coefficients are constant with respect to frequency for the infinite resistive sheet.

Now consider the combination of the infinite dielectric slab with an embedded resistive sheet for various permittivities and resistivities. Again, the slab is 0.25 in. thick with a resistive sheet of vanishing thickness embedded half way between (see Figure 3). Figures 8 through 13 illustrate the results of varying relative permittivities $\epsilon_r = 3, 6, 9$ and resistivity $R = 50, 100, 200, 400 \Omega/\text{sq}$.

As the resistivity of the embedded sheet is increased, the reflection at normal incidence is decreased across the frequency band. This decrease in reflection becomes less pronounced for larger values of permittivity. This trend is to be expected since the dielectric slab and the resistive sheet reflection coefficients act as if they were in

Figure 4: Reflection coefficients for infinite dielectric slab.

Figure 5: Transmission coefficients for infinite dielectric slab.

Figure 6: Reflection coefficients for infinite resistive sheet.

Figure 7: Transmission coefficients for infinite resistive sheet.

Figure 8: Reflection coefficients for infinite dielectric slab $\epsilon_r = 3$ with embedded resistive sheet.

Figure 9: Transmission coefficients for infinite dielectric slab $\epsilon_r = 3$ with embedded resistive sheet.

Figure 10: Reflection coefficients for infinite dielectric slab $\epsilon_r = 6$ with embedded resistive sheet.

Figure 11: Transmission coefficients for infinite dielectric slab $\epsilon_r = 6$ with embedded resistive sheet.

Figure 12: Reflection coefficients for infinite dielectric slab $\epsilon_r = 9$ with embedded resistive sheet.

Figure 13: Transmission coefficients for infinite dielectric slab $\epsilon_r = 9$ with embedded resistive sheet.

parallel with one another: as the permittivity increases, it tends to mask the effect of the resistive sheet.

In this simple analysis, we have illustrated the effect of an embedded resistive sheet on specular scattering at normal incidence. Preliminary indications are that the specular reflection from a material body may be reduced by the addition of a resistive sheet inside the scatterer. However, if the slab thickness becomes small, the reflection from an electrically thin dielectric slab may be very small compared to the reflection from any sheet of reasonable resistivity. In this case, the addition of a resistive sheet may substantially increase the overall reflection coefficient.

CHAPTER III

INTEGRAL EQUATION FORMULATION

The solution to the general scattering problem requires the formulation of a set of mathematical expressions that accurately describe the nature of the system with the appropriate boundary conditions. As previously mentioned, we have opted to use a combined Volume Integral Equation (VIE) – Surface Integral Equation (SIE) formulation with the Method of Moments (MoM). In general, this approach involves using the volume equivalence theorem to replace the inhomogeneous scatterer with equivalent electric and magnetic currents radiating in free space. The resistive boundary condition is modeled by a resistive sheet and equivalent surface currents. As discussed in Appendix A, this SIE formulation may also be used to approximate a thin dielectric shell. The VIE-SIE formulation is presented in detail below.

A general dielectric/ferrite volume V bounded by surface S is shown in Figure 14. An embedded resistive sheet S_R is also illustrated, with resistance $\overline{\overline{R}}$. The dyad $\overline{\overline{R}}$ represents the anisotropic surface resistance of the sheet and is given by (in general)

$$\overline{\overline{R}}(\vec{r}) = \hat{t}\hat{t}R_{tt}(\vec{r}) + \hat{t}\hat{b}R_{tb}(\vec{r}) + \hat{b}\hat{t}R_{bt}(\vec{r}) + \hat{b}\hat{b}R_{bb}(\vec{r}), \quad (3.1)$$

for the tangential $\hat{t}$ and binormal $\hat{b}$ components of the resistive boundary condition (i.e., $\hat{n} \times \hat{t} = \hat{b}$).

Figure 14: Penetrable dielectric/ferrite material with resistive boundary condition: original problem.

In general, S_R may be an open or closed surface of nonhomogeneous resistivity. Also, S_R may be a sub-region of the boundary surface S (e.g., a grounded slab). In the limit as $\overline{\overline{R}} \rightarrow 0$, the surface becomes perfectly conducting. The boundary conditions on S_R are as follows [27]:

The tangential electric field is continuous:

$$\hat{n} \times (\vec{E}^+ - \vec{E}^-) = \vec{0}. \quad (3.2)$$

The tangential magnetic field is discontinuous:

$$\hat{n} \times \hat{n} \times \vec{E} = -\overline{\overline{R}} \cdot \hat{n} \times (\vec{H}^+ - \vec{H}^-). \quad (3.3)$$

The $+/-$ signs indicate either side of the thin resistive boundary and $\hat{n}$ is the surface normal on the $+$ side of the boundary.

Consider the inhomogeneous material in the presence of *impressed* source currents $(\vec{J}^i, \vec{M}^i)$ as illustrated in Figure 14. The complex permittivity and complex permeability of the material body are given for time-harmonic fields ($e^{j\omega t}$ assumed and suppressed):

$$\epsilon(\vec{r}) = \epsilon'(\vec{r}) - j\epsilon''(\vec{r}) = \epsilon'(\vec{r}) - j\frac{\sigma(\vec{r})}{\omega}, \quad (3.4)$$

$$\mu(\vec{r}) = \mu'(\vec{r}) - j\mu''(\vec{r}), \quad (3.5)$$

where $\sigma(\vec{r})$ is the conductivity of the material body.

The volume equivalence theorem is used to replace the dielectric/ferrite material with equivalent electric and magnetic currents $(\vec{J}^V, \vec{M}^V)$, while surface equivalence is used to replace the resistive boundary surface by an equivalent electric surface current $(\vec{J}^R)$. These currents radiate the following fields in *free space* as illustrated in Figure 15:

$$\begin{aligned} (\vec{E}^i, \vec{H}^i) &= \text{the free space fields of } (\vec{J}^i, \vec{M}^i), \\ (\vec{E}^J, \vec{H}^J) &= \text{the free space fields of } \vec{J}^V, \\ (\vec{E}^M, \vec{H}^M) &= \text{the free space fields of } \vec{M}^V, \\ (\vec{E}^R, \vec{H}^R) &= \text{the free space fields of } \vec{J}^R. \end{aligned}$$

In general,

$$\begin{aligned} \vec{E} &= \vec{E}^i + \vec{E}^s \\ &= \vec{E}^i + \vec{E}^J + \vec{E}^M + \vec{E}^R, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \vec{H} &= \vec{H}^i + \vec{H}^s \\ &= \vec{H}^i + \vec{H}^J + \vec{H}^M + \vec{H}^R. \end{aligned} \quad (3.7)$$

Using the equivalent currents, the volume integral equations can be written as

$$\vec{E} = \vec{E}^i + \vec{E}^J + \vec{E}^M + \vec{E}^R$$

Figure 15: Penetrable dielectric/ferrite material with resistive boundary condition: equivalent problem.

$$= \frac{\vec{J}^V}{j\omega(\epsilon - \epsilon_o)} \quad \text{in } V, \quad (3.8)$$

$$\begin{aligned} \vec{H} &= \vec{H}^i + \vec{H}^J + \vec{H}^M + \vec{H}^R \\ &= \frac{\vec{M}^V}{j\omega(\mu - \mu_o)} \quad \text{in } V. \end{aligned} \quad (3.9)$$

This yields two coupled field integral equations for the three unknown currents $(\vec{J}^V, \vec{M}^V, \vec{J}^R)$. The third equation comes from the boundary condition on the resistive surface:

$$\begin{aligned} \hat{n} \times \hat{n} \times \vec{E} &= -\overline{\overline{R}} \cdot \hat{n} \times (\vec{H}^+ - \vec{H}^-) \\ &= -\overline{\overline{R}} \cdot \vec{J}^R \quad \text{on } S_R, \end{aligned} \quad (3.10)$$

which may be rewritten as

$$(\vec{E} = \overline{\overline{R}} \cdot \vec{J}^R) \Big|_{\tan S_R} \quad \text{on } S_R. \quad (3.11)$$

where $\tan S_R$ indicates that only the tangential components of the vector equation are enforced. Thus, the VIE-SIE formulation results in two volume equations and one surface equation for the three vector unknowns.

The free space scattered fields $(\vec{E}^s, \vec{H}^s)$ due to any electric and magnetic current $(\vec{J}, \vec{M})$ can be found using vector and scalar potentials [24]:

$$\vec{E}^s(\vec{r}) = -j\omega\vec{A}(\vec{r}) - \nabla\Phi(\vec{r}) - \frac{1}{\epsilon_o}\nabla \times \vec{F}(\vec{r}), \quad (3.12)$$

$$\vec{H}^s(\vec{r}) = -j\omega\vec{F}(\vec{r}) - \nabla\Psi(\vec{r}) + \frac{1}{\mu_o}\nabla \times \vec{A}(\vec{r}). \quad (3.13)$$

The vector potentials are given by

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \vec{J}(\vec{r}') G(\vec{r}, \vec{r}') d\vec{r}', \quad (3.14)$$

$$\vec{F}(\vec{r}) = \frac{\epsilon_o}{4\pi} \int \vec{M}(\vec{r}') G(\vec{r}, \vec{r}') d\vec{r}'; \quad (3.15)$$

The scalar potentials are given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_o} \int \rho^e(\vec{r}') G(\vec{r}, \vec{r}') d\vec{r}', \quad (3.16)$$

$$\Psi(\vec{r}) = \frac{1}{4\pi\mu_o} \int \rho^m(\vec{r}') G(\vec{r}, \vec{r}') d\vec{r}', \quad (3.17)$$

where the electric and magnetic charge distributions (ρ^e, ρ^m) are related to the electric and magnetic current distributions $(\vec{J}, \vec{M})$ by the equations of continuity:

$$\rho^e(\vec{r}') = -\frac{1}{j\omega} \nabla \cdot \vec{J}(\vec{r}'), \quad (3.18)$$

$$\rho^m(\vec{r}') = -\frac{1}{j\omega} \nabla \cdot \vec{M}(\vec{r}'). \quad (3.19)$$

The free space Green's function is given by

$$G(\vec{r}, \vec{r}') = \frac{e^{-jkR}}{R}, \quad (3.20)$$

for wave number $k = \omega\sqrt{\mu_o\epsilon_o}$ and $R = |\vec{r} - \vec{r}'|$.

The “curl” terms in Equations (3.12) and (3.13) may be simplified by using a vector identity (and noting that the integrals are defined in the Cauchy principal value sense) [28, 29]:

$$\nabla \times \vec{A} = \frac{\mu_o}{4\pi} \int \vec{J}(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}', \quad (3.21)$$

and

$$\nabla \times \vec{F} = \frac{\epsilon_o}{4\pi} \int \vec{M}(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}', \quad (3.22)$$

for

$$\nabla' G(\vec{r}, \vec{r}') = (\vec{r} - \vec{r}') (1 + jkR) \frac{e^{-jkR}}{R^3}. \quad (3.23)$$

With the governing equations and boundary conditions stated in terms of a set of unknowns, a MoM solution must be formulated to solve the problem numerically. The first step in this solution will be to outline an appropriate set of basis functions. This discussion will be carried out in the next chapter.

CHAPTER IV

BASIS FUNCTIONS

Discretization of the system of equations involves the expansion of the unknown currents with an appropriate set of basis functions. For the problem at hand, we find it convenient to expand the currents in terms of subsectional basis functions that are supported only on the individual cells of the discretized body. At least two types of subsectional basis functions must be specified: a volume basis function for $(\vec{J}^V, \vec{M}^V)$ and a surface basis function for $\vec{J}^R$. Also, since *vector* integral equations have been formulated, these basis functions must support vector quantities; i.e.,

$$\vec{J}^R(\vec{r}) \approx \sum_{n=1}^M I_n \vec{f}_n^S(\vec{r}) \quad \text{on } S_R, \quad (4.1)$$

$$\vec{J}^V(\vec{r}) \approx \sum_{n=1}^N J_n \vec{f}_n^V(\vec{r}) \quad \text{in } V, \quad (4.2)$$

$$\vec{M}^V(\vec{r}) \approx \sum_{n=1}^N M_n \vec{f}_n^V(\vec{r}) \quad \text{in } V. \quad (4.3)$$

The choice of subsectional basis functions is coupled to the method with which we wish to discretize, or mesh, the body. Since we wish to study complex three-dimensional bodies, the individual cells must be able to accurately model complex shapes. With this in mind, we have chosen to use tetrahedral cells to model the material volume and triangular cells to model the resistive boundary.

Triangular and tetrahedral cells have been used previously with MoM solutions to calculate the scattering from lossless and lossy homogeneous bodies [25, 28] as well as inhomogeneous dielectric bodies [26]. Here, we propose to couple the two

methods and extend the formulation to include three-dimensional dielectric/ferrite materials with embedded resistive sheets.

Special basis functions have been developed [25, 26] for use with triangular and tetrahedral modeling. These functions have particular characteristics that make them useful in the MoM formulation. Each type of vector basis function will be discussed below.

4.1 Triangular vector basis functions

For the pair of triangles shown in Figure 16, define using the notation of Rao, et al. [25] a surface vector basis function

$$\vec{f}_n^S(\vec{r}) = \begin{cases} \frac{\ell_n}{2A_n^+} \vec{\rho}_n(\vec{r}) & \vec{r} \in T_n^+, \\ \frac{\ell_n}{2A_n^-} \vec{\rho}_n(\vec{r}) & \vec{r} \in T_n^-, \\ \vec{0} & \text{otherwise,} \end{cases} \quad (4.4)$$

where ℓ_n is the length of the common edge between the triangles and $A_n^\pm$ is the area of the corresponding triangle. The $+/-$ convention indicates the reference direction of the current through the triangular pair: the vector $\vec{\rho}_n$ is taken from the free vertex of the triangle to the opposite adjacent edge of the triangular pair for the “positive” triangle and away from the adjoining edge to the free vertex for the “negative” triangle. Some of the important properties associated with this basis function are listed here from [25]:

1. “The current has no component normal to the boundary (excluding the common edge) of the surface formed by the triangular pair T_n^+ and T_n^- , and hence no line charge exists along this boundary.”

Figure 16: Triangular cell pair $T_n^\pm$ [25].

2. "The component of the current normal to the n th edge is constant and continuous across the edge This result, together with (1), implies that *all* edges of T_n^+ and T_n^- are free of line charges."
3. "The surface divergence of $\vec{f}_n^S$, which is proportional to the surface charge density of the basis function, is

$$\nabla_S \cdot \vec{f}_n^S = \begin{cases} +\frac{\ell_n}{A_n^+} & \vec{r} \in T_n^+, \\ -\frac{\ell_n}{A_n^-} & \vec{r} \in T_n^-, \\ 0 & \text{otherwise,} \end{cases} \quad (4.5)$$

.... The charge density is thus constant in each triangle, the total charge associated with the triangle pair T_n^+ and T_n^- is zero, and the basis functions for the charge evidently have the form of *pulse doublets*."

4. "The *moment* of $\vec{f}_n^S$ is given by $(A_n^+ + A_n^-)\vec{f}_n^{S,avg}$ where

$$(A_n^+ + A_n^-)\vec{f}_n^{S,avg} \equiv \int_{T_n^+ + T_n^-} \vec{f}_n^S dS \quad (4.6)$$

$$= \frac{\ell_n}{2}(\vec{\rho}_n^{c+} + \vec{\rho}_n^{c-}) \quad (4.7)$$

$$= \ell_n(\vec{r}_n^{c+} - \vec{r}_n^{c-}) \quad (4.8)$$

and $\vec{\rho}_n^{c\pm}$ is the vector between the free vertex and the centroid of $T_n^\pm$... and $\vec{r}_n^{c\pm}$ is the vector from the origin $\mathcal{O}$ to the centroid of $T_n^\pm$."

The importance of properties (3) and (4) will become apparent when the contributions to the MoM matrix from the individual cells are calculated.

These vector basis functions are constructed such that the normal component of the current is continuous across all *non-boundary* edges of the triangular facets. If S_R is an open surface, no basis functions are defined across the edges discretizing the boundary curve of the surface since no current is allowed to flow off these edges. If

S_R is a closed surface, no boundary curve exists and the basis functions are defined across all triangular facet edges. In this way, only dipole pairs are specified in the surface expansion.

Rao describes these basis functions as similar to the standard scalar “rooftop” functions which support a linear variation in one direction and are constant in the transverse direction. He goes on to show that these vector bases will support any constant field quantity within the cell (surface or volume). Other details of these basis functions can be found in [30].

4.2 Tetrahedral vector basis functions

The tetrahedral vector basis function is a simple extension of the triangular vector basis function to three dimensions. Consider now the pair of tetrahedrons shown in Figure 17. Using the notation of [26], define a volume vector basis function

$$\vec{f}_n^V(\vec{r}) = \begin{cases} \frac{a_n}{3V_n^+} \vec{\phi}_n(\vec{r}) & \vec{r} \in \mathcal{T}_n^+ \\ \frac{a_n}{3V_n^-} \vec{\phi}_n(\vec{r}) & \vec{r} \in \mathcal{T}_n^- \\ \vec{0} & \text{otherwise} \end{cases} \quad (4.9)$$

where a_n is the area of the common face between the tetrahedrons and $V_n^\pm$ is the volume of the corresponding tetrahedron. A similar set of properties to that of the triangular basis function applies to the tetrahedral basis functions [26]:

1. “ $\vec{f}_n^V(\vec{r})$ has no component normal to any face except the common face of the conjoined pair $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$.”
2. “The component of $\vec{f}_n^V(\vec{r})$ normal to the n th face is constant and continuous across the face”

Figure 17: Tetrahedral cell pair $\mathcal{T}_n^\pm$ [26].

3. “The divergence of the basis function, is

$$\nabla \cdot \vec{f}_n^V(\vec{r}) = \begin{cases} +\frac{a_n}{V_n^+} & \vec{r} \in \mathcal{T}_n^+, \\ -\frac{a_n}{V_n^-} & \vec{r} \in \mathcal{T}_n^-, \\ 0 & \text{otherwise,} \end{cases} \quad (4.10)$$

.... The charge density ... is constant in each tetrahedron.”

4. “The *moment* of $\vec{f}_n^V$ is

$$\int_{\mathcal{T}_n^\pm} \vec{f}_n^V dS = \frac{a_n}{3} \vec{\wp}_n^{c\pm} \quad (4.11)$$

where $\vec{\wp}_n^{c\pm}$ is the vector between the free vertex and the centroid of $\mathcal{T}_n^\pm$.”

As in the case with the triangular surface basis functions, the tetrahedral basis functions are defined in dipole pairs to insure flux continuity across material interfaces. However, for the dipole pairs that are separated by the resistive boundary condition, the electric flux density experiences a jump discontinuity equal to the equivalent surface charge associated with the boundary (see Appendix A). For these basis functions, we find it necessary to define pairs of monopoles. These monopole bases have the same functional support in each tetrahedron as the dipole basis functions, but do not insure flux continuity across the resistive boundary. No explicit boundary condition on the normal component of these monopoles is enforced. However, the correct jump discontinuity should be calculated since the tangential components of the boundary condition and Maxwell’s equations are enforced on both sides of the resistive sheet.

CHAPTER V

METHOD OF MOMENTS SOLUTION

Given the governing mathematical equations discussed in Chapter III and the basis functions and discretization introduced in Chapter IV, a set of linearly independent equations that can be displayed in matrix form using the Method of Moments may now be formulated. First, the unknown currents are expanded using the chosen basis functions of Chapter IV. On the surface S_R , let

$$\vec{J}^R(\vec{r}) \approx \sum_{n=1}^M I_n \vec{f}_n^S(\vec{r}). \quad (5.1)$$

where, in order to satisfy the condition of current continuity, M is the number of non-boundary triangular facet edges [25].

In the material volume V , rather than directly expand the unknown volumetric currents $(\vec{J}^V, \vec{M}^V)$, the intermediate flux densities are expanded following the procedure of [26]:

$$\vec{D}(\vec{r}) \approx \sum_{n=1}^N D_n \vec{f}_n^V(\vec{r}), \quad (5.2)$$

$$\vec{B}(\vec{r}) \approx \sum_{n=1}^N B_n \vec{f}_n^V(\vec{r}), \quad (5.3)$$

with N being the number of tetrahedral cell faces. Whereas on the resistive surface the normal component of the current is required to be continuous across non-boundary edges, in the material volume the normal component of the flux densities is required to be continuous across material interfaces (but not across the resistive

boundary). The intermediate unknown flux densities are related to the unknown volumetric currents by

$$\vec{J}^V(\vec{r}) \approx j\omega \sum_{n=1}^N D_n \kappa^e(\vec{r}) \vec{f}_n^V(\vec{r}), \quad (5.4)$$

$$\vec{M}^V(\vec{r}) \approx j\omega \sum_{n=1}^N B_n \kappa^m(\vec{r}) \vec{f}_n^V(\vec{r}), \quad (5.5)$$

where the *contrast ratios* $\kappa^e(\vec{r})$ and $\kappa^m(\vec{r})$ are given by

$$\kappa^e(\vec{r}) = \frac{\epsilon(\vec{r}) - \epsilon_o}{\epsilon(\vec{r})}, \quad (5.6)$$

$$\kappa^m(\vec{r}) = \frac{\mu(\vec{r}) - \mu_o}{\mu(\vec{r})}. \quad (5.7)$$

The three equations (3.8), (3.9), and (3.10) are now rewritten in terms of the intermediate unknowns:

$$\begin{aligned} \vec{E}^i(\vec{r}) &= \frac{\vec{D}(\vec{r})}{\epsilon(\vec{r})} + j\omega \vec{A}(\vec{r}) + \nabla \Phi(\vec{r}) + \frac{1}{\epsilon_o} \nabla \times \vec{F}(\vec{r}) \\ &+ j\omega \vec{A}^R(\vec{r}) + \nabla \Phi^R(\vec{r}), \end{aligned} \quad (5.8)$$

$$\begin{aligned} \vec{H}^i(\vec{r}) &= \frac{\vec{B}(\vec{r})}{\mu(\vec{r})} + j\omega \vec{F}(\vec{r}) + \nabla \Psi(\vec{r}) - \frac{1}{\mu_o} \nabla \times \vec{A}(\vec{r}) \\ &- \frac{1}{\mu_o} \nabla \times \vec{A}^R(\vec{r}), \end{aligned} \quad (5.9)$$

$$\begin{aligned} (\vec{E}^i(\vec{r}) &= \vec{\bar{R}} \cdot \vec{J}^R(\vec{r}) + j\omega \vec{A}(\vec{r}) + \nabla \Phi(\vec{r}) + \frac{1}{\epsilon_o} \nabla \times \vec{F}(\vec{r}) \\ &+ j\omega \vec{A}^R(\vec{r}) + \nabla \Phi^R(\vec{r})) \Big|_{\tan S_R}, \end{aligned} \quad (5.10)$$

where $\tan S_R$ indicates that only the tangential components of the vector equation are enforced.

The set of equations is now “tested” to generate a system of P equations in P unknowns where $P = (2N + M)$. Following the procedure of Schaubert, et al. [26], and Rao [30], a Galerkin testing procedure (test functions same as basis functions) is employed where suitable approximations are made later to calculate the individual

matrix elements. Define the “symmetric” inner products:

$$\langle \vec{a}, \vec{b} \rangle_s = \int_s \vec{a} \cdot \vec{b} ds, \quad (5.11)$$

$$\langle \vec{a}, \vec{b} \rangle_v = \int_v \vec{a} \cdot \vec{b} dv, \quad (5.12)$$

where s is defined over the triangular pair $T_n^\pm$ and v is defined over the tetrahedral pair $T_n^\pm$. Then, testing the above three equations results in

$$\begin{aligned} \langle \vec{E}^i(\vec{r}), \vec{f}_m^V \rangle_v &= \langle \frac{\vec{D}(\vec{r})}{\epsilon(\vec{r})}, \vec{f}_m^V \rangle_v + \langle j\omega \vec{A}(\vec{r}), \vec{f}_m^V \rangle_v \\ &+ \langle \nabla \Phi(\vec{r}), \vec{f}_m^V \rangle_v + \langle \frac{1}{\epsilon_o} \nabla \times \vec{F}(\vec{r}), \vec{f}_m^V \rangle_v \\ &+ \langle j\omega \vec{A}^R(\vec{r}), \vec{f}_m^V \rangle_v + \langle \nabla \Phi^R(\vec{r}), \vec{f}_m^V \rangle_v, \end{aligned} \quad (5.13)$$

$$\begin{aligned} \langle \vec{H}^i(\vec{r}), \vec{f}_m^V \rangle_v &= \langle \frac{\vec{B}(\vec{r})}{\mu(\vec{r})}, \vec{f}_m^V \rangle_v + \langle j\omega \vec{F}(\vec{r}), \vec{f}_m^V \rangle_v \\ &+ \langle \nabla \Psi(\vec{r}), \vec{f}_m^V \rangle_v - \langle \frac{1}{\mu_o} \nabla \times \vec{A}(\vec{r}), \vec{f}_m^V \rangle_v \\ &- \langle \frac{1}{\mu_o} \nabla \times \vec{A}^R(\vec{r}), \vec{f}_m^V \rangle_v, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \langle \vec{E}^i(\vec{r}), \vec{f}_m^S \rangle_s &= \langle \vec{R}(\vec{r}) \cdot \vec{J}^S, \vec{f}_m^S \rangle_s + \langle j\omega \vec{A}(\vec{r}), \vec{f}_m^S \rangle_s \\ &+ \langle \nabla \Phi(\vec{r}), \vec{f}_m^S \rangle_s + \langle \frac{1}{\epsilon_o} \nabla \times \vec{F}(\vec{r}), \vec{f}_m^S \rangle_s \\ &+ \langle j\omega \vec{A}^R(\vec{r}), \vec{f}_m^S \rangle_s + \langle \nabla \Phi^R(\vec{r}), \vec{f}_m^S \rangle_s. \end{aligned} \quad (5.15)$$

These equations can then be represented in matrix form as follows:

$$\begin{bmatrix} R_{mn} & U_{mn} & W_{mn} \\ V_{mn} & S_{mn} & Y_{mn} \\ X_{mn} & Z_{mn} & T_{mn} \end{bmatrix} \begin{bmatrix} I_n \\ D_n \\ B_n \end{bmatrix} = \begin{bmatrix} E_m^R \\ E_m \\ H_m \end{bmatrix} \quad (5.16)$$

The sub-matrix elements are given as follows:

$$R_{mn} = j\omega \langle \vec{A}_n^R, \vec{f}_m^S \rangle_s + \langle \nabla \Phi_n^R, \vec{f}_m^S \rangle_s - \langle \vec{R}_n \cdot \vec{f}_n^S, \vec{f}_m^S \rangle_s \quad (5.17)$$

$$S_{mn} = j\omega \langle \vec{A}_n, \vec{f}_m^V \rangle_v + \langle \nabla \Phi_n, \vec{f}_m^V \rangle_v + \frac{1}{\epsilon_n} \langle \vec{f}_n^V, \vec{f}_m^V \rangle_v \quad (5.18)$$

$$T_{mn} = j\omega \langle \vec{F}_n, \vec{f}_m^V \rangle_v + \langle \nabla \Psi_n, \vec{f}_m^V \rangle_v + \frac{1}{\mu_n} \langle \vec{f}_n^V, \vec{f}_m^V \rangle_v \quad (5.19)$$

$$U_{mn} = j\omega \langle \vec{A}_n, \vec{f}_m^S \rangle_s + \langle \nabla \Phi_n, \vec{f}_m^S \rangle_s \quad (5.20)$$

$$V_{mn} = j\omega \langle \vec{A}_n^R, \vec{f}_m^V \rangle_v + \langle \nabla \Phi_n^R, \vec{f}_m^V \rangle_v \quad (5.21)$$

$$W_{mn} = \frac{1}{\epsilon_o} \langle \nabla \times \vec{F}_n, \vec{f}_m^S \rangle_s \quad (5.22)$$

$$X_{mn} = -\frac{1}{\mu_o} \langle \nabla \times \vec{A}_n^R, \vec{f}_m^V \rangle_v \quad (5.23)$$

$$Y_{mn} = \frac{1}{\epsilon_o} \langle \nabla \times \vec{F}_n, \vec{f}_m^V \rangle_v \quad (5.24)$$

$$Z_{mn} = -\frac{1}{\mu_o} \langle \nabla \times \vec{A}_n, \vec{f}_m^V \rangle_v \quad (5.25)$$

$$E_m^R = \langle \vec{E}^i, \vec{f}_m^S \rangle_s \quad (5.26)$$

$$E_m = \langle \vec{E}^i, \vec{f}_m^V \rangle_v \quad (5.27)$$

$$H_m = \langle \vec{H}^i, \vec{f}_m^V \rangle_v \quad (5.28)$$

The following types of inner products must be calculated to fill the above matrices:

- $\langle \vec{P}, \vec{f}_m^S \rangle_s$ where $\vec{P} = \{\vec{A}_n^R, \vec{A}_n, \vec{E}^i\}$,
- $\langle \nabla \Upsilon_n, \vec{f}_m^S \rangle_s$ where $\Upsilon = \{\Phi_n^R, \Phi_n\}$,
- $\langle \vec{R}_n \cdot \vec{f}_n^S, \vec{f}_m^S \rangle_s$,
- $\langle \vec{Q}, \vec{f}_m^V \rangle_v$ where $\vec{Q} = \{\vec{A}_n^R, \vec{A}_n, \vec{F}_n, \vec{E}^i, \vec{H}^i\}$,
- $\langle \nabla \Omega_n, \vec{f}_m^V \rangle_v$ where $\Omega = \{\Phi_n^R, \Phi_n, \Psi_n\}$,
- $\langle \vec{f}_n^V, \vec{f}_m^V \rangle_v$,
- $\langle \nabla \times \vec{F}_n, \vec{f}_m^S \rangle_s$,
- $\langle \nabla \times \vec{S}_n, \vec{f}_m^V \rangle_v$ where $\vec{S} = \{\vec{A}_n^R, \vec{A}_n, \vec{F}_n\}$.

Each of these inner products will be discussed below.

- $\langle \vec{P}, \vec{f}_m^S \rangle_s$ The inner product is defined over two adjacent triangles:

$$\langle \vec{P}, \vec{f}_m^S \rangle_s = \frac{\ell_m}{2} \left[\frac{1}{A_m^+} \int_{T_m^+} \vec{P} \cdot \vec{\rho}_m^+ ds + \frac{1}{A_m^-} \int_{T_m^-} \vec{P} \cdot \vec{\rho}_m^- ds \right]. \quad (5.29)$$

Assuming $\vec{P}$ is slowly varying over the cells, the integral may be approximated by the value of the integrand at the centroids of the triangles (point matching). Then, from [30, p. 43],

$$\langle \vec{P}, \vec{f}_m^S \rangle_s \approx \frac{\ell_m}{2} \left[\vec{P}(\vec{r}_m^{c+}) \cdot \vec{\rho}_m^{c+} + \vec{P}(\vec{r}_m^{c-}) \cdot \vec{\rho}_m^{c-} \right]. \quad (5.30)$$

- $\langle \nabla \Upsilon_n, \vec{f}_m^S \rangle_s$ The inner product may be written as

$$\langle \nabla \Upsilon_n, \vec{f}_m^S \rangle_s = \int_s \nabla \Upsilon_n \cdot \vec{f}_m^S ds \quad (5.31)$$

$$= \oint_c \Upsilon_n \vec{f}_m^S \cdot d\vec{\ell} - \int_s \Upsilon_n \nabla \cdot \vec{f}_m^S ds, \quad (5.32)$$

where the vector identity

$$\nabla \cdot (\Upsilon \vec{f}) = \vec{f} \cdot (\nabla \Upsilon) + \Upsilon (\nabla \cdot \vec{f}) \quad (5.33)$$

and the divergence theorem have been used. Because of the nature of the basis function along the edge of the cell, the closed line integral is zero [30, p.42]. Then, approximating the integral by the value of the integrand at the centroid of the region,

$$\langle \nabla \Upsilon_n, \vec{f}_m^S \rangle_s \approx \ell_m \left[\Upsilon(\vec{r}_m^{c-}) - \Upsilon(\vec{r}_m^{c+}) \right]. \quad (5.34)$$

- $\langle \overline{\vec{R}}_n \cdot \vec{f}_n^S, \vec{f}_m^S \rangle_s$ The inner product over two adjacent triangles may be written as

$$\langle \overline{\vec{R}}_n \cdot \vec{f}_n^S, \vec{f}_m^S \rangle_s = \int_{T_m^+} \overline{\vec{R}}_n \cdot \vec{f}_n^S \cdot \vec{f}_m^S ds + \int_{T_m^-} \overline{\vec{R}}_n \cdot \vec{f}_n^S \cdot \vec{f}_m^S ds. \quad (5.35)$$

If m and n do not lie on the same observation triangle T^p , then the contribution to the above inner product is zero. Otherwise, for the same observation and source

triangle $T^p = T^q$, the integration over one triangle may be expressed analytically as

$$\begin{aligned} \int_{T^p} \bar{\bar{R}}_n \cdot \bar{f}_n^S \cdot \bar{f}_m^S ds &= \pm \frac{\ell_m \ell_n}{4A^p} \left[\frac{3}{4} (\bar{r}^{cp} \cdot \bar{\bar{R}}_n \cdot \bar{r}^{cp}) + (\bar{r}_m \cdot \bar{\bar{R}}_n \cdot \bar{r}_n) \right. \\ &+ \frac{1}{12} [(\bar{r}_1 \cdot \bar{\bar{R}}_n \cdot \bar{r}_1) + (\bar{r}_2 \cdot \bar{\bar{R}}_n \cdot \bar{r}_2) + (\bar{r}_3 \cdot \bar{\bar{R}}_n \cdot \bar{r}_3)] \\ &\left. - (\bar{r}_m \cdot \bar{\bar{R}}_n \cdot \bar{r}^{cp}) + (\bar{r}^{cp} \cdot \bar{\bar{R}}_n \cdot \bar{r}_n) \right], \end{aligned} \quad (5.36)$$

where $m, n = 1, 2, 3$ in the “local” coordinate system described in Figure 18. The $+/-$ sign is chosen according to the current reference directions of $\bar{f}_m^S$ and $\bar{f}_n^S$ in T^p : “the positive sign is selected if the reference directions of edges m and n are both directed either into or out of T^p ; the negative sign is selected otherwise” [30].

• $\langle \bar{Q}, \bar{f}_m^V \rangle_v$ The inner product over two adjacent tetrahedrons may be written as

$$\langle \bar{Q}, \bar{f}_m^V \rangle_v = \frac{a_m}{3} \left[\frac{1}{V_m^+} \int_{T_m^+} \bar{Q} \cdot \bar{\wp}_m^+ dv + \frac{1}{V_m^-} \int_{T_m^-} \bar{Q} \cdot \bar{\wp}_m^- dv \right], \quad (5.37)$$

which again may be approximated at the centroids of the cells [26, p. 84] by

$$\langle \bar{Q}, \bar{f}_m^V \rangle_v \approx \frac{a_m}{3} \left[\bar{Q}(\bar{r}_m^{c+}) \cdot \bar{\wp}_m^{c+} + \bar{Q}(\bar{r}_m^{c-}) \cdot \bar{\wp}_m^{c-} \right]. \quad (5.38)$$

• $\langle \nabla \Omega_n, \bar{f}_m^V \rangle_v$ The inner product may be expressed as

$$\langle \nabla \Omega_n, \bar{f}_m^V \rangle_v = \int_v \nabla \Omega_n \cdot \bar{f}_m^V dv \quad (5.39)$$

$$= \oint_s \Omega_n \bar{f}_m^V \cdot \hat{n} ds - \int_v \Omega_n \nabla \cdot \bar{f}_m^V dv, \quad (5.40)$$

where again the vector identity (5.33) and the divergence theorem have been used.

Then, from [26, p. 84]

$$\langle \nabla \Omega_n, \bar{f}_m^V \rangle_v \approx \begin{cases} a_m [\Omega(\bar{r}^{c-}) - \Omega(\bar{r}^{c+})] & T_m^+, T_m^- \in v, \\ \pm a_m [\Omega(\bar{r}^{c\pm} \pm \frac{1}{3} \bar{\wp}^{c\pm}) - \Omega(\bar{r}^{c\pm})] & T_m^\mp \notin v. \end{cases} \quad (5.41)$$

Figure 18: Local indexing for source triangle T^q and observation triangle T^p [30].

- $\langle \vec{f}_n^V, \vec{f}_m^V \rangle_v$ For the tetrahedral pair $\mathcal{T}_m^\pm$,

$$\langle \vec{f}_n^V, \vec{f}_m^V \rangle_v = \int_{\mathcal{T}_m^+} \vec{f}_n^V \cdot \vec{f}_m^V dv + \int_{\mathcal{T}_m^-} \vec{f}_n^V \cdot \vec{f}_m^V dv. \quad (5.42)$$

From [26, p. 84], for m and n in the same tetrahedron $\mathcal{T}^p$,

$$\begin{aligned} \int_{\mathcal{T}^p} \vec{f}_n^V \cdot \vec{f}_m^V dv &= \pm \frac{a_m a_n}{9V^p} \left[\frac{8}{5} |\vec{r}^{cp}|^2 - (\vec{r}_m + \vec{r}_n) \cdot \vec{r}^{cp} + (\vec{r}_m \cdot \vec{r}_n) \right. \\ &\quad - \frac{1}{10} (\vec{r}_1 \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_3 + \vec{r}_1 \cdot \vec{r}_4 + \vec{r}_2 \cdot \vec{r}_3 \\ &\quad \left. + \vec{r}_2 \cdot \vec{r}_4 + \vec{r}_3 \cdot \vec{r}_4) \right], \end{aligned} \quad (5.43)$$

where the same local convention described for the triangular pair above has been used here for the tetrahedral pair.

- $\langle \nabla \times \vec{F}_n, \vec{f}_m^S \rangle_s$ Following the procedure above,

$$\begin{aligned} \langle \nabla \times \vec{F}_n, \vec{f}_m^S \rangle_s &= \frac{\ell_m}{2} \left[\frac{1}{A_m^+} \int_{\mathcal{T}_m^+} (\nabla \times \vec{F}_n)^+ \cdot \vec{\rho}_m^+ ds \right. \\ &\quad \left. + \frac{1}{A_m^-} \int_{\mathcal{T}_m^-} (\nabla \times \vec{F}_n)^- \cdot \vec{\rho}_m^- ds \right], \end{aligned} \quad (5.44)$$

which can again be approximated at the centroids of the cells by

$$\approx \frac{\ell_m}{2} \left[\nabla \times \vec{F}_n(\vec{r}_m^{c+}) \cdot \vec{\rho}_m^{c+} + \nabla \times \vec{F}_n(\vec{r}_m^{c-}) \cdot \vec{\rho}_m^{c-} \right]. \quad (5.45)$$

Here, the curl term is given by the integral

$$\nabla \times \vec{F}_n = \frac{\epsilon_o}{4\pi} \int \vec{M}(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}', \quad (5.46)$$

and

$$\nabla' G(\vec{r}, \vec{r}') = (\vec{r} - \vec{r}') (1 + jkR) \frac{e^{-jkR}}{R^3}. \quad (5.47)$$

• $\langle \nabla \times \vec{S}_n, \vec{f}_m^V \rangle_v$ Again, following the procedure outlined above

$$\begin{aligned} \langle \nabla \times \vec{S}_n, \vec{f}_m^V \rangle_v &= \frac{a_m}{2} \left[\frac{1}{V_m^+} \int_{T_m^+} (\nabla \times \vec{S}_n)^+ \cdot \vec{\wp}_m^+ dv \right. \\ &\quad \left. + \frac{1}{V_m^-} \int_{T_m^-} (\nabla \times \vec{S}_n)^- \cdot \vec{\wp}_m^- dv \right], \end{aligned} \quad (5.48)$$

which can again be approximated at the centroids of the cells by

$$\begin{aligned} \langle \nabla \times \vec{S}_n, \vec{f}_m^V \rangle_v &\approx \frac{a_m}{2} \left[\nabla \times \vec{S}_n(\vec{r}_m^{c+}) \cdot \vec{\wp}_m^{c+} \right. \\ &\quad \left. + \nabla \times \vec{S}_n(\vec{r}_m^{c-}) \cdot \vec{\wp}_m^{c-} \right], \end{aligned} \quad (5.49)$$

where specifically for $\nabla \times \vec{S}$ and $\vec{S} = \{\vec{A}_n^R, \vec{A}_n, \vec{F}_n\}$ the following integrals are expressed:

$$\nabla \times \vec{A}_n^R = \frac{\mu_o}{4\pi} \int \vec{J}^R(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}' \quad (5.50)$$

$$\nabla \times \vec{A}_n = \frac{\mu_o}{4\pi} \int \vec{J}^V(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}' \quad (5.51)$$

$$\nabla \times \vec{F}_n = \frac{\epsilon_o}{4\pi} \int \vec{M}^V(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}' \quad (5.52)$$

The calculation of these integrals as well as the other potential integrals found in the inner products are left to Appendices B, C, and D.

A computer code based on this MoM formulation has been written to analyze the electromagnetic scattering from non-homogeneous penetrable materials. Computations performed with the Material Fan Blade (MFB) FORTRAN code [31] will be presented for various geometries in order to assess the accuracy, versatility, and usefulness of the formulation. Specifically, we are interested in three geometries: the material coated sphere, the grounded slab, and the material fan blade. These results will be presented in the Chapters VII, VIII, and IX. First, however, certain mesh considerations involving the number of unknowns, sampling, and discretization must be discussed. These will be presented in the next chapter.

CHAPTER VI

MESH CONSIDERATIONS

Although meshing the geometry is an important part of the solution to scattering from penetrable material bodies, we will spend very little time here discussing the actual mesh dynamics. Several commercial mesh generation packages with varying degrees of effectiveness and cost are available. We have used the package COSMOS GEOSTAR [32] to generate most of the meshed geometries in this dissertation. The reader is referred to the GEOSTAR user manual for specifics concerning this piece of software. Rather than focusing on mesh mechanics, we will briefly discuss some mesh considerations with respect to the MoM solution discussed above. Other issues involving mesh connectivity (i.e., how the mesh is defined in terms of nodes, edges, faces, and cells) are left to Appendix E.

6.1 Number of Unknowns

To determine the number of MoM unknowns associated with a meshed geometry, each “piece” of the geometry must be considered separately. For the resistive sheet comprised of N_{rf} triangular facets, one unknown is associated with every non-boundary edge on the sheet. For a sheet with N_{be} boundary edges, the total number of unknowns M associated with current on the sheet is given by

$$M = \frac{3 * N_{rf} - N_{be}}{2}. \quad (6.1)$$

For the material volume meshed with N_c tetrahedral cells, one unknown is associated with every face, including the boundary faces N_{bf} on the exterior surface enclosing the volume. In addition, two unknowns are associated with all tetrahedral faces adjacent to the resistive sheet. In the event that the resistive sheet is only partially embedded, we define N_{pmf} to be the number of tetrahedral faces associated with “positive” monopole faces, and N_{nmf} to be the number of tetrahedral faces associated with “negative” monopole faces. The number of unknowns N associated with *either* the dielectric portion or the ferrite portion of the material is then given by

$$N = \frac{4 * N_c + N_{pmf} + N_{nmf} + N_{bf}}{2}. \quad (6.2)$$

In the event that the resistive sheet and the exterior boundary surface overlap (e.g., a clad material), the resistive faces are not included in the count for N_{bf} . The total number of unknowns P for the scattering problem is then simply the sum of the above portions in accordance with the material characteristics.

For scatterers comprised of many triangular and/or tetrahedral cells, the number of unknowns may be approximated:

$$M = \frac{3}{2} * N_{rf} \quad \text{for large } N_{rf} \quad (6.3)$$

$$N = 2 * N_c \quad \text{for large } N_c \quad (6.4)$$

6.2 Sampling

In the MoM solution, each unknown represents a surface or volume current over some subsectional region of the scatterer. We shall define the sampling to be the spacing between unknowns throughout the scatterer. To calculate the sampling on meshed surfaces or sheets, the surface area S_R is first divided by the number

of (resistive) facets N_{rf} to obtain the average triangular facet area a . For regular (equilateral) triangles of edge length ℓ_s , this area is given by [33]

$$a = \frac{\sqrt{3}}{4}\ell_s^2. \quad (6.5)$$

Define ℓ_s to be the *average edge length* for the meshed surface. The average height h_s of each triangle is quickly found to be

$$h_s = \frac{\sqrt{3}}{2}\ell_s. \quad (6.6)$$

The unknown surface currents have been expanded in terms of triangular dipole pairs: i.e., each basis function extends over two adjacent triangles. Therefore, in order to compare with the sampling associated with other methods (e.g., brick modeling with pulse bases), a natural sampling parameter is the spatial extent of the basis. Thus, define the *sampling parameter* p_s on the surface to be

$$p_s = 2h_s \quad (6.7)$$

$$= \sqrt{3}\ell_s. \quad (6.8)$$

Similarly, a sampling parameter p_v for the meshed volume may be calculated. First, obtain an average tetrahedral cell volume v by dividing the total meshed volume V by the number of tetrahedral cells N_c . For regular tetrahedrons of edge length ℓ_v , the volume is given by [33]

$$v = \frac{\sqrt{2}}{12}\ell_v^3. \quad (6.9)$$

The average height associated with each tetrahedral cell is then given by

$$h_v = \frac{\sqrt{6}}{3}\ell_v. \quad (6.10)$$

Noting again that the tetrahedral basis functions are defined over two adjacent cells, define the sampling parameter p_v to be twice the average tetrahedral height:

$$p_v = 2h_v \quad (6.11)$$

$$= \frac{2\sqrt{6}}{3}\ell_v. \quad (6.12)$$

The above analysis offers only a rough guideline to characterize sampling: we have assumed equal size equilateral triangles and equal size regular tetrahedrons for the mesh. A number of other factors including the relative sizes of the elements and their aspect ratios must also be considered when meshing a body. Schaubert et al. [26] have commented that they obtained the best results by keeping the triangular faces close to equilateral. Also, care was taken to insure that the relative sizes of the cells remained approximately the same. We have tried to follow these guidelines when meshing the geometries presented in this dissertation. However, the use of automatic mesh generators does not usually allow the luxury of maintaining such strict guidelines. One must be expected to compromise in order to avoid the tedium of generating complex three-dimensional meshes “by hand.”

6.3 Triangles versus Quadrilaterals

In this section, a simple comparison is made between triangular and quadrilateral surface patching. Computations performed with the MFB code [31] will now be compared to computations performed with ESP4 [34]. As discussed previously, MFB uses linear vector basis functions (constant in the transverse direction) over triangular cells with essentially point testing. ESP4 uses piecewise sinusoidal basis functions (constant in the transverse direction) over quadrilateral cells with Galerkin testing (test function same as basis function).

Consider the perfectly conducting regular pentagon with $1\lambda_o$ edges illustrated in Figure 19. This geometry was chosen so as not to favor either triangular or quadrilateral patches. For a plane wave incident broadside to the plate, the backscattered cross-polarized fields should be ideally zero for all polarizations. Therefore, as one

Figure 19: Illustration of $1\lambda_o$ edge regular pentagon.

measure of convergence, cross-polarized field levels will be compared using MFB and ESP4 as a function of number of unknowns.

Figure 20 illustrates the backscattered echo area (σ_{xy}) for an incident x -polarized plane wave with y -polarized receive at broadside to the PEC pentagonal plate. Figure 21 illustrates the backscattered echo area (σ_{yx}) for an incident y -polarized plane wave with x -polarized receive. The co-polarized backscatter levels at normal incidence for this geometry are $\sigma_{xx} = 13.7$ dB for an x -polarized field and $\sigma_{yy} = 13.9$ dB for a y -polarized field.

Note that neither method gives monotonic convergence (lower cross-pol level) as the number of unknowns is increased. This is to be expected since in general the Method of Moments does not make any guarantees for convergence (monotonic convergence can be guaranteed using the Method of Least Squares). Both MFB and ESP4 provide comparable results as the number of unknowns are increased.

Figure 22 presents computation times for both the MFB and ESP4 calculations on a Tektronix XD-88 workstation operating at 17 MIPS for the above cases. As illustrated, MFB requires considerably less computation time than ESP4 for the same number of modes. This is not surprising given that ESP4 uses Galerkin testing and MFB uses point testing. For the pentagonal plate at broadside, the convergence is comparable. No generalizations for other geometries or other aspect angles should be made from this limited comparison.

Figure 20: Cross-polarized backscatter echo area σ_{xy} for $1 \lambda_o$ PEC pentagonal plate.

Figure 21: Cross-polarized backscatter echo area σ_{yx} for $1 \lambda_o$ PEC pentagonal plate.

Figure 22: Computation time for cross-polarized backscatter echo area for $1 \lambda_o$ PEC pentagonal plate.

CHAPTER VII

MATERIAL COATED RESISTIVE SPHERE

One of the few canonical problems available to test the three-dimensional scattering from a material body in the presence of a resistive sheet is the coated sphere. The eigenfunction solution for scattering from a material coated resistive sphere is presented in Appendix F. This solution is very general in that it reduces to the eigenfunction solutions for scattering from a perfectly conducting sphere, a resistive spherical shell (or the thin dielectric sheet approximation), a solid dielectric/ferrite sphere, two concentric material spheres, and the material coated perfectly conducting sphere. This solution will be used to test the accuracy of the MoM code described above. The geometry for the problem is illustrated in Figure 23.

We have chosen to study spheres of radius 1 m at a frequency of 15 MHz; i.e., $a = 0.10 \lambda_o$, where λ_o is the free space wavelength. It necessary to study this relatively small scattering size in order to maintain a sufficient sampling criteria as well as to keep the number of unknowns manageable. As shall be seen, the number of unknowns increases rapidly for material scatterers even at these small electrical sizes.

Since the scattering for targets is volume dominated at low frequencies, we have chosen to model the spheres with equal volume polyhedrons meshed with triangular and tetrahedral cells. An 88 facet triangular mesh is illustrated in Figure 24. Assuming regular (equilateral) triangular facets, an average edge length of approximately $\ell_s = 0.057 \lambda_o$ is obtained for this mesh. If we define a surface sampling parameter

Figure 23: Material coated sphere.

Figure 24: Typical mesh for material coated sphere.

p_s to be the spatial extent of the basis function across two adjacent equilateral triangular facets (see Chapter V), then this parameter may be readily calculated for the 88 facet triangular mesh to be $p_s = 0.100 \lambda_o$.

7.1 Conducting/Resistive Sphere

A bistatic calculation using this mesh is compared to the eigenfunction solution for a PEC sphere of radius $a = 0.10 \lambda_o$ for both $\hat{\theta}$ ($\phi = 0^\circ$ cut) and $\hat{\phi}$ ($\phi = 90^\circ$

cut) polarizations in Figures 25 and 26, respectively. (Note the difference in scales between the two polarizations.) The agreement is good in both cases with less than 0.2 dB difference between the MoM calculation and the eigenfunction solution. Similar results using the same mesh are shown in Figures 27 and 28 for a resistive spherical shell of $200 \Omega/\text{sq}$. In both cases, the MoM solutions required 132 unknowns.

A swept frequency calculation is shown in Figure 29 for a thin dielectric spherical shell to illustrate the effects of sampling. Results are shown for a sphere using the thin dielectric sheet approximation with $Z_S = -j942.5$ for various frequencies represented by varying the wavenumber-radius product ka . The echo area is normalized to cross section area πa^2 . This case was chosen to compare to the previously published eigenfunction results of Andreassen [35].

For the swept frequency MoM calculation, two meshes were chosen: a 144 triangular facet (216 unknowns) sparse mesh and a 372 triangular facet (558 unknowns) dense mesh. As demonstrated in Figure 29, the MoM calculation for the sparse mesh agrees very closely with the eigenfunction result up to about $ka = 1.8$. At this electrical size, the sampling is $p_s = 0.222 \lambda_o$ or $\ell_s = 0.129 \lambda_o$. Reasonable results for backscatter (within 1 dB) are obtained for this mesh past $ka = 3.5$ which corresponds to $\ell_s = 0.250 \lambda_o$ or a sampling parameter of $p_s = 0.433 \lambda_o$. (A sampling parameter of $p_s = 0.5 \lambda_o$ corresponds to $ka = 4$ for the sparse mesh.) By increasing the number of triangular facets to 372, very close agreement between the MoM calculation and the eigenfunction solution may be obtained well past $ka = 3$. Note that $\hat{\theta}$ and $\hat{\phi}$ polarizations calculated by the MoM for both the sparse and dense meshes were essentially identical.

Figure 25: Bistatic echo area of PEC sphere with radius $a = 0.10 \lambda_0$ for $\hat{\theta}$ polarization.

Figure 26: Bistatic echo area of PEC sphere with radius $a = 0.10 \lambda_0$ for $\hat{\phi}$ polarization.

Figure 27: Bistatic echo area of resistive spherical shell $R = 200 \, \Omega/\text{sq.}$ with radius $a = 0.10 \, \lambda_o$ for $\hat{\theta}$ polarization.

Figure 28: Bistatic echo area of resistive spherical shell $R = 200 \Omega/\text{sq.}$ with radius $a = 0.10 \lambda_o$ for $\hat{\phi}$ polarization.

Figure 29: Swept frequency backscatter echo area of thin dielectric spherical shell $Z_S = -j942.5 \, \Omega/\text{sq.}$

7.2 Solid Material Sphere

Results for various solid spheres are presented next. A mesh of 243 tetrahedral cells was used to obtain $\ell_v = 0.053 \lambda_o$ or $p_v = 0.086 \lambda_o$. These numbers must be multiplied by the square root of the relative permittivities and permeabilities in order to obtain the sampling in the material.

Figures 30 and 31 give the results for bistatic scattering from a solid dielectric sphere with relative permittivity $\epsilon_r = 2.50 - j1.25$ and radius $a = 0.10 \lambda_o$ for $\hat{\theta}$ and $\hat{\phi}$ polarizations, respectively. The 243 tetrahedral cell model results in a sampling of $p_v = 0.144 \lambda$ with an average edge length of $\ell_v = 0.088 \lambda$ in the material. The results are very good, with less than 0.10 dB difference between the MoM solution and the exact solution across the pattern. Similar results are shown in Figures 32 and 33 for a solid ferrite sphere with relative permeability $\mu_r = 1.60 - j0.80$ and radius $a = 0.10 \lambda_o$, and a sampling of $p_v = 0.115 \lambda$ and an average edge length $\ell_v = 0.070$. In both cases, the MoM solutions required 558 unknowns.

Figures 34 and 35 give the results for bistatic scattering from a solid material sphere with relative permittivity $\epsilon_r = 2.50 - j1.25$, relative permeability $\mu_r = 1.60 - j0.80$, and radius $a = 0.10 \lambda_o$ for $\hat{\theta}$ and $\hat{\phi}$ polarizations, respectively. Using the 243 tetrahedral cell model discussed above results in a sampling in the material of $p_v = 0.192 \lambda$ and an average edge length $\ell_v = 0.118 \lambda$. Again, the results are very good with less than 0.10 dB difference between the MoM and the exact solutions. For this case with both dielectric and ferrite material, 1116 unknowns were required.

A swept frequency calculation for the solid dielectric sphere with relative permittivity $\epsilon_r = 2.50 - j1.25$ is illustrated in Figure 36. A mesh of 243 tetrahedral cells was used. Once again, the echo area is normalized to cross section area πa^2 and k is in terms of free-space wavelengths. Both $\hat{\theta}$ and $\hat{\phi}$ polarization calculations

Figure 30: Bistatic echo area of solid dielectric sphere $\epsilon_r = 2.50 - j1.25$ with radius $a = 0.10 \lambda_0$ for $\hat{\theta}$ polarization.

Figure 31: Bistatic echo area of solid dielectric sphere $\epsilon_r = 2.50 - j1.25$ with radius $a = 0.10 \lambda_0$ for $\hat{\phi}$ polarization.

Figure 32: Bistatic echo area of solid ferrite sphere $\epsilon_r = 1.60 - j0.80$ with radius $a = 0.10 \lambda_0$ for $\hat{\theta}$ polarization.

Figure 33: Bistatic echo area of solid ferrite sphere $\mu_r = 1.60 - j0.80$ with radius $a = 0.10 \lambda_o$ for $\hat{\phi}$ polarization.

Figure 34: Bistatic echo area of solid material sphere $\epsilon_r = 2.50 - j1.25$ $\mu_r = 1.60 - j0.80$ with radius $a = 0.10 \lambda_o$ for $\hat{\theta}$ polarization.

Figure 35: Bistatic echo area of solid material sphere $\epsilon_r = 2.50 - j1.25$ $\mu_r = 1.60 - j0.80$ with radius $a = 0.10 \lambda_o$ for $\hat{\phi}$ polarization.

Figure 36: Swept frequency backscatter echo area of solid dielectric sphere $\epsilon_r = 2.50 - j1.25$.

are shown to demonstrate the depolarization effect of insufficient sampling in the penetrable material.

The MoM calculation agrees precisely with the eigenfunction result up to about $ka = 0.8$. At this electrical size, the sampling is $p_v = 0.183 \lambda$ or $\ell_v = 0.112 \lambda$ in the material. Schaubert et al. [26] reported “useful results for scattered field quantities often can be obtained when the edges of the triangular faces are longer than 0.25 wavelengths in the dielectric medium.” Here, an average edge length of $\ell_v = 0.25 \lambda$ ($p_v = 0.408 \lambda$) corresponds to $ka = 1.7$. These results are consistent with their findings in that the backscattered results are within approximately 1 dB of the exact solution.

7.3 Material Coated Sphere

Several coated sphere scatterers that involve conductors in the immediate presence of dielectric/ferrite materials are considered next. Two mesh geometries are used: a sparse mesh with a 32 triangular facet conductor embedded in a 175 tetrahedral cell spherical volume, and a dense mesh with a 72 triangular facet conductor embedded in a 340 tetrahedral cell spherical volume. The sphere consists of a $a = 0.05 \lambda_o$ radius conductor with a $t = 0.05 \lambda_o$ thick material coating, resulting in two concentric spheres with an exterior radius of $b = 0.10 \lambda_o$. In each mesh configuration, one layer of tetrahedrons was used to model the material thickness.

Bistatic pattern results are shown in Figures 37 and 38 for a dielectric coated PEC with relative permittivity $\epsilon_r = 2.00 - j0.200$ for $\hat{\theta}$ and $\hat{\phi}$ polarizations, respectively. For the sparse mesh, the sampling in the dielectric is $p_v = 0.130 \lambda$ with $\ell_v = 0.080 \lambda$, and for the dense mesh the sampling is $p_v = 0.104 \lambda$ with $\ell_v = 0.064 \lambda$.

Figure 37: Bistatic echo area of dielectric coated PEC sphere $\epsilon_r = 2.00 - j0.200$ with radius $a = 0.05 \lambda_o$ and coating thickness $t = 0.05 \lambda_o$ for $\hat{\theta}$ polarization.

Figure 38: Bistatic echo area of dielectric coated PEC sphere $\epsilon_r = 2.00 - j0.200$ with radius $a = 0.05 \lambda_0$ and coating thickness $t = 0.05 \lambda_0$ for $\hat{\phi}$ polarization.

The results are good: approximately 0.5 dB difference between MoM and exact solutions for the dense mesh, and 0.8 dB difference for the sparse mesh. The dense mesh required 908 unknowns while the sparse mesh required only 458 unknowns.

We note that the results for the dielectric coated PEC are not quite as good as for the solid dielectric case or the PEC alone. The presence of a conductor (perfect or non-perfect) results in a discontinuity of the normal flux densities in the material. This discontinuity is modeled with the addition of the monopole basis functions introduced in Chapter IV which are treated as additional unknowns. Still, the presence of the conductor results in increased field variation within the material. Thus, it is not surprising to learn that increased sampling is needed in order to obtain good results when a conductor is added to the material.

We also note that while good results were obtained for material thicknesses of $0.05 \lambda_o$ and $0.0375 \lambda_o$ for a PEC sphere of radius $a = 0.05 \lambda_o$, problems were encountered for smaller material thicknesses. The MoM solution became very sensitive to the mesh configuration. We feel that the above meshes provided insufficient sampling to account for the increased field variation within the thinner material layer. Decreasing the mesh size to increase sampling and adding an additional layer of tetrahedrons resulted in poor mesh geometries (i.e., elongated tetrahedral cells, relatively small volumes) that did not improve the calculated results.

Bistatic results for a ferrite coated sphere using the same two meshes are presented for $\hat{\theta}$ and $\hat{\phi}$ polarizations in Figures 39 and 40. For a relative permeability of $\mu_r = 1.50 - j0.15$, a $p_v = 0.113 \lambda$ sampling with $\ell_v = 0.069 \lambda$ was obtained for the sparse mesh and a $p_v = 0.090 \lambda$ sampling with $\ell_v = 0.055 \lambda$ was obtained for the dense mesh. Again, the results are good with a 0.5 dB difference between MoM and exact solutions for the dense mesh and a 1.0 dB difference for the sparse mesh. In both cases, the difference is largest in the forward scattering region.

Figure 40: Bistatic echo area of ferrite coated PEC sphere $\epsilon_r = 1.50 - j0.15$ with radius $a = 0.05 \lambda_o$ and coating thickness $t = 0.05 \lambda_o$ for $\hat{\phi}$ polarization.

The next set of coated spheres involves both dielectric and ferrite material with $\epsilon_r = 2.00 - j0.200$ and $\mu_r = 1.50 - j0.15$. For these cases, the sparse mesh has a sampling of $p_v = 0.160 \lambda$ sampling with $\ell_v = 0.098 \lambda$ and requires 868 unknowns, while the dense mesh has a sampling of $p_v = 0.128 \lambda$ sampling with $\ell_v = 0.078 \lambda$ and requires 1700 unknowns. Bistatic results are presented in Figures 41 and 42. Again, the results are similar to the above cases: approximately 0.5 dB difference between MoM and exact solutions for the dense mesh and 1.0 dB difference for the sparse mesh.

The final sets of sphere data involve dielectric coatings over resistive spherical shells. The previously discussed geometry is again used with the dense mesh of 72 triangular facets on the resistive sheet and 340 tetrahedral volume cells in the dielectric. Figures 43 and 44 present the $\hat{\theta}$ and $\hat{\phi}$ polarization bistatic results for a resistive sphere of $R = 200 \Omega/\text{sq.}$ with a dielectric coating of permittivity $\epsilon_r = 2.00 - j0.200$. The results are similar to the dielectric coated PEC sphere, as expected.

In order to demonstrate the limiting case where the resistance becomes very large, bistatic results for a thick dielectric spherical shell are presented. The geometry of the previous coated resistive sphere is used here with relative permittivity $\epsilon_r = 2.00 - j0.200$, except now the resistivity of the sheet is set to the order of $10^7 \Omega/\text{sq.}$ In effect, the resistive sheet disappears leaving a thick dielectric shell. The results, presented in Figures 45 and 46, are very good with only 0.3 dB difference across the patterns.

Figure 41: Bistatic echo area of material coated PEC sphere $\epsilon_r = 2.00 - j0.200$ $\mu_r = 1.50 - j0.15$ with radius $a = 0.05 \lambda_o$ and coating thickness $t = 0.05 \lambda_o$ for $\hat{\theta}$ polarization.

Figure 42: Bistatic echo area of material coated PEC sphere $\epsilon_r = 2.00 - j0.200$ $\mu_r = 1.50 - j0.15$ with radius $a = 0.05 \lambda_o$ and coating thickness $t = 0.05 \lambda_o$ for $\hat{\phi}$ polarization.

Figure 43: Bistatic echo area of dielectric coated resistive spherical shell $R = 200 \, \Omega/\text{sq.}$ $\epsilon_r = 2.00 - j0.200$ with radius $a = 0.05 \lambda_o$ and coating thickness $t = 0.05 \lambda_o$ for $\hat{\theta}$ polarization.

Figure 45: Bistatic echo area of thick dielectric spherical shell $\epsilon_r = 2.00 - j0.200$ with radius $a = 0.05 \lambda_0$ and coating thickness $t = 0.05 \lambda_0$ for $\hat{\theta}$ polarization.

MoM computation times are summarized in the following tables for all of the sphere cases generated in this chapter. Table 2 presents the matrix fill times, matrix solve times, and complete computation times for the bistatic cases (two co-polarized calculations, two cross-polarized calculations, 91 far-zone angles) on a Tektronix XD-88 workstation operating at 17 MIPS with 16 MRAM. Table 3 presents the same data for computation on a Cray Y-MP supercomputer.

Table 2: Computation times for material coated sphere geometries on XD-88.

triangular facets	tetrahedral cells	unknowns	time (seconds)		
			fill	solve	complete
88	0	132	3.8	1.3	9.9
144	0	216	10.0	6.2	24.0
0	243	558	74.5	145.6	240.5
0	243	1116	156.7	1273.9	1455.9
32	175	458	44.8	76.9	138.0
72	340	904	167.8	662.5	863.7
32	175	868	xxx	xxx	xxx
72	340	1700	xxx	xxx	xxx

Table 3: Computation times for material coated sphere geometries on Cray Y-MP.

triangular facets	tetrahedral cells	unknowns	time (seconds)		
			fill	solve	complete
88	0	132	0.56	0.10	5.5
144	0	216	1.5	0.34	9.1
0	243	558	10.6	4.1	17.5
0	243	1116	23.9	28.2	55.3
32	175	458	6.4	2.4	11.5
72	340	904	24.2	15.5	44.6
32	175	868	14.1	13.8	30.8
72	340	1700	53.5	93.5	152.5

CHAPTER VIII

GROUNDED SLAB

This chapter provides results of MoM calculations for the grounded slab. Three material configurations are considered: the PEC ground plate, the dielectric slab, and the grounded dielectric slab. Several mesh geometries are also studied. A typical mesh is shown in Figure 47 with its accompanying coordinate system. This particular meshed geometry is comprised of two layers of 389 tetrahedral cells over a 72 facet triangular plate. Two polarizations are defined: horizontal polarization designates the electric field polarized in the x - z plane, and vertical polarization designates the electric field polarized normal to the x - z plane. Backscatter calculations are performed by sweeping θ (measured from the z axis) in the x - z plane ($\phi = 0^\circ$ cut).

8.1 Ground Plate

First consider a 2 in. by 2 in. PEC plate at 2 GHz. Electrically, this corresponds roughly to a $1/3 \lambda_o$ by $1/3 \lambda_o$ plate. Three mesh geometries are considered: a 72 triangular facet (96 unknowns) mesh, a 98 triangular facet (133 unknowns) mesh, and a 128 triangular facet (176 unknowns) mesh. Sampling parameters and computation times for these mesh configurations are listed in Table 4. Computation time corresponds to total CPU time used to calculate 181 sample points (360° every 2°) for two co-polarized calculations and two cross-polarized calculations on a Tektronix XD-88 workstation operating at 17 MIPS with 16 MRAM. Backscatter calculations

Figure 47: Typical mesh and coordinate system for grounded slab.

Table 4: Mesh parameters and computation times for ground plate configurations.

triangular facets	unknowns	ℓ_s (λ_o)	p_s (λ_o)	time (s)
72	96	0.061	0.105	19.7
98	133	0.052	0.090	32.2
128	176	0.045	0.078	50.8

for horizontal and vertical co-polarized fields are shown in Figures 48 and 49, respectively (note the difference in scales). As illustrated in these figures, the MoM calculations have converged for the three mesh configurations.

8.2 Dielectric Slab

A lossless dielectric slab $\epsilon_r = 2.60$ of dimensions 2 in. by 2 in. by 3/4 in. is considered next. At 2 GHz, this corresponds to a slab of approximately $1/3 \lambda_o$ by $1/3 \lambda_o$ by $1/8 \lambda_o$. Three mesh geometries are again considered: a 389 tetrahedral cell (898 unknowns) mesh forming two layers of tetrahedrons, a 474 tetrahedral cell (1092 unknowns) mesh forming three layers of tetrahedrons, and a 571 tetrahedral cell (1324 unknowns) mesh forming three layers of tetrahedrons. Table 5 lists sampling parameters and computation times. For these cases, calculations were performed on a Cray Y-MP supercomputer. Backscatter calculations for horizontal and vertical polarizations are shown in Figures 50 and 51, respectively. As illustrated in these figures, the MoM calculations have converged as there is practically no difference in the calculations for the three cases.

solid line — 72 triangular facets, 96 unknowns

dashed line — 98 triangular facets, 133 unknowns

dot-dashed line — 128 triangular facets, 176 unknowns

Figure 48: Backscatter echo area of 2 in. by 2 in. PEC plate for horizontal polarization at 2 GHz.

Figure 49: Backscatter echo area of 2 in. by 2 in. PEC plate for vertical polarization at 2 GHz.

solid line — 389 tetrahedral cells, 898 unknowns

dashed line — 474 tetrahedral cells, 1092 unknowns

dot-dashed line — 571 tetrahedral cells, 1324 unknowns

Figure 50: Backscatter echo area of 2 in. by 2 in. by 3/4 in. dielectric slab $\epsilon_r = 2.60$ for horizontal polarization at 2 GHz.

solid line — 389 tetrahedral cells, 898 unknowns
dashed line — 474 tetrahedral cells, 1092 unknowns
dot-dashed line — 571 tetrahedral cells, 1324 unknowns

Figure 51: Backscatter echo area of 2 in. by 2 in. by 3/4 in. dielectric slab $\epsilon_r = 2.60$ for vertical polarization at 2 GHz.

Table 5: Mesh parameters and computation times for dielectric slab configurations.

tetrahedral cells	unknowns	ℓ_v (λ)	p_v (λ)	time (s)
389	898	0.110	0.180	69.9
474	1092	0.103	0.168	102.4
571	1324	0.097	0.158	153.1

8.3 Grounded Dielectric Slab

Finally, consider a lossless dielectric slab $\epsilon_r = 2.60$ of dimensions 2 in. by 2 in. by 3/4 in. backed by a PEC ground plate. Again, three mesh geometries are considered: a 72 triangular facet mesh for the PEC plate with 389 tetrahedral cells for the dielectric slab (994 unknowns), a 72 triangular facet mesh for the PEC plate with 474 tetrahedral cells for the dielectric slab (1188 unknowns), and a 98 triangular facet mesh for the PEC plate with 571 tetrahedral cells for the dielectric slab (1457 unknowns). Table 6 lists sampling parameters and Cray computation times for the various configurations.

Backscatter calculations for horizontal and vertical polarizations are shown in Figures 52 and 53, respectively. The results for the grounded slab are good, with only about 1 dB variation between the cases (away from the nulls), but not as good as either the PEC plate or the dielectric slab alone. This should be expected since the addition of the PEC to the slab causes increased field variation in the dielectric that must be modeled by the linear vector basis functions. We do note, however, that increasing the mesh density beyond the 571 element mesh by adding more tetrahedral

Table 6: Mesh parameters and computation times for grounded dielectric slab configurations.

triangular facets	tetrahedral cells	unknowns	ℓ_v (λ)	p_v (λ)	time (s)
72	389	994	0.110	0.180	83.2
72	474	1188	0.103	0.168	118.0
98	1571	1457	0.097	0.158	183.1

cells actually degraded the results. These very dense meshes resulted in elongated tetrahedrals and very small relative cell sizes. Most likely, the irregular cells resulted in more singular integrals that were improperly treated by the numerical integration schemes. For this reason, care must be taken when meshing geometries.

solid line — 72 triangular facets, 389 tetrahedral cells, 994 unknowns
dashed line — 72 triangular facets, 474 tetrahedral cells, 1188 unknowns
dot-dashed line — 98 triangular facets, 571 tetrahedral cells, 1457 unknowns

Figure 52: Backscatter echo area of 2 in. by 2 in. by 3/4 in. grounded dielectric slab $\epsilon_r = 2.60$ for horizontal polarization at 2 GHz.

solid line — 72 triangular facets, 389 tetrahedral cells, 994 unknowns
dashed line — 72 triangular facets, 474 tetrahedral cells, 1188 unknowns
dot-dashed line — 98 triangular facets, 571 tetrahedral cells, 1457 unknowns

Figure 53: Backscatter echo area of 2 in. by 2 in. by 3/4 in. grounded dielectric slab $\epsilon_r = 2.60$ for vertical polarization at 2 GHz.

CHAPTER IX

MATERIAL FAN BLADE

This final chapter of results is concerned with the material fan blade. The general blade geometry, designated here as the SR3 blade, is shown in Figure 54 and has typical dimensions of 23.5 cm by 10.0 cm by 0.5 cm. The MoM calculations include results for both conducting and dielectric blade configurations. Three types of blade models are represented: “flat” which is a planform of the blade’s contour, “thin” which is a three-dimensional representation of the blade with zero thickness, and “thick” which is a three-dimensional representation of the blade with thickness. The numerical descriptions of these blades were meshed from the parameterizations described in [7]. Figure 55 illustrates the coordinate system and a typical mesh for the blade geometry.

Backscatter calculations and measurements are compared at 2 GHz for the different blade models and material configurations. Two polarizations are defined: horizontal polarization designates the electric field polarized in the y - z plane, and vertical polarization designates the electric field polarized normal to the y - z plane. Backscatter calculations are performed by sweeping θ (measured from the z axis) in the y - z plane ($\phi = 90^\circ$ cut). Note that the blade cord has an approximate 10° pitch at the base which increases along the span.

Figure 54: Illustration of the SR3 blade — approximate height of 23.5 cm.

9.1 Conducting Blade

The first set of figures compares measurements to MoM calculations for three different conducting blade models. Figures 56 and 57 illustrate comparisons between calculated and measured flat conducting SR3 blades for vertical and horizontal polarizations, respectively. Note that the agreement is good.

Figures 58 and 59 illustrate comparisons between calculated and measured results for vertical and horizontal polarizations using the conducting thin SR3 blade. Figures 60 and 61 compare these results for the thick SR3 blade. Overall, the agreement is fairly good. The most significant discrepancy occurs for the thick conducting blade near grazing incidence with horizontal polarization. This disagreement is believed to be due to model alignment error and minor shape differences between the physical and numerical geometric models.

Figure 55: Typical mesh and coordinate system for the SR3 blade.

Figure 56: Backscatter echo area for flat PEC SR3 blade for vertical polarization at 2 GHz.

Figure 57: Backscatter echo area for flat PEC SR3 blade for horizontal polarization at 2 GHz.

Figure 58: Backscatter echo area for thin PEC SR3 blade for vertical polarization at 2 GHz.

Figure 59: Backscatter echo area for thin PEC SR3 blade for horizontal polarization at 2 GHz.

Figure 60: Backscatter echo area for thick PEC SR3 blade for vertical polarization at 2 GHz.

Figure 61: Backscatter echo area for thick PEC SR3 blade for horizontal polarization at 2 GHz.

9.2 Dielectric Blade

This next set of figures compares measurements to MoM calculations for two dielectric blade models. Figures 62 and 63 illustrate the comparisons between calculated and measured thin dielectric SR3 blades for vertical and horizontal polarizations, respectively. The thin dielectric model uses the thin sheet approximation (discussed in Appendix A) with $Z_S = -j3521 \text{ } \Omega/\text{sq}$. This resistance corresponds to a dielectric constant of $\epsilon_r = 4$ for a material thickness $t = 0.85 \text{ mm}$. Figures 64 and 65 illustrate these comparisons for the thick dielectric SR3 blade. Note that the thin blade calculation for horizontal polarization can be in significant error near the angles of 150° and 330° since only the tangential current component is used in the thin sheet approximation. A significant normal current component to the surface (which is being neglected) can exist near these angles. Also, as indicated in the conducting case, the disagreement between calculation and measurement for the thick blade horizontal polarization is thought to be due to model alignment error and minor shape differences between the physical and numerical geometric models.

Figure 62: Backscatter echo area for thin dielectric SR3 blade $\epsilon_r = 4$ ($Z_S = -j3521 \Omega/\text{sq.}$) for vertical polarization at 2 GHz.

Figure 63: Backscatter echo area for thin dielectric SR3 blade $\epsilon_r = 4$ ($Z_S = -j3521 \Omega/\text{sq.}$) for horizontal polarization at 2 GHz.

Figure 64: Backscatter echo area for thick dielectric SR3 blade $\epsilon_r = 4$ for vertical polarization at 2 GHz.

Figure 65: Backscatter echo area for thick dielectric SR3 blade $\epsilon_r = 4$ for horizontal polarization at 2 GHz.

Table 7: Computation times for material fan blade geometries.

geometry	triangular facets	tetrahedral cells	unknowns	time (s)
flat conducting	318	0	451	348
flat dielectric	318	0	450	348
thin conducting	318	0	450	348
thin dielectric	318	0	450	348
thick conducting	598	0	897	99
thick dielectric	0	1365	3248	1752

The CPU run times for the calculations performed in this chapter are contained in Table 7. Each CPU time entry involved 181 sample points (360° every 2°) for two co-polarized calculations and two cross-polarized calculations. The flat and thin model calculations were done on a Tektronix XD-88 workstation operating at 17 MIPS with 16 MRAM. The thick model calculations were done on a Cray Y-MP computer.

CHAPTER X

CONCLUSIONS

This dissertation has detailed the Method of Moments (MoM) formulation for the electromagnetic scattering by an inhomogeneous penetrable material with an embedded resistive sheet. This formulation is an extension of previously published work [25, 26] to include a conductive/resistive sheet in the immediate presence of a material scatterer. Volume and surface equivalence were used to replace the materials with unknown electric and magnetic currents radiating in free space, leading to a coupled field integral equation formulation. Triangular surface facets and tetrahedral volume cells were used to discretize the scatterer. Appropriate linear vector basis functions were then chosen to expand the unknown current over the resistive boundary surface and the unknown electric and magnetic flux densities in the material volume. A MoM formulation then transformed this set of equations into a matrix equation.

The triangular surface facets and tetrahedral volume cells allowed for greater flexibility in the geometric modeling of the material body. Also, the MoM formulation was very general in that it allowed for a variety of material configurations: open or closed conducting surfaces, open or closed resistive (thin dielectric) surfaces, solid dielectric/ferrite material volumes, embedded conducting/resistive surfaces in material volumes, and partially embedded conducting/resistive surfaces (cladded materials).

Several geometries and material configurations were considered in order to evaluate the method's performance. The MoM solution for the material coated resistive sphere was compared to a new eigenfunction solution for bistatic patterns and swept frequency backscatter calculations. Also, the grounded slab was considered and backscatter patterns for the material fan blade were compared to measurements. Good results were obtained in all of these cases.

No significant problems were encountered in the calculations of scattering from resistive surfaces or material volumes individually. However, some special considerations needed to be addressed when a conductive sheet was placed in the immediate presence of a thin material volume. For these cases, the scattering calculations became very sensitive to the mesh configuration. Without the conductor, no difficulties arose calculating the scattered fields from a given mesh configuration for a thin material. In the presence of the conductor, the same mesh provided very poor results. We feel that these meshes provided insufficient sampling to account for the increased field variation within the thin material in the presence of the conducting surface.

Decreasing the mesh size to increase sampling was not always satisfactory since poor mesh geometries sometimes resulted. Elongated tetrahedral cells or relatively small cell volumes can result in more singular integrals that are not treated properly by the numerical integration schemes. This can lead to ill-conditioned matrices and poor calculated results. Care should be exercised when discretizing the geometry to insure a "good" mesh.

The only other area of major concern with this formulation is of no surprise: penetrable volume scattering leads to large numbers of unknowns, large matrices, and long computation times. To alleviate problems encountered in storing and inverting large matrices, iterative techniques [36, 37, 38, 39] might be employed to solve the MoM matrix equation. The conjugate gradient method [36, 40, 41, 42, 43]

coupled with preconditioning techniques [37, 44, 45] has demonstrated some success in this area. Also, parametric elements long used in the FEM world have been recently applied to MoM formulations [46] to reduce the number of unknowns. The parametric elements are based on triangular and tetrahedral “parent” elements that are shaped using a quadratic or cubic Jacobian transformation. The new curved elements can potentially better model the shape of the scatterer. Also, by using the shape function as a basis function, a quadratic or cubic expansion may better represent the unknown current or field. Thus, by more accurately modeling the scatterer and the unknown currents, the parametric elements may reduce the number of unknowns. However, this reduction comes with an increased cost in matrix fill time.

APPENDIX A

PENETRABLE BOUNDARY CONDITIONS

The resistive boundary condition (RBC) is used to describe the electromagnetic scattering from a specific type of thin resistive material. One should note that the *penetrable* RBC should not be confused with the *impenetrable* Leontovich impedance boundary condition (IBC), perhaps more appropriately known as the Schukin-Leontovich boundary condition [47]. Also, the RBC should not be confused with the sheet impedance approximation for thin dielectrics [35, 48]. The RBC will be derived from a general expansion of the fields in a thin lossy dielectric slab as outlined by Rojas [49], and the normal and tangential boundary conditions will be given. Also, the sheet impedance approximation will be discussed and demonstrated to have the same functional form as the RBC. Thus, the RBC is shown to reduce to the sheet impedance approximation under certain conditions if the resistance is replaced by a complex impedance.

A.1 Resistive Boundary Condition

Begin with the thin material slab illustrated in Figure 66. The material is assumed to be lossy with conductivity σ and have relative permittivity ϵ_r and relative permeability μ_r . Weinstein (Vaynshteyn) [50] has approximated the boundary conditions for a very thin slab using a Taylor series expansion on the thickness t [51].

Figure 66: Thin material slab.

If the thickness t satisfies the condition

$$k_o \sqrt{\mu_r \epsilon_r} \frac{t}{2} \ll 1, \quad (\text{A.1})$$

(for the free space wave number $k_o = \omega \sqrt{\mu_o \epsilon_o}$ and time-harmonic $e^{j\omega t}$ assumed and suppressed) then the boundary conditions of $\mathcal{O}(t)$ on the sheet may be written [50, p.301]

$$\begin{aligned} E_z^+ - E_z^- &= -jk_o Z_o (\mu_r - 1) \frac{t}{2} [H_x^+ + H_x^-] \\ &+ \left(\frac{1}{\epsilon_r} - 1 \right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial z} + \frac{\partial E_y^-}{\partial z} \right), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} E_x^+ - E_x^- &= jk_o Z_o (\mu_r - 1) \frac{t}{2} [H_z^+ + H_z^-] \\ &+ \left(\frac{1}{\epsilon_r} - 1 \right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial x} + \frac{\partial E_y^-}{\partial x} \right), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned}
H_z^+ - H_z^- &= jk_o Y_o (\epsilon_r - 1) \frac{t}{2} [E_x^+ + E_x^-] \\
&+ \left(\frac{1}{\mu_r} - 1 \right) \frac{t}{2} \left(\frac{\partial H_y^+}{\partial z} + \frac{\partial H_y^-}{\partial z} \right), \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
H_x^+ - H_x^- &= -jk_o Y_o (\epsilon_r - 1) \frac{t}{2} [E_z^+ + E_z^-] \\
&+ \left(\frac{1}{\mu_r} - 1 \right) \frac{t}{2} \left(\frac{\partial H_y^+}{\partial x} + \frac{\partial H_y^-}{\partial x} \right), \tag{A.5}
\end{aligned}$$

where the characteristic impedance of free space is $Z_o = 1/Y_o = \sqrt{\mu_o/\epsilon_o}$. For lossy material, ϵ_r and μ_r may be complex:

$$\epsilon_r = \epsilon_r' - j\epsilon_r'' = \epsilon_r' - j\frac{\sigma}{\omega\epsilon_o}, \tag{A.6}$$

$$\mu_r = \mu_r' - j\mu_r''. \tag{A.7}$$

For the resistive boundary, the slab is assumed to be made of conducting material only; i.e.,

$$\mu_r = 1, \tag{A.8}$$

$$\epsilon_r = 1 - \frac{j\sigma}{\omega\epsilon_o}. \tag{A.9}$$

Then, (A.2) through (A.5) reduce to

$$E_z^+ - E_z^- = \left(\frac{1}{1 - \frac{j\sigma}{\omega\epsilon_o}} - 1 \right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial z} + \frac{\partial E_y^-}{\partial z} \right), \tag{A.10}$$

$$E_x^+ - E_x^- = \left(\frac{1}{1 - \frac{j\sigma}{\omega\epsilon_o}} - 1 \right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial x} + \frac{\partial E_y^-}{\partial x} \right), \tag{A.11}$$

$$H_z^+ - H_z^- = jk_o Y_o \left(-\frac{j\sigma}{\omega\epsilon_o} \right) \frac{t}{2} [E_x^+ + E_x^-] \tag{A.12}$$

$$H_x^+ - H_x^- = -jk_o Y_o \left(-\frac{j\sigma}{\omega\epsilon_o} \right) \frac{t}{2} [E_z^+ + E_z^-]. \tag{A.13}$$

First, note that

$$jk_o Y_o \left(-\frac{j\sigma}{\omega\epsilon_o} \right) = \sigma. \tag{A.14}$$

Also,

$$t \left(\frac{1}{1 - \frac{j\sigma}{\omega\epsilon_o}} - 1 \right) = t \left(\frac{\omega\epsilon_o}{\omega\epsilon_o - j\sigma} - 1 \right) \quad (\text{A.15})$$

$$= t \left(\frac{j\sigma}{\omega\epsilon_o - j\sigma} \right) \quad (\text{A.16})$$

$$= t \left(\frac{-1}{1 - \frac{\omega\epsilon_o}{j\sigma}} \right). \quad (\text{A.17})$$

Assuming that $\sigma \gg \omega\epsilon_o$, (A.17) may be written using a binomial expansion as

$$-t \left(1 - \frac{\omega\epsilon_o}{j\sigma} \right)^{-1} \approx -t + -t \left(\frac{\omega\epsilon_o}{j\sigma} \right) + -t \left(\frac{\omega\epsilon_o}{j\sigma} \right)^2 + \dots \quad (\text{A.18})$$

In the limit as $t \rightarrow 0$ and $\sigma \rightarrow \infty$ such that the product $t\sigma$ remains finite, (A.18) becomes

$$-t \left(1 - \frac{\omega\epsilon_o}{j\sigma} \right)^{-1} \rightarrow 0, \quad (\text{A.19})$$

and

$$\frac{1}{t\sigma} \rightarrow R, \quad (\text{A.20})$$

where R is defined to be the resistance of the sheet. Then, (A.10) through (A.13) reduce to

$$E_z^+ = E_z^-, \quad (\text{A.21})$$

$$E_x^+ = E_x^-, \quad (\text{A.22})$$

$$H_z^+ - H_z^- = \left(\frac{1}{R} \right) E_x, \quad (\text{A.23})$$

$$H_x^+ - H_x^- = - \left(\frac{1}{R} \right) E_z, \quad (\text{A.24})$$

which is the stated resistive boundary condition.

Equations (A.21) through (A.24) express the tangential components of the resistive boundary condition. The corresponding normal components can be found

readily by applying Maxwell's curl equations in source-free space:

$$\nabla \times \vec{E} = -jk_o Z_o \vec{H}, \quad (\text{A.25})$$

$$\nabla \times \vec{H} = +jk_o Y_o \vec{E}. \quad (\text{A.26})$$

For the electric fields $\vec{E}^+$, the magnetic fields on the plus side of the sheet may be written

$$H_x^+ = -\frac{1}{jk_o Z_o} \left(\frac{\partial E_z^+}{\partial y} - \frac{\partial E_y^+}{\partial z} \right), \quad (\text{A.27})$$

$$H_y^+ = -\frac{1}{jk_o Z_o} \left(\frac{\partial E_x^+}{\partial z} - \frac{\partial E_z^+}{\partial x} \right), \quad (\text{A.28})$$

$$H_z^+ = -\frac{1}{jk_o Z_o} \left(\frac{\partial E_y^+}{\partial x} - \frac{\partial E_x^+}{\partial y} \right). \quad (\text{A.29})$$

A similar set can be derived for the magnetic fields on the minus side of the sheet.

From (A.21) and (A.22),

$$H_y^+ = -\frac{1}{jk_o Z_o} \left(\frac{\partial E_x^+}{\partial z} - \frac{\partial E_z^+}{\partial x} \right), \quad (\text{A.30})$$

$$= -\frac{1}{jk_o Z_o} \left(\frac{\partial E_x^-}{\partial z} - \frac{\partial E_z^-}{\partial x} \right), \quad (\text{A.31})$$

$$= H_y^-. \quad (\text{A.32})$$

Or, if the sheet separates to dissimilar media, the more general result is given by

$$\hat{n} \cdot (\vec{B}^+ - \vec{B}^-) = 0, \quad (\text{A.33})$$

i.e., the normal component of the magnetic flux density is continuous across the resistive sheet (as expected).

Now consider the normal components of the electric field on both sides of the sheet:

$$E_y^+ = \frac{1}{jk_o Y_o} \left(\frac{\partial H_x^+}{\partial z} - \frac{\partial H_z^+}{\partial x} \right), \quad (\text{A.34})$$

$$E_y^- = \frac{1}{jk_o Y_o} \left(\frac{\partial H_x^-}{\partial z} - \frac{\partial H_z^-}{\partial x} \right). \quad (\text{A.35})$$

The difference of the normal components of the electric fields may be written as

$$E_y^+ - E_y^- = \frac{1}{jk_o Y_o} \left[\frac{\partial}{\partial z} (H_x^+ - H_x^-) - \frac{\partial}{\partial x} (H_z^+ - H_z^-) \right]. \quad (\text{A.36})$$

Using the resistive boundary condition in (A.23) and (A.24), (A.36) may be rewritten

$$E_y^+ - E_y^- = \frac{1}{jk_o Y_o} \left[\frac{\partial}{\partial z} \left(-\frac{E_z}{R} \right) - \frac{\partial}{\partial x} \left(\frac{E_x}{R} \right) \right]. \quad (\text{A.37})$$

Applying the generalized form of Ohm's Law for a surface current $\vec{J}$

$$\vec{J} = \bar{\sigma} \vec{E} \quad (\text{A.38})$$

$$= \frac{1}{R} \vec{E}, \quad (\text{A.39})$$

(where $\bar{\sigma} \equiv t\sigma$) and allowing for different media on either side of the plate, the difference in the normal component of the electric flux density may be expressed as

$$D_y^+ - D_y^- = -\frac{1}{j\omega} \left[\frac{\partial}{\partial z} J_z + \frac{\partial}{\partial x} J_x \right]. \quad (\text{A.40})$$

Finally, noting on the surface the equation of continuity

$$\nabla \cdot \vec{J} = -j\omega\rho \quad (\text{A.41})$$

may be used to relate the surface current $\vec{J}$ to the surface charge ρ , (A.40) may be written

$$\hat{n} \cdot (\vec{D}^+ - \vec{D}^-) = \rho, \quad (\text{A.42})$$

i.e., the normal component of the electric flux density is discontinuous across the resistive sheet by an equivalent surface charge.

The boundary conditions on the resistive sheet can be summarized as follows:

$$\hat{n} \times (\vec{E}^+ - \vec{E}^-) = \vec{0} \quad (\text{A.43})$$

$$\hat{n} \times \hat{n} \times \vec{E} = -R\hat{n} \times (\vec{H}^+ - \vec{H}^-) \quad (\text{A.44})$$

$$= -R\vec{J} \quad (\text{A.45})$$

$$\hat{n} \cdot (\vec{D}^+ - \vec{D}^-) = -\frac{1}{j\omega} \nabla \cdot \vec{J} \quad (\text{A.46})$$

$$= \rho \quad (\text{A.47})$$

$$\hat{n} \cdot (\vec{B}^+ - \vec{B}^-) = 0. \quad (\text{A.48})$$

A.2 Thin Dielectric Sheet Approximation

The thin dielectric sheet approximation may be derived readily from the above expressions. In this case, assume that the slab is made of dielectric material only; i.e.,

$$\mu_r = 1, \quad (\text{A.49})$$

$$\epsilon_r = \epsilon'_r - j\epsilon''_r. \quad (\text{A.50})$$

Now, (A.2) through (A.5) reduce to

$$E_z^+ - E_z^- = \left(\frac{1}{\epsilon_r} - 1\right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial z} + \frac{\partial E_y^-}{\partial z} \right), \quad (\text{A.51})$$

$$E_x^+ - E_x^- = \left(\frac{1}{\epsilon_r} - 1\right) \frac{t}{2} \left(\frac{\partial E_y^+}{\partial x} + \frac{\partial E_y^-}{\partial x} \right), \quad (\text{A.52})$$

$$H_z^+ - H_z^- = jk_o Y_o (\epsilon_r - 1) \frac{t}{2} [E_x^+ + E_x^-], \quad (\text{A.53})$$

$$H_x^+ - H_x^- = -jk_o Y_o (\epsilon_r - 1) \frac{t}{2} [E_z^+ + E_z^-]. \quad (\text{A.54})$$

For ϵ_r remaining finite while $t \rightarrow 0$,

$$t \left(\frac{1}{\epsilon_r} - 1 \right) \rightarrow 0. \quad (\text{A.55})$$

Also,

$$jk_o Y_o (\epsilon_r - 1) = j\omega(\epsilon - \epsilon_o) \quad (\text{A.56})$$

Thus, in the limit as $t \rightarrow 0$, let

$$\frac{1}{j\omega(\epsilon - \epsilon_o)t} \rightarrow Z_S, \quad (\text{A.57})$$

where Z_S is defined to be the thin sheet impedance. Then, (A.51) through (A.54) reduce to

$$E_z^+ = E_z^-, \quad (\text{A.58})$$

$$E_x^+ = E_x^-, \quad (\text{A.59})$$

$$H_z^+ - H_z^- = \left(\frac{1}{Z_S} \right) E_x, \quad (\text{A.60})$$

$$H_x^+ - H_x^- = - \left(\frac{1}{Z_S} \right) E_z, \quad (\text{A.61})$$

provided that the partial derivatives of the normal component of the electric field are small compared to the tangential components of the electric field at the surface of the sheet. This condition together with (A.1) are equivalent to the conditions given by Andreassen [35] for the thin dielectric spherical shell. Note that the thin sheet approximation also may be derived very simply using the volume equivalence theorem [48].

APPENDIX B

LOCALIZED COORDINATE SYSTEMS

In the calculation of the various inner products used to fill the impedance matrix, several types of integrals over triangular surfaces and tetrahedral volumes must be computed. In order to expedite these computations, local coordinate systems are developed according to [26, 30] based on some work done in finite elements by Zienkiewicz [52]. These localized coordinate systems will be introduced below.

B.1 Triangular local coordinates

Consider the triangular region defined illustrated in Figure 67. A local system of “area” coordinates within this triangular cell T^q is defined with the following relationships from [30]:

$$A^q \equiv A_1 + A_2 + A_3, \quad (\text{B.1})$$

where A_q is the total area of the triangular cell and A_i for $i = 1, 2, 3$ are the sub-areas defined by the surface vector basis functions $\vec{\rho}_i$. Now define the normalized area coordinates:

$$\xi \equiv A_1/A^q, \quad (\text{B.2})$$

$$\eta \equiv A_2/A^q, \quad (\text{B.3})$$

$$\zeta \equiv A_3/A^q. \quad (\text{B.4})$$

Figure 67: Local triangular coordinate system [30].

Then by this definition

$$\xi + \eta + \zeta = 1, \quad (\text{B.5})$$

and in the localized coordinate system the following position vectors

$$\vec{r}_1 = (\xi, \eta, \zeta) = (1, 0, 0), \quad (\text{B.6})$$

$$\vec{r}_2 = (0, 1, 0), \quad (\text{B.7})$$

$$\vec{r}_3 = (0, 0, 1), \quad (\text{B.8})$$

may be specified such that a “source” position vector may be expressed as

$$\vec{r}' = \xi \vec{r}_1 + \eta \vec{r}_2 + \zeta \vec{r}_3. \quad (\text{B.9})$$

A general surface integration over the triangular cell may then be written as

$$\begin{aligned} \int_{T^q} g(\vec{r}') d\vec{r}' &= \\ 2A^q \int_0^1 \int_0^{1-\eta} g(\xi \vec{r}_1 + \eta \vec{r}_2 + (1-\xi-\eta)\vec{r}_3) d\xi d\eta. \end{aligned} \quad (\text{B.10})$$

The area of the triangle may be found for vertex vectors

$$\vec{r}_1 = x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z}, \quad (\text{B.11})$$

$$\vec{r}_2 = x_2 \hat{x} + y_2 \hat{y} + z_2 \hat{z}, \quad (\text{B.12})$$

$$\vec{r}_3 = x_3 \hat{x} + y_3 \hat{y} + z_3 \hat{z}, \quad (\text{B.13})$$

as

$$A = \frac{1}{2!} |(\vec{r}_1 - \vec{r}_2) \times (\vec{r}_2 - \vec{r}_3)|, \quad (\text{B.14})$$

$$= \frac{1}{2!} |(\vec{r}_2 - \vec{r}_3) \times (\vec{r}_3 - \vec{r}_1)|, \quad (\text{B.15})$$

$$= \frac{1}{2!} |(\vec{r}_3 - \vec{r}_1) \times (\vec{r}_1 - \vec{r}_2)|. \quad (\text{B.16})$$

The centroid of the triangle $\vec{r}^c$ is given by

$$\vec{r}^c = \frac{1}{4} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3). \quad (\text{B.17})$$

B.2 Tetrahedral local coordinates

The extension to the tetrahedron is straight forward. For a tetrahedral cell $\mathcal{T}^q$ of volume V^q , the position vector $\vec{r}'$ may be written

$$\vec{r}' = \xi \vec{r}_1 + \eta \vec{r}_2 + \zeta \vec{r}_3 + \gamma \vec{r}_4. \quad (\text{B.18})$$

Then any volume integral over the tetrahedral cell may be written as

$$\int_{\mathcal{T}^q} g(\vec{r}') d\vec{r}' = 6V^q \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} g(\xi \vec{r}_1 + \eta \vec{r}_2 + \zeta \vec{r}_3 + (1-\xi-\eta-\zeta)\vec{r}_4) d\xi d\eta d\zeta. \quad (\text{B.19})$$

The volume of the tetrahedron may be found for the vertex vectors

$$\vec{r}_1 = x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z}, \quad (\text{B.20})$$

$$\vec{r}_2 = x_2 \hat{x} + y_2 \hat{y} + z_2 \hat{z}, \quad (\text{B.21})$$

$$\vec{r}_3 = x_3 \hat{x} + y_3 \hat{y} + z_3 \hat{z}, \quad (\text{B.22})$$

$$\vec{r}_4 = x_4 \hat{x} + y_4 \hat{y} + z_4 \hat{z}, \quad (\text{B.23})$$

as the determinant

$$V = \frac{1}{3!} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}. \quad (\text{B.24})$$

The centroid of the tetrahedron $\vec{r}^c$ is given by

$$\vec{r}^c = \frac{1}{4} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4). \quad (\text{B.25})$$

APPENDIX C

CALCULATION OF POTENTIAL INTEGRALS

The calculation of the potential integrals and the curls of these potential integrals over triangular and tetrahedral cells will now be discussed. Using the local coordinate systems described in Appendix B, the vector integrals will be transformed into equivalent scalar integrals. For point matching, the “mutual” terms can be integrated numerically using a special Gaussian quadrature developed for simplex shapes (see Appendix D). Since the “self” terms involve testing the field in the source region, care must be taken when evaluating these integrals. Specifics with regard to setting up the potential integrals and the curls of these integrals are provided below.

C.1 Triangular potential integrals

Consider the following two types of potential integrals:

$$\tilde{A}_n^R(\tilde{r}_m^{c\pm}) = \frac{\mu_o}{4\pi} \int_s \tilde{f}_n^S(\tilde{r}') \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\tilde{r}', \quad (\text{C.1})$$

$$\Phi_n^R(\tilde{r}_m^{c\pm}) = -\frac{1}{4\pi j\omega\epsilon_o} \int_s \nabla'_S \cdot \tilde{f}_n^S(\tilde{r}') \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\tilde{r}', \quad (\text{C.2})$$

where

$$R_m^\pm \equiv |\tilde{r}_m^{c\pm} - \tilde{r}'|, \quad (\text{C.3})$$

and s is the subsectional surface supporting the basis function $\tilde{f}_n^S$.

Although the matrix elements indexed on mn are associated with the “edges” of the source and observation triangles (due to the definition of the triangular basis

functions), the integration domains exist over the corresponding source faces and the observation points are located at the centroids $\vec{r}^{c\pm}$ of triangular faces. As noted by Rao [30], “face-by-face computation of integrals results in up to a nine-fold increase in efficiency in filling the matrix Z over the direct edge-by-edge approach.”

Therefore, consider the source triangle T^q and the observation triangle T^p illustrated in Figure 68. The potential integrals associated with this pair are from (3.14), (3.16), and (5.1)

$$\vec{A}_i^{Rpq} = \frac{\mu_o}{4\pi} \int_{T^q} \left(\frac{\ell_i}{2A^q} \right) \vec{\rho}_i \frac{e^{-jkR^p}}{R^p} d\vec{r}', \quad (\text{C.4})$$

$$\Phi_i^{Rpq} = -\frac{1}{4\pi j\omega\epsilon_o} \int_{T^q} \left(\frac{\ell_i}{A^q} \right) \frac{e^{-jkR^p}}{R^p} d\vec{r}', \quad (\text{C.5})$$

with

$$R^p \equiv |\vec{r}^p - \vec{r}'|, \quad (\text{C.6})$$

and $i = 1, 2, 3$. These integrals represent the contributions due to the i th basis function of source T^q at the centroid of the observation T^p .

Using the localized coordinate systems described in Appendix B,

$$\vec{\rho}_i = \pm(\vec{r}' - \vec{r}_i) \quad (\text{C.7})$$

$$= \pm(\xi\vec{r}_1 + \eta\vec{r}_2 + (1 - \xi - \eta)\vec{r}_3 - \vec{r}_i) \quad (\text{C.8})$$

$$i = 1, 2, 3$$

where the $+/-$ sign indicates the current reference direction of $\vec{\rho}_i$ in the cell. Then, these integrals may be rewritten as [30]

$$\vec{A}_i^{Rpq} = \pm \frac{\mu_o \ell_i}{4\pi} (\vec{r}_1 I_\xi^{pq} + \vec{r}_2 I_\eta^{pq} + \vec{r}_3 I_\zeta^{pq} - \vec{r}_i I^{pq}), \quad (\text{C.9})$$

$$\Phi_i^{Rpq} = \mp \frac{\ell_i}{2\pi j\omega\epsilon_o} I^{pq}, \quad (\text{C.10})$$

Figure 68: Local indexing for source triangle T^q and observation triangle T^p [30].

where

$$I^{pq} = \int_0^1 \int_0^{1-\eta} \frac{e^{-jkR^p}}{R^p} d\xi d\eta, \quad (\text{C.11})$$

$$I_\xi^{pq} = \int_0^1 \int_0^{1-\eta} \xi \frac{e^{-jkR^p}}{R^p} d\xi d\eta, \quad (\text{C.12})$$

$$I_\eta^{pq} = \int_0^1 \int_0^{1-\eta} \eta \frac{e^{-jkR^p}}{R^p} d\xi d\eta, \quad (\text{C.13})$$

$$I_\zeta^{pq} = I^{pq} - I_\xi^{pq} - I_\eta^{pq}. \quad (\text{C.14})$$

Thus, three *scalar* integrals must be calculated to fill up to nine impedance matrix elements [30].

When the observation point lies in the region of integration (self term), care must be taken in evaluating the above expressions. Provided that the observation point does not lie on an edge of the triangular region, these potential integrals are *convergent* as defined by Kellogg [53, p. 147]; i.e., the integrals reduce to their principal values (in the Cauchy sense). To evaluate these integrals, the integrands are rewritten as the sum of a singular term and a non-singular term. The principal value of the singular integral is calculated analytically, while the non-singular integral is calculated numerically. For details of this procedure, see Wilton, et al. [54].

The curl of the vector potential integral also must be calculated:

$$\nabla \times \vec{A}_n^R(\vec{r}_m^c) = \frac{\mu_o}{4\pi} \int_s \vec{J}^R(\vec{r}') \times \nabla' G(\vec{r}, \vec{r}') d\vec{r}', \quad (\text{C.15})$$

with

$$\nabla' G(\vec{r}, \vec{r}') = (\vec{r} - \vec{r}')(1 + jkR_m^\pm) \frac{e^{-jkR_m^\pm}}{(R_m^\pm)^3}. \quad (\text{C.16})$$

Substituting in the basis function, (C.15) becomes

$$\nabla \times \vec{A}_i^{Rpq} = \frac{\mu_o}{4\pi} \int_{T_q} \left(\frac{\ell_i}{2A^q} \right) \vec{\rho}_i \times (\vec{r}^{cp} - \vec{r}') (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\vec{r}'. \quad (\text{C.17})$$

From Equation (C.7),

$$\vec{\rho}_i \times (\vec{r}^{cp} - \vec{r}') = (\vec{\rho}_i \times \vec{r}^{cp}) - (\vec{\rho}_i \times \vec{r}') \quad (\text{C.18})$$

$$= \pm \left((\vec{r}' - \vec{r}_i) \times \vec{r}^{cp} - (\vec{r}' - \vec{r}_i) \times \vec{r}' \right) \quad (\text{C.19})$$

$$= \pm \left((\vec{r}' \times \vec{r}^{cp}) - (\vec{r}_i \times \vec{r}^{cp}) + (\vec{r}_i \times \vec{r}') \right) \quad (\text{C.20})$$

$$= \pm \left((\vec{r}^{cp} \times \vec{r}_i) + (\vec{r}_i \times \vec{r}') - (\vec{r}^{cp} \times \vec{r}') \right) \quad (\text{C.21})$$

$$= \pm \left((\vec{r}^{cp} \times \vec{r}_i) + (\vec{r}_i - \vec{r}^{cp}) \times \vec{r}' \right). \quad (\text{C.22})$$

Then, using the area coordinates for T^q , (C.17) may be written [30]

$$\begin{aligned} \nabla \times \vec{A}_i^{Rpq} &= \pm \frac{\mu_o \ell_i}{4\pi} \left[(\vec{r}^{cp} \times \vec{r}_i) J^{pq} \right. \\ &\quad \left. + (\vec{r}_i - \vec{r}^{cp}) \times (\vec{r}_1 J_\xi^{pq} + \vec{r}_2 J_\eta^{pq} + \vec{r}_3 J_\zeta^{pq}) \right], \end{aligned} \quad (\text{C.23})$$

where

$$J^{pq} = \int_0^1 \int_0^{1-\eta} (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta, \quad (\text{C.24})$$

$$J_\xi^{pq} = \int_0^1 \int_0^{1-\eta} \xi (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta, \quad (\text{C.25})$$

$$J_\eta^{pq} = \int_0^1 \int_0^{1-\eta} \eta (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta, \quad (\text{C.26})$$

$$J_\zeta^{pq} = J^{pq} - J_\xi^{pq} - J_\eta^{pq}. \quad (\text{C.27})$$

Note that no self terms involving the above integrals must be calculated for the MoM formulation described in Chapter V: these expressions are used to calculate the magnetic field $\vec{H}$ tested in the material volume due to the electric current $\vec{J}^R$ on the resistive surface.

C.2 Tetrahedral potential integrals

First, the volume integrals involved in the calculation of the vector magnetic potential $\vec{A}$ and the scalar electric potential Φ will be considered. The vector electric

potential $\vec{F}$ and the scalar magnetic potential Ψ can then be found easily using duality. From (3.14), (3.16), and (5.4),

$$\vec{A}_n(\vec{r}_m^{c\pm}) = \frac{j\omega\mu_o}{4\pi} \int_v \kappa^e(\vec{r}') \vec{f}_n^V(\vec{r}') \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}', \quad (\text{C.28})$$

$$\Phi_n(\vec{r}_m^{c\pm}) = -\frac{1}{4\pi\epsilon_o} \int_v \nabla' \cdot (\kappa^e(\vec{r}') \vec{f}_n^V(\vec{r}')) \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}', \quad (\text{C.29})$$

where again

$$R_m^\pm \equiv |\vec{r}_m^{c\pm} - \vec{r}'|. \quad (\text{C.30})$$

and v is the subsectional surface supporting the basis function $\vec{f}_n^V$. Using the properties of the vector volume basis function,

$$\begin{aligned} \vec{A}_n(\vec{r}_m^{c\pm}) &= \frac{j\omega\mu_o}{4\pi} \left[\frac{\kappa_n^{e+} a_n}{3V_n^+} \int_{T_n^+} \vec{\wp}_n^+ \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}' \right. \\ &\quad \left. + \frac{\kappa_n^{e-} a_n}{3V_n^-} \int_{T_n^-} \vec{\wp}_n^- \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}' \right], \end{aligned} \quad (\text{C.31})$$

and

$$\begin{aligned} \Phi_n(\vec{r}_m^{c\pm}) &= -\frac{1}{4\pi\epsilon_o} \left[\frac{\kappa_n^{e+} a_n}{V_n^+} \int_{T_n^+} \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}' - \frac{\kappa_n^{e-} a_n}{V_n^-} \int_{T_n^-} \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}' \right] \\ &\quad + \frac{(\kappa_n^{e+} - \kappa_n^{e-})}{4\pi\epsilon_o} \int_{a_n} \frac{e^{-jkR_m^\pm}}{R_m^\pm} d\vec{r}'. \end{aligned} \quad (\text{C.32})$$

Equations (C.31) and (C.32) require integrations over the individual source tetrahedrons as well as a surface integration over the intervening triangular face. The surface integration is performed according to the method outlined above. This leaves for the source-observation tetrahedron pair the following types of volume integrals:

$$\vec{A}_i^{pq} = \frac{j\omega\mu_o}{4\pi} \int_{T^q} \kappa_i^{e,q} \left(\frac{a_i}{3V^q} \right) \vec{\wp}_i \frac{e^{-jkR^p}}{R^p} d\vec{r}', \quad (\text{C.33})$$

$$\Phi_i^{pq} = -\frac{1}{4\pi\epsilon_o} \int_{T^q} \kappa_i^{e,q} \left(\frac{a_i}{V^q} \right) \frac{e^{-jkR^p}}{R^p} d\vec{r}', \quad (\text{C.34})$$

with

$$R^p \equiv |\vec{r}^p - \vec{r}'|, \quad (\text{C.35})$$

and $i = 1, 2, 3, 4$. These integrals represent the contributions due to the i th basis function of source $\mathcal{I}^q$ at the centroid of the observation $\mathcal{T}^p$.

Again, using the localized coordinate systems described in Appendix B,

$$\vec{\wp}_i = \pm(\vec{r}' - \vec{r}_i) \quad (\text{C.36})$$

$$= \pm(\xi\vec{r}_1 + \eta\vec{r}_2 + \zeta\vec{r}_3 + (1 - \xi - \eta - \zeta)\vec{r}_4 - \vec{r}_i) \quad (\text{C.37})$$

$$i = 1, 2, 3, 4$$

where the $+/-$ sign indicates the reference flux direction of $\vec{\wp}_i$ in the cell. Then, (C.33) and (C.34) may be expressed as

$$\vec{A}_i^{pq} = \pm \frac{j\omega\mu_o a_i \kappa_i^{e,q}}{2\pi} (\vec{r}_1 \mathcal{I}_\xi^{pq} + \vec{r}_2 \mathcal{I}_\eta^{pq} + \vec{r}_3 \mathcal{I}_\zeta^{pq} + \vec{r}_4 \mathcal{I}_\gamma^{pq} - \vec{r}_i \mathcal{I}^{pq}), \quad (\text{C.38})$$

$$\Phi_i^{pq} = \mp \frac{3a_i \kappa_i^{e,q}}{2\pi\epsilon_o} \mathcal{I}^{pq}, \quad (\text{C.39})$$

where

$$\mathcal{I}^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \frac{e^{-jkR^p}}{R^p} d\xi d\eta d\zeta, \quad (\text{C.40})$$

$$\mathcal{I}_\xi^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \xi \frac{e^{-jkR^p}}{R^p} d\xi d\eta d\zeta, \quad (\text{C.41})$$

$$\mathcal{I}_\eta^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \eta \frac{e^{-jkR^p}}{R^p} d\xi d\eta d\zeta, \quad (\text{C.42})$$

$$\mathcal{I}_\zeta^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \zeta \frac{e^{-jkR^p}}{R^p} d\xi d\eta d\zeta, \quad (\text{C.43})$$

$$\mathcal{I}_\gamma^{pq} = \mathcal{I}^{pq} - \mathcal{I}_\xi^{pq} - \mathcal{I}_\eta^{pq} - \mathcal{I}_\zeta^{pq}. \quad (\text{C.44})$$

Here, four *scalar* integrals must be calculated to fill up to sixteen impedance matrix elements [26].

In the case of the self terms, these potential integrals are convergent (as defined by Kellogg [53, p. 147]) provided that the observation point does not lie on an edge of

the tetrahedral region. Thus, the integrals are well defined in the Cauchy principal value sense [55] and are evaluated using the procedures of Wilton, et al. [54].

The “curl” term associated with the tetrahedral basis functions can be evaluated in a similar way:

$$\nabla \times \vec{A}_n(\vec{r}_m^c) = \frac{\mu_o}{4\pi} \int_v \vec{J}_n(\vec{r}') \times \nabla' G(\vec{r}_m^c, \vec{r}') d\vec{r}', \quad (\text{C.45})$$

with

$$\nabla' G(\vec{r}_m^c, \vec{r}') = (\vec{r}_m^c - \vec{r}') (1 + jkR_m^\pm) \frac{e^{-jkR_m^\pm}}{(R_m^\pm)^3}. \quad (\text{C.46})$$

Substituting in the basis function,

$$\begin{aligned} \nabla \times \vec{A}_i^{pq} &= \frac{j\omega\mu_o}{4\pi} \int_{T_q} \kappa^e(\vec{r}') \left(\frac{a_i}{3V_q} \right) \vec{\wp}_i \times (\vec{r}^{cp} - \vec{r}') \\ &\quad (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\vec{r}'. \end{aligned} \quad (\text{C.47})$$

Again, using the localized coordinate system described in Appendix B,

$$\vec{\wp}_i = \pm(\vec{r}' - \vec{r}_i) \quad i = 1, 2, 3, 4 \quad (\text{C.48})$$

and

$$\vec{r}' = \xi\vec{r}_1 + \eta\vec{r}_2 + \zeta\vec{r}_3 + \gamma\vec{r}_4, \quad (\text{C.49})$$

then from (C.22) and the volume coordinates,

$$\begin{aligned} \nabla \times \vec{A}_i^{pq} &= \frac{j\omega\mu_o a_i \kappa_i^{e,q}}{2\pi} \left[(\vec{r}^{cp} \times \vec{r}_i) \mathcal{J}^{pq} \right. \\ &\quad \left. + (\vec{r}_i - \vec{r}^{cp}) \times (\vec{r}_1 \mathcal{J}_\xi^{pq} + \vec{r}_2 \mathcal{J}_\eta^{pq} + \vec{r}_3 \mathcal{J}_\zeta^{pq} + \vec{r}_4 \mathcal{J}_\gamma^{pq}) \right], \end{aligned} \quad (\text{C.50})$$

where

$$\mathcal{J}^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} (1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta d\zeta, \quad (\text{C.51})$$

$$\mathcal{J}_\xi^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \xi(1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta d\zeta, \quad (\text{C.52})$$

$$\mathcal{J}_\eta^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \eta(1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta d\zeta, \quad (\text{C.53})$$

$$\mathcal{J}_\zeta^{pq} = \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} \zeta(1 + jkR^p) \frac{e^{-jkR^p}}{(R^p)^3} d\xi d\eta d\zeta, \quad (\text{C.54})$$

$$\mathcal{J}_\gamma^{pq} = \mathcal{J}^{pq} - \mathcal{J}_\xi^{pq} - \mathcal{J}_\eta^{pq} - \mathcal{J}_\zeta^{pq}. \quad (\text{C.55})$$

Although no self terms must be calculated for the curl integrals according to the MoM formulation described in Chapter V, singular integrands are still encountered since the vector unknowns $\vec{J}$ and $\vec{M}$ have the same functional support in the material volume. Yaghjian [29] has shown that these integrals are convergent and well defined in the Cauchy principal value sense. Since the singularity goes as $1/R^3$ and is therefore highly localized, these principal value integrals are approximated by simply removing the singular point (at the centroid of the tetrahedrons for point matching) and numerically integrating the rest of the region.

APPENDIX D

GAUSSIAN QUADRATURE OVER TRIANGLES AND TETRAHEDRONS

The MoM solution proposed in this dissertation requires the evaluation of integrals involving special vector basis functions over triangles and tetrahedrons. In Appendix C of this manuscript, these vector integrals were expressed as separable scalar integrals of potential functions over these regions. Given that the singularities associated with the “self terms” can be removed and dealt with separately, this appendix will focus upon the numerical integration of analytic functions over triangular and tetrahedral regions.

Much work has been done on the problem of numerical integration over simplexes [56, 57, 58, 59] for various orders of integration (e.g., linear, quadratic, cubic). Stroud [58] defines the n -dimensional simplex as such:

“A simplex T_n has $n+1$ vertices and is bounded by $n+1$ hyperplanes $[(n-1)$ -dimensional surface]; each hyperplane passes through n vertices. T_n is the analogue of the triangle in two dimensions and the tetrahedron in three dimensions.”

We will restrict ourselves to simplexes of orders $n = 2, 3$; i.e., the triangle and the tetrahedron. Here, only a few of the Gaussian nodes and weights as presented in [52, 60] will be listed (please note corrections). Other orders and schemes may be found in the above references.

Using the local coordinate system defined in Appendix B, an integral over a triangular region T of area A may be approximated numerically as

$$\int_T g(\vec{r}') d\vec{r}' = 2A \int_0^1 \int_0^{1-\eta} g(\xi, \eta) d\xi d\eta \quad (\text{D.1})$$

$$\approx 2A \sum_{i=1}^N g(\xi_i, \eta_i) w_i, \quad (\text{D.2})$$

where (ξ_i, η_i) are the nodes and w_i are the corresponding weights of the quadrature scheme. Table 8 lists a few sets of nodes and weights for various Gaussian quadrature rules over a triangular region.

Similarly, an integral over a tetrahedral region T of volume V may be approximated numerically as

$$\int_T g(\vec{r}') d\vec{r}' = 6V \int_0^1 \int_0^{1-\eta} \int_0^{1-\eta-\zeta} g(\xi, \eta, \zeta) d\xi d\eta d\zeta \quad (\text{D.3})$$

$$\approx 6V \sum_{i=1}^N g(\xi_i, \eta_i, \zeta_i) w_i. \quad (\text{D.4})$$

Table 9 lists a few sets of nodes and weights for various Gaussian quadrature rules over a tetrahedral region.

Table 8: Gaussian nodes and weights for integration over triangular region.

order	triangular coordinates	weights $2w_i$
linear	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
quadratic	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{1}{3}$
	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$
	$0, \frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}$
cubic	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$-\frac{27}{48}$
	$\frac{3}{5}, \frac{1}{5}, \frac{1}{5}$	$\frac{25}{48}$
	$\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$	$\frac{25}{48}$
	$\frac{1}{5}, \frac{1}{5}, \frac{3}{5}$	$\frac{25}{48}$
cubic	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{27}{60}$
	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{8}{60}$
	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{8}{60}$
	$0, \frac{1}{2}, \frac{1}{2}$	$\frac{8}{60}$
	$1, 0, 0$	$\frac{3}{60}$
	$0, 1, 0$	$\frac{3}{60}$
	$0, 0, 1$	$\frac{3}{60}$

Table 9: Gaussian nodes and weights for integration over tetrahedral region.

order	tetrahedral coordinates	weights $6w_i$
linear	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	1
quadratic	$\alpha, \beta, \beta, \beta$	$\frac{1}{4}$
	$\beta, \alpha, \beta, \beta$	$\frac{1}{4}$
	$\beta, \beta, \alpha, \beta$	$\frac{1}{4}$
	$\beta, \beta, \beta, \alpha$	$\frac{1}{4}$
	$\alpha = 0.58541020$ $\beta = 0.13819660$	
cubic	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	$-\frac{4}{5}$
	$\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$	$\frac{9}{20}$
	$\frac{1}{6}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}$	$\frac{9}{20}$
	$\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6}$	$\frac{9}{20}$
	$\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}$	$\frac{9}{20}$

APPENDIX E

CONNECTIVITY

The accurate geometric modeling of complex three-dimensional objects not only necessitates a robust meshing algorithm, but also requires a sophisticated connectivity scheme to identify the relationships between nodes, edges, faces, and cells. These relationships are especially important for the volumetric MoM solution presented in this dissertation. Here, dipole modes over adjacent volumetric cells, monopole modes for cells touching the resistive sheet, and surface dipole modes on the sheet itself have been specified. The connectivity algorithm used to define these relationships is summarized in this appendix.

The mesh program used to subdivide the three-dimensional material body is the commercial code COSMOS GEOSTAR [32]. No further discussion of this code is necessary except to note the output: GEOSTAR provides indexed node locations (x, y, z) and the nodal indexing for cells and boundaries. The cells of interest are tetrahedral volumes and triangular surfaces. Triangular surface facets (used to define the resistive sheet) must be associated with faces of corresponding tetrahedrals cells if these sheet triangles are in contact with the volume.

The node locations, cell indices, and boundaries are input to a preprocessor which assigns connectivity parameters to the geometry. See the MFB User Manual [31] for the specific input requirements. From this input set, the geometry is completely characterized in terms of node, edge, face, and volumetric cell indices.

Figure 69: Edge index orientation.

These indices are then used to identify monopole and dipole expansions in the volume and on the resistive surface.

Starting with pairs of *node indices* k , *edge indices* j are arbitrarily assigned to all adjacent nodes that are used to form individual cells. *Edge directions* are also assigned as shown in Figure 69.

A set of three nodes may be then connected to form a face. All faces are assigned *face indices* i and normal directions n_i . Faces associated with boundary surfaces such as the resistive sheet or the surface bounding the volume are assigned normals in a regular fashion (e.g., all normals pointing to one side, outward, etc.). Interior faces associated with other volumetric cells are assigned arbitrary surface normals.

Figure 70: Face index orientation.

With each surface normal, a contour C_i is assigned using the right hand rule with respect to the normal. If the contour direction is consistent with the individually assigned edge direction, then that edge is defined to be positive; otherwise, the edge is defined to be negative (see Figure 70). These designations are used in the assignment of dipole modes on the resistive sheet.

When an edge is associated with a triangular basis function (i.e., it is a non-boundary edge of the resistive sheet), the corresponding monopole mode (one half of the dipole mode) associated with each face adjacent to that edge is assigned to be positive or negative according to the sign of the edge length and the face contour as indicated in Figure 71. In this way, a current flow is specified and surface dipole orientations are defined that are globally consistent across the surface.

Figure 71: Surface dipole current direction assignment.

In a similar fashion, positive and negative “areas” are defined for the faces of the volumetric cells. Every face in the meshed body is assigned a surface normal according to the rules described above. Each tetrahedral cell is then constructed from four triangular *face indices* i and assigned a *cell index* h . If an individual triangular face normal is such that it points out of the tetrahedral cell, then that face is assigned a positive area; otherwise it is assigned a negative area (see Figure 72).

Each triangular face in the volume is associated with a volumetric dipole mode that spans two adjacent tetrahedral cells. The corresponding monopole mode (one half of the dipole mode) is assigned a positive or negative direction according to the sign of the face area as indicated in Figure 73. In this way, flux densities are assigned to each pair of tetrahedral cells forming a dipole and global consistency is achieved.

Figure 72: Cell index orientation.

Figure 73: Volume dipole current direction assignment.

APPENDIX F

THE MATERIAL COATED RESISTIVE SPHERE – EIGENFUNCTION SOLUTION

Very few canonical problems exist for electromagnetic scattering from three-dimensional material objects. Even fewer solutions can be found involving the resistive boundary condition. Therefore, the eigenfunction solution for plane wave scattering from a material coated resistive sphere is presented. This solution is very general in that it reduces to the eigenfunction solutions for scattering from a perfectly conducting sphere [24, p. 292–294], a resistive spherical shell (or the thin dielectric sheet approximation) [35], a solid dielectric/ferrite sphere [24, p. 297], two concentric material spheres [61], and the material coated perfectly conducting sphere [62]. This solution provides an excellent test case for those researchers wishing to validate numerical codes.

The geometry for the material coated resistive sphere is illustrated in Figure 74. An incident plane wave polarized in the $\hat{x}$ direction and traveling in the minus $\hat{z}$ direction is assumed as shown

$$\vec{E}^i = \hat{x} E_o e^{jkz} \quad (\text{F.1})$$

$$= \hat{x} E_o e^{jkr \cos \theta}, \quad (\text{F.2})$$

where $k = \omega\sqrt{\mu\epsilon}$ is the wave number of the exterior region and time-harmonic fields $e^{j\omega t}$ are assumed and suppressed. The sphere is comprised of three regions:

$$a < r < b \quad \text{material region 1 } (\mu_1, \epsilon_1)$$

Figure 74: Material coated resistive sphere.

$r = a$ resistive surface R

$r < a$ material region 2 (μ_2, ϵ_2)

The relative permittivities and permeabilities are allowed to be complex in order to model loss.

The solution is expressed as a sum of TE and TM components transverse to the radial direction $\hat{r}$:

$$\vec{E} = \vec{E}^{TE} + \vec{E}^{TM}, \quad (\text{F.3})$$

$$\vec{H} = \vec{H}^{TE} + \vec{H}^{TM}, \quad (\text{F.4})$$

where the vector potentials are chosen such that

$$\vec{E}^{TM} = -\frac{\nabla \times \vec{F}}{\epsilon}, \quad (\text{F.5})$$

$$\vec{E}^{TE} = \frac{\nabla \times \nabla \times \vec{A}}{j\omega\epsilon\mu}, \quad (\text{F.6})$$

and

$$\vec{H}^{TE} = \frac{\nabla \times \nabla \times \vec{F}}{j\omega\epsilon\mu}, \quad (\text{F.7})$$

$$\vec{H}^{TM} = \frac{\nabla \times \vec{A}}{\mu}. \quad (\text{F.8})$$

In spherical coordinates, the total fields may be written

$$E_r = \frac{1}{j\omega\epsilon\mu} \left[\frac{\partial^2}{\partial r^2} + k^2 \right] A_r, \quad (\text{F.9})$$

$$E_\theta = \frac{1}{j\omega\epsilon\mu} \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} A_r - \frac{1}{\epsilon r \sin \theta} \frac{\partial F_r}{\partial \phi}, \quad (\text{F.10})$$

$$E_\phi = \frac{1}{j\omega\epsilon\mu} \frac{1}{r \sin \theta} \frac{\partial^2}{\partial r \partial \phi} A_r + \frac{1}{\epsilon r} \frac{\partial F_r}{\partial \theta}, \quad (\text{F.11})$$

and

$$H_r = \frac{1}{j\omega\epsilon\mu} \left[\frac{\partial^2}{\partial r^2} + k^2 \right] F_r, \quad (\text{F.12})$$

$$H_\theta = \frac{1}{j\omega\epsilon\mu} \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} F_r + \frac{1}{\mu r \sin \theta} \frac{\partial A_r}{\partial \phi}, \quad (\text{F.13})$$

$$H_\phi = \frac{1}{j\omega\epsilon\mu} \frac{1}{r \sin \theta} \frac{\partial^2}{\partial r \partial \phi} F_r - \frac{1}{\mu r} \frac{\partial A_r}{\partial \theta}. \quad (\text{F.14})$$

Begin by noting that the radial components of the incident field may be written as

$$E_r^i = \cos \phi \sin \theta E_o e^{jkr \cos \theta} \quad (\text{F.15})$$

$$= -\frac{E_o \cos \theta}{jkr} \frac{\partial}{\partial \theta} (e^{jkr \cos \theta}). \quad (\text{F.16})$$

From [24, p. 290], the incident field may be expanded using spherical wave functions:

$$e^{jr \cos \theta} = \sum_{n=0}^{\infty} j^n (2n+1) j_n(r) P_n(\cos \theta), \quad (\text{F.17})$$

where j_n are spherical Bessel functions and P_n are Legendre polynomials of the first kind. Thus,

$$E_r^i = -\frac{E_o \cos \phi}{jkr} \frac{\partial}{\partial \theta} \left\{ \sum_{n=0}^{\infty} j^n (2n+1) j_n(kr) P_n(\cos \theta) \right\} \quad (\text{F.18})$$

$$= -\frac{E_o \cos \phi}{jkr} \sum_{n=0}^{\infty} j^n (2n+1) j_n(kr) \frac{\partial}{\partial \theta} P_n(\cos \theta). \quad (\text{F.19})$$

Noting the following properties of the associated Legendre polynomials,

$$\frac{\partial}{\partial \theta} P_n(\cos \theta) = P_n^1(\cos \theta), \quad (\text{F.20})$$

and

$$P_0^1(\cos \theta) = 0, \quad (\text{F.21})$$

Equation (F.19) may be written as

$$E_r^i = -\frac{E_o \cos \phi}{jkr} \sum_{n=1}^{\infty} j^n (2n+1) j_n(kr) P_n^1(\cos \theta). \quad (\text{F.22})$$

Also from [24, p. 268], define the Schelkunoff-type spherical Bessel functions to be

$$\hat{B}_n(kr) = kr b_n(kr) \quad (\text{F.23})$$

$$= \sqrt{\frac{\pi kr}{2}} B_{n+\frac{1}{2}}(kr), \quad (\text{F.24})$$

then the incident electric field may be expressed

$$E_r^i = j \frac{E_o \cos \phi}{(kr)^2} \sum_{n=1}^{\infty} j^n (2n+1) \hat{J}_n(kr) P_n^1(\cos \theta). \quad (\text{F.25})$$

Similarly, the incident magnetic field may be written

$$H_r^i = -j \frac{E_o \sin \phi}{\eta (kr)^2} \sum_{n=1}^{\infty} j^n (2n+1) \hat{J}_n(kr) P_n^1(\cos \theta), \quad (\text{F.26})$$

for characteristic impedance η of the exterior medium.

The vector potential functions may now be constructed

$$A_r^i = j \sqrt{\mu \epsilon} \cos \phi \sum_{n=1}^{\infty} A_n \hat{J}_n(kr) P_n^1(\cos \theta), \quad (\text{F.27})$$

$$F_r^i = j \sqrt{\mu \epsilon} \sin \phi \sum_{n=1}^{\infty} F_n \hat{J}_n(kr) P_n^1(\cos \theta), \quad (\text{F.28})$$

for unknowns A_n and F_n . Substituting Equation (F.27) into Equation (F.9) and noting from [24, p. 268] that

$$\left[\frac{\partial^2}{\partial r^2} + k^2 - \frac{n(n+1)}{r^2} \right] \hat{B}_n = 0, \quad (\text{F.29})$$

the radial component of the incident electric field may be expressed as

$$E_r^i = \frac{1}{j \omega \mu \epsilon} \sum_{n=1}^{\infty} j \sqrt{\mu \epsilon} \cos \phi \frac{n(n+1)}{r^2} A_n \hat{J}_n(kr) P_n^1(\cos \theta) \quad (\text{F.30})$$

$$= \frac{\cos \phi}{kr^2} \sum_{n=1}^{\infty} n(n+1) A_n \hat{J}_n(kr) P_n^1(\cos \theta). \quad (\text{F.31})$$

Comparing this equation to Equation (F.25),

$$A_n = j \frac{E_o j^n (2n+1)}{kn(n+1)}. \quad (\text{F.32})$$

Following a similar procedure,

$$F_n = -j \frac{E_0 j^n (2n+1)}{\eta k n (n+1)} \quad (\text{F.33})$$

$$= -\frac{A_n}{\eta}. \quad (\text{F.34})$$

The fields in all regions are now ready to be expanded in terms of spherical wave functions.

In the exterior region (μ, ϵ) , the total fields are expressed in terms of a standing wave represented by the spherical Bessel function $\hat{J}_n(kr)$ and a divergent wave represented by the spherical Hankel function of the second kind $\hat{H}_n^{(2)}(kr)$:

$$A_r = j\sqrt{\mu\epsilon} \cos \phi \sum_{n=1}^{\infty} [A_n \hat{J}_n(kr) + b_n \hat{H}_n^{(2)}(kr)] P_n^1(\cos \theta), \quad (\text{F.35})$$

$$F_r = j\sqrt{\mu\epsilon} \sin \phi \sum_{n=1}^{\infty} [F_n \hat{J}_n(kr) + c_n \hat{H}_n^{(2)}(kr)] P_n^1(\cos \theta). \quad (\text{F.36})$$

Similarly, in material region (μ_1, ϵ_1) of the sphere, the spherical Neumann function $\hat{N}_n(kr)$ may be used:

$$A_r^1 = j\sqrt{\mu_1\epsilon_1} \cos \phi \sum_{n=1}^{\infty} [d_n \hat{J}_n(k_1 r) + p_n \hat{N}_n(k_1 r)] P_n^1(\cos \theta), \quad (\text{F.37})$$

$$F_r^1 = j\sqrt{\mu_1\epsilon_1} \sin \phi \sum_{n=1}^{\infty} [e_n \hat{J}_n(k_1 r) + q_n \hat{N}_n(k_1 r)] P_n^1(\cos \theta). \quad (\text{F.38})$$

Likewise, in material region (μ_2, ϵ_2) of the sphere,

$$A_r^2 = j\sqrt{\mu_2\epsilon_2} \cos \phi \sum_{n=1}^{\infty} f_n \hat{J}_n(k_2 r) P_n^1(\cos \theta), \quad (\text{F.39})$$

$$F_r^2 = j\sqrt{\mu_2\epsilon_2} \sin \phi \sum_{n=1}^{\infty} g_n \hat{J}_n(k_2 r) P_n^1(\cos \theta). \quad (\text{F.40})$$

In Equations (F.35) through (F.40) b_n , c_n , d_n , e_n , f_n , g_n , p_n , and q_n are all unknowns. Using Equations (F.9) through (F.14), the fields in each region may be expressed in terms of their TE and TM components. At the boundary $r = a$, the

resistive boundary condition (discussed at length in Appendix A) is applied:

$$\hat{n} \times (\vec{E}^1 - \vec{E}^2) = \vec{0}, \quad (\text{F.41})$$

$$\hat{n} \times \hat{n} \times \vec{E} = -R\hat{n} \times (\vec{H}^1 - \vec{H}^2), \quad (\text{F.42})$$

where $(\vec{E}^1, \vec{H}^1)$ are the fields in region 1, $(\vec{E}^2, \vec{H}^2)$ are the fields in region 2, $\hat{n}$ is the surface normal at $r = a$ pointing into region 1 ($\hat{n} = \hat{r}$ in this case), and R is the resistance. At the boundary $r = b$, continuity of the tangential $\vec{E}$ and $\vec{H}$ fields is applied:

$$\hat{n} \times (\vec{E}^1 - \vec{E}^2) = \vec{0}, \quad (\text{F.43})$$

$$\hat{n} \times (\vec{H}^1 - \vec{H}^2) = \vec{0}. \quad (\text{F.44})$$

Using the orthogonality of the spherical wave functions, each boundary results in four equations (two tangential polarizations $\hat{\theta}$ and $\hat{\phi}$) for each TE and TM component. However, eight of these equations are redundant and the TE and TM equations are not coupled. Thus, two independent sets of four equations in four unknowns remain: one set for the TE components,

$$\eta_2 g_n \hat{J}_n(k_2 a) = \eta_1 [e_n \hat{J}_n(k_1 a) + q_n \hat{N}_n(k_1 a)], \quad (\text{F.45})$$

$$j\eta_2 g_n \hat{J}_n(k_2 a) = R \{ [e_n \hat{J}_n'(k_1 a) + q_n \hat{N}_n'(k_1 a)] - g_n \hat{J}_n'(k_2 a) \}, \quad (\text{F.46})$$

$$\eta [F_n \hat{J}_n(kb) + c_n \hat{H}_n^{(2)}(kb)] = \eta_1 [e_n \hat{J}_n(k_1 b) + q_n \hat{N}_n(k_1 b)], \quad (\text{F.47})$$

$$[F_n \hat{J}_n'(kb) + c_n \hat{H}_n^{(2)'}(kb)] = [e_n \hat{J}_n'(k_1 b) + q_n \hat{N}_n'(k_1 b)], \quad (\text{F.48})$$

and one set for the TM components,

$$f_n \hat{J}_n'(k_2 a) = [d_n \hat{J}_n'(k_1 a) + p_n \hat{N}_n'(k_1 a)], \quad (\text{F.49})$$

$$f_n \hat{J}'_n(k_2 a) = R \left\{ j/\eta_1 \left[d_n \hat{J}_n(k_1 a) + p_n \hat{N}_n(k_1 a) \right] - (j/\eta_2) f_n \hat{J}_n(k_2 a) \right\}, \quad (\text{F.50})$$

$$\left[A_n \hat{J}'_n(kb) + b_n \hat{H}_n^{(2)'}(kb) \right] = \left[d_n \hat{J}'_n(k_1 b) + p_n \hat{N}'_n(k_1 b) \right], \quad (\text{F.51})$$

$$\eta_1 \left[A_n \hat{J}_n(kb) + b_n \hat{H}_n^{(2)}(kb) \right] = \eta \left[d_n \hat{J}_n(k_1 b) + p_n \hat{N}_n(k_1 b) \right]. \quad (\text{F.52})$$

Solving for the scattered fields yields

$$b_n = U_n A_n, \quad (\text{F.53})$$

$$c_n = V_n F_n, \quad (\text{F.54})$$

with

$$U_n = \frac{\eta_1 \hat{J}_n(kb) r'_n - \eta \hat{J}'_n(kb) r_n}{\eta \hat{H}_n^{(2)'}(kb) r_n - \eta_1 \hat{H}_n^{(2)}(kb) r'_n}, \quad (\text{F.55})$$

$$V_n = \frac{\eta \hat{J}_n(kb) s'_n - \eta_1 \hat{J}'_n(kb) s_n}{\eta_1 \hat{H}_n^{(2)'}(kb) s_n - \eta \hat{H}_n^{(2)}(kb) s'_n}. \quad (\text{F.56})$$

Here,

$$r_n = \beta_n \hat{N}_n(k_1 b) - \alpha_n \hat{J}_n(k_1 b), \quad (\text{F.57})$$

$$r'_n = \beta_n \hat{N}'_n(k_1 b) - \alpha_n \hat{J}'_n(k_1 b), \quad (\text{F.58})$$

$$s_n = v_n \hat{N}_n(k_1 b) - \nu_n \hat{J}_n(k_1 b), \quad (\text{F.59})$$

$$s'_n = v_n \hat{N}'_n(k_1 b) - \nu_n \hat{J}'_n(k_1 b), \quad (\text{F.60})$$

where

$$\alpha_n = \hat{N}'_n(k_1 a) - (\eta_2/\eta_1) t_n \hat{N}_n(k_1 a), \quad (\text{F.61})$$

$$\beta_n = \hat{J}'_n(k_1 a) - (\eta_2/\eta_1) t_n \hat{J}_n(k_1 a), \quad (\text{F.62})$$

$$\nu_n = \hat{N}_n(k_1 a) - (\eta_2/\eta_1) w_n \hat{N}'_n(k_1 a), \quad (\text{F.63})$$

$$v_n = \hat{J}_n(k_1 a) - (\eta_2/\eta_1) w_n \hat{J}'_n(k_1 a), \quad (\text{F.64})$$

and

$$t_n = \frac{jR\hat{J}'_n(k_2a)}{\eta_2\hat{J}'_n(k_2a) + jR\hat{J}_n(k_2a)}, \quad (\text{F.65})$$

$$w_n = \frac{-jR\hat{J}_n(k_2a)}{\eta_2\hat{J}_n(k_2a) - jR\hat{J}'_n(k_2a)}. \quad (\text{F.66})$$

The coefficients of (F.55) and (F.56) may be substituted directly into Richmond's equations for the far-zone scattered field [62], repeated here for convenience

$$E_\theta^s = \frac{E_o \cos \phi e^{-jkr}}{jkr} \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{n(n+1)} \left[\frac{V_n P_n^1}{\sin \theta} + U_n \sin \theta P_n^{1'} \right], \quad (\text{F.67})$$

$$E_\phi^s = \frac{E_o \sin \phi e^{-jkr}}{jkr} \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{n(n+1)} \left[V_n \sin \theta P_n^{1'} + \frac{U_n P_n^1}{\sin \theta} \right], \quad (\text{F.68})$$

where $P_n^1 \equiv P_n^1(\cos \theta)$ and $P_n^{1'}$ indicates differentiation with respect to $\cos \theta$.

One can readily verify that when the resistance $R \rightarrow 0$ the solution reduces to Richmond's solution [62] for the ferrite coated perfectly conducting sphere with the following substitutions into Richmond's notation:

$$t_n = w_n = 0, \quad (\text{F.69})$$

$$\alpha_n = \hat{N}'_n(k_1a), \quad (\text{F.70})$$

$$\beta_n = \hat{J}'_n(k_1a), \quad (\text{F.71})$$

$$\nu_n = \hat{N}_n(k_1a), \quad (\text{F.72})$$

$$v_n = \hat{J}_n(k_1a), \quad (\text{F.73})$$

$$r_n = \hat{J}'_n(k_1a)\hat{N}_n(k_1b) - \hat{N}'_n(k_1a)\hat{J}_n(k_1b) = R_n(k_1b), \quad (\text{F.74})$$

$$r'_n = \hat{J}'_n(k_1a)\hat{N}'_n(k_1b) - \hat{N}'_n(k_1a)\hat{J}'_n(k_1b) = R'_n(k_1b), \quad (\text{F.75})$$

$$s_n = \hat{J}_n(k_1a)\hat{N}_n(k_1b) - \hat{N}_n(k_1a)\hat{J}_n(k_1b) = S_n(k_1b), \quad (\text{F.76})$$

$$s'_n = \hat{J}_n(k_1a)\hat{N}'_n(k_1b) - \hat{N}_n(k_1a)\hat{J}'_n(k_1b) = S'_n(k_1b), \quad (\text{F.77})$$

For the special case of the resistive (thin dielectric) spherical shell with radius $r = a = b = r_1$ and $(\epsilon_1, \mu_1) = (\epsilon_2, \mu_2) = (\epsilon, \mu)$,

$$t_n = \frac{jR/\eta \hat{J}_n'(kr_1)}{\hat{J}_n'(kr_1) + jR/\eta \hat{J}_n(kr_1)}, \quad (\text{F.78})$$

$$w_n = \frac{-jR/\eta \hat{J}_n(kr_1)}{\hat{J}_n(kr_1) - jR/\eta \hat{J}_n'(kr_1)}, \quad (\text{F.79})$$

with

$$\alpha_n = \frac{\hat{N}_n'(kr_1) \hat{J}_n'(kr_1) + jR/\eta}{\hat{J}_n'(kr_1) + jR/\eta \hat{J}_n(kr_1)}, \quad (\text{F.80})$$

$$\beta_n = \frac{[\hat{J}_n'(kr_1)]^2}{\hat{J}_n'(kr_1) + jR/\eta \hat{J}_n(kr_1)}, \quad (\text{F.81})$$

$$v_n = \frac{[\hat{J}_n(kr_1)]^2}{\hat{J}_n(kr_1) - jR/\eta \hat{J}_n'(kr_1)}, \quad (\text{F.82})$$

$$\nu_n = \frac{\hat{N}_n(kr_1) \hat{J}_n(kr_1) + jR/\eta}{\hat{J}_n(kr_1) - jR/\eta \hat{J}_n'(kr_1)}. \quad (\text{F.83})$$

Also,

$$r_n = -1, \quad (\text{F.84})$$

$$r_n' = \frac{-jR/\eta \hat{J}_n'(kr_1)}{\hat{J}_n'(kr_1) + jR/\eta \hat{J}_n(kr_1)}, \quad (\text{F.85})$$

$$s_n = \frac{-jR/\eta \hat{J}_n(kr_1)}{\hat{J}_n(kr_1) - jR/\eta \hat{J}_n'(kr_1)}, \quad (\text{F.86})$$

$$s_n' = 1. \quad (\text{F.87})$$

Substituting into (F.55) and (F.56) yields

$$U_n = \frac{-[\hat{J}_n'(kr_1)]^2}{\hat{J}_n'(kr_1) \hat{H}_n^{(2)'}(kr_1) + R/\eta}, \quad (\text{F.88})$$

$$V_n = \frac{-[\hat{J}_n(kr_1)]^2}{\hat{J}_n(kr_1) \hat{H}_n^{(2)}(kr_1) + R/\eta}, \quad (\text{F.89})$$

which is the result of Andreasen [35] with

$$R/\eta = \frac{1}{j(\kappa_e - 1)kd}, \quad (\text{F.90})$$

where κ_e is the relative dielectric constant, k is the free-space wavenumber, d is the thickness of the thin shell, and noting

$$kr_1 b_n(kr_1) \equiv \hat{B}_n, \quad (\text{F.91})$$

$$\frac{d[kr_1 b_n(kr_1)]}{d(kr_1)} = \hat{B}'_n. \quad (\text{F.92})$$

The results for other degenerative spherical eigenfunction solutions may also be obtained quite easily.

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