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Construction of spherical t-designs

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The Ohio State University, 1989

U·M·I

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CONSTRUCTION OF SPHERICAL t -DESIGNS

DISSERTATION

**Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University**

By

BELA BAJNOK, M.Ed., M.S.

*** * * * ***

The Ohio State University

1989

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I. INTRODUCTION

The following concept of spherical t -designs was introduced by Delsarte, Goethals and Seidel in 1977 [4].

Let d and t be positive integers and let S^{d-1} be the unit sphere in $\mathbb{R}^d$. A finite subset X of S^{d-1} is called a spherical t -design, if and only if

$$(1.1) \quad \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) d\omega(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

holds for all polynomials $f(x)=f(x_1, x_2, \dots, x_d)$ of degree at most t . Here $|S^{d-1}|$ denotes the surface area of S^{d-1} .

There are several different but equivalent definitions of spherical t -designs. One equivalent definition is that X is a spherical t -design, if and only if

$$(1.2) \quad \sum_{x \in X} f(x) = 0$$

for all homogeneous harmonic polynomials $f(x)$ with $1 \leq \deg f(x) \leq t$, cf.[4].

In 1984 Seymour and Zaslavsky proved the existence of spherical t -designs for any t and d , but for sufficiently large n , where $n=|X|$, cf.[10]. It is unknown how large n needs to be. Delsarte, Goethals and Seidel give a necessary lower bound

$$(1.3) \quad n \geq \begin{cases} 2 \binom{\frac{t-1}{2} + d - 1}{d - 1} & \text{if } t \text{ is odd,} \\ \binom{\frac{t}{2} + d - 1}{d - 1} + \binom{\frac{t}{2} + d - 2}{d - 1} & \text{if } t \text{ is even.} \end{cases}$$

The spherical t -design X is called tight, if and only if equality holds in (1.3) with $|X|=n$. The theory of tight t -designs is well developed, Bannai and Damerell in [2] and [3] proved that tight spherical t -designs exist for $d \geq 3$ if and only if $t=1,2,3,4,5,7$ or 11 . All tight t -designs are known, except for $t=4,5$ and 7 . But since spherical designs can be used for numerical integration (and applied in physics, astronomy, etc.), it is also of interest to explicitly construct spherical designs for all possible values of t , d and n .

For $t=2$ (1.3) gives $n \geq d+1$. In 1988 Y. Mimura [8] proved that no spherical 2-design exists if d is odd and $n=d+2$, and he gave an explicit construction for the remaining cases.

For $t \geq 3$, some sporadic examples of spherical t -designs are known for specific values of t , d and n (mostly by using the orbits of a finite subgroup G of $O(d)$), but no general construction has been given.

We can easily see that if X is a spherical t -design, then X is a spherical s -design for every $s=1,2, \dots, t$. Furthermore, if t is odd and X is a spherical $(t-1)$ -design which is antipodal (i.e. $X=-X$) then X is a spherical t -design. In our case, although some of the sets constructed will not be antipodal, still, every polynomial of odd degree will vanish on them (i.e. (1.2) will automatically hold if $\deg(f)$ is odd). Therefore we will be interested in constructing spherical t -designs for $t=5,7,9, \dots$ only.

In chapter 2 of this dissertation, we will study $\text{Harm}_d(t)$, the vector space of all homogeneous, harmonic polynomials of degree t . We will be particularly interested in $\text{Harm}_d^*(t)$, a subspace of $\text{Harm}_d(t)$, formed by those polynomials of $\text{Harm}_d(t)$, that contain all variables with even exponents only.

In chapter 3, we will construct a set of size m ($m > t^{d/2}$, but arbitrary), on which all polynomials of $\text{Harm}_d(t) \setminus \text{Harm}_d^*(t)$ will vanish. After that, we restrict our attention to the relatively few polynomials of $\text{Harm}_d^*(t)$.

In chapter 4, we will give an explicit construction of spherical 5-designs for (arbitrary d and) $n \geq n_5$ (for n_5 see (4.27) and (4.58)) The constructed design will be the union of two

spherical 3-designs X_1 and X_2 , such that the polynomials of $\text{Harm}_d^*(4)$ will vanish on $X_1 \cup X_2$.

In chapter 5, we will explicitly construct spherical 7-designs, as unions of spherical 5-designs, such that all polynomials of $\text{Harm}_d^*(6)$ will vanish on the union. For the size of the design see (5.44). For the sake of being able to give the coordinates of the points explicitly, we have to assume that n is divisible by $4d(d-1)$. Also, our construction might contain repeated points.

This idea of constructing spherical t -designs seems to work for arbitrary t , but as t increases, the computations get more and more tedious.

In the last chapter, we will discuss the case $d=3$. Using t -designs on the interval $[-1,1]$ (a concept introduced by Tchebyshev), we will explicitly construct spherical t -designs on the regular sphere S^2 . The question yet to be answered is how this construction generalizes for $d \geq 4$.

II. HARMONIC POLYNOMIALS

A polynomial $f(x) \in \mathbb{R}[x_1, x_2, \dots, x_d]$ is called harmonic, if it satisfies Laplace's equation

$\Delta f(x) = 0$, where

$$(2.1) \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_d^2}.$$

The set of all homogeneous, harmonic polynomials of degree s forms a vector space $\text{Harm}_d(s)$ with

$$(2.2) \quad \dim \text{Harm}_d(s) = \binom{s+d-1}{d-1} - \binom{s+d-3}{d-1}.$$

A basis for $\text{Harm}_d(s)$ can be explicitly computed as follows.

Let $m_0, m_1, \dots, m_{d-2}$ be nonnegative integers such that

$$0 \leq m_{d-2} \leq \dots \leq m_1 \leq m_0 = s.$$

For $k=0, 1, \dots, d-3$ let $r_k = (x_{k+1}^2 + \dots + x_d^2)^{1/2}$, and define

$$(2.3) \quad g_k(x_{k+1}, \dots, x_d) = (r_k)^{m_k - m_{k+1}} \cdot C_{m_k - m_{k+1}}^{m_{k+1} + (d-2-k)/2}(x_{k+1}/r_k),$$

where C_n^v is the n -th degree Gegenbauer (or ultraspherical) polynomial defined by the generating function

$$(2.4) \quad (1 - 2xt + t^2)^{-v} = \sum_{n=0}^{\infty} C_n^v(x) t^n \quad (v \neq 0).$$

For properties of Gegenbauer polynomials see [5].

Here we will only need the case when $2v$ is an integer. Then we have

$$(2.5) \quad C_n^{p+1/2}(x) = \frac{2^p p!}{(2p)!} \frac{d^p}{dx^p} P_{n+p}(x) \quad \text{and}$$

$$(2.6) \quad C_n^{p+1}(x) = \frac{1}{2^p p!(n+p+1)} \frac{d^{p+1}}{dx^{p+1}} T_{n+p+1}(x),$$

where

$$(2.7) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

is the Legendre polynomial of degree n , and

$$(2.8) \quad T_n(x) = \operatorname{Re}[x + i(1 - x^2)^{1/2}]^n$$

is the Tchebychev polynomial of degree n .

Set $h_1(z) = \operatorname{Re}(z)$ and $h_2(z) = \operatorname{Im}(z)$ (z is a complex number).

For a fixed choice of the integers

$0 \leq m_{d-2} \leq \dots \leq m_1 \leq m_0 = s$ and $\mu = 1$ or 2 ($\mu = 1$ if $m_{d-2} = 0$), define the polynomial

$$(2.9) \quad f_{m_0, m_1, \dots, m_{d-2}, \mu}(x_1, x_2, \dots, x_d) = \\ = h_\mu \left[(x_{d-1} + ix_d)^{m_{d-2}} \right] \cdot \prod_{k=0}^{d-3} g_k(x_{k+1}, \dots, x_d).$$

Then the set

$$(2.10) \quad \Phi(s) = \{f_{m_0, m_1, \dots, m_{d-2}, \mu}\}$$

forms a basis for $\operatorname{Harm}_d(s)$. (For a complete proof see [5].)

We will be interested in a subspace of $\operatorname{Harm}_d(s)$, formed by all polynomials $f \in \operatorname{Harm}_d(s)$ containing all the variables $x_1, x_2, \dots, x_d$ with even exponents only. Denote this subspace by $\operatorname{Harm}_d^*(s)$. (Note that $\operatorname{Harm}_d^*(s) = \{0\}$, if s is odd.)

Theorem 2.1.

(1) Let $f_{m_0, m_1, \dots, m_{d-2}, \mu}(x) \in \Phi(s)$.

Then $f \in \text{Harm}_d^*(s)$ if and only if $m_0, m_1, \dots, m_{d-2}$ are all even and $\mu=1$.

(2) Let $\Phi^*(s) = \Phi(s) \cap \text{Harm}_d^*(s)$.

Then $\Phi^*(s)$ is a basis for $\text{Harm}_d^*(s)$.

Proof.

(1) It is easy to see that $C_n^V(x)$ is an even function if n is even and an odd function if n is odd. Hence $g_k(x_{k+1}, \dots, x_d)$ has all of $x_{k+2}, \dots, x_d$ with even exponents only; and x_{k+1} with even exponents only if $\deg(g_k)$ is even, and odd exponents only if $\deg(g_k)$ is odd. So $f \in \text{Harm}_d^*(s)$ if and only if $m_0 - m_1, m_1 - m_2, \dots, m_{d-3} - m_{d-2}$ and m_{d-2} are all even and $\mu=1$.

(2) Clearly $\Phi^*(s)$ is linearly independent. We have to show that it generates $\text{Harm}_d^*(s)$.

As we saw in (1) above, every $f \in \Phi'(s) = \Phi(s) \setminus \Phi^*(s)$ has some x_i with odd exponents only. Therefore no nontrivial linear combination of $\Phi'(s)$ is in $\text{Harm}_d^*(s)$.

Corollary 2.2.

Suppose s is even.

(3) Then

$$(2.11) \quad \dim \text{Harm}_d^*(s) = \binom{s/2 + d - 2}{d - 2}.$$

(4) For $s \geq 4$ we have

$$(2.12) \quad \dim \text{Harm}_d^*(s) = \sum_{k=0}^{s/2-2} \binom{d}{k+2} \binom{s/2-2}{k}.$$

Proof.

(3) There are exactly

$$(2.13) \quad \binom{(s/2 + 1) + (d - 2) - 1}{d - 2}$$

ways of choosing the even integers $0 \leq m_{d-2} \leq \dots \leq m_1 \leq m_0 = s$.

(4) We need to prove the combinatorial identity

$$(2.14) \quad \binom{s/2 + d - 2}{d - 2} = \sum_{k=0}^{s/2-2} \binom{d}{d-k-2} \binom{s/2-2}{k}.$$

Let A_1 be a set of d elements, A_2 be a set of $s/2-2$ elements, and suppose that A_1 and A_2 are disjoint. For $k=0, 1, \dots, s/2-2$, let

$$H_1(k) = \{ C_1 \mid A_1 \supseteq C_1 \text{ and } |C_1| = d-2-k \},$$

$$H_2(k) = \{ C_2 \mid A_2 \supseteq C_2 \text{ and } |C_2| = k \} \text{ and}$$

$$H = \{ C \mid A_1 \cup A_2 \supseteq C \text{ and } |C| = d-2 \}.$$

Then

$$(2.15) \quad \begin{aligned} |H| &= \binom{s/2 + d - 2}{d - 2}, \\ |H_1(k)| &= \binom{d}{d-k-2}, \\ |H_2(k)| &= \binom{s/2-2}{k}, \end{aligned}$$

and there is a one-to-one correspondence between H and

$$(2.16) \quad \bigcup_{k=0}^{s/2-2} [H_1(k) \times H_2(k)].$$

So in particular for $s=2, 4, 6$ and 8 we get

$$(2.17) \quad \dim \text{Harm}_d^*(2) = d - 1,$$

$$(2.18) \quad \dim \text{Harm}_d^*(4) = \binom{d}{2},$$

$$(2.19) \quad \dim \text{Harm}_d^*(6) = \binom{d}{2} + \binom{d}{3} \quad \text{and}$$

$$(2.20) \quad \dim \text{Harm}_d^*(8) = \binom{d}{2} + 2\binom{d}{3} + \binom{d}{4}.$$

We can also find the following bases:

$$\Psi^*(2) = \{x_i^2 - x_{i+1}^2 \mid 1 \leq i \leq d-1\},$$

$$\Psi^*(4) = \{x_i^4 - 6x_i^2x_j^2 + x_j^4 \mid 1 \leq i < j \leq d\},$$

$$\Psi^*(6) = \{x_i^6 - 15x_i^4x_j^2 + 15x_i^2x_j^4 - x_j^6 \mid 1 \leq i < j \leq d\} \cup$$

$$\cup \{2(x_i^6 + x_j^6 + x_k^6) - 15(x_i^4x_j^2 + \dots + x_j^2x_k^4) + 180x_i^2x_j^2x_k^2 \mid 1 \leq i < j < k \leq d\},$$

$$\Psi^*(8) = \{x_i^8 - 28x_i^6x_j^2 + 70x_i^4x_j^4 - 28x_i^2x_j^6 + x_j^8 \mid 1 \leq i < j \leq d\} \cup$$

$$\cup \{-x_i^8 + x_j^8 + 14(x_i^6x_j^2 - x_i^2x_j^6 + x_i^6x_k^2 - x_j^6x_k^2) + 210(x_i^2x_j^4x_k^2 - x_i^4x_j^2x_k^2) \mid$$

$$\mid 1 \leq i < j < k \leq d \text{ or } 1 \leq j < k < i \leq d\} \cup$$

$$\cup \{3(x_i^8 + x_j^8 + x_k^8 + x_p^8) - 28(x_i^6x_j^2 + \dots + x_k^2x_p^6) +$$

$$+ 210(x_i^4x_j^2x_k^2 + \dots + x_j^2x_k^2x_p^4) - 3780(x_i^2x_j^2x_k^2x_p^2) \mid 1 \leq i < j < k < p \leq d\}.$$

Now let $\Phi'(s) = \Phi(s) \setminus \Phi^*(s)$, then $\Psi(s) = \Phi'(s) \cup \Psi^*(s)$ is a basis for $\text{Harm}_d(s)$. We will use the notations:

$$(2.21) \quad F(t) = \bigcup_{s=1}^t \Psi(s), \quad \text{and}$$

$$(2.22) \quad F^*(t) = \bigcup_{s=1}^t \Psi^*(s).$$

We will associate matrices with spherical designs in the following way.

For a set $X = \{\mathbf{u}_k = (u_{1k}, u_{2k}, \dots, u_{dk}) \in \mathbb{R}^d \mid k=1, 2, \dots, n\}$ we consider the $d \times n$ matrix U with column vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

For a polynomial $f(x_1, x_2, \dots, x_d)$ we define

$$(2.23) \quad \sum_U f = \sum_{k=1}^n f(u_{1k}, \dots, u_{dk}).$$

Then X is a spherical t -design, if and only if

$$(2.24) \quad \sum_{i=1}^d u_{ik}^2 = 1, \quad \text{for } 1 \leq k \leq n \quad \text{and}$$

$$(2.25) \quad \sum_U f = 0 \quad \text{for all } f \in F(t).$$

III. A TRIGONOMETRIC LEMMA

In this section, as a first step for constructing spherical t -designs, we will construct a matrix A with

$$(3.1) \quad \sum_A f = 0 \quad \text{for all } f \in F'(t),$$

where $F'(t) = F(t) \setminus F^*(t)$.

Let $m \in \mathbb{N}$ and define the $d \times m$ matrix $A = A(d, m, t) = (a_{ik})$ as follows:

(a) if d is even, $k=1, 2, \dots, m$, and $e=0, 1, 2, \dots, d/2-1$, then

$$(3.2) \quad a_{ik} = \begin{cases} \cos(\frac{2\pi}{m} t^e k) & \text{if } i = 2e + 1 \\ \sin(\frac{2\pi}{m} t^e k) & \text{if } i = 2e + 2 \end{cases}$$

(b) if d is odd, $k=1, 2, \dots, m$, and $e=0, 1, 2, \dots, (d-1)/2-1$, then

$$(3.3) \quad a_{ik} = \begin{cases} \cos(\frac{2\pi}{m} t^e k) & \text{if } i = 2e + 1 \\ \sin(\frac{2\pi}{m} t^e k) & \text{if } i = 2e + 2 \\ (1/\sqrt{2}) \cos(\frac{2\pi}{m} \frac{m}{2} k) = (-1)^k / \sqrt{2} & \text{if } m \text{ is even and } i = d \\ 0 & \text{if } m \text{ is odd and } i = d \end{cases}$$

Lemma 3.

Let A be the $d \times m$ matrix defined above, and let $f = x_1^{t_1} x_2^{t_2} \dots x_d^{t_d} \in \mathbb{R}[x_1, x_2, \dots, x_d]$, where at least one of $t_1, \dots, t_d$ is odd and $t_1 + \dots + t_d = t$.

Suppose $m > t^{d/2}$. Then

$$(3.4) \quad \sum_A f = 0.$$

Proof.

(a) Suppose d is even.

Let $k=1,2,\dots,m$, $e=0,1,2,\dots,d/2-1$, $i=2e+1$, $j=2e+2$, $s_e=t_i+t_j$, $\alpha=\alpha_{e,k}=(2\pi/m)t^e k$.

Using the trigonometric identities

$$(3.5) \quad \begin{aligned} \sin x \sin y &= 1/2 [\cos(x-y) - \cos(x+y)], \\ \sin x \cos y &= 1/2 [\sin(x+y) + \sin(x-y)], \\ \cos x \cos y &= 1/2 [\cos(x+y) + \cos(x-y)], \end{aligned}$$

we see that

$$(3.6) \quad \begin{aligned} (a_{ik})^{t_i} (a_{jk})^{t_j} &= (\cos \alpha)^{t_i} (\sin \alpha)^{t_j} \\ &= \lambda_{e,0} \cos(0) + \lambda_{e,1} \cos(\alpha) + \lambda_{e,2} \cos(2\alpha) + \dots + \lambda_{e,s_e} \cos(s_e \alpha) + \\ &\quad + \mu_{e,1} \sin(\alpha) + \mu_{e,2} \sin(2\alpha) + \dots + \mu_{e,s_e} \sin(s_e \alpha), \end{aligned}$$

where the coefficients $\lambda_{e,0}, \dots, \lambda_{e,s_e}$ and $\mu_{e,1}, \dots, \mu_{e,s_e}$ do not depend on k .

Furthermore,

if t_j is odd, then $\lambda_{e,0} = \lambda_{e,1} = \dots = \lambda_{e,s_e} = 0$,

if t_j is even and t_i is odd, then $\mu_{e,1} = \mu_{e,2} = \dots = \mu_{e,s_e} = 0$, and $\lambda_{e,0} = \lambda_{e,2} = \dots = \lambda_{e,s_e-1} = 0$,

if both t_i and t_j are even, then $\mu_{e,1} = \mu_{e,2} = \dots = \mu_{e,s_e} = 0$, and $\lambda_{e,1} = \lambda_{e,3} = \dots = \lambda_{e,s_e-1} = 0$.

In particular,

$$(3.7) \quad \lambda_{e,0} = 0, \text{ unless both } t_i \text{ and } t_j \text{ are even.}$$

Using (3.5) again, we can write

$$(3.8) \quad f(a_{1k}, \dots, a_{dk}) = \prod_{i=1}^d (a_{ik})^{t_i} = \sum_{c=0}^{t^{d/2}} v_c \cos(c \frac{2\pi}{m} k) + \sum_{c=0}^{t^{d/2}} \omega_c \sin(c \frac{2\pi}{m} k),$$

where the coefficients $v_0, v_1, \dots, v_{t^{d/2}}$, and $\omega_0, \omega_1, \dots, \omega_{t^{d/2}}$ do not depend on k and are 0, unless c has the form $c = c_0 + c_1 t + c_2 t^2 + \dots + c_{d/2-1} t^{d/2-1}$, where

$c_e = 0, \pm 1, \pm 2, \dots, \pm s_e$. Since

$c \leq s_0 + s_1 t + \dots + s_{d/2-1} t^{d/2-1} \leq t^{d/2}$ and $m > t^{d/2}$; $c \equiv 0 \pmod{m}$ implies $c = 0$, which gives

$c_e = 0$ for all $e = 0, 1, 2, \dots, d/2 - 1$. Therefore we have

$$(3.9) \quad v_0 = \lambda_{0,0} \lambda_{1,0} \dots \lambda_{\frac{d}{2}-1,0},$$

hence $v_0 = 0$ by (3.7) and our assumption that at least one exponent is odd.

Using the identities

$$(3.10) \quad \sum_{k=1}^m \cos\left(\frac{2\pi}{m} ck\right) = \begin{cases} m & \text{if } c \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{k=1}^m \sin\left(\frac{2\pi}{m} ck\right) = 0,$$

we get

$$(3.11) \quad \sum_A f = \sum_{k=1}^m \prod_{i=1}^d (a_{ik})^{t_i} = \sum_{c=1}^{t^{d/2}} v_c \left[\sum_{k=1}^m \cos\left(\frac{2\pi}{m} ck\right) \right] + \sum_{c=1}^{t^{d/2}} \omega_c \left[\sum_{k=1}^m \sin\left(\frac{2\pi}{m} ck\right) \right] = 0.$$

(b) Let d be odd.

If m is odd, or if m is even and t_d is even, then the result follows from (a).

If m is even and t_d is odd, then

$$(3.12) \quad (a_{dk})^{t_d} = (1/\sqrt{2})^{t_d} (-1)^k = (1/\sqrt{2})^{t_d} \cos\left(\frac{2\pi}{m} \frac{m}{2} k\right),$$

and (3.9) holds again, since $m > t^{d/2}$ implies

$$(3.13) \quad \frac{m}{2} > (t-1)t^{\frac{d-1}{2}-1}.$$

According to (3.1) which follows from our Lemma, we can very well use the matrix A to construct spherical t-designs. Using the identities of (3.5), we can calculate the following:

$$(3.14) \quad \sum_A x_i^2 = \begin{cases} 0 & \text{if } d \text{ is odd, } m \text{ is odd and } i = d \\ \frac{m}{2} & \text{otherwise} \end{cases}$$

$$(3.15) \quad \sum_A x_i^4 = \begin{cases} \frac{m}{4} & \text{if } d \text{ is odd, } m \text{ is even and } i = d \\ 0 & \text{if } d \text{ is odd, } m \text{ is odd and } i = d \\ \frac{3m}{8} & \text{otherwise} \end{cases}$$

$$(3.16) \quad \sum_A x_i^2 x_j^2 = \begin{cases} \frac{m}{8} & \text{if } i = 2e + 1, j = 2e + 2, \\ & \text{where } e = 0, 1, 2, \dots, \left[\frac{d}{2}\right] - 1 \\ 0 & \text{if } d \text{ is odd, } m \text{ is odd, } 1 \leq i, j \leq d \\ & \text{and } i = d \text{ or } j = d \\ \frac{m}{4} & \text{otherwise} \end{cases}$$

$$(3.17) \quad \sum_A x_i^6 = \begin{cases} \frac{m}{4} & \text{if } d \text{ is odd, } m \text{ is even and } i = d \\ 0 & \text{if } d \text{ is odd, } m \text{ is odd and } i = d \\ \frac{5m}{16} & \text{otherwise} \end{cases}$$

$$(3.18) \quad \sum_A x_i^4 x_j^2 = \begin{cases} \frac{m}{16} & \text{if } i = 2e + 1, j = 2e + 2, \\ & \text{where } e = 0, 1, 2, \dots, \left[\frac{d}{2}\right] - 1 \\ 0 & \text{if } d \text{ is odd, } m \text{ is odd, } 1 \leq i, j \leq d \\ & \text{and } i = d \text{ or } j = d \\ \frac{3m}{16} & \text{otherwise} \end{cases}$$

$$(3.19) \quad \sum_A x_i^2 x_j^2 x_k^2 = \begin{cases} \frac{m}{16} & \text{if } i = 2e + 1, j = 2e + 2 < k \leq d, \\ & \quad d \text{ even, or } d \text{ odd and } m \text{ even;} \\ & \quad \text{or } i = 2e + 1, j = 2e + 2 < k \leq d - 1, \\ & \quad \quad d \text{ odd and } m \text{ odd;} \\ & \quad \text{or } i < j = 2e + 1, k = 2e + 2; \\ & \quad \quad \text{where } e = 0, 1, 2, \dots, \left\lfloor \frac{d}{2} \right\rfloor - 1 \\ 0 & \text{if } d \text{ is odd, } m \text{ is odd, } 1 \leq i < j < k = d \\ \frac{m}{8} & \text{otherwise} \end{cases}$$

From these formulas we can compute the values of the polynomials $f \in \Psi^*(s)$ for $s=2,4$ and 6 on the matrix A.

IV. EXPLICIT CONSTRUCTION OF SPHERICAL 4- AND 5-DESIGNS

4.1. The structure of the construction

According to section 2, we have to construct a matrix U , which satisfies both (2.24) and (2.25).

In (2.25), for $t=5$ we have $F(5)=F'(5) \cup F^*(5)$, where $F^*(5)=F^*(4)=\Psi^*(2) \cup \Psi^*(4)$, with

$$\Psi^*(2)=\{f_2(x_i, x_j)=x_i^2-x_{i+1}^2 \mid 1 \leq i \leq d-1\},$$

$$\Psi^*(4)=\{f_4(x_i, x_j)=x_i^4-6x_i^2x_j^2+x_j^4 \mid 1 \leq i < j \leq d\}.$$

To construct U , we can very well use the matrix A , defined in section 3.

From (3.1), (3.14)-(3.16) we get that if

$$(4.1) \quad m > 2^d,$$

then

$$(4.2) \quad \sum_A f = 0 \quad \text{for any } f \in F'(5)$$

$$(4.3) \quad \sum_A f_2(x_i, x_{i+1}) = \begin{cases} \frac{m}{2} & \text{if } d \text{ is odd and } m \text{ is odd and } i = d-1 \\ 0 & \text{otherwise} \end{cases}$$

$$(4.4) \quad \sum_A f_4(x_i, x_j) = \begin{cases} 0 & \text{if } i = 2e+1, j = 2e+2, e = 0, 1, 2, \dots, \left[\frac{d}{2}\right] - 1 \\ -\frac{7m}{8} & \text{if } d \text{ is odd, } m \text{ is even and } 1 \leq i < j = d \\ \frac{3m}{8} & \text{if } d \text{ is odd, } m \text{ is odd and } 1 \leq i < j = d \\ -\frac{3m}{4} & \text{otherwise} \end{cases}$$

So we see that (4.1) implies that the matrix $(2/d)^{1/2} \cdot A$ defines a spherical (2- and) 3-design if d is even, or d is odd and m is even. (If both d and m is odd, then $f_2(x_i, x_{i+1}) = x_i^2 - x_{i+1}^2$ does not vanish on A for $i=d-1$.) To construct spherical (4- and) 5-designs, we need to solve the following two problems:

(i) (4.3) and (4.4) show that the value of $f_2(x_i, x_{i+1})$ and $f_4(x_i, x_j)$ on A depends on the choice of i and j . We will construct a matrix D (with d rows) such that

$$(4.5) \quad \sum_D f = 0 \quad \text{for any } f \in F''(5),$$

where $F''(5) = F(5) - \{f_4(x_i, x_j) \mid 1 \leq i < j \leq d\}$, and

$$(4.6) \quad \sum_D f_4(x_i, x_j) = -S$$

will not depend on i, j . Roughly speaking D will have the form $D = (A \mid B)$, for some matrix B defined later.

(ii) Then we will construct a matrix C such that

$$(4.7) \quad \sum_C f = 0 \quad \text{for any } f \in F''(5), \text{ and}$$

$$(4.8) \quad \sum_C f_4(x_i, x_j) = S \quad \text{for any } 1 \leq i < j \leq d.$$

Then (4.5)-(4.8) give that the matrix $U = (D \mid C)$ satisfies (2.25).

We will construct the matrices B and C as follows.

Suppose $(h_4 =) h \in \mathbb{R}$ satisfies

$$(4.9) \quad 1/2 < h < 1.$$

Define $a, b \in \mathbb{R}$ to be the unique solutions of the following system:

$$(4.10) \quad \begin{cases} 0 < a < b \\ a^2 + b^2 = 1 \\ a^4 + b^4 = h. \end{cases}$$

(4.9) implies that a and b are well defined.

Let $x=x(h) \in \mathbb{R}^d$ denote the point with coordinates $(0,0, \dots, 0,a,b)$.

Let H be the 8-element group

$$(4.11) \quad H = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}.$$

Let H act on x by permuting and/or negating the last two (nonzero) coordinates of x . Let $X(h)=x^H$.

Let G be a subgroup of S_d , the symmetric group on d elements. Let G act on $\mathbb{R}^d$ by permuting the d coordinates of the points. Now define the matrix $M(x(h),G)$ with column vectors $X(h)^G$.

In order to avoid repeated points in our design, instead of taking each point k times ($k \in \mathbb{N}$), we will define the matrix

$$(4.12) \quad M_k(x(h),G) = (M(x(h_1),G) \mid M(x(h_2),G) \mid \dots \mid M(x(h_k),G)),$$

where

$$(4.13) \quad h_i = h + \frac{k+1-2i}{k} \delta,$$

with $\delta = \min\{h-1/2; 1-h\}$ and $i=1,2, \dots, k$.

Then (4.9) implies that $1/2 < h_i < 1$, so $M_k(x(h),G)$ is well defined, and we have

$$h_1 + h_2 + \dots + h_k = kh.$$

Let's illustrate here how we can use the matrix $M_k(x(h),G)$ to construct spherical 5-designs by addressing problem (ii) mentioned above. (To solve (i) we will have to consider the

cases d even and d odd separately.)

Let $C_k(h) = M_k(x(h), S_d)$. In this matrix in each pair of rows a_i and b_i are underneath each other exactly twice for each $i=1, 2, \dots, k$, and the matrix has size $d \times 4d(d-1)k$. We can easily check that

$$(4.14) \quad \sum_{C_k(h)} f_4(x_i, x_j) = -24k + 8(d+2)kh \quad \text{for all } 1 \leq i < j \leq d.$$

Therefore we have

Proposition 4.1.

Suppose $k \in \mathbb{N}$ and $S \in \mathbb{R}$ satisfy

$$(4.15) \quad 4(d-4)k < S < 8(d-1)k.$$

Define $C_k(h)$ as above, with

$$(4.16) \quad (h_4 =) h = \frac{S + 24k}{8k(d+2)}.$$

Then (4.15) implies (4.9), so $C = C_k(h)$ is well defined, and satisfies both (4.7) and (4.8). 🍏

4.2. Explicit construction. d even.

In this section let d be even. We will first construct the matrix $D = (A \mid B)$ of section 4.1 (i).

According to (4.2)-(4.4), if m satisfies (4.1), then we have

$$(4.17) \quad \sum_A f = 0 \quad \text{for any } f \in F''(5), \text{ and}$$

$$(4.18) \quad \sum_A f_4(x_i, x_j) = \begin{cases} 0 & \text{if } i = 2e + 1, j = 2e + 2, \\ & \text{for } e = 0, 1, 2, \dots, \frac{d}{2} - 1 \\ -\frac{3m}{4} & \text{otherwise.} \end{cases}$$

Now we will construct the matrix B explicitly as follows.

Let G be the cyclic subgroup of S_d generated by the permutation $(135\dots d-1)(246\dots d)$.

Then for $h \in \mathbb{R}$ satisfying (4.5) the matrix $M(x(h), G)$ is

$$\begin{pmatrix} a - a & a - a & b \dots - b & 0 & \dots & \dots & 0 \\ b & b - b & -b & a \dots - a & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & a \dots - b & 0 & 0 \\ 0 & \dots & \dots & 0 & b \dots - a & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & a \dots - b & \\ 0 & \dots & \dots & \dots & 0 & b \dots - a & \end{pmatrix},$$

and has size $d \times 4d$.

Let p be a positive integer, and define $B_p(h) = M_p(x(h), G)$, with G as above.

We can check easily, that

$$(4.19) \quad \sum_{B_p(h)} f = 0 \quad \text{for all } f \in F''(5), \text{ and}$$

$$(4.20) \quad \sum_{B_p(h)} f_4(x_i, x_j) = \begin{cases} -24p + 32ph & \text{if } i = 2e + 1, j = 2e + 2, \\ & \text{for } e = 0, 1, \dots, \frac{d}{2} - 1 \\ 8ph & \text{otherwise} \end{cases}$$

Now suppose we have

$$(4.21) \quad \frac{m}{p} < 4d^2.$$

Then $h \in \mathbb{R}$ defined by

$$(4.22) \quad h = 1 - \frac{m}{8d^2 p}$$

satisfies (4.9).

Therefore the $d \times (m+4dp)$ matrix

$$(4.23) \quad D = D(m, p) = \left(\sqrt{\frac{2}{d}} A \parallel B_p(h) \right)$$

is well defined, and we have the following

Proposition 4.2.1.

Let d, m, p be positive integers, d even. Suppose m and p satisfy (4.1) and (4.21). Define the matrix $D = D(m, p) = ((d_{ik}))$ as in (4.23) ($i=1, 2, \dots, d, k=1, 2, \dots, m+4dp$). Then

$$(4.24) \quad \sum_{i=1}^d d_{ik}^2 = 1 \quad \text{for all } k,$$

$$(4.25) \quad \sum_D f = 0 \quad \text{for all } f \in F'(5),$$

$$(4.26) \quad \sum_D f_4(x_i, x_j) = 8p - \frac{4m}{d^2} = -S \quad \text{for all } 1 \leq i < j \leq d.$$

This solves problem (i).

Using Propositions 4.1 and 4.2.1, we can now construct spherical 5-designs, when d is even.

Theorem 4.2.2.

Let d and n be positive integers, d even. Suppose n satisfies

$$(4.27) \quad n > \max\{2^d(d+2)/d+4(d+2)(d^2+1)/(d+1), 4(d-1)(d+2)(d^3-d^2+4)/(d^2+5d+2)\}.$$

Let

$$(4.28) \quad k = \left\lceil \frac{n}{4d(d+2)(d-1)} \right\rceil + 1,$$

$$(4.29) \quad p = \left\lceil \frac{n - 4d(d-1)k}{4d(d+1)} \right\rceil + 1,$$

$$(4.30) \quad m = n - 4d(d-1)k - 4dp.$$

Define $S \in \mathbb{R}$ with (4.26) and $h \in \mathbb{R}$ with (4.16). Then the matrix

$U = U(k, p, m) = (D(m, p) \parallel C_k(h))$ is well defined, has size $d \times n$, and defines a spherical 5-design.

Proof.

We need to show that (4.27) implies that the chosen parameters satisfy (4.1), (4.21) and (4.15), the hypotheses of Propositions 4.1 and 4.2.1, and therefore U satisfies (2.24) and (2.25).

To show (4.1):

$$\begin{aligned} m &= n - 4d(d-1)k - 4dp \\ &\geq n - 4d(d-1)k - 4d[n - 4d(d-1)k]/[4d(d+1)] - 4d \\ &= dn/(d+1) - 4d^2(d-1)k/(d+1) - 4d \\ &\geq dn/(d+1) - 4d^2(d-1)n/[4d(d+2)(d-1)(d+1)] - 4d^2(d-1)/(d+1) - 4d \\ &= dn/(d+2) - 4d(d^2+1)/(d+1) \\ &> d(d+2)2^d/d(d+2) + 4d(d+2)(d^2+1)/[(d+1)(d+2)] - 4d(d^2+1)/(d+1) \\ &= 2^d. \end{aligned}$$

To show (4.21):

$$\begin{aligned} m/p &= [n - 4d(d-1)k]/p - 4d \\ &< 4d(d+1)[n - 4d(d-1)k]/[n - 4d(d-1)k] - 4d \\ &= 4d^2. \end{aligned}$$

To show (4.15):

$$\begin{aligned} S/k &= (4m - 8d^2p)/(d^2k) \\ &= 4n/(d^2k) - 8(d+2)p/(dk) - 16(d-1)/d. \end{aligned}$$

(1) To show $S/k < 8(d-1)$, we approximate from above:

$$\begin{aligned} S/k &< 4n/(d^2k) - 8(d+2)[n - 4d(d-1)k]/[4d(d+1)k] - 16(d-1)/d \\ &= 2n/[d(d+1)k] - 8(d-1)/(d+1) \\ &< 8d(d+2)(d-1)/[d(d+1)] - 8(d-1)/(d+1) \\ &= 8(d-1). \end{aligned}$$

(2) To show $S/k > 4(d-4)$, we approximate from below:

$$\begin{aligned} S/k &\geq 4n/(d^2k) - 8(d+2)\{[n - 4d(d-1)k]/[4d(d+1)k] + 1/k\} - 16(d-1)/d \\ &= [2n - 8(d+1)(d+2)]/[d(d+1)k] - 8(d-1)/(d+1) \\ &\geq [2n - 8(d+1)(d+2)]4d(d+2)(d-1)/\{[n + 4d(d+2)(d-1)]d(d+1)\} - 8(d-1)/(d+1) \\ &= 8(d^2 + d - 1)/(d+1) - 32(d+2)^2(d^2 + 1)(d-1)/\{[n + 4d(d+2)(d-1)](d+1)\}, \end{aligned}$$

and since $n > 4(d-1)(d+2)(d^3 - d^2 + 4)/(d^2 + 5d + 2)$ says that the second term is smaller than $4(d^2 + 5d + 2)/(d+1)$ in absolute value, we get

$$S/k > 8(d^2 + d - 1)/(d+1) - 4(d^2 + 5d + 2)/(d+1) = 4(d-4).$$

We can also see that the design has no repeated points. 🍏

4.3. Explicit construction, d odd

Although similar to section 4.2, the case when d is odd is more complicated. We need to handle the following two special cases first:

(A) n is odd and $d \nmid n$,

(B) n is even.

Then we can easily combine (A) and (B) to get a construction for a general n .

(4.4) tells us that in both cases $f_4(x_i, x_j)$ can have three different values on A , depending on i and j . So to solve problem (i) of section 4.1 we will need to do two steps.

(A) Suppose n is odd and $n=dn'$ for some $n' \in \mathbb{N}$.

Let m be an odd integer. Then the $d \times m$ matrix $A=A(d,m,5)$ defined with (3.3) has only zeros in its last row. Define the matrices $A_1, A_2, \dots, A_d$ by permuting the rows of A cyclically, i.e. if the row vectors of A are $a_1, a_2, \dots, a_d$, then define $A_1=A$ and A_i with row vectors $a_i, a_{i+1}, \dots, a_d, a_1, \dots, a_{i-1}$ ($i=2,3,\dots,d$).

Define the $d \times dm$ matrix $A'=(A_1 | A_2 | \dots | A_d)$.

For $1 \leq i < j \leq d$, let $A'_{i,j}$ be the $2 \times dm$ matrix, consisting of the i -th and j -th rows of A' . Then for each $e=0,1,2, \dots, (d-1)/2-1$, the rows a_{2e+1} and a_{2e+2} appear in $A'_{i,j}$ underneath each other exactly ϵ times; where $\epsilon=1$, if i and j are consecutive (i.e. $i=1,2, \dots, d-1$ and $j=i+1$, or $i=d$ and $j=1$), and $\epsilon=0$ otherwise.

Therefore, from (4.2), (4.3) and (4.4) we get

Proposition 4.3.1.

Suppose m is an odd integer, satisfying

$$(4.31) \quad m > 2^{d-1}.$$

Then

$$(4.32) \quad \sum_{A'} f = 0 \quad \text{for any } f \in F''(5),$$

$$(4.33) \quad \sum_{A'} f_4(x_i, x_j) = \begin{cases} \frac{d-3}{2} \left(-\frac{3m}{4} \right) + 2\frac{3m}{8} = -\frac{3(d-5)}{8}m & \text{if} \\ i = 1, 2, \dots, d-1 \text{ and } j = i+1, \text{ or } i = d \text{ and } j = 1 \\ (d-2) \left(-\frac{3m}{4} \right) + 2\frac{3m}{8} = -\frac{3(d-3)}{4}m & \text{otherwise} \end{cases}$$

Now we will define the matrix B' as follows.

Let G' be the cyclic subgroup of S_d generated by the permutation $(123\dots d)$.

For $h \in \mathbb{R}$ satisfying (4.9) the matrix $M(x(h), G')$ is

$$\begin{pmatrix} a - a & a - a & b \dots - b & 0 & \dots & \dots & \dots & \dots & \dots & 0 & b \dots - a \\ b & b - b - b & a \dots - a & a \dots - b & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & b \dots - a & a \dots - b & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & b \dots - a & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & a \dots - b & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & b \dots - a & a \dots - b & \dots & \dots \end{pmatrix},$$

and has size $d \times 8d$.

Let p be a positive integer, and define $B'_p(h) = M_p(x(h), G')$, with G' as above.

We can check easily, that

$$(4.34) \quad \sum_{B'_p(h)} f = 0 \quad \text{for all } f \in F''(5), \text{ and}$$

$$(4.35) \quad \sum_{B'_p(h)} f_4(x_i, x_j) = \begin{cases} -24p + 40ph & \text{if } i = 1, 2, \dots, d-1 \text{ and } j = i+1 \\ & \text{or } i = d \text{ and } j = 1 \\ 16ph & \text{otherwise} \end{cases}$$

Now suppose $p \in \mathbb{N}$ satisfies

$$(4.36) \quad \frac{m}{p} < 8(d-1).$$

Then $h \in \mathbb{R}$ defined by

$$(4.37) \quad h = 1 - \frac{m}{16(d-1)p}$$

satisfies (4.9).

Therefore the $d \times (md + 8dp)$ matrix

$$(4.38) \quad D' = D'(m, p) = \left(\sqrt{\frac{2}{d-1}} A' \parallel B'_p * (h) \right)$$

is well defined, and we have

Proposition 4.3.2.

Let d, m, p be positive integers, d odd. Suppose m and p satisfy (4.31) and (4.36). Define the matrix $D' = D'(m, p) = ((d'_{ik}))$ as in (4.38) ($i=1, 2, \dots, d, k=1, 2, \dots, dm+8dp$). Then

$$(4.39) \quad \sum_{i=1}^d d'^2_{ik} = 1 \quad \text{for all } k,$$

$$(4.40) \quad \sum_{D'} f = 0 \quad \text{for all } f \in F''(5),$$

$$(4.41) \quad \sum_{D'} f_4(x_i, x_j) = 16p - \frac{2(2d-5)}{(d-1)^2} m = -S \quad \text{for all } 1 \leq i < j \leq d.$$

This answers problem (i). We will take care of problem (ii) with the same matrix $C_k(h)$ as in Proposition 4.1 (but with different k and h).

Theorem 4.3.3.

Let d and n be integers, d odd. Suppose $n = dn'$, where n' is also an odd integer, which satisfies

$$(4.42) \quad n' > \max \left\{ \frac{d^2-4}{(d-1)^2} 2^{d-1} + \frac{4(d^2-4)(d^2+1)}{d(d-1)^2}, \frac{4(d^2-4)(d^3-2d^2+d-4)}{d(d-4)(d-1)} \right\}.$$

Let

$$(4.43) \quad k = \left\lceil \frac{(d-4)n'}{4(d-1)(d^2-4)} \right\rceil + 1,$$

$$(4.44) \quad p = \left\lceil \frac{n' - 4(d-1)k}{8d} \right\rceil + 1,$$

$$(4.45) \quad m = n' - 4(d-1)k - 8p.$$

Define $S \in \mathbb{R}$ with (4.41) and $h \in \mathbb{R}$ with (4.16). Then the matrix

$U' = U'(k, p, m) = (D'(m, p) \parallel C_k(h))$ is well defined, has size $d \times n$, and defines a spherical 5-design.

Proof.

We see that m is odd, since n' is odd. We have to show that (4.42) implies that the chosen parameters satisfy (4.31), (4.36) and (4.15), the hypotheses of Propositions 4.1 and 4.3.2, and therefore U' satisfies (2.24) and (2.25).

To show (4.31):

$$\begin{aligned} m &= n' - 4(d-1)k - 8p \\ &\geq n' - 4(d-1)k - [n' - 4(d-1)k]/d - 8 \\ &= (d-1)n'/d - 4(d-1)^2k/d - 8 \\ &\geq (d-1)n'/d - 4(d-1)^2(d-4)n'/[4d(d-1)(d-2)(d+2)] - 4(d-1)^2/d - 8 \\ &= (d-1)^2n'/[(d^2-4) - 4(d^2-4d+1)/d] \\ &> (d-1)^2\{(d^2-4)2^{d-1}/(d-1)^2 + 4(d^2-4)(d^2+1)/[d(d-1)^2]\}/(d^2-4) - 4(d^2+1)/d \\ &= 2^{d-1}. \end{aligned}$$

To show (4.36):

$$\begin{aligned} m/p &= [n' - 4(d-1)k]/p - 8 \\ &< 8d[n' - 4(d-1)k]/[n' - 4(d-1)k] - 8 \\ &= 8(d-1). \end{aligned}$$

To show (4.15):

$$\begin{aligned} S/k &= [2(2d-5)m - 16(d-1)^2p]/(d-1)^2k \\ &= 2(2d-5)n'/[(d-1)^2k] - 16(d^2-4)p/[(d-1)^2k] - 8(2d-5)/(d-1). \end{aligned}$$

(1) To show $S/k < 8(d-1)$, we approximate from above:

$$\begin{aligned}
S/k &< 2(2d-5)n'/[(d-1)^2k] - 16(d^2-4)[n'-4(d-1)k]/[8d(d-1)^2k] - 8(2d-5)/(d-1) \\
&= 2(d-4)n'/[d(d-1)k] - 8(d-4)/d \\
&< 8(d^2-4)/d - 8(d-4)/d \\
&= 8(d-1).
\end{aligned}$$

(2) To show $S/k > 4(d-4)$, we approximate from below:

$$\begin{aligned}
S/k &\geq 2(2d-5)n'/[(d-1)^2k] - 16(d^2-4)[n'-4(d-1)k]/[8d(d-1)^2k] - 16(d^2-4)/[(d-1)^2k] \\
&\quad - 8(2d-5)/(d-1) \\
&= [2(d-1)(d-4)n' - 16d(d^2-4)]/[d(d-1)^2k] - 8(d-4)/d \\
&\geq 4(d-1)(d^2-4)[2(d-1)(d-4)n' - 16d(d^2-4)]/\{d(d-1)^2[(d-4)n' + 4(d-1)(d^2-4)]\} \\
&= 8(d-1) - 32(d^2-4)^2(d^2+1)/\{d(d-1)[(d-4)n' + 4(d-1)(d^2-4)]\},
\end{aligned}$$

and since $n > 4(d^2-4)(d^3-2d^2+d-4)/[d(d-1)(d-4)]$ says that the second term is smaller than $4(d+2)$ in absolute value, we get

$$S/k > 8(d-1) - 4(d+2) = 4(d-4).$$

We can also see that the design has no repeated points. 🍏

(B) Now suppose that d is odd and n is even.

As in (A), $f_4(x_i, x_j)$ takes three different values on A , depending on i and j . First we will construct a matrix A'' , on which $f_4(x_i, x_j)$ takes only two different values. Then we will construct the matrices B'' and C'' so that $U'' = (A'' \mid B'' \mid C'')$ will satisfy (2.25).

Let K_{d-1} be the complete graph on the $d-1$ vertices $\{1, 2, \dots, d-1\}$. Since $d-1$ is even, K_{d-1} can be decomposed into $d-2$ perfect matchings: $P_1, P_2, \dots, P_{d-2}$. List the edges in $P_1, P_2, \dots, P_{d-2}$ in any order. Without loss of generality assume that the edge set of P_1 is $E(P_1) = \{(1, 2), (3, 4), \dots, (d-2, d-1)\}$, where (i, j) denotes an edge of K_{d-1} .

For $P = \{(i_1, i_2), \dots, (i_{d-2}, i_{d-1})\} \in \{P_1, P_2, \dots, P_{d-2}\}$ define

$$\pi(P) = \begin{pmatrix} 1 & 2 & \dots & d-1 & d \\ i_1 & i_2 & \dots & i_{d-1} & d \end{pmatrix},$$

a permutation in S_d , the symmetric group on p elements (the last element is fixed). Then $\pi(P_1)$ is the identity.

Let m be an even integer, satisfying (4.1). Let $a_1, a_2, \dots, a_d$ be the row vectors of the $d \times m$ matrix A defined with (3.3). Define the matrices $A_1, A_2, \dots, A_{d-2}$ to be row permutations of A determined by $\pi(P_1), \pi(P_2), \dots, \pi(P_{d-2})$.

Now define the $d \times (d-2)m$ matrix $A'' = (A_1 | A_2 | \dots | A_{d-2})$.

For $1 \leq i < j \leq d-1$, let $A''_{i,j}$ be the $2 \times (d-2)m$ matrix, consisting of the i -th and j -th rows of A'' . Then for each $e=0,1,2, \dots, (d-1)/2-1$, the rows a_{2e+1} and a_{2e+2} appear in $A''_{i,j}$ underneath each other exactly once.

Therefore, from (4.2), (4.3) and (4.4) we get

Proposition 4.3.4.

Suppose m is an even integer satisfying (4.1). Then

$$(4.46) \quad \sum_{A''} f = 0 \quad \text{for any } f \in F''(5), \text{ and}$$

$$(4.47) \quad \sum_{A''} f_4(x_i, x_j) = \begin{cases} -\frac{3m}{4}(d-3) & \text{if } 1 \leq i < j \leq d-1 \\ -\frac{7m}{8}(d-2) & \text{if } 1 \leq i < j = d \end{cases}$$

Now we will define the matrix B'' as follows.

Let G'' be the cyclic subgroup of S_d generated by the permutation $(123\dots d-1)$.

Then for $h_1 \in \mathbb{R}$ satisfying (4.9) the matrix $M(x(h_1), G'')$ is

$$\begin{pmatrix} a & -a & a & -a & b & \dots & -b & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & a & \dots & -b & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & a & \dots & -b & \dots \\ b & b & -b & -b & a & \dots & -a & b & \dots & -a & \dots & b \dots \dots -a \end{pmatrix},$$

and has size $d \times 8(d-1)$.

Let p be a positive integer, and define $B''_p(h_1) = M_p(x(h_1), G'')$, with G'' as above.

We can check easily, that

$$(4.48) \quad \sum_{B''_p(h_1)} f = 0 \quad \text{for all } f \in F'(5),$$

$$(4.49) \quad \sum_{B''_p(h_1)} f_2(x_i, x_{i+1}) = \begin{cases} 0 & \text{for } i = 1, 2, \dots, d-2 \\ -4(d-2)p & \text{for } i = d-1, \end{cases}$$

$$(4.50) \quad \sum_{B''_p(h_1)} f_4(x_i, x_j) = \begin{cases} 8h_1p & \text{if } 1 \leq i < j \leq d-1 \\ -24p + 4(d+6)h_1p & \text{if } 1 \leq i < j = d. \end{cases}$$

Finally, for $h_2 \in \mathbb{R}$ satisfying (4.9), define the $d \times 4(d-1)(d-2)p$ matrix

$$C''_p(h_2) = \begin{pmatrix} C_p(h_2) \\ \hline 0 \dots \dots \dots 0 \end{pmatrix},$$

where $C_p(h_2)$ was defined in section 4.1 to be $M_p(x(h_2), S_{d-1})$, of size $(d-1) \times 4(d-1)(d-2)$.

We can check easily, that

$$(4.51) \quad \sum_{C''_p(h_2)} f = 0 \quad \text{for all } f \in F'(5),$$

$$(4.52) \quad \sum_{C''_p(h_2)} f_2(x_i, x_{i+1}) = \begin{cases} 0 & \text{for } i = 1, 2, \dots, d-2 \\ 4(d-2)p & \text{for } i = d-1, \end{cases}$$

and using (4.14):

$$(4.53) \quad \sum_{C''_p(h_2)} f_4(x_i, x_j) = \begin{cases} -24p + 8(d+1)h_2p & \text{if } 1 \leq i < j \leq d-1 \\ 4(d-2)h_2p & \text{if } 1 \leq i < j = d. \end{cases}$$

Now we will combine the matrices A'' , B'' and C'' to construct spherical 5-designs.

Proposition 6.

Let m and p be positive integers, m even, satisfying (4.1). Suppose

$$(4.54) \quad 4d^2(d-4)/(3d-10) < m/p < 2d^2(d-1)/(d-2).$$

Let

$$(4.55) \quad h_1 = \frac{3}{d+2} + \frac{4d-8}{8d^2(d+2)} \frac{m}{p}, \text{ and} \\ h_2 = \frac{3}{d+2} + \frac{3d-10}{8d^2(d+2)} \frac{m}{p}.$$

Then the matrix

$$(4.56) \quad U'' = U''(m, p) = \left(\sqrt{\frac{2}{d}} A'' \mid B''_p(h_1) \mid C''_p(h_2) \right)$$

is well defined, has size $d \times [(d-2)m + 4d(d-1)p]$, and defines a spherical 5-design.

Proof.

(4.54) implies that with $D_1 = 3d-10$, $D_2 = 4(d-2)$ and $v=1$ or 2 ,

$4d^2(d-4)/D_v < m/p < 8d^2(d-1)/D_v$, hence

$4d^2(d-4) + 24d^2 < 24d^2 + D_v m/p < 8d^2(d-1) + 24d^2$, from which we get

$1/2 < h_v < 1$, so $B''_p(h_1)$ and $C''_p(h_2)$ are well defined. U'' clearly satisfies (2.24).

Using (4.46), (4.48), (4.49), (4.51) and (4.52), every $f \in F''(5)$ vanishes on U'' . Finally, using (4.47), (4.50) and (4.53) we can check that for h_1 and h_2 defined above $f_4(x_i, x_j)$ is also 0 on U'' :

(1) For $1 \leq i < j \leq d-1$:

$$-3(d-3)m/d^2 + 8h_1p - 24p + 8(d+1)h_2p = 0, \text{ and}$$

(2) For $1 \leq i < j = d$:

$$-7(d-2)m/(2d^2) - 24p + 4(d+6)h_1p + 4(d-2)h_2p = 0.$$

Therefore U'' also satisfies (2.25), hence defines a spherical 5-design. 🍏

Theorem 4.3.6.

Let d and n be integers, d odd and n even. Suppose

$$(4.57) \quad n > \max \left\{ \frac{d^2 - 2}{d} 2^d + 4(d-1)(d^2 - 2); \frac{4(d^2 - 2)(d^3 - 3d^2 - 5d + 10)}{d-2} \right\}.$$

Let p be the unique integer in the interval

$$\left(\frac{n}{2d^3 - 4d}, \frac{n}{2d^3 - 4d} + d - 2 \right],$$

for which $n \equiv 4d(d-1)p \pmod{d-2}$, and let $m = [n - 4d(d-1)p]/(d-2)$.

Then the matrix $U'' = U''(m, p)$ defined with (4.56) is well defined, has size $d \times n$, and defines a spherical 5-design.

Proof.

Since $4d(d-1)$ and $d-2$ are relatively prime, p is well defined. Also, m is even, since n is even. We have to show that the chosen parameters satisfy (4.1) and (4.54), the hypotheses of Proposition 4.3.5.

To show (4.1):

$$m = [n - 4d(d-1)p]/(d-2)$$

$$\begin{aligned}
&\geq \{n-4d(d-1)[n/(2d^3-4d)+d-2]\}/(d-2) \\
&= dn/(d^2-2)-4d(d-1) \\
&> d(d^2-2)2^d/[d(d^2-2)]+4d(d-1)(d^2-2)/(d^2-2)-4d(d-1) \\
&= 2^d.
\end{aligned}$$

To show (4.54):

$$\begin{aligned}
m/p &= [n-4d(d-1)p]/[(d-2)p] \\
&= n/[(d-2)p]-4d(d-1)/(d-2).
\end{aligned}$$

(1) To show that $m/p < 2d^2(d-1)/(d-2)$, we approximate from above:

$$\begin{aligned}
m/p &< 2d(d^2-2)/(d-2)-4d(d-1)/(d-2) \\
&= 2d^2 \\
&< 2d^2(d-1)/(d-2).
\end{aligned}$$

(2) To show that $m/p > 4d^2(d-4)/(3d-10)$, we approximate from below:

$$\begin{aligned}
m/p &\geq n/\{[n/(2d^3-4d)+d-2](d-2)\}-4d(d-1)/(d-2) \\
&= 2d^2-4d^2(d^2-2)^2/[n+2d(d-2)(d^2-2)],
\end{aligned}$$

and since $n > 4(d^2-2)(d^3-3d^2-5d+10)/(d-2)$ says that the second term is smaller than $2d^2(d-2)/(3d-10)$ in absolute value, we get that

$$m/p > 2d^2-2d^2(d-2)/(3d-10) = 4d^2(d-4)/(3d-10).$$

We can also see that the design has no repeated points. 🍏

(C) Now we can combine (A) and (B) to construct spherical 5-designs for a general n (d is odd). We will define the matrix U as follows.

(1) If n is even, satisfying (4.57), let $U=U''$, as defined in (B).

(2) If n is odd, let n' be the smallest odd integer, satisfying (4.42), and let $n''=n-dn'$. Then n'' is even, and suppose also that n'' satisfies (4.57). Define $U=(U' \mid U'')$, where U' is the $d \times (dn')$ matrix of (A), and U'' is the $d \times n''$ matrix of (B).

Theorem 4.3.7.

Let d and n be positive integers, d odd. Suppose n satisfies

$$(4.58) \quad n > \max \left\{ \frac{d^2 - 2}{d} 2^d + 4(d-1)(d^2 - 2); \frac{4(d^2 - 2)(d^3 - 3d^2 - 5d + 10)}{d - 2} \right\} + \\ + \max \left\{ \frac{d^2 - 4}{(d-1)^2} 2^{d-1} + \frac{4(d^2 - 4)(d^2 + 1)}{d(d-1)^2}; \frac{4(d^2 - 4)(d^3 - 2d^2 + d - 4)}{d(d-4)(d-1)} \right\}.$$

Then the matrix U defined above is well defined, has size $d \times n$, and gives a spherical 5-design.

V. EXPLICIT CONSTRUCTION OF SPHERICAL 6- AND 7-DESIGNS

5.1. The structure of the construction

For $t=7$ we have $F(7)=F'(7) \cup F^*(7)$, where $F^*(7)=F^*(6)=\Psi^*(2) \cup \Psi^*(4) \cup \Psi^*(6)$, with

$$\Psi^*(2)=\{f_2(x_i, x_{i+1})=x_i^2-x_{i+1}^2 \mid 1 \leq i \leq d-1\},$$

$$\Psi^*(4)=\{f_4(x_i, x_j)=x_i^4-6x_i^2x_j^2+x_j^4 \mid 1 \leq i < j \leq d\},$$

$$\begin{aligned} \Psi^*(6)=\{x_i^6-15x_i^4x_j^2+15x_i^2x_j^4-x_j^6 \mid 1 \leq i < j \leq d\} \cup \{f_6(x_i, x_j, x_k)= \\ =2(x_i^6+x_j^6+x_k^6)-15(x_i^4x_j^2+ \dots +x_j^2x_k^4)+180x_i^2x_j^2x_k^2 \mid 1 \leq i < j < k \leq d\}. \end{aligned}$$

The value of f_2 and f_4 on A was given in (4.3) and (4.4). For simplicity let m be odd if d is odd, then from (3.17)-(3.19) for $f_6(x_i, x_j, x_k)$ we get

$$(5.1) \quad \sum_A f_6(x_i, x_j, x_k) = \begin{cases} 0 & \text{if } i = 2e + 1, j = 2e + 2 < k \leq d \text{ if } d \text{ even} \\ & \text{or } i = 2e + 1, j = 2e + 2 < k \leq d - 1 \text{ if } d \text{ odd} \\ & \text{or } i < j = 2e + 1, k = 2e + 2 \\ & \text{where } e = 0, 1, 2, \dots, \left\lfloor \frac{d}{2} \right\rfloor - 1 \\ -\frac{5m}{8} & \text{if } d \text{ is odd and } i = 2e + 1, j = 2e + 2, k = d \\ & \text{where } e = 0, 1, 2, \dots, \left\lfloor \frac{d}{2} \right\rfloor - 1 \\ -\frac{15m}{8} & \text{if } d \text{ is odd and } k = d \text{ but } i \text{ and } j \text{ are not paired} \\ \frac{15m}{2} & \text{otherwise} \end{cases}$$

(i) According to (4.3), (4.4) and (5.1), the values of $f_2(x_i, x_{i+1})$, $f_4(x_i, x_j)$ and $f_6(x_i, x_j, x_k)$ depend heavily on the choice of the indices i, j and k . We were able to overcome this

difficulty for f_2 and f_4 by solving problem (i) of section 4.1. Here we will use the brutal method of permuting all rows of A . Addressing the problem this way works for arbitrary t . So define the $d \times m$ matrices $A_1, A_2, \dots, A_{d!}$ to be the $d!$ different row permutations of A , and let $A^* = \gamma(A_1 | A_2 | \dots | A_{d!})$, where $\gamma = (2/d)^{1/2}$ if d is even, and $\gamma = [2/(d-1)]^{1/2}$ if d is odd.

Proposition 5.1.

Let d and m be positive integers with m odd if d odd, satisfying

$$(5.2) \quad m > 6^{d/2}.$$

Define $F'(7) = F(7) \setminus \{f_4(x_i, x_j) \mid 1 \leq i < j \leq d\} \setminus \{f_6(x_i, x_j, x_k) \mid 1 \leq i < j < k \leq d\}$. Then

$$(5.3) \quad \sum_{A^*} f = 0 \quad \text{for any } f \in F'(7)$$

$$(5.4) \quad \sum_{A^*} f_4(x_i, x_j) = \begin{cases} -\frac{3(d-2)(d-2)!}{d} m & \text{if } d \text{ is even} \\ -\frac{3(2d-7)(d-2)!}{2(d-1)} m & \text{if } d \text{ is odd} \end{cases}$$

$$(5.5) \quad \sum_{A^*} f_6(x_i, x_j, x_k) = \begin{cases} \frac{60(d-4)(d-2)!}{d^2} m & \text{if } d \text{ is even} \\ \frac{15(4d^2 - 35d + 68)(d-3)!}{(d-1)^2} m & \text{if } d \text{ is odd.} \end{cases}$$

Proof.

(5.3): Lemma 3 implies that every $f \in F'(7)$ is 0 on A , hence on A^* . Clearly, every antisymmetrical polynomial also vanishes on A^* . Therefore all $f \in F'(7)$ is 0 on A^* .

(5.4): We will say that the indices i and j are paired, if $i=2e+1, j=2e+2$ for some $e=0,1,\dots,[d/2]-1$. We will use (4.4). If i and j are paired, then the value of f_4 is 0 on A .

(a) If d is even, then i and j are not paired $d(d-2)(d-2)!$ times in A^* , in which case f_4 takes $-3m/d^2$ on A .

(b) If d is odd, then i and j are not paired $(d-1)(d-3)(d-2)!$ times, in which case f_4 is $-3m/(d-1)^2$ on A ; and $j=d$ $(d-1)!$ times, when f_4 takes $3m/[2(d-1)^2]$ on A .

(5.5): We will use (5.1).

(a) If d is even, then none of the indices i, j, k are paired $d(d-2)(d-4)(d-3)!$ times, in which case f_6 takes $60m/d^3$ on A . If two of them are paired, then the value is 0.

(b) If d is odd, then the value of f_6 on A^* is

$$\begin{aligned} & (d-1)(d-3)(d-5)(d-3)! [60m/(d-1)^3] + 3(d-1)(d-3)! [-5m/(d-1)^3] + \\ & + 3(d-1)(d-3)(d-3)! [-15m/(d-1)^3] = \\ & = 15m(4d^2 - 35d + 68)(d-3)!/(d-1)^2. \end{aligned}$$

(ii) The second step is to use the matrix $C_k(h)$ of 4.1 to construct the matrix

$V = (A^* \mid C_k(h))$, such that f_4 will be 0 on V , and V will give a spherical 5-design. To construct $C_k(h)$ we will use Proposition 4.1. If we choose k to satisfy (4.15), where

$$(5.6) \quad S = - \sum_{A^*} f_4(x_i, x_j) = \begin{cases} \frac{3d(d-2)(d-2)!}{d} m & \text{if } d \text{ is even} \\ \frac{3(2d-7)(d-2)!}{2(d-1)} m & \text{if } d \text{ is odd} \end{cases}$$

then h defined by (4.16) satisfies (4.9), hence $C = C_k(h)$ is well defined, and f_4 vanishes on V .

(iii) Finally we will construct a matrix $M_p(h_6)$, such that f_6 will vanish on the matrix

$U = (V \mid M_p(h_6))$, and U will define a spherical 7-design. The entries of $M_p(h_6)$ will be explicitly given if we allow repeated points in our design, and the existence of such matrix will be proven only if we do not.

The matrix $M_p(h_6)$ will be constructed as follows.

Let h_6 be a positive real number, and suppose that the following system has a solution:

$$(5.7) \quad \begin{cases} a_1^2 + a_2^2 + \dots + a_d^2 = 1 \\ a_1^4 + a_2^4 + \dots + a_d^4 = h_4 = \frac{3}{d+2} \\ a_1^6 + a_2^6 + \dots + a_d^6 = h_6. \end{cases}$$

(Note that $1/d \leq h_4 \leq 1$, so the first two equations have a common solution.)

For a fixed nonnegative solution define $x = x(h_6) = (a_1, a_2, \dots, a_d) \in \mathbb{R}^d$.

Define the 2^d -element group H by

$$(5.8) \quad H = \left\{ \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \pm 1 \end{pmatrix} \right\}.$$

Let H act on $\mathbb{R}^d$ by negating (some of) the coordinates of the points. Let S_d , the symmetric group on d elements, act on $\mathbb{R}^d$ by permuting the coordinates of the points. Then define the $d \times 2^d d!$ matrix $M = M(x(h_6)) = M(h_6)$ with column vectors $(x^h)^g$, for each $h \in H$ and $g \in S_d$.

Finally, if p is a positive integer, then let $M_p(h_6) = (M(h_6) \mid M(h_6) \mid \dots \mid M(h_6))$ (with p blocks).

Then we have

$$(5.9) \quad \sum_M f_4 = 2^{d+1} (d-2)! [-3 + (d+2)h_4], \quad \text{and}$$

$$(5.10) \quad \sum_M f_6 = 3 \cdot 2^{d+1} (d-3)! [30 - 15(d+4)h_4 + (d+4)(d+8)h_6].$$

So if we choose $h_4 = 3/(d+2)$ as in (5.7), then f_4 vanishes on M . Furthermore, if we let

$$(5.11) \quad T = - \sum_V f_6 = - \sum_{A^*} f_6 - \sum_{C_k(h)} f_6,$$

and choose $p \in \mathbb{N}$ and $h_6 \in \mathbb{R}$ such that

$$(5.12) \quad h_6 = \frac{15}{(d+2)(d+4)} + \frac{T}{3 \cdot 2^{d+1} (d+4)(d+8)(d-3)!p},$$

then f_6 will vanish on U , and U will define a spherical 7-design. We will also be able to find an explicit solution for (5.7) with h_6 as in (5.12).

5.2. Explicit construction

First we will determine a sufficient condition for h_6 with which the system of equations in (5.7) has a solution, and we will construct such a solution explicitly as follows. We will have 3 cases depending on the remainder when d divided by 3. Easy calculation proves

Proposition 5.2.1.

Suppose that $d > 2$ and define

$$(1) \quad x_1 = x_2 = \dots = x_{(d+2)/3} = 3/(d+2), \text{ and } x_{(d+5)/3} = \dots = x_d = 0, \text{ if } d \equiv 1 \pmod{3},$$

$$(0) \quad x_1 = \dots = x_{(d-3)/3} = 3/(d+3), \quad x_{d/3} = 3/(d+3) + r_0, \quad x_{(d+3)/3} = 3/(d+3) - r_0, \text{ and} \\ x_{d/3+2} = \dots = x_d = 0, \quad \text{if } d \equiv 0 \pmod{3},$$

$$(2) \quad x_1 = \dots = x_{(d-2)/3} = 3/(d+4), \quad x_{(d+1)/3} = 3/(d+4) + r_2, \quad x_{(d+4)/3} = 3/(d+4) - r_2, \text{ and} \\ x_{(d+7)/3} = \dots = x_d = 0, \quad \text{if } d \equiv 2 \pmod{3},$$

where $r_0 = [3/\{2(d+2)(d+3)\}]^{1/2}$ and $r_2 = [3/\{(d+2)(d+4)\}]^{1/2}$.

Then all x_i 's are nonnegative,

$$x_1 + x_2 + \dots + x_d = 1,$$

$$x_1^2 + x_2^2 + \dots + x_d^2 = 3/(d+2),$$

and

$$x_1^3 + x_2^3 + \dots x_d^3 = \begin{cases} \frac{9}{(d+2)^2} & \text{if } d \equiv 1 \pmod{3} \\ \frac{9(d+5)}{(d+3)^2(d+2)} & \text{if } d \equiv 0 \pmod{3} \\ \frac{9(d+8)}{(d+4)^2(d+2)} & \text{if } d \equiv 2 \pmod{3} \end{cases}$$

Lemma 5.2.2.

The system

$$(5.13) \quad \begin{cases} 0 < x < y < z \\ x + y + z = 1 \\ x^2 + y^2 + z^2 = a \\ x^3 + y^3 + z^3 = b \end{cases}$$

has a (unique) solution for x, y and z if and only if

- (i) $1/3 < a < 1$,
- (ii) $b > (3a-1)/2$ and
- (iii) $a - 2/9 - \sqrt{2(3a-1)^{3/2}}/18 < b < a - 2/9 + \sqrt{2(3a-1)^{3/2}}/18$.

Proof.

(5.13) is equivalent with

$$(5.14) \quad \begin{cases} 0 < x < y < z \\ x + y + z = 1 \\ xy + xz + yz = \frac{1-a}{2} \\ xyz = \frac{2b-3a+1}{6} \end{cases}$$

and (5.14) has a solution if and only if the polynomial $f(X) = X^3 - X^2 + (1-a)X/2 - (2b-3a+1)/6$ has three distinct positive roots (x, y and z). Let u and v be the two critical points of $f(X)$, i.e. where $f'(X) = 0$. Then $f(X)$ has three distinct positive roots, if and only if

(i)' u and v are two distinct positive numbers (W.L.O.G. assume that $u < v$),

(ii)' $f(0) < 0$ and

(iii)' $f(u) > 0$ and $f(v) < 0$.

Then an easy calculation shows that (i) $\Leftrightarrow$ (i)', (ii) $\Leftrightarrow$ (ii)' and (iii) $\Leftrightarrow$ (iii)'. $\clubsuit$

So, in particular, we got that the system

$$(5.15) \quad \begin{cases} 0 < x < y < z \\ x + y + z = 1 \\ x^2 + y^2 + z^2 = \frac{1}{2} \\ x^3 + y^3 + z^3 = \frac{1}{4} + r \end{cases}$$

has a (unique) solution, if and only if $0 < r < 1/18$.

Corollary 5.2.3.

For $\mu=2,3$ or 4 the system

$$(5.16) \quad \begin{cases} 0 < x < y < z \\ x + y + z = \frac{6}{d + \mu} \\ x^2 + y^2 + z^2 = \frac{18}{(d + \mu)^2} \\ x^3 + y^3 + z^3 = \frac{54}{(d + \mu)^3} + r^* \end{cases}$$

has a (unique) solution, if and only if $0 < r^* < 12/(d + \mu)^3$. $\clubsuit$

Now suppose r^* is fixed (defined later).

Define the numbers $a_1, a_2, \dots, a_d$ as follows:

$$(5.17) \quad a_i = \begin{cases} \sqrt{x_i} & \text{if } i = 3, 4, \dots, d-1 \\ \sqrt{x} & \text{if } i = 1 \\ \sqrt{y} & \text{if } i = 2 \\ \sqrt{z} & \text{if } i = d, \end{cases}$$

where the x_i 's were defined in Proposition 5.2.1 and x , y and z are defined with (5.16).

Then using Proposition 5.2.1 and Corollary 5.2.3 we get

Proposition 5.2.4.

Suppose h_6 satisfies

$$(5.18) \quad T_1 \leq h_6 < T_2, \text{ where}$$

$$(5.19) \quad T_1 = \begin{cases} \frac{9}{(d+2)^2} & \text{if } d \equiv 1 \pmod{3} \\ \frac{9(d+5)}{(d+3)^2(d+2)} & \text{if } d \equiv 0 \pmod{3} \\ \frac{9(d+8)}{(d+4)^2(d+2)} & \text{if } d \equiv 2 \pmod{3} \end{cases}$$

and

$$(5.20) \quad T_2 = \begin{cases} \frac{9}{(d+2)^2} + \frac{12}{(d+2)^3} = \frac{3(3d+10)}{(d+2)^3} & \text{if } d \equiv 1 \pmod{3} \\ \frac{9(d+5)}{(d+3)^2(d+2)} + \frac{12}{(d+3)^3} = \frac{3(3d^2+28d+53)}{(d+2)(d+3)^3} & \text{if } d \equiv 0 \pmod{3} \\ \frac{9(d+8)}{(d+4)^2(d+2)} + \frac{12}{(d+4)^3} = \frac{3(3d^2+40d+104)}{(d+2)(d+4)^3} & \text{if } d \equiv 2 \pmod{3} \end{cases}$$

Let $r^* = h_6 - T_1$. Then the numbers $a_1, a_2, \dots, a_d$ defined with (5.17) satisfy (5.7).

To make the construction complete, we have to choose the parameters m , k and p (depending on d and n), such that m satisfies (5.2), k satisfies (4.15), where S is given by (5.6), and h_6 defined with (5.12) satisfies (5.18). Of course we also need

$$(5.21) \quad n = d! \cdot m + 4d(d-1)k + d! \cdot 2^d p,$$

since $U=(A^* \mid C_k(h) \mid M_p(h_6))$ has size $d \times [d! \cdot m + 4d(d-1)k + d! \cdot 2^d p]$. We will choose m in terms of d and n , k in terms of m , and

$$(5.22) \quad p = [n - d! \cdot m - 4d(d-1)k] / (d! \cdot 2^d).$$

Then (5.22) implies (5.21). The choice of k has to guarantee that $p \in \mathbb{N}$. (4.15) is equivalent with

$$(5.23) \quad S/[8(d-1)] < k < S/[4(d-4)].$$

Suppose $4d(d-1) \mid n$, and choose k to be the unique integer in the interval

$$(5.24) \quad \left(\frac{S}{8(d-1)}, \frac{S}{8(d-1)} + (d-2)! \cdot 2^{d-2} \right],$$

for which p defined in (5.22) is an integer. Then (5.23) holds, if

$$(5.25) \quad \frac{S}{8(d-1)} + (d-2)! \cdot 2^{d-2} < \frac{S}{4(d-4)}, \quad \text{i. e.}$$

$$(5.26) \quad S > \frac{d-4}{d+2} (d-1)! \cdot 2^{d-1},$$

where S is given by (5.6). Therefore we need

$$(5.27) \quad m > \begin{cases} \frac{d(d-4)(d-1)}{6(d-2)(d+2)} 2^d & \text{if } d \text{ is even} \\ \frac{(d-4)(d-1)}{3(2d-7)(d+2)} 2^d & \text{if } d \text{ is odd.} \end{cases}$$

Instead of (5.27), we will assume the more simple

$$(5.28) \quad m > d2^d/6.$$

Clearly (5.28) implies (5.27) for every d .

Now we will compute T from (5.11) explicitly. First we can easily check that

$$(5.29) \quad \sum_{C_k(h)} f_6 = -12(d+4)k + 12(3d+2)kh,$$

where h is given by (4.16), and k is defined by (5.24) with S given in (5.6) (depending on m). Therefore, using (5.5), (5.6), (5.24) and (5.29) we can calculate

$$\begin{aligned}
 (5.30) \quad T &= -\sum_V f_6 = -\sum_{A^*} f_6 - \sum_{C_k^{(h)}} f_6 \\
 &= -\sum_{A^*} f_6 + 12(d+4) - 12(3d+2)k \frac{S+24k}{8k(d+2)} \\
 &= -\sum_{A^*} f_6 - \frac{3(3d+2)}{2(d+2)}S + \frac{12(d-1)(d-2)}{d+2}k \\
 &= -\sum_{A^*} f_6 - \frac{3(3d+2)}{2(d+2)}S + \frac{12(d-1)(d-2)}{d+2} \left(\frac{S}{8(d-1)} + k_0 \right) \\
 &= -\sum_{A^*} f_6 - 3S + \frac{12(d-1)(d-2)}{d+2}k_0 \\
 &= \begin{cases} -\frac{60(d-4)(d-2)!}{d^2}m + \frac{3(d-2)(d-2)!}{d}m + \frac{12(d-1)(d-2)}{d+2}k_0 & \text{if } d \text{ is even} \\ -\frac{15(4d^2-35d+68)(d-3)!}{(d-1)^2}m + \frac{3(2d-7)(d-2)!}{2(d-1)}m + \\ \quad + \frac{12(d-1)(d-2)}{d+2}k_0 & \text{if } d \text{ is odd} \end{cases} \\
 &= \begin{cases} -\frac{3(3d-10)(d+8)(d-2)!}{d^2}m + \frac{12(d-1)(d-2)}{d+2}k_0 & \text{if } d \text{ is even} \\ -\frac{3(6d^3+d^2-275d+638)(d-3)!}{2(d-1)^2}m + \frac{12(d-1)(d-2)}{d+2}k_0 & \text{if } d \text{ is odd} \end{cases}
 \end{aligned}$$

where

$$(5.31) \quad k = S/[8(d-1)] + k_0, \text{ for some}$$

$$0 < k_0 \leq (d-2)! \cdot 2^{d-2}.$$

Therefore, using (5.12), (5.22), (5.30) and (5.31), we get

$$(5.32) \quad h_6 = A - \frac{(Bm - Ck_0) \cdot d! \cdot 2^d}{n - d!m - S_0 dm / 2 - 4d(d-1)k_0},$$

where

$$(5.33) \quad S_0 = \frac{S}{m} = \begin{cases} \frac{3d(d-2)(d-2)!}{d} & \text{if } d \text{ is even} \\ \frac{3(2d-7)(d-2)!}{2(d-1)} & \text{if } d \text{ is odd} \end{cases}$$

$$(5.34) \quad A = \frac{15}{(d+2)(d+4)}$$

$$(5.35) \quad B = \begin{cases} \frac{(3d-10)(d+8)(d-2)}{2^{d+1}(d+4)(d+8)d^2} & \text{if } d \text{ is even} \\ \frac{(6d^3 + d^2 - 275d + 638)}{2^{d+2}(d+4)(d+8)(d-1)^2} & \text{if } d \text{ is odd} \end{cases}$$

$$(5.36) \quad C = \frac{(d-1)(d-2)}{2^{d-1}(d+2)(d+4)(d+8)(d-3)!}$$

By Proposition 5.2.4 we need $T_1 \leq h_6 < T_2$, where T_1 and T_2 were given in (5.19) and (5.20). For m that means

$$(5.37) \quad m \leq \frac{(A - T_1)n + [C \cdot d! \cdot 2^d - (A - T_1)4d(d-1)] \cdot k_0}{B \cdot d! \cdot 2^d + (A - T_1)d! + (A - T_1)S_0 d / 2} \text{ and}$$

$$(5.38) \quad m > \frac{(A - T_2)n + [C \cdot d! \cdot 2^d - (A - T_2)4d(d-1)] \cdot k_0}{B \cdot d! \cdot 2^d + (A - T_2)d! + (A - T_2)S_0 d / 2}.$$

Now using the bounds for k_0 given in (5.31), from (5.37) and (5.38) we get

$$(5.39) \quad (an+c)/(as+b) < m \leq a'n/(a's+b),$$

where

$$(5.39) \quad a=A-T_2, (a \approx 6/d^2)$$

$$(5.40) \quad a'=A-T_1, (a' \approx 6/d^2, \text{ and } a'-a \approx 12/d^3)$$

$$(5.41) \quad b=B \cdot 2^d \cdot d!, (b \approx 3(d-1)!/2)$$

$$(5.42) \quad c=[C \cdot 2^d \cdot d! - (A-T_2)4d(d-1)] \cdot 2^d \cdot (d-2)!, (c \approx 2^{d+1}d!) \text{ and}$$

$$(5.43) \quad s=S_0 d/2 + d!, (s \approx d!).$$

With these preparations and notations we can prove

Theorem 5.2.5.

Let d and n be positive integers with $4d(d-1)|n$.

Using the notations of (5.6), (5.33)-(5.36) and (5.39)-(5.43), suppose that

$$(5.44) \quad n > \max\{(2as+2b+c)(a's+b)/[b(a'-a)], (6^{d/2}+2)(a's+b)/a', (d2^d/6+2)(a's+b)/a'\}$$

(this number is approximately $6^{d/2}(d+1)!/4$).

Let

$$(5.45) \quad m = \begin{cases} \left\lceil \frac{a'n}{a's+b} \right\rceil & \text{if } d \text{ is even, or } d \text{ is odd and } \left\lceil \frac{a'n}{a's+b} \right\rceil \text{ is odd} \\ \left\lceil \frac{a'n}{a's+b} \right\rceil - 1 & \text{if } d \text{ is odd and } \left\lceil \frac{a'n}{a's+b} \right\rceil \text{ is even} \end{cases}$$

Let k be the unique integer in the interval

$$\left(\frac{S}{8(d-1)}, \frac{S}{8(d-1)} + (d-2)! \cdot 2^{d-2} \right],$$

for which $n-d! \cdot m \equiv 4d(d-1)k \pmod{d! \cdot 2^d}$, and let $p=[n-d! \cdot m-4d(d-1)k]/(d! \cdot 2^d)$.

Then the matrix $U=(A^* \mid C_k(h) \mid M_p(h_6))$ is well defined, has size $d \times n$, and defines a spherical 7-design.

Proof.

We see that m is odd, if d is odd. We need to show, that m satisfies (5.2), k satisfies (4.15) and h_6 defined with (5.12) satisfies (5.18). We have already shown that k satisfies (4.15), if (5.28) holds for m , and h_6 satisfies (5.18), if (5.39) holds for m . So we need to show that (5.44) implies that m defined by (5.45) satisfies (5.2), (5.28) and (5.39).

To show (5.2):

$$\begin{aligned} m &> a'n/(a's+b)-2 \\ &> a'(6^{d/2}+2)(a's+b)/[a'(a's+b)]-2 \\ &= 6^{d/2}. \end{aligned}$$

To show (5.28):

$$\begin{aligned} m &> a'n/(a's+b)-2 \\ &> a'(d2^{d/6}+2)(a's+b)/[a'(a's+b)]-2 \\ &= d2^{d/6}. \end{aligned}$$

To show (5.39):

- (1) Clearly $m \leq a'n/(a's+b)$.
- (2) To approximate from below, we know that $m > a'n/(a's+b)-2$, and $n > (2as+2b+c)(a's+b)/[b(a'-a)]$ implies that $a'n/(a's+b)-2 > (an+c)(as+b)$. 🍏

With Theorem 5.2.5 we have completed the construction of spherical 7-designs.

Remarks.

- (1) In our construction above, we allowed points with (integral) multiplicity. The argument below shows that we can avoid this. A matrix $M_P(h_6)$ with no repeated columns exists satisfying (5.9) and (5.10), with $h_4=3/(d+2)$ and h_6 given by (5.12). That means that

we have to find p distinct solutions to (5.7), or with $x_i=(a_i)^2$ ($i=1,2, \dots, d$) to the system:

$$(5.46) \quad \begin{cases} 0 < x_1 < x_2 < \dots < x_d \\ x_1 + x_2 + \dots + x_d = 1 \\ x_1^2 + x_2^2 + \dots + x_d^2 = h_4 = \frac{3}{d+2} \\ x_1^3 + x_2^3 + \dots + x_d^3 = h_6 \end{cases}$$

where h_6 is given by (5.12).

Let W_d be the pointset in $\mathbb{R}^d$, satisfying

$$(5.47) \quad \begin{cases} 0 < x_1 < x_2 < \dots < x_d \\ x_1 + x_2 + \dots + x_d = 1 \\ x_1^2 + x_2^2 + \dots + x_d^2 = h_4 = \frac{3}{d+2} \end{cases}$$

Let $f(x)=f(x_1, x_2, \dots, x_d)=(x_1)^3+(x_2)^3+ \dots +(x_d)^3$. Suppose the points P and Q lie in the connected space W_d , and $f(P)<h_6$, $f(Q)>h_6$. Then, by the Intermediate Value Theorem, if $d>3$, then there are p distinct (actually: infinitely many) points $R_1, R_2, \dots, R_p$ all in W_d , with $f(R_i)=h_6$ for all $i=1,2, \dots, p$. Hence

$M_p(h_6)=(M(R_1) \mid M(R_2) \mid \dots \mid M(R_p))$ satisfies (5.9) and (5.10).

- (2) The argument above does not work when $d=3$, since W_3 is a circle, and then there are only two disjoint paths between P and Q . However, for

$$(5.48) \quad h_4=3/(d+2) \text{ and}$$

$$(5.49) \quad h_6=15/[(d+2)(d+4)]$$

we get that the system

$$(5.50) \quad \begin{cases} 0 < x_1 < x_2 < x_3 \\ x_1 + x_2 + x_3 = 1 \\ x_1^2 + x_2^2 + x_3^2 = h_4 = \frac{3}{d+2} \\ x_1^3 + x_2^3 + x_3^3 = h_6 = \frac{15}{(d+2)(d+4)} \end{cases}$$

has a (unique) solution by Lemma 5.2.2. Therefore $M(h_6)$ defines a spherical 7-design on $2^3 \cdot 3! = 48$ points. (For $d=3$ see also section 6.)

VI. SPHERICAL t-DESIGNS ON THE REGULAR SPHERE S^2

6.1. Spherical t-designs on the circle S^1

In this section let d be 2.

According to (1.3), if X is a t -design on S^1 , then $|X|=n \geq t+1$. Below we give a construction of spherical t -designs on S^1 for every $t \in \mathbb{N}$ and $n \geq t+1$. Namely, we will show that a regular n -gon with $n \geq t+1$ is a t -design on S^1 .

By (2.9) and (2.10), the set

$$(6.1) \quad \Phi_2(s) = \{ \operatorname{Re}(x + iy)^s, \operatorname{Im}(x + iy)^s \}$$

forms a basis for $\operatorname{Harm}_2(s)$. Therefore the matrix

$$(6.2) \quad U = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

defines a spherical t -design on S^1 , if and only if

$$(6.3) \quad x_k^2 + y_k^2 = 1 \quad \text{for all } k=1, 2, \dots, n$$

$$(6.4) \quad \sum_{k=1}^n \operatorname{Re}(x_k + \sqrt{-1} y_k)^s = 0 \quad \text{for all } s = 1, 2, \dots, t \quad \text{and}$$

$$(6.5) \quad \sum_{k=1}^n \operatorname{Im}(x_k + \sqrt{-1} y_k)^s = 0 \quad \text{for all } s = 1, 2, \dots, t.$$

For $k=1, 2, \dots, n$ choose

$$(6.6) \quad \begin{aligned} x_k &= \cos(2k\pi/n), \text{ and} \\ y_k &= \sin(2k\pi/n). \end{aligned}$$

Then (6.3) clearly holds. Also,

$$(6.7) \quad (x_k + \sqrt{-1}y_k)^s = \cos\left(\frac{2\pi}{n}ks\right) + \sin\left(\frac{2\pi}{n}ks\right),$$

hence, using (3.10),

$$(6.8) \quad \sum_{k=1}^n \operatorname{Im} (x_k + \sqrt{-1}y_k)^s = \sum_{k=1}^n \sin\left(\frac{2\pi}{n}ks\right) = 0, \text{ and}$$

$$(6.9) \quad \sum_{k=1}^n \operatorname{Re} (x_k + \sqrt{-1}y_k)^s = \sum_{k=1}^n \cos\left(\frac{2\pi}{n}ks\right) = 0, \text{ because}$$

$$1 \leq s \leq t < n.$$

Here we mention a theorem of Y. Hong [6], which says that if X is a spherical t -design on S^1 with $|X|=n$, then

- (i) if $t+1 \leq n \leq 2t+1$, then X must be a regular n -gon,
- (ii) if $n=2t+2$, then X must be a union of two regular n -gons,
- (iii) for $n \geq 2t+3$ there are infinitely many spherical t -designs on S^1 , which are not the unions of regular n_i -gons with $n_i \geq t+1$.

6.2. t -designs on the interval $[-1,1]$

A subset $X=\{x_1, x_2, \dots, x_{n(t)}\}$ of the interval $[-1,1]$ is called a t -design on $[-1,1]$, if and only if

$$(6.10) \quad \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{n(t)} \sum_{k=1}^{n(t)} f(x_k)$$

holds for all polynomials of degree at most t .

The existence of such designs was proved by Seymour and Zaslavsky in a very general context [10].

Putting $f(x)=x^s$ in (6.10), we get that X is t -design on $[-1,1]$, if and only if

$$(6.11) \quad x_1^s + x_2^s + \dots + x_{n(t)}^s = \begin{cases} \frac{n(t)}{s+1} & \text{if } 1 \leq s \leq t \text{ and } s \text{ is even} \\ 0 & \text{if } 1 \leq s \leq t \text{ and } s \text{ is odd} \end{cases}$$

This implies that $x_1, x_2, \dots, x_{n(t)}$ are the $n(t)$ roots of an even polynomial, hence the set X is symmetric with respect to the origin. In particular, if $n(t)$ is odd, then one of the points is the origin.

By a Fisher-type inequality, we get $n(t) \geq t$. The t -design X is called tight, if $|X|=t$. A famous problem of Tchebyshev (1873) is to find all values of t , for which tight t -designs exist on $[-1,1]$. (Examples were found for $t \leq 7$.) The problem was solved by S.N. Bernstein in 1937, he proved that tight t -designs on $[-1,1]$ exist only for $t=1,2,3,4,5,6,7$ and 9. (Namely he proved that $n(t) > ct^2$, for some positive constant c , which proved the nonexistence for $t \geq 11$) For a simplified version of Bernstein's proof see [7].

For $t=8$ we get $n(8) \geq 9$, the 9-design clearly is also an 8-design. Unfortunately, for $t \geq 10$, no construction has been given, even the smallest $n(t)$ has not been determined yet.

6.3. Spherical t -designs on S^2

In this section let $d=3$. We will construct spherical t -designs on the regular sphere, using t -designs on $[-1,1]$.

According to (2.3), (2.9) and (2.10), the set

$$(6.12) \quad \Phi_3(s) = \{f_{s, m_1, \mu}(x, y, z) \mid m_1 = 0, 1, 2, \dots, s \text{ and } \mu = 1, 2\}$$

$$= \left\{ h_\mu \left[(y + iz)^{m_1} \right] \cdot C_{s-m_1}^{m_1+1/2}(x) \mid m_1 = 0, 1, 2, \dots, s \text{ and } \mu = 1, 2 \right\}$$

forms a basis for $\text{Harm}_3(s)$, where $h_1(z) = \text{Re}(z)$, $h_2(z) = \text{Im}(z)$ and C_n^v is the n -th degree Gegenbauer (ultraspherical) polynomial, defined by (2.4), here we have $v = m_1 + 1/2$, so we will just need (2.5).

We will construct a matrix U satisfying (2.24) and (2.25) as follows.

Suppose the set $X = \{x_1, x_2, \dots, x_{n(t)}\}$ is a t -design on the interval $[-1, 1]$ (see section 6.2). Let m be an integer with $m \geq t+1$. For $k=1, 2, \dots, m$ and $j=1, 2, \dots, n(t)$, let

$$(6.13) \quad y_{j,k} = (1-x_j^2)^{1/2} \cos(2k\pi/m) \text{ and}$$

$$z_{j,k} = (1-x_j^2)^{1/2} \sin(2k\pi/m).$$

Then define the $3 \times m$ matrix U_j :

$$(6.14) \quad U_j = \begin{pmatrix} x_j & x_j & \dots & x_j \\ y_{j1} & y_{j2} & \dots & y_{jm} \\ z_{j1} & z_{j2} & \dots & z_{jm} \end{pmatrix},$$

and the $3 \times n(t)m$ matrix

$$(6.15) \quad U = (U_1 \mid U_2 \mid \dots \mid U_{n(t)}).$$

Theorem 6.

Let t and m be positive integers, with $m \geq t+1$. Suppose X is a t -design on the interval $[-1, 1]$. Then the matrix U defined above gives a spherical t -design on S^2 .

Proof.

By (6.13), the matrix U clearly satisfies (2.24). To prove (2.25), we need to show that $f_{s, m_1, \mu}$ vanishes on U , for every $s=1, 2, \dots, t$, $m_1=0, 1, \dots, s$ and $\mu=1, 2$.

(a) Suppose $m_1=0$.

Then $\mu=1$, and by (6.12), we get

$$f_{s,m_1,\mu}(x) = C_s^{1/2}(x).$$

By (2.5), $C_s^{1/2}(x) = P_s(x)$, the Legendre polynomial of degree s . Therefore,

$$(6.17) \quad \sum_U f_{s,m_1,1} = m \cdot \sum_{j=1}^{n(t)} P_s(x_j),$$

and since $\deg P_s(x) = s \leq t$, and X is a t -design on $[-1,1]$, by (6.10) we have

$$(6.18) \quad \sum_{j=1}^{n(t)} P_s(x_j) = \frac{n(t)}{2} \int_{-1}^1 P_s(x) dx = 0.$$

(b) Now suppose $m_1 \geq 1$.

Then, using (6.12) and (6.13) we have

$$(6.19) \quad f_{s,m_1,\mu}(x_j, y_{j,k}, z_{j,k}) = F_{s,m_1}(x_j) \cdot h_{\mu}[\cos(2km_1\pi/m) + i \sin(2km_1\pi/m)],$$

where

$$(6.20) \quad F_{s,m_1}(x_j) = (1 - x_j^2)^{m_1/2} \cdot C_{s-m_1}^{m_1+1/2}(x_j).$$

Hence we get

$$(6.21) \quad \sum_U f_{s,m_1,\mu} = \left(\sum_{j=1}^{n(t)} F_{s,m_1}(x_j) \right) \cdot h_{\mu} \left(\sum_{k=1}^m \cos\left(\frac{2\pi}{m} km_1\right) + i \sum_{k=1}^m \sin\left(\frac{2\pi}{m} km_1\right) \right) = 0$$

by (3.10), since $1 \leq m_1 \leq s \leq t < m$.

Remarks.

- (1) Consider the matrix $A = A(d, m, t)$ defined by (3.2) and (3.3). Suppose m is even if d is odd.

Let us "normalize" A , i.e. multiply each entry by $(2/d)^{1/2}$. Call this matrix $A^\#(d)$. Clearly $A^\#(d)$ satisfies (2.24), and defines a spherical 2-design, since by (3.14) $f_2(x_i) = x_i^2 - x_{i+1}^2$ vanishes on $A^\#(d)$ for every $i=1, 2, \dots, d-1$.

Also, for $d=3$ and $t=3$ by Lemma 3, if $m > 3^{3/2}$, i.e. $m \geq 6$, then $A^\#(3) = A^\#$ defines a spherical 3-design on S^2 . But (3.15) and (3.16) gives that $A^\#$ is never a spherical 4-design.

For $m=4$, $A^\#$ is the regular tetrahedron, and for $m=6$ it is a regular octahedron. (No other regular polyhedron appears). So the regular tetrahedron is a spherical 2-design (but not a 3-design) on S^2 , and the octahedron is a 3-design (but not a 4-design) on S^2 .

- (2) We can calculate the smallest size $N(t)$ of spherical t -designs on S^2 that we constructed. Namely, we have

$$N(t) = (t+1)n(t).$$

Using Bernstein's result, that tight t -designs on $[-1, 1]$ exist if (and only if) $t \leq 7$ or $t=9$, for these values of t we have:

$$N(2)=6, N(3)=12, N(4)=20, N(5)=30, N(6)=42, N(7)=56 \text{ and } N(9)=90.$$

It is interesting to note, that for $t=7$, the system (6.11), on which our construction of spherical 7-designs is based here is identical (after introducing new variables) with the system (5.50). However, the two spherical 7-designs constructed are different.

- (3) It would be interesting to determine $M(t)$, the size of the a smallest spherical t -design on S^2 . From (1.3), we have the lower bound

$$(6.23) \quad M(t) \geq \begin{cases} (t/2 + 1)^2 & \text{if } t \text{ is even} \\ (t/2 + 1)^2 - 1/4 & \text{if } t \text{ is odd.} \end{cases}$$

A t -design X , for which equality holds in (6.23) with $|X|=M(t)$, is called tight. All tight t -designs on S^2 are known:

for $t=2$ $M(2)=4$, and the regular tetrahedron is a spherical 2-design on S^2 ,

for $t=3$ $M(3)=6$, and the regular octahedron is a spherical 3-design on S^2 and

for $t=5$ $M(5)=12$, and the regular icosahedron is a spherical 5-design on S^2 .

No other tight t -design exist on S^2 . The cube is a 3-design (but not a 4-design), and the dodecahedron is a 5-design (but not a 6-design).

Other values of $M(t)$ are not yet determined.

However, from remark (3) and (6.23) we have:

$$9 \leq M(4) \leq 20,$$

$$16 \leq M(6) \leq 42,$$

$$20 \leq M(7) \leq 56,$$

$$25 \leq M(8) \leq 90, \text{ and}$$

$$30 \leq M(9) \leq 90.$$

We remark here that $M(9) \leq 60$ by a paper of W. Neutsch [9].

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