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Envelope radar cross section analysis of faired composite bodies

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The Ohio State University, 1988

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ENVELOPE RADAR CROSS SECTION ANALYSIS OF FAIRED COMPOSITE BODIES

A Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Barend Jacobus Erasmus Taute, B.Sc., Hons.B.Sc., M.Eng.

* * * * *

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Dedicated to my parents,
Willem and Susara Taute.

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CHAPTER I

INTRODUCTION

In radar cross section analysis of complex structures, one needs a computer model of the structure. One way to model scattering structures for radar cross section studies is to numerically define a set of flat panels or curved patches. However, apart from the problems with storage and bookkeeping for complex surfaces, these models present difficulties such as "waviness" in applications where continuity of surface derivatives is required [1]. This is particularly relevant in high frequency radar cross section (RCS) analyses. Such problems do not exist when canonical shapes, such as disks, cone frustrums, ellipsoids and ogives are used in that exact analytic equations are used to define the surfaces and to compute their derivatives. In these models, however, many wedge junctions can be introduced; whereas, streamlined aerodynamic structures usually have relatively smooth junctions. As an example, Figure 1(a) shows a structure composed of only basic shapes and Figure 1(b) shows a more realistic model where the wedge junctions had been faired. Fairing is similar to "sanding off" knife edge junctions or "filling in" sharp interior junctions to yield surfaces with slope continuity. Wedge junctions may, in some cases, be good enough for practical purposes, but grossly incorrect scattering results may also be obtained. This is especially true in edge caustic directions where wedges have high radar cross sections. Wedges also do not model the reflected fields from faired junctions.

Figure 1: Comparison of a model with wedge junctions and a model with faired junctions.

Barger [1,2] extended the class of shapes with the advantage of using analytic representations. The approach used by Barger defines smooth transition surfaces between beginning and ending cross-sectional contours. These surfaces are particularly relevant to aerodynamic structures. Campbell [3] analyzed such surfaces between elliptic initial and final contours and showed the variety of transition surfaces that can be obtained. These surfaces, however, do not particularly address the problem of fairing, which requires smooth matching of a faired surface to canonical shapes.

In this study, an analytic method will be described whereby wedge junctions between canonical shapes can be faired. The method involves defining elliptic

contours that match tangentially to the surfaces on both sides of a junction. These ellipses will be defined as a function of the parameter that describes the cross section of the body. The family of ellipses around the cross section then defines a **faired shape**. Such shapes will be described for junctions between shapes along a central axis (i.e., for isolated scattering bodies), and for junctions where two bodies intersect. The next objective is to compute the high frequency radar cross section (RCS) envelope for such faired composite bodies.

In a high frequency analysis, scattering bodies are assumed to be large in terms of the wavelength of an incident electromagnetic field. When this is true, the far zone backscattered field (or RCS) can be found by vector addition of contributions from distinct scattering centers. The techniques for RCS computations involve then, in general, computation of reflected fields from doubly curved surfaces, computation of diffracted fields from surface discontinuities, and other special techniques in caustic regions where the previous techniques fail. For reflected fields, the Geometrical Optics (GO) [4] field equations give accurate results. For diffracted fields, the Uniform Geometrical Theory of Diffraction (UTD), as presented by Kouyoumjian and Pathak [5], is efficient and accurate for this application. In caustic regions, fields are computed using a Physical Optics (PO) analysis, as well as the work on GO Equivalent Line Currents by Volakis and Peters [6], and on Equivalent Edge Currents by Ryan and Peters [7,8]. The work by Burnside, Peters and Chu [9] is of particular relevance here in that they applied the UTD to treat diffraction from junctions between canonical shapes, including smooth junctions. Measured results verified that a first order UTD analysis is sufficient to identify the major characteristics of an RCS pattern. The work by Chiang and Marhefka [10] on cylinders, and by Ebihara and Marhefka [11] on cone frustrums, further validated and extended the UTD approach. Therefore, in the analysis of

faired composite bodies, the above mentioned techniques can be employed with confidence.

The RCS pattern of complex structures as a function of incidence angle has, in general, many oscillations due to scattering contributions that add up anywhere from in phase to out of phase. In the **envelope technique**, the envelope of this RCS pattern is obtained by adding individual contributions **in phase** [12,13]. As an illustration, Figure 2 shows both the oscillating (σ_{PO}) and the envelope (σ_{env}) RCS for a circular disk, obtained from a Physical Optics (PO) analysis. Note that the envelope technique can use larger angular increments for data points than would be required for conventional methods. This inherent efficiency can be employed in order to compute full volumetric patterns of the RCS of faired composite bodies. The emphasis is then on shorter computation time and on identifying only the major characteristics of an RCS pattern, which is obtained from a first order analysis.

The usefulness of the **envelope technique** has been demonstrated by Pistorius [13] for a variety of circular canonical shapes and flat plate structures. The study by Pistorius used mostly PO solutions to obtain the RCS envelopes for canonical shapes such as disks, cylinders and cone frustrums. These solutions are accurate only close to caustic directions and do not take into account the wedge angles at the ends of a basic shape or the polarization of the incident field. Furthermore, only principal plane analyses were done and side structures were modelled by flat plates.

In this study, the envelope technique will be extended to treat more general structures. Firstly, bodies of elliptical (as opposed to circular) cross section will be used. The bodies include geometries with a main body and side bodies, such as aircraft wings, intersecting it. The side bodies will be modelled using canonical

Figure 2: RCS pattern and envelope for a circular disk with radius $a = 5$ wavelengths at 1 GHz.

and faired shapes, rather than flat plates. Secondly, GO reflected fields will be computed for a more general class of doubly curved surfaces, including the faired shapes described above. Thirdly, UTD diffracted fields will be computed for all junctions in the surface. Therefore, the actual wedge angle at a junction will be incorporated in the solution. This solution is valid over a wider angular range than PO results and automatically includes the effect of incident field polarization. In caustic regions, where UTD solutions become unbounded, the specular values from PO and Equivalent Current solutions will be applied. Finally, full volumetric patterns of the RCS envelope will be computed.

The rest of this study is organized as follows:

- **Chapter II** gives the electromagnetic theory relevant to this study. The basic principles of a high frequency, ray optical solution for backscattered fields are described. General equations for GO and UTD fields are given and specialized for far zone backscattering. Limiting field values for the various caustic regions are given or derived. Finally, the envelope technique is explained.
- **Chapter III** describes isolated scattering bodies with faired junctions, and how to obtain the surface characteristics needed for scattering computations on these surfaces. Equations are given for both canonical shapes and the newly defined faired shapes. Finally, a “matched ogive” shape is introduced.
- **Chapter IV** discusses field computations for isolated bodies. The coordinates for a full volumetric RCS pattern are described here. Efficient methods for finding specular reflection points on curved surfaces are discussed, and the necessary methods for computing UTD diffracted fields from junctions

are given. Computed and measured results are given for a test body, and the results demonstrate all the first order scattering mechanisms.

- **Chapter V** introduces geometries with a side body which intersects a main body. The procedures for finding contours of intersection and fairing of the junction between the two bodies are described. Computing surface derivatives on this faired shape is also discussed.
- **Chapter VI** describes field computation aspects that are particular to structures with intersecting side bodies. Those include computing GO fields from the faired intersection and finding diffracted fields from numerically defined junction contours. Computed results are presented to illustrate the versatility of the methods.
- **Chapter VII** gives a summary and conclusions about the work presented in this study and future possibilities.

CHAPTER II

ELECTROMAGNETIC THEORY

In this chapter, a discussion is given of the electromagnetic (EM) techniques used to compute the radar cross section (RCS) envelope of the bodies defined in the following chapters. Previously developed techniques are given without proof and then specialized for far-field backscatter.

2.1 Discussion of the Approach

The scattering bodies used in this study are composed of analytically defined basic shapes with elliptic cross sections. A representative example of such a scattering body is shown in Figure 3. The body consists of three shapes: a doubly curved surface (S1: ogive section); a singly curved surface (S2: elliptic cone frustum) and a flat surface (S3: elliptic disk). These and other canonical shapes will be described in detail in Chapter 3.

The RCS analysis in this study is applied to convex bodies (i.e. no cavities, inlets etc.) with perfectly conducting surfaces, immersed in a lossless, homogeneous, isotropic medium (free space). The incident field is a plane wave of arbitrary polarization and angle of incidence. The aim is then to find a high frequency (HF) solution for the RCS envelope of the scattering body. In an HF solution, ray optical techniques can be employed. This means that the propagation of the incident and scattered fields are traced along straight lines in homogeneous free space. The lines

Figure 3: Cross section of typical isolated scattering body.

satisfy the generalized Fermat's principle and the field intensity is governed by the conservation of energy flux in a tube of rays [4]. (In other words, Fermat's principle states that the ray transit time between a source and observer must be a minimum. For reflection/diffraction, the ray trajectory includes a reflection/diffraction point on the surface.)

At high frequencies, various scattering mechanisms such as reflection and diffraction can be seen as localized phenomena. Scattered field contributions come from points on the body, called "scattering centers," which can be identified using traditional HF techniques. The total scattered field would then be the vector sum of contributions from all possible scattering centers on the body. The total RCS envelope is found by adding the scalar magnitudes of backscattered fields or powers rather than the complex field values.

The HF solution will only be valid if the geometry also satisfies the following conditions:

- The body should be several wavelengths long.
- The surface radius of curvature should be larger than $\lambda/4$ everywhere.
- Scattering centers for diffraction should be more than λ apart.

where λ is the free space wavelength of the incident field.

The HF solution used in this work consists of first order scattering mechanisms. By “first order” is meant scattering mechanisms which include only one interaction with the scattering body between the incident ray and the scattered ray. These three mechanisms, indicated in Figure 3, are:

1. Reflection from a doubly curved surface ($\mathbf{E}^{GO}$).
2. Diffraction from a discontinuity in surface radius of curvature ($\mathbf{E}_1^d$).
3. Diffraction from a wedge ($\mathbf{E}_2^d$).

Geometrical Optics (GO) [4,5] is used to find an expression for the reflected fields, and the Uniform Geometrical Theory of Diffraction (UTD) [5] is used to compute diffracted fields. Diffracted fields compensate for step discontinuities in reflected fields and make the total scattered field continuous. These solutions are used everywhere except at caustics (a congruence of paraxial rays) where both solutions become unbounded. Three types of caustics associated with scattering from a body are shown in Figure 4 and will be discussed next.

The first type of caustic occurs in the backscatter direction broadside to the flat face of a disk (direction 1 in Figure 4). This is the only direction in which backscattered reflected fields from the disk will be observed. However, since the

1. Flat surface caustic
2. Straight line caustic
3. Ring caustic

Figure 4: Caustic directions for a scattering body.

surface of the disk is flat, the whole surface contributes to the backscattered reflected field which violates the scattering center nature of the HF solution. The Physical Optics (PO) procedure [14] gives accurate results for specular reflection regions and will be used to find the value for the backscattered field at such **flat surface caustics**.

The second type of caustic occurs in the direction of specular reflection from the flat sides of the cone frustum (direction 2 in Figure 4). In these directions, a whole line contributes to the reflected field. A GO Equivalent Line Current analysis [6] will be applied to find the backscattered fields at such **straight line caustics**.

The third type of caustic is due to specular diffraction from a flat elliptic ring edge, such as junction J2 in Figure 3, and occurs in direction 3 as illustrated in Figure 4. In this case the whole ring junction contributes to the backscattered

diffracted field. An equivalent edge current analysis explained by Ryan and Peters [7] will be used to find the backscattered field for such ring caustics.

A fourth type of caustic, due to reflection from a whole ring on a toroid, will also be treated using the GO Equivalent Line Current analysis.

Note that first order analysis has its limitations. In certain directions, higher order mechanisms such as double diffractions and creeping waves may have significant contributions to the total RCS. For intersecting bodies, other effects such as double reflections and various reflection/diffraction combinations may also be significant. To deal with these mechanisms in a general way requires a major extension of this work, and will not be discussed here.

2.2 Notation and Conventions With Respect To Electromagnetic Fields

In this study, vector quantities with complex amplitudes are indicated by boldface symbols (eg. $\mathbf{E}$), unit vectors are indicated by a hat (eg. $\hat{n}$) and dyadic quantities by a double bar (eg. $\overline{\overline{\mathbf{D}}}$). In addition, the following symbols are commonly used in electromagnetic field equations:

ω = angular frequency

λ = the free space wavelength of the incident field

k = $2\pi/\lambda$ = the wave number of EM fields

$\mathbf{E}$ = complex electric field vector

E = complex electric field amplitude

U = scalar magnitude of a co-polarized backscattered
electric field for unit magnitude incident field

s' = distance from a far-field source

s = distance from a scattering point to

Figure 5: Incident and scattered field coordinates.

a far-field observer

$\hat{s}'$ = incident field propagation direction

$\hat{s}$ = scattered field propagation direction, and

σ = radar cross section (RCS) quantity.

Throughout this study, the $e^{j\omega t}$ time convention (where t represents time) is assumed and suppressed.

The incident and scattered fields are defined in the XYZ-coordinate system, in terms of the spherical unit vectors, $\hat{\theta}$ and $\hat{\phi}$. The geometry for these fields is shown in Figure 5. The incident field is a plane wave with polarization perpendicular to

$\hat{\mathbf{s}}'$, such that

$$\mathbf{E}^i = [\mathbf{e}_\theta \hat{\boldsymbol{\theta}} + \mathbf{e}_\phi \hat{\boldsymbol{\phi}}] e^{-j\mathbf{k}\cdot\mathbf{s}'} \quad (2.1)$$

where $|\mathbf{e}_\theta|^2 + |\mathbf{e}_\phi|^2 \triangleq 1$. The backscattered field from an arbitrary target can be written as

$$\mathbf{E}^{bs} = [E_\theta^{bs} \hat{\boldsymbol{\theta}} + E_\phi^{bs} \hat{\boldsymbol{\phi}}] \frac{e^{-j\mathbf{k}\cdot\mathbf{s}}}{s} . \quad (2.2)$$

This field will in general have polarization different from $\mathbf{E}^i$. The co-polarized backscattered field amplitude is defined as

$$E_{co}^{bs} = (E_\theta^{bs} \mathbf{e}_\theta + E_\phi^{bs} \mathbf{e}_\phi) \frac{e^{-j\mathbf{k}\cdot\mathbf{s}}}{s} . \quad (2.3)$$

Finally, the variable U is the scalar magnitude of the co-polarized backscattered field, such that

$$U \triangleq |E_{co}^{bs}| . \quad (2.4)$$

The following sections give the equations for computing the backscattered fields due to reflection and diffraction, and the values in caustic regions.

2.3 The Geometrical Optics Field

The geometrical optics (GO) field, which is an integral part of the asymptotic HF scattering solution, is described by Kouyoumjian and Pathak in [4,5]. The results are summarized in the following.

2.3.1 GO Reflected Field Equation

Luneburg [15] and Kline [16] developed an asymptotic high frequency technique for solving Maxwell's equations in which the fields are expanded in inverse powers of the angular frequency, ω . For large ω , in a source free homogeneous isotropic medium, the electric field intensity can be expressed as

$$\mathbf{E}(\mathbf{r}) \sim e^{-j\mathbf{k}\cdot\mathbf{r}} \sum_{m=0}^{\infty} \frac{\mathbf{E}_m(\mathbf{r})}{(-j\mathbf{k})^m} \quad (2.5)$$

Figure 6: Astigmatic ray tube.

where $\mathbf{r}$ is the position vector of the field point and $k = \omega/c$ (c is the speed of light in the medium). Substituting Equation (2.5) in the vector wave equation, satisfied by $\mathbf{E}$, yields the eikonal equation, $\nabla\psi = 1$, and the transport equation, $[\nabla^2\psi + 2\nabla\psi \cdot \nabla]\mathbf{E}_m = -\nabla^2\mathbf{E}_{m-1}$ with $\mathbf{E}_{-1} = 0$. The surfaces of constant ψ are called wave fronts, and the family of wave fronts describe a system of associated rays which are straight lines in a homogeneous medium. The rays are everywhere orthogonal to the wave fronts in an isotropic medium. A tube of such rays propagating in the direction $\hat{s}$ is shown in Figure 6. Integrating the transport equation for $m = 0$ from some reference point, $s = 0$, to a field point at s yields the leading term in Equation (2.5). This term is known as the Geometrical Optics (GO) or ray-optical field, and is given by

$$\mathbf{E}(s) = \mathbf{E}_0(0)e^{-jk\psi(0)} \sqrt{\frac{\rho_1 \rho_2}{(\rho_1 + s)(\rho_2 + s)}} e^{-jks} \quad (2.6)$$

where $\rho_{1,2}$ are the principal radii of curvature of the wave front surface at s . In general, the ray tube is **astigmatic**: the rays do not converge to a single focal point. This property is taken care of in Equation (2.6) where ρ_1 and ρ_2 can have different magnitudes. The square root quantity in (2.6) is known as the ray spread factor. This factor represents the spreading of energy and is a consequence of the

conservation of energy in a ray tube. At points in the ray tube where $s = -\rho_{1,2}$, the spread factor becomes infinite and the GO solution is not valid. Such a congruence of rays (1-2 and 3-4 in Figure 6) is known as a caustic, and $s = -\rho_{1,2}$ are also referred to as caustic distances. Note that, as one passes through a caustic line, $(\rho_{1,2} + s)$ changes sign. The square root then naturally introduces the correct phase change such that

$$\sqrt{-|\rho_{1,2} + s|} = \sqrt{|\rho_{1,2} + s|} e^{-j\pi/2} \quad (2.7)$$

In the absence of a source, $\mathbf{E}$ should also satisfy $\nabla \cdot \mathbf{E} = 0$. This leads to $\hat{\mathbf{s}} \cdot \mathbf{E} = 0$, which implies that $\mathbf{E}$ is polarized transverse to the ray direction.

Now consider an electric field, $\mathbf{E}^i$, with direction vector $\hat{\mathbf{s}}'$ incident on a curved surface as shown in Figure 7. Let Q_r , the point of reflection, be the reference point $s = 0$. Applying the boundary condition, $\hat{\mathbf{n}} \times [\mathbf{E}^i + \mathbf{E}^r] = 0$, on the perfectly conducting surface at Q_r yields the reflected field which is given by

$$\mathbf{E}^r(s) \sim \mathbf{E}^i(Q_r) \cdot \overline{\overline{\mathbf{R}}} \sqrt{\frac{\rho_1^r \rho_2^r}{(\rho_1^r + s)(\rho_2^r + s)}} e^{-jks} \quad (2.8)$$

where

$$\begin{aligned} \overline{\overline{\mathbf{R}}} &= \text{dyadic reflection coefficient} \\ &= \hat{\mathbf{e}}_{\perp} \hat{\mathbf{e}}_{\perp} - \hat{\mathbf{e}}_{\parallel}^i \hat{\mathbf{e}}_{\parallel}^r \end{aligned} \quad (2.9)$$

The reflected field propagates in the direction $\hat{\mathbf{s}}$ which satisfies the law of reflection, $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = -\hat{\mathbf{s}}' \cdot \hat{\mathbf{n}}$, and is in the same plane as $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$. The rest of the unit vectors are defined by (see Figure 7):

$$\hat{\mathbf{e}}_{\perp} = \frac{\hat{\mathbf{s}}' \times \hat{\mathbf{n}}}{|\hat{\mathbf{s}}' \times \hat{\mathbf{n}}|} \quad (2.10)$$

$$\hat{\mathbf{e}}_{\parallel}^i = \hat{\mathbf{s}}' \times \hat{\mathbf{e}}_{\perp}, \text{ and} \quad (2.11)$$

$$\hat{\mathbf{e}}_{\parallel}^r = \hat{\mathbf{s}} \times \hat{\mathbf{e}}_{\perp} \quad (2.12)$$

(a) Side view of a reflection point on a curved surface.

(b) Principal directions and curvature.

Figure 7: Reflection by a curved surface.

Let $R_{1,2}$ be the principal radii of curvature of the reflecting surface at Q_r , and $\hat{U}_{1,2}$ the corresponding principal directions. The reflected field caustic distances $\rho_{1,2}^r$ for the case of a spherical incident wave front (of which the plane wave is a special case) is given by [5]

$$\begin{aligned} \frac{1}{\rho_{1,2}^r} = & \frac{1}{s'} + \frac{1}{\cos \theta^i} \left[\frac{\sin^2 \theta_2}{R_1} + \frac{\sin^2 \theta_1}{R_2} \right] \pm \\ & \pm \sqrt{\frac{1}{\cos^2 \theta^i} \left[\frac{\sin^2 \theta_2}{R_1} + \frac{\sin^2 \theta_1}{R_2} \right]^2 - \frac{4}{R_1 R_2}}. \end{aligned} \quad (2.13)$$

Here s' is the radius of curvature of the incident wave front, θ^i is the angle of incidence and $\theta_{1,2}$ are the angles between $\hat{s}'$ and $\hat{U}_{1,2}$, respectively. Therefore, one obtains that

$$\cos \theta^i = -\hat{s}' \cdot \hat{n}, \text{ and} \quad (2.14)$$

$$\sin^2 \theta_{1,2} = 1 - (\hat{s}' \cdot \hat{U}_{1,2})^2. \quad (2.15)$$

2.3.2 Backscattered GO Reflected Field

In the case of far-field backscatter, s and s' are large and $\hat{s} = -\hat{s}'$. At field points away from the caustics, Equation (2.8) reduces to

$$\mathbf{E}^r(s) \sim -\mathbf{E}^i(Q_r) \sqrt{\rho_1^r \rho_2^r} \frac{e^{-jks}}{s}. \quad (2.16)$$

Furthermore, for backscattered reflected fields, $\theta^i = 0$ and $\theta_{1,2} = 90^\circ$, so that $\rho_{1,2}^r = R_{1,2}/2$. Therefore, the reflected field amplitude at far-field backscatter points is given by

$$U^r = \frac{\sqrt{|R_1 R_2|}}{2s}. \quad (2.17)$$

Note that U^r is independent of the frequency. It is also evident that if either R_1 or R_2 or both become infinite, then the GO solution for the backscattered reflected field would be invalid in that the solution would be infinite. This is the case for backscattered reflection off the surface of a cone frustum and a disk.

2.4 Diffracted Fields

2.4.1 High Frequency Diffracted Field Solution

Geometrical Optics gives an accurate high frequency approximation for the incident and reflected fields, but it does not account for discontinuities in the surface. Surface discontinuities such as abrupt changes in slope or radius of curvature cause step discontinuities in the GO incident and reflected fields along directions called shadow boundaries. Figure 8(a) shows the cross section view of a slope discontinuity or wedge, with its two faces denoted 'o' and 'n', respectively. The o-face blocks the incident and reflected fields along the lines ISB_o and RSB_o , respectively. The "lit" and "shadow" sides of the incident and reflected fields are also indicated in Figure 8(a). The reflection boundary (RB) for a discontinuity in radius of curvature, i.e. a smooth junction, is shown in Figure 8(b). This is the direction along which the reflected field will have a step discontinuity. Maxwell's equations, however, require continuity of both the normal and tangential components of the electric field along source free boundaries such as ISB_o , RSB_o and RB. Therefore, it is necessary to add to the HF solution corrections, called **diffracted fields**, which compensate for the discontinuities and make the total HF field continuous.

It can be expected [4] that the diffracted fields will also propagate along ray paths satisfying Fermat's principle. Keller [17] extended the GO solution to include such rays that diffract from a curved edge (see Figure 9(a)). This led to Keller's law of edge diffraction [4,18]:

If the incident ray strikes the edge obliquely, making an angle β_0 with the edge, then diffracted rays lie on the surface of a cone with half angle equal to β_0 .

(a)

(b)

Figure 8: Shadow boundaries for GO fields due to: (a) Slope discontinuity (wedge) (b) Discontinuity in surface radius of curvature (smooth junction)

Mathematically, one obtains that

$$\hat{\mathbf{e}} \cdot \hat{\mathbf{s}}' = \hat{\mathbf{e}} \cdot \hat{\mathbf{s}} = \cos \beta_0 \quad (2.18)$$

where $\hat{\mathbf{e}}$ is the edge tangent vector at the point of diffraction, Q_E . This relationship is used to find diffraction points on a curved edge. The diffracted field has one caustic at the point of diffraction and another one determined by the curvature of the edge. This causes the ray optical equation for diffraction (for slow varying $\mathbf{E}^i$ at Q_E) to reduce to give [4]

$$\mathbf{E}^d(s) \sim \mathbf{E}^i(Q_E) \cdot \overline{\overline{\mathbf{D}}} \sqrt{\frac{\rho^{ec}}{(\rho^{ec} + s)s}} e^{-jks} \quad (2.19)$$

where ρ^{ec} is the second edge caustic distance for the diffracted ray, and $\overline{\overline{\mathbf{D}}}$ is the dyadic diffraction coefficient. Equation (2.19) could also have been derived differently, where it is shown to be the leading term in an asymptotic approximation of the diffracted field [5].

For a spherical incident wave front, ρ^{ec} is given by [5]

$$\frac{1}{\rho^{ec}} = \frac{1}{s'} - \frac{\hat{\mathbf{n}}_e \cdot (\hat{\mathbf{s}}' - \hat{\mathbf{s}})}{|\mathbf{a}_e| \sin^2 \beta_0} \quad (2.20)$$

where

$\hat{\mathbf{n}}_e$ = outward normal to the edge, and

$|\mathbf{a}_e|$ = absolute value of the

edge radius of curvature at Q_E .

Keller developed the Geometrical Theory of Diffraction (GTD) [17] to find an expression for $\overline{\overline{\mathbf{D}}}$. Keller's solution, however, yielded an accurate solution for the diffracted field only for field points away from the shadow boundaries. Kouyoumjian and Pathak [5] extended the solution to make it uniform; i.e., continuous across the shadow boundaries. This solution is known as the Uniform Geometrical Theory of Diffraction (UTD) and will be described next.

2.4.2 UTD Wedge Diffraction Coefficient

The three dimensional UTD diffraction coefficient for diffraction by a wedge is given in detail in [5,9]. It differs from Keller's version by including a transition function, $F(x)$, which enforces continuity of the total field across the shadow boundaries. (Keller's version is obtained by setting $F(x) = 1$ in the following equations.)

The geometry for UTD wedge diffraction at a point, Q_E , on an edge, is given in Figure 9(b) and (c). The interior wedge angle, WA, is defined in the plane perpendicular to the edge vector, $\hat{e}$. The vectors $\hat{s}'$ and $\hat{e}$ define the incident plane, and the vectors $\hat{s}$ and $\hat{e}$ define the diffraction plane. The o-face of the wedge is taken as a reference for measuring the angles, ϕ'_0 and ϕ_0 , which specify these planes, respectively. The n-face of the wedge is at the exterior wedge angle specified by $n\pi$, where

$$n = \frac{2\pi - \text{WA}}{\pi} . \quad (2.21)$$

The angle, β_0 , between $\hat{s}'$ and $\hat{e}$, is the same as the angle between $\hat{s}$ and $\hat{e}$. The unit vectors $\hat{\phi}'_0$ and $\hat{\beta}'_0$ are perpendicular to $\hat{s}'$, and the unit vectors $\hat{\phi}_0$ and $\hat{\beta}_0$ are perpendicular to $\hat{s}$, so that

$$\hat{\phi}'_0 = \frac{\hat{e} \times \hat{s}'}{|\hat{e} \times \hat{s}'|} \quad (2.22)$$

$$\hat{\beta}'_0 = \hat{s}' \times \hat{\phi}'_0 \quad (2.23)$$

$$\hat{\phi}_0 = \frac{\hat{s} \times \hat{e}}{|\hat{s} \times \hat{e}|} , \text{ and} \quad (2.24)$$

$$\hat{\beta}_0 = \hat{s} \times \hat{\phi}_0 . \quad (2.25)$$

Now, the diffraction dyad and its associated edge fixed unit vectors are given by

$$\overline{\overline{D}} = -\hat{\beta}'_0 \hat{\beta}_0 D_s - \hat{\phi}'_0 \hat{\phi}_0 D_h . \quad (2.26)$$

$D_{h,s}$ represents, respectively, the scalar diffraction coefficients for acoustically hard and soft incidence of the electric field on the edge. In terms of polarization: D_h refers to the case where $\mathbf{E}^i$ is perpendicular to the incident plane containing $\hat{\mathbf{e}}$ and $\hat{\mathbf{s}}'$; whereas, D_s refers to the case where $\mathbf{E}^i$ is parallel to the incident plane. These diffraction coefficients are expressed by

$$D_{h,s} = \left[\frac{-e^{-j\pi/4}}{2n\sqrt{2\pi k \sin \beta_0}} \right] [D_1 + D_2 \pm (D_3 + D_4)] \quad (2.27)$$

where

$$D_1 = \cot \left(\frac{\pi + \beta^-}{2n} \right) F[kL^i a^+(\beta^-)] \quad (2.28)$$

$$D_2 = \cot \left(\frac{\pi - \beta^-}{2n} \right) F[kL^i a^-(\beta^-)] \quad (2.29)$$

$$D_3 = \cot \left(\frac{\pi + \beta^+}{2n} \right) F[kL^r a^+(\beta^+)] \quad (2.30)$$

$$D_4 = \cot \left(\frac{\pi - \beta^+}{2n} \right) F[kL^r a^-(\beta^+)] \quad (2.31)$$

$$\beta^\pm = \phi_0 \pm \phi'_0 \quad (2.32)$$

$$a^\pm(\beta) = 2 \cos^2 \left(\frac{2n\pi N^\pm - \beta}{2} \right), \text{ and} \quad (2.33)$$

$$N^\pm = \text{nearest integer to } \left[\frac{\beta \pm \pi}{2\pi n} \right]. \quad (2.34)$$

For an incident spherical wave front with radius of curvature, s' , the L-parameters for the o- and n-faces of the wedge are given by

$$L^i = \frac{s s'}{(s + s')} \sin^2 \beta_0, \text{ and} \quad (2.35)$$

$$L^r = \frac{s(\rho_e^r + s)\rho_1^r \rho_2^r}{\rho_e^r(\rho_1^r + s)(\rho_2^r + s)} \sin^2 \beta_0 \quad (2.36)$$

where

$$\frac{1}{\rho_e^r} = \frac{1}{s'} - \frac{2(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e)(\hat{\mathbf{s}}' \cdot \hat{\mathbf{n}})}{|\mathbf{a}_e| \sin^2 \beta_o} \quad (2.37)$$

Figure 10: Wedge transition function.

$\rho_{1,2}^r$ = principal radii of curvature of the reflected
 wave front at Q_E on the o/n-face, and
 $\hat{n}$ = outward surface normal on the o/n-face.

The transition function, $F(x)$, contains a Fresnel integral which can be computed in series form [5] and is given by

$$F(x) = 2j\sqrt{x}e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau \quad (2.38)$$

where $F(-|x|) = F^*(|x|)$, such that $*$ means that the complex conjugate is taken. Since $F(x)$ is complex, it alters the phase of the diffraction coefficient, making it behave in a non-ray optical fashion. This is especially true in transition regions, defined by $|x| < 2\pi$, around the various shadow boundaries. Outside the transition regions, $F(x) \approx 1$ so that the diffraction coefficient reduces to Keller's form. A plot of the phase and magnitude of $F(x)$ is given in Figure 10.

There are four possible shadow boundaries for general incidence on a wedge: n-face ISB; o-face ISB; n-face RSB and o-face RSB. The four shadow boundaries may

not all exist within the exterior wedge angle, and in cases such as grazing incidence and 180° wedge angles, two may coincide. Each shadow boundary represents a discontinuity in either the incident or reflected field. Correspondingly, each of the four terms in the UTD diffraction coefficient has peak amplitude and discontinuous phase at their respective shadow boundaries (when they exist). The diffracted field discontinuity at each shadow boundary is just enough to compensate for the discontinuity in either the incident or reflected fields to make the total field continuous. This continuity is achieved by making use of the characteristics of $F(x)$ at the shadow boundaries.

In Keller's version of the diffraction dyad, $F(x)$ was taken to be identical to one. Consequently, the cotangent terms caused the diffraction coefficient to be infinite on the shadow boundaries. On the other hand, $F(x)$ goes to zero on the shadow boundaries, and the products of the cotangents and $F(x)$ yield a finite but discontinuous field across the various shadow boundaries.

2.4.3 UTD Smooth Junction Diffraction Coefficient

A smooth junction is defined here as a junction on the surface of the scattering body where the slope is continuous, but the surface radii of curvature are discontinuous. A step discontinuity in surface radius of curvature results in a step discontinuity in the GO reflected fields along the reflection boundary, RB, as shown in Figure 8(b).

Chu [9] used the UTD wedge diffraction coefficient, Equation (2.27), to find a diffraction coefficient for this case where $WA = 180^\circ$ and $n = 1$. There are no incident shadow boundaries, and correspondingly terms D_1 and D_2 cancel; while, terms D_3 and D_4 compensate for the reflected field discontinuity. Equation (2.27)

then reduces to

$$D_{h,s} = \pm \left[\frac{-e^{-j\pi/4}}{\sqrt{2\pi k} \sin \beta_0} \right] \tan \left(\frac{\beta^+}{2} \right) \{ F[kL^{ro}a(\beta^+)] - F[kL^{rn}a(\beta^+)] \} . \quad (2.39)$$

Kouyoumjian and Pathak [19] extended this solution to improve its accuracy outside the transition regions by adding multiplication factors. The smooth junction diffraction coefficients are then given by

$$D_{h,s}^{sm} = f_{h,s} D_{h,s} \quad (2.40)$$

in which

$$f_h = \frac{1 + \cos \phi'_0 \cos \phi'_0}{\left(1 + \frac{\sin \phi_0}{\sin \phi'_0} \right) \cos^2 \left(\frac{\phi_0 - \phi'_0}{2} \right)} , \text{ and} \quad (2.41)$$

$$f_s = \frac{\sin \phi'_0 \sin \phi_0}{\left(1 + \frac{\sin \phi_0}{\sin \phi'_0} \right) \cos^2 \left(\frac{\phi_0 - \phi'_0}{2} \right)} . \quad (2.42)$$

2.4.4 Backscattered UTD Diffracted Field

In the case of far-field backscatter, where $\hat{s} = \hat{s}'$ and $s, s' \rightarrow \infty$, the parameters for computing the diffraction coefficients, $D_{h,s}$ and $D_{h,s}^{sm}$, become

$$\begin{aligned} \beta_0 &= 90^\circ \\ \sin \beta_0 &= 1 \\ \hat{\beta}'_0 &= \hat{e} \\ \hat{\beta}_0 &= -\hat{e} \\ \phi_0 &= \phi'_0 \\ \hat{\phi}_0 &= \hat{\phi}'_0 \\ &= \hat{\beta}'_0 \times \hat{s}' \\ L^i &\rightarrow \infty \end{aligned}$$

$$F[\mathbf{k}L^i\mathbf{a}^\pm] \rightarrow 1$$

$$L^r \rightarrow \frac{\rho_1^r \rho_2^r}{\rho_e^r} \quad (2.43)$$

$$\rho_e^r \rightarrow -\frac{|\mathbf{a}_e|}{2(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e)(\hat{\mathbf{s}}' \cdot \hat{\mathbf{n}})} , \text{ and} \quad (2.44)$$

$$\rho^{ec} \rightarrow -\frac{|\mathbf{a}_e|}{2\hat{\mathbf{n}}_e \cdot \hat{\mathbf{s}}'} . \quad (2.45)$$

In order to compute the co-polarized backscattered diffracted field, the incident field which is given in ray-fixed coordinates in (2.1) must be expressed in edge-fixed coordinates. This is done through a coordinate rotation, such that

$$\mathbf{E}^i = [e_{\beta'_0}^i \hat{\beta}'_0 + e_{\phi'_0}^i \hat{\phi}'_0] e^{-j\mathbf{k}\mathbf{s}'} \quad (2.46)$$

where

$$\begin{bmatrix} e_{\beta'_0}^i \\ e_{\phi'_0}^i \end{bmatrix} = \begin{bmatrix} \hat{\beta}'_0 \cdot \hat{\theta} & \hat{\beta}'_0 \cdot \hat{\phi} \\ \hat{\phi}'_0 \cdot \hat{\theta} & \hat{\phi}'_0 \cdot \hat{\phi} \end{bmatrix} \begin{bmatrix} e_{\theta}^i \\ e_{\phi}^i \end{bmatrix} . \quad (2.47)$$

The diffracted field in edge-fixed coordinates is

$$\mathbf{E}^d = [E_{\beta_0}^d \hat{\beta}_0 + E_{\phi_0}^d \hat{\phi}_0] \frac{e^{-j\mathbf{k}\mathbf{s}}}{s} \quad (2.48)$$

where

$$E_{\beta_0}^d = -e_{\beta'_0}^i D_s \sqrt{\rho^{ec}} , \text{ and} \quad (2.49)$$

$$E_{\phi_0}^d = -e_{\phi'_0}^i D_h \sqrt{\rho^{ec}} . \quad (2.50)$$

The backscattered diffracted field in ray-fixed coordinates is given by

$$\mathbf{E}^d = [E_{\theta}^d \hat{\theta} + E_{\phi}^d \hat{\phi}] \frac{e^{-j\mathbf{k}\mathbf{s}}}{s} \quad (2.51)$$

where

$$\begin{bmatrix} E_{\theta}^d \\ E_{\phi}^d \end{bmatrix} = \begin{bmatrix} \hat{\theta} \cdot \hat{\beta}_0 & \hat{\theta} \cdot \hat{\phi}_0 \\ \hat{\phi} \cdot \hat{\beta}_0 & \hat{\phi} \cdot \hat{\phi}_0 \end{bmatrix} \begin{bmatrix} E_{\beta_0}^d \\ E_{\phi_0}^d \end{bmatrix} . \quad (2.52)$$

Figure 11: Backscattered diffracted field coordinate vectors.

Finally, the co-polarized diffracted field is expressed by

$$U^d = |e_\theta E_\theta^d + e_\phi E_\phi^d| \frac{1}{s} . \quad (2.53)$$

The coordinate rotations between the edge-fixed and ray-fixed coordinate systems simplify greatly for backscattering. Figure 11 shows a view along the incident field ray direction to illustrate the relative directions of unit vectors for backscattering. Let us assume that

$$F1 = \hat{\beta}'_0 \cdot \hat{\theta} , \text{ and} \quad (2.54)$$

$$F2 = \hat{\beta}'_0 \cdot \hat{\phi} . \quad (2.55)$$

Then, the coordinate rotations become

$$\begin{bmatrix} e_{\beta'_0}^i \\ e_{\phi'_0}^i \end{bmatrix} = \begin{bmatrix} +F1 & +F2 \\ -F2 & +F1 \end{bmatrix} \begin{bmatrix} e_\theta^i \\ e_\phi^i \end{bmatrix} , \text{ and} \quad (2.56)$$

$$\begin{bmatrix} E_\theta^d \\ E_\phi^d \end{bmatrix} = \begin{bmatrix} -F1 & -F2 \\ -F2 & +F1 \end{bmatrix} \begin{bmatrix} E_{\beta'_0}^d \\ E_{\phi'_0}^d \end{bmatrix} . \quad (2.57)$$

2.5 Field Magnitudes in Caustic Regions

In this section, the backscattered field values are obtained at the three types of caustics described in Section 2.1 and shown in Figure 4. The three types are flat surface, straight line and elliptic ring caustics. In all three cases the caustic occurs due to a whole surface or line contributing to the backscattered diffracted field in a specular direction. Both the GO and UTD solutions fail in these caustic regions in that their magnitudes become unbounded:

- For the first two types one or both of the principal radii of curvature ($R_{1,2}$) on a surface are infinite, so that the backscattered GO field (Equation (2.17)) off the surface becomes unbounded. In the UTD diffracted field solution, variable L' for the term that compensates in the backscatter direction will also be infinite so that the corresponding transition function becomes unity. As a result, there is no compensation for the cotangent term that goes to infinity at that angle, yielding an unbounded diffraction solution in the caustic region.
- In the case of a ring caustic, the incident vector, $\hat{s}'$, is perpendicular to the edge normal, $\hat{n}_e$. Therefore, the second edge caustic, ρ^{ec} , and thus the spread factor and diffracted field solution become infinite.

In the caustic regions where the HF solutions fail, accurate results can be obtained by finding equivalent currents on the surface and using the radiation integral to find the backscattered field. These field values will be used to determine the UTD results in caustic regions. This is done since the GO backscattered fields for flat shapes occur only at the exact specular angles, so that the diffracted field at a caustic will include the specular GO field.

The UTD diffraction solution fails gracefully around the caustic region, so that the correction is necessary in a small region around the actual far-field caustic. In the envelope solution for RCS, however, the emphasis is on identifying correct peak and low values for the envelope, and not so much on the exact values close to the peaks. Therefore, only the specular field values will be obtained and then used to limit diffracted fields at and around the caustic regions.

2.5.1 Flat Surface Caustic

The well known Physical Optics (PO) solution for the peak RCS of a flat plate is given in [14, p116] by

$$\sigma_{flat}^c = \frac{4\pi\mathcal{A}^2}{\lambda^2} \quad (2.58)$$

where $\mathcal{A}$ is the surface area of the plate. Expressed as a field quantity, the specular limit in the caustic region of a flat surface is

$$U_{flat}^c = \frac{\mathcal{A}}{\lambda_s} . \quad (2.59)$$

For an elliptical disk with major and minor half-axes ‘a’ and ‘b’, the surface area is

$$\mathcal{A} = \pi ab . \quad (2.60)$$

2.5.2 Straight Line Caustic

For the case of specular backscatter from a straight line, Chu [9] modified the UTD diffraction coefficient by using an average spread factor to obtain a continuous and finite but non-uniform diffracted field through the caustic region. Ebihara [11] incorporated the equivalent line current solution to modify the UTD transition function for this case. However, in the specular direction both these solutions reduce to the PO result given by Ross [20] for a circular cone frustum. Since only

Figure 12: Circular cone frustum geometry.

the specular value is of interest in the envelope method, this value will be used to limit the diffracted field in the specular direction from the cone frustum.

The specular value for the RCS of a circular cone frustum is

$$\sigma_{line}^c = \frac{8\pi}{9\lambda} \frac{[a_2^{3/2} - a_1^{3/2}]^2}{\sin^2 \gamma \cos \gamma} \quad (2.61)$$

where $a_{1,2}$ are the radii of the circular edges of the cone frustum, and γ is the flare angle as shown in Figure 12.

Now consider an elliptic cone frustum. Assume that the ellipses at its two ends have the same aspect ratio. Then the illuminated line on the surface for a longitudinal cut plane, α (see Figure 13), can be regarded locally as part of a circular cone frustum with edge radii equal to the radii of curvature of the elliptical edges at α . If $a_{e1,2}$ are these radii of curvature and γ is the local flare angle, then the specular field magnitude for the straight line at α is given by

$$U_{cfrus}^c = \sqrt{\frac{2}{\lambda \cos \gamma}} \left| \frac{(a_{e2})^{3/2} - (a_{e1})^{3/2}}{3 \sin \gamma} \right| \frac{1}{s}. \quad (2.62)$$

Figure 13: Specular line on an elliptic cone frustum.

Note that, if d is the length of the line, and t is the length of the cone frustum (see Figure 13), then

$$\cos \gamma = \frac{t}{d}, \text{ and} \quad (2.63)$$

$$\sin \gamma = \frac{\sqrt{d^2 - t^2}}{d}. \quad (2.64)$$

An equation for the edge radius of curvature, $a_e(a,b)$, is given in Appendix A. The same result can be used for the specular field from a cone, in which case either a_{e1} or a_{e2} will be zero.

For a cylinder, where $a_{e1} = a_{e2}$ and $\gamma = 0$, Equation (2.62) cannot be used directly due to the singularities in the numerator and denominator. Taking the limit as $a_{e2} \rightarrow a_{e1} = a_e$ and applying L'Hospital's rule yields

$$U_{cyl}^c = \sqrt{\frac{1}{2\lambda}} d \frac{\sqrt{a_e}}{s} \quad (2.65)$$

2.5.3 Ring Caustic for Diffraction

Backscatter in the direction perpendicular to the plane of an elliptic ring edge can be treated using a combination of UTD diffraction theory and the radiation integral. The diffraction theory specifies the diffraction at each point on the ring and the radiation integral sums these contributions. The method is known as the equivalent edge current method. This method, introduced by Ryan and Peters [7] for a circular ring, has been extended by Chu [9, pp 227 - 231], for the case of an elliptic ring. The elliptic ring solution will be derived in this section, and simplified to find the specular backscatter field for use in the envelope method.

The geometry for an elliptic ring edge in the YZ-plane is shown in Figure 14. A unit magnitude incident plane wave propagates in the $-\hat{x}$ direction and its magnitude at the edge is given by

$$\mathbf{E}_{x=0}^i = [e_\theta \hat{\theta} + e_\phi \hat{\phi}] \quad (2.66)$$

$$= [e_\theta \hat{y} - e_\phi \hat{z}] . \quad (2.67)$$

Next, the diffracted field needs to be found in the backscatter direction.

A diffraction point corresponding to the angle, α , on the elliptic ring in Figure 14 can be treated locally as part of a straight edge. For such a point, the UTD diffracted field in the backscatter direction is finite. Now, if this field is equated to the field radiated by a two-dimensional current filament, then a solution can be found for the equivalent current, at the point, that would produce a backscattered field of the same magnitude as the diffracted field. In general, both electric and magnetic currents are excited. These equivalent currents are expressed by Chu [9, p40] as

$$I_e(\alpha) = -\frac{\hat{e} \cdot \mathbf{E}^i}{Z_0 \sin \beta_0} D_s \sqrt{\frac{8\pi}{k}} e^{-j\pi/4} , \text{ and} \quad (2.68)$$

Figure 14: Geometry for an elliptical ring edge.

$$\text{Im}(\alpha) = -\frac{(\hat{\mathbf{e}} \times \hat{\mathbf{s}}') \cdot \mathbf{E}^i}{\sin \beta_0} D_h \sqrt{\frac{8\pi}{k}} e^{-j\pi/4} \quad (2.69)$$

where both the edge vector, $\hat{\mathbf{e}}$, and the diffraction coefficients, $D_{h,s}$, are functions of α . For a given α , the ellipse parameter is given by (see Appendix A)

$$\nu = \tan^{-1} \left[\frac{a}{b} \tan \alpha \right] \quad (2.70)$$

where 'a' and 'b' are the major and minor half-axis lengths of the elliptic edge.

The angles and vectors in Equation (2.69) for on-axis backscatter is given by

$$\sin \beta_0 = 1 \quad (2.71)$$

$$\hat{\mathbf{s}}' = -\hat{\mathbf{x}}, \text{ and} \quad (2.72)$$

$$\hat{\mathbf{e}} = \frac{a \sin \nu \hat{\mathbf{y}} - b \cos \nu \hat{\mathbf{z}}}{\sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu}}. \quad (2.73)$$

In computing the diffraction coefficient, it is assumed that all the transition functions are unity. This will be valid except for the special case where the ring caustic direction coincides with the specular reflection direction for one of the faces of the edge, as for a disk. In that special case the flat surface caustic amplitude (2.59) will be used in the treatment of the ring caustic. The diffraction coefficients are now given by

$$D_{h,s} = \left[\frac{-e^{-j\pi/4}}{\sqrt{2\pi k} \sin \beta_0} \right] \mathcal{D}_{h,s} , \quad (2.74)$$

where

$$\mathcal{D}_{h,s} = \frac{1}{2n} \left\{ 2 \cot \left(\frac{\pi}{2n} \right) \pm \left[\cot \left(\frac{\pi + 2\phi'_0}{2n} \right) + \cot \left(\frac{\pi - 2\phi'_0}{2n} \right) \right] \right\} . \quad (2.75)$$

Let us define the variable Δ as

$$\Delta = \sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu} . \quad (2.76)$$

Then, the equivalent edge currents reduce to give

$$I_e(\nu) = \frac{1}{\Delta} [e_\phi a \sin \nu + e_\theta b \cos \nu] \left(\frac{2j}{kZ_0} \right) \mathcal{D}_s , \text{ and} \quad (2.77)$$

$$I_m(\nu) = \frac{1}{\Delta} [e_\phi B \cos \nu - e_\theta A \sin \nu] \left(\frac{-2j}{k} \right) \mathcal{D}_h . \quad (2.78)$$

The backscattered field is the sum of the radiation integrals for I_e and I_m and can be represented as

$$\mathbf{E}^s = (E_\theta^e + E_\theta^m) \hat{\theta} + (E_\phi^e + E_\phi^m) \hat{\phi} . \quad (2.79)$$

The field components in integral form are

$$E_\theta^e = -j\omega\nu \left(\frac{e^{-jks}}{4\pi s} \right) \int I_e e^{jk\mathbf{r}' \cdot \hat{\mathbf{r}}} \hat{\theta} \cdot d\mathbf{l} , \text{ and} \quad (2.80)$$

$$E_\phi^e = -j\omega\nu \left(\frac{e^{-jks}}{4\pi s} \right) \int I_e e^{jk\mathbf{r}' \cdot \hat{\mathbf{r}}} \hat{\phi} \cdot d\mathbf{l} \quad (2.81)$$

$$E_{\theta}^m = -jk \left(\frac{e^{-jks}}{4\pi s} \right) \int I_m e^{jk\mathbf{r}' \cdot \hat{\mathbf{r}}} \hat{\boldsymbol{\phi}} \cdot d\mathbf{l} \quad (2.82)$$

$$E_{\phi}^m = jk \left(\frac{e^{-jks}}{4\pi s} \right) \int I_m e^{jk\mathbf{r}' \cdot \hat{\mathbf{r}}} \hat{\boldsymbol{\theta}} \cdot d\mathbf{l} \quad (2.83)$$

and integration is around the full elliptic ring. These integrals can be evaluated by first noting that

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}} \quad (2.84)$$

$$\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}} \quad (2.85)$$

$$y = a \cos \nu \quad (2.86)$$

$$z = b \sin \nu \quad (2.87)$$

$$d\mathbf{l} = -\hat{\mathbf{e}} d\mathbf{l} \quad (2.88)$$

$$= -\hat{\mathbf{e}} \sqrt{dy^2 + dz^2} \quad (2.89)$$

$$= -\hat{\mathbf{e}} \Delta d\nu, \text{ and} \quad (2.90)$$

$$\mathbf{r}' \cdot \hat{\mathbf{r}} = 0. \quad (2.91)$$

In general, the integral equations (2.80) and (2.81) will have to be computed by numerical integration. However, it may be the case that $\mathcal{D}_{h,s}(n, \phi'_0)$ varies slowly as a function of ν around the ellipse, compared to the other variables in the integral. This is true for a circular cone frustum but gets more and more inaccurate as 'a/b', the aspect ratio of the ellipse, deviates more and more from one, and as the flare angle, γ , gets bigger. However, an average value $\mathcal{D}_{h,s}^{ave}$ may be computed at the ellipse parameter $\nu = 45^\circ$, and then assumed to be a constant. The benefit of doing this is that $\mathcal{D}_{h,s}^{ave}$ may then be factored out of the integrands in (2.80) through (2.83). The remaining integrals can be easily evaluated in terms of elliptic integrals, described in Appendix B. Let us define the following elliptic integrals:

$$\mathcal{I}_1(a, b) = \int_{-\pi}^{\pi} \frac{\sin^2 \nu}{\Delta} d\nu \quad (2.92)$$

$$\mathcal{I}_2(a, b) = \int_{-\pi}^{\pi} \frac{\cos^2 \nu}{\Delta} d\nu, \text{ and} \quad (2.93)$$

$$\mathcal{I}_3(a, b) = \int_{-\pi}^{\pi} \frac{\sin \nu \cos \nu}{\Delta} d\nu. \quad (2.94)$$

$$\equiv 0 \quad (2.95)$$

Note that $\mathcal{I}_3$ is zero due to the integration of an even function times an uneven function over a symmetric interval. Also, note that, if $b = a$, then the first two integrals reduce to

$$\mathcal{I}_{1,2} = \frac{\pi}{a}. \quad (2.96)$$

Finally, the components of the backscattered field are

$$E_{\theta}^s = e_{\theta} \left(\frac{e^{-jks}}{2\pi s} \right) (b^2 \mathcal{I}_2 \mathcal{D}_h - a^2 \mathcal{I}_1 \mathcal{D}_s) \quad (2.97)$$

$$E_{\phi}^s = e_{\phi} \left(\frac{e^{-jks}}{2\pi s} \right) (a^2 \mathcal{I}_1 \mathcal{D}_h - b^2 \mathcal{I}_2 \mathcal{D}_s) \quad (2.98)$$

and the ring caustic field value used to limit the UTD diffracted fields in the caustic region is

$$U_{ring}^c = |E_{\theta}^s e_{\theta} + E_{\phi}^s e_{\phi}|. \quad (2.99)$$

2.6 Ring Caustic for Reflection

For geometries such as the faired body in Figure 1(b) where a faired shape matches to a disk, the surface of the faired shape becomes flat in the plane perpendicular to the central axis of the body. Therefore, one of the principal radii of curvature of the faired shape will become infinite right at the junction, so that the GO field solution becomes unbounded there. An equivalent problem is that of computing the reflected fields from an elliptic toroid, with incidence parallel to the axis of the toroid, as shown in Figure 15. The Figure shows both the front view as

Figure 15: Geometry for reflection from a toroid in the ring caustic direction.

seen by an observer at the source and a side view of the toroid. The dotted line on the toroid indicates the line that reflects the incident field for on-axis incidence.

The GO Equivalent Line Currents for reflection from a surface such as this are given by [6]

$$I_e(\alpha) = -\frac{\hat{\mathbf{e}} \cdot \mathbf{E}^i}{Z_0 \sin \beta_0} \sqrt{\rho_1^r} \sqrt{\frac{8\pi}{k}} e^{-j\pi/4}, \text{ and} \quad (2.100)$$

$$I_m(\alpha) = -\frac{(\hat{\mathbf{e}} \times \hat{\mathbf{s}}') \cdot \mathbf{E}^i}{\sin \beta_0} \sqrt{\rho_1^r} \sqrt{\frac{8\pi}{k}} e^{-j\pi/4} \quad (2.101)$$

where $\sqrt{\rho_1^r} = \sqrt{R/2}$, and R is the radius of curvature of the toroid in the plane perpendicular to the edge vector, $\hat{\mathbf{e}}$. For on-axis incidence, $\sin \beta_0 = 1$. Note that R will change as a function of α around the elliptic reflection contour. As before, one can assume that R varies relatively slowly inside the integral sign, so that it

can be set equal to an average value, R^{ave} , and then taken out of the integral. The resulting integral can be computed in a way similar to what was done in the previous section. In this case, the caustic value for the GO field reduces to give

$$U_{GOring}^c = \left| \frac{1}{s} \sqrt{\frac{k}{2\pi}} \sqrt{\frac{R^{ave}}{2}} (b^2 I_2 + a^2 I_1) \right|. \quad (2.102)$$

The next section describes the computation of the envelope RCS.

2.7 Envelope Description of Radar Cross Section

Once all the desired field magnitudes are computed, the total backscattered field is the vector sum of contributions from all the scattering centers. In a first order analysis, only direct reflected and diffracted fields are included. If N is the number of scattering centers then

$$E_{co,tot}^{bs} = \sum_{i=1}^N E_{co,i}^{bs}. \quad (2.103)$$

An angular plot of $E_{co,tot}^{bs}$ will in general have peaks, and many high frequency oscillations. This happens because contributions from different scattering centers add or subtract depending on changes in phase as a function of angle. The phase changes are due to the relative locations of the scattering centers and also due to the scattering mechanisms involved. To show such a pattern accurately will require data points at small angular increments and accurate determination of the locations of scattering centers. In addition, if the frequency changes, the positions of nulls will vary greatly while the envelope of the pattern stays fairly constant. This is illustrated in Figure 16 which show plots of computed HF backscattered fields for a circular disk at 1 and 2 GHz. Note that the peaks change—due to the specular values that depend on frequency.

The envelope of the backscattered field results from adding all the scattering contributions in phase; i.e. adding scalar magnitudes instead of complex magni-

Figure 16: Backscattered fields for a circular disk with radius $a = 0.5$ meter at 1 and 2 GHz.

tudes. Since the phase of E_{co}^{bs} contains information about both the mechanism and the location of a scattering center, addition of field amplitudes yields a maximum possible envelope, such that

$$U_{max} = \sum_{i=1}^N U_i . \quad (2.104)$$

On the other hand, addition of backscattered powers yields a more realistic average envelope [12], which is given by

$$U_{ave} = \sqrt{\sum_{i=1}^N (U_i)^2} . \quad (2.105)$$

This is particularly true for large bodies that may be non-rigid, where the locations of scattering centers have some randomness over time. A good example is an aircraft wing which moves up and down during a normal flight. This average also appears to be reasonably close to the traditional median valued curve.

Note that the field envelope is inversely proportional to the far-field distance parameter, s . In order to eliminate this dependency, results are presented as radar cross section (RCS) envelopes where

$$\begin{aligned} \sigma &\triangleq \lim_{s \rightarrow \infty} 4\pi s^2 \left| \frac{E^{bs}}{E^i} \right| && [\text{meter}^2] \\ &= 4\pi s^2 U^2 && [\text{meter}^2] \\ &= 10 \log(4\pi s^2 U^2) && [\text{dBsm}] \\ &= 10 \log(4\pi s^2 U^2 / \lambda^2) && [\text{dB}/\lambda^2] \end{aligned} \quad (2.106)$$

2.8 Summary

The approach for computing the envelope of the high frequency RCS of scattering bodies has been described in this Chapter. The solution includes GO reflected fields from doubly curved surfaces and UTD diffracted fields from discontinuities

in surface slope or surface radius of curvature. Solutions were also found for the field values in caustic regions, where both these solutions are invalid in that they become infinite.

The next Chapter deals with the definition of the surfaces of isolated scattering bodies, composed of canonical and faired shapes. Equations are also given for finding the surface characteristics which are needed for RCS computations.

CHAPTER III

DEFINING SURFACES: ISOLATED BODIES WITH FAIRED JUNCTIONS

In this Chapter, the equations are given for describing the surfaces of isolated scattering bodies. The term “isolated” refers to a body with shapes centered around only one axis, although it need not be a surface of revolution. In Chapter V, the surfaces will be extended to include side bodies that intersect a main body.

A typical body is centered around an axis and composed of a series connections with canonical shapes of elliptic cross section. The canonical shapes used here are **disks** (flat surfaces), **cones**, **cylinders** and **cone frustrums** (singly curved surfaces) and **ellipsoids** and **ogives** (doubly curved surfaces). The junctions between bodies are defined in a plane perpendicular to the central axis.

Singly curved surfaces are assumed to have a constant aspect ratio (ratio between the half-axis lengths of the elliptic cross sections). In other words, the surfaces are not “ruled surfaces.”

It is assumed that there is perfect contour matching at the junctions between the shapes. However, there may be a slope discontinuity across a junction, forming a wedge. **Faired shapes**, which replace sharp junctions with smooth surfaces, will also be described.

The equations for ellipses and various functions of elliptic curves are used frequently throughout the chapter. These equations and functions are listed in

Figure 17: General surface with elliptic cross section.

Appendix A for easy reference.

First, let us look at a description of a general surface of elliptic cross section. The equations for specific canonical shapes will be given later.

3.1 Equations for a General Surface with Elliptic Cross Section

Figure 17 shows the outline of a general body with elliptic cross section in the XYZ-coordinate system. The body is centered around the X-axis, and its beginning and ending points are indicated by x_1 and x_2 . For a given x -location, the Y- and Z-axis half-lengths are denoted by $A(x)$ and $B(x)$, respectively. The angle, α , is defined in the YZ-plane, starting on the positive Y-axis and increasing toward the positive Z-axis. The distance from the center of an elliptical cross section to a point on the surface is indicated by ρ . The surface is now uniquely

defined by the variables x and α , such that

$$\mathbf{r}(x, \alpha) = [x, Y(x, \alpha), Z(x, \alpha)] , \text{ with} \quad (3.1)$$

$$Y(x, \alpha) = \rho \cos \alpha \quad (3.2)$$

$$= A(x) \cos \nu \quad (3.3)$$

$$Z(x, \alpha) = \rho \sin \alpha \quad (3.4)$$

$$= B(x) \sin \nu . \quad (3.5)$$

The ellipse angular parameter, ν , is given by

$$\nu = \tan^{-1} \left[\frac{A(x)}{B(x)} \tan \alpha \right] . \quad (3.6)$$

Two independent variables are needed to define points on a surface. When working with different shapes in a series connection, it is convenient to use the same variables throughout the body. It is also more convenient to match real angles and coordinates, rather than a variable with no physical meaning, such as ν . For these reasons the independent variables x and α will be used to describe all the shapes in this chapter.

In order to compute surface characteristics, it is necessary to compute the first and second partial derivatives of $\mathbf{r}$ with respect to x and α . Those are given by

$$\mathbf{r}_x = [1, Y_x, Z_x] \quad (3.7)$$

$$\mathbf{r}_\alpha = [0, Y_\alpha, Z_\alpha] \quad (3.8)$$

$$\mathbf{r}_{xx} = [0, Y_{xx}, Z_{xx}] \quad (3.9)$$

$$\mathbf{r}_{x\alpha} = [0, Y_{x\alpha}, Z_{x\alpha}] , \text{ and} \quad (3.10)$$

$$\mathbf{r}_{\alpha\alpha} = [0, Y_{\alpha\alpha}, Z_{\alpha\alpha}] . \quad (3.11)$$

The subscripts x and α denote partial derivatives.

Let us now define the **aspect ratio**, Γ_A , and the **slope ratio**, Γ_S , as

$$\Gamma_A \triangleq \frac{A(x)}{B(x)}, \text{ and} \quad (3.12)$$

$$\Gamma_S \triangleq \frac{A_x(x)}{B_x(x)}. \quad (3.13)$$

For **canonical shapes**, it is mainly $A(x)$ and $B(x)$ that are different for the different shapes. Also, if the aspect ratio is constant with respect to x , then the derivatives are given by:

$$\mathbf{r}_x = [1, A_x \cos \nu, B_x \sin \nu] \quad (3.14)$$

$$\mathbf{r}_\alpha = [0, -A \sin \nu \nu_\alpha, B \cos \nu \nu_\alpha] \quad (3.15)$$

$$\mathbf{r}_{xx} = [0, A_{xx} \cos \nu, B_{xx} \sin \nu] \quad (3.16)$$

$$\mathbf{r}_{x\alpha} = [0, -A_x \sin \nu \nu_\alpha, B_x \cos \nu \nu_\alpha], \text{ and} \quad (3.17)$$

$$\mathbf{r}_{\alpha\alpha} = [0, -A(\cos \nu \nu_\alpha^2 + \sin \nu \nu_{\alpha\alpha}), -B(\sin \nu \nu_\alpha^2 - \cos \nu \nu_{\alpha\alpha})] \quad (3.18)$$

The terms ν_α and $\nu_{\alpha\alpha}$ are found by taking partial derivatives of ν with respect to α in Equation (3.6). Expressions for ν_α and $\nu_{\alpha\alpha}$ are given in Appendix A. The surface derivatives for each shape is found by using its particular expressions for A , A_x , A_{xx} , B , B_x , and B_{xx} in (3.14) to (3.18).

Next, a discussion will be given on finding surface characteristics for the general surface described in this section.

3.2 Computing Surface Characteristics for a General Surface

In order to compute diffracted and reflected fields off a surface, one needs to compute the following surface characteristics:

$$\hat{\mathbf{r}}_x = \text{X-directed tangent vector,}$$

- $\hat{r}_\alpha$ = α -directed tangent vector,
- $\hat{n}$ = Outward directed normal vector,
- $\hat{U}_{1,2}$ = Principal direction vectors, and
- $R_{1,2}$ = Principal radii of curvature.

All of the above can be computed for a general surface using the derivatives given in (3.7) to (3.11). The procedure is straight-forward and described in various references on Differential Geometry, such as [22] and [23]. The necessary equations are given in Appendix D.

The normal and tangent vectors for an arbitrary point P on a surface are shown in Figure 18(a). The tangents are $\hat{r}_x$ and $\hat{r}_\alpha$, and the normal, $\hat{n}$, is equal to the cross product of $\hat{r}_\alpha$ and $\hat{r}_x$.

The principal radii of curvature, $R_{1,2}$, are the maximum and minimum values of surface radius of curvature for surface curves through a point. Such curves through a point, P, are shown in Figure 18(a) and (b). These curves are orthogonal at P and their tangent vectors, $\hat{U}_{1,2}$, are the principal directions. In general, $\hat{U}_{1,2}$ do not coincide with the X- and α -tangent vectors.

3.3 Equations for Canonical Shapes

Canonical shapes are the basic building blocks for composite surfaces. In this report, each canonical shape is defined by its beginning and ending points, x_1 and x_2 , and four constant parameters, P_1 , P_2 , P_3 and P_4 , which will be described below.

(b) Principal directions and principal radii of curvature

Figure 18: Surface characteristics on a general surface.

3.3.1 Cone Frustrum, Cylinder, Cone and Disk

The cylinder, cone and disk are all special cases of a cone frustrum. These shapes are shown in Figure 19(a) through (d). The constant parameters for the cone frustrum in Figure 19(a) are

$$\begin{aligned}
 P_1 &= a_1 = \text{Y-axis half-length at } x_1 \\
 P_2 &= b_1 = \text{Z-axis half-length at } x_1 \\
 P_3 &= a_2 = \text{Y-axis half-length at } x_2, \text{ and} \\
 P_4 &= b_2 = \text{Z-axis half-length at } x_2.
 \end{aligned}
 \tag{3.19}$$

The cone frustrum surface is described using

$$A(x) = a_1 + m_y(x - x_1) \tag{3.20}$$

$$B(x) = b_1 + m_z(x - x_1) \tag{3.21}$$

in Equations (3.1) to (3.5) where

$$m_y = \frac{a_2 - a_1}{x_2 - x_1}, \text{ and} \tag{3.22}$$

$$m_z = \frac{b_2 - b_1}{x_2 - x_1}. \tag{3.23}$$

Surface derivatives are computed using

$$A_x = m_y \tag{3.24}$$

$$A_{xx} = 0 \tag{3.25}$$

$$B_x = m_z, \text{ and} \tag{3.26}$$

$$B_{xx} = 0 \tag{3.27}$$

in Equations (3.14) to (3.17). For the cone frustrum, the aspect ratio is taken to be the same at the two end points. This choice yields a constant aspect ratio and

slope ratio as a function of x , so that

$$\Gamma_A^{cfrus} = \Gamma_S^{cfrus} = \frac{a_1}{b_1} = \frac{a_2}{b_2} . \quad (3.28)$$

The cylinder, cone and disk are now defined as special cases of the cone frustum:

- **Cylinder:** The parameters at the two ends are equal, such that $a_1 = a_2$ and $b_1 = b_2$.
- **Cone:** The parameters at one of the two ends are zero, such that either $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$, and $x_1 \neq x_2$.
- **Disk:** The surface of the disk can be seen as that of a cone with zero length, such that either $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$, and $x_1 = x_2$.

3.3.2 Ellipsoid

An ellipsoid is shown in Figure 20(a). The constant parameters for the ellipsoid are

$$\begin{aligned} P_1 &= x_0 = \text{X-axis center of the ellipsoid} \\ P_2 &= a = \text{Y-axis half-length at } x_0 \\ P_3 &= b = \text{Z-axis half-length at } x_0, \text{ and} \\ P_4 &= c = \text{X-axis half-length at } x_0. \end{aligned} \quad (3.29)$$

The ellipsoid surface is described using

$$A(x) = a T(x - x_0) , \text{ and} \quad (3.30)$$

$$B(x) = b T(x - x_0) \quad (3.31)$$

in Equations (3.1) to (3.5) where

$$T(x - x_0) = \frac{\sqrt{c^2 - (x - x_0)^2}}{c} . \quad (3.32)$$

(a) Cone Frustum

(b) Cylinder

(c) Cone

(d) Disk

Figure 19: Cone frustum, cylinder, cone and disk geometry.

(a) Ellipsoid geometry.

(b) Matching an ellipsoid to a cone frustum.

Figure 20: Ellipsoid geometry.

Surface derivatives are computed using

$$A_x = a T_x \quad (3.33)$$

$$A_{xx} = a T_{xx} \quad (3.34)$$

$$B_x = b T_x, \text{ and} \quad (3.35)$$

$$B_{xx} = b T_{xx} \quad (3.36)$$

in Equations (3.14) to (3.17) where

$$T_x = \frac{-x - x_0}{c \sqrt{c^2 - (x - x_0)^2}}, \text{ and} \quad (3.37)$$

$$T_{xx} = \frac{-c}{(c^2 - (x - x_0)^2)^{3/2}}. \quad (3.38)$$

It follows from (3.30) and (3.31), and (3.33) and (3.35), that the aspect ratio and the slope ratio are constant and equal:

$$\Gamma_A^{elld} = \Gamma_S^{elld} = \frac{a}{b}. \quad (3.39)$$

It is possible to find an ellipsoid that will match smoothly to the end of a cone frustrum, since the aspect and slope ratios for the cone frustrum and the ellipsoid are both constant and equal. One way to define such an ellipsoid is by specifying the end point, x_t , for the ellipsoid on the x-axis. The YX-plane contour for a cone frustrum with an ellipsoid at one end is shown in Figure 20(b).

Let x_1 denote the x-coordinate at the end of the cone frustrum. The slope of the cone frustrum surface is m_y and its Y-axis half-length at x_1 is a_1 . The intersection of the cone frustrum slope line and the X-axis is denoted by x_{max} . A matched ellipsoid can be found only if x_t lies between x_1 and x_{max} .

The problem is now to find the parameters for the ellipse that will go through point $P_1(x_1, a_1)$ on the cone frustrum, with slope m_y , and point $P_2(x_t, 0)$ on the

X-axis, with slope $-\infty$. A procedure for finding the parameters a, c and x_0 for this ellipse, which has one axis parallel to the X-axis, is described in Appendix C. The remaining parameter for the ellipsoid, b , is equal to a divided by the aspect ratio for the cone frustum.

3.3.3 Ogive

An ogive is shown in Figure 21. The constant parameters for the ogive are

$$\begin{aligned} P_1 &= x_0 = \text{X-axis center of the ogive} \\ P_2 &= a = \text{Y-axis half-length at } x_0 \\ P_3 &= b = \text{Z-axis half-length at } x_0, \text{ and} \\ P_4 &= c = \text{X-axis half-length at } x_0. \end{aligned} \tag{3.40}$$

The ogive is different from the ellipsoid in that the contours in the YX- and ZX-planes are circular and not elliptic. The radii of these circles, R_A and R_B , respectively, are uniquely defined when the parameters a, b and c are given. The ogive surface is then described using

$$A(x) = \sqrt{R_A^2 - (x - x_0)^2} - R_A + a, \text{ and} \tag{3.41}$$

$$B(x) = \sqrt{R_B^2 - (x - x_0)^2} - R_B + b \tag{3.42}$$

in Equations (3.1) to (3.5) where

$$R_A = \frac{a^2 + c^2}{2a}, \text{ and} \tag{3.43}$$

$$R_B = \frac{b^2 + c^2}{2b}. \tag{3.44}$$

The surface derivatives are found using

$$A_x = \frac{-(x - x_0)}{(R_A^2 - (x - x_0)^2)^{\frac{1}{2}}} \tag{3.45}$$

(a) 3D view

(b) XY plane

(c) XZ plane

Figure 21: Ogive geometry.

$$A_{xx} = \frac{-R_A^2}{(R_A^2 - (x - x_0)^2)^{\frac{3}{2}}} \quad (3.46)$$

$$B_x = \frac{-(x - x_0)}{(R_B^2 - (x')^2)^{\frac{1}{2}}}, \text{ and} \quad (3.47)$$

$$B_{xx} = \frac{-R_B^2}{(R_B^2 - (x - x_0)^2)^{\frac{3}{2}}}. \quad (3.48)$$

It can be seen from (3.41) and (3.42) that the aspect ratio for the ogive is a function of x . Therefore, Equations (3.14) through (3.17) will not be correct for computing the surface derivatives for the ogive; the derivatives of ν with respect to x , found by taking derivatives in Equation (3.6), must be included in the solution. The slope ratio for the ogive is also a function of x , different from that for the aspect ratio, so that

$$\Gamma_A^{ogv}(x) \neq \Gamma_S^{ogv}(x) \neq \frac{a}{b}. \quad (3.49)$$

Therefore, if an ogive section matches the contour of one of the other canonical shapes above, then the slopes will not match all around the junction contour. The only exception is for matching a cylinder at the center of the ogive, where $x' = 0$. At this cross section, the aspect ratio is equal to a/b and the slope is zero around the contour so that both the contour and the slopes can be matched to a cylinder.

A method for matching an ogive-like shape to a cone frustum will be described in a later section.

3.3.4 Examples

Figure 22 shows three representative examples of composite bodies that can be defined using the canonical shapes which were described previously. Body (a) is a combination of a cone, cylinder, cone frustum and a disk. Body (b) is a

combination of a half of an ogive, a cylinder section, and half of an ellipsoid. Body (c) is a cone frustrum with parts of ellipsoids smoothly matched to its two ends.

The problem of fairing the sharp junctions between shapes, such as those in Figure 22(a), will be considered next.

3.4 Definition of Faired Shapes

Fairing of wedge junctions is achieved by taking away parts of the surrounding surfaces and replacing it with a shape that matches smoothly to the original shapes on both sides of the junction. For wedge angles of less than 180° , fairing is much like using sand paper to smooth the joint. For wedge angles of larger than 180° , fairing can be seen as filling up the surface angle with putty, and spreading it out to yield a smooth transition between shapes. In this study, the fairing is performed by fitting elliptic curves to define the transition between the surfaces on the two sides of the junction. These elliptic curves can be defined at each angle, α , around the body.

Let us define the Q-axis as the axis in the YZ-plane which makes an angle, α , with the Y-axis, as shown in Figure 23(a). Figure 23(b) shows the contour of a longitudinal cut, in the QX-plane, through a body with a sharp junction. The junction is at the point, P. Also shown is the elliptic curve, $Q^\alpha(x)$, which starts at P_1 to the left of P and ends at P_2 to the right of P, and provides slope continuity at the junction. The family of elliptic curves for all α angles around the body describes a **faired shape**, which is given by

$$\mathbf{r}(x, \alpha) = [x, Q^\alpha(x) \cos \alpha, Q^\alpha(x) \sin \alpha] . \quad (3.50)$$

The procedure for finding these elliptic transition curves is described next.

(a) Cone, cylinder, cone frustum and disk.

(b) Matched ellipsoid, cone frustum, matched ellipsoid.

(c) Half an ogive, cylinder, half an ellipsoid.

Figure 22: Examples of composite bodies using canonical shapes.

(a) YZ cross section

(b) XQ cross section

Figure 23: Geometry for fairing of a sharp junction.

3.4.1 Defining Elliptic Transition Curves

The configuration for the elliptic transition curve is shown in Figure 23(b). The following parameters are defined:

- x_{p1} = initial x-coordinate for the faired shape
- x_{p2} = final x-coordinate for the faired shape
- $d_{1,2}$ = distance between P and $P_{1,2}$
- $S_{1,2}$ = slopes of the original surfaces at $P_{1,2}$
- $\xi_{1,2}$ = the angles that the extended slope lines
through $P_{1,2}$ make with the +X-axis.

The problem is to find an ellipse that will match the slopes and the points on both sides of the junction. This ellipse is uniquely defined by the five parameters $[a, b, x_0, q_0, \xi]$, where ξ is the angle that one of the ellipse axes makes with the X-axis; x_0 and q_0 are the coordinates for the center of the ellipse, and a and b are the major and minor axis half-lengths. The problem, however, has only four constraints; i.e., two points and two slopes. It is therefore necessary to fix one parameter in order to find a unique solution for the matching ellipse. The easiest parameter to fix is ξ , and the procedure for finding the ellipse parameters for a given ξ is described in Appendix C.

It is shown in Appendix C that the choice of ξ has an effect on whether or not a matching ellipse can be found for a particular set of distances d_1 and d_2 . Also, since d_1 and d_2 may change as a function of α for a particular junction, it is necessary to avoid abrupt changes in the matching ellipses as α changes. The following criterion for choosing ξ will ensure that an ellipse can always be found and that changes as a function of α around the body will be smooth:

Figure 24: Smooth transition of fairing ellipses as the fairing distances change.

- For $d_1 \geq d_2$, choose $\xi = \xi_1$.
- For $d_2 > d_1$, choose $\xi = \xi_2$.

With these choices, if there were a continuous change from $d_1 > d_2$ through $d_1 = d_2$ to $d_1 < d_2$, then the matching ellipses will change smoothly from ellipses with one axis parallel to the tangent line through P_1 , to a circle, to ellipses with one axis parallel to the tangent line through P_2 . This smooth progression of matching ellipses is illustrated in Figure 24, where d_1 is changed, while d_2 is kept constant.

3.4.2 Examples

Figure 25 shows two different faired bodies based on the original body in Figure 22(a). The differences between the two bodies, which shows the versatility of this approach, are due to choosing different fairing distances at each of the junctions.

Figure 25: Faired body with different fairing distances.

The next section deals with computing surface derivatives on faired shapes.

3.4.3 Surface Derivatives on Faired Shapes

The faired shape is described by

$$\mathbf{r}(x, \alpha) = [x, q \cos \alpha, q \sin \alpha] \quad (3.51)$$

where $q = Q^\alpha(x)$ as previously defined. Figure 26(a) shows a point $P(x_0, \alpha_0)$ on a faired shape, as well as the contour lines for constant x and α through P . Surface derivatives on the faired shape are given by

$$\mathbf{r}_x = [1, q_x \cos \alpha, q_x \sin \alpha] \quad (3.52)$$

$$\mathbf{r}_\alpha = [0, (q_\alpha \cos \alpha - q \sin \alpha), (q_\alpha \sin \alpha + q \cos \alpha)] \quad (3.53)$$

$$\mathbf{r}_{xx} = [0, q_{xx} \cos \alpha, q_{xx} \sin \alpha] \quad (3.54)$$

$$\mathbf{r}_{x\alpha} = [0, (q_{x\alpha} \cos \alpha - q_x \sin \alpha), (q_{x\alpha} \sin \alpha + q_x \cos \alpha)] \quad (3.55)$$

$$\mathbf{r}_{\alpha\alpha} = \left[0, \begin{pmatrix} q_{\alpha\alpha} \cos \alpha - 2q_\alpha \sin \alpha \\ -q \cos \alpha \end{pmatrix}, \begin{pmatrix} q_{\alpha\alpha} \sin \alpha + 2q_\alpha \cos \alpha \\ -q \sin \alpha \end{pmatrix} \right] \quad (3.56)$$

- **Finding q_x and q_{xx} :**

Figure 26(b) shows the surface contour through P in the QX-plane. This contour is part of an ellipse for which the parameters are known. The derivatives q_x and q_{xx} can therefore be computed analytically, using the equations in Appendix A.

- **Finding q_α , $q_{x\alpha}$ and $q_{\alpha\alpha}$:**

Figure 26(c) shows the surface contour through P in the YZ-plane. Surface derivatives in this plane depend on the ellipses that define the contours for α values adjacent to α_0 . The derivatives q_α , $q_{x\alpha}$ and $q_{\alpha\alpha}$ are found numerically by computing q and q_x at angles on both sides of α_0 . **Centered finite-divided-difference formulas** [24] for two data points on each side of α_0 are used to numerically compute these derivatives. The formulas for first and second derivatives of a function F are

$$\begin{aligned} D_1(F, \alpha) &= \text{First derivative of } F \text{ with respect to } \alpha \\ &= \frac{-F_2 + 8F_1 - 8F_{-1} + F_{-2}}{12h}, \text{ and} \end{aligned} \quad (3.57)$$

$$\begin{aligned} D_2(F, \alpha) &= \text{Second derivative of } F \text{ with respect to } \alpha \\ &= \frac{-F_2 + 16F_1 - 30F_0 + 16F_{-1} - F_{-2}}{12h^2} \end{aligned} \quad (3.58)$$

where

$$h = \alpha\text{-increment in radians, and}$$

Figure 26: Surface derivatives on faired shapes in the XYZ-coordinate system.

F_i = Value of function F evaluated at angle $(\alpha_0 + i h)$.

Using this notation, the required α derivatives are

$$q_\alpha = D_1(q, \alpha) \quad (3.59)$$

$$q_{x\alpha} = D_1(q_x, \alpha) , \text{ and} \quad (3.60)$$

$$q_{\alpha\alpha} = D_2(q, \alpha) . \quad (3.61)$$

Numerical derivatives, and especially second derivatives, can become unstable as a result of round-off errors in the computer. This happens in particular when large and almost equal numbers are subtracted to compute the derivatives. Figure 26(b) shows a region where this problem may occur, i.e., in the region where the slope, q_x , is large and goes through a change from minus infinity to plus infinity. From a different perspective, assume that the surface derivatives are used to compute surface curvature in this region. The figure shows that the variation of the surface is approximately parallel to the q -axis. If x were taken as the independent variable, then small steps in x result in large steps on the surface. Therefore, x is not a very desirable parameter for describing surface curvature in this region. Stable and more accurate results are obtained when q is used as the independent variable.

In order to use q as the independent variable in describing the surface, it is necessary to define a QAX-coordinate system, with basis vectors $\{\hat{q} , \hat{\alpha} , \hat{x}\}$ fixed by the angle α_0 at P. The unit vectors, $\hat{q}$ and $\hat{\alpha}$, are in the YZ-plane, as shown in Figure 27(a), and $\hat{x}$ is the same as before. The surface is now expressed as a function of q and α :

$$\mathbf{r}(q, \alpha) = [q , A(q, \alpha) , X(q, \alpha)]_{QAX} . \quad (3.62)$$

Figure 27: Surface derivatives on faired shapes in the QAX-coordinate system.

$A(q, \alpha)$ is the independent coordinate in the $\hat{\alpha}$ -direction, as shown in Figure 27(b). This coordinate is given in terms of α by

$$A(q, \alpha) = q \tan(\alpha - \alpha_0) . \quad (3.63)$$

For the surface contour through point P, $q = q_0$ is taken. $X(q, \alpha)$ is the x -coordinate for surface points in the AX-plane through P. In this coordinate system, the surface derivatives at P are given by

$$\mathbf{r}_q = [1 , 0 , X_q]_{QAX} \quad (3.64)$$

$$\mathbf{r}_\alpha = [0 , q_0 , X_\alpha]_{QAX} \quad (3.65)$$

$$\mathbf{r}_{qq} = [0 , 0 , X_{qq}]_{QAX} \quad (3.66)$$

$$\mathbf{r}_{q\alpha} = [0 , 1 , X_{q\alpha}]_{QAX} , \text{ and} \quad (3.67)$$

$$\mathbf{r}_{\alpha\alpha} = [0 , 0 , X_{\alpha\alpha}]_{QAX} . \quad (3.68)$$

The derivatives along the A-axis were found by taking partial derivatives of Equation (3.63) and evaluating those at (q_0, α_0) .

- **Finding X_q and X_{qq} :**

The contour line in the QX-plane for fixed α is the same as before. Given X and q , the derivatives with respect to q can be found using the equations in Appendix A.

- **Finding X_α , $X_{q\alpha}$ and $X_{\alpha\alpha}$:**

A different approach is needed to find the contour line in the XA-plane for fixed q . The value of X_i at a given angle $\alpha_i = (\alpha_0 + i h)$ is found from the matching ellipse equation for this angle. First, the length, q_i , must be found for a surface point along the constant q -plane, as shown in Figure 27(b). This

length is given by

$$q_i = \frac{q_0}{\cos(i h)} . \quad (3.69)$$

X_i is found from substituting q_i in the ellipse equation at this angle. With X_i and q_i known, the derivatives X_α , $X_{q\alpha}$ and $X_{\alpha\alpha}$ can be computed numerically, similar to what was done previously.

The surface characteristics, using these derivatives in the QAX-coordinate system, are found using the formulas in Appendix D for the XYZ-coordinate system. The only difference is that the resulting vector directions will be in the QAX-coordinate system. A straight-forward coordinate rotation is used to transform these vectors to the XYZ-coordinate system.

3.5 Matched Ogive

It was pointed out previously that it is impossible to match an ogive smoothly to a cone frustum with an elliptical cross section. In this section, an ogive-like shape that matches smoothly to a cone frustum is defined. The shape is similar to the faired shape in the previous section, in that contours will be defined to describe the surface as a function of α .

The Matched Ogive shape is defined by circular contours at each angle α around the body. The shape is determined by x_t , the x-coordinate of its tip. Figure 28 shows the geometry for a longitudinal cross section of the body at a given α angle. The point, $P(x_1, a_2)$, is on the end of the cone frustum, and the slope of the cone frustum through P is S . The problem is to find the center, (x_o, q_o) , and radius, R_o , of a circle that will go through point $T(x_t, 0)$ on the x-axis and point P on the cone frustum, and has a slope equal to S at P. Such a circle is described by the

function

$$(x - x_o)^2 + (q - q_o)^2 = R_o^2 . \quad (3.70)$$

First, define line L_1 through P and perpendicular to the cone frustrum surface. The slope of L_1 is $S_2 = -1/S$. In order for the slope of the circle to be equal to S at P, the center of the circle must lie on the L_1 line, so that

$$q_o = a_2 + S_2(x_o - x_1) . \quad (3.71)$$

Substituting (3.71) for q_o into (3.70) yields an equation with two unknowns; i.e., x_o and R_o . Evaluating this equation at the points, P and T, and eliminating R_o , yields

$$x_o = \frac{x_1^2 - a_2^2 + 2a_2S_2x_1 - x_t^2}{2(x_1 + a_2S_2 - x_t)} . \quad (3.72)$$

Finally, q_o is found from (3.71), and R_o is found from substituting x_o and q_o in (3.70).

Note that q_o must be below the x-axis. Otherwise, the shape will have an indented point instead of a sharp point. This requirement sets a lower limit, x_{min} , for the tip of the matched ogive, as shown in Figure 28. This minimum value is achieved when $q_o = 0$, and is given by

$$x_{min} = x_{o,min} + R_{o,min} \quad (3.73)$$

where

$$x_{o,min} = a_2S + x_1 , \text{ and} \quad (3.74)$$

$$R_{o,min} = \sqrt{(x_{o,min} - x_1)^2 + a_2^2} . \quad (3.75)$$

The upper limit for the tip is found by extending the tangent line through P and finding its intersection, x_{max} , with the x-axis. This limit is also shown in Figure 28.

Figure 29: Example of a cone frustum with matched ogives at its ends.

The circle that describes the matched ogive for every α is a special case of an ellipse, with parameters

$$[a, b, x_0, q_0, \xi] = [R_o, R_o, x_o, q_o, 0]. \quad (3.76)$$

Surface derivatives can therefore be computed in the same way as for the faired shape.

An example of a cone frustum with matched ogives on both ends is shown in Figure 29.

3.6 Summary

The analytic descriptions for surfaces of isolated scattering bodies were described in this Chapter. The bodies were defined as a series connection of canonical shapes with elliptic cross section. A faired shape was also defined. This shape replaces a wedge junction between canonical shapes with a smooth surface. Equations were given for computing surface derivatives for all these shapes. The surface derivatives are used for computing the surface characteristics needed in

scattered field computations. Techniques for finding reflected and diffracted fields from isolated scattering bodies will be described in the next Chapter.

CHAPTER IV

FIELD COMPUTATIONS FOR ISOLATED BODIES

This Chapter describes techniques for finding the first order, high frequency, RCS envelope of an isolated scattering body. Contributions to the total RCS include GO reflected fields from all doubly curved surfaces and UTD diffracted fields from all the junctions between shapes. PO limits are applied in caustic regions where UTD diffracted fields and GO reflected fields become unbounded. Full volumetric RCS patterns of scattering bodies are studied by displaying computed results on an RCS map where gray scale plots are used to distinguish between different RCS levels.

Computed results are given for an ellipsoid and two test bodies. The first test body has a wedge junction. In the second test body, this wedge junction is faired. Results for individual mechanisms are given as the techniques are developed. Finally, measured and computed results for the total RCS of the test bodies are compared.

4.1 Full Volumetric Patterns of the RCS of Scattering Bodies

Figure 30(a) shows the definition of angles and vector directions for backscattered fields. The incident field, $\mathbf{E}^i$, propagates in the $\hat{\mathbf{s}}'$ direction, and the backscattered field, $\mathbf{E}^{bs}$, propagates in the $\hat{\mathbf{s}}$ direction, where $\hat{\mathbf{s}} = -\hat{\mathbf{s}}'$. In terms of the

spherical angles, θ and ϕ , the backscatter direction is given by

$$\hat{s}(\theta, \phi) = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \quad (4.1)$$

or

$$\hat{s}(\theta, \phi) = s_x \hat{x} + s_y \hat{y} + s_z \hat{z} . \quad (4.2)$$

A full volumetric pattern of the RCS of a scattering body is obtained by displaying its RCS for the whole spherical range of angles; i.e., $0 < \theta < 180^\circ$ and $-180^\circ < \phi < 180^\circ$, as shown in Figure 30(b). The vertical axis represents θ and has N_θ divisions, where each division represents $\Delta\theta = \frac{180}{N_\theta}$ degrees. The horizontal axis represents ϕ , and has N_ϕ divisions where each division represents $\Delta\phi = \frac{360}{N_\phi}$ degrees. The center of each angular box on the map corresponds to a backscatter direction, $\hat{s}(\theta, \phi)$. Alternatively, each box can be defined by the indices, (i_θ, i_ϕ) , where the relationships between angles and indices are given by

$$\begin{aligned} \theta &= i_\theta \Delta\theta - \frac{1}{2} \Delta\theta & i_\theta &= 1, \dots, N_\theta \\ \phi &= -180^\circ + i_\phi \Delta\phi - \frac{1}{2} \Delta\phi & i_\phi &= 1, \dots, N_\phi . \end{aligned} \quad (4.3)$$

Each box in the map represents an “angular slot” in space, centered at the angle corresponding to (i_θ, i_ϕ) , as shown in Figure 30(b). When displaying full volumetric RCS patterns, each box will be given a shade of gray on a scale corresponding to the RCS, $\sigma(i_\theta, i_\phi)$, at or near the center of the angular slot. Note that the center of the map represents the RCS in the direction of the positive X-axis; i.e., where $\hat{s}(90^\circ, 0^\circ) = \hat{x}$. Also note that, in space, the right and left sides of the map are connected; i.e., $\hat{s}(90^\circ, 180^\circ)$ and $\hat{s}(90^\circ, -180^\circ)$ are both equal to $-\hat{x}$.

For bodies with an elliptic cross section, there is a quarter hemisphere symmetry in the geometry. Figure 30(a) shows how the body can be divided into four similar parts, numbered I, II, III and IV. Due to this symmetry, there will also

be symmetry in the volumetric RCS pattern. Note that backscattered fields from part I will only go in directions corresponding to the quarter hemisphere in which it lies. This hemisphere is represented by the lower right quadrant of the RCS map in Figure 30(b). Fields from the other parts of the body will then be mirror images of those in the first quadrant, as indicated on the map. Therefore, it is only necessary to compute field values for angles in the first quadrant of the map. The angles in the other quadrants with the same RCS are found from the following equivalence relationships:

$$\begin{aligned}
 (\theta, \phi) \text{ in quadrant I} &\Leftrightarrow (\theta, -\phi) \text{ in quadrant II} \\
 &\Leftrightarrow (180^\circ - \theta, -\phi) \text{ in quadrant III, and} \\
 &\Leftrightarrow (180^\circ - \theta, \phi) \text{ in quadrant IV.}
 \end{aligned}$$

As an illustration, the shaded blocks in Figure 30(b) indicate angles in the four quadrants for which the scattering body will have the same RCS.

The following sections discuss the techniques used for finding the RCS patterns for individual scattering mechanisms.

4.2 Geometrical Optics Field Computations

For GO reflected fields from doubly curved surfaces, the main problem is to find the backscatter reflection points on the surface. At a reflection point, the surface equations are used to find the principal radii of curvature for the surface. These radii are then used to compute the backscattered GO field as described in Chapter II. Note that GO-field magnitudes are independent of frequency (as long as the high frequency approximation is valid).

Next, a closed form solution is given for reflection points on an ellipsoid. For the remaining surfaces, reflection points will be found using a numerical search algorithm.

4.2.1 Reflection from an Ellipsoid

An ellipsoid, defined by the parameters $[x_0, a, b, c]$, is shown in Figure 31. In order to compute the GO reflected fields from this ellipsoid, it is necessary to compute the reflection point, $P(x_r, \alpha_r)$, for a given backscatter direction; i.e., the point where the direction of the surface normal, $\hat{n}$, is equal to the direction of the backscatter vector, $\hat{s}$. The equations given by Choi [25, p 43] are used in the following analysis to find this reflection point. First, let the backscatter direction be given by $\hat{s} = \hat{x}s_x + \hat{y}s_y + \hat{z}s_z$. Then, $\hat{s}$ can also be defined in terms of the spherical angles, α and β , illustrated in Figure 31, such that

$$\hat{s}(\alpha, \beta) = \cos \beta \hat{x} + \sin \beta \cos \alpha \hat{y} + \sin \beta \sin \alpha \hat{z} \quad (4.4)$$

where

$$\beta = \cos^{-1} s_x, \text{ and} \quad (4.5)$$

$$\alpha = \tan^{-1} \left[\frac{s_z}{s_y} \right]. \quad (4.6)$$

Now, let the ellipse angular parameters in the YZ- and XZ-planes be v and u , respectively. At the reflection point, these parameters are given by

$$v_r = \tan^{-1} \left[\frac{b}{a} \tan \alpha \right], \text{ and} \quad (4.7)$$

$$u_r = \tan^{-1} \left[\frac{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}{c} \tan \beta \right]. \quad (4.8)$$

Figure 31: Backscattered reflection point on an ellipsoid.

Finally, the parameters for the reflection point, $P(x_r, \alpha_r)$, are given by

$$x_r = x_0 + c \cos u_r, \text{ and} \quad (4.9)$$

$$\alpha_r = \tan^{-1} \left[\frac{b}{a} \tan v_r \right]. \quad (4.10)$$

Computed Results:

Figure 32 shows the volumetric RCS pattern for an ellipsoid, where only GO reflected fields are included. The parameters of the ellipse are $[a, b, c] = [.2, .5, 1]$ meters. The lay-out of the figure is described next, since it will be frequently used to display RCS.

The RCS map in Figure 32 was discussed in Section 4.1. Note that the rectangular box above the RCS map shows a range of gray scales and the dB-

values they represent. The vertical cursor line on the map represents a variation of θ for a constant value of ϕ . Let this constant value be ϕ_v , where $\phi_v = 90^\circ$ in this example. A plot of the RCS envelope as a function of θ , along the line $\phi = \phi_v$, is shown to the right of the map. Similarly, the horizontal cursor line represents $\theta = \theta_v$, where $\theta_v = 90^\circ$ in this example, and the RCS envelope along this line is shown below the map. The crossing point of the two cursor lines represent the incidence direction, (θ_v, ϕ_v) . This direction will be called the **Front View** direction. Note that the angles for this view direction are given on the left side of the display as θ and ϕ . Now, if the backscatter vector is $\hat{s}(\theta_v, \phi_v)$, then an observer at the source will see the scattering body as it is displayed in the **Front** box. This is indicated by the back ends of arrows (crosses within circles, indicating the incidence vector direction) on the perimeter of this box. The shade of gray on the RCS map, at the point where the two cursor lines cross, corresponds to the RCS for the Front View direction. This feature is used in order to interpret results on the map, by being able to look at the scattering body as seen from a specific incidence direction. The **Side** view of the ellipsoid would be seen by an observer to the right of the Front view direction, looking in the $-\hat{\phi}_v$ -direction. For this view, illumination is from the left of the box, as indicated by the arrows on the left side of the box. The **Top** view of the ellipsoid would be seen by an observer that moved up from the Front view direction and looks down from right above the ellipsoid, in the $\hat{\theta}_v$ -direction. For this view, illumination is from the bottom of the box, as indicated by the arrows on the bottom edge of the box.

The polarization and frequency of the incident field are given in the top left hand corner of the display. Note that a polarization of $(\theta, \phi) = (1, 0)$ means that the incident electric field is theta polarized (i.e., $(e_\theta, e_\phi) = (1, 0)$, as defined in Chapter II); while, $(\theta, \phi) = (0, 1)$ implies phi polarization. The vertical distance

Figure 32: Volumetric RCS pattern for an ellipsoid.

between the two “+” markers next to the Top view indicates the length of one free space wavelength, λ , relative to the size of the body.

Now, note that the RCS pattern for the ellipsoid, which includes only GO reflected fields, is smooth and continuous over the full volumetric pattern. The low RCS values correspond to reflections from the narrow ends of the ellipsoid where the surface radii of curvature are small. The peak RCS values correspond to reflection from the flat faces of the ellipsoid, such as shown in the Front view, where the radii of curvature are large.

The next section describes a search algorithm for finding reflection points on faired surfaces and ogives.

4.2.2 Newton-Raphson Search for Reflection Points

A doubly curved surface with elliptic cross section is shown in Figure 33. The surface is described by $\mathbf{r}(x, \alpha) = \hat{x}x + \hat{y}Y(x, \alpha) + \hat{z}Z(x, \alpha)$. Assume that the first and second derivatives of $\hat{\mathbf{r}}$ with respect to x and α are known everywhere on the surface; i.e., that Y_x , Y_α , Y_{xx} , $Y_{x\alpha}$, $Y_{\alpha\alpha}$, and Z_x , Z_α , Z_{xx} , $Z_{x\alpha}$, $Z_{\alpha\alpha}$ are known. Let $\hat{\mathbf{s}}(\theta, \phi)$ be a given backscattered reflection direction. The reflection point, $P(x_r, \alpha_r)$, is the point where the outward surface normal, $\hat{\mathbf{n}}$, is equal to $\hat{\mathbf{s}}$; i.e., where $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = 1$. Using the normal is, however, not a very sensitive test for closeness to the desired point of reflection. The reason is that the inner product of $\hat{\mathbf{s}}$ with the normal will be very close to one even when the two vector directions are a few degrees apart. An alternative search can be conducted by searching for the point where the inner products of $\hat{\mathbf{s}}$ with the two parametric tangents, $\hat{\mathbf{r}}_x$ and $\hat{\mathbf{r}}_\alpha$, are zero [26]. The zero crossings of these inner products are more sensitive indicators of closeness to the reflection point. This search procedure starts with an initial guess, (x, α) , and then uses a two dimensional Newton-Raphson method

Figure 33: Search for reflection points on a doubly curved surface.

[27] to determine a step size, $(\Delta x, \Delta \alpha)$, which will decrease both inner products.

The two inner product functions are

$$f_i(x, \alpha) = \hat{\mathbf{t}}_i(x, \alpha) \cdot \hat{\mathbf{s}} \quad i = 1, 2 \quad (4.11)$$

where

$$\hat{\mathbf{t}}_1(x, \alpha) = \hat{\mathbf{r}}_x = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}Y_x + \hat{\mathbf{z}}Z_x}{\sqrt{1 + Y_x^2 + Z_x^2}}, \text{ and} \quad (4.12)$$

$$\hat{\mathbf{t}}_2(x, \alpha) = \hat{\mathbf{r}}_\alpha = \frac{\hat{\mathbf{y}}Y_\alpha + \hat{\mathbf{z}}Z_\alpha}{\sqrt{Y_\alpha^2 + Z_\alpha^2}}. \quad (4.13)$$

The parameters (x_r, α_r) will define a specular reflection point if $f_i(x_r, \alpha_r) = 0$.

Closeness to the actual specular point can be evaluated by checking whether

$$f_1^2 + f_2^2 \leq \cos^2\left(\frac{\pi}{2} - \epsilon\right) \quad (4.14)$$

where ϵ is the maximum allowable angle error between the vectors, $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$. Note that the GO field usually varies quite smoothly and slowly over a surface. Therefore, it is only necessary to find the surface point for which the normal direction is somewhere within the angular slot of interest. Using this criterion, one can set $\epsilon = \text{minimum}(\frac{\Delta\theta}{2}, \frac{\Delta\phi}{2})$.

Now consider a point, $P(x, \alpha)$, on the surface where (4.14) is not satisfied. Since the surface is smooth, the f_i functions can be expanded in a Taylor series in the neighborhood of (x, α) . Retaining only the first order terms of Δx and $\Delta \alpha$ yields linear approximations of the two functions, such that, for $i = 1, 2$,

$$f_i(x + \Delta x, \alpha + \Delta \alpha) \approx f_i(x, \alpha) + \frac{\partial f_i(x, \alpha)}{\partial x} \Delta x + \frac{\partial f_i(x, \alpha)}{\partial \alpha} \Delta \alpha. \quad (4.15)$$

Setting $f_i(x + \Delta x, \alpha + \Delta \alpha)$ equal to the desired value of zero yields two linear equations with Δx and $\Delta \alpha$ as unknowns. Note that Δx and $\Delta \alpha$ are then the increments in x and α which will tend to decrease f_i . These increments are found

by simultaneously solving the two equations in (4.15). The process is then repeated by using $x^{new} = x + \Delta x$ and $\alpha^{new} = \alpha + \Delta \alpha$, until (4.14) is satisfied.

Note that the partial derivatives of f_1 and f_2 , used in the Taylor expansion, are given by

$$\frac{\partial f_1}{\partial x} = \hat{s} \cdot \left\{ \frac{[0, Y_{xx}, Z_{xx}]}{\sqrt{1 + Y_x^2 + Z_x^2}} + [1, Y_x, Z_x] \frac{-(Y_x Y_{xx} + Z_x Z_{xx})}{(1 + Y_x^2 + Z_x^2)^{\frac{3}{2}}} \right\} \quad (4.16)$$

$$\frac{\partial f_1}{\partial \alpha} = \hat{s} \cdot \left\{ \frac{[0, Y_{x\alpha}, Z_{x\alpha}]}{\sqrt{1 + Y_x^2 + Z_x^2}} + [1, Y_x, Z_x] \frac{-(Y_x Y_{x\alpha} + Z_x Z_{x\alpha})}{(1 + Y_x^2 + Z_x^2)^{\frac{3}{2}}} \right\} \quad (4.17)$$

$$\frac{\partial f_2}{\partial x} = \hat{s} \cdot \left\{ \frac{[0, Y_{x\alpha}, Z_{x\alpha}]}{\sqrt{Y_\alpha^2 + Z_\alpha^2}} + [0, Y_\alpha, Z_\alpha] \frac{-(Y_\alpha Y_{x\alpha} + Z_\alpha Z_{x\alpha})}{(Y_\alpha^2 + Z_\alpha^2)^{\frac{3}{2}}} \right\} \quad (4.18)$$

and

$$\frac{\partial f_2}{\partial \alpha} = \hat{s} \cdot \left\{ \frac{[0, Y_{\alpha\alpha}, Z_{\alpha\alpha}]}{\sqrt{Y_\alpha^2 + Z_\alpha^2}} + [0, Y_\alpha, Z_\alpha] \frac{-(Y_\alpha Y_{\alpha\alpha} + Z_\alpha Z_{\alpha\alpha})}{(Y_\alpha^2 + Z_\alpha^2)^{\frac{3}{2}}} \right\} \quad (4.19)$$

Discussion:

The search for reflection points for all aspect angles will be more efficient if the reflection point for one incident angle is used as the initial guess for the next angle. This can be done by starting at one end of the body and then incrementing θ and ϕ in such a way that the reflection points will follow a continuous path over the surface of the body. At each reflection point, the GO field is computed and stored in an array corresponding to entries in the RCS map. Figure 33 illustrates this procedure for a doubly curved shape between x_1 and x_2 on the X-axis, and is explained as follows:

1. Find the normal vector, $\hat{n}$, at the point $P(x_1, 0)$ on the surface. Compute the spherical angles (θ, ϕ) corresponding to the backscatter direction at this point (note that θ will be equal to 90°). Now find the indices, i_θ and i_ϕ , corresponding to θ and ϕ . This is the first reflection point.

2. Next, keep i_θ constant, increment i_ϕ and search for the corresponding reflection point on the surface. Then keep on incrementing i_ϕ until x_2 is reached. Reflection points will lie along contour 1 on the surface.
3. Now, decrement i_θ and search for the reflection point at x_2 . This could be at a ϕ angle different from the previous one.
4. Next, keep i_θ constant and decrement i_ϕ . Keep on decreasing i_ϕ until x_1 is reached. The points will lie along contour 2 on the surface.
5. Now, decrement i_θ again and search for the reflection point at x_1 . This could be at a ϕ value different from the previous one.
6. Repeat steps 2 to 5 until the whole part I of the body has been covered.

Once this procedure is completed, the remaining angles in quadrant one of the RCS map will have no GO-field contributions from this particular shape. As explained earlier, it is only necessary to find GO-fields in one quarter hemisphere; fields for other aspect angles are found by using the symmetry of the body.

The search method described above is convenient and efficient for shapes on isolated bodies. This is the case since the beginning and ending points for the shapes are defined at constant X-coordinates, so that it is easy to determine when the end of the shape is reached.

4.2.3 GO Reflected Field Results for Test Bodies

In the rest of this Chapter, computed results are given for the unfaired and the faired test bodies shown in Figure 34(a) and (b), respectively. The unfaired body is composed of three shapes: a matched ogive (S_1), a cone frustum (S_2), and a disk (S_3). There are two junctions in this body: a smooth junction (J_1) and

a wedge junction (J_2). For the faired body the wedge junction has been replaced by a faired shape (S_{f3}). Therefore, this body has two additional smooth junctions (J_{f2} and J_{f3}). Both test bodies are bodies of revolution, and their dimensions are given in Figure 34.

Computed Results:

Computations of the full volumetric RCS patterns were done using $N_\theta = 31$ and $N_\phi = 64$, so that $\Delta\theta = 5.8^\circ$ and $\Delta\phi = 5.6^\circ$.

1. Matched Ogive:

Figure 35 shows the volumetric RCS pattern of the GO reflected fields from the matched ogive in both test bodies (Shapes S_1 and S_{f1}). Due to the sharp point of the ogive, there are no backscattered reflected fields for incidence in the negative X-axis direction (center of the RCS map). A source moving away from the X-axis in any direction will reach an angle (about 30° from the X-axis) where the incident direction is perpendicular to the surface at the tip. The front view for such an incidence direction is illustrated in the figure. GO-reflected fields are observed from that angle onward, until the end of the ogive is reached (73° from the X-axis). At this angle, corresponding to the normal direction on the side of the cone frustum, the GO-fields stop. Note that the magnitude of the reflected fields decrease toward the center of the map. This is due to the small radius of surface curvature near the tip of the ogive. The “doughnut” pattern of these fields is typical for shapes centered around the X-axis. This particular pattern is circular, since the body is circular. For bodies with elliptic cross section, the pattern will also have an elliptic looking shape.

(a) UNFAIRED TEST BODY

(b) FAIRED TEST BODY

Figure 34: Geometry for various test bodies.

Figure 35: Volumetric pattern of the GO reflected fields from shape S_1 : Matched Ogive.

For the unfaired test body, the matched ogive is the only doubly curved shape for which GO fields need to be computed. For the faired body, on the other hand, GO fields must also be computed for the faired shape (S_{f3}). These fields will be discussed next.

2. Faired Shape:

Figure 36 shows the volumetric RCS pattern of the GO reflected fields off the faired shape, S_{f3} , alone. Again, starting out from the center of the RCS map, one reaches the angles where the normals to the cone frustum surface correspond to the backscattered field directions. Reflected fields are observed starting at these angles and going around the body, all the way to the $-X$ -axis. The increase in the field magnitude, observed close to the $-X$ -axis direction, is due to the flatness of the faired shape as it matches smoothly to the surface of the disk. For reflection in the direction of the $-X$ -axis (i.e., for $\theta = 90^\circ$ and $\phi = \pm 180^\circ$) the GO field equation becomes infinite due to the faired shape being flat in the plane perpendicular to the X -axis. Here, the reflected field was set equal to the limiting value computed by the GO Equivalent Line Current method, described in Chapter II.

Next, the computation of diffracted fields from junctions will be discussed.

4.3 Diffracted Field Computations

Diffracted fields from all junctions in scattering bodies are included in the solution for the total RCS pattern. The junctions are either wedges or smooth junctions, and have elliptical contours in planes perpendicular to the X -axis. In order to compute the diffracted fields from these junctions, it is first necessary to find the point(s) on each junction for which diffraction is in the backscatter direc-

Figure 36: Volumetric pattern of the GO reflected fields from shape S_{f3} : Faired Shape.

tion. Then (see Chapter II), the following variables are needed at each diffraction point:

1. $\hat{e}$, the edge vector
2. $\hat{n}_e$, the outward normal to the edge
3. a_e , the edge radius of curvature
4. WA, the wedge angle
5. ϕ'_0 , the incident angle for diffraction
6. $R_{1,2}^{o,n}$, the principal radii of curvature on the o- and n-face surfaces, and
7. $\hat{U}_{1,2}^{o,n}$, the principal directions on the o- and n-face surfaces.

Figure 37 shows the definition of the o- and n-faces and the direction of the edge vector for a diffraction point, $P(x_d, \alpha_d)$, on an elliptic junction. The equations for computing the edge vector, normal and radius of curvature at P are given in Appendix A, and computation of the o- and n-face principal directions and radii of curvature are discussed in Chapter III. The following sections describe how diffraction points are found, as well as the computation of WA and ϕ'_0 at a diffraction point.

4.3.1 Diffraction Points

Figure 37 illustrates the geometry for finding diffraction points. For backscattered diffracted fields, diffraction points are the points on an elliptic junction where the incident vector direction is perpendicular to the edge vector. These points are

Figure 37: Diffraction point geometry.

found by setting $\hat{\mathbf{e}} \cdot \hat{\mathbf{s}}' = 0$, where

$$\hat{\mathbf{s}}' = \hat{\mathbf{x}}s'_x + \hat{\mathbf{y}}s'_y + \hat{\mathbf{z}}s'_z, \text{ and} \quad (4.20)$$

$$\hat{\mathbf{e}} = \frac{\hat{\mathbf{y}}(-a \sin \nu) + \hat{\mathbf{z}}b \cos \nu}{\sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu}}. \quad (4.21)$$

This yields the value of the ellipse parameter at the diffraction point, given by

$$\nu_d = \tan^{-1} \left[\frac{b}{a} \frac{s'_z}{s'_y} \right]. \quad (4.22)$$

Note that, for $\nu = (\nu_d + 180^\circ)$, the edge vector is $-\hat{\mathbf{e}}$, and that $-\hat{\mathbf{e}} \cdot \hat{\mathbf{s}}' = 0$. Therefore, there are two points on the elliptic junction from which backscattered diffracted fields can come, shown as P_1 and P_2 in Figure 37. The α angles for these two

points are given by

$$\alpha_{d1} = \tan^{-1} \left[\frac{b^2}{a^2} \frac{s'_z}{s'_y} \right], \text{ and} \quad (4.23)$$

$$\alpha_{d2} = \alpha_{d1} \pm 180^\circ \quad (4.24)$$

where use has been made of the relationship between α and ν .

One needs to determine whether both diffraction points are illuminated by the incident field. A necessary test, that a diffraction point be illuminated from the outside of the scattering body, will be discussed in the next section where the wedge and incident angles are computed.

4.3.2 Wedge and Incident Angles

Let us define the **incident plane** as the plane through a diffraction point and perpendicular to the edge vector at that point. Figure 38(a) shows the contour of a wedge junction in the incident plane. Since $\hat{s}'$ is perpendicular to the edge vector, it must lie in this plane. Furthermore, since the edge vector is a tangent to both the o- and n-face surfaces, the surface normals at the edge on both sides must also lie in this plane. These normals are indicated by $\hat{n}_o$ and $\hat{n}_n$. Note that the x -directed tangents on the two faces do not necessarily lie in this plane. Therefore, let us define two tangent vectors, $\hat{t}_o$ and $\hat{t}_n$, in this plane. These vectors are shown in Figure 38(a), and are given by

$$\hat{t}_o = \hat{e} \times \hat{n}_o, \text{ and} \quad (4.25)$$

$$\hat{t}_n = \hat{e} \times \hat{n}_n. \quad (4.26)$$

Wedge Angle:

The wedge angle is the interior angle between the two faces of the wedge. First, let

(a) Finding the wedge angle

(b) Finding the incident angle

Figure 38: Incident plane through a diffraction point.

us compute ψ , the angle between the two surface tangents in the incident plane, such that

$$\psi = \cos^{-1}(\hat{\mathbf{t}}_o \cdot \hat{\mathbf{t}}_n), \quad 0 < \psi < 180^\circ. \quad (4.27)$$

Now it must be determined whether the wedge angle is equal to $180^\circ + \psi$ or $180^\circ - \psi$.

This is done as follows:

$$\text{If } (\hat{\mathbf{t}}_n \times \hat{\mathbf{t}}_o) \cdot \hat{\mathbf{e}} \begin{cases} > 0 \\ \leq 0 \end{cases} \text{ then } \text{WA} = \begin{cases} 180^\circ - \psi \\ 180^\circ + \psi \end{cases}. \quad (4.28)$$

Incident Angle:

The incident angle, ϕ'_0 , is the angle between the incident vector and the o-face tangent, as shown in Figure 38(b). This angle is found in a way similar to the wedge angle. Let

$$\phi^* = \cos^{-1}(-\hat{\mathbf{s}}' \cdot \hat{\mathbf{t}}_o), \quad 0 < \phi^* < 180^\circ. \quad (4.29)$$

$$\text{If } (\hat{\mathbf{t}}_o \times \hat{\mathbf{s}}') \cdot \hat{\mathbf{e}} \begin{cases} \geq 0 \\ < 0 \end{cases} \text{ then } \phi'_0 = \begin{cases} \phi^* \\ 360^\circ - \phi^* \end{cases}. \quad (4.30)$$

With this approach, ϕ'_0 will always be positive.

As mentioned before, a diffraction point must be illuminated from outside the scattering body. This will be the case if

$$\phi'_0 \leq 360^\circ - \text{WA}. \quad (4.31)$$

4.3.3 Field Magnitudes in Caustic Regions

The test bodies in Figure 34 are examples of scatterers for which all four types of caustics, described in Chapter II, occur. The magnitudes of diffracted fields in the caustic directions is illustrated in Figure 39. Note that the caustic directions for incidence in the plane of the paper are:

- The flat surface caustic, in the $-X$ -axis direction.
- The straight line caustics, in directions perpendicular to the sides of the cone frustrum.
- The ring caustic, in the $+X$ -axis direction.

First, consider diffracted fields from the four diffraction points, P_{11} , P_{12} , P_{21} and P_{22} , on the unfaired test body.

1. For the flat disk at the end of the body, diffracted fields from points P_{21} and P_{22} contribute to the total field around the caustic direction. If the specular backscattered field for the disk is U_{flat}^c , then the diffracted fields from P_{21} and P_{22} are set equal to $\frac{1}{2}U_{flat}^c$ in the caustic region, so that they will add up to the caustic value.
2. For the straight line at the top of the figure, diffracted fields from points P_{11} and P_{21} contribute to the total field in the caustic direction. In this case the caustic field magnitude is

$$U^c = U_l^c + \frac{1}{2}U_1^{GO} \quad (4.32)$$

where U_l^c is the line caustic value for the straight line on the surface of the cone frustrum and U_1^{GO} is the GO reflected field from the ogive at P_{11} . The factor of $\frac{1}{2}$ is included because, right at P_{11} , an incident field illuminates the ogive only to the one side of the reflection point (the other side being replaced by the cone frustrum surface). In this case, diffracted fields from points P_{11} and P_{21} are set equal to $\frac{1}{2}U^c$ in the caustic region, so that they will add up to the caustic value. A similar analysis applies to the straight line caustic at the bottom of the figure.

(a) Unfaired body

(b) Faired body

Figure 39: Magnitudes of diffracted fields in caustic directions.

3. For the ring caustic in the X-axis direction, U_{ring1}^c is the ring caustic magnitude for the smooth junction. Diffracted fields from points P_{11} and P_{12} are set equal to $\frac{1}{2}U_{ring1}^c$ in this caustic region. A similar analysis applies to the wedge junction.

Now, consider diffracted fields from the six diffraction points, P_{11} , P_{12} , P_{21} , P_{22} , P_{31} and P_{32} on the faired test body.

1. For the $-X$ -axis direction, the caustic field value is

$$U^c = U_{flat}^c + \frac{1}{2}U_{GO3}^c \quad (4.33)$$

where U_{flat}^c is the flat surface caustic value for the disk and U_{GO3}^c is the GO field from the faired shape in the $-X$ -axis direction, computed using the GO Equivalent Line Current method. Again, the factor of $\frac{1}{2}$ is used since only half of an equivalent toroid, at the ring-shaped junction, is illuminated by the incident wave (the other half being covered by the disk). In this case, diffracted fields from points P_{31} and P_{32} are set equal to $\frac{1}{2}U^c$ in the caustic region.

2. For the straight line at the top of the faired body, diffracted fields from P_{11} and P_{21} contribute to the total field in the caustic direction. In this case the caustic field value is

$$U^c = \frac{1}{2}U_1^{GO} + U_l^c + \frac{1}{2}U_2^{GO} \quad (4.34)$$

where U_l^c is the line caustic value, U_1^{GO} is the GO reflected field from the ogive at P_{11} and U_2^{GO} is the GO reflected field from the faired shape at P_{21} . Diffracted fields from points P_{11} and P_{21} are set equal to $\frac{1}{2}U^c$ in this caustic region. A similar analysis applies to the straight line caustic at the bottom of the figure.

3. The ring caustic in the +X-axis direction is treated in the same way as was done for the unfaired body.

Note that, if the RCS envelope were computed using addition of powers, then fields from diffraction points in each of the above caustic directions should be set equal to $\frac{1}{\sqrt{2}}$ times the caustic value, so that the total field will have the correct magnitude.

Also note that the peaks in caustic regions can be very narrow. Therefore, when computing the envelope, the field magnitude in an angular box will be set equal to the caustic value if the caustic direction lies within approximately an angular increment from the center of the box.

4.3.4 Computed Diffracted Field Results for the Test Bodies

1. Smooth Junction Diffraction: Unfaired Body Junction J_1

Figure 40 shows the RCS map for diffraction from junction J_1 on the unfaired test body. Comparing this map with that for the ogive section in Figure 35, one sees that the peak of the diffraction pattern is at the angle (73° from the X-axis) where the GO fields stop. Also note that the ring caustic value, at the center of the map, is below the -40 dB level for this smooth junction.

The RCS map for junction J_{f1} on the faired body will be the same as that in Figure 40, except that the peak values are lower.

2. Wedge Diffraction: Unfaired Body Junction J_2

The RCS map for diffraction from junction J_2 on the unfaired test body, together with two line plots along the lines $\theta_v = 90^\circ$ and $\phi_v = -100^\circ$, are shown in Figure 41. The following features are worth noting:

Figure 40: Smooth Junction Diffraction: Unfaired Body Junction J_1

Figure 41: Wedge Diffraction: Unfair Body Junction J₂

- The peaks at $\theta = 90^\circ$ and $\phi = \pm 180^\circ$ represent the RCS for incidence perpendicular to the flat surface of the disk, where the diffracted fields were limited to the flat surface caustic value.
- There is a discontinuity in the ϕ -line plot at $\phi = -90^\circ$, which is explained as follows: Let us move along the line $\theta_v = 90^\circ$, starting at the left end of the map at $\phi = -180^\circ$ and increasing ϕ . For incidence angles between -180° and -90° , two diffraction points are illuminated on junction, J_2 ; one on the near end of the junction and one on the far end of the junction. This can be seen in the Front view of the body, which is at $(\theta_v, \phi_v) = (90^\circ, -100^\circ)$. Now, note that $\phi = -90^\circ$ is a shadow boundary for the diffraction point at the far end of the body. Therefore, the discontinuity in the diffracted field at this angle indicates that the contribution from the far end was still of significant magnitude when it was shadowed by the near end. This implies that double diffraction should be included in a more complete theoretical solution for the geometry.
- The ring of high RCS values in the center of the map corresponds to incidence directions perpendicular to the cone frustum surface, where line caustic values were used.
- The center of the map represents incidence along the $+X$ -axis, such that the whole junction J_2 is illuminated. In this direction, the ring caustic value was used.
- There is a difference in the rate of drop-off, from the ring caustic value, for directions along vertical and horizontal lines through the center of the map. This is due to the polarization of the incident wave and the way it interacts with the wedge junction. Note that, for the RCS map

in Figure 41, the incident field is ϕ polarized, as indicated on the figure. Also note that, for incidence close to the $+X$ -axis, junction J_2 is a “trailing edge;” i.e., the electric field travels along the body and then hits the wedge discontinuity. Now, if one moves along a vertical line through the center of the map, the incident electric field is parallel to the surface of the body. This will tend to “short out” the field, yielding a low diffracted field return when the wave strikes the wedge. On the other hand, if one moves along a horizontal line through the center of the map, the electric field is perpendicular to the surface of the body, so that it can be nonzero on the surface of the body. In this case, the wedge discontinuity causes stronger diffraction close to the $+X$ -axis, as can be seen on the map.

3. Smooth Junction Diffraction: Faired Body Junction J_{f2}

Diffraction due to junction J_{f2} on the faired test body is shown in Figure 42. The pattern is essentially the same as that for junction J_{f1} , with peaks in the line caustic directions such as illustrated by the front view direction.

4. Smooth Junction Diffraction: Faired Body Junction J_{f3}

Figure 43 shows the RCS for junction J_{f3} on the faired test body, where the caustic value was applied on the $-X$ -axis.

Figure 42: Smooth Junction Diffraction: Paired Body Junction J_{f2}

Figure 43: Smooth Junction Diffraction: Faired Body Junction J_{f3}

4.4 Total RCS Envelope Patterns for the Test Bodies

In this section, the total RCS envelopes for the unfaired and faired bodies will be compared with each other and with measured results. Unless otherwise stated, it is assumed that field addition was used in finding the computed total RCS envelopes. The computed patterns will be discussed first.

4.4.1 Computed Results

The RCS maps for the two test bodies are shown in Figure 44(a) and (b). Comparing the two RCS maps, one notes the “hole” in the center of the RCS map for the faired body, representing RCS values below the minimum of the gray scale, as opposed to that for the unfaired body. This difference is due to the smooth junctions on the faired body, which reduced the RCS in the region where the unfaired body has a large ring caustic value. Note that the RCS map for the faired body is approximately symmetric around the center of the map. This is because the faired body has only smooth junctions, for which diffractions are much less dependent on the polarization than is the case for wedge junctions. Further differences between the RCS for the two bodies can be seen on the plots in Figure 45, where $\theta = 90^\circ$ and ϕ is varied. Note that the peak values are lower for the faired body. Firstly, the values at $\pm 180^\circ$ are lower due to the smaller area of the flat disk for the faired body. Secondly, the values at $\pm 73^\circ$ are smaller due to the shorter sections of the cone frustrum that contribute to the total field in line caustic regions. Thirdly, the ring caustic value is much lower for the faired body, due to the smooth junctions. On the other hand, the RCS for the faired body shows an increase in the region where GO fields from the faired shape contribute to the total field; i.e., for $|\phi| > 73^\circ$.

(a) UNFAIRED TEST BODY

(b) FAIRED TEST BODY

Figure 44: Comparison of the computed total RCS maps of the faired and the unfaired test bodies.

Figure 45: Comparison of the computed total RCS line plots of the faired and the unfaired test bodies, for phi polarization.

4.4.2 Measured Results

Measurements on the two test bodies were performed in the ElectroScience Laboratory Compact Range. Measurements were done as a function of the ϕ angle, with $\theta = 90^\circ$, for both phi and theta polarizations. The Front View direction, which will be indicated on the pattern plots, corresponds to the incidence direction $(\theta_v, \phi_v) = (90^\circ, 0^\circ)$. Angular increments of $\Delta\phi = 0.5^\circ$ were used.

The measured results include vector subtraction of the range background and calibration with respect to a 6" sphere. Each measurement as a function of aspect angle, ϕ , and frequency, f , involved the following steps:

1. Taking a measurement ($E^t(\phi, f)$) with the body mounted on the pedestal.
2. Taking a background measurement ($E^{tb}(\phi, f)$) with the body removed from its mount.
3. Taking a measurement ($E^s(\phi, f)$) with a 6 inch sphere mounted on the pedestal.
4. Taking a background measurement ($E^{sb}(\phi, f)$) with the sphere removed from its mount.

These values were then used to calibrate the measured result for the test bodies, such that

$$E^{cal}(\phi, f) = [E^t(\phi, f) - E^{tb}(\phi, f)] \frac{E^{s, exact}(f)}{[E^s(\phi, f) - E^{sb}(\phi, f)]} \quad (4.35)$$

where $E^{s, exact}(f)$ is the computed exact RCS value for the 6 inch sphere.

Two kinds of measurements were taken. First, pattern measurements were taken at fixed frequencies (5, 10 GHz) and polarization (theta or phi). These patterns are compared with computed results. Second, swept frequency (2 – 18 GHz)

Figure 46: Unfaired body, measured and computed total RCS for phi polarization at 10 GHz.

measurements were taken at various angles. Those are used to analyze magnitudes in the caustic regions as a function of frequency, and to analyze the time domain impulse response of the targets by taking an inverse Fourier Transform of the results. The measured results are discussed next.

1. Unfaired body: Total RCS for phi and theta polarization

Measured and computed results for the unfaired test body at 10 GHz are compared in Figure 46 (for phi polarization) and Figure 47 (for theta polarization). There is good agreement between measured and computed results everywhere, except in the region close to zero degrees. The differences there are due to tip diffraction and creeping waves around the back of the body.

Figure 47: Unfaired body, measured and computed total RCS for theta polarization at 10 GHz.

This can be seen from the impulse response plots for $\phi = 30^\circ$ incidence. Figure 48(a) shows the impulse response for phi polarization, with a scale drawing of the body superimposed on the plot. Note that 1 nanosecond on the plot corresponds to 5.905 inches of two-way travel of a radar pulse; i.e., pulses that are one nanosecond apart on the plot correspond to first order scattering centers that are 5.905 inches apart on the body. Pulse 1 on the plot corresponds to a combination of tip diffraction, smooth junction diffraction and the GO reflected field from the ogive section. Pulse 2 corresponds to the trailing edge diffraction from junction J_2 , which is particularly strong for phi polarization. The later pulses are the double diffraction terms which

were not included in the computed results.

Figure 48(b) shows the impulse response for theta polarization. Note that the trailing edge diffraction is much smaller in this case, due to "shorting out" of the electric field on the surface of the body, as discussed earlier.

2. Unfaired body: Adding powers vs. adding fields

In Figure 46, field addition was used in obtaining the total envelope. That result followed the peaks of the measured results quite well, as would be expected for such a rigid body. Figure 49 shows the same measured result compared to an envelope where power addition was used. This yields an average value of the total field which is below the envelope of the peaks especially in areas where more than one mechanism are dominant; eg. in the region $30^\circ < |\phi| < 73^\circ$, where both the GO field and the smooth junction diffraction are strong. Power addition would therefore be more appropriate for non-rigid scatterers where the scattering centers can move relative to each other; i.e., for aircraft wings during flight, to yield average or median valued curves.

3. Faired body: Total RCS at 10 GHz for theta and phi polarization

Measured and computed results for the faired body are shown in Figures 50 and 51 for theta and phi polarization, respectively. Note that, again, there is good agreement between the peaks of the measured patterns and the computed envelope. There is much less difference between the measured theta and phi polarization results than was the case for the unfaired body, since there are no wedge junctions. The differences are now due to higher order effects, such as creeping waves and double diffraction, which can be different for the two polarizations.

(a) Horizontal polarization

(b) Vertical polarization

Figure 48: Unfaired body, impulse responses for phi and theta polarizations at $\phi = 30^\circ$ incidence.

Figure 49: Unfaired body: Comparing measured and computed total RCS results, using power addition for finding the computed envelope.

4. Comparison of impulse responses for the faired and unfaired bodies in the ring caustic region:

Figure 52(a) and (b) show, respectively, the impulse responses for the unfaired and faired bodies in the ring caustic region; i.e. for $\phi = 0^\circ$ incidence. A scale drawing of each body is also superimposed on the plots as an aid to identifying the scattering mechanisms. For both bodies, pulse 1 represents tip diffraction, pulse 2 represents diffraction at the junction between the ogive section and the cone frustrum, and pulse 4 represents creeping waves around the back. Pulse 3 for the unfaired body is the return from the wedge junction, which has a large value in the ring caustic region. On the other

POLARIZATION:
 $(\theta, \phi) = (0, 1)$

FREQUENCY:
 10.00 GHz

FRONT VIEW:
 $\theta = 90.0 \text{ deg}$
 $\phi = 0.0 \text{ deg}$

FAIRED BODY
 PHI POLARIZATION

Figure 50: Faired body: Measured and computed results at 10 GHz for phi polarization.

POLARIZATION:
 $(\theta, \phi) = (1., 0.)$

FREQUENCY:
 10.00 GHz

FRONT VIEW:
 $\theta = 90.0 \text{ deg}$
 $\phi = 0.0 \text{ deg}$

FAIRED BODY
 THETA POLARIZATION

Figure 51: Faired body: Measured and computed results at 10 GHz for theta polarization.

(a) Unfaired body

(b) Faired body

Figure 52: Impulse responses for the faired and unfaired test bodies in the ring caustic regions.

hand, for the faired body, pulse 3 is the return from the smooth junction at the beginning of the faired shape. As expected, this term is much smaller than that for the unfaired body. Note that its magnitude is comparable to those for the tip and creeping wave terms.

5. Unfaired body: Magnitudes in caustic regions as a function of the frequency

In this section, the RCS magnitudes in the flat surface, straight line and ring caustic regions, due to the mechanisms discussed previously, are studied as a function of the frequency. Results are obtained from taking frequency sweep measurements at the angles corresponding to the caustics, such as for the ring caustic in Figure 52(a). Individual mechanisms are then isolated by time-windowing the impulse responses around the specific pulses (pulse 3 in Figure 52(a)) that represent those mechanisms, and then smoothing the magnitude responses by subtracting contributions that are not centered around those pulses. Figure 53 shows three sets of plots that represent the three caustic mechanisms; i.e., flat surface caustic, straight line caustic and ring caustic. For each one the original measured magnitude, a smoothed magnitude and a computed magnitude is shown. Note that there is good agreement between the measured and computed results. Also note that, as expected from the theoretical solutions, the RCS due to the flat surface caustic value is proportional to $1/\lambda^2$, the RCS due to the straight line caustic is proportional to $1/\lambda$ and the RCS due to the ring caustic is a constant with respect to the frequency.

Figure 53: Unfaired body, comparing computed and measured magnitudes in caustic regions.

6. RCS for both bodies at a lower frequency

Figures 54 and 55 show comparisons of computed and measured results for the unfaired and faired bodies, respectively, for phi polarization at 5 GHz. At this frequency, scattering points are closer in terms of the wavelength than was the case at 10 GHz. This results in broader peaks in the caustic regions and fewer oscillations in the measured patterns. The whole body is also smaller in terms of the wavelength, so that creeping waves and other higher order mechanisms will also be stronger. This is particularly apparent for the faired body which has a smooth path for creeping waves around the back. For this body, in the region around $\phi = 0^\circ$ where first order mechanisms have low magnitude, the difference between the measured and computed results are much greater than was the case at 10 GHz.

4.4.3 Conclusions

1. The usefulness of the envelope method was illustrated in that it can be used to efficiently compute the full volumetric RCS envelope pattern of isolated scattering bodies. Displaying the results on an RCS map is a convenient way to interpret this pattern.
2. It was shown that the computed RCS envelope gives a good approximation of the envelope for the RCS of an isolated scattering body. This is especially true in regions where first order mechanisms are dominant.
3. The envelope results were computed for angular increments of about 5.5° . This could be done independent of the frequency, for the whole RCS map. On the other hand, the measured results required much smaller angular increments. The $.5^\circ$ increment that was used may be smaller than what was

POLARIZATION:
 $(\theta, \phi) = (0., 1.)$

FREQUENCY:
 5.00 GHz

FRONT VIEW:
 $\theta = 90.0 \text{ deg}$
 $\phi = 0.0 \text{ deg}$

THETA IS FIXED

UNFAIRED BODY
 PHI POLARIZATION

Figure 54: Unfaired body: Comparison between measured and computed results at 5 GHz.

POLARIZATION:
 $(\theta, \phi) = (0, 1)$

FREQUENCY:
 5.00 GHz

FRONT VIEW:
 $\theta = 90.0 \text{ deg}$
 $\phi = 0.0 \text{ deg}$

FAIRED BODY
 PHI POLARIZATION

Figure 55: Faired body: Comparison between measured and computed results at 5 GHz.

actually necessary, but for higher frequencies even smaller increments will be required in order to plot all the peaks and nulls in the pattern. This illustrates the computational efficiency of the envelope method.

4.5 Summary

The computation of first order, high frequency, RCS envelopes of isolated scattering bodies was discussed in this Chapter. The procedure includes computing diffracted fields from all junctions on the surface, reflected fields from all doubly curved surfaces, and applying limits in caustic regions. Techniques were described whereby the symmetry of a body can be exploited in order to minimize computations when full volumetric RCS patterns are computed.

It was shown that the computed first order RCS envelopes for the two test bodies compared well with measured results. Discrepancies were due to higher order mechanisms which were not included in the computed solution, but are significant in regions where there are no dominant first order mechanisms.

In the next Chapter, the geometries will be extended to include structures consisting of a main body with side bodies intersecting it.

CHAPTER V

DEFINING SURFACES: INTERSECTING BODIES WITH FAIRED JUNCTIONS

In this Chapter, a description is given of the surfaces of generic scattering structures consisting of a main body and a side body which intersects the main body. The main and side bodies are individually composed of the canonical shapes and faired shapes as was defined for isolated bodies in Chapter III.

Figure 56 shows the outline for a main body, S_1 , intersected by a side body, S_2 . The main body is defined in the XYZ-coordinate system, and the side body is defined in the $X'Y'Z'$ -coordinate system. It is assumed here that the side body fully intersects the main body. A description will be given of how to find the contour where the two bodies intersect. Note that the two bodies intersect at a sharp surface angle. A method will be described whereby this junction can be replaced by a faired shape that matches smoothly to both bodies. The computation of surface derivatives on the faired shapes will also be discussed.

5.1 Coordinate System for a Side Body

The side body is defined in the $X'Y'Z'$ -coordinate system, which can have arbitrary orientation and displacement with respect to the main XYZ-coordinate system. The primed coordinate system, shown in Figure 57(a), and its associated unit vectors, $\{\hat{x}', \hat{y}', \hat{z}'\}$, are uniquely specified by the parameters $[x_0, y_0, z_0, \theta, \phi, \zeta]$,

Figure 56: Main body with an intersecting side body.

where:

$[x_0, y_0, z_0]$ = the coordinates for the origin of the
 $X'Y'Z'$ -coordinate system at O'

(θ, ϕ) = orientation angles for the X' -axis, and

ζ = rotation angle for the Z' -axis.

The angles θ and ϕ are the spherical coordinate angles in the XYZ-coordinate system. The unit vectors in the spherical coordinate system are given by

$$\hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \quad (5.1)$$

$$\hat{\theta} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta, \text{ and} \quad (5.2)$$

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi. \quad (5.3)$$

The X' -axis is defined to be in the direction of the $-\hat{r}$ unit vector. The Z' -axis is defined by the angle, ζ , where ζ is the counter-clockwise rotation of the Z' -axis around the X' -axis with respect to the $-\hat{\theta}$ unit vector. This definition of ζ is illustrated in Figure 57(b) for a view looking toward the origin in the $-\hat{r}$ -direction. Finally, the Y' -axis is defined by the unit vector $\hat{y}' = \hat{z}' \times \hat{x}'$. Using this definition, the unit vectors in the $X'Y'Z'$ -coordinate system are given by

$$\hat{x}' = -\hat{r} \quad (5.4)$$

$$\hat{y}' = -\cos \zeta \hat{\phi} + \sin \zeta \hat{\theta}, \text{ and} \quad (5.5)$$

$$\hat{z}' = -\sin \zeta \hat{\phi} - \cos \zeta \hat{\theta}. \quad (5.6)$$

In finding the contour of intersection between the main and side bodies, it will be necessary to perform a transformation of points between the two coordinate systems. The transformations between the coordinates of a point $P(x, y, z)$ in the

LOOKING ALONG THE
RADIAL LINE, TOWARDS THE ORIGIN

Figure 57: Definition of the primed coordinate system for side bodies.

XYZ-coordinate system and those for the same point, $P(x', y', z')$, in the $X'Y'Z'$ -coordinate system, are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \text{ and} \quad (5.7)$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}. \quad (5.8)$$

Note that l_{ij} , $i, j = 1, 2, 3$, are the direction cosines, given by

$$l_{ij} = \hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}'_j, \quad i, j = 1, 2, 3 \quad (5.9)$$

where

$$\begin{aligned} \hat{\mathbf{p}}_1 &= \hat{\mathbf{x}} & \hat{\mathbf{p}}_2 &= \hat{\mathbf{y}} & \hat{\mathbf{p}}_3 &= \hat{\mathbf{z}} \\ \hat{\mathbf{p}}'_1 &= \hat{\mathbf{x}}' & \hat{\mathbf{p}}'_2 &= \hat{\mathbf{y}}' & \hat{\mathbf{p}}'_3 &= \hat{\mathbf{z}}'. \end{aligned} \quad (5.10)$$

Points in both coordinate systems will also be expressed in the cylindrical coordinates $[x, \rho, \alpha]$. The relationships between the rectangular and cylindrical coordinates in the XYZ-coordinates system are illustrated in Figure 58 and are given by

$$\rho = \sqrt{y^2 + z^2} \quad (5.11)$$

$$\alpha = \tan^{-1}(y/z) \quad (5.12)$$

$$y = \rho \cos \alpha, \text{ and} \quad (5.13)$$

$$z = \rho \sin \alpha \quad (5.14)$$

where x is the same in both coordinate systems. Similar equations are used in the primed coordinate system. A point, $P(x, y, z)$, can therefore be expressed in four

Figure 58: Definition of cylindrical coordinates.

equivalent ways as

$$\begin{aligned}
 P(x, y, z) &\Leftrightarrow P(x, \rho, \alpha) \quad (\text{rectangular} \leftrightarrow \text{cylindrical in } XYZ) \\
 &\Leftrightarrow P(x', y', z') \quad (\text{original} \leftrightarrow \text{prime coordinates}) \\
 &\Leftrightarrow P(x', \rho', \alpha') \quad (\text{rectangular} \leftrightarrow \text{cylindrical in } X'Y'Z')
 \end{aligned}$$

Equations (5.7) and (5.8) can also be used for transforming the coordinates of a vector direction from the original to the primed coordinate system, and vice versa. In order to do this, the displacement coordinates, $[x_0, y_0, z_0]$, must be set equal to zero.

The next section describes a way to find the contour where two bodies intersect.

5.2 Geometry for Intersecting Bodies

5.2.1 Finding the Contour of Intersection

Figure 59 shows a cross sectional cut through the main body and part of a side body that intersects it. The main body is denoted by S_1 , and points on the main body are described by the function $S_1(x, \alpha)$. The side body is denoted by S_2 , and points on the side body are described by the function $S_2(x', \alpha')$. The contour of

Figure 59: Finding a contour of intersection.

intersection will be defined as a function of the α' angle in the primed coordinate system, and is denoted by $C^{\alpha'}$.

A point on the contour of intersection must be on both surfaces simultaneously. Therefore, the search for a contour of intersection is a search for (x', α') and (x, α) such that

$$C^{\alpha'}(x', x, \alpha) = S_2(x', \alpha') = S_1(x, \alpha) . \quad (5.15)$$

It would not be a simple task to find the contour of intersection analytically for two arbitrary surfaces. Therefore, an algorithm will be described whereby this contour can be found iteratively. The algorithm describes the search for a coordinate, x'_p , given α' , where the two bodies intersect. Before listing the steps in the algorithm, let us look at the results of one iteration of the algorithm, shown in Figure 59. For a given α' angle, the search is conducted in the $Q'X'$ -plane. Let us represent x'_{p1} as the first attempt at finding the intersection point. $P'_1(x', y', z')$ is the corresponding

point on S_2 and $P'_1(x, \rho_2, \alpha)$ is the same point in XYZ-cylindrical coordinates. Now let $P(x, y, z)$ be the point on the contour of the main body which has the same (x, α) -coordinates as P'_1 , and $P(x, \rho_1, \alpha)$ is the same point in XYZ-cylindrical coordinates. If the two points were both on the contour of intersection, then ρ_1 would be equal to ρ_2 . It can be seen that this is not true for the given choice of x'_p . The next best choice for x'_p is x'_{p2} , the projection of point P onto the X' -axis. P'_2 , the point on the side body corresponding to x'_{p2} , is already much closer to the actual point of intersection at $C^{\alpha'}$. The above process can be repeated until a point x'_p is found for which ρ_1 is close enough to ρ_2 .

Due to the nature of this iterative search, it is necessary to specify a maximum distance error, ε , between ρ_1 and ρ_2 . This error defines the required closeness of the contour to the actual intersection of the two bodies. The steps of the search algorithm are now as follows:

1. Select x'_{init} , the initial X' -coordinate where the search process on the side body begins.
2. Select the angle α' to fix the $Q'X'$ -plane in which the point of intersection is to be found.
3. Set $x'_p = x'_{init}$.
4. Find the coordinates for the point P' on the side body in XYZ-cylindrical coordinates:

$$\begin{aligned}
 P' &= S_2(x'_p, \alpha') \\
 &= P'(x'_p, y', z') \\
 &\Leftrightarrow P'(x, y, z) \\
 &\Leftrightarrow P'(x, \rho_2, \alpha)
 \end{aligned}$$

5. Use (x, α) for point P' and compute the coordinates for point P on the main body in XYZ-cylindrical coordinates:

$$\begin{aligned} P &= S_1(x, \alpha) \\ &= P(x, y, z) \\ &\Leftrightarrow P(x, \rho_1, \alpha) \end{aligned}$$

6. Compare the ρ -coordinates of points P and P' :

- (a) If $|\rho_1 - \rho_2| > \epsilon$, then find the projection of point P on the X' -axis and use that coordinate for the next iteration:

$$\begin{aligned} P(x', y', z') &\Leftrightarrow P(x, y, z) \\ x'_p &= x' \end{aligned}$$

Go back to step 4 until P and P' are close enough.

- (b) If $|\rho_1 - \rho_2| \leq \epsilon$, then P and P' are close enough, and their coordinates can be stored. Set $x'_{init} = x'_p$ for use with the next α' angle.

Discussion:

- The search algorithm is relatively insensitive to the initial value x'_{init} . This is seen in Figure 59 where one iteration through the algorithm yielded a point much closer than the first point, and each subsequent point will be closer to the actual intersection. The intersection point can typically be reached within ten iterations, depending on the complexity of the geometry and the magnitude of ϵ . If the algorithm does not converge to a point within about twenty-five iterations, then it usually means that the point of intersection does not exist; i.e., that the side body does not intersect the main body for the given angle α' .

- Convergence of the algorithm is improved by setting the initial value for the next α' equal to the final value for the previous α' angle.
- It may appear in Figure 59 that the search is done along a fixed cross section of the main body. This, however, is not true for general bodies where each point, $S_2(x', \alpha')$, for fixed α' can have a different set of coordinates, (x, α) .

5.2.2 Examples

Examples of intersecting bodies are shown in Figure 60(a) and (b). The first example is that of a cone frustrum intersecting an ogive/ellipsoid combination at an angle of 90° . This geometry will be used later to illustrate variations in the fairing of the junction between the ogive and the cylinder. The second example shows an ellipsoid intersecting a cone frustrum at a somewhat unusual angle, to show the ability of the search algorithm to find the contour of intersection.

The next section deals with fairing of the junctions between intersecting bodies.

5.3 Geometry for Intersecting Bodies With Faired Junctions

The junction where the side body intersects the main body is a sharp junction with an exterior wedge angle of less than 180° . The aim here is to define a faired shape around the junction that will match smoothly to both the side and main bodies. The configuration for such a faired shape is shown in Figure 61. An ellipse matching technique, similar to the fairing of sharp junctions (described in Chapter III) will be used to define this faired shape.

5.3.1 Finding Parameters for the Faired Shape

Let the beginning of the faired shape be defined by the contour, $C_2^{\alpha'}$, on the side body and its end by the contour, $C_1^{\alpha'}$, on the main body. An efficient and

(a) Cone frustum intersecting an ogive/ellipsoid combination.

(b) Ellipsoid intersecting a cone frustum.

Figure 60: Examples of intersecting bodies.

Figure 61: Fairing the junction where two bodies intersect.

Figure 62: Defining beginning and ending contours for the faired shape.

flexible way for defining these contours is illustrated in Figure 62, and will be explained next.

First, let us define a cylindrical magnification of the main body. The surface MS_1 is a version of the main body, magnified M times in the direction perpendicular to the X -axis. Therefore, if $P(x, \rho, \alpha)$ were a point on the main body, then $P_M(x, M\rho, \alpha)$ is a point on the surface, MS_1 . The contour, $C_2^{\alpha'}$, on the side body can now be found as the intersection of this magnified main body with the side body.

Next, let us define a cylindrical magnification of the side body. The surface $M'S_2$ is a version of the side body, magnified M' times in the direction perpendicular to the x' -axis. Therefore, if $P(x', \rho', \alpha')$ were a point on the side body, then $P_{M'}(x', M'\rho', \alpha')$ is a point on the surface, $M'S_2$. The contour, $C_1^{\alpha'}$, on the main

body can now be found as the intersection of this magnified side body with the main body.

The $C_1^{\alpha'}$ and $C_2^{\alpha'}$ contours are both functions of the α' angle in the primed coordinate system. Therefore, for a given α' , one can define the $Q'X'$ -plane which goes through two known points: P_1 on the main body and P_2 on the side body, as shown in Figure 61.

If one finds the surface slopes at points P_1 and P_2 in this plane, denoted by $q'_{x', 1,2}$, then one can define an elliptic curve that will match the surfaces smoothly at both points. Such ellipses are shown in Figure 62. The family of ellipses, as a function of α' around the side body, defines the faired shape that matches the surface of the side body to the surface of the main body. Finding these slopes is discussed next.

The slope in the $Q'X'$ -plane at P_2 is found from the equations that describe the surface of the side body. The tangent vector, $\hat{t}_2$, in Figure 61 indicates the orientation of this slope line.

Some manipulation is needed to find the slope in the $Q'X'$ -plane at P_1 on the main body. First, find the surface normal, $\hat{n}$, from the surface equations for the main body, and transform the coordinates of $\hat{n}$ to the prime coordinate system. Next, let us find $\hat{b}$, the normal to the $Q'X'$ -plane, where

$$\hat{b} = \sin \alpha' \hat{y}' - \cos \alpha' \hat{z}' . \quad (5.16)$$

The normal vectors $\hat{n}$ and $\hat{b}$ are both perpendicular to $\hat{t}_1$, the surface tangent in the $Q'X'$ -plane, so that

$$\hat{t}_1 = \frac{\hat{b} \times \hat{n}}{|\hat{b} \times \hat{n}|} \quad (5.17)$$

$$= \hat{x} t_{1x'} + \hat{y} t_{1y'} + \hat{z} t_{1z'} . \quad (5.18)$$

Figure 63: Tangents and slopes in the $Q'X'$ -coordinate system.

As shown in Figure 63, the slope at P_1 can now be computed as

$$q'_{x'1} = \frac{t_{1q'}}{t_{1x'}} \quad (5.19)$$

where

$$t_{1q'} = \frac{t_{1y'}}{\cos \alpha'} = \frac{t_{1z'}}{\sin \alpha'} \quad (5.20)$$

With the slopes at points P_1 and P_2 known, everything is known to define the matching ellipses at each α' angle around the surface. The procedure for finding these ellipses is described in Appendix C and discussed in Section 3.4.1.

The faired shape for smoothing of a junction between two canonical shapes was defined in Chapter III as a function of the independent parameters, x and α . This could be done since all the ellipses around the shape started and stopped at the same X -coordinates. Also, each point on the X -axis defined only one point on the faired shape. The faired shape for intersecting bodies, however, cannot be defined in the same way. First, each ellipse for this faired shape can start and stop at X' -coordinates different from the adjacent one. Furthermore, there may be a range of X' -coordinates that define two points on the matching ellipse. This

Figure 64: Cross section of a faired intersection.

problem is particularly apparent in Figure 64 where the ellipse starts at x'_2 and stops at x'_1 , but also extends beyond x'_1 . In the region $x' > x'_1$, each X' -coordinate describes two points on the ellipse, so that x' is an ambiguous parameter. These ambiguities in surface definition can be overcome by using the ellipse angular parameter, ν' , as the independent parameter, instead of x' . Now, in addition to knowing the X' -coordinates at the start and stop points of the ellipse, each ellipse will be uniquely defined by the parameters $[a, b, x'_0, q'_0, \xi, \nu'_1, \nu'_2]$. Points on the ellipse are then defined by

$$\mathbf{r}(\nu', \alpha') = [x'(\nu'), q'(\nu') \cos \alpha', q'(\nu') \sin \alpha'] , \quad (5.21)$$

where $\nu'_2 < \nu' < \nu'_1$ and both x' and q' are functions of ν' .

5.3.2 Examples

Examples of intersecting bodies with faired junctions are shown in Figure 65 and Figure 66. The original bodies are the same as those shown in Figure 60. The ogive-cylinder body is shown in Figure 65(a) and (b) with two different faired shapes. In case (a), M is large and M' is small; in case (b), M is small and M' is large. These two cases illustrate the range of different faired shapes that can be defined by the method described above.

Figure 66 shows a faired shape for the cone frustrum-ellipsoid body. This example shows the ability of the faired shape to smoothly match to a main body even from an unusual angle.

The next section deals with the computation of surface characteristics on such faired shapes.

5.3.3 Surface Characteristics on the Faired Shape

The surface derivatives needed for computing surface characteristics on the faired shape are found in a way similar to what was described in Section 3.4.3 for faired shapes on isolated bodies. The basic difference is that the ellipses at each α' angle are now functions of ν' , rather than x' . Therefore, the surface is defined in terms of the ν' parameter for the ellipse at each α' angle. Note that, due to the changing of the ellipses as a function of α' , the parameter ν' is a "floating" parameter with respect to α' . Therefore, given ν' , it will be necessary to find the X' -coordinate corresponding to ν' , and then use x' and α' as the independent parameters for computing surface derivatives.

In the $Q'X'$ -plane, the derivatives $q'_{x'}$ and $q'_{x'\alpha'}$ are computed analytically as a function of ν' , using the equations given in Appendix A.

(a) Main body magnification larger than side body magnification.

(b) Side body magnification larger than main body magnification.

Figure 65: Examples of faired shapes with different magnification factors.

Figure 66: Example of a faired intersection.

The derivatives with respect to α' ($q'_{\alpha'}$, $q'_{x'\alpha'}$ and $q'_{\alpha'\alpha'}$) are found numerically, as described in Section 3.4.3 for the faired shape on isolated bodies. In order to keep x' constant when data points are taken at angles adjacent to α' , it is necessary to compute the corresponding ellipse parameter, ν' , for these ellipses (the procedure is described in Section A.3). Then the values q' and $q'_{x'}$ can be computed at the adjacent angles and used for the numerical computation of derivatives.

Section 3.4.3 also describes the computation of surface derivatives in the $Q'A'X'$ -coordinate system when the slope, $q'_{x'}$, gets large. In this case, q' is taken as the independent variable, instead of x' . This is especially important for the fairing of intersecting bodies, for two reasons. Firstly, it is more common for the slope, $q'_{x'}$, to go through the $\pm\infty$ -change. This is the case for the faired shape in Figure 64. Secondly, if the angle between the side and main bodies is less than 90° , it can happen that the ellipses at the incremental angles next to α' do not extend to the coordinate x' . This problem is illustrated in an exaggerated way in Figure 67. Three adjacent ellipses, numbered -1, 0 and +1, are shown on a faired surface. Surface derivatives are needed at a point, P, on ellipse number 0. The line of constant x' through P cuts through ellipse number +1, but misses ellipse number -1 altogether. On the other hand, the contour of constant q' can still be found on all three ellipses.

5.4 Summary and Conclusions

A systematic way for describing geometries with a side body intersecting a main body was given. It was shown that the contour of intersection between the two bodies can be found in a relatively simple way by an iterative search algorithm. The junction between the two bodies can be faired by defining a surface composed of elliptic contours around the side body. A flexible method was given for defining

Figure 67: x' -contour vs. q' -contour for surface derivatives.

the contours where this faired shape starts on the side body and ends on the main body. The faired shape matches smoothly to both the main and side bodies. Surface derivatives can be found on the faired shape and used to compute the surface characteristics.

The techniques described above were applied to only one side body, but, in principle, there is no limit to the number of side bodies that can be added to the main body. The only restriction here is that the side bodies do not overlap where they intersect the main body. As an illustration, various views of the unfaired model of an airplane is shown in Figure 68. The fuselage consists of part of an ellipsoid with a matched ogive as the nose section and a flat disk at the tail end. The wings and tail were composed of cone frustrums with ellipsoids matched to the ends, and the cockpit is an ellipsoid. Figure 69 shows the same airplane with all the side bodies faired into the main body. The wedge junction at the rear end was also faired. Such geometries are easy to describe and modify in an interactive way, using the methods described in this chapter.

The next Chapter describes techniques for computing scattered fields from structures with main and side bodies.

Figure 68: Various views of an unfaired model of an airplane.

Figure 69: Various views of a faired model of an airplane.

CHAPTER VI

FIELD COMPUTATIONS FOR INTERSECTING BODIES

This Chapter describes the computation of scattered fields from structures consisting of a main body and a side body intersecting it. As was done for isolated scattering bodies, GO reflected fields are computed for all doubly curved surfaces, UTD diffracted fields are computed for all junctions in the surface, and PO and equivalent current results are applied in caustic regions.

Those shapes and junctions on the side body that are similar to the ones for isolated scattering bodies will be called **regular shapes and junctions**. Backscattered fields for these shapes and junctions are computed in a way similar to what was done for isolated bodies.

The junctions at the contours of intersection are not elliptic, as was the case for junctions on isolated bodies, and are also not symmetric in the four quadrants of the primed coordinate system. Furthermore, the faired shape at an intersection, in general, does not begin and end in a constant x' plane. Therefore, special consideration will be given to the computation of backscattered fields from such shapes and junctions.

Shadowing of diffraction and reflection points by other parts of the body is a more serious computational problem for structures with intersecting bodies. To treat shadowing in a general way is beyond the scope of this study. However, some aspects of shadowing can be dealt with in a fairly simple manner. Those include

making sure that diffractions are only computed for exterior angles of incidence (as was done for isolated scattering bodies), and excluding scattering points on the main body which are within the contour of intersection of a side body.

The computational techniques will be illustrated for an ellipsoid with a cone frustrum intersecting it. Field computations for the main body are discussed first.

6.1 Field Computations for the Main Body

For shapes and junctions on the main body, field computations are done in the same way as was described for isolated scattering bodies. In addition, it is necessary to exclude those scattering points on the main body which fall within the contour of intersection of a side body. This is a minimum requirement for shadowing of scattering centers on the main body, and does not include all possibilities where scattering points on the main body are shadowed by the side body. Testing for points within the contour of intersection is described in the following section.

6.1.1 Excluding Points Within a Contour of Intersection

The contour of intersection is a set of points defined as a function of the α' angle. Therefore, the test for excluding scattering points inside the contour requires a numerical search. Figure 70 shows a contour of intersection, $C^{\alpha'}(x', x, \alpha)$, in the $X\alpha$ -plane. One needs to determine whether a scattering point, $P_s(x_s, \alpha_s)$, is inside or outside the contour of intersection. First, the extremes for x and α on the contour are found. Those are denoted by x_{min} , x_{max} , α_{min} and α_{max} , as shown in Figure 70, and are computed only once for a given side body. If either x_s or α_s or both fall outside of the respective extremes, then P_s must be outside the contour of intersection. If both coordinates fall inside the extremes, then it is necessary to find the α angles where the line $x = x_s$ crosses the contour. These angles,

Figure 70: Checking for points inside a contour of intersection.

denoted by α_1 and α_2 , are determined through a search around the contour. (A linear interpolation between the α values for the points on the two sides of $x = x_s$ approximates α at a crossing.) Now, if $\alpha_1 < \alpha_s < \alpha_2$, then P_s is within the contour of intersection. Note that, for correct testing, care must be taken that α , α_1 and α_2 are all within the same range (i.e., not $\pm 360^\circ$ apart).

6.1.2 Computed Results

Figure 71 shows the test bodies which will be used in this chapter to illustrate field computations for the main and side bodies. The main body is an ellipsoid and the side body is a cone frustum with a disk at the end. The junction where the two bodies intersect is unfaired in Figure 71(a), and faired in Figure 71(b). The parameters for these bodies are summarized as follows (all dimensions are in inches):

1. Ellipsoid: $[x_0, a, b, c] = [10, 4, 5, 6]$

2. Cone frustrum: $[x_1, x_2, a_1, b_1, a_2, b_2] = [5, -11, 2, 3, 1, 1.5]$
3. Disk: $[x_1, x_2, a_1, b_1, a_2, b_2] = [-11, -11, 2, 3, 0, 0]$
4. Side body offset vector: $[x_0, y_0, z_0] = [10, 0, 1]$
5. Side body orientation: $[\theta, \phi, \xi] = [70, 90, 90]$, and
6. Magnification factors: $[M, M'] = [1.5, 1.6]$.

Computations will be done for vertical polarization at 10 GHz.

The GO reflected fields from the ellipsoid in the unfaired test body is shown in Figure 72. Note that the “hole” in the RCS pattern corresponds to points on the ellipsoid that fall within the contour where the cone frustrum intersects the main body. The pattern for the faired test body is the same as this pattern, except that the hole in the RCS pattern will be larger since the area of intersection is larger.

6.2 Field Computations for the Regular Parts of a Side Body

Field computations for junctions and shapes on the side body can be done in the primed coordinate system, in the same way as was described for isolated scattering bodies. Once the computations are done, the results are transformed to the corresponding angles in the main coordinate system. The effect of this transformation is that the relative position of the corresponding RCS pattern will be shifted on the map display.

Also note that, for the shape next to the junction with the main body, it is necessary to take into account that the end of the shape is now defined by a general contour, and not by an ellipse at a constant X' -coordinate. Therefore, if it is a doubly curved surface, GO reflected fields from this last shape must stop at the contour. On the other hand, if it is a singly curved surface, then PO limits for this

Figure 71: Geometry for test bodies.

Figure 72: GO fields from the ellipsoid in the unfaired body.

shape must take into account that the shape has different end points as a function of the α' angle.

The next section describes the coordinate transformation for fields from side bodies.

6.2.1 Coordinate Transformation for Fields on Side Bodies

The spherical coordinate angles, (θ', ϕ') , and the corresponding angular indices, $(i_{\theta'}, i_{\phi'})$, are defined in the $X'Y'Z'$ -coordinate system. Fields for the side body are computed and stored in an array, $U'(i_{\theta'}, i_{\phi'})$, over the full range of values for $i_{\theta'}$ and $i_{\phi'}$. The corresponding field magnitudes in the XYZ-coordinate system are then found by starting with the indices (i_{θ}, i_{ϕ}) and finding the equivalent primed indices $(i_{\theta'}, i_{\phi'})$, where the field magnitudes are stored. This is achieved by transforming incident vectors to the primed coordinate system, as follows:

$$\begin{aligned}
 \hat{s}(i_{\theta}, i_{\phi}) &= \hat{s}(\theta, \phi) && \text{(use Equation (4.3))} \\
 &= \hat{x}s_x + \hat{y}s_y + \hat{z}s_z && \text{(use Equation (4.1))} \\
 &= \hat{x}s_{x'} + \hat{y}s_{y'} + \hat{z}s_{z'} && \text{(vector from XYZ to X'Y'Z')} \\
 &= \hat{s}(\theta', \phi') && \text{(derived from Equations (4.1) and (4.2))} \\
 &= \hat{s}(i_{\theta'}, i_{\phi'}) && \text{(derived from Equation (4.3))}
 \end{aligned}$$

Now, the scattered field magnitude is $U(i_{\theta}, i_{\phi}) = U'(i_{\theta'}, i_{\phi'})$.

Diffracted field results for the wedge junction between the cone frustrum and the disk discussed next.

6.2.2 Computed Results

Figure 73 shows computed results for the diffracted fields from the wedge junction between the cone frustrum and the disk on the unfaired test body. The angle where the cursor lines cross correspond to incidence in the direction perpendicular to the face of the disk, as seen in the front view. The peak at this angle

Figure 73: UTD fields from the wedge junction at the end of the cone frustum on the unfaired test body.

is therefore the flat surface caustic value. Note that the S-shaped curve of peak values corresponds to incidence perpendicular to the sides of the cone frustum. The RCS magnitude along this curve changes because of the elliptic cross section of the cone frustum, which causes the line caustic values to change as a function of the α' angle on the side body.

The next section discusses the the GO reflected fields from a faired shape at the intersection between the main and side bodies.

6.3 GO Reflected Fields from a Faired Intersection

A Newton-Raphson search for the reflection points works well for a faired shape at the junction between the two canonical shapes in that the independent parameters, x and α , are defined in the same way everywhere on the shape, and because the shape has beginning and ending points at planes of constant x . For a faired shape at the junction between two intersecting bodies, however, the independent parameters are α' and $\nu^{x'}$, where $\nu^{x'}$ is the ellipse parameter in the x' direction. In this case, $\nu^{x'}$ is defined differently for each α' angle. Therefore, a Newton-Raphson search for reflection points is not suitable in this case. GO fields for this shape can be found by taking a large number of points on the shape, finding the reflected field direction at each point, and then computing the reflected field for that direction. Since there is no symmetry in this faired shape, the whole shape must be included in the computations. This procedure for computing backscattered reflected fields, which may be called **stepping over the surface**, is described in the next section.

Figure 74: Geometry for taking surface steps to fill a GO-field array.

6.3.1 Surface Steps for Finding Reflected Field Directions

Instead of starting with a desired reflected field direction, this method starts with a point on the surface, finds the corresponding reflected field direction and stores the GO-field for that direction. Then, a step is taken on the surface and the procedure repeated at the next point. If enough points are taken on the surface, then all the angles which have GO-field contributions from this particular shape will have non-zero fields stored in the field array.

The geometry for this stepping procedure is shown in Figure 74. The main challenge with this approach is to define the surface steps small enough to include all angular slots with GO-fields from this shape, without having the steps so small

Figure 75: Incrementing of the real angle versus the ellipse parametric angle.

that computation time is significantly increased. Note that the whole surface will be covered if α' is incremented from 0 to 360°, and $\nu^{\alpha'}$ from its beginning to its ending values at a given α' angle.

The α' step size should be a function of a/b , the aspect ratio of the shape (this aspect ratio can be determined at the beginning contour of the faired shape) in order to minimize the number of increments in the $Y'Z'$ -plane. This can be done by incrementing the ellipse angular parameter, $\nu^{\alpha'}$, in stead of α' . The reason is illustrated in Figure 75, which shows the elliptic cross section of a shape at a given X' -coordinate. The Y' - and Z' -axis half-lengths of the ellipse are a and b , respectively. The right top quarter of the contour shows the distribution of surface points and normal directions when constant increments in α' are taken. The angular spacing between normals are much larger for points close to the thin end of the ellipse than for those near the top of the ellipse. The left top quarter of the contour shows the distribution of surface points and normals when constant

increments in $\nu^{\alpha'}$ are taken. In this case, there are more even spacings between the normal directions for surface points. Therefore, incrementing $\nu^{\alpha'}$ reduces the number of increments necessary in order to include all reflected field directions in the stepping procedure.

Let us define $\Delta = \text{minimum}(\Delta\theta, \Delta\phi)$, where $\Delta\theta$ and $\Delta\phi$ are the angular increments used in the RCS map. An estimate of the minimum number of increments in $\nu^{\alpha'}$, so that no angular slot is 'skipped' in the stepping process, is then $\frac{360^\circ}{\Delta}$. A similar estimate can be made for the number of increments in $\nu^{x'}$, for fixed α' . Figure 74 shows a progression of normal vectors for fixed α' on the side body. Let us define $\psi(\alpha')$ as the angle between the normals, $\hat{n}_1$ and $\hat{n}_2$, at the two ends of the shape. The magnitude of $\psi(\alpha')$ is found from $\psi(\alpha') = \cos^{-1}(\hat{n}_1 \cdot \hat{n}_2)$. Then, the minimum number of increments in $\nu^{x'}$ should be $\frac{\psi}{\Delta}$.

Reflected fields obtained using this procedure is discussed in the next section.

6.3.2 Computed Results

Figure 76 shows computed GO reflected fields from the faired shape at the intersection between the ellipsoid and the cone frustrum in the faired test body. Note the S-shape end contour of the GO-fields correspond to the beginning of the faired shape where it matches smoothly to the cone frustrum surface. The contour around the "hole" in the pattern, around $(\theta, \phi) = (70^\circ, 90^\circ)$, correspond to the end of the fairing where it matches smoothly to the surface of the ellipsoid.

The next section discusses the computation of diffracted fields from the junctions at the beginning and ending contours for this faired shape.

Figure 76: GO reflected fields from the faired shape at the intersection between two bodies.

6.4 Diffracted Field Computations for Junctions at Intersections

The junction at the intersection between the main and side bodies, in general, does not have a flat elliptic contour. The same is true for the beginning and ending contours for the faired shape at an intersection. These contours are defined as a sequence of points in a three dimensional coordinate system. The edge characteristics for such junctions, needed for diffracted field computations, are computed using a numerical method. This method, as well as a numerical search for diffraction points on the contour, are discussed next.

6.4.1 Edge Parameters on a Numerically Defined Edge

Figure 77(a) shows a projection of the three dimensional contour of a numerically defined edge. It is necessary to compute the edge vector, $\hat{e}$, edge radius of curvature, a_e , and the vector normal to the edge, $\hat{n}_e$, at every point on the contour. The procedure described here fits a circle through three consecutive points on the contour. The radius of the circle is then used as the edge radius of curvature, the tangent to the circle is used as the edge vector, and the normal to the circle is used as the normal to the edge. First, let us look at the definition of a plane in three dimensional space.

Theory: A plane in three dimensional space.

Let $\mathbf{r}(x, y, z) = L\hat{x} + M\hat{y} + N\hat{z}$ be a vector, where L , M and N are the 'direction numbers' of the vector. A plane, $\Psi(x, y, z)$, perpendicular to this vector, is described by the equation

$$\Psi(x, y, z) = Lx + My + Nz - T = 0 , \quad (6.1)$$

(a) Edge contour

(b) Fitting a circle through three points

Figure 77: Numerically defined edge contour.

where the parameter T is found by substituting the coordinates of a point on the plane in Equation (6.1).

Finding the circle through three points in space:

One needs the edge characteristics at P_0 on the contour in Figure 78(a). First, let us find points on the contour on both sides of P_0 . Denote the three consecutive points by $P_{-1}(x_{-1}, y_{-1}, z_{-1})$, $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$. These points are reproduced in Figure 78(b), where the plane of the paper contains all three points. Now, let us define the remaining vectors and points in this plane. The vectors $\mathbf{r}_a$ and $\mathbf{r}_b$ connect the outside points with P_0 , and are given by

$$\mathbf{r}_a = \hat{\mathbf{x}}(x_0 - x_{-1}) + \hat{\mathbf{y}}(y_0 - y_{-1}) + \hat{\mathbf{z}}(z_0 - z_{-1}) \quad (6.2)$$

$$\triangleq \hat{\mathbf{x}}L_a + \hat{\mathbf{y}}M_a + \hat{\mathbf{z}}N_a, \text{ and} \quad (6.3)$$

$$\mathbf{r}_b = \hat{\mathbf{x}}(x_1 - x_0) + \hat{\mathbf{y}}(y_1 - y_0) + \hat{\mathbf{z}}(z_1 - z_0) \quad (6.4)$$

$$\triangleq \hat{\mathbf{x}}L_b + \hat{\mathbf{y}}M_b + \hat{\mathbf{z}}N_b. \quad (6.5)$$

The vector, $\mathbf{N}$, is the normal to the paper, given by

$$\mathbf{N} = \mathbf{r}_a \times \mathbf{r}_b \quad (6.6)$$

$$\triangleq \hat{\mathbf{x}}L_n + \hat{\mathbf{y}}M_n + \hat{\mathbf{z}}N_n. \quad (6.7)$$

The points P_a and P_b are at the centers between the outside points and P_0 , and are given by

$$P_a = \left(\frac{x_0 + x_{-1}}{2}, \frac{y_0 + y_{-1}}{2}, \frac{z_0 + z_{-1}}{2} \right), \text{ and} \quad (6.8)$$

$$P_b = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}, \frac{z_0 + z_1}{2} \right). \quad (6.9)$$

Now, the circle must lie in the plane that contains the three points, P_{-1} , P_0 and P_1 . The first criterion for the circle is that it must lie in the plane perpendicular

to the vector $\mathbf{N}$ which also contains the point, P_0 . This plane is given by

$$\Psi_n(x, y, z) \triangleq L_n x + M_n y + N_n z - T_n = 0 \quad (6.10)$$

where T_n is found by substituting the coordinates of P_0 into (6.10).

Figure 78(b) also shows two lines, C_a and C_b . These lines, $C_{a,b}$, are in the plane Ψ_n , are perpendicular to vectors $\mathbf{r}_{a,b}$ and go through points, $P_{a,b}$. Therefore, the center of the circle, (x_c, y_c, z_c) , must be at the point where these two lines cross. The two lines described above are actually the crossings between the plane Ψ_n and the planes $\Psi_{a,b}$ which go through the points $P_{a,b}$ and are perpendicular to the vectors $\mathbf{r}_{a,b}$. Therefore, the center of the circle can be found by solving for the point where these three planes cross (eg., using matrix inversion). Note that the planes $\Psi_{a,b}$ are given by

$$\Psi_a(x, y, z) \triangleq L_a x + M_a y + N_a z - T_a = 0 \quad (6.11)$$

$$\Psi_b(x, y, z) \triangleq L_b x + M_b y + N_b z - T_b = 0 \quad (6.12)$$

where $T_{a,b}$ are found by substituting the coordinates of $P_{a,b}$ into Equations (6.11) and (6.12), respectively.

Now, let $\mathbf{A}$ be the vector between the center of the circle and P_0 , such that

$$\mathbf{A} = \hat{\mathbf{x}}(x_0 - x_c) + \hat{\mathbf{y}}(y_0 - y_c) + \hat{\mathbf{z}}(z_0 - z_c) . \quad (6.13)$$

Then, the edge characteristics are given by

$$a_e = |\mathbf{A}| \quad (6.14)$$

$$\hat{\mathbf{n}}_e = \frac{\mathbf{A}}{a_e} , \text{ and} \quad (6.15)$$

$$\hat{\mathbf{e}} = \hat{\mathbf{n}}_e \times \left(\frac{\mathbf{N}}{|\mathbf{N}|} \right) . \quad (6.16)$$

6.4.2 Search for Diffraction Points

Diffraction points on the contour are found by looking for points where the edge vector is perpendicular to the incident vector. In other words, if $G = \hat{s}' \cdot \hat{e}$, then G is equal to zero at diffraction points. The diffraction points are then found by computing G as a function of α' at points along the contour. Since G may not be exactly equal to zero at any point, diffraction points are found by testing for a sign change in G . If G changed sign between two consecutive points on the contour, then the diffraction point is taken as the known point closest to the zero crossing.

6.4.3 Computed Results

Figure 78 shows diffracted field results for the sharp junction between the ellipsoid and the cone frustum in the unfaired test body. First, note that the S-shaped contour of peak values correspond to the line caustic limiting values for the cone frustum. Next, note that the peak values around $(\theta, \phi) = (70^\circ, 90^\circ)$ correspond to the reflection shadow boundaries for GO reflected fields from the ellipsoid, right at the contour of intersection. The large region of low diffraction is the region where this contour is not visible to the incident wave (i.e., the ϕ'_0 angles at the junction were tested and found to be inside the body).

The regions of low diffracted fields around the point where the cursor lines cross correspond to a special case where the co-polarized backscattered diffracted field goes to zero. This case occurs when the electric field vector makes an angle of 45° with the edge vector at the diffraction point, and when the ϕ' angle is equal to half the exterior wedge angle. For such angles and incident polarization, one can show that double reflected GO field from the two faces of the wedge is cross polarized with respect to the incident field. Therefore, the diffracted

Figure 78: Diffracted fields from the interior wedge junction where two bodies intersect.

field components that compensate for the doubly reflected field will also be cross polarized. When the incidence direction bisects the two faces of the wedge, such that the backscatter direction is also equal distances away from the reflection shadow boundaries, a cancellation of terms occurs, yielding the low diffraction returns.

Figure 79 shows diffracted field results for the smooth junction where the faired shape matches to the main body. The ring of peak values correspond to the reflection shadow boundaries along the contour of intersection on the main body. The Front view of the structure shows an incident direction along such a shadow boundary. Note that the Side view is helpful to see that the incident direction is normal to the surface of the ellipsoid at the junction with the faired shape.

Figure 80 shows diffracted field results for the smooth junction between the faired shape and the side body. As before, the S-shaped curve corresponds to the line caustic directions for the cone frustrum. The Front view of the body illustrates this at an arbitrary chosen angle on the map. The Side and Top views are helpful for seeing that incidence is perpendicular to the side of the cone frustrum.

6.5 Summary and Discussion

This Chapter described field computations for intersecting bodies. It was shown that computations for junctions and shapes on the side bodies, which are similar to those used for isolated bodies, can be applied in the same way, with the exception that a coordinate rotation is necessary. Special techniques were described for computing the reflected and diffracted fields for the faired shape where a side body intersects a main body.

Computed results for the two structures were given to illustrate the techniques. Note that the methods are not limited to structures with only one intersecting

Figure 79: Diffracted fields from the smooth junction where the faired shape enters the main body.

Figure 80: Diffracted fields from the smooth junction between the faired shape and the side body.

body. However, for bodies with many intersecting side bodies, the shadowing of scattering centers becomes a more serious problem. Since such shadowing is not included in this study, no detailed discussion will be given for such cases. Still, some insight can be gained from an RCS map for a modelled airplane. As an example, Figure 81 shows the total RCS map for an airplane. The large values where the cursor lines cross (in the Front view direction) corresponds to the line caustic values for the cone frustrum in the tail section. The peak values at the top and bottom of the map occur for θ equal to 0° and 180° . In those directions, the line caustics for the flat sides of the wings cause the high returns. The low section toward the top of the map corresponds to incidence in the direction which an observer on the ground would see when the airplane flies in the direction of the observer; i.e., a view of the bottom of the airplane as seen from about 30° below the direction of flight. In this direction, there are no major scattering centers illuminated by the incident field. On the other hand, if the observer were to move straight up, to a point above the direction of flight, then both the tail section and the cockpit will be visible, resulting in the higher RCS seen in the lower central part of the map.

Figure 81: Total RCS map for an airplane.

CHAPTER VII

SUMMARY AND CONCLUSIONS

This Chapter gives a summary of the work done in this dissertation, some general conclusions and suggestions for future work.

1. Analytic description of faired composite bodies:

A class of radar scattering targets can be modelled by connecting canonical shapes with elliptic cross sections together. These canonical shapes include disks, cone frustrums, ellipsoids and ogives. The types of bodies considered in this study include isolated bodies, composed of a connection of canonical shapes along a central axis, and structures with a main body and side bodies intersecting it. This class of structures, however, can introduce wedge junctions between shapes; whereas, streamlined aerodynamic shapes generally have relatively smooth junctions. This dissertation presented a method whereby such wedge junctions can be faired analytically. Fairing is similar to "sanding off" knife edge junctions or "filling in" sharp interior junctions to yield smooth surfaces with slope continuity. It was shown that this method produces realistically looking bodies, such as the faired airplane shown in Figure 69. The input data that defines the fairing is straight forward and easy to implement, so that a wide range of faired shapes can be defined with only small changes to the input data. Furthermore, since the faired shapes

are defined analytically, the surface derivatives which are necessary for RCS computations can be accurately computed.

2. Extension of the envelope technique for computing high frequency radar cross section (RCS) envelopes:

A high frequency analysis can be applied to compute the RCS of electrically large scattering bodies. In such an analysis, the total RCS can be found from vector addition of contributions from distinct scattering centers, such as diffraction and reflection points. The RCS patterns obtained this way show then, in general, many oscillations due to the scattered field contributions adding in or out of phase. The envelope technique, as used in this study, is an extension of the work by Pistorius [13]. In this technique, field contributions are added in phase. The resulting total envelope pattern yields either a maximum or an average RCS for a given body, depending on whether fields or powers are added. In this study, first order RCS envelopes for the faired bodies described above are computed using Geometrical Optics for reflected fields, the Uniform Geometrical Theory of Diffraction for diffracted fields, and Physical Optics and Equivalent Line Current techniques to treat fields in caustic regions. Good agreement was obtained between computed envelope patterns and the peaks of the oscillations in measured results for isolated scattering bodies. The few discrepancies in the results were due to higher order mechanisms which were not included in the solution.

3. Full volumetric RCS patterns:

Computation of the envelope RCS pattern for a scattering body can be done using much larger angular increments than would be required for conventional RCS patterns. This inherent efficiency was exploited in order to com-

pute full volumetric RCS patterns for various isolated and intersecting scattering bodies. These envelope RCS patterns were displayed on gray scale maps, which are invaluable for interpreting the results.

4. Suggestions for future work:

- (a) The technique used for fairing the junctions between canonical shapes of elliptic cross section can be extended to apply to more general shapes, such as super-ellipsoids.
- (b) Higher order terms can be added to these solutions in order to more accurately compute the RCS envelope. Such terms include double diffraction and creeping wave mechanisms.
- (c) More general shadowing algorithms can be added to these solutions, in order to more accurately predict the RCS patterns of intersecting structures.

APPENDIX A

ELLIPSE EQUATIONS AND FUNCTIONS

In this appendix equations pertaining to a planar ellipse such as in Figure 82 are given. The ellipse is represented in the XQ -coordinate system here, although it is used in various coordinate systems in the main text. The $X'Q'$ -coordinate system which alligns with the ellipse axes is also defined for convenience. The

Figure 82: Ellipse geometry

relationship between coordinates in the two systems are given by

$$x = x' \cos \xi - q' \sin \xi \quad (\text{A.1})$$

$$q = x' \sin \xi + q' \cos \xi \quad (\text{A.2})$$

$$x' = x \cos \xi + q \sin \xi, \text{ and} \quad (\text{A.3})$$

$$q' = -x \sin \xi + q \cos \xi. \quad (\text{A.4})$$

In the XQ-coordinate system, the ellipse is uniquely defined by the parameters $[a, b, x_0, q_0, \xi]$. In the $X'Q'$ -coordinate system the ellipse is uniquely defined by the parameters $[a, b, x'_0, q'_0]$. These parameters are defined as follows:

$$r_0 = \text{vector indicating the origin of the ellipse}$$

$$= (x_0, q_0) \text{ in the XQ-coordinate system}$$

$$= (x'_0, q'_0) \text{ in the } X'Q'\text{-coordinate system}$$

$$a = x'\text{-axis half length}$$

$$b = q'\text{-axis half length, and}$$

$$\xi = \text{rotation angle of the ellipse } X'\text{-axis}$$

with respect to the X-axis.

In order to define points on the ellipse, the ellipse is expressed both in terms of rectangular (x, q) and polar (r', ν) or (r', δ) coordinates, and the relationships between these are given where necessary for use in the main text. The polar

parameters are defined by

$$\begin{aligned} \mathbf{r} &= \text{vector to a point on the ellipse} \\ &= \mathbf{r}_0 + \mathbf{r}' \end{aligned} \quad (\text{A.5})$$

$\mathbf{r}'$ = vector from the center of the
ellipse to a point on the contour

ν = ellipse angular parameter... $0 < \nu < 2\pi$, and

δ = angle between $\mathbf{r}'$ and the $+X'$ -axis... $0 < \delta < 2\pi$.

A.1 Ellipse in the $X'Q'$ -Coordinate System

- In polar coordinates:

$$\mathbf{r}'(x', q') = (r' \cos \delta, r' \sin \delta) \quad (\text{A.6})$$

$$= (a \cos \nu, b \sin \nu) \quad (\text{A.7})$$

where

$$r' = \sqrt{(a \cos \nu)^2 + (b \sin \nu)^2}, \text{ and} \quad (\text{A.8})$$

$$\nu = \tan^{-1} \left[\frac{a}{b} \tan \delta \right]. \quad (\text{A.9})$$

- In rectangular coordinates:

$$\frac{(x' - x'_0)^2}{a^2} + \frac{(q' - q'_0)^2}{b^2} = 1 \quad (\text{A.10})$$

or else as a biquadratic equation:

$$A'(x')^2 + B'x' + C'(q')^2 + D'q' + 1 = 0 \quad (\text{A.11})$$

where, if

$$E'_0 = \frac{(x'_0)^2}{a^2} + \frac{(q'_0)^2}{b^2} - 1 \quad (\text{A.12})$$

then

$$A' = \frac{1}{a^2} \frac{1}{E'_0} \quad (\text{A.13})$$

$$B' = \frac{-2x'_0}{a^2} \frac{1}{E'_0} \quad (\text{A.14})$$

$$C' = \frac{1}{b^2} \frac{1}{E'_0} \quad (\text{A.15})$$

$$D' = \frac{-2q'_0}{b^2} \frac{1}{E'_0} \quad (\text{A.16})$$

Alternatively, for given $[A', B', C', D']$, the ellipse parameters $[x'_0, q'_0, a, b]$ can be found as follows :

$$E'_0 = \frac{1}{\frac{B'^2}{4A'} + \frac{D'^2}{4C'} - 1} \quad (\text{A.17})$$

so that

$$x'_0 = \frac{-B'}{2A'} \quad (\text{A.18})$$

$$q'_0 = \frac{-D'}{2C'} \quad (\text{A.19})$$

$$a = \sqrt{\frac{1}{A'E'_0}} , \text{ and} \quad (\text{A.20})$$

$$b = \sqrt{\frac{1}{C'E'_0}} . \quad (\text{A.21})$$

A.2 Equations in the XQ-Coordinate System

The coordinates (x, q) of the ellipse as a function of ν are given by

$$x = x_0 + a \cos \xi \cos \nu - b \sin \xi \sin \nu , \text{ and} \quad (\text{A.22})$$

$$q = q_0 + a \sin \xi \cos \nu + b \cos \xi \sin \nu . \quad (\text{A.23})$$

For $\xi \neq 0$, a fifth parameter that couples x and q is needed in the following biquadratic equation:

$$Ax^2 + Bx + Cq^2 + Dq + Exq + 1 = 0 . \quad (\text{A.24})$$

Figure 83: Points on the ellipse for a given x coordinate

This equation is found by substituting (A.3) and (A.4) into (A.11), so that, for given $[A', B', C', D']$,

$$A = A' \cos^2 \xi + C' \sin^2 \xi \quad (\text{A.25})$$

$$B = B' \cos \xi - D' \sin \xi \quad (\text{A.26})$$

$$C = A' \sin^2 \xi + C' \cos^2 \xi \quad (\text{A.27})$$

$$D = B' \sin \xi + D' \cos \xi, \text{ and} \quad (\text{A.28})$$

$$E = 2 \sin \xi \cos \xi (A' - C'). \quad (\text{A.29})$$

A.3 Finding Points on the Ellipse

For given ν , a unique point can be found on the ellipse, using (A.7) and (A.5). On the other hand, if one of the rectangular parameters x or q is given, then there is a choice of two possible points on opposite sides of the ellipse, as shown in Figure 83 for fixed x . In such cases additional information such as a previous point close to the required one (indicated by subscript *prev*) is needed to find a unique

point on the ellipse. The following analysis is done for a given x -coordinate, but could be done in a similar way if a q -coordinate were given.

- Given x and ν_{prev} , find ν

Let

$$\mathcal{A} = a \cos \xi \quad (\text{A.30})$$

$$\mathcal{B} = -b \sin \xi, \text{ and} \quad (\text{A.31})$$

$$\mathcal{D} = x - x_0. \quad (\text{A.32})$$

then (A.22) can be rewritten as

$$\mathcal{A} \cos \nu = \mathcal{D} - \mathcal{B} \sin \nu. \quad (\text{A.33})$$

Squaring (A.33), and some manipulation, yields a quadratic equation in terms of $\sin \nu$:

$$(\mathcal{B}^2 + \mathcal{A}^2) \sin^2 \nu - (2\mathcal{D}\mathcal{B}) \sin \nu + (\mathcal{D}^2 - \mathcal{A}^2) = 0 \quad (\text{A.34})$$

Solving (A.34) yields two solutions, $(\sin \nu^s)_{1,2}$ —select the one which is closest to $(\sin \nu_{prev})$. Taking the inverse sinusoid yields ν^s . Now, since the inverse sinusoid is defined for $-90^\circ < \nu^s < 90^\circ$, another solution could be $\nu_2^s = (180 - \nu^s)$ —select ν^s or ν_2^s for which the cosine is closest to $(\cos \nu_{prev})$.

- Given x and q_{prev} , find q :

Writing (A.24) in terms of powers of q yields two solutions for q , given by

$$q_{1,2} = \frac{-(D + Ex) \pm \sqrt{(D + Ex)^2 - 4(Ax^2 + Bx + 1)}}{2C} \quad (\text{A.35})$$

Again, the value closest to q_{prev} is chosen.

A.4 Derivatives on the Contour of the Ellipse

The subscripts x , q and δ will denote partial derivatives with respect to x , q and δ , respectively.

- **Derivatives of ν with respect to δ**

Taking the first and second derivatives with respect to δ in (A.9) yields:

$$\nu_\delta = \frac{ab}{b^2 \cos^2 \delta + a^2 \sin^2 \delta}, \text{ and} \quad (\text{A.36})$$

$$\nu_{\delta\delta} = \frac{2ab \cos \delta \sin \delta [b^2 - a^2]}{[b^2 \cos^2 \delta + a^2 \sin^2 \delta]^2}. \quad (\text{A.37})$$

- **Derivatives of q with respect to x :**

- In rectangular coordinates, when the coordinates x and q are given:

Taking first and second derivatives with respect to x in (A.24) yields

$$q_x = -\frac{[2Ax + B + Eq]}{[2Cq + D + Ex]}, \text{ and} \quad (\text{A.38})$$

$$q_{xx} = -\frac{2[A + Cq_x^2 + Eq_x]}{[2Cq + D + Ex]} \quad (\text{A.39})$$

- In polar coordinates, when the parameter ν is given:

First, x and q are found using (A.22) and (A.23). Secondly, taking first and second derivatives with respect to x in (A.22) and simplifying yields

$$\nu_x = \frac{-1}{[a \cos \xi \sin \nu + b \sin \xi \cos \nu]}, \text{ and} \quad (\text{A.40})$$

$$\nu_{xx} = (x - x_0)\nu_x^3. \quad (\text{A.41})$$

Finally, taking first and second derivatives with respect to x in (A.23) yields

$$q_x = [-a \sin \xi \sin \nu + b \cos \xi \cos \nu]\nu_x, \text{ and} \quad (\text{A.42})$$

$$q_{xx} = -[q - q_0]\nu_x^2 + [-a \sin \xi \sin \nu + b \cos \xi \cos \nu]\nu_{xx}. \quad (\text{A.43})$$

• Derivatives of x with respect to q

-- In polar coordinates, when the coordinate ν is given:

First, x and q are found using (A.22) and (A.23). Secondly, taking first and second derivatives with respect to q in (A.23) and simplifying yields

$$\nu_q = \frac{1}{[-a \sin \xi \sin \nu + b \cos \xi \cos \nu]}, \text{ and} \quad (\text{A.44})$$

$$\nu_{qq} = (q - q_0)\nu_q^3. \quad (\text{A.45})$$

Finally, taking first and second derivatives with respect to q in (A.22) yields

$$x_q = -[a \cos \xi \sin \nu + b \sin \xi \cos \nu]\nu_q, \text{ and} \quad (\text{A.46})$$

$$x_{qq} = -[x - x_0]\nu_q^2 - [a \cos \xi \sin \nu + b \sin \xi \cos \nu]\nu_{qq}. \quad (\text{A.47})$$

A.5 Tangent, Normal and Radius of Curvature

For simplicity, the ellipse is taken to be in the $X'Q'$ -coordinate system where $\mathbf{r}'(\nu)$ denotes a point on the ellipse contour as a function of ν . A theoretical explanation of the method used to find the tangent vector, normal vector and radius of curvature is found in references [22, pp 61-66], and [23, pp 1-15].

Let $\mathbf{r}'_\nu$ denote the first derivative of the position vector $\mathbf{r}'$ with respect to ν , and Δ the magnitude of this vector, such that

$$\mathbf{r}'_\nu = (-a \sin \nu, b \cos \nu), \text{ and} \quad (\text{A.48})$$

$$\Delta \triangleq |\mathbf{r}'_\nu| \quad (\text{A.49})$$

$$= \sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu}. \quad (\text{A.50})$$

The tangent vector for given ν and the associated normal vector perpendicular to it is given by

$$\hat{\mathbf{t}}'(\nu) = \frac{\mathbf{r}'_\nu}{|\mathbf{r}'_\nu|} \quad (\text{A.51})$$

$$= \frac{(-a \sin \nu, b \cos \nu)}{\Delta}, \text{ and} \quad (\text{A.52})$$

$$\hat{\mathbf{n}}'(\nu) = \frac{(b \cos \nu, a \sin \nu)}{\Delta}. \quad (\text{A.53})$$

The ellipse radius of curvature, $a_e(\nu)$, is found by evaluating the magnitude of the curvature vector $\mathbf{k}'$ as follows:

$$\mathbf{k}'(\nu) = \frac{d\hat{\mathbf{t}}'}{d\nu} \frac{1}{|\mathbf{r}'_\nu|} \quad (\text{A.54})$$

$$= -ab \frac{(b \cos \nu, a \sin \nu)}{\Delta^3} \frac{1}{\Delta}, \text{ and} \quad (\text{A.55})$$

$$a_e(\nu) = \frac{1}{|\mathbf{k}'|} \quad (\text{A.56})$$

$$= \frac{\Delta^3}{ab}. \quad (\text{A.57})$$

APPENDIX B

ELLIPTIC INTEGRAL COMPUTATION

In this Appendix the computation of elliptic integrals is described. The two integrals to be computed are as follows:

$$\mathcal{I}_1(a, b) = \int_0^{2\pi} \frac{\sin^2 \nu}{\sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu}} d\nu, \text{ and} \quad (\text{B.1})$$

$$\mathcal{I}_2(a, b) = \int_0^{2\pi} \frac{\cos^2 \nu}{\sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu}} d\nu. \quad (\text{B.2})$$

Due to the $\pi/2$ periodicity of the area under the $\sin^2 \nu$ and $\cos^2 \nu$ curves, the integrals need only be computed over a range from 0 to $\pi/2$ and the results multiplied by four to obtain the integral over the full range. Let

$$\sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu} = b \sqrt{1 - K \sin^2 \nu} \quad (\text{B.3})$$

$$K = \frac{b^2 - a^2}{b^2} \quad (\text{B.4})$$

$$= k^2 \dots \text{for } b \geq a. \quad (\text{B.5})$$

A complete elliptic integral of the type

$$I(k, A, B) \triangleq \int_0^{\pi/2} \frac{A \cos^2 \nu + B \sin^2 \nu}{\sqrt{1 - k^2 \sin^2 \nu}} d\nu \dots\dots -1 < k < 1 \quad (\text{B.6})$$

can be computed using Landen's transformation, as is done in subroutine CEL2 in the SSP package of mathematical functions [28].

- **Case where $b \geq a$**

In this case, k lies within the allowable range for using (B.6), so that

$$\mathcal{I}_1(a, b) = \frac{4}{b} I(k, 0, 1) , \text{ and} \quad (\text{B.7})$$

$$\mathcal{I}_2(a, b) = \frac{4}{b} I(k, 1, 0) . \quad (\text{B.8})$$

- **Case where $b < a$**

In this case, $|k| > 1$ and the integrals will have to be rewritten in order to use the subroutine CEL2 for the computations. By using the transformation $\nu' = (\pi/2 - \nu)$ in (B.1) and (B.2), it can be shown that

$$\mathcal{I}_1(a, b) = \mathcal{I}_2(b, a) , \text{ and} \quad (\text{B.9})$$

$$\mathcal{I}_2(a, b) = \mathcal{I}_1(b, a) . \quad (\text{B.10})$$

where the right hand sides of Equations (B.9) and (B.10) can be computed using the definitions in (B.7) and (B.8).

APPENDIX C

ELLIPTIC TRANSITION CURVES

The problem is to find the equation for an elliptic curve that will connect two points on two lines, with slope continuity at the junctions. Such an ellipse in the XQ -coordinate system is shown in Figure 84. In order to uniquely define an ellipse of general orientation, one needs five boundary conditions to solve for the five ellipse parameters: $[a, b, x_0, q_0, \xi]$. Therefore, if only four conditions are given (two points and two slopes), one of the five parameters needs to be fixed. It is assumed here that ξ is fixed. This problem can then be solved in the $X'Q'$ -coordinate system where the ellipse axes are parallel to the X' and Q' axes.

PROBLEM:

Figure 84: Geometry for a matching ellipse.

Find the parameters $[a, b, x_0, q_0]$ for the ellipse in the XQ-coordinate system that satisfies the following conditions:

1. Match point $P1(x_1, q_1)$ and slope S_1 at $P1$;
2. Match point $P2(x_2, q_2)$ and slope S_2 at $P2$;
3. The rotation angle for the ellipse axis is ξ .

SOLUTION:

• **Converting to the $X'Q'$ -coordinate system:**

Points $(x_{1,2}, q_{1,2})$ in XQ can be converted to points $(x'_{1,2}, q'_{1,2})$ in $X'Q'$ by using equations (A.3) and (A.4).

A slope in XQ can be expressed as the tangent of an angle, such that

$$S = dq/dx = \tan(\gamma) . \quad (C.1)$$

The same slope in $X'Q'$ is then

$$S' = \frac{dq'}{dx'} = \tan(\gamma - \xi) \quad (C.2)$$

$$= \frac{S - \tan \xi}{1 + S \tan \xi} \quad (C.3)$$

where use has been made of trigonometric identity 5.36 in [29, p 15]. Now $S_{1,2}$ can be converted to $S'_{1,2}$ using (C.3).

• **Solving for the ellipse parameters in the $X'Q'$ -coordinate system:**

The ellipse as a biquadratic equation is given by (see Appendix A)

$$A'(x')^2 + B'x' + C'(q')^2 + D'q' + 1 = 0 \quad (C.4)$$

Taking the first derivative with respect to x in (C.4) yields

$$2A'x' + B' + 2C'q'S' + D'S' = 0 \quad (C.5)$$

Inserting the two sets of points and derivatives in (C.4) and (C.5) gives four independent equations with the four unknowns $[A', B', C', D']$. In matrix notation one finds that

$$\begin{bmatrix} (x'_1)^2 & x'_1 & (q'_1)^2 & q'_1 \\ (x'_2)^2 & x'_2 & (q'_2)^2 & q'_2 \\ 2x'_1 & 1 & 2q'_1 S'_1 & S'_1 \\ 2x'_2 & 1 & 2q'_2 S'_2 & S'_2 \end{bmatrix} \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \quad (C.6)$$

$[A', B', C', D']$ are found through matrix inversion or Gaussian elimination (eg. using subroutine SIMQ in the SSP package [28]).

- **Finding the ellipse parameters in the XQ-coordinate system:**

- $[a, b, x'_0, q'_0]$ can be computed from $[A', B', C', D']$ using Equations (A.17) and (A.18)–(A.21).
- $[x_0, q_0]$ can be computed from $[x'_0, q'_0, \xi]$ using (A.1) and (A.2).
- $[A, B, C, D, E]$ can be computed from $[A', B', C', D', \xi]$ using (A.25)–(A.29).
- The initial and final values of the ellipse parameter ν at P_1 and P_2 can be found by first computing the real angles $\delta_{1,2}$ shown in Figure 84:

$$\delta_{1,2} = \tan^{-1} \left[\frac{q'_{1,2} - q'_0}{x'_{1,2} - x'_0} \right]. \quad (C.7)$$

Now, $\nu_{1,2}$ are found by using Equation (A.9).

- **Existence of solutions:**

For a given set of points and slopes, there is a limited range of angles ξ for which an ellipse can be found. The problem is illustrated in Figure 85. P_0 is the point where the extensions of the slope lines through points P_1 and P_2

Figure 85: Geometry for a matching ellipse with one axis parallel to a slope line.

cross. The angles that the two slope lines make with the X-axis are denoted by $\xi_{1,2}$ and the distances from P_0 to $P_{1,2}$ by $d_{1,2}$.

The restrictions on ξ depend on the relative lengths of d_1 and d_2 . In Figure 85, d_2 was chosen to be much shorter than d_1 and a matching ellipse with $\xi = \xi_1$ is shown. Now let us choose $\xi = \xi_2$; i.e. one of the ellipse axes is parallel to the slope line through P_2 and the other goes through P_2 . It can be intuitively seen that it would be impossible to find an ellipse that reaches P_1 and matches the slope line tangentially. Note that it may still be possible to solve Equation (C.6), but the resulting biquadratic equation may describe a parabola or hyperbola. If ξ is increased from ξ_1 to ξ_2 , then somewhere between the two angles there will be a dividing line where it becomes impossible to find a matching ellipse. There does not seem to be an easy way to find this dividing line, but in general one can say that:

If d_2 is shorter than d_1 then one should choose ξ closer to ξ_1 than to ξ_2 , and vice versa.

Figure 86: Degenerate case for finding a matching ellipse.

• **Degenerate case:**

There is a special case where the matrix in Equation (C.6) becomes singular so that $[A', B', C', D']$ cannot be found using the method outlined above. This case is illustrated in Figure 86 for a problem where $d_1 = d_2$. Let $\xi = \xi_0$ where $\xi_0 = (\xi_1 + \xi_2)/2$, and let L_0 denote the line through P_0 that makes an angle ξ_0 with the X-axis. This choice results in $S'_1 = -S'_2$ and $x'_1 = x'_2$ in the prime coordinate system. The effect of this symmetry is the loss of one degree of freedom, which makes the problem under-determined. Physically it means that there are an infinite number of ellipses, centered along the line L_0 , which will match the points and slopes. The simplest case would be a circle ($a = b$), and then ellipses that are increasingly longer ($a > b$) or flatter ($a < b$). In this case, a unique ellipse can be found if an additional parameter is fixed. One such a parameter that is easy to use is the eccentricity, ϵ , of the ellipse, defined as follows:

$$\epsilon \triangleq \frac{\sqrt{a^2 - b^2}}{a} \dots \dots \text{for } a > b \quad (C.8)$$

so that

$$b = a\sqrt{1 - \epsilon^2} \quad (C.9)$$

The ellipse can then be expressed as

$$\frac{(x' - x'_0)^2}{1} + \frac{(q' - q'_0)^2}{1 - \epsilon^2} = a^2. \quad (C.10)$$

The center of the ellipse must be on L_0 , such that

$$q'_0 = \frac{q'_1 + q'_2}{2}. \quad (C.11)$$

Taking the first derivative with respect to x' in Equation (C.10) and evaluating this equation at P_1 yields a solution for x'_0 :

$$x'_0 = x'_1 + \frac{(q'_1 - q'_0)S'_1}{1 - \epsilon^2} \quad (C.12)$$

Now 'a' is found from Equation (C.10) and 'b' from Equation (C.9), and the rest of the parameters can be found in the same way as above for the regular solution. Note that $\epsilon = 0$ represents a circle, $0 < \epsilon < 1$ represents a whole range of ellipses, $\epsilon = 1$ represents a hyperbola and $\epsilon > 1$ represents a parabola.

DISCUSSION:

In a problem where the relative lengths of d_1 and d_2 changes continuously, it is necessary to ensure that there will be a smooth transition between the ellipses that define the cases where $d_1 < d_2$ and those where $d_1 > d_2$. Such continuity can be achieved by the following scheme:

1. For $d_1 \geq d_2$, choose $\xi = \xi_1$.
2. For $d_2 > d_1$, choose $\xi = \xi_2$.

The reason why this scheme works will now be explained.

Let us choose $d_1 > d_2$ and therefore $\xi = \xi_1$. Define $L_{q1,q2}$ as the lines through the center of the ellipse and points $P_{1,2}$ (see Figure 85). Due to the above choice of ξ , the line L_{q1} will be perpendicular to the slope line L_1 ; while, L_{q2} will be at some angle with respect to L_2 . Now, let us keep d_2 constant and decrease d_1 . In the limiting case where $d_1 = d_2$, the symmetry of the problem suggests that the center of the ellipse must be on the line L_0 as shown in Figure 86 and therefore line L_{q2} will also be perpendicular to line L_2 . This can only be true if this ellipse is actually a circle. Similarly, if the starting point were $d_2 > d_1$ and d_2 were decreased until $d_2 = d_1$, the limiting ellipse would also be this circle. Therefore, if there were a continuous change from $d_1 > d_2$, through $d_1 = d_2$, to $d_1 < d_2$, then

the matching ellipses will change smoothly from ellipses with one axis parallel to L_1 , to a circle, to ellipses with one axis parallel to L_2 .

APPENDIX D

EQUATIONS FROM DIFFERENTIAL GEOMETRY

The purpose of this Appendix is to give a summary of the results from differential geometry which are necessary for defining surface properties. The theory is based on that given by Struik [23] and Lipschutz [22]. The required surface properties are the tangent vectors, normal vector, principal radii of curvature and principal directions.

D.1 Defining the surface

Consider an analytically defined surface, S , in a three-dimensional Euclidean space with basis $\{\hat{x}, \hat{y}, \hat{z}\}$, as shown in Figure 87. Let $r(x,p)$ be a regular parametric representation of S , where x and p are independent parameters, so

Figure 87: Geometry for a surface S with a parametric representation $r(x,p)$.

that

$$\mathbf{r}(\mathbf{x}, p) = [\mathbf{x} , Y(\mathbf{x}, p) , Z(\mathbf{x}, p)] . \quad (\text{D.1})$$

Assume that the first and second partial derivatives of $\mathbf{r}(\mathbf{x}, p)$ with respect to $\mathbf{x}$ and p exist and are continuous at every $(\mathbf{x}, p)$. These derivatives are given by

$$\mathbf{r}_{\mathbf{x}} = [1 , Y_{\mathbf{x}} , Z_{\mathbf{x}}] \quad (\text{D.2})$$

$$\mathbf{r}_p = [0 , Y_p , Z_p] \quad (\text{D.3})$$

$$\mathbf{r}_{\mathbf{x}\mathbf{x}} = [0 , Y_{\mathbf{x}\mathbf{x}} , Z_{\mathbf{x}\mathbf{x}}] \quad (\text{D.4})$$

$$\mathbf{r}_{\mathbf{x}p} = [0 , Y_{\mathbf{x}p} , Z_{\mathbf{x}p}] , \text{ and} \quad (\text{D.5})$$

$$\mathbf{r}_{pp} = [0 , Y_{pp} , Z_{pp}] . \quad (\text{D.6})$$

D.2 Tangents and normals

The two parametric tangents, $\hat{\mathbf{r}}_{\mathbf{x}}$ and $\hat{\mathbf{r}}_p$, and the outward surface normal, $\hat{\mathbf{n}}$, are given by

$$\hat{\mathbf{r}}_{\mathbf{x}} = \frac{\mathbf{r}_{\mathbf{x}}}{|\mathbf{r}_{\mathbf{x}}|} \quad (\text{D.7})$$

$$= \frac{[1 , Y_{\mathbf{x}} , Z_{\mathbf{x}}]}{\sqrt{1 + Y_{\mathbf{x}}^2 + Z_{\mathbf{x}}^2}} \quad (\text{D.8})$$

$$\hat{\mathbf{r}}_p = \frac{\mathbf{r}_p}{|\mathbf{r}_p|} \quad (\text{D.9})$$

$$= \frac{[0 , Y_p , Z_p]}{\sqrt{Y_p^2 + Z_p^2}} , \text{ and} \quad (\text{D.10})$$

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_p \times \mathbf{r}_{\mathbf{x}}}{|\mathbf{r}_p \times \mathbf{r}_{\mathbf{x}}|} \quad (\text{D.11})$$

$$= \frac{[Y_{\mathbf{x}}Z_p - Z_{\mathbf{x}}Y_p , -Z_p , Y_p]}{\sqrt{(Y_{\mathbf{x}}Z_p - Z_{\mathbf{x}}Y_p)^2 + Z_p^2 + Y_p^2}} . \quad (\text{D.12})$$

D.3 Curvature

Let s be the arc length parameter along an arbitrary curve, $\mathbf{r}(s)$, on the surface S . The tangent to the curve is given by $\hat{\mathbf{t}}(s) = d\mathbf{r}(s)/ds$. The curvature vector, $\mathbf{k} = d\hat{\mathbf{t}}(s)/ds$, is a vector pointing in the direction of change in $\hat{\mathbf{t}}(s)$ with respect to s . The magnitude of the curvature vector, $\kappa = |\mathbf{k}|$, is known as the **curvature** of the curve at s and its reciprocal, $R = 1/\kappa$, as the **radius of curvature** of the curve at s .

D.4 Principal directions and radii of curvature

It can be shown [22,23] that, from the family of curves that run through a point on a surface, there exist two curves with orthogonal tangent directions, $\hat{\mathbf{U}}_{1,2}$, for which the curvatures, $\kappa_{1,2}$, are at a maximum and a minimum respectively (unless the surface is flat or spherical, in which cases the curvature is constant for all possible curves). These two tangent directions are called **principal directions** and the corresponding curvatures the **principal curvatures**. The parametric tangents, $\hat{\mathbf{r}}_x$ and $\hat{\mathbf{r}}_p$, describe a tangent plane. A vector, $\mathbf{U}$, in the tangent plane can be expressed in terms of its coordinates, dx and dp , in the x - and p -tangent directions, such that

$$\mathbf{U}(dx, dp) = dx \hat{\mathbf{r}}_x + dp \hat{\mathbf{r}}_p . \quad (D.13)$$

The principal curvatures and directions are found through computation of the coefficients of the first and second fundamental forms. The first fundamental form is defined by

$$d\mathbf{r} \cdot d\mathbf{r} = E dx^2 + 2F dx dp + G dp^2 \quad (D.14)$$

where

$$E = d\mathbf{r}_x \cdot d\mathbf{r}_x = 1 + Y_x^2 + Z_x^2 \quad (D.15)$$

$$F = d\mathbf{r}_x \cdot d\mathbf{r}_p = Y_x Y_p + Z_x Z_p, \text{ and} \quad (\text{D.16})$$

$$G = d\mathbf{r}_p \cdot d\mathbf{r}_p = Y_p^2 + Z_p^2. \quad (\text{D.17})$$

Also define $H = |\mathbf{r}_x \times \mathbf{r}_p| = \sqrt{EG - F^2}$. Then, one obtains the second fundamental form, which is given by

$$-d\mathbf{r} \cdot d\hat{\mathbf{n}} = -d^2\mathbf{r} \cdot \hat{\mathbf{n}} \quad (\text{D.18})$$

$$= e dx^2 + 2f dx dp + g dp^2 \quad (\text{D.19})$$

where

$$e = \mathbf{r}_{xx} \cdot \hat{\mathbf{n}} = (-Y_{xx}Z_p + Z_{xx}Y_p)/H \quad (\text{D.20})$$

$$f = \mathbf{r}_{xp} \cdot \hat{\mathbf{n}} = (-Y_{xp}Z_p + Z_{xp}Y_p)/H, \text{ and} \quad (\text{D.21})$$

$$g = \mathbf{r}_{pp} \cdot \hat{\mathbf{n}} = (-Y_{pp}Z_p + Z_{pp}Y_p)/H. \quad (\text{D.22})$$

Also define $h = \sqrt{eg - f^2}$. The principal curvatures are now found by first computing the Gaussian and average curvatures, which are given by

$$\begin{aligned} A &= \text{average curvature} \\ &= \frac{(\kappa_1 + \kappa_2)}{2} \end{aligned} \quad (\text{D.23})$$

$$= \frac{Eg - 2fF + eG}{2H^2}, \text{ and} \quad (\text{D.24})$$

$$\begin{aligned} K &= \text{Gaussian curvature} \\ &= \kappa_1 \kappa_2 \end{aligned} \quad (\text{D.25})$$

$$= \frac{h^2}{H^2}. \quad (\text{D.26})$$

Finally, by computing the variable T as

$$T = \sqrt{A^2 - K} \quad (\text{D.27})$$

$$= \frac{\kappa_1 - \kappa_2}{2} \quad (\text{D.28})$$

one finds that

$$\kappa_{1,2} = A \mp T, \text{ and} \quad (D.29)$$

$$R_{1,2} = 1/\kappa_{1,2}. \quad (D.30)$$

The components of the principal directions are found by setting $\kappa = \kappa_{1,2}$ and solving for dx/dp in the following matrix equation:

$$\left[\begin{bmatrix} e & f \\ f & g \end{bmatrix} - \kappa \begin{bmatrix} E & F \\ F & G \end{bmatrix} \right] \begin{bmatrix} dx \\ dp \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (D.31)$$

or

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} dx \\ dp \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (D.32)$$

where

$$p_1 = e - \kappa E \quad (D.33)$$

$$p_2 = f - \kappa F \quad (D.34)$$

$$p_3 = g - \kappa G, \text{ and} \quad (D.35)$$

$$dx/dp = (-p_2/p_1), \text{ or} \quad (D.36)$$

$$dx/dp = (-p_3/p_2). \quad (D.37)$$

In most cases, either (D.36) or (D.37) can be used, but in some cases either p_1 or p_2 may be very small while the other one is larger. In these cases, numerical error can be avoided by the proper choice of using either (D.36) or (D.37). When dx/dp is known, the particular principal direction is found from

$$\mathbf{U} = \mathbf{U}(dx/dp, 1). \quad (D.38)$$

Three special cases will be considered next.

1. Umbilical point:

At umbilical points, the curvature is the same in all directions; i.e. $\kappa_1 = \kappa_2$.

For this case, one finds that

$$\frac{e}{E} = \frac{g}{G} = \frac{f}{F} . \quad (D.39)$$

When this happens, one can take

$$\mathbf{U}_1 = \mathbf{U}(1, 0) , \text{ and} \quad (D.40)$$

$$\mathbf{U}_2 = \mathbf{U}(0, 1) . \quad (D.41)$$

2. Parametric lines = lines of principal curvature:

In this special case one finds that $f = 0$ and $F = 0$, and the principal directions are given in (D.40) and (D.41).

3. Only one parametric line = a line of principal curvature:

In order to identify this case, let us first eliminate κ from (D.31). The resulting equation is

$$(Fg - Gf)dx^2 + (Eg - Ge)dx dp + (Ef - eF)dp^2 = 0 . \quad (D.42)$$

- If $(Fg - Gf) = 0$ then $dp = 0$ is a solution to (D.42) and

$$\mathbf{U}_1 = \mathbf{U}(1, 0) , \text{ and} \quad (D.43)$$

$$\mathbf{U}_2 = \mathbf{U}\left(1, \frac{dp}{dx}\right) \quad (D.44)$$

where

$$\frac{dp}{dx} = -\frac{(eG - Eg)}{(Ef - eF)} \quad (D.45)$$

- If $(Ef - eF) = 0$ then $dx = 0$ is a solution to (D.42) and

$$U_2 = U(0, 1), \text{ and} \quad (D.46)$$

$$U_1 = U\left(\frac{dx}{dp}, 1\right) \quad (D.47)$$

where

$$\frac{dx}{dp} = -\frac{(eG - Eg)}{(Fg - fG)} \quad (D.48)$$

In the above cases, one actually needs to find only the first principal direction. The other can then be found by taking the vector product with the normal vector $\hat{n}$. This approach is advisable in cases where the denominators of Equations (D.45) and (D.48) can become very small.

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