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**Analysis of the current density distribution due to surface
electrode stimulation of the human body**

Cilliers, Petrus Johannes, Ph.D.

The Ohio State University, 1988

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ANALYSIS OF THE CURRENT DENSITY DISTRIBUTION
DUE TO SURFACE ELECTRODE STIMULATION OF THE
HUMAN BODY.

A Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Petrus J. Cilliers, B.Sc(Ing), B.Eng(Hons), M.S.

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Advisor
Department of Electrical
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DEDICATION

To my wife Jeanne and my sons André and Leon.

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I would like to extend my sincere gratitude to my adviser, Prof. Herman R. Weed. His guidance, personal interest and encouragement throughout this project was invaluable. I would also like to thank the members of my committee, Dr. Karl Kornacker and Dr. W.D. Burnside for their suggestions, comments and encouragement.

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Soli Deo Gloria

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CHAPTER I

INTRODUCTION.

This dissertation presents the use of mathematical modelling to investigate the rôle of electrode properties and electrode configuration in the current density distribution resulting from electrical stimulation by surface electrodes.

1.1 Motivation.

The transcutaneous electrical stimulation of nerves and muscles is used in several disciplines for both research and routine clinical therapy. These include Functional Neuromuscular Stimulation (FNS) for rehabilitation [50,58], Transcutaneous Electrical Nerve Stimulation (TENS) for the suppression of pain [54] and Peripheral Nerve Stimulation (PNS) [37] for evoked muscle reflexes. Square voltage or current pulses in the range of 10 to 100 volts or 10 to 100 mA are frequently used for these purposes. The desired effect is achieved by the selective stimulation of a particular excitable tissue in the body. The primary application of interest in the present study is the use of surface electrode stimulation in FNS. The feasibility of this method for restoring functional movement to the paralyzed limbs of individuals who have suffered spinal cord injury has been demonstrated by various researchers [17,50,6,91,86]. The use of surface electrodes in this application is still much an art and the parameters for selecting and placing the electrodes are often determined by trial and error.

Several electrode systems are commonly used for FNS to couple electrical energy from the stimulator to the target nerves: (a) totally implanted electrodes, (with power coupled inductively through the skin), (b) percutaneous and subcutaneous electrodes (with lead wires protruding through the skin) and (c) transcutaneous or surface electrodes. Implanted and percutaneous wire electrodes provide an obvious advantage for the localization of the resulting field. Because of their closer proximity to the target nerves, these electrodes can also be operated at lower current levels. On the other hand, surface electrodes provide a number of benefits which have often made them the preferred means of transferring the externally applied current to the body. These include:

- Ease of application and removal: the patient can be trained to do it;
- No need for surgical invasion to apply (or remove) the electrodes;
- Less problems with biological compatibility;
- Lower risk of irritation and infection due to the fact that the skin is not perforated;
- Greater robustness: electrodes and connectors don't break as easily;
- Versatility: electrodes can easily be repositioned when required by changes in muscle properties or other factors;
- Lower cost of materials and of electrode placement.
- Patient preference.

The electric field, which couples the externally applied current with the nervous system, must be of sufficient intensity and have the correct orientation with

respect to the direction of the nerve to effect the desired stimulation. Furthermore, the selectivity of the electrode configuration for the depolarization of a particular group of nerves, and the strength of muscle contraction following electrical stimulation of a given frequency and intensity, are both closely related to the spatial distribution of the electric field. It is also known that the application of electrical currents to the body can result in thermal and electrochemical damage to the tissue due to localized heating and the formation of electrochemical products in the region close the electrodes where the current density is the highest. The risk of such damage can potentially be minimized by the appropriate placement of the electrodes and by using electrodes that keep the current density at the electrode/skin interface below the threshold for damage [10].

The potential for harm is often inferred from indirect observations. The current density distribution of surface electrodes is very difficult to measure in situ and thus it is important to develop a theoretical model which allows determination of the surface current density as a function of the geometry of the electrode, geometry of the tissue and the conductivity of the electrode and tissues.

A quantitative assessment of the factors that govern the current density and electric field distribution can be applied to the design and improvement of the electrodes and systems used for electrical stimulation and to provide a better understanding of the risk of thermal damage with currently used electrode systems.

As a necessary prelude to finding the most efficient electrode configuration for a particular application, including electrode size, spacing, positioning and electrode material, it is logical and desirable to conduct a theoretical investigation – by the application of engineering principles – of the physical factors that govern the current and electric field distribution.

The results presented here may be useful in other cases where electrical currents are applied to the body via surface electrodes such as electrosurgery, defibrillation, external cardiac pacing, and electroconvulsion shock therapy.

1.2 Problem Definition.

The theoretical basis of transcutaneous FNS is incomplete due to the difficulty of in situ measurements and the limitations of present models and experimental methods used in the characterization and quantification of the relation between the electrode configuration and the resulting electric field and current density distribution. This leads to a lack of understanding of, among others, the fundamental limitations of the technique and of potential methods to improve present clinical practice.

This work is concerned with the characterization by modelling and measurement of the current density and electric field in biological tissue as it applies to electrical stimulation by means of surface electrodes. Specifically, the effects of electrode shape, electrode conductivity, and the conductivity of the underlying tissues on the field distribution are analyzed by means of approximate mathematical models which are verified by both in-vitro and in-vivo measurements.

1.3 Approach.

In principle, the spatial distribution of the electric field can be predicted from a complete knowledge of the electrode configuration (including electrode material, size, shape and position) and the electrical properties of the body tissues. The body constitutes a volume conductor of complex geometry characterized by its conductivity and permittivity. There is a wide range of values for the values of conductivity and permittivity in the literature. Exact data for the low-frequency

conductivity of living human tissue is very difficult to obtain because the conductivity and permittivity of the body tissues are typically inhomogeneous, anisotropic, and exhibit not only a subject-to-subject variability, but also vary with the physiological condition of the tissues. This complicates the modelling of the field distribution in the body tissues. In practice, the accuracy of modelling is constrained by the accuracy and variability of the data, the availability of an efficient method to calculate the current density field, and by the ability to experimentally verify the accuracy of the predictions.

In this study, approximations are made to the complicated geometry of the body and the complex structure of the tissues in order to obtain a tractable mathematical model. For example, when the current density on the electrode surface is the parameter of interest, it is inefficient to include the whole body in the model. On the other hand, if the field at a point some distance away from the electrodes is required, the size, shape and conductivity of the electrodes is of little consequence, and they can be approximated by point sources without much loss of accuracy. In this study a transition region is derived beyond which it would be appropriate to replace the actual electrode configuration with a similar configuration of point sources.

The effects of interest are studied by means of two models: 1) A resistive electrode model for predicting the current density distribution on the surface of a class of electrodes frequently used in FNS, and

2) A multiple electrode model for predicting the current density distribution on and in the near zone of combinations of surface electrodes.

Although each model stresses a different aspect of the problem, they are interrelated and based on the common approximations of quasi-stationarity and linearity which in each case allows one to express the mathematical modelling in terms of

solutions to Laplace's equation subject to appropriate boundary conditions. The rectangular resistive electrode model is used for an investigation of the following effects on the current density distribution:

- The ratio of the electrode conductivity to that of the underlying medium,
- Variations in the electrode thickness,
- Variations in the spatial distribution of electrode conductivity.

It is shown how the occurrence of hot spots depends on the feed configuration and on the ratio between the conductivity of the electrode and that of the tissues to which it is applied. A new technique to make the current distribution more uniform than that of conventional surface electrodes is suggested on the basis of the rectangular resistive electrode model. The measured conductivity of some commonly used surface electrodes is used in the model to predict their current density when applied to the human body. Also described in this section are some experiments to measure the current density close to the electrode surface. These include measurements using an electrochemical staining method and a noninvasive technique based on measurement of the potential distribution on the back of the electrode when it is applied to the human body.

The scope of this work includes techniques of electrophysiological volume conductor field modelling, measurement of model parameters, verification of parameters predicted by the models and an analysis of measuring techniques. The modelling is based on approximations that are amenable to solution of the field problem by numerical methods. The goal of this work is to provide new information on a number of effects and measuring methods that relate to transcutaneous electrical

stimulation and thus to extend the basis on which further work in this field could be done.

1.4 Organization of the Dissertation.

The following paragraphs give a brief outline of the specific chapters:

Chapter 1 Introduction - The motivation and significance of the present work is discussed and the specific problems that are addressed are defined. A brief description of the solution methods and supporting measurements is provided.

Chapter 2 Theory and Background - Relevant aspects of the physiological basis of electrical stimulation are reviewed. The threshold for damage by electrical stimulation is discussed together with previous work on the characterization of the current distribution of surface electrodes. A review of existing neuroelectric models that quantify the relation between the extracellular electric field distribution and nerve excitation, is given as a justification of the need for models to calculate the extracellular field. Some background is presented on the structure and properties of biological tissue, particularly skin and muscle tissue, which are considered the most significant in the present application.

Previous work in electrophysiological volume conductor modelling is reviewed. The basis for the assumptions of quasi-stationarity and linearity is discussed, followed by a summary of the fundamental equations that are used in subsequent chapters. Modelling techniques and the relative merits of various solution methods are discussed.

Chapter 3 Conductive Rubber Electrode Model: Rectangular Resistive Electrodes - Chapter three presents the modelling of the current density distribution of commonly used rectangular resistive surface electrodes by means of a hybrid numerical technique.

Chapter 4 Multiple Electrode Models: Circular Disk Electrodes -

Chapter four considers the modelling of the current density distribution of a combination of several coplanar surface electrodes using various numerical techniques and a comparison with alternate models including the use of point electrodes.

Chapter 5 Summary and Conclusions - A summary of accomplishments is considered together with an overview of their significance and limitations. Suggestions for future work are presented.

CHAPTER II

BACKGROUND AND THEORY.

This chapter presents a review of the theories, experiments, and computer simulations relevant to the problem of electrical stimulation by means of surface electrodes. The purpose of the review is to provide the background and basis for the analysis developed in later chapters and provide an indication of the complexity of the problem. The physiological basis of nerve stimulation by surface electrodes is reviewed. The results of both animal and human experiments which indicate the relation between the electrode configuration and the efficiency of stimulation are discussed. Theoretical models and numerical methods to predict the current density and electric field distribution are reviewed.

2.1 Previous Current Density Models.

Several analog, analytic, and numerical models have been developed to study the impedance and current distribution of surface electrodes, and to develop better electrodes [8,95,68,87]. The occurrence of a current density concentration at the edge of a circular metal disk electrode applied to a volume conductor has been demonstrated in theory, in physical models [68], and in the use of such electrodes for stimulation [49,53,5] and defibrillation.

In general the theoretical problem of finding the current density due to highly conducting electrodes is one with *mixed boundary conditions*, in which the potential

is specified over part of the boundary (on the electrodes) and its normal derivative over the remainder of the boundary. This type of problem requires the solution of *dual integral equations*, the theory of which is complicated and not well developed [40]. Analytic solutions to this problem have been obtained only for a number of special cases. The first was given by Weber (in 1872) [93]. He found the potential field for an isolated disk electrode on the surface of a uniform half-space as well as for a pair of electrodes on opposite sides of a uniform conducting slab.

Rush et al. [74] gave the field in a conducting slab with disk electrodes on opposite sides of the slab in an analysis of the current distribution for defibrillation electrodes.

A text by Sneddon [81] gives several analytic solutions to the mixed boundary value problem. Only one of these is for a geometry that is a reasonable approximation to the problem in the present study, namely the case of two coplanar disks on a homogeneous half-space (originally by Kobayashi [48]). This solution is used in Chapter 4 as a basis for comparison with the work in the present text.

Kim [44] obtained a solution by a two-dimensional finite element method for an inhomogeneous body model and demonstrated the same general tendencies that are observed in analytic models, i.e., a peaking of the current density toward the outer perimeter of the source electrode and a skewing of the current distribution towards a nearby electrode. He coined the term ‘leading edge effect’ to describe the latter phenomenon.

Wiley et al. [95] approximated the conducting body by a resistive sheet and obtained an analytical solution for the current density by using a distributed equivalent-circuit model. He defined a characteristic length and showed that its magnitude governs the current distribution under the electrode: when the characteristic length is less than the electrode radius, the current density is peaked

toward the edge of the electrode.

Several methods to improve the uniformity of the current density at the skin-electrode interface have been proposed. These include the use of rounded edges, an electrode coating of varying resistivity or of varying thickness [24,68,42] and the segmentation of the electrode into concentric rings [96].

Brennen [8] analyzed the current density distribution of conductive rubber electrodes by the use of a (one-dimensional) lossy cable equation. The Brennen model is only applicable to the center region of long strip electrodes and fails to account for the rise of current towards the edge which could occur in cases where the electrode conductivity is high compared to that of the underlying tissue.

Reddy et al. [68] used a physical model to investigate the surface current density of metal and resistive electrodes and found that resistive electrodes exhibit a peak in the current density near the feed point.

The mathematical model for a resistive electrode which will be discussed in Chapter 3 of this dissertation is considered an improvement on existing models for rectangular conductive rubber electrodes.

2.2 Applications of Surface Electrodes.

Surface electrodes are used in many clinical applications, among others for transcutaneous electrical stimulation, external cardiac pacing, defibrillation and electrosurgery. Transcutaneous electrical stimulation is applied for pain control [10], muscle conditioning [83], peripheral nerve stimulation [37] and functional stimulation [91,50,58]. In external cardiac pacing the heart is stimulated at regular intervals to allow control of the heart rate when the intrinsic cardiac rate is insufficient [92]. Defibrillation is used to depolarize all cardiac muscles simultaneously by an electric shock in an attempt to restore normal cardiac rhythm [74]. In

electrosurgery radio-frequency currents are applied by a small electrode for cutting and coagulating tissue while the return current is collected by a large dispersive surface electrode [75]. The electrodes used in each of these applications are operated at various levels of current and frequency. Table 1 gives typical values of the current, frequency and pulse duration used.

Table 1: Characteristics of surface electrode applications.

Application	Waveform	Peak Current	Duration	Frequency
TENS,FES	Rectangular	20–150 mA	10–500 μ s	10–100Hz
External Pacing	Rectangular	50–200 mA	10–40 ms	40–100Hz
Defibrillation	Exponential	Up to 50 A	5–10 ms	–
	Damped			
	Sinusoid			
Electrosurgery	RF Burst	0.25–1 A	1–10 s	0.5–2 MHz.

All these applications share a common difficulty: The current distribution on the skin surface is not constant but is characterized by peaks that occur either along the periphery or in the region where the external feed wire is connected to the electrode. The current density is always higher at the surface than anywhere else in the body [85]. The location of the hot spots depends on the location and geometry of the electrode, its conductive properties and the properties of the skin and subcutaneous tissue. Models representing highly conductive electrodes in direct contact with the body have predicted an edge peaked current density [44,95,96]. In resistive electrodes a peak in the feed region was predicted by Brennen [8] who used a simplified one-dimensional model and by Reddy [68] who performed mea-

surements on a physical model consisting of a sponge filled with conducting gel [68]. When a pair of electrodes are close together, the edge effect can be worsened by an increase in the current distribution along the edge that is nearest to the second electrode [43,44].

2.3 Conductive Rubber Electrodes.

Conductive rubber electrodes are used extensively for TENS and FNS since their pliability allows them to easily conform to the body contours and hence improve the electrode/skin contact. They are manufactured by impregnating silicon rubber with a conductive agent, typically carbon, resulting in a material with a conductivity that is about 10 to 20 times that of muscle tissue and significantly less than that of metal electrodes. Conductive rubber electrodes are relatively inert compared to metal electrodes. Conductive rubber electrodes can be used for extended periods of time with or without electrode gel. Despite these advantages, such electrodes are not exempt from causing electrode burns due to uneven distribution of the current [5]. When applied to the human body, these electrodes can have a current distribution that is significantly different from that of metal electrodes. This is due to the resistive potential drop along the electrode as the current spreads from the feedpoint to the electrode edges. The closest canonical problem which crudely approximates the surface electrodes problem of interest, namely that of an infinitely conducting disk electrode, is known to have a current density singularity at the edge of the electrode. The question then arose whether conductive rubber electrodes could have the same nonuniform current density distribution and if so what the factors are that determine the surface current density distribution.

The current density threshold for discomfort in normal subjects depends on the stimulation frequency and is relatively insensitive to electrode surface area [58]. In paralyzed individuals the current pulses required for functional movement can be up to 100 mA and more, which when applied to a small area of the skin, has the potential for producing irritation and more severe harmful effects.

2.4 Indirect Evidence for a Non-uniform Surface Current Density Distribution.

Although the surface current density distribution produced by passing current into the body by means of surface electrodes is very difficult to measure *in situ*, there is indirect evidence for its nonuniformity. This includes electrode burns, in-vivo thermographic and potential distribution measurements as well as measurements on simulated tissue. Each of these is now briefly discussed.

1. *Electrode Burns* : The most compelling evidence for the non-uniformity of the current density distribution is that when electrode burns occur, they are frequently located close to the edge of the electrode [5]. Since the heating and power density is proportional to the square of the current density, this points to a nonuniformity of the current density distribution. This might however be caused by poor electrode contact and not by some inherent property of the electrode-skin contact.
2. *Thermographic Measurements* : Thermographic measurements have been used to determine the in-vivo temperature distribution at the site of dispersive surface electrodes used in electrosurgery. In one such study, Aubry-Frize et al. [4] inferred a non-uniform heat distribution on the electrode from a model that relates the temperature elevation of the skin to the electrode geometry.

3. *Potential Distribution Measurements in Resistive Electrodes* : In resistive electrodes, such as conductive rubber electrodes, the spreading of the current from the feedpoint out toward the edges of the electrode and into the tissues causes a potential drop along the back of the electrode that can be related to the current density distribution at the electrode-skin interface. However, the inferred current density distribution is highly dependent on the parameters of the model used in deriving the relation between the potential distribution and the current density distribution. Brennen [8] first used this method to characterize the current distribution of conductive rubber electrodes.

4. *In Vitro Measurements Using Simulated Body Tissue* : Using a physical model consisting of surface electrodes applied to a gel-soaked sponge that simulated muscle tissue, Reddy and Webster [68] inferred a non-uniform distribution of current close to the electrode from measurements with a voltage gradient probe.

2.5 Detrimental Effects of the Non-uniform Current Density Distribution of Surface electrodes.

The passing of current through a volume conductor causes heating and the formation of electrochemical products. The extent of thermal damage to the skin and the quantity of electrochemical reaction products are directly related to the charge per unit area of the electrode surface delivered by the stimulus [9]. Due to the high current levels required for transcutaneous electrical stimulation (typically 10-150 mA) and the concomitant danger of thermal damage to the tissues, the minimization of the power dissipated at the electrode/skin interface is a major concern in the application of electrical stimulation by means of surface electrodes [10].

This problem is made worse by the fact that the current of most surface electrodes is not uniformly distributed over the surface of the electrode. Areas of high current density under the electrodes have also been linked to stinging pain sensations some subjects have experienced with surface electrode stimulation [7].

In addition to tissue heating, there are other effects associated with the electrode material and with the current distribution which can cause tissue damage. These will be discussed briefly, before the problem of thermal damage is considered in greater detail.

In dry electrodes, i.e., electrodes used without a gel or other coupling medium, dielectric breakdown of the stratum corneum can occur if the applied voltage is sufficiently high [32]. The place where such breakdown occurs is usually a function of the structure of the superficial layer of the skin (the stratum corneum). However, it is more likely to occur at charge concentration points on the surface of the electrode.

It has also been shown that tissue irritation can result from the use of an unbalanced waveform, i.e., one that allows a residual charge to accumulate at the electrode/tissue interface [63,59]. If prolonged, this can result in chemical burns and tissue necrosis. Although such effects have been associated with areas of high current density, the problem is not unique to surface electrode stimulation and can be reduced by using appropriate waveforms and inert electrode materials.

To prevent thermal damage, the electrodes should provide adequate surface area to minimize the average current density and should have a uniform current distribution, i.e., no "hot spots".

The peak current at the threshold of thermal damage due to stimulation by rectangular current pulses can be calculated by equating the average power density of the stimulus to the burning pain threshold [10]. The average power density is

given by

$$P = \frac{I^2 r f t_p}{A^2} \quad (\text{W/cm}^2) \quad (2.1)$$

where

$$\begin{aligned} I &= \text{peak current (Ampere);} \\ r &= \text{specific resistance } (\Omega\text{-cm}^2); \\ f &= \text{frequency (Hz);} \\ t_p &= \text{pulse width (sec);} \\ A &= \text{electrode contact area (cm}^2\text{).} \end{aligned}$$

The burning pain threshold for radiant heat as developed by Hardy et al. [35] is $\theta_r = 1.0 \text{ W/cm}^2$. Equating this to (2.1) and rearranging gives

$$I = A \left(\frac{\theta_r}{r t_p f} \right)^{\frac{1}{2}} \quad (\text{Amp}). \quad (2.2)$$

In general, the current density is nonuniform, and the thermal threshold may in fact be much lower than estimated by this approximate formula. This equation is plotted in Figure 1 with r as a parameter for $f = 20 \text{ Hz}$ and $t_p = 300 \mu\text{s}$. Burton et al. [10] give $r = 47 \text{ k}\Omega\text{-cm}^2$ as a typical value for carbon rubber electrodes at 20 Hz .

2.6 Electrical Properties of Skin and Subcutaneous Tissues.

A brief discussion of the electrical properties of skin and subcutaneous tissues is given here to justify the assumptions made in the formulation of the approximate models considered in Chapters 3 and 4 and to provide some typical values for the parameters of these models. The skin anatomy is considered to show the layered structure of the superficial tissues.

Figure 1: Peak square-wave current at the threshold of thermal damage as a function of electrode surface area. (Redrawn from Burton et al. [10])

The conductivity of body tissues has been measured by many researchers [14,21,18,25]. There is a great variability in the electrical properties of body tissues reported in the literature due to differences in species and measurement methods.

The various limbs and tissues of the body have a complex geometry. In addition, biological tissue is by the very nature of its cellular structure inhomogeneous and anisotropic. In the interpretation of measurements of the electrical properties, the inhomogeneity and anisotropy should be taken into account.

The following aspects of the electrical parameters of the tissues are considered: linearity, frequency dependence, anisotropy and time invariance.

Introduction.

The dimensions and electrical properties (conductivity and dielectric constant) of the skin and subcutaneous tissue layers are important parameters in the analysis of the current density distribution of surface electrodes. In order to formulate appropriate models, it is desirable to have numerical values for the electrical properties of the skin and subcutaneous tissues over the range of frequencies and currents typically used in FES. For instance, it is important to determine whether the ratio of capacitive to resistive current is sufficiently small throughout the frequency spectrum of the stimulating pulses to justify the neglect of the capacitive current components. There is a great deal of variability in the data reported in the literature, due to variations in the experimental methods used by various investigators. Although many studies have been done on animal tissues, there is very little available on the in situ properties of human tissues. The data from animal studies cannot always be applied directly to human tissue, but some general trends can be inferred from animal studies. In the present context, the focus is on the parameters of human tissues. Where such data is lacking, corresponding data from

animal studies are given.

2.6.1 Anatomy.

Anatomically the skin consists of three principal layers: the stratum corneum, the epidermis and the dermis (See Figure 2). The stratum corneum is composed of several keratin layers. These are mostly dehydrated non-living cells. The thickness of the stratum corneum is from 0.04 to 0.08 mm on various parts of the body. Its thickness is approximately uniform except where it is penetrated by hair and pores. Its electrical resistivity is much higher than that of the subcutaneous tissues and it contributes the largest part of the capacitive reactance of the skin impedance. Grimnes [34] has shown that the sweat pores provide the ionic current pathways that determine the resistive component of the skin impedance. Removal of the stratum corneum by skin stripping reduces the skin impedance to an almost pure resistance which is on the order of one tenth of the that of the intact skin impedance [87]. Underneath the stratum corneum is the living non-vascular epidermis and the dermis which consists of dense connective tissue. The bulk of the subcutaneous tissue consists of muscle tissue, except when there is a substantial layer of fat between the skin and the muscle tissue. The motor nerves, which form the target of the electrical stimulation in FNS, are typically buried between several muscle layers. Muscle fibers form the bulk of the tissue between the skin and the peripheral nerves (see Figure 3).

2.6.2 Linearity.

Skin resistance has been shown to be non-linear [21,84,87,88]. Several mechanisms have been identified: With potentials on the order of -10 to -50 volts with respect to a reference electrode placed elsewhere on the body, the electro-osmotic transport of water through the pores of the skin leads to a reversible increase in

THE STRUCTURE OF THE SKIN

Figure 2: The structure of human skin. (Adapted from Woodburne [98])

Figure 3: Cross section through human upper thigh, halfway between the hip and the knee. (From Eycleshymer [19])

the electrolytic conductance [31]. At higher potentials irreversible dielectric breakdown of the stratum corneum can occur [32]. The skin resistance also decreases with the continued passage of direct currents, especially at the cathode [84]. The magnitude of the skin resistance is a function of the applied current density.

It is well-known that the membranes of tissue fibers are characterized by active transport processes and have nonlinear electrical properties. The nonlinear properties of nerve membranes have been incorporated in the Frankenhaeuser-Huxley and other nerve models [20,57].

On the other hand, very little has been documented about the nonlinearities in the macroscopic conductivity of tissues such as fat and muscle except that the macroscopic electrical properties of muscle tissue are approximately linear when the current density is below the excitation level [78]. Above the excitation level, the contraction and concomitant changes in the physical structure of the excited muscles might give rise to non-linearities in the electrical properties of the tissue. In practice, when an increase in the applied current is observed in terms of an increased force of contraction, non-linearities in the electrical properties of the muscle are probably masked by the marked non-linearity in the recruitment curve. In human experiments performed with pulsed currents it was found that the impedance of an electrode applied to the human quadriceps is approximately linear up to the threshold of muscle contraction.

In electrophysiological volume conductor models, the tissue properties are usually assumed to be linear in order to obtain the field distribution by using linear equations. From the correlation between such models and in vivo measurements, it can be inferred that the error introduced by assuming the linearity of the conductivity of the body tissues, is within the range of the experimental error in measurement.

Table 2: Averages of the ratio of capacitive to resistive current for various frequencies and body tissues

Tissue	0.1 kHz	1 kHz	10 kHz	Reference
Muscle †	0.04	0.06	0.24	In vivo dogs [78]
Fatty Tissue †	0.01	0.03	0.15	"
Stratum Corneum *	1.40	3.30	8.30	Human forearm [99]

Notes: †* These data have a typical standard deviation of 30%.

In practice, the effect of the nonlinearity of the skin impedance can be reduced by the use of current controlled stimulation pulses, rather than constant voltage pulses.

2.6.3 Frequency Dependence of Tissue Conductivity and Dielectric Constant.

Both the conductivity, σ , and dielectric constant, ϵ , of most body tissues are dispersive in the frequency range DC to 10kHz [78]. Averages of the ratio of capacitive to resistive current ($2\pi f\sigma/\epsilon$) for various frequencies, f , and body tissues are given in Table 2. This illustrates that muscle tissue is almost purely resistive at frequencies up to 10kHz. A frequency of 10kHz is considered as an upper limit to the range of frequencies of interest because the predominant part of the energy of signals used for functional electrical stimulation is associated with frequencies below 10kHz [11]. However, for the stratum corneum, the complex conductivity given by

$$\hat{\sigma} = \sigma + j2\pi f\epsilon \quad (2.3)$$

goes from a predominantly resistive value at low frequencies to a predominantly capacitive value at higher frequencies. Over the same range, its magnitude in-

creases about tenfold. Since muscle forms the bulk of the conducting medium, the body is modelled as a purely resistive medium for a first approximation of the low-frequency current distribution.

2.6.4 Anisotropy.

Due to the layered structure of the stratum corneum, its electrical properties are markedly different in the directions parallel and perpendicular to the surface of the skin [99]. Grimnes [34] found that the stratum corneum can be modelled as a stratified structure with the resistance of the layers decreasing with increasing depth into the tissue. He also came to the conclusion that the sweat ducts provide the dominant ionic path for current flow into the skin [34]. Hence the skin conductivity in a direction perpendicular to the skin is much lower than in the tangential direction. In skeletal muscle tissue both the conductivity and permittivity is about 3 to 10 times higher in a direction perpendicular to the muscle fibers than in the direction parallel to the fibers [18], the exact figure depending on the measurement method, frequency, and the particular muscle.

For muscle fibers the anisotropy is correlated with the direction of the fibers. The muscle fiber structure can be approximated by tubular membranes of low conductivity surrounded by fluids of higher conductivity. The membranes have a common orientation in a given muscle, which results in a high conductivity (σ_h) in the axial direction parallel to the fibers and a lower conductivity (σ_l) in the direction transverse to the fibers. Most peripheral muscles have their fibers running approximately parallel to the surface of the skin.

A summary of reported values for muscle conductivity obtained with the four-electrode method is given in Table 3.

Table 3: Muscle conductivity values reported in the literature

Parallel to muscle fibers σ_h in S/m	Transverse to muscle fibers σ_l in S/m	Anisotropy (per author) σ_h/σ_l
$0.33 < \sigma_h < 0.80$	$0.04 < \sigma_l < 0.15$	$2.04 < \sigma_h/\sigma_l < 15.33$

Gielen [25] has demonstrated that the measured values of muscle conductivity depend on the dimensions of the measuring device and has developed a model to explain this observation in terms of the cellular structure of muscle tissue. He cautioned against the use of macroscopic conductivity data in models to predict the fields in regions of microscopic dimensions such as in the immediate surroundings of a particular nerve.

In order to represent the body by a deterministic mathematical model it has to be described in terms of a finite number of homogeneous regions.

2.6.5 Time Invariance.

The electrical impedance of human skin is not in general time-invariant. This is mainly due to the variance of the electrical properties of the stratum corneum. Grimnes [33] found that the magnitude of the impedance of pregelled ECG skin electrodes attached to the human forearm showed a marked long-term decrease. In another study Grimnes [34] found that there is a significant change in the conductive component of the skin impedance with variation in the rate of perspiration. This trend was attributed partly to electrolyte spread on the skin (which increases the effective electrode area) and partly to electrolyte penetration into the skin. In experiments done for this study, a similar trend was observed for conductive rub-

ber FES electrodes applied to a human quadriceps muscle (both with and without conductive gel).

2.7 Mathematical Modelling.

The use of mathematical models in volume conductor studies have received considerable attention in the literature, partly due to the invasive nature and concomitant difficulty of any experimental determination of the internal current density and electric field distribution.

2.7.1 Quasi-stationarity.

A fundamental assumption in a great number of the biological volume conductor models reported in the literature, is that of quasi-stationarity. In general, the calculation of time-varying electromagnetic fields in a conducting medium involves a solution of the time-dependent Maxwell's equations subject to certain boundary conditions. Plonsey and Heppner [64] have shown that a quasi-static model, where Maxwell's equations reduce to the Laplace (or Poisson) equation can be used for studies in electrophysiology where the sources are of bioelectric origin, such as action potentials.

Cilliers and Weed [11] showed that the conditions for quasi-stationarity also apply when the sources are pulses of the kind that are used in electrical stimulation by surface electrodes.

The basis for the quasi-static approximation is that the wavelength of the applied stimulus is large compared to the dimensions of the volume conductor. The implication of the quasi-static approximation is that there is no propagation of the electric field and that the time variation of the potential field at all points is identical with that of the applied source. The validity of this approximation

was tested by measuring the time course of the electric field in the tissues of an intact animal during the application of $300\mu\text{s}$ current pulses to a pair of surface electrodes. It was observed that the capacitance of the electrode-skin interface causes a difference between the waveform of the potential and that of the applied current, but that the internal electric field had the same waveform as the applied current. In order to justify the use of the quasi-stationary approximation in the field analysis in subsequent chapters it seems sufficient to assume that the electrical stimuli are applied by means of *current* sources.

2.7.2 The Fundamental Equations.

The quasistatic current density satisfies the equation

$$\nabla \cdot \mathbf{J} = I_v \quad (2.4)$$

where ∇ is the divergence operator, $\mathbf{J}$ is the current density (A m^{-2}), and I_v is the volume current source density (A m^{-3}) [65].

The electric field $\mathbf{E}$ (V m^{-1}) is related to the current density by

$$\mathbf{J} = \bar{\sigma} \mathbf{E} \quad (2.5)$$

where $\bar{\sigma}$ is the conductivity tensor (S m^{-1}).

Since $\nabla \times \mathbf{E}$ is zero in a quasistatic field, the electric field $\mathbf{E}$ can be derived from a scalar potential V (Volt) by

$$\mathbf{E} = -\nabla V . \quad (2.6)$$

Combining (2.4), (2.5), and (2.6) gives

$$\nabla \cdot (\bar{\sigma} \nabla V) = -I_v . \quad (2.7)$$

If the anisotropy of $\bar{\sigma}$, referred to the principal axes of an orthogonal coordinate system, leads to a diagonal matrix with elements σ_x , σ_y , and σ_z , then the components of $\mathbf{J}$ and $\mathbf{E}$, along the same principal axes, are related by

$$J_x = \sigma_x E_x, J_y = \sigma_y E_y, \text{ and } J_z = \sigma_z E_z \quad (2.8)$$

and (2.7) may be written

$$\frac{\partial}{\partial x} \left(\sigma_x \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(\sigma_y \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\sigma_z \frac{\partial V}{\partial z} \right) = -I_v. \quad (2.9)$$

For a homogeneous, anisotropic medium the conductivities σ_x , σ_y , and σ_z are constant and the problem to be solved is

$$\sigma_x \frac{\partial^2 V}{\partial x^2} + \sigma_y \frac{\partial^2 V}{\partial y^2} + \sigma_z \frac{\partial^2 V}{\partial z^2} = -I_v \quad (2.10)$$

and if the medium is both homogeneous and isotropic with scalar conductivity σ , the equation becomes the Poisson equation

$$\nabla^2 V = \frac{-I_v}{\sigma}. \quad (2.11)$$

In regions of the volume conductor where there are no impressed sources, $-I_v$ is set to zero and (2.11) reduces to the Laplace equation

$$\nabla^2 V = 0. \quad (2.12)$$

The solution of (2.11) and (2.12) are well known and may be obtained by numerical methods such as the finite element method or by analytic methods such as separation of variables in a number of coordinate systems. When the anisotropies can be fit into the rectangular or cylindrical coordinate system, (2.10) can be converted into an equivalent Poisson equation by means of an appropriate coordinate transformation [80, p. 269].

With surface electrodes, the source current density is zero everywhere except on the electrode surface, where it can be expressed in terms of the surface current density. Using this approach, the solution of (2.11) in a uniform half-space can be expressed as

$$V(\mathbf{r}) = \frac{1}{2\pi\sigma} \int_S \frac{J^s(\mathbf{r}')}{R} ds' \quad (2.13)$$

where R is the distance from the source point at $\mathbf{r}'$ to the field point at $\mathbf{r}$, S is the surface of the electrode and J^s is the surface current density on the electrode. In general, J^s is not known a priori, and must first be obtained by treating (2.13) as an integral equation for J^s and then calculating V at points within the volume conductor. If the electrode is assumed to be a point source with current I , (2.13) becomes the Green's function for a conducting half-space

$$V(\mathbf{r}) = \frac{I(\mathbf{r}')}{2\pi\sigma R} . \quad (2.14)$$

The moment method [36] with subsectional basis functions and collocation can be used to reduce the integral equation of (2.13) to a set of algebraic equations. The moment method approach is described in detail in the context of a specific set of subsectional basis functions in Chapters 3 and 4.

When the volume conductor is of finite extent and/or is partially homogeneous, a solution of Laplace's equation is added to (2.13) or (2.14) to satisfy the boundary conditions and/or the continuity conditions imposed at the inhomogeneity surfaces.

The two parameters of interest in the forward problem are the current density on the electrode surface and the electric field distribution within the conducting medium. The physiological relevance of these parameters have been outlined before.

In general, the current density on the electrode is required in the expressions for the electric field within the medium, or need to be numerically approximated as a first step in the solution of the forward problem.

2.8 Analytic Solutions by Analogy to Electrostatics.

Potential problems in electrostatics and volume conduction situations inside body tissues are equivalent in the sense that they lead to the same mathematical formulation. Analytic equations for the electric field in a homogeneous volume conductor can be derived from known solutions of the electrostatic potential field in cases of equivalent geometry. The three dimensional potential field of a charged conductor in a homogeneous dielectric medium constitutes an electrostatic problem which is amenable to an analytic solution when the medium is unbounded or when the boundaries of the medium and conductors can be described in appropriate coordinate systems. For example, the analytic equation approach can be used to determine the current density and electric field of a monopolar highly conducting electrode, in the shape of an oblate spheroid, applied to the human body by using the analogy with the electrostatic charge distribution on an electrode of similar shape in free space. The analytic solution for this electrostatic problem is given by Smythe [80, p. 124].

Multiple Electrodes.

The case of multiple highly conducting electrodes falls in a class of mixed boundary value problems for which very few analytic solutions exist. The mixed boundary value problem arises from the fact that on the part of the surface where the electrode is applied, the potential is specified (Dirichlet boundary condition); while on the rest of the surface, the normal current density is set to zero (Homo-

geneous Neumann boundary condition).

Only the case of two disk electrodes has been treated with some success [48]. The current density on each electrode depends on the electrode separation. When a pair of electrodes are separated by more than 2 electrode diameters, the current density on each electrode is within 5% of that of an isolated electrode. However, when the electrodes are closely spaced, the current density is skewed towards the other electrode. More detail on this analysis is given in Chapter 4.

2.9 Volume Conductor Modelling.

The class of problems which deal with the potential distribution due to known source distributions are known as “forward potential problems”. Solutions to these problems provide the tools for estimating the nature of the sources from a given potential distribution or solving the “inverse potential problem”. In the study of muscle excitation by surface electrode stimulation the “inverse problem” would comprise the specification of the source distribution (electrode configuration) in order to produce a given field distribution. In the present study only some aspects of the related “forward problem” are investigated.

Most of the literature on volume conductor modelling has focused on the solution of the forward and inverse problems of electrocardiology and the diagnosis of nerve condition from measurements taken on the limb surface. A number of models related to the current distribution due to surface electrodes, particularly as it applies to defibrillation [73], electrosurgery [94] and electroconvulsion shock therapy [72] have been published.

A review of methods for forward field problems in electrophysiology is given by Heringa [38].

When compared to models pertaining to biological sources the following aspects of surface electrode stimulation require special consideration:

1. the sources are external current pulses rather than the currents associated with the depolarization of excitable tissue (action potentials)
2. the current levels and frequency content of external sources are much higher than biological sources, and
3. the source configuration is a controllable parameter; whereas with biological sources, the source characteristics are the unknown parameters which one would like to determine from the potential field distribution.

All of these aspects affect the choice of the modelling method.

2.10 Analytical Solutions vs. Numerical Approximation.

As in electrophysiological modelling in general, there are two diverging approaches to the solution of the forward problem associated with surface electrode stimulation. These are discussed in the next section.

2.10.1 Analytic Solutions.

Analytic solutions can be obtained by making enough simplifying assumptions about both the electrode and medium configuration that the problem can be fit into some orthogonal coordinate system or becomes amenable to a transformation that would permit an analytic solution.

The main advantage of this approach is the relative ease of obtaining numerical results and hence the suitability of the method to study the effect of variations in the pertinent features of the problem such as changes in the geometry and electrical properties of the electrode and of the medium.

As an extreme example of this approach one might assume that the medium is a half-space with linear, homogeneous, and isotropic conductivity and the electrodes are point sources. Even this simple model would allow one to compare the field distribution of monopolar and dipolar electrodes and to study the effects of electrode spacing on the depth of penetration of the current.

The disadvantage is that the results are not very accurate or directly applicable to any specific "real world" situation, such as the excitation of a specific nerve in say the arm of a particular individual. Even with idealized models one finds that in all but the simplest configurations, the real difficulties arise with the necessity of reducing the formal solution to a state that permits numerical computation.

2.10.2 Finite Element Numerical Approximations.

The basis of this approach is the discretization of the medium and electrode into a mesh of homogeneous "cells" connected by a set of "nodes". The potential field is obtained by a discretization of (2.11) leading to the finite difference method [47,15], or a variational formulation such as the finite element method [100].

The finite difference method is conceptually simple, but very demanding in terms of computational effort and storage requirements. It is not particularly suited to the accommodation of curved boundaries and Neumann boundary conditions. The finite element method is more versatile, but significantly more difficult and costly to implement.

Advantages : One advantage of the numerical methods is that the number of assumptions are minimized so that the results should be a very accurate, provided that the errors due to the numerical method have been minimized. It allows one to include all the known aspects of a particular application such as the actual com-

plicated geometry, anatomy and electrical properties (including inhomogeneities, anisotropy and nonlinearities) of the whole body of a specific individual. In principle, by including enough elements, this approach can give a prediction of the field that is as accurate as the data used in the model.

Disadvantages : These methods require a calculation of the numerical values of the field at each mesh point in the structure in order to determine the field at any particular point of interest. This leads to the number of calculations for a three-dimensional model quickly becoming intractable as the number of elements increases. The use of these methods is also hampered by the difficulty of obtaining the precise anatomical and electrical data required to construct a representative three-dimensional model. It is very demanding in terms of programming effort, computation time and computer memory requirements. In practice, one is also limited by the accuracy of measurements to verify the model.

Existing Models : Three-dimensional finite element models of the whole body do exist such as the one developed by Kim [43]. His model however was not detailed enough to accurately model thin layers such as the skin. Although one could conceptually add more elements to model such fine structures, it would make the model computationally inefficient. In cases where the geometry is intricate but fixed and the number of electrode configurations are limited such as in spinal cord stimulation by implanted electrodes, the finite element method has been proven to be useful [12].

However, in all the applications of interest in this study, there are variations in both the geometry and electrical properties of the tissues associated with different limbs, individual differences, and with the movement and physical condition of the

limbs of a particular individual. Although the numerical methods are particularly suited to study this kind of variation, the effort of reprogramming the configuration for each case would be prohibitive.

2.10.3 Previous Volume Conductor and Electrode Models.

Brennen [8] modelled the current density distribution on the surface of practical surface electrodes by approximating the electrode by an infinite conducting strip with current density variation only in one dimension. His model did not account for edge effects.

Wiley and Webster [94] analyzed the current distribution on a circular disk electrode, showing the origin of the perimeter burn problem which had been observed in the use of electrosurgical dispersive electrodes. They suggest a segmentation of the disk into annular rings to improve the current distribution.

Reddy and Webster [68] used a model with a varying thickness of resistive coating on the surface of a circular metal electrode to design a surface electrode with almost uniform current density (as confirmed by in vitro measurements).

Rush et al [73] used a model comprising disk electrodes on an infinite conducting slab to study the influence of electrode size, shape and location on the current distribution in the human torso due to external defibrillation.

The electric field in planar stratified multilayer medium has been studied in the theory of geophysical exploration. Stefanescu [82] derived equations for the case of a point source on a three-layer medium. Van Nostrand [89] gives a review of the literature in this field and presents the equations for the potential due to a point source on the surface of an N -layer medium.

In this work a technique for modelling conductive rubber electrodes is presented which has as its prime parameter the ratio between the electrode conduc-

tivity and the conductivity of the underlying medium. This model accounts for edge effects as well as feed point effects. An extension of the model which incorporates spatial variations in the thickness or conductivity of the electrode provides the basis of a design to make the current distribution more uniform is discussed in Chapter 3.

2.10.4 Modelling versus Measurements.

An enormous number of difficult and invasive experiments would be required in order to obtain conclusions of general value on the current density and field distribution of surface electrodes given the total range of variability of the parameters involved [60]. A more theoretical approach seems, therefore, indicated. This approach utilizes the fact that the electrical properties (conductivity and dielectric constant) of various tissues and the arrangement of tissues of different properties in the body, together with a given electrode configuration, determine uniquely the spatial distribution of the resulting electric field within the body.

In order to obtain accurate predictions of the field distribution in a specific limb of the body, the indicated approach would be to develop a three dimensional finite element model which in principle can represent the complex geometry and variability of the electrical properties to any desired degree of accuracy. However, such a model was considered impractical due to several factors:

- The lack of sufficient computing power to solve the large systems of equations of three-dimensional finite elements models.
- The cost of a suitable finite element program with interactive mesh generation and graphical output capabilities.

- The lack of accurate and detailed information on the electrical properties of the body tissues such as would be required for such a model.
- The lack of experimental and theoretical data against which the predictions of such a model could be compared to verify its accuracy and validity.
- The prohibitive cost of running a large number of simulations to derive some generalization from results obtained for specific electrode configurations on a specific body limb.

The chosen approach was to use simple models utilizing canonical geometries and simplified approximations of actual tissue properties to derive some basic guidelines for the relation between the electrode configuration and the resulting surface current density and field distribution. The models are tailored to the investigation of specific aspects of electrical stimulation by means of surface electrodes. Because of the low frequency of the applied sources, the capacitive current resulting from the dielectric properties of the body are small compared to the resistive currents. The body itself is approximated by a uniform resistive half-space. The effect of electrode conductivity on the surface current distribution is examined by using a resistive electrode model for rectangular electrodes of variable thickness or conductivity.

2.10.5 Surface Current Density Solutions.

For the resistive surface electrode problem, the surface current density is determined by a new hybrid solution method which combines a finite difference solution for the field in the electrode with an integral formulation of Laplace's equation which is solved by means of the Method of Moments. One of the advantages of this approach over other methods such as a complete finite difference or finite element

model is that the integral equation formulation provides an analytical relationship between the current distribution and the parameters of the model, and thus provides insight into the effects of the various parameters on the current distribution. In finite element and finite difference models such insight can only be obtained through numerical experimentation involving a large number of costly computer runs and cannot be generalized for other parameter sets.

2.11 Physiological Background - Nerve Excitation.

The physiological basis for electrical stimulation of a nerve is discussed in this section.

2.11.1 Stimulation Threshold: The Strength-Duration Curve.

The inside of excitable tissues such as nerve or muscle cells is actively polarized at a potential of about -70mV relative to the extracellular medium, when the cells are in their resting state. By applying an electric field of sufficient strength and correct orientation to the cell, a depolarization of the membrane can be achieved. The threshold for nerve cells is much lower than that for muscle cells, so that when the nerve is intact, muscle stimulation is actually achieved by stimulating the nerve innervating the muscle. This is also desirable since it leads to a more efficient and coordinated contraction of the muscle than can be achieved by directly stimulating muscle fibers. Depolarization of the nerve results in the initiation of an action potential which propagates along the nerve to the motor endpoints where the stimulus is transferred to the muscle fibers.

The threshold levels of pulse duration or amplitude to excite nerve or muscle cells can be expressed via the characteristic strength-duration curve, an example of which is shown in Figure 4.

Figure 4: Strength-Duration curves of the four major excitatory responses for M. Triceps Brachii of healthy males as obtained with 20 Hz monophasic current pulses. (A) sensory threshold, (B) motor threshold, (C) pain threshold, (D) maximal tolerable pain response. (Adapted from Alon et al. [2])

Geddes et al. [22] have shown that such curves can be approximated by

$$I = \frac{I_{\infty}}{1 - e^{-d/\tau}} . \quad (2.15)$$

The basic phenomenon reflected by this curve is the following: as the duration d of the stimulating pulse is decreased, higher current levels are required to reach threshold.

The strength-duration curve is usually characterized by the *rheobase* (I_{∞}) and the *chronaxie* (d_c) which is related to the membrane time constant (τ) by $d_c = \tau \ln 2$. The rheobase is the minimum DC current required to produce stimulation. In practice, it is determined from an extrapolation of the data obtained with long pulses. The chronaxie is the minimum pulse duration required to activate the nerve with the pulse amplitude at twice the rheobase level.

The rheobase is independent of the pulse duration, but is a function of the electrode size and its location with respect to the excitable tissue. It also depends on the direction of the electric field with respect to the target nerve. On the other hand, the membrane time constant τ , depends only on the type of tissue being stimulated and determines the shape of the curve for a particular excitable tissue and a particular current waveshape.

In general, the depolarization threshold for a fixed pulse waveform and repetition frequency, is a function of the spatial distribution of the field in the conducting medium surrounding the nerve and the interaction between the nerve and the extracellular field.

2.11.2 Muscle Force: The Recruitment Curve.

Not all nerve fibers in a nerve bundle have the same threshold. Larger motor fibers have a lower threshold than fibers with a smaller diameter. When the current

amplitude is increased beyond the motor threshold, more and more nerve fibers are recruited, resulting in an increase in the muscle force. When all the fibers have been recruited, a further increase in the applied current does not produce any increase in muscle strength. The relationship between the intensity of stimulation (which in the case of stimulation by rectangular pulses can be expressed in terms of either the pulse width or the pulse amplitude), and the force produced can be expressed via what has become known as a *recruitment curve*. A typical example of such a curve, obtained with various monophasic waveforms applied directly to a nerve by means of a cuff electrode, is shown in Figure 5 [27].

The recruitment curve depends on the following parameters:

- The stimulating waveform, including such parameters as the pulse frequency, cathodic pulse shape, anodic pulse shape, pulse duration, pulse amplitudes and the interpulse period between the cathodic and the anodic pulse and whether the current or voltage is held constant during primary pulse;
- The strength and condition of the particular “target nerve;”
- The electrode geometry and the location of the electrode with respect to the nerve, and
- The direction of the electric field relative to that of the nerve.

2.11.3 Electrode to Nerve Coupling.

The work on the prediction of the electric field due to transcutaneous stimulation is based on the view that the electrode to nerve coupling can be treated in terms of two conceptually different aspects of the problem:

Figure 5: Recruitment curves for different waveshapes applied to an exposed cat gastrocnemius muscle. (Adapted from Gorman et al. [27])

1. the coupling factor(s) between the applied current and the field distribution close to the target nerve , and
2. the relationship between the field along the nerve and the excitation threshold.

Unless the electrodes are at a distance from the nerves that is comparable to their diameters, the effect of the nerves on the field distribution can be neglected, since they constitute a minute part of the conducting medium.

On the other hand, the nerve-field interaction depends on the particular kind of nerve (myelinated or unmyelinated), the size of the nerve and its (nonlinear) membrane properties. This interaction is not specifically addressed in the present work, but has been the object of several studies which will be discussed in this section.

The electric field is considered to act as the “intermediate” parameter between the electrode and the nerve.

These two aspects can each be studied separately, but need to be combined to obtain any information on the excitation of a particular nerve from a prediction of the electric field due to remote electrode stimulation.

The first of these aspects is a problem that is almost entirely physical and mathematical and can be analyzed by field-theory concepts that have been widely developed and successfully applied in the study of volume conductor problems.

The interaction between the nerve and the extracellular field has been studied by theoretical models which are based on observations obtained with voltage clamp experiments on an isolated nerve. The nerve models are usually derived from the well known Hodgkin-Huxley [39] and Frankenhaeuser-Huxley [20] equations which

represent the excitation mechanisms of unmyelinated and myelinated nerve fibers, respectively.

McNeal [57] developed one such model to analyze the subthreshold stimulation of myelinated nerve fibers by a spherical electrode in close proximity to the nerve. He suggested that the model could also be applied to excitation by remote, e.g. surface electrodes, if the electric field in the proximity of the nerve could be related to the current and configuration of the remote electrodes.

Rattay [66] extended the McNeal model to include suprathreshold stimulation of both myelinated and unmyelinated fibers. His analysis allows one to determine the threshold and the site of excitation from the external potential and its second derivative along the direction of the nerve, which he calls the activating function responsible for the excitation. He also indicates that the influence of the axon on the extracellular potential field is not essential.

Reilly et al. [69] developed a spatially extended nonlinear node (SENN) model based on that of McNeal. It includes nonlinear Frankenhaeuser-Huxley conductances at each of several adjacent nodes and allows the simulation of a variety of electrode/neuron geometries of practical interest. Their results correlate well with strength-duration data from sensory experiments.

2.11.4 Experiments.

This section presents an overview of experiments which indicate some of the stimulation parameters that are affected by the electrode configuration.

Milner et al. [58] did experiments on normal human subjects to determine optimal stimulation parameters for lower limb stimulation by means of surface electrodes. They centered their electrodes over the motor points which were determined as the sites that gave the maximal contraction with a small (1/4 inch

diameter) monopolar search electrode and found that both the maximum force and the pain threshold were fairly independent of electrode size and that the total current was the main factor determining force and pain threshold. On the quadriceps muscle group they found that two active electrode sites were more satisfactory than one and speculated that the electric field shaping produced by this configuration resulted in more effective motor point stimulation or even some deep muscle stimulation. Alon [2] studied the effect of electrode size on the pain threshold level for normal subjects and found that the total charge, rather than the current density, was the limiting factor on subject tolerance of stimulation.

Vossius [90] has also used arrays of electrodes to produce more effective stimulation than could be achieved with monopolar surface electrodes.

Grandjean and Mortimer [28] studied the effectiveness of different electrodes on muscle recruitment by experiments on the cat gastrocnemius muscle. They noted that the following factors are affected by electrode placement (and hence by the spatial distribution of the electric field):

- 1) Selectivity as evaluated by the spillover of stimulation to adjacent muscles;
- 2) Stimulation threshold and maximum force produced;
- 3) Controllability as reflected by the change in shape and slope of the recruitment curve.

Gorman and Mortimer [27] used experiments on cats to show that the nerves are excitable over most of their length but the threshold depends on where along the nerve the excitation is applied. The motor points, which are functionally described as the points of lowest threshold for motoric stimulation, are anatomically located close to the point where the nerve enters the muscle.

Rushton [70] found that excitation depends on the angle between the nerve and the direction of the external field such that no excitation occurs when the field

is perpendicular to the axis of the nerve, i.e. when adjacent nodes on the nerve are at the same potential. This is consistent with the predictions of the Rattay and Reilly models and is taken as the working hypothesis for the present work.

CHAPTER III

THE SURFACE CURRENT DENSITY DISTRIBUTION OF A RECTANGULAR RESISTIVE ELECTRODE ON A RESISTIVE HALF-SPACE.

3.1 Introduction.

This chapter presents an investigation of the spatial distribution of the surface current density produced by rectangular resistive surface electrodes applied to the human body. A mathematical model is developed to predict the surface current density distribution of a wide range of electrode designs. The model is more general and more accurate than earlier models for rectangular resistive electrodes. Two experimental methods are presented for the measurement of the current density distribution. The two experimental methods presented and evaluated here are:

- An in-vivo method whereby the current density distribution can be inferred from a non-invasive measurement of the voltage distribution produced on such an electrode while passing current to a human subject.
- An in-vitro method utilizing an artificial conducting medium for forming an image of the surface current density distribution.

Rectangular resistive surface electrodes may be used as electrosurgical dispersive electrodes, external cardiac pacing electrodes or for functional electrical stimulation. The approach used here allows one to model several electrode shapes

ranging from square to long strip electrodes, and electrodes that have different thicknesses on different parts of the electrode and thus provides an extension of earlier models which dealt only with approximate solutions for uniform thickness electrodes [8,95].

The model implements a novel hybrid solution method where a finite difference solution for the fields in the electrode is combined with a three dimensional moment method solution for the fields in the body. The known singularity at the edge of an (infinitely conducting) electrode is specifically incorporated into the basis functions used for the moment method solution. This improves the accuracy of the solution and accelerates the numerical convergence.

It applies to electrodes of arbitrary size, width-to-length ratio, feed geometry, and thickness and can also handle electrodes that do not have a uniform thickness or uniform conductivity. The body is modelled as a uniform half-space. The half-space approximation is considered reasonable in situations where the dimensions of the electrodes are small compared to the depth and radius of curvature of the body part to which the electrodes are applied.

The effects of electrode size and shape, and of the feed configuration on the current density at the electrode-body interface are predicted and improvements in the design of surface electrodes, such as to make the current density distribution more uniform, are suggested. Present standards regarding safe operating levels for current density are based on the average current density which is only equal to the true current density if the electrodes have a constant current density. By demonstrating the possibility of (and range of parameters which lead to) hot spots either at the edge or close to the feed region, even with a perfect electrode-body contact, it is shown that the assumption of constant current density is not universally applicable. Design guidelines are provided to reduce the likelihood of hot

spots in the situations where such might otherwise occur.

Background.

In previous attempts to characterize the current density distribution of rectangular electrodes, restrictive assumptions about the electrode geometry and the nature of the underlying medium were made. Brennen [8] used a simple one-dimensional model in which it was assumed that the electrode is of infinite extent and the body a thin conducting slab. The solution presented by Brennen was based on the assumption that the central part of a long strip electrode on the human body can be equivalent to a lossy cable. This solution applied only to regions far removed from the feed point and electrode edges and failed to predict the edge-effect. Rubinstein et al. [71] did a two-dimensional analysis for a metal electrode on a conducting halfspace. The results of the model used in the present chapter gives an improved match to data measured by Brennen and reduces to that obtained by Rubinstein et al. when the electrode conductivity is taken to be much larger than that of the body.

In this chapter Laplace's equation is solved in two different regions, the electrode and the body, using two different approaches. A moment method (MM) solution using the Green's function for the potential of a point source on a conducting half-space is used for the solution of the field in the body. The field in the electrode is obtained from a single layer discretization of the electrode that is similar to that used in the finite difference (FD) method. The solution is obtained by imposing continuity of the boundary conditions on the common boundary, i.e. at the electrode-body interface. The combination of these two methods will be referred to as the hybrid FD/MM solution method. The hybrid FD/MM model can be tailored to a wide range of electrode geometries and permits a more accu-

rate representation of some practical electrodes than is currently available in the literature. Commonly used electrodes will be considered for sample calculations.

The FD approach used here explicitly models boundary conditions in the feed region and along the electrode edges. As a first order approximation and in order to limit the computational effort to a tractable level, the body is modelled as a uniform resistive half-space. The accurate representation of the different tissue layers and organs of the body requires a computationally intensive finite-element approach which is only practical for two-dimensional (2-D) models of the body [45,46]. The simplified analytic model is significantly less costly to implement and more accurate than 2-D models for predicting the current distribution in the vicinity of a three-dimensional electrode.

In the present analysis the electrode polarization impedance and contact layer impedance due to electrode gel is assumed to be negligible.

The conductivity and thickness values used for modelling the electrodes are taken from data measured off commercially available electrodes. The accuracy of the model is verified by comparison with computed results in the literature and with both in-vitro and in-vivo measurements of the current distribution of typical conductive rubber electrodes.

The results indicate that when the electrode material is highly conductive compared to the underlying medium, the current distribution approaches that of a metal electrode, exhibiting an increase in the current density toward the electrode perimeter (the so called "edge effect"). On the other hand, if the electrode material has a conductivity of the same order of magnitude as the body, the current distribution can exhibit a peak or "hot spot" close to the feed point. Furthermore, by shaping the electrode thickness and/or conductivity as a function of position and by correctly choosing the feed configuration, the nonuniformity of the cur-

Figure 6: Resistive surface electrode on a uniform half-space.

rent density distribution can be reduced under conditions where a hot spot might otherwise occur.

3.2 Formulation of the Model.

The physical setup shown in Figure 6, which represents a resistive electrode applied to a volume conductor such as the human body, is modelled by a rectangular electrode made out of a resistive material (such as conductive silicone rubber) with a uniform conductivity, σ_e , on the surface of a uniform half-space with conductivity, σ_b . Current is supplied to the electrode by means of a wire which attaches to the electrode by a metal pin. The region where the wire makes contact with the electrode is called the feed region and is modelled by a thickening of the electrode. The external boundary condition is specified by imposing a constant potential on those parts of the electrode in contact with the feed pin. In the case of an edge feed extending beyond the part of the electrode in contact with the body, the external boundary condition is given in terms of the fraction of the applied current entering each part of the feed region. The direction perpendicular to the plane of the electrodes is designated the z -direction, with increasing values

of z corresponding to increasing depth into the medium.

The electrode is designated as region 1 and the conducting half-space as region 2. Each region is taken to have a uniform conductivity (i.e. linear, homogeneous and isotropic). The current density distribution is derived from a solution of Laplace's equation for the potential distribution. In each region the solution can be reduced to a relation between the distribution of the z -directed component of the current density, $J_z(x, y)$, and the potential distribution, $\Psi(x, y)$, at the electrode-body interface. The electrode-body contact surface forms the common boundary between regions 1 and 2. An approximate solution for $J_z(x, y)$ is obtained by matching the potential distribution for region 1 to that of region 2 at a discrete number of points on the common boundary. The parameters which determine the solution in region 1 are the electrode thickness distribution, the feed configuration, and the conductivity of the electrode material.

The common boundary surface is subdivided into rectangular subdomains of dimensions $\Delta x \times \Delta y$. The subdomains are called surface patches. Initially a constant current density is assumed on each surface patch. Later a refinement of the model is introduced where the known edge singularity is included in the edge patches. The electrode is subdivided into rectangular prisms, one on each surface patch. The prisms are called elements of the region 1 model. Each element is initially assumed to have the same height, T , equal to the thickness of the electrode. Later T will be taken as a variable parameter in order to model an electrode of variable thickness.

Figure 7 shows the elements of the mathematical model for the electrode on the half-space. The 3×3 square grid is used only for illustrative purposes. In actual computations more surface patches are used and patch dimensions are chosen so that the width and length of the electrode are multiples of Δx and Δy respectively.

Figure 7: Model for resistive electrode on a uniform half-space.

The z -directed component of the current density of the n -th surface patch is taken as

$$J_{z,n}(x,y) = I_{z,n} P_n(x,y)/A , \quad (3.1)$$

where

$$\begin{aligned} I_{z,n} &= z\text{-directed current through the } n\text{-th patch,} \\ P_n(x,y) &= \text{current density distribution on the } n\text{-th patch,} \\ A &= \text{area of surface patch } (= \Delta x \times \Delta y). \end{aligned}$$

In order to introduce the modelling approach a simplified model is first used where the back-to-front voltage drop across the electrode is neglected so that the voltage at the center of the n -th source patch, V_n , can be taken to be equal to the voltage at the center and on the back of the n -th element. The external boundary condition for the first model is specified by giving the fraction, I_{in} , of the input current into each of the elements in the feed region. This is termed the current feed formulation. A more realistic boundary condition specification, called the voltage feed formulation is introduced later together with an explicit incorporation of the back-to-front voltage drop across the electrode.

Region 1 Relations.

The parameters which determine the relations between the voltage distribution in the electrode and the surface current distribution in region 1 are the electrode thickness, the feed configuration, and the conductivity of the electrode material.

In region 1 Laplace's equation is solved by a finite difference approximation of the continuity equation which is given by

$$\nabla \cdot \mathbf{J} = \frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z = 0 , \quad (3.2)$$

Figure 8: Rectangular prism element used in formulation of finite difference scheme.

where $\mathbf{J}$ is the current density vector and J_x, J_y , and J_z are its rectangular components.

In the finite difference scheme the differentials in (3.2) are replaced by difference quotients. This is equivalent to discretizing the electrode into finite elements as shown in Figure 7 and solving for the currents and voltages in a resistive network where the nodes are connected by resistors R_e . The value of R_e depends on the electrode thickness, T , the conductivity, σ_e , and the ratio $\Delta x/\Delta y$.

Consider a typical element of the electrode (See Figure 8). The FD approximation of (3.2) applied to a point at the center of the element can be written as

$$\frac{1}{\Delta x} \frac{\Delta I_x}{\Delta y \Delta z} + \frac{1}{\Delta y} \frac{\Delta I_y}{\Delta x \Delta z} + \frac{1}{\Delta z} \frac{\Delta I_z}{\Delta x \Delta y} = 0, \quad (3.3)$$

where

$$\Delta I_x = I_x^+ - I_x^-, \quad (3.4a)$$

$$\Delta I_y = I_y^+ - I_y^-, \quad (3.4b)$$

$$\Delta I_z = I_z - I_{in}, \quad (3.4c)$$

and $\Delta x, \Delta y, \Delta z (= T)$ are the dimensions of the element. The superscripts $-$ and $+$ designate components of the current entering and leaving the element, respectively.

Let V_o denote the voltage at the center of the element and $V_x^-, V_x^+, V_y^-, V_y^+$ the voltages at the centers of the four adjacent elements. From $\mathbf{J} = \sigma_e \mathbf{E}$, and the finite difference approximation for the components of $\mathbf{E} = -\nabla V$ parallel to the electrode surface, i.e.,

$$E_x = -\Delta V_x / \Delta x, \quad E_y = -\Delta V_y / \Delta y. \quad (3.5)$$

Then one can write

$$I_x^+ = -\sigma_e (V_o - V_x^+) \Delta y \Delta z / \Delta x, \quad (3.6a)$$

$$I_x^- = -\sigma_e (V_x^- - V_o) \Delta y \Delta z / \Delta x, \quad (3.6b)$$

$$I_y^+ = -\sigma_e (V_o - V_y^+) \Delta x \Delta z / \Delta y, \text{ and} \quad (3.6c)$$

$$I_y^- = -\sigma_e (V_y^- - V_o) \Delta x \Delta z / \Delta y. \quad (3.6d)$$

For edge elements one of these current components is set to zero and for corner elements two of these are set to zero. Substituting (3.6a) thru (3.6d) in (3.3) and rearranging terms results in

$$I_z = \sigma_e T \left\{ \alpha (V_x^- + V_x^+) + \frac{1}{\alpha} (V_y^- + V_y^+) - 2V_o \left(\alpha + \frac{1}{\alpha} \right) \right\}, \quad (3.7)$$

where $\alpha = \Delta y / \Delta x$. This form holds for an element surrounded by four other elements. The general form can be written

$$I_z = \sigma_e T \left\{ \alpha V_x + \frac{1}{\alpha} V_y - V_o \left(\alpha \beta_x + \frac{1}{\alpha} \beta_y \right) \right\} + I_{in}. \quad (3.8)$$

The parameters V_x, V_y, β_x , and β_y of the specific forms for each of the nine possible element locations defined in Figure 9 are given in Table 4. The sum $\beta_x + \beta_y$ correspond to the number of 'neighbors' of a particular element. The input current I_{in} is set to zero on elements that do not fall in the feed region.

Figure 9: Definition of the element locations referenced in Table 4

Table 4: Equations used in the computation of the P -matrix.

General form				
$I_z = \sigma_e T \left\{ \alpha V_x + \frac{1}{\alpha} V_y - V \left(\alpha \beta_x + \frac{1}{\alpha} \beta_y \right) \right\} + I_{in}$				
$\alpha = \Delta y / \Delta x$				
Location	V_x	V_y	β_x	β_y
1	V_x^+	V_y^+	1	1
2	$V_x^- + V_x^+$	V_y^+	2	1
3	V_x^-	V_y^+	1	1
4	V_x^+	$V_y^+ + V_y^-$	1	2
5	$V_x^- + V_x^+$	$V_y^+ + V_y^-$	2	2
6	V_x^-	$V_y^+ + V_y^-$	1	2
7	V_x^+	V_y^-	1	1
8	$V_x^- + V_x^+$	V_y^-	2	1
9	V_x^-	V_y^-	1	1

For N elements, there are N equations of the form given in (3.8). These N equations can be written in matrix form as

$$[I] = \sigma_e T[P][V] + [I_{in}] , \quad (3.9)$$

where the elements of $[I]$ are the N unknown surface currents $I_{z,n}$, $[P]$ is a $N \times N$ coefficient matrix with dimensionless elements, $[V]$ has as elements the node voltages $V_{o,n}$, and $[I_{in}]$ is the source vector with nonzero entries in locations corresponding to elements in the feed region. Notice that σ_e and T appear as a product in the scalar coefficient of $[P]$, which suggests that a modification of the electrode conductivity will affect the current density distribution in the same way as a similar change in the electrode thickness. Later when the back-to-front voltage drop is included in the solution, it will be shown that this approximation holds for a wide range of electrode conductivities. The algorithm for the computation of the elements of the P -matrix is given in Section 3.5.

Region 2 Relations.

The current distribution on the surface of the conducting half-space (region 2) is obtained from an approximate numerical solution of the following integral equation:

$$\Psi(x, y, z) = \frac{1}{2\pi\sigma_b} \iint \frac{J_z(x', y', 0)}{R} dx' dy' , \quad (3.10)$$

where

$$\begin{aligned} \Psi(x, y, z) &= \text{voltage at field point } (x, y, z) \text{ in volt,} \\ J_z(x', y', 0) &= \text{surface current density at source point } (x', y', 0) \text{ in A/m}^2, \\ \sigma_b &= \text{body (half-space) conductivity in S/m, and} \\ R &= \sqrt{(x - x')^2 + (y - y')^2 + z^2}. \end{aligned}$$

Here R is the distance between the source and field points. The function $1/(2\pi\sigma_b R)$ is the Green's function associated with a point source on the surface of a resistive half-space.

For a numerical solution by means of the method of moments, the surface current density is expanded in a set of N subsectional basis functions or source patches with unknown coefficients

$$J_z(x', y', 0) = \sum_{n=1}^N \frac{I_n}{A} P_n(x', y') , \quad (3.11)$$

where I_n , short for $I_{z,n}$ is the (unknown) current that enters the half-space through the n -th source patch (with area A), and $P_n(x', y')$ is the two-dimensional current distribution on the n -th source patch. The n -th patch is centered on (x_n, y_n) . The next step in the moment method is to obtain the inner products of both sides of (3.10) and a set of test functions $t_m(x, y)$. By choosing as test functions the following set

$$t_m(x, y) = \delta(x - x_m) \delta(y - y_m) , \quad m = 1, 2, \dots, N \quad (3.12)$$

where $\delta(x)$ is the Dirac delta function, and evaluating the integral

$$\iint \Psi(x, y, 0) t_m(x, y) dx dy \quad (3.13)$$

over the surface of the field patch centered on (x_m, y_m) , one obtains a set of N equations which are given by

$$V_m = \sum_{n=1}^N I_n Z_{m,n} , \quad m = 1, 2, \dots, N , \quad (3.14)$$

where

$$Z_{m,n} = \frac{1}{2\pi\sigma_b A_n} \int_{x_n-\Delta x/2}^{x_n+\Delta x/2} dx' \int_{y_n-\Delta y/2}^{y_n+\Delta y/2} dy' \frac{P_n(x', y')}{R_m} , \quad (3.15)$$

with

$$R_m = \sqrt{(x' - x_m)^2 + (y' - y_m)^2} . \quad (3.16)$$

In matrix notation the system of N linear equations represented by (3.14) can be written as

$$[V] = [Z_b][I] . \quad (3.17)$$

The elements $Z_{m,n}$ of the matrix $[Z_b]$ have dimensions of impedance. Hence the matrix $[Z_b]$ is termed the 'impedance matrix'.

It is convenient to write the impedance matrix $[Z_b]$ as

$$[Z_b] = \frac{1}{\sigma_b a} [Q] , \quad (3.18)$$

where

$$\begin{aligned} \frac{1}{\sigma_b a} &= \text{normalized body impedance in } \Omega, \\ \sigma_b &= \text{body conductivity in S/m,} \\ a &= \text{characteristic length} = \sqrt{\ell w} \text{ in m,} \\ \ell, w &= \text{electrode dimensions in m, and} \\ [Q] &= N \times N \text{ matrix with dimensionless elements.} \end{aligned}$$

The length a is a reference or characteristic length, so chosen to facilitate comparison of this form with the solution for a disk electrode of radius, a . The elements of $[Q]$ are given by

$$Q_{m,n} = \frac{a \Delta x}{2\pi A_n} D_{m,n} , \quad (3.19)$$

where $D_{m,n}$, defined by

$$D_{m,n} = \int_{L_x - \frac{1}{2}}^{L_x + \frac{1}{2}} d\zeta \int_{\alpha(L_y - \frac{1}{2})}^{\alpha(L_y + \frac{1}{2})} d\eta \frac{P_n(\zeta, \eta)}{\sqrt{\zeta^2 + \eta^2}} , \quad (3.20)$$

which is dimensionless and also independent of the patch dimensions. This form follows from (3.15) by use of the substitutions

$$\begin{aligned} \zeta &= L_x + 2(x' - x_n)/\Delta x , \quad \eta = L_y + 2(y' - y_n)/\Delta y , \\ L_x &= (x_m - x_n)/\Delta x , \quad L_y = (y_m - y_n)/\Delta y , \text{ and } \alpha = \Delta y/\Delta x . \end{aligned} \quad (3.21)$$

3.3 Solution for the Current Feed Formulation.

The formal solution for the unknown currents $[I]$ is obtained by substituting (3.17) in (3.9) and using the expression for $[Z_b]$ given in (3.18) to give

$$\left\{ [U] - \frac{\sigma_e T}{\sigma_b a} [P][Q] \right\} [I] = [I_{in}] , \quad (3.22)$$

where $[U]$ is the identity matrix with unit elements on the main diagonal and zero's everywhere else. This constitutes a system of linear equations for which the solution $[I]$ is obtained by standard numerical methods such as Gaussian elimination.

The equation given in (3.22) explicitly contains the ratio $(\sigma_e T)/(\sigma_b a)$ which characterizes the form of the solution. For example, if this ratio is small compared to unity, the second term of the coefficient matrix in (3.22) becomes negligible and the solution takes the following form

$$[I] \approx [I_{in}] , \quad (3.23)$$

which implies that the current enters the body mostly in the feed region, giving rise to a feed region 'hot spot'. The conditions that are conducive to this form of the solution are

- Low electrode-to-body conductivity ratio $(\sigma_e/\sigma_b \text{ small})$, or
- A thin electrode $(T/a \text{ small})$.

This prediction is verified by the numerical results computed for various values of these parameters.

3.4 Voltage Feed Formulation.

The current feed formulation is conceptually simple but somewhat artificial, since it is not known a priori how the externally applied current is distributed over

the feed region. In the voltage feed formulation a constant voltage is specified on all parts of the feed region that are in contact with the feed wire. This together with the specification that the normal derivative of the voltage is zero elsewhere on the back of the electrode constitutes a mixed boundary value problem, but is in effect a more realistic feed region model. The voltage feed formulation also permits the modeling of a metal foil attached to the central part of the back of the resistive electrode as a means of modifying the current density distribution.

At the same time as being a more realistic feed model, the voltage feed formulation includes an explicit representation of the back-to-front voltage drop across the electrode. The extent of the back-to-front voltage drop is important in the use of the voltage distribution on the back of the electrode (which can be measured non-invasively) to estimate the voltage distribution on the electrode-body interface (which can be used to compute the surface current density.) The back-to-front voltage drop is caused by the perpendicular current density component, I_z . It is assumed that the parallel components of the current in the electrode cause a linear change in I_z between the back ($z = -T$) and the front ($z = 0$) of the electrode. For example on an element in the feed region the z -directed component of the current density is taken as

$$\begin{aligned} J_z(z) &= \frac{1}{A} \left[I_z \left(\frac{T+z}{T} \right) - I_{in} \left(\frac{z}{T} \right) \right] \\ &= \frac{I_z}{A} \left[1 + \left(1 - \frac{I_{in}}{I_z} \right) \frac{z}{T} \right], \end{aligned} \quad (3.24)$$

where $A = \Delta x \Delta y$.

Let V_f, V_c , and V_s denote the voltage on the back, center, and front respectively of a particular element, and $[V_f], [V_c]$, and $[V_s]$ the corresponding matrices which contain the values of these voltages on all N elements. Integrating the z -directed component of the electric field between the back and front of the electrode

gives an expression for the back-to-front voltage drop, i.e.,

$$\begin{aligned}
V_f - V_s &= \int_{-T}^0 E_z(z) dz \\
&= \int_{-T}^0 J_z(z) / \sigma_e dz \\
&= \frac{T}{2A\sigma_e} (I_z + I_{in}) .
\end{aligned} \tag{3.25}$$

The relation between V_c and V_s is found likewise and is given by

$$V_c - V_s = \frac{T}{8A\sigma_e} (3I_z + I_{in}) . \tag{3.26}$$

For elements that are not in the feed region I_{in} is set to zero. In matrix form these relations can be written as

$$[V_f] = [V_s] + \frac{1}{2a\sigma_e} [T] \{ [I_z] + [I_{in}] \} , \text{ and} \tag{3.27}$$

$$[V_c] = [V_s] + \frac{1}{8a\sigma_e} [T] \{ 3[I_z] + [I_{in}] \} , \tag{3.28}$$

where $a = \sqrt{\ell w}$ and $[T]$ is a diagonal matrix with entries equal to the thickness at the center of each element multiplied by a/A . Since the expressions for region 1 are derived from the application of $\nabla \cdot \mathbf{J} = 0$ at the *center* of each element, it is more accurate to replace the voltage matrix $[V]$ in (3.44) with the matrix $[V_c]$ to yield:

$$[I] = \sigma_e a [P] [V_c] + [I_{in}] . \tag{3.29}$$

Likewise it is more accurate to replace $[V]$ in (3.17) with $[V_s]$, the voltage on the *surface* of region 2, to yield:

$$[V_s] = \frac{1}{\sigma_b a} [Q] [I] . \tag{3.30}$$

In equations (3.27) thru (3.30) the known quantities are $[P]$, $[Q]$, $[T]$, and $[V_f]$ and the unknowns are $[I]$, $[I_{in}]$, $[V_e]$, and $[V_s]$. An essential step in eliminating all but

one of the unknowns from the combination of these four equations, is to write $[I_{in}]$ in terms of the other variables. For this purpose (3.27) is written as

$$[I_{in}] = [F] \left\{ 2\sigma_e a[T]^{-1} ([V_f] - [V_s]) - [I] \right\} , \quad (3.31)$$

where $[F]$ is a diagonal matrix with unit entries in locations that correspond to feed region elements and zero's elsewhere. The notation $[T]^{-1}$ is used for the inverse of $[T]$. By eliminating $[V_c]$ between (3.28) and (3.29) one obtains

$$\left\{ [U] - \frac{3}{8}[P][T] \right\} [I] = \sigma_e a[P][V_s] + \left\{ [U] + \frac{1}{8}[P][T] \right\} [I_{in}] . \quad (3.32)$$

By substituting $[I_{in}]$ from (3.31) and $[V_s]$ from (3.30) the final result can be written as

$$[X][I] = \sigma_e a[Y][V_f] , \quad (3.33)$$

where

$$\begin{aligned} [X] &= [U] - \frac{\sigma_e}{\sigma_b}[P][Q] + \frac{3}{8}[P][T] + \left\{ [U] + \frac{1}{8}[P][T] + \frac{\sigma_e}{\sigma_b}[Y][Q] \right\} [F] , \\ [Y] &= 2[T]^{-1} + \frac{1}{4}[P] . \end{aligned} \quad (3.34)$$

For a thin electrode of constant thickness, T , and for large σ_e/σ_b , (3.33) can be reduced to

$$-\frac{\sigma_e}{\sigma_b}[P][Q][I] = \frac{2\sigma_e A}{T}[V_f] , \quad (3.35)$$

which has the same form as (3.22) for large σ_e/σ_b and indicates the following approximate relationship between I_{in} and V_f on the elements in the feed region:

$$V_f = I_{in}R_s , \quad R_s = \frac{T}{2\sigma_e A} . \quad (3.36)$$

This value of R_s is exactly the back-to-front resistance of an element of thickness, T , and surface area, A , for an assumed linear change in current, I_z , from the back to the front of the electrode. This result provides an internal check on the accuracy of the voltage feed formulation.

In numerical computations it was found that the current density distribution obtained from the voltage feed formulation predicted a slightly lower feed region peak than results obtained from the current feed formulation with an assumed equal division of feed current between the feed region elements.

Estimate of the Back-to-Front Voltage Drop.

A simple computation based on (3.25) is given here to provide an estimate of how the back-to-front voltage drop might affect the current density distribution predicted from the voltage distribution on the back of the electrode. Consider an element with $\Delta x = \Delta y = 5$ mm, $T = 1$ mm, $\sigma_e = 20$ S/m. Assume that the current leaving the element is $I_z = 10$ mA. First take the element to be in the feed region and have an input current of $I_{in} = 25$ mA. Let ΔV_z denote the back-to-front voltage drop across the element and ΔV_x the transverse voltage drop from the center of this element to the center of an adjacent one. From (3.25) one finds that

$$\Delta V_z = \frac{T}{2\sigma_e \Delta x \Delta y} (I_{in} + I_z) = 35 \text{ mV} . \quad (3.37)$$

Assuming a transverse current of $I_x = (I_{in} - I_z)/4$ to each adjacent element, ΔV_x is given by

$$\Delta V_x = \frac{\Delta x}{\sigma_e \Delta y T} I_x = 200 \text{ mV} \quad (3.38)$$

Hence, for a feed region element, the back-to-front voltage drop, ΔV_z , is about 20% of the transverse voltage drop, ΔV_x . Next consider a similar sized edge element outside the feed region. Setting $I_{in} = 0$ in (3.25), yields $\Delta V_z = 10$ mV. Assuming $I_x = I_z/2$ (e.g. for a corner element where $I_x = I_y = I_z/2$) yields $\Delta V_x = 250$ mV. In this case ΔV_z is 4% of ΔV_x .

It thus appears that it is acceptable to neglect the back-to-front voltage drop in the use of the (measured) voltage distribution on the back of the electrode as

an estimate of the voltage distribution on the front of the electrode. This would at worst lead to a slight underestimation of the current density peak in the feed region by overestimating the voltage drop from a feed-region element to an adjacent non feed region element.

3.5 Computation of the Elements of the P -matrix .

The P -matrix is the coefficient matrix in the region 1 relations. The element $P_{m,n}$ of the P -matrix is a dimensionless number associated with the contribution of the voltage on the n -th node (at the center of the n -th prism) to the current I_z exiting the electrode through the m -th patch. In the computation of $P_{m,n}$ the index m is called the ‘reference’ index and n the ‘test’ index. For an electrode with length ℓ and width w , subdivided into an array of $N_x \times N_y$ rectangular patches such that

$$N_x = \ell/\Delta x, N_y = w/\Delta y \quad (3.39)$$

the location of the m -th [n -th] patch on the electrode is described by the indices (i, j) [k, l] such that

$$\begin{aligned} m &= i + N_y(j - 1) , n = k + N_y(l - 1) , \\ i \text{ and } k &= 1, 2, \dots, N_x; j \text{ and } l = 1, 2, \dots, N_y . \end{aligned} \quad (3.40)$$

The indices k, l, i , and j are each incremented in nested loops (in the given order) and the corresponding values of m and n computed from (3.40). The values of $P_{m,n}$ and the neighbor counters β_x and β_y are assigned according to the algorithm given in Table 5. At the end of each pass through all the values of n , the elements on the diagonal of the P -matrix, $P_{m,m}$, are assigned and the ‘neighbor’ counters set to zero.

Table 5: Algorithm for the computation of the elements of the P -matrix.

Condition	Test criterion ⁽¹⁾	$P_{m,n}$ ⁽²⁾	β_x, β_y ⁽³⁾
x^- neighbor	$i = k - 1$ and $j = l$	α	$\beta_x + 1 \rightarrow \beta_x$
x^+ neighbor	$i = k + 1$ and $j = l$	α	$\beta_x + 1 \rightarrow \beta_x$
y^- neighbor	$i = k$ and $j = l - 1$	$1/\alpha$	$\beta_y + 1 \rightarrow \beta_y$
y^+ neighbor	$i = k$ and $j = l + 1$	$1/\alpha$	$\beta_y + 1 \rightarrow \beta_y$
End of pass	$i = N_x$ and $j = N_y$	$-(\beta_x \alpha + \beta_y / \alpha)$	$0 \rightarrow \beta_x, 0 \rightarrow \beta_y$
<p>Notes:</p> <p>(1) $N_x = \ell / \Delta x$, $N_y = w / \Delta y$; w, ℓ = electrode width, length.</p> <p>(2) $m = i + N_y(j - 1)$, $n = k + N_y(l - 1)$</p> <p>$i, k = 1, 2, \dots, N_x$; $j, l = 1, 2, \dots, N_y$; $m, n = 1, 2, \dots, N (= N_x N_y)$</p> <p>(3) Read '$\beta + 1 \rightarrow \beta$' as 'assign $\beta + 1$ to β'.</p>			

Table 6: Example P -matrix for $N_x = N_y = 3$ and $\alpha = 1$

-2	1	0	1	0	0	0	0	0
1	-1	1	0	1	0	0	0	0
0	1	-2	0	0	1	0	0	0
1	0	0	-1	1	0	1	0	0
0	1	0	1	-4	1	0	1	0
0	0	1	0	1	-1	0	0	1
0	0	0	1	0	0	-2	1	0
0	0	0	0	1	0	1	-1	1
0	0	0	0	0	1	0	1	-2

Table 6 gives an example of a P -matrix. In this example only the center element is surrounded by four neighbors. Notice that the P -matrix is symmetric, banded, and sparse. This is typical for finite difference solutions.

3.6 Variable Electrode Thickness.

An extension of the model to accomodate variable electrode thickness was motivated by the fact that the feed region of electrodes with a pin feed is usually somewhat thicker than the rest of the electrode. In an attempt to develop a more accurate representation of the feed region of such electrodes, it was found that the increased thickness in the feed region also produced a reduction in the height of the feed region current density peak. This observation indicated that it might be possible to find some ideal thickness profile that would produce a constant current density distribution. However, for the rectangular electrode model, it was not possible to formulate the problem in such a way that the ideal thickness profile could be determined either analytically or numerically. On the other hand, this provided the impetus to develop the resistive disk electrode model of Chapter 6, for which it was possible to derive an analytic expression of the ideal thickness profile.

This section introduces a simple modification of the rectangular electrode model to include the specification of an arbitrary thickness profile. In the next section it is shown that under certain conditions the use of a variable electrode conductivity can give the same effect as a variable thickness.

Let T_i and T_j denote the thickness at the centers of two adjacent elements (say two x^+ -neighbor elements) of the discretized model of the electrode. Consider the component of the current leaving the x^+ -face of the i -th element, as defined in (3.6a). Assuming a linear variation in the thickness between adjacent elements,

one can use the replacement

$$\Delta z = \frac{1}{2}(T_i + T_j) \quad (3.41)$$

for the thickness at the interface between the elements. The modified form of (3.6a) is then given by

$$I_x^{i,j} = -\frac{\sigma_e}{2} \frac{\Delta y}{\Delta x} (T_i + T_j)(V_i - V_j) . \quad (3.42)$$

Likewise, the current between two y^+ -neighbor elements (3.6c) becomes

$$I_y^{i,j} = -\frac{\sigma_e}{2} \frac{\Delta x}{\Delta y} (T_i + T_j)(V_i - V_j) . \quad (3.43)$$

Substituting these equations (and the corresponding forms for x^- and y^- neighbors) in (3.3) one can write the resulting set of N equations in matrix form as

$$[I] = \sigma_e a[P][V] + [I_{in}] , \quad (3.44)$$

where the P -matrix now incorporates the thickness variation through normalized elements such as

$$\begin{aligned} P_{i,j} &= \frac{\alpha}{2a}(T_i + T_j) \text{ if } i \text{ and } j \text{ are } x\text{-neighbors} \\ P_{i,j} &= \frac{1}{2a\alpha}(T_i + T_j) \text{ if } i \text{ and } j \text{ are } y\text{-neighbors} \end{aligned} , \quad (3.45)$$

where $a = \sqrt{\Delta x \Delta y}$ is the characteristic length and $\alpha = \Delta y / \Delta x$. In the limiting case where $T_i = T_j = T$, (3.44) reduces to the simpler form for a constant thickness electrode given in (3.9).

3.7 Variable Electrode Conductivity.

As an alternate to varying the electrode thickness, the current distribution can also be controlled by varying the electrode conductivity. Here variable electrode conductivity is considered for an electrode with a constant thickness T to

show the equivalence between varying electrode conductivity and varying electrode thickness.

Let σ_i and σ_j denote the electrode conductivity at the centers (x_i, y_i) and (x_j, y_i) of two adjacent elements (in this case two x^+ -neighbors). For a linear change in the conductivity, the conductivity as a function of position can be written as

$$\sigma(x) = \frac{1}{x_i - x_j} [\sigma_i(x - x_j) - \sigma_j(x - x_i)] , \quad x_j \leq x \leq x_i . \quad (3.46)$$

The voltage difference between the centers of the two elements is obtained from an integration of the x -component of the electric field, i.e.,

$$V_j - V_i = - \int_{x_j}^{x_i} E_x dx = - \int_{x_j}^{x_i} \frac{J_x}{\sigma(x)} dx \approx - \frac{I_x^+}{T \Delta y} \int_{x_j}^{x_i} \frac{dx}{\sigma(x)} . \quad (3.47)$$

Using Dwight [16, 90.1] to evaluate the integral gives

$$\begin{aligned} V_j - V_i &= \frac{\Delta x}{\Delta y} \frac{1}{T(\sigma_j - \sigma_i)} \ln \left(\frac{\sigma_j}{\sigma_i} \right) I_x^+ \\ &= \frac{\Delta x}{\Delta y} \frac{2}{T} \frac{I_x^+}{(\sigma_j + \sigma_i)} \left[1 + \frac{1}{3} \left(\frac{s-1}{s+1} \right)^2 \frac{1}{5} \left(\frac{s-1}{s+1} \right)^4 + \dots \right] , \end{aligned} \quad (3.48)$$

where $s = \sigma_j/\sigma_i$. The series expansion of $\ln(s)$ follows from Dwight [16, 601.7].

Neglecting the higher order terms in the series gives the result:

$$V_j - V_i = - \frac{\Delta x}{\Delta y} \frac{2}{T} \frac{I_x^+}{(\sigma_i + \sigma_j)} , \quad (3.49)$$

which has the same form as (3.42), but with conductivity and thickness interchanged. This result can also be obtained by substituting the average conductivity

$$\sigma_{av} = \frac{1}{2}(\sigma_i + \sigma_j) \quad (3.50)$$

for σ_e in (3.6a). The range of σ_i/σ_j for which the error due to this approximation is less than say 5% can be estimated by setting the first higher order term

Figure 10: Surface patch geometry used in the evaluation of the elements of the impedance matrix $[Z_b]$ for rectangular electrodes.

in (3.48) equal to 0.05. The result is

$$\frac{1}{2.3} < \sigma_j / \sigma_i < 2.3 . \quad (3.51)$$

Thus within these bounds a variation in electrode thickness (for a constant conductivity electrode) is equivalent to a similar variation in electrode conductivity (for a constant thickness electrode). In practice it might be possible and preferable to vary both conductivity and thickness to obtain a more uniform current density distribution.

3.8 Computation of the Half-space Impedance Matrix $[Z_b]$.

Introduction.

This section gives the detail of the numerical evaluation of the elements of the impedance matrix $[Z_b]$ defined in (3.15). Initially a constant current density is assumed on all the surface patches, including those along the edge of the electrode. In this case the elements of the impedance matrix can be expressed in closed form. Later a modified form of the impedance matrix is defined. The modification involves replacing the edge patches by modified patches which incorporate the known edge singularity. These patches will be termed singular edge and corner patches. For those elements of $[Z_b]$ which involve a source patch along the edge, the corresponding surface integral can be reduced to one requiring only a one-dimensional numerical integration. The advantage of the approach using singular edge patches is that it produces more accurate results than with ordinary constant current density edge patches. This will be demonstrated by means of the numerical results.

Definitions.

Each element of the impedance matrix corresponds to the mutual impedance between two surface patches. The geometry used in this derivation is shown in Figure 10. Each of the N patches has dimensions $\Delta x \times \Delta y$. The n -th patch is centered at (x_n, y_n) . The center of the source patch is at distances L_x and L_y along the rectangular axes from the center of a coplanar field patch. The mutual impedance between the n -th source patch and m -th field patch is given by the ratio V_m/I_n , where V_m is the voltage at the center of the m -th field patch due to the current, I_n , entering the body through the n -th source patch when the

current through all other patches is set to zero. The diagonal terms of $[Z_b]$ are all identical and correspond to the self-impedance of each patch. The self-impedance is defined as the ratio of the voltage at the center of the patch to the current entering the body through the same patch.

If all the surface patches are taken to have constant current density $[Z_b]$ is a symmetric matrix. This is because $D_{n,m} = D_{m,n}$, which can be demonstrated as follows: From (3.20) it follows that

$$D_{n,m} = \int_{-L_x - \frac{1}{2}}^{-L_x + \frac{1}{2}} d\zeta \int_{-L_y - \frac{1}{2}}^{-L_y + \frac{1}{2}} d\eta \frac{P_m}{\sqrt{\zeta^2 + \eta^2}} . \quad (3.52)$$

By using the transformation:

$$\zeta' = -\zeta , \quad \eta' = -\eta \quad (3.53)$$

together with the fact that $P_m = P_n$ for constant current density patches, this expression reduces to that for $D_{m,n}$. However, when all source patches are not identical, as when the edge patches are taken to have a singular current density distribution, $[Z_b]$ would no longer be symmetric since then there would be combinations of m and n for which $P_n \neq P_m$.

Constant Current Density Surface Patches.

For constant current density surface patches, the source distribution is taken as

$$P_n(x', y') = \begin{cases} 1 & \text{for } |x' - x_n| < \Delta x/2, \\ & |y' - y_n| < \Delta y/2 \\ 0 & \text{otherwise} \end{cases} . \quad (3.54)$$

Then $D_{m,n}$ is given by

$$D_{m,n} = \int_{L_x - \frac{1}{2}}^{L_x + \frac{1}{2}} d\zeta \int_{\alpha(L_y - \frac{1}{2})}^{\alpha(L_y + \frac{1}{2})} d\eta \frac{1}{\sqrt{\zeta^2 + \eta^2}} . \quad (3.55)$$

Note that upper and lower limits of each integral are either both positive, both negative or symmetric since L_x and L_y are multiples of Δx and Δy respectively. The quadratic form of the integrand in (3.20) results in a positive value for the integral irrespective of the signs of the limits. The case where both limits are negative can be written in terms of an equivalent expression with both limits positive, because of the symmetry of the matrix $[D]$. When either L_x or L_y are zero the limits are termed semi-symmetric. When both L_x and L_y are zero the limits are symmetric and the corresponding elements lie on the main diagonal of $[D]$ and hence define the self-impedance terms of $[Z_b]$.

Analytic Forms.

For constant current density source patches the integral for $D_{m,n}$ can be expressed in closed form by utilizing Dwight [16, Eq. 200.01] such that

$$\int dy (x^2 + y^2)^{-\frac{1}{2}} = \sinh^{-1}|y/x|, \quad (3.56)$$

and [16, Eq. 731.0]

$$\int \frac{\sinh^{-1}t}{t^2} dt = - \left\{ \frac{\sinh^{-1}t}{t} + \ln \left(\frac{1 + \sqrt{1+t^2}}{t} \right) \right\}. \quad (3.57)$$

Since $\sinh^{-1}t = \ln(t + \sqrt{1+t^2})$ for $t > 0$, it is convenient to define the function

$$F(x, y) = x \ln \left(\frac{y+s}{x} \right) + y \ln \left(\frac{x+s}{y} \right); \quad x > 0, y > 0, s = \sqrt{x^2 + y^2}. \quad (3.58)$$

This allows one to write the integral on x as

$$\int_{x_1}^{x_2} \sinh^{-1}|y/x| dx = |F(x_2, y) - F(x_1, y)|. \quad (3.59)$$

Table 7 gives a summary of the resulting equations for $Z_{m,n}$.

Table 7: Expressions for the elements $Z_{m,n}$ of the impedance matrix $[Z_b]$ for constant current density source patches.

General form:		
$Z_{m,n} = \frac{1}{\sigma_b a} Q_{m,n} , \quad Q_{m,n} = \frac{1}{2\pi} \frac{a \Delta x}{A_n} D_{m,n} , \quad a = \sqrt{\ell w}$ $D_{m,n} = \int_{x_1}^{x_2} d\zeta \int_{y_1}^{y_2} d\eta (\zeta^2 + \eta^2)^{-\frac{1}{2}}$		
Condition	Limits (1)	$D_{m,n}$ (2)
General terms	$x_2 > x_1 > 0 , \quad y_2 > y_1 > 0$	$ F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1) $
X-neighbors	$y_1 = -y_2 = \frac{\alpha}{2} , \quad x_2 > x_1 > 0$	$2 F(x_2, y_2) - F(x_1, y_2) $
Y-neighbors	$x_1 = -x_2 = \frac{1}{2} , \quad y_2 > y_1 > 0$	$2 F(x_2, y_2) - F(x_2, y_1) $
Self-terms	$x_1 = -x_2 , \quad y_1 = -y_2 = \frac{\alpha}{2}$	$4 F(x_2, y_2) $
Far-field terms	$\frac{L_x^2 + (\alpha L_y)^2}{1 + \alpha^2} > 10$	$\frac{\alpha}{\sqrt{L_x^2 + (\alpha L_y)^2}}$
Definitions: (1) : $x_1 = L_x - 1/2, \quad x_2 = L_x + 1/2, \quad L_x = x_m - x_n ,$ $y_1 = \alpha(L_y - 1/2), \quad y_2 = \alpha(L_y + 1/2), \quad L_y = y_m - y_n , \quad \alpha = \Delta y / \Delta x .$ (2) : $F(x, y) = x \ln \left(\frac{y+s}{x} \right) + y \ln \left(\frac{x+s}{y} \right) ; \quad x > 0, y > 0, s = \sqrt{x^2 + y^2} .$ See Figure 10 for the definitions of $\Delta x, \Delta y, x_m, x_n, y_m, y_n, \ell,$ and w .

Figure 11: Comparison of the exact and approximate forms of $D_{m,n}$ for constant current density surface patches.

Approximate Form.

When the source and field patches are separated by a distance that is large compared to the size of the patches, the integral can be approximated by

$$D_{m,n} \approx \frac{\alpha}{\sqrt{L_x^2 + \alpha(L_y)^2}}, \quad (3.60)$$

This approximate form is obtained by assuming that $L_x \gg 1, L_y \gg 1$ so that $\sqrt{\zeta^2 + \eta^2} \approx \sqrt{L_x^2 + (\alpha(L_y)^2)}$ over the range of integration in (3.20). The same result is obtained by assuming that the current (of strength $I_n = \Delta x \Delta y$) is concentrated in a point at the center of the source patch.

Figure 11 shows the approximate and exact forms of $D_{m,n}$ as a function of normalized distance $R/\Delta x$ for $L_y = 0$ and $\Delta x = \Delta y$. For $R > 3\sqrt{(\Delta x)^2 + (\Delta y)^2}$,

the difference between the exact and approximate forms is less than 1%. The approximate form not only serves to verify the accuracy of the exact form, but its use leads to significant savings in computation time.

3.9 Modification of the MM Solution for the Half-Space to Incorporate the Edge Condition.

This section presents the derivation of a modification for some elements of the impedance matrix $[Z_b]$ in (3.17) to resolve a known anomaly in the solution. The anomaly was first pointed out by Wilton et al. [97] who have shown that when pulse basis functions and point matching are used in a two-dimensional MM solution for an infinitely conducting strip electrode, the computed current density is typically (20 to 40%) too high at the matching point closest to the edge, the exact amount depending on the width of the basis functions. This anomalous behaviour of the MM solution was resolved by their use of a special edge basis function. In the present case the same approach was used but adapted to the three-dimensional problem.

Background.

One of the ways to test the accuracy of the region 2 equations (and the computer code written to implement these equations) is to consider the special case of an infinitely conducting electrode. For such an electrode it is safe to assume that the whole electrode is at the same voltage level. However, even for this special case, there is no known exact solution with which to compare the MM solution which follows from (3.17) with all elements of $[V]$ taken to be constant. There are however two related problems for which there are exact solutions. These are the circular disk electrode [40] and the two-dimensional (infinitely long) strip electrode [81]. In both these cases the electrode is assumed to be infinitely thin and the quasistatic

surface current density distribution is of the form

$$J_z/J_{ave} = \frac{c}{\sqrt{1 - (x/a)^2}}, \quad (3.61)$$

where J_{ave} is the average current density, x the distance from the center of the electrode, a the distance from the center to the edge. The constant c is $1/2$ (or $2/\pi$) for the disk (or strip) electrode. Note that the current density has a singularity at the edge $x = a$. This singularity, which is characteristic of the static electric field near the edge of any flat plate [41] is assumed to occur also at the edge of an infinitely conducting rectangular electrode. To account for this behaviour in the moment method solution it is incorporated into the current density distribution assumed on the source patches at the edge and corners of the rectangular electrode.

Edge and Corner Patch Formulation.

To simplify the analysis consider only square patches of side dimension Δ . The special patches which incorporate the edge singularity will be termed singular patches. Two kinds of special patches will be considered: edge patches and corner patches. Figure 12 gives a diagram of the singular edge and corner patches together with the ordinary constant current density patches used on the rest of the electrode.

For edge patches the current density distribution $P_n(\zeta, \eta)$, as used in (3.20) is replaced by

$$P_n^e(\zeta, \eta) = \frac{1}{\sqrt{2}\sqrt{1 - 2\zeta/\Delta}}, \quad (3.62)$$

where ζ is the distance from the center of the patch to the singular edge. This function has the same behaviour as (3.61) at the edge and is simpler to integrate over the remainder of the patch. The factor $\sqrt{2}$ is a normalization factor to give a unit current through the patch. The reason for using rounded corner patches is because the numerical integration of (3.20) using a product of two edge distributions

Figure 12: The three different current density patches used in the moment method formulation for a rectangular electrode.

as defined in (3.62) presented numerical instability. Because of the double singularity at the corner the integral strongly depended on the order of the quadrature. The singularities associated with the singular edge and corner points are avoided by using a product rule with Gauss-Legendre quadrature which does not sample the integrand at the edges of the domain of integration. Since there is no known exact solution for the static fields at a 90° corner of a rectangular electrode on a conducting half-space which could be used for the corner patches, a special patch is chosen with a rounded corner. (The perfectly square corner is a theoretical problem. In practice any corner has some extent of rounding). The current distribution on the rounded part of the corner patch is taken to be the same as that of a disk electrode with radius Δ . On the rest of the corner patch it is taken as the product of two edge distributions, i.e.

$$P_n^c(\zeta, \eta) = \begin{cases} \frac{0.96}{\sqrt{2}\sqrt{1 - (2r/\Delta)^2}} \text{ for } r = \sqrt{\zeta^2 + \eta^2} \leq \Delta/2 \text{ and } \zeta, \eta > 0, \\ \frac{0.96}{\sqrt{2}\sqrt{1 - 2\zeta/\Delta}\sqrt{1 - 2\eta/\Delta}} \text{ otherwise.} \end{cases} \quad (3.63)$$

For the computation of the self-impedance terms the singularity associated with the matching point at the center of the patch is removed by first implementing the rectangular to polar coordinate transformation

$$\zeta = r \cos \theta, \quad \eta = r \sin \theta. \quad (3.64)$$

The Jacobian of this transformation exactly cancels the match point singularity and results in an integral that can be done numerically (See Appendix A).

Application.

The modified Z -matrix was obtained by multiplying those elements of the matrix that involve edge and corner patches by a correction factor. The correction

factor is essentially the ratio of the integrals with ordinary and singular source patches. Appendix A gives the detail of the equations used for the modified Z -matrix. The modified Z -matrix formulation was used for all the numerical results in Section 3.10 except those that illustrate the improvement due to the modification.

3.10 Numerical Results .

A computer code has been developed using the preceding formulation to determine the surface current density distribution and the surface voltage distribution for various electrode parameters. Numerical results are presented to demonstrate the accuracy of the code, and to illustrate the effects of variations in the electrode conductivity, electrode thickness and feed configuration. The perfectly conducting square electrode is considered first, then a perfectly conducting rectangular electrode. The solution for the perfectly conducting electrode is a good approximation when the ratio σ_e/σ_b is on the order of 1000. Next some examples of conductive rubber electrodes presented. The effect of variations in σ_e/σ_b is presented for both a center fed and an edge fed electrode. Finally the effects of variations in the thickness of the electrode and in the extent of the feed region are shown. Additional numerical results on the voltage distribution along the back of the electrode appear together with corresponding experimental results in the next section.

3.10.1 Perfectly Conducting Square Plate Electrode.

The computed results for an infinitely conducting square plate are presented in order to demonstrate the accuracy of the code. No exact solution is available, but numerical data on the capacitance of a square plate in free space have been obtained via the method of moments using collocation and square constant charge

Figure 13: Capacitance of a square plate in free space as computed with constant charge density patches and with singular edge patches.

Figure 14: Normalized surface current density distribution for a square plate of infinite conductivity on a resistive half-space.

Table 8: Normalized capacitance of a square plate in pF/m.

No. of subareas	Method 1	Method 2
N	C/w	C/w
9	36.81	39.79
16	37.75	40.15
25	38.32	40.34
36	38.74	40.39
47	39.00	40.45
100	39.53	40.54
121	39.64	40.55
256	40.00	40.61
<p>Note:</p> <p>Method 1 = constant charge patches</p> <p>Method 2 = singular edge patches</p>		

subdomains [36, p. 28]. These data can be compared to the resistance of a square plate of infinite conductivity on a resistive half-space by use of the relation

$$\frac{1}{R} = \frac{1}{2} \frac{\sigma}{\epsilon_0} C, \quad (3.65)$$

where R is the resistance of a square plate on a half-space of conductivity σ and C the capacitance of the same plate in free space with permittivity ϵ_0 [80, p. 258].

Table 8 and Figure 13 illustrate the convergence rate of the square plate capacitance obtained from (3.65) and the value of R computed from the singular edge patch formulation. The solution for singular edge patches is compared with Harrington's solution for ordinary square constant charge patches. From this graph it appears that the asymptotic result for $N \rightarrow \infty$ is 40.66 pF/m.

Harrington [36, p. 28] gives 40pF/m as a good estimate of the capacitance of a square plate (the true capacitance is obtained by multiplying this number by the side length in meters). This value is obtained as an extrapolation of the curve of capacitance versus inverse number of unknowns and is confirmed by the moment method solution of Rao et al. [67] who used triangular patches with constant charge density. The asymptotic result computed here (40.66pF/m) is within 0.07% of the asymptotic value of 40.69pF/m obtained by Okon and Harrington [62] by the use of a nonuniform subdivision of a square plate into continuously graded constant charge subdomains.

A capacitance of 40.66pF/m is equivalent to a normalized resistance value of $R\sigma w=0.4355$. The normalized resistance can also be written as $R\sigma\sqrt{A}$, where A is the surface area of the electrode. The corresponding value of $R\sigma\sqrt{A}$ for a circular disk electrode on a uniform resistive half-space ($R=1/(4\sigma a)$) is 0.4431. This is within 1.7% of the value for the square electrode. This suggests that $R\sigma\sqrt{A}=0.44$ is a useful approximation for a perfectly conducting electrode of any shape or size

Table 9: Computation times in CPU seconds of modified MM solution for a square electrode on a VAX 8550.

Operation	$N = 121$	$N = 256$
Computation of $[Z_b]$	3	6
Solving for $[I]$	5	28

applied to a resistive half-space.

Note that the approximation with the singular edge patches converges faster and is closer to the asymptotic result than the ordinary pulse approximation for all values of N . This is taken as an indication that the modified solution presented here is more accurate than the solution with ordinary edge patches.

Figure 14 shows the normalized surface current density along the centerline of square plate of infinite conductivity compared with the exact solution for an infinitely conducting circular disk with diameter equal to the side length of the square plate. The markers show the computed current density distribution at the centers of the subdomains. The exact result for a circular disk is shown for comparison. Notice how the solution with ordinary pulse basis functions overestimates the current density near the edge. At the center of the edge patch the ordinary solution gives a value of J_z/J_{ave} that is 35% above that of the modified solution.

Computation time.

Table 9 gives typical computation times for finding the elements of the impedance matrix $[Z_b]$ and for solving the system of equations for an infinitely conducting square plate on a VAX 8550. The numerical integration involved in finding the elements of $[Z_b]$ corresponding to singular edge patches was performed using a product rule with Gaussian quadratures of order 2 and 3 respectively. For

edge patches this gave an accuracy within 0.5% of that obtainable with much higher order quadratures. For corner patches a 10 x 10 product rule using Gaussian quadrature gave sufficient accuracy. The numerical integration used for edge patches resulted in the computation of $[Z_b]$ taking about twice as long as for ordinary constant current density edge patches. The subroutine SIMQ of the SSP subroutine library was used for solving the system of equations. The computation of the the elements of the electrode impedance matrix $[Z_e]$ takes about one fifth of the time for the $[Z_b]$ -matrix. The computation time could possibly be reduced by using the symmetry of the problem. The $[Z_e]$ -matrix is symmetrical, but the $[Z_b]$ matrix is not since the voltage due to an edge patch at the center of some other test patch is not the same as the voltage at the edge due to a (nonsingular) central patch.

Display Format.

The results in the remainder of this section are presented in the form of surface plots, where the height of the surface plot is proportional to the normalized current density J_z/J_{ave} at the center of each subdomain. Contours are drawn on the surface plots at unit intervals of J_z/J_{ave} . The dark line contours are drawn at the level of the average current density ($J_z/J_{ave} = 1$). The actual extent of the electrode is shown by a line drawn around the base of the surface plot. (The edge of the surface plot corresponds to the centers of the subareas along the electrode perimeter). The surface plots were produced using subroutines developed by the National Center for Atmospheric Research (NCAR), Boulder, Colorado, and adapted by the author for use on a VAX computer.

Figure 15: J_z/J_{ave} for an infinitely conducting rectangular plate on a resistive half-space. (a) ordinary MM solution with constant current density edge patches, (b) modified MM solution with singular edge patches.

3.10.2 Comparison of Ordinary and Modified MM solution.

Figure 15 illustrates the effect of incorporating the edge condition into the moment method solution for a rectangular electrode. It shows the solution for ordinary rectangular constant current density basis functions (a) together with the modified solution where the edge singularity is incorporated into the basis functions used along the edge of the electrode. The electrode is taken as an infinitely conducting plate with dimensions 100 x 50 mm. With the 11 x 23 grid of rectangular subareas used here the edge of the surface plots come within 2 mm from the edge of the electrode. In these plots the spacing between contours is 0.25. These results indicate the extent of the difference between the ordinary and modified solutions. The modified solution results in a lower normalized current density distribution both at the center and along the edges. At the center the difference is about 5%, but within 2% from the edge of the electrode, the difference between the two solutions ranges from 30 to 40% with larger differences occurring closer to the corners.

3.10.3 Finite Conductivity Electrodes.

The numerical results presented in the following sections illustrate the use of the code for the prediction of the current density distribution for various electrode geometries and feed configurations. The specific examples are based on the dimensions and properties of two commercially available conductive rubber electrodes (Medtronic Model #3793 and #3791 TENS electrodes as shown in Figure 16. The conductivity of the electrode material used for these electrodes was determined experimentally and found to be in the range 10 to 20 S/m with a uniformity of about 5% for any specific electrode. The same electrodes are used for in-vitro and in-vivo measurements of the surface current density. The experimental results are

Figure 16: Medtronic Conductive rubber electrodes used in experiments and sample computations.

presented in a later section. The singular edge patches shown in Figure 12 are used for all numerical results except where noted otherwise.

3.10.4 Variations in σ_e/σ_b .

Figure 17 shows the predicted surface current density distribution for a Medtronic #3791 edge fed electrode for various values of σ_e/σ_b . The dots on the thickness profile indicate the location of the patches through which current enters the electrode. It is assumed that 2/3 of the current enters the electrode through the two central patches and the remainder through the two outer patches. This feed model represents the current entering the electrode from the extended part of the electrode designed to hold the feed pin. For the computed results, a 10x7 grid with rectangular subdomains and singular edge patches is assumed. As σ_e/σ_b increases from 10 to 100 the feed region peak disappears and the peak current density shifts to the perimeter of the electrode, ultimately approaching the current distribution predicted for an infinitely conducting electrode.

Figure 18 shows the predicted surface current density distribution for a Medtronic #3793 center fed electrode. In this case an almost uniform current density distribution is obtained for $\sigma_e/\sigma_b = 100$. These results will be used to compare with the in-vitro measurements presented in the next section.

3.10.5 Thickness and Feed Variations.

Figure 19 illustrates the predicted surface current density for various modifications of a center fed rectangular electrode and the resulting effects on the surface current density distribution. In each case the electrode thickness profile and feed configuration is shown on the left and the corresponding normalized surface current density (J_z/J_{ave}) on the right. The dots on the thickness profile indicate the

Figure 17: J_z/J_{ave} for Medtronic #3791 edge fed electrode for various values of σ_e/σ_b : (a) $\sigma_e/\sigma_b=100$, (b) $\sigma_e/\sigma_b=30$, (c) $\sigma_e/\sigma_b=10$.

Figure 18: J_z/J_{ave} for Medtronic #3793 center fed electrode for various values of σ_e/σ_b : (a) $\sigma_e/\sigma_b = 8$, (b) $\sigma_e/\sigma_b = 20$, (c) $\sigma_e/\sigma_b = 100$, (d) $\sigma_e/\sigma_b = 1000$.

Figure 19: J_z/J_{ave} for various modifications of a center fed rectangular resistive electrode. The figures on the left represent electrode thickness and feed configuration, those on the right the normalized surface current density distribution for $\sigma_e/\sigma_b = 10$.

location of the feed points. These points are assumed to be at the same voltage. These points represent points in contact with a feed pin or a feed plate. Except where noted otherwise, the electrode material is assumed to have a conductivity of 10 S/m and the half-space representing the body, a conductivity of 1 S/m. For the computed results a 15x7 grid with rectangular subdomains is assumed.

Figure 19 (a) models the standard Medtronic 3793 conductive rubber electrode (100 x 48 mm) with a non-symmetric pin feed. A constant thickness of 1 mm is assumed except along the center line and in the feed region. The part between the feed pin and the electrode is taken to be 3 mm thick. The predicted current density distribution has a feed point peak with a maximum level of $J_z/J_{ave} = 7$. The maximum to minimum current density ratio (J_{max}/J_{min}) is 8.2/0.15.

In Figure 19 (b) a modified conductivity (thickness) profile is introduced. (The effect of changing conductivity or thickness is the same). This thickness profile is similar in shape to the voltage distribution corresponding to a uniform current density distribution. At the center the conductivity (thickness) is 15 times the value at the corners. Such an electrode could possibly be constructed by stacking several layers of conductive material with decreasing size and increasing conductivity. Notice the reduction in the feed region peak of the surface current density. For this configuration $J_{max}/J_{min} = 1.5/0.5$. Approximately the same results are obtained with a linear taper from the center to the perimeter.

In Figure 19 (c) a different feed configuration is presented. The feed model represents a rectangular constant potential plate affixed to the central part of the back of the electrode. In practice the plate could be a flexible metal foil. A uniform electrode thickness of 1 mm is assumed. This configuration also produces a reduction in the feed region peak compared to the standard configuration. Here $J_{max}/J_{min} = 2.3/0.3$.

3.11 Experimental results.

Two experimental methods are discussed for the determination of the surface current density of resistive surface electrode. The first is a measurement of the voltage distribution on the back of the electrode, a method that is indirect but non-invasive and can potentially be used for in-vivo measurements. The second, discussed in the next section, is an electrochemical in-vitro imaging technique.

3.11.1 Voltage Distribution on the Back of the Electrode.

This section presents a discussion of the use of the voltage distribution on the back of the electrode as a means of noninvasively determining the surface current density distribution. First a short description of the theoretical justification for this method is given. Then the experimental method for measuring such data is discussed and finally some examples of measured data are presented together with corresponding computed values.

Theory.

For resistive electrodes, the spreading of current from the feed point through the electrode and into the body causes an ohmic voltage drop which can be measured on the back of the electrode. Brennen [8] first suggested measuring the voltage distribution on the back of the electrode to derive the parameters of a model to predict the current density distribution. He considered a long strip electrode and for points not too close to the feed point or edges, he modelled the electrode as a lossy cable described by the exponential current density distribution:

$$J_z(x) = J_0 e^{-x/L} , \quad (3.66)$$

where x is the distance from the feed point along the center of the electrode, J_0 the feed point current density distribution and L the characteristic length.

Integrating this expression Brennen obtained the following theoretical expression for the voltage distribution.

$$V(x) = V_0(1 - e^{-x/L}) . \quad (3.67)$$

The value of L was determined for various frequencies of the applied signal by matching this model to the voltage measured along the midline of the back of Medtronic #3792 electrodes (200 x 38 x 0.9 mm) with the electrodes applied to both sides of the spine of a human subject.

In the present (hybrid) model the complete solution for the current density distribution (including feed point and edges) can be derived from the voltage distribution by substituting the voltage measured on the back of the electrode (at the centers of each of the subareas of a chosen grid) in the right-hand side of the matrix equation which is given by

$$[Z_b][I] = [V] \quad (3.68)$$

and then solving for the current density distribution $[I]$.

Since the elements of $[V]$ actually represent the voltage distribution on the front of the electrode, the accuracy of this method depends partly on how well the voltage distribution on the back of the electrode approximates the voltage distribution on the front of the electrode and partly on how well the properties of the underlying medium are represented in the body impedance matrix, $[Z_b]$. For the resistive half-space model of the body, the electrical characteristics are completely represented by the effective body conductivity, σ_b . The effective body conductivity was computed from

$$R = \frac{1}{2\sigma_b a} , \quad (3.69)$$

which is twice the theoretical resistance of a single monopolar perfectly conducting electrode of radius, a , applied to a uniform half-space of conductivity, σ_b . The numerical value of the effective body conductivity was determined from (3.69) and measurements of the impedance of two identical metal disk electrodes applied to symmetrical locations on the body. The result was in the range $\sigma_b = 0.01$ to 0.03 S/m.

In order to illustrate the accuracy of the hybrid model the measured data presented here are compared with predicted values for $[V]$ derived from the measured electrode properties and a value of σ_b chosen to give a good fit to the measured data. The measured data includes some of Brennen's results for the #3792 electrode as well in vivo measurements obtained by this author. The experimental method for this last set of measurements is described in the next section.

Experimental Method.

The experimental setup for measuring the voltage distribution on the back of the electrode is shown in Figure 20. The measurements were performed with two identical electrodes on symmetrical locations on the thighs of male and female volunteers. The electrodes were applied with a thin layer of electrode gel and held in place with special adhesive straps that fit over the electrode, while allowing the feed wire to protrude. A plexiglass template with a rectangular array of holes was placed against the back of the electrode. The voltage relative to the feed pin was measured by connecting a digital multimeter between the feed pin and a test probe. The sharp ended tip of the test probe was pushed through the holes in the template to make contact with the electrode. The tip of the test probe was spring loaded so that the force applied to the electrode (and hence the electrode deformation) could be kept the same at each measuring point. The measured data presented here was

obtained with a steady AC current of 1 mA at 500 Hz applied to the electrode. This current was well below the sensation threshold. The voltage between the two electrodes applied to the body was also measured. The voltage between the feed pin and some central point within the body (corresponding to a point at infinity in the model) was taken as one half the voltage between the two electrodes which is designated by V_0 . Let $V_p(x, y)$ designate the voltage measured between the feed pin and the point on the back of the electrode with local coordinates (x, y) . The desired voltage distribution $V(x, y)$ is then given by

$$V(x, y) = \frac{1}{2}V_0 - V_p(x, y) . \quad (3.70)$$

Examples and Discussion.

The first example is taken from Brennen's measurements on the Medtronic #3792 strip electrode. The other two examples are based on measurements this author obtained with the Medtronic #3791 edge fed electrode and the #3793 center fed electrode.

Brennen's Strip Electrode. Figure 21 gives Brennen's data for the voltage distribution $V_p(x)$ measured between the feed pin and points on the midline on the back of a Medtronic # 3792 electrode together. Also shown are numerical data from Brennen's 1D cable model and the 3D hybrid model described in this chapter. Note the improved fit of the 3D model.

The sensitivity of $V_p(x)$ to variations in σ_e/σ_b (as predicted by means of the hybrid model) is shown in Figure 22. The general shape of $V_p(x)$ remains the same even if σ_e/σ_b changes by a factor of 10. However, a change of 20% produces a significantly different curve. Significance is judged here on the difference between

Figure 20: Experimental setup for measuring the voltage distribution on the back of a conductive rubber electrode on a human subject.

Figure 21: Voltage distribution for the Medtronic # 3792 electrode applied to human skin.

3792 ELECTRODE
VOLTAGE DISTRIBUTION

Figure 22: Predicted voltage distribution Medtronic # 3792 electrode for various values of σ_e / σ_b .

Figure 23: Predicted current density distribution for Medtronic # 3792 electrode with $\sigma_e = 9.35 \text{ S/m}$ and (a) $\sigma_e/\sigma_b = 100$, (b) $\sigma_e/\sigma_b = 1000$.

the measured data in Figure 21 and the best-fit curve of the hybrid model. The predicted current density distributions corresponding to $\sigma_e/\sigma_b = 100$ and 1000 are given in Figure 23. From these data it seems that the measurement of the voltage distribution on the back of the electrode is sufficiently sensitive to distinguish between a feed peaked and an edge peaked current density distribution.

Edge and Center Fed Rectangular Electrodes. Figure 24 and 25 give in vivo measured data for the two electrodes in Figure 16, together with computed values based on an electrode conductivity of $\sigma_e = 10$ S/m. The value of σ_b for the best match between the measured and computed voltage distribution was in each case determined interactively, but fell well within the range of measured values of the effective body conductivity. The close correlation between measured and predicted values is quite surprising, given the simplicity of the model and the fact that the range between the minimum and maximum values of $V_p(x,y)$ is only 5 to 10 % of the voltage difference V_0 between the two electrodes. In the numerical results, the values of $V(x,y)$ were found to be inversely proportional to the assumed value of the effective body conductivity, σ_b . On the other hand, the voltage distribution relative to the feed point, $V_p(x,y)$, was found to be a function of the ratio σ_e/σ_b . The current density distributions corresponding to the voltage distributions shown in Figure 24 and Figure 25 are similar to those shown in Figure 17 (c) and Figure 18 (d), respectively. For comparison, Figure 26 shows the predicted voltage distribution for the #3793 electrode with (a) $\sigma_b = 1$ and (b) for an assumed constant current density distribution. For $\sigma_b = 1$, the predicted current density has a strong feed-region peak. Note the similar shape of (a) and the voltage distributions predicted for $\sigma_b \approx 0.01$ for which the predicted current density distribution has no feed region peak and a strong edge-peak. In (b) the

Figure 24: Measured and predicted voltage distribution for Medtronic #3791 electrode.

Figure 25: Measured and predicted voltage distribution for Medtronic #3793 electrode.

Voltage
Distribution
#3793
Electrode

(a) $\sigma_e = 10 \text{ S/m}$, $\sigma_b = 1 \text{ S/m}$

(b) Constant current density
distribution
 $\sigma_b = 0.016 \text{ S/m}$

Figure 26: Predicted voltage distribution for electrode with dimensions of Medtronic #3793 electrode with (a) $\sigma_e/\sigma_b = 10.0$ and (b) constant current density and body conductivity $\sigma_b = 0.16$.

body conductivity was assumed to be 0.016 S/m in order to facilitate comparison with the results shown in Figure 25.

3.11.2 In vitro Measurement of the Current Density Distribution.

This section describes an experimental technique for the in vitro measurement of the current density distribution of conductive rubber surface electrodes. The method was suggested by Doug Gray of Medtronic Neuro Inc. [30]. It is based on an electrochemical reaction which causes an image of the current density distribution on a mixture containing potassium iodide (KI) and starch. The first part of this section gives the detail of the the experimental method, the preparation of the KI-mixture, and the electrochemical reaction. This is followed by an example of measured data obtained by this method for the two conductive rubber electrodes shown in Figure 16.

Experimental Setup.

Figure 27 shows the experimental setup for measuring the current density distribution. The test electrode is placed on top of the gelled KI-mixture and used as the anode (positive electrode) of an electrochemical cell. A DC current source supplies the current to the test electrode. The cathode, which will be referred to as the return electrode, is a sheet of aluminum foil at the bottom of the plastic tray. On one side the return electrode extends above the KI-mixture. A lead with an alligator clip connected to this point provides electrical contact with the rest of the circuit. The current, I , is monitored with a digital multimeter. The voltage, V , between the test and return electrodes is also measured in order to estimate the conductivity, σ , of the KI-mixture. The formula for computing the conductivity is derived from the exact resistance of an infinitely conducting square electrode on a

Figure 27: Experimental setup for the in vitro measurement of current density distribution.

Figure 28: Operational amplifier current source.

Table 10: Composition of the Potassium Iodide (KI)-mixture.

Ingredients	% by weight
KI	0.16
Starch	0.20
Gelatin	0.70
Bacteriacide	0.01
H ₂ O	98.93

resistive half-space (See Section 3.10) and is given by

$$\sigma = 0.44/(R\sqrt{A}) , \quad (3.71)$$

where $R = V/I$ and A is the surface area of the electrode.

In order to limit the time for the dispersion of the image, all electrodes, including the reference electrodes, are energized at the same time. This is accomplished by using an array of operational amplifier current sources, each with the circuit given in Figure 28.

Preparation of the KI-mixture.

Table 10 gives the ingredients for the KI-mixture. The total volume of the mixture should be sufficient to give a depth of at least the width of the largest electrode. The bacteriacide prevents molding of the mixture if it is not used for some period of time.

The mixture is prepared as follows:

- 1) Weigh all ingredients accurately.
- 2) Add the KI to cold H₂O while stirring.

- 3) Add starch and bacteriacide.
- 4) Heat the mixture to boiling point while stirring constantly.
- 5) Add the gelatin while stirring.
- 6) When all the gelatin is dissolved, remove from heat and pour into the plastic container lined with the return electrode.
- 7) Let cool at room temperature, then store in a refrigerator until used.

Experimental Procedure.

The procedure for obtaining a current density image is as follows:

- 1) Clean the electrodes with 100% ethanol.
- 2) Apply a thin and even layer of electrode gell to each electrode surface (This facilitates the removal of the electrode).
- 3) Place the electrodes on the gelled KI-mixture. Be careful not to trap any air bubbles under the electrode.
- 4) Check the current in each of the electrode circuits by shorting the current sources to the return electrode through a milliammeter.
- 5) Connect the electrodes to the current source(s). Disconnect after the predetermined time.
- 6) Remove the electrodes, taking care not to disturb the surface of the KI-mixture.
- 7) Take a photograph of the image from a point directly above the surface of the KI-mixture.

How the Current Density Image is Formed.

The KI-mixture is originally translucent. When a DC current is passed through the medium, the oxidation at the anode of the iodide (I^-) in the KI

produces I_3^- ions according to the reaction

The I_3^- ions react with the starch molecules, causing the mixture to turn dark blue. The intensity of the color has a spatial distribution which is proportional to the charge density distribution which is approximately equal to the time integral of the current density distribution for short exposure times. The exposure time is defined as the time during which a DC current is applied to the medium. While the current is applied the discoloration is mostly confined to the immediate proximity of the anode. Because of the diffusion of the colored molecules, the image gradually disperses. An image of the current density distribution on the surface of the electrode is obtained by quickly removing the electrode and taking a photograph.

Exposure Parameters.

The current and exposure time is computed from

$$It = qA , \tag{3.72}$$

where

- I = current (ampere),
- t = exposure time (seconds),
- q = average charge density (Coulomb/m²), and
- A = electrode area (m²).

Example and Discussion.

Figure 29 shows a photograph of an image formed by the method described here. Figure 30 is a key to identify the various electrodes. The dimensions of the two larger electrodes are given in Figure 16. The smaller electrodes which were

Figure 29: Photograph of the current density image for the # 3791 and #3793 conductive rubber electrodes.

Figure 30: Key to Figure 29

Table 11: Parameters for current density image in Figure 29.

Electrode	A (cm ²)	q (C/m ²)	I (mA)	R (k Ω)
# 3793	46	1200	11.55	.23
# 3791	18	600	4.46	.59
a	1	1200	2.00	1.10
b	1	600	1.00	2.20
c	1	300	0.50	4.40
d	1	150	0.25	8.80
e	1	75	0.13	17.60
f	1	38	0.06	35.50

used to give reference exposures all are 1cm x 1cm. All the electrodes have a conductivity of about 10 S/m. The surface area of each of the electrodes and the exposure parameters are given in Table 11.

The image corresponds to average charge densities ranging from 100 to 1200 C/m². This range was determined by trial and error. At the low end the image is barely visible due to the translucency of the KI-mixture. At the high end it is saturated and also has noticable diffusion of the color away from the electrode. The discoloration was found to extend to a depth of at most 1.0 mm.

For both the larger electrodes one can see a feed region peak as well as the edge effect associated with the increase in the current density along the perimeter of the electrode. This compares well with the current density distribution predicted by the model (See Figure 17 (a) and Figure 18 (b)). For the # 3793 electrode the edge closest to the return electrode had a higher current density than the opposite edge. Because of the short distance between this electrode and the return electrode this can be attributed to the leading edge effect.

In addition to the expected regions of high current density on the larger electrodes, there were also dark lines traversing the low current density regions. An examination of the electrode surfaces revealed that these regions corresponded to parts of the electrode that had a rougher surface finish than the rest of the electrode. From this observation it could be concluded that surface finish plays as important a role in the current density distribution as the feed configuration.

The reference electrodes provide a qualitative scale of current density distribution. However, because of the edge effect, none of these electrodes had a uniform current density as would have been desirable for a true reference scale.

When tried with brass and silver metal electrodes it was found that the discoloration of the KI-mixture does not take place as with conductive rubber electrodes,

but that the electrode itself is oxidized. This observation might indicate that the conductive rubber electrodes are much less reactive than certain metal electrodes. Whether this is also the case when conductive rubber electrodes are compared with more inert metal electrodes such as stainless steel and platinum, remains to be investigated.

Comments.

The experimental method presented here demonstrated the existence of feed point peaks, the edge effect and the leading edge effect and thus provides a qualitative verification of the current density distribution predicted by the hybrid model. Due to the inherent limitations of the experimental method (edge-effect on reference electrodes, dispersion of the image, and limited dynamic range) it would be difficult to use it for quantitative measurements.

3.12 Chapter Summary.

This chapter presented a model for computing the surface current density distribution of rectangular resistive electrodes applied to a uniform resistive half-space model of the body. The relation between the ratio of the electrode to body conductivity (the conductivity ratio) and the surface current density distribution was demonstrated by means of computations for some specific conductive rubber electrodes. The possibility of a feed region hot spot was shown to occur when the conductivity ratio was less than 30. Experimental measurements obtained with an in vitro staining technique provided a qualitative confirmation for the predictions of the model. The model was used to evaluate the sensitivity of an indirect method for assessing the surface current density, namely the measurement of the voltage distribution on the back of the electrode. Measurements performed

by this method could be closely matched to voltage distributions predicted by the body by the judicious choice of the effective body conductivity. The value of the conductive ratio at which the best match was obtained corresponds to an almost uniform current density distribution.

CHAPTER IV

MULTIPLE ELECTRODES.

4.1 Introduction.

This chapter presents an analysis of the quasistatic current density distribution due to multiple coplanar surface electrodes applied to a homogeneous resistive half-space. The consideration of multiple electrodes not only permits one to estimate the range of validity of results obtained with monopolar electrode models but also to address a number of interesting questions relating specifically to the fields and interaction of multiple electrodes. The results are obtained by means of an efficient three dimensional moment method solution for the quasistatic field problem.

An electrode arrangement in which two or more circular disks are positioned on the surface of the body is widely used in the transmission of electrical energy to regions within the body. Although the particular application of interest here is the selective stimulation of peripheral nerves, the results also apply to the stimulation of other internal organs such as the heart and to the coupling of energy to implanted passive devices by electrodes on the surface of the body [76].

In the case of nerve stimulation there is evidence that the spatial distribution of the current density due to surface electrode stimulation combined with the natural spatial distribution of nerve fibers in a nerve bundle and of nerve bundles in a muscle might create a more physiological recruitment of muscle fibers than

with electrodes implanted proximal to a motor endpoint [56]. This could possibly be due to the fact that nerves are only stimulated if the electric field parallel to the nerve is of sufficient strength [70]. The analysis of multiple electrode fields is aimed at investigating the extent to which the current density distribution can be tailored to selectively stimulate particular nerves.

The results of this chapter provide some guidelines in answering the following questions pertaining to the use and modelling of surface electrode stimulation:

- How does the surface current distribution on each electrode depend on the distance to the other electrodes ?
- What is the current required to achieve a given electric field at a particular depth below the surface and how does this depend on the electrode configuration (i.e. electrode size and spacing) ?
- What is the difference between the fields produced by monopolar and bipolar electrodes ?
- How is the impedance between electrodes related to the electrode configuration ?
- At what distance from the electrodes is it permissible to neglect the actual current distribution on the surface of the electrodes and model the electrodes by point sources and pairs of electrodes by point dipoles ?

In Chapter 3 the surface current density was determined for monopolar electrodes. The return electrode was assumed to be large and at such a distance from the primary electrode that it had no effect on the surface current distribution. The modelling of the three dimensional current distribution of several interacting surface electrodes is complex and not well established in the literature, even

for the apparently simple case of electrodes applied to the surface of a uniform (homogeneous, linear, and isotropic) medium.

In this chapter the electrodes are taken as circular disks of infinite conductivity and the body as a uniform resistive half-space. Since this constitutes a much simplified model of human stimulation by means of multiple surface electrodes in which the inhomogeneities and anisotropy of the body is neglected, the results should be interpreted qualitatively rather than quantitatively. Despite its simplicity, this model requires a solution that is significantly more complex than that for a monopolar disk electrode. The difficulty derives from the fact that it reduces to a mixed boundary value problem which requires special numerical techniques for its solution.

One method of reducing the computational effort required to solve for the field distribution of multiple surface electrodes, is by approximating the problem to a two-dimensional one. This has often been done when the finite element method is used to obtain a numerical solution. In order to demonstrate the limitations of two-dimensional models, this chapter also includes a comparison between the results of a three-dimensional model of two coplanar disk electrodes and a related two-dimensional model. It will be shown that the extent of the so called “leading edge effect” is overestimated by two-dimensional models.

4.2 Previous Work.

Kim et al. [44,45] coined the term “leading edge effect” to describe the skewing of the current density on a surface electrode towards the direction of a proximal return electrode. Their results were obtained by means of two-dimensional finite element models which included some inhomogeneities of the body conductivity.

The difference between monopolar and multipolar electrode stimulation was addressed by Comte [13] who obtained expressions for the far-fields of monopolar and bipolar configurations of point electrodes. He used the term “stimulus spread” to describe the distance at which a given current will produce a desired field strength. He did not consider the range of validity of these approximations, but observed that “Larger electrodes require more current ... ” (to achieve a given field strength near the electrode).

An electrolytic tank solution of the electric field associated with two circular disks on the surface of a uniform resistive half-space was presented by Gold and Schuder [26]. The electrolytic tank solution was only obtained for the field in the plane perpendicular to the plane of the disks and passing through their centers and did not deal with the current density distribution on the surface of the disk electrodes. The data is presented in normalized form to facilitate extrapolation to any two-disk geometry. However, extrapolation to points other than those in the central plane, or to more than two disks is not possible from the given data.

Kobayashi [48] solved the electrostatic problem of two identical coplanar disk electrodes using an integral representation of the potential in terms of the so called “Kobayashi potentials” and four infinite sets of unknown coefficients. The solution he formulated converges rapidly (i.e. only a small number of terms in the series are significant) due to the fact that he used entire basis functions derived from the eigenfunction solution of Laplace’s equation in the integral representation of the potential. However, his results cannot be simply extended to more than two disks, nor to electrodes with different radii, and the expressions for the current distribution at points away from the electrodes present severe computational difficulty. A short outline of Kobayashi’s approach is given here together with a derivation of expressions for the surface current density distribution.

The numerical solution presented here is more general than those available in the literature. It adds to the numerical results of Kim et al. by deriving a quantitative relation between the extent of skewing of the near-field current density and the distance to a proximal electrode for the case of two circular disk electrodes. This makes it possible to determine the minimum electrode spacing for which the axisymmetric approximation of the near-field current distribution is a reasonable one. Results are also presented which indicate that the leading edge effect is overestimated by two-dimensional models.

Likewise this work improves on that of Comte by determining the range beyond which it is reasonable to approximate the electrodes by point sources in the computation of the far-field.

4.3 Theoretical Background.

The problem of finding the quasistatic field distribution of multiple disk electrodes is a difficult one and it is not amenable to a closed form solution as in the case of a single disk electrode [80, p. 124]. Approximate solutions can be obtained by various numerical methods. In the present chapter the moment method (MM) [36] is used to solve an integral equation for the current density distribution of coplanar disk electrodes. The current density is expanded in two-dimensional subsectional “pulse-like” basis functions and the unknown coefficients are determined by the method of collocation, with the constant voltage on each of the electrodes being the boundary condition that is enforced in an “averaged” sense. The solution obtained by this method will be referred to as the MM solution. It will be shown to closely approximate the exact solution in a number of test cases.

The solution for two disks based on the Kobayashi potentials will be referred to as the Kobayashi solution. The Kobayashi solution will be used to show the

extent of the leading edge effect as a function of electrode separation, to compute the interelectrode impedance for a bipolar electrode consisting of two identical disk electrodes, to compare the extent of the leading edge effect in two- and three-dimensional models, and to check the accuracy of the MM solution.

The MM solution converges slower than the Kobayashi solution (i.e. requires a larger number of basis functions for the same accuracy) but can be simply extended to more than two coplanar disk electrodes of different radii. For the Kobayashi solution the computation of the electric field at points away from the surface of the electrode is very difficult. It requires the numerical evaluation of improper integrals with oscillatory integrands, where the period of oscillation depends on the location of the observation point. For the present model such computations are simple and efficient once the set of unknown coefficients, which constitute the solution for the surface current density, has been obtained.

4.4 Formulation of the Integral Equation.

This section presents a formulation of the integral equation for the current density distribution. The integral equation is derived from a Green's function that satisfies Laplace's equation for a point source on a resistive half-space. The numerical solution of the integral equation is presented in the next section.

Consider a configuration of coplanar disk electrodes on the surface of a uniform half-space as shown in Figure 31. The electrode locations are specified in the cylindrical coordinate system (r, χ, z) . The direction perpendicular to the plane of the electrodes is designated the z -direction with increasing values of z corresponding to increasing depth into the medium. For a given surface current distribution $J_z(r', \chi', 0)$ the potential distribution can be expressed by the integral which is

Figure 31: Multiple coplanar disk electrode geometry. The k -th electrode has its center at (p_k, θ_k) .

Figure 32: Definition of source and field points in the global coordinate system.

given by

$$\Psi(r, \phi, z) = \frac{1}{2\pi\sigma} \iint \frac{J_z(r', \chi', 0)}{R} r' dr' d\chi', \quad (4.1)$$

where

$\Psi(r, \chi, z)$ = potential at field point (r, χ, z) in volt,

$J_z(r', \chi', 0)$ = surface current density at source point $(r', \chi', 0)$ in A/m²,

σ = body conductivity in S/m,

$R = |\mathbf{r} - \mathbf{r}'|$ = distance between the source and field points,

The source and field points are defined in Figure 32. The integral in (4.1) is taken over the whole $z=0$ surface. The subscript z in the notation for the surface current density is used to be consistent with the fact that the surface current density is equal to the z -directed component of the current density on the surface of the electrodes for perfectly conducting electrodes.

Equation (4.1) together with the boundary conditions

$$\Psi|_{z=0} = V_k \text{ on the } k\text{-th electrode} \quad (4.2a)$$

$$J_z|_{z=0} = 0 \text{ everywhere except on electrodes} \quad (4.2b)$$

constitute an integral equation formulation for the surface current density distribution.

4.5 Numerical Solution Procedure.

The first step in obtaining a numerical solution to the integral equation (4.1) by means of the Method of Moments is to expand the surface current density in terms of a set of N basis functions. In the present formulation a set of subsectional basis functions are used. Each disk is divided into N_ϕ equiangular sections in the ϕ direction and N_ρ sections of equal length in the ρ direction, where (ρ, ϕ) are polar coordinates in a disk centered coordinate system. This subdivision of the disk is

Figure 33: Surface patch numbering scheme for two disk electrodes.

Figure 34: The two-dimensional pulse basis functions used for disk electrodes.

Figure 35: Relation between the disk centered coordinate system and the global coordinate system.

such that it exactly covers the surface of the disk and gives results that are at least as accurate as a subdivision into quadrilateral subdomains as used by Okon et al. [62] or triangular subdomains as used by Rao et al. [67]. The subdomains are called surface patches. For K disks, the total number of surface patches is $N = KN_\rho N_\phi$. On each surface patch there is a basis function with a constant current density. With this expansion, the surface current density in (4.1) is then written as

$$J_z(r', \chi', 0) = \sum_{n=1}^N \frac{I_n}{A_n} P_n(r', \chi') , \quad (4.3)$$

where

$$\begin{aligned} I_n &= \text{current through the } n\text{-th patch,} \\ A_n &= \text{the surface area of the } n\text{-th patch,} \\ P_n &= \text{the basis function on the } n\text{-th patch.} \end{aligned}$$

For computation it is most convenient to express the basis functions in a disk centered coordinate system as shown in Figure 34. The relation between the disk centered coordinate systems and the global coordinate system is defined in Figure 35.

The disk on which the source point is located is called the source disk. The solution obtained by this division into subsectional basis functions will be referred to as the Moment Method (MM) solution. The source patches are numbered using the indices (i, j, k) such that the n -th patch is associated with (i, j, k) through the relation

$$\begin{aligned} n &= i + (j - 1)N_\phi + (k - 1)N_\rho N_\phi \\ i &= 1, 2, \dots, N_\phi , \quad j = 1, 2, \dots, N_\rho , \quad k = 1, 2, \dots, K . \end{aligned} \quad (4.4)$$

The patches are numbered in the direction of increasing ϕ , increasing ρ , and increasing electrode number (in that order), as shown in Figure 33. In the coordinate

system centered on the k -th source disk, the basis functions are defined by

$$P_n(\rho', \phi') = \begin{cases} 1 & , \text{ for } (\rho', \phi') \text{ on the } n\text{-th patch,} \\ 0 & , \text{ elsewhere.} \end{cases} \quad (4.5)$$

The potential at the disk centered field point (ρ, ϕ) can now be written as

$$\Psi(\rho, \phi, z) = \frac{1}{2\pi\sigma} \sum_{n=1}^N \frac{I_n}{A_n} \iint_{B_n} \frac{\rho' d\rho' d\phi'}{R} , \quad (4.6)$$

where B_n is the region of the n -th patch which is located on the k -th disk and defined by

$$\begin{aligned} |\rho' - \rho_n| &\leq \Delta\rho/2 , \quad |\phi' - \phi_n| \leq \Delta\phi/2 \\ \Delta\rho &= 1/N_\rho , \quad \Delta\phi = 2\pi/N_\phi . \end{aligned} \quad (4.7)$$

The center of the n -th patch is located at

$$\rho_n = \rho_i = (i - 0.5)\Delta\rho , \quad \phi_n = \phi_j = (j - 0.5)\Delta\phi . \quad (4.8)$$

The area of the n -th patch is given by

$$A_n = \rho_n \Delta\rho \Delta\phi . \quad (4.9)$$

The unknown coefficients I_n are solved by the method of collocation which implements a set of N weight functions defined by

$$w_m(\rho, \phi) = \frac{1}{\rho} \delta(\rho - \rho_m) \delta(\phi - \phi_m) , \quad m = 1, 2, \dots, N , \quad (4.10)$$

where $\delta(\cdot)$ is the Dirac delta function and (ρ, ϕ) is the field point. Using the inner product defined by

$$\langle f, g \rangle = \iint f(\rho, \phi) g(\rho, \phi) \rho d\rho d\phi \quad (4.11)$$

the continuity of the potential at $z=0$ is enforced by requiring that

$$\langle \Psi, w_m \rangle = V_m , \quad m = 1, 2, \dots, N . \quad (4.12)$$

This amounts to matching the potential given by (4.6) to the specified potential V_m at the center of each patch. The expression for the potential at the center of the m -th patch is given by

$$V_m = \Psi(\rho_m, \phi_m, z) = \frac{1}{2\pi\sigma} \sum_{n=1}^N \frac{I_n}{A_n} \iint_{B_n} \frac{\rho' d\rho' d\phi'}{R_m}, \quad (4.13)$$

for $m=1, 2, \dots, N$. Here R_m is the distance between (ρ_m, ϕ_m) (the center of the m -th patch on the field disk) and the source point on the n -th source patch. From Figure 35 it can be seen that R_m is given by

$$R_m = \sqrt{\tilde{\rho}_m^2 + \rho'^2 - 2\tilde{\rho}_m \rho' \cos(\tilde{\phi}_m - \phi')}, \quad (4.14)$$

where

$$\tilde{\rho}_m = \sqrt{X_m^2 + Y_m^2}, \quad (4.15)$$

$$\tilde{\phi}_m = \tan^{-1}(Y_m/X_m), \quad (4.16)$$

$$X_m = p_f \cos \theta_f - p_s \cos \theta_s + \rho_m \cos \phi_m, \quad (4.17)$$

$$Y_m = p_f \sin \theta_f - p_s \sin \theta_s + \rho_m \sin \phi_m. \quad (4.18)$$

Using matrix notation, the N linear equations given by (4.13) can be written as

$$[V] = [Z][I] = \frac{1}{2\pi\sigma} [A][Q][I], \quad (4.19)$$

where

$$[V] = N \text{ element excitation or potential vector,}$$

$$[Z] = \text{order } N \text{ square impedance matrix,}$$

$$[I] = N \text{ element current or solution vector,}$$

$$[Q] = \text{order } N \text{ square coefficient matrix, and}$$

$$[A] = \text{order } N \text{ diagonal matrix with elements } 1/A_n.$$

Once $[Z]$ is known and $[V]$ is specified, the solution vector $[I]$ can found from (4.19) using standard techniques of matrix algebra.

4.6 Computation of the Impedance Matrix.

The computation of the elements of the impedance matrix comprises the most demanding part of the MM solution. The equations and approximations used for the computation of these elements are derived in the next two sections.

The elements of the impedance matrix are defined by

$$Z_{m,n} = \frac{V_m}{I_n} = \frac{1}{2\pi\sigma A_n} Q_{m,n} , \quad (4.20)$$

where V_m is the potential at the center of the m -th field patch due to the current from the n -th source patch when the current through all the other patches is set to zero. The surface integrals, which comprise the elements of the matrix $[Q]$, are given by

$$Q_{m,n} = \int_{\phi_n^-}^{\phi_n^+} \int_{\rho_n^-}^{\rho_n^+} \frac{\rho' d\rho' d\phi'}{R_m} . \quad (4.21)$$

The limits of integration are

$$\rho_n^\pm = \rho_n \pm \Delta\rho/2 , \quad \phi_n^\pm = \phi_n \pm \Delta\phi/2 , \quad . \quad (4.22)$$

The elements of $[Z]$ which form the main diagonal are the self-impedance terms. The off-diagonal elements of $[Z]$ are the mutual impedance terms.

4.6.1 Computation of the Mutual Impedance Terms.

For the mutual impedance terms the integral on ρ' in (4.21) can be expressed in closed form. The integral on ϕ' is performed using numerical integration. The mutual impedance terms can be expressed as

$$Z_{m,n} = \frac{1}{2\pi\sigma A_n} \int_{\phi_n^-}^{\phi_n^+} f_{m,n}(\phi') d\phi' , \quad (4.23)$$

where $f_{m,n}(\phi')$, given by the equation

$$f_{m,n}(\phi') = \int_{\rho_n^-}^{\rho_n^+} \frac{\rho' d\rho'}{R_m} , \quad (4.24)$$

can be expressed in closed form by the use of Dwight [16, 380.011, 380.001]. The result can be written as

$$f_{m,n}(\phi') = F(\rho_n^+) - F(\rho_n^-) , \quad (4.25)$$

where

$$F(\rho') = R_m(\rho') + b_m \ln 2 |R_m(\rho') + \rho' - b_m| , \quad (4.26)$$

$$b_m = \rho_m \cos(\phi_m - \phi') , \quad (4.27)$$

and $\ln(\cdot)$ is a natural logarithm. This form applies when $|\rho_m - \rho'| > \Delta\rho/2$ and $|\phi_m - \phi'| > \Delta\phi/2$. A different form is required when $\phi' = \phi_m$, which occurs when ϕ_m and ϕ_n lie on the same radial line of the same disk. For this special case the integral on ρ' reduces to

$$f_{m,n}(\phi_m) = \int_{\rho_n^-}^{\rho_n^+} \frac{\rho' d\rho'}{|\rho_m - \rho'|} , \quad (4.28)$$

which can be expressed in closed form as

$$f_{m,n}(\phi_m) = a \left\{ \Delta\rho + \rho_m \ln \left(\frac{\rho_m - \rho_n^+}{\rho_m - \rho_n^-} \right) \right\} , \quad (4.29)$$

where

$$a = \begin{cases} -1 & , \rho_m > \rho_n^+ \\ +1 & , \rho_m < \rho_n^- \end{cases} . \quad (4.30)$$

Approximate Form of Mutual Impedance Terms.

For

$$R_m > 3 \max[\rho_n \Delta\phi, \Delta\rho] , \quad (4.31)$$

that is, when R_m is greater than 3 times the maximum dimension of the n -th patch, the approximation given by

$$Z_{m,n} = \frac{1}{2\pi\sigma R_{m,n}} \quad (4.32)$$

$$R_{m,n} = \sqrt{\tilde{\rho}_m^2 + \rho_n - 2\tilde{\rho}_m \rho_n \cos(\tilde{\phi}_m - \phi_n)} . \quad (4.33)$$

Figure 36: Exact versus point source approximation in the computation of the mutual impedance terms.

is within 1% of the exact result determined by the numerical evaluation of (4.21). This is the expression for the potential at a distance $R_{m,n}$ from a unit current point source on the surface of a half-space. Figure 36 shows a comparison of the exact and approximate forms for a patch with dimensions $\Delta\rho=0.2$ and $\rho_n\Delta\phi=0.27$.

4.6.2 Computation of the Self-impedance Terms.

The self-impedance terms of the MM impedance matrix are defined by

$$Z_{n,n} = \frac{V_n}{I_n}, \quad (4.34)$$

where V_n is the potential at the center of the n -th source patch and I_n the current through the same patch. Consider the source patch centered on (ρ_n, ϕ_n) . Without loss of generality, one can take $\phi_n = 0$. To compute $Z_{n,n}$, one needs to evaluate the integral

$$Q_{n,n} = \int_{\rho_n^-}^{\rho_n^+} \int_{\phi_n^-}^{\phi_n^+} \frac{\rho' d\rho' d\phi'}{R_n}, \quad (4.35)$$

where

$$\rho_n^\pm = \rho_n \pm \Delta\rho/2, \quad \phi_n^+ = -\phi_n^- = \Delta\phi/2, \quad R_n = \sqrt{\rho_n^2 + \rho'^2 - 2\rho_n\rho' \cos \phi'} . \quad (4.36)$$

in (4.21) becomes zero when the source and field points coincide. This causes the integrand to become singular. Because of the singularity a special approach is required for the computation of the self-impedance terms. This section presents the details of an approach which removes the singularity by the use of a coordinate transformation. The resulting expressions for the self-impedance terms require only a single numerical integration over a finite range. A second method in which a rectangular approximation of the source patches is implemented, is also presented. The rectangular approximation is less accurate than the ‘exact’ method, but has the advantage that it results in closed form expressions for the self-impedance terms. It becomes more accurate as the number of subareas are increased. The method used for the rectangular approximation is easily extended to obtain closed form expressions for the potential at any point on the surface of the source patch (See Appendix B).

The singularity in the integrand of (4.35) which occurs when the source point (ρ', ϕ') coincides with the field point $(\rho_n, 0)$ prohibits the direct evaluation of this equation by numerical integration. However, the singularity can be removed by using a transformation to a patch centered polar coordinate system in which the field point forms the origin of the coordinate system. This approach is similar to that of Lean et al. [52] who used it in the evaluation of singular surface integrals over triangular and rectangular domains. The transformation is shown in Figure 37

and is defined by

$$\left. \begin{aligned} \gamma &= R , \\ \gamma \cos \psi &= \rho' \cos \phi' - \rho_n , \\ \gamma \sin \psi &= \rho' \sin \phi' . \end{aligned} \right\} \quad (4.37)$$

The incremental distances $d\rho' d\phi'$ and $d\gamma d\psi$ are related by

$$d\gamma d\psi = J_s d\rho' d\phi' , \quad (4.38)$$

where J_s , defined by

$$J_s = \frac{\partial(\gamma, \psi)}{\partial(\rho', \phi')} = \begin{vmatrix} \frac{\partial\gamma}{\partial\rho'} & \frac{\partial\gamma}{\partial\phi'} \\ \frac{\partial\psi}{\partial\rho'} & \frac{\partial\psi}{\partial\phi'} \end{vmatrix} = \frac{\rho'}{\gamma} . \quad (4.39)$$

is the Jacobian of the transformation. With this transformation the integral in (4.35) can be written as

$$Q_{n,n} = \iint_{\mathcal{R}} d\gamma d\psi , \quad (4.40)$$

where $\mathcal{R}$, the domain of the integral, now needs to be expressed in terms of appropriate limits on γ and ψ .

Using the symmetry about the radial line through the center of the source patch and the subdivision of the domain into three regions as shown in Figure 38, the integral in (4.35) can be subdivided as follows:

$$Q_{n,n} = 2 \left\{ \iint_{\mathcal{R}_1} d\gamma d\psi + \iint_{\mathcal{R}_2} d\gamma d\psi + \iint_{\mathcal{R}_3} d\gamma d\psi \right\} , \quad (4.41)$$

where the integral over the i -th region is given by

$$q_i = \iint_{\mathcal{R}_i} d\gamma d\psi = \int_{\psi_a}^{\psi_b} d\psi \int_0^{r(\psi)} d\gamma = \int_{\psi_a}^{\psi_b} r(\psi) d\psi . \quad (4.42)$$

Here $r(\psi)$ is the length of the radius vector between the center of the patch and the edge of the patch at polar angle ψ . The integral in (4.42) has no singularities and

Figure 37: Transformation used for removing the singularity in the integral for the self-impedance terms.

Figure 38: Definition of the regions over which the various parts of the integral for the self-impedance terms are determined.

can be performed by standard numerical integration methods. All that remains in order to compute the self-impedance terms is to derive expressions for $r(\psi)$ and the limits ψ_a and ψ_b for each region $\mathcal{R}_i$.

Integration Limits (ψ_a and ψ_b) and Expressions for $r(\psi)$.

This section presents the derivation of the equations for the limits and the integrand used in the numerical computation of the self-impedance integral. In each of the three regions defined in Figure 38 a different set of equations apply. The centermost patches require special consideration because these have a different geometry than the generic form of the other patches.

Region $\mathcal{R}_1$: ($0 < \psi < \psi_1$).

Consider the geometry shown in Figure 39(a). The limiting angle ψ_1 is obtained from the following expressions that pertain to the triangle OPQ:

$$\psi_1 = \pi - \sin^{-1} \left(\frac{\rho_b}{\gamma_1} \sin \Delta\phi/2 \right) , \quad (4.43)$$

$$\gamma_1 = \sqrt{\rho_n^2 + \rho_b^2 - 2\rho_n\rho_b \cos \Delta\phi/2} . \quad (4.44)$$

Note that $\psi_1 > \pi/2$ for $\rho_b \cos \Delta\phi/2 < \rho_n$, otherwise $\psi_1 \leq \pi/2$.

The expression for $r(\psi)$ in region $\mathcal{R}_1$, denoted by $r_1(\psi)$, is obtained from a consideration of triangle OPR, for which

$$\begin{aligned} \rho_b^2 &= \rho_n^2 + r_1^2 - 2r_1\rho_n \cos(\pi - \psi) , \\ &= \rho_n^2(1 - \sin^2 \psi) + \rho_n^2 \sin^2 \psi + r_1^2 + 2r_1\rho_n \cos \psi , \\ &= (r_1 + \rho_n \cos \psi)^2 + \rho_n^2 \sin^2 \psi . \end{aligned} \quad (4.45)$$

Hence

$$r_1(\psi) = -\rho_n \cos \psi + \sqrt{\rho_b^2 - \rho_n^2 \sin^2 \psi} . \quad (4.46)$$

(a)

(b)

(c)

(d)

Figure 39: Geometry for computing the integral for the self-impedance terms: (a) Region $\mathcal{R}_1$, (b) Region $\mathcal{R}_2$, (c) Region $\mathcal{R}_3$, (d) centermost patches.

Region $\mathcal{R}_2$: $(\psi_1 < \psi < \psi_2)$.

To find the second limiting angle, ψ_2 , consider the geometry shown in Figure 39(b). From the geometry of triangle OPQ it follows that

$$\psi_2 = \sin^{-1} \left(\frac{\rho_a}{\gamma_2} \sin \Delta\phi/2 \right) , \quad (4.47)$$

$$\gamma_2 = \sqrt{\rho_n^2 + \rho_a^2 - 2\rho_n\rho_a \cos \Delta\phi/2} . \quad (4.48)$$

Note that $\psi_2 > \pi/2$.

The expression for $r_2(\psi)$ is obtained from triangle OPR by a similar method as that used for finding $r_1(\psi)$. The result is

$$r_2(\psi) = \sqrt{\rho_n^2 + \alpha^2 - 2\rho_n\alpha \cos \Delta\phi/2} , \quad (4.49)$$

$$\alpha = \frac{\rho_n \sin \psi}{\sin(\psi - \Delta\phi/2)} . \quad (4.50)$$

Region $\mathcal{R}_3$: $(\psi_2 < \psi < \pi)$.

The approach for finding $r_3(\psi)$ is similar to that used for finding $r_1(\psi)$. From a consideration of the triangle OPR in Figure 39(c) it follows that

$$\begin{aligned} \rho_a^2 &= \rho_n^2 + r_3^2 - 2r_3\rho_n \cos(\pi - \psi) , \\ &= \rho_n^2(1 - \sin^2 \psi) + \rho_n^2 \sin^2 \psi + r_3^2 + 2r_3\rho_n \cos \psi , \\ &= (r_3 + \rho_n \cos \psi)^2 + \rho_n^2 \sin^2 \psi . \end{aligned} \quad (4.51)$$

Hence

$$r_3(\psi) = -\rho_n \cos \psi - \sqrt{\rho_a^2 - \rho_n^2 \sin^2 \psi} . \quad (4.52)$$

Note that the first term in (4.52) is positive since $\psi \geq \psi_2 > \pi/2$. The argument of the square root remains positive if one chooses $\Delta\rho$ correctly. Section 4.6.3 gives a criterion for the selection of $\Delta\rho$.

Centermost Patches.

For the centermost patches shown in Figure 39(d) $\rho_n = \Delta\rho/2, \rho_a = 0, \rho_b = \Delta\rho$, and $\psi_2 = \pi$, which implies that there is no region $\mathcal{R}_3$ for these patches. However, the expressions for ψ_1 , $r_1(\psi)$ and $r_2(\psi)$ given in (4.43), (4.46), and (4.49) remain valid.

Example.

Figure 40 shows an example of the integrand of (4.42), i.e., the function $r(\psi')$ versus ψ' , for an edge patch on a disk subdivided into 45 patches ($N_\rho = 5$ and $N_\phi = 9$). Because it is slowly varying, this function can easily be integrated using standard methods of numerical integration. The use of a 5-point Gauss-Legendre quadrature scheme over each of the three subregions was found to give better than 0.1% accuracy in the computation of the integral as compared to higher order integration schemes.

4.6.3 Optimum Relation Between N_ρ and N_ϕ .

Recall that the dimensions of the source patches are defined by

$$\Delta\rho = 1/N_\rho, \quad \Delta\phi = 2\pi/N_\phi. \quad (4.53)$$

Note that N_ϕ and N_ρ cannot be picked arbitrarily. For a particular choice of N_ϕ , the value of N_ρ is constrained by the geometry shown in Figure 39(b). The minimum value of $\Delta\rho$ (and hence the maximum value of N_ρ) is obtained when PQ is tangent to the arc with radius ρ_a at Q, i.e.,

$$\rho_n \cos \Delta\phi/2 \geq \rho_a \geq \rho_n - \Delta\rho/2. \quad (4.54)$$

The worst case condition occurs for the patches closest to the perimeter of the disk. For these patches $\rho_n = 1 - \Delta\rho/2$. Substituting this value in (4.54) and rearranging

Figure 40: Integrand and integration limits used in the computation of a typical self-impedance term.

Table 12: Optimal patch size parameters for 'exact' analysis.

$\Delta\rho = \frac{1}{N_\rho}$, $\Delta\phi = \frac{2\pi}{N_\phi}$,		
N_ϕ	$N_\rho(\text{max})$	$N = N_\rho N_\phi$
5	3	15
6	4	24
8	7	56
10	10	100
12	15	180
15	23	345
20	41	820

terms gives

$$\Delta\rho = \frac{1}{N_\rho} \geq \frac{2(1 - \cos \Delta\phi/2)}{2 - \cos \Delta\phi/2} = \frac{2(1 - \cos \pi/N_\phi)}{2 - \cos \pi/N_\phi} . \quad (4.55)$$

Table 12 gives a number of values for N_ρ and N_ϕ that satisfy this criterion.

4.6.4 Rectangular Patch Approximation.

By the use of the transformation

$$2c = \Delta\rho , \quad 2d = \rho_n \Delta\phi , \quad x = \rho - \rho_n , \quad y = \rho_n(\phi - \phi_n) \quad (4.56)$$

the basis functions shown in Figure 34 are approximated by rectangular patches dimensions $2c \times 2d$ as shown in Figure 41. Taking $\phi_n = 0$ in (4.56) leads to no loss of generality. For the rectangular patch approximation the surface integral that occurs in the evaluation of the self-impedance terms can then be expressed in closed form by means of the transformation

$$x = \zeta \cos \psi , \quad y = \zeta \sin \psi , \quad (4.57)$$

which removes the singularity in the integrand and reduces the expression for $Q_{n,n}$ to

$$Q_{n,n} = 4 \left\{ \int_0^{\tan^{-1}d/c} d\psi \int_0^{a/\cos\psi} d\zeta + \int_{\tan^{-1}d/c}^{\pi/2} d\psi \int_0^{a/\sin\psi} d\zeta \right\} . \quad (4.58)$$

Each of these integrals can be expressed in closed form. The result is

$$Q_{n,n} = 4c \{ \ln \tan(\pi/4 + \gamma/2) + d/c \ln \tan(\pi/2 - \gamma/2) \} \quad (4.59)$$

$$\gamma = \tan^{-1}d/c . \quad (4.60)$$

This integral can be reduced to the expression for the self-impedance terms given in Table 7 (See Appendix B).

Figure 41: Rectangular patch approximation.

The maximum error of the rectangular patch approximation is proportional to d/c , which reaches its maximum value for patches close to perimeter of the disk. For example, for $N_\rho=5$ and $N_\phi=8$, the maximum value of d/c is 3.6, which gives a maximum error of 0.5%. In general, the larger N_ϕ/N_ρ , the more accurate is the rectangular patch approximation. The use of the rectangular patch approximation can potentially reduce the CPU time for the computation of the elements of the impedance matrix.

4.7 Current Density at an Arbitrary Field Point.

Once the coefficients I_n in the expression for the surface current density are known, the potential Ψ at any point in the half-space is completely specified by (4.1) and (4.67). The current density $\mathbf{J}$ and the electric field $\mathbf{E}$ at any other point in the resistive half-space is then obtained from

$$\mathbf{J} = \sigma \mathbf{E} = -\sigma \nabla \Psi . \quad (4.61)$$

It is convenient to write the components of the current density vector $\mathbf{J}$ as

$$\mathbf{J} = J_\rho \hat{\rho} + J_\phi \hat{\phi} + J_z \hat{z} , \quad (4.62)$$

where $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ are unit vectors in a disk centered coordinate system as shown in Figure 42.

In this coordinate system, the components of $\mathbf{J}$ are given by

$$J_\rho = -\sigma \frac{\partial \Psi}{\partial \rho} , \quad J_\phi = -\sigma \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} , \quad J_z = -\sigma \frac{\partial \Psi}{\partial z} . \quad (4.63)$$

For each disk, substitution of (4.1) in (4.63) results in expressions of the form

$$J_\xi = \sum_{n=1}^N I_n J_\xi^n , \quad (4.64)$$

Figure 42: Disk centered coordinate system used in the computation of the current density components.

where $\xi \in \{\rho, \phi, z\}$ and J_ξ^n is the general form of the ξ -component of the current density produced by a unit current entering the body through the n -th source patch (on the k -th disk). (The relation between n and k is defined in (4.4)). The specific expressions for the components J_ξ^n are

$$J_\rho^n = \frac{1}{2\pi A_n} \iint_{B_n} \frac{\rho - \rho' \cos(\phi - \phi')}{R^3} \rho' d\rho' d\phi' , \quad (4.65)$$

$$J_\phi^n = \frac{1}{2\pi A_n} \iint_{B_n} \frac{\rho \rho' \sin(\phi - \phi')}{R^3} \rho' d\rho' d\phi' , \quad (4.66)$$

$$J_z^n = \frac{1}{2\pi A_n} \iint_{B_n} \frac{z}{R^3} \rho' d\rho' d\phi' , \quad (4.67)$$

where

$$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + (\rho')^2 - 2\rho\rho' \cos(\phi - \phi') + z^2} , \quad (4.68)$$

with $\mathbf{r}$ and $\mathbf{r}'$ the radius vector to the source point $(\rho', \phi', 0)$ and the field point (ρ, ϕ, z) respectively. The domain B_n spans the surface of the n -th patch. For the computation of each of the J_ξ^n the following three regions are considered:

1. Field point $\mathbf{r}$ on the source patch,
2. Field point $\mathbf{r}$ in the *far-field* of the source patch, and
3. Field point $\mathbf{r}$ *in between* the previous two regions.

The approach used for the evaluation of $\mathbf{J}$ in each of these regions is discussed in the next three sections.

Field Point on the Source Patch.

When $z = 0$ and $\mathbf{r}$ on B_n , the tangential components of $\mathbf{J}$ (J_ϕ, J_ρ) are zero and the perpendicular component (J_z) is equal to the assumed uniform current density due to a unit current through the source patch, i.e.,

$$J_\rho^n = J_\phi^n = 0 , \quad J_z^n = \frac{1}{A_n} . \quad (4.69)$$

Field Point in the Far Field of the Source Patch.

By definition points in the far field of the source patch are at such a distance from the source patch that sufficient accuracy of the field components is obtained by assuming that the total current through the source patch is concentrated at the center of the source patch. The threshold distance beyond which the far field approximation applies is denoted by R_{thr} and depends on the size of the source patch. For 1% accuracy R_{thr} is given by

$$R_{thr} > 3 \max[\Delta\rho, \rho_n \Delta\phi] . \quad (4.70)$$

For $R > R_{thr}$ the integral expressions for the current density components reduce to

$$J_\rho^n \approx \frac{1}{2\pi} \frac{\rho - \rho_n \cos(\phi - \phi_n)}{R_n^3} , \quad (4.71)$$

$$J_\phi^n \approx \frac{1}{2\pi} \frac{\rho \rho_n \sin(\phi - \phi_n)}{R_n^3} , \quad (4.72)$$

$$J_z^n \approx \frac{1}{2\pi} \frac{z}{R_n^3} , \quad (4.73)$$

where

$$R_n = \sqrt{\rho^2 + \rho_n^2 - 2\rho\rho_n \cos(\phi - \phi') + z^2} . \quad (4.74)$$

Field Point in the Near Field but not on the Source Patch.

In the intermediate range the current density components are determined by the numerical evaluation of the integrals in (4.65) thru (4.67). Each current density component can be reduced to the form

$$J_\xi^n(\rho, \phi, z) = \frac{1}{2\pi A_n} \int_{\phi_n^-}^{\phi_n^+} f_\xi^n(\phi') d\phi' , \quad (4.75)$$

where $\xi \in \{\rho, \phi, z\}$ and $\phi_n^\pm = \phi_n \pm \Delta\phi/2$.

The closed form expressions for the integral on ρ' ($f_{\xi}^n(\phi')$) are given in Table 13. Since the $f_{\xi}^n(\phi')$ are slowly varying functions of ϕ' it was found that a 5-point Gauss-Legendre quadrature gave sufficient accuracy for the numerical integration of the integral on ϕ' .

Disk Centered Field Coordinates.

When the current density components due to the k -th disk electrode is computed, a local cylindrical coordinate system is used with its origin at the center of the k -th disk, i.e., at $(p_k, \theta_k, 0)$. The relation between the source electrode centered field coordinates (ρ, ϕ, z) and the global coordinates (r, χ, z) is given by

$$\rho \cos \phi = r \cos \chi - p_k \cos \theta_k , \quad (4.76)$$

$$\rho \sin \phi = r \sin \chi - p_k \sin \theta_k , \quad (4.77)$$

from which it follows that

$$\rho = \sqrt{r^2 + p_k^2 - 2rp_k \cos(\chi - \theta_k)} , \quad (4.78)$$

$$\phi = \sin^{-1} \left[\frac{p_k}{r} \sin(\chi - \theta_k) \right] . \quad (4.79)$$

Rectangular Current Density Components.

The current density components due to several electrodes are added by converting the from cylindrical to rectangular components. The relation between the tangential rectangular components (J_x, J_y) and the tangential cylindrical components (J_ρ, J_ϕ) is given by

$$\begin{bmatrix} J_x \\ J_y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} J_\rho \\ J_\phi \end{bmatrix} . \quad (4.80)$$

The perpendicular component, J_z , is not affected by this transformation.

Table 13: Expressions for the current density components at field point (ρ, ϕ, z) due to a unit current through the source patch centered on (ρ_n, ϕ_n) .

$J_{\xi}^n(\rho, \phi, z) = \frac{1}{2\pi A_n} \int_{\phi_n^-}^{\phi_n^+} f_{\xi}^n(\phi') d\phi' ,$ $f_{\xi}^n(\phi') = \int_{\rho_n^-}^{\rho_n^+} (\cdot) d\rho' = g_{\xi}^n(\phi', \rho_n^+) - g_{\xi}^n(\phi', \rho_n^-) ,$ $\phi_n^{\pm} = \phi_n \pm \Delta\phi/2 , \quad \rho_n^{\pm} = \rho_n \pm \Delta\rho/2 , \quad A_n = \rho_n \Delta\rho \Delta\phi .$		
Component (ξ)	Condition	Expressions for ρ' -integral $g_{\xi}^n(\phi', \rho')$
ρ	$d \neq 0$	$\rho h_1(\rho') - \cos(\phi - \phi') h_2(\rho')$
	$d = 0$	$\rho h_3(\rho') - h_4(\rho')$
ϕ	$d \neq 0$	$\rho \sin(\phi - \phi') h_2(\rho')$
	$d = 0$	0
z	$d \neq 0$	$z h_1(\rho')$
	$d = 0$	0
<p>Definitions:</p> <p>(1) : $b = -2\rho \cos(\phi - \phi') , \quad c = \rho^2 + z^2 , \quad d = b^2 - 4c .$</p> <p>(2) : $h_1 = 2 \frac{b\rho' + 2c}{dR} , \quad h_2 = -\frac{(2b^2 - 4c)\rho' + 2bc}{dR} + \ln 2R + 2\rho' + b ,$ where $R = \sqrt{\rho^2 + (\rho')^2 + z^2 - 2\rho\rho' \cos(\phi - \phi')} .$</p> <p>(3) : $h_3 = \frac{1}{R} \left(\frac{b}{4R} - 1 \right) , \quad h_4 = \frac{1}{R} \left(\frac{-b^2}{8R} + b + R \ln R \right) ,$ when $d = 0$, and hence $z = 0, \phi' = \phi$, and $R = \rho - \rho' .$</p>		
<p>References:</p> <p>The expressions for h_1, h_2, h_3 and h_4 are derived from (4.65) thru (4.67) using Equations 14.291, 14.292, 14.74, and 14.75 from [79] respectively.</p>

Figure 43: Geometry for the Kobayashi solution: Identical coplanar disk electrodes in a bipolar configuration.

4.8 Kobayashi Solution.

This section presents the derivation of an alternative expression for the surface current density for the specific case of a bipolar configuration of two coplanar unit radius disk electrodes. The result is used for checking the MM solution. It is also used for demonstrating the dependence of the leading edge effect and the interelectrode impedance on the electrode spacing. These results are not as accurately given by the MM solution.

The derivation is based on the solution of Laplace's equation for the electrostatic potential in terms of Kobayashi potentials [48] and hence will be referred to as the Kobayashi solution.

4.8.1 Electrostatic Potential.

Kobayashi derived expressions for the electrostatic potential distribution due to two coplanar unit radius disk electrodes in free space as shown in Figure 43. Using the eigenfunction solutions for Laplace's equation in cylindrical coordinates as basis functions, the potential for two disk electrodes is expressed by the equation

$$\begin{aligned} \Psi = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (A_m^n \cos m\phi + B_m^n \sin m\phi) \int_0^{\infty} J_m(\rho u) J_{m+2n+\frac{1}{2}}(u) e^{-uz} u^{-\frac{1}{2}} du \\ & + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\bar{A}_m^n \cos m\bar{\phi} + \bar{B}_m^n \sin m\bar{\phi}) \int_0^{\infty} J_m(\bar{\rho} u) J_{m+2n+\frac{1}{2}}(u) e^{-uz} u^{-\frac{1}{2}} du, \end{aligned} \quad (4.81)$$

where (ρ, ϕ) and $(\bar{\rho}, \bar{\phi})$ are related by (See Figure 43)

$$\bar{\phi} = \pi - \phi - \psi, \quad \psi = \sin^{-1}(p/\rho \sin \phi), \quad \bar{\rho} = (p^2 + \rho^2 - 2p\rho \cos \phi)^{\frac{1}{2}}. \quad (4.82)$$

The expansion functions in (4.81) are known as Kobayashi potentials. These functions satisfy the following boundary condition

$$\left. \frac{\partial \Psi}{\partial z} \right|_{z=0} = 0 \text{ for } \rho > 1 \text{ and } \bar{\rho} > 1 . \quad (4.83)$$

The four sets of unknown coefficients $A_m^n, B_m^n, \bar{A}_m^n$, and $\bar{B}_m^n$ are solved by using the orthogonality of the expansion functions and the specified potential on each of the electrodes.

For two electrodes of opposite polarity, held at +1 and -1 volt respectively (bipolar configuration), the symmetry of the field with respect to the line connecting the centers of the electrodes results in

$$\bar{A}_m^n = -A_m^n , \quad \bar{B}_m^n = B_m^n = 0 . \quad (4.84)$$

Kobayashi calculated numerical values of A_m^n for the bipolar configuration for various values of the electrode spacing p . The numerical values used here in the computation of the surface current density and the impedance between the electrodes are given in Table 14.

4.8.2 Surface Current Density.

For the electrostatic problem, Kobayashi defined the charge distribution on the surface of the electrodes as

$$q = -\frac{1}{4\pi} \left. \frac{\partial \Psi}{\partial z} \right|_{z=0-}^{z=0+} \quad (4.85)$$

and showed that for two electrodes of opposite polarity, held at +1 and -1 volt respectively, the total charge on each electrode is given by

$$Q = \int_{-\pi}^{\pi} d\phi \int_0^a q r dr = \frac{2a}{\pi} A_0^0 , \quad (4.86)$$

Table 14: Coefficients A_m^n used in the Kobayashi solution for the surface current density of two unit radius coplanar disks with center-to-center spacing p .

p	A_0^0	A_1^0	A_2^0	A_3^0	A_4^0
2.4	1.43043	0.42417	0.20724	0.10146	0.04989
3.0	1.29208	0.21172	0.07620	0.02739	0.00980
10.0	1.06827	0.001371	0.00138	0.00014	0.0
p	A_0^1	A_1^1	A_2^1	A_3^1	A_4^1
2.4	-0.04335	-0.03023	-0.01245	0.0	0.0
3.0	-0.01435	-0.00687	-0.00193	0.0	0.0
10.0	-0.00023	0.0	0.0	0.0	0.0

where A_0^0 is the coefficient of the first term in the series expansion for the potential distribution and $a=1$ the radius of the electrode.

For the analogous quasistatic volume conductor problem the current density on the surface of the electrode is defined as

$$J_z = -\hat{\sigma} \left. \frac{\partial \Psi}{\partial z} \right|_{z=0+}, \quad (4.87)$$

where $\hat{\sigma}$ is the complex conductivity of the halfspace $z \geq 0$. The total current through each electrode is analogous to the total charge in the electrostatic solution and is given by

$$I = 4a\hat{\sigma}A_0^0. \quad (4.88)$$

Using Ψ as defined in (4.81) in (4.87) and normalizing J_z to the average current density ($J_{ave} = I/\pi a^2$) gives

$$\frac{J_z(\rho, \phi, z)}{J_{ave}} \sqrt{\frac{\pi}{8}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A_m^n}{A_0^0} \{Q(\rho, \phi) - Q(\bar{\rho}, \bar{\phi})\}, \quad (4.89)$$

where

$$Q(\rho, \phi) = \cos m\phi \int_0^{\infty} J_m(\rho u) J_{m+2n+\frac{1}{2}}(u) e^{-uz} u^{\frac{1}{2}} du. \quad (4.90)$$

For $z=0$ the integrals in (4.90) can be evaluated in closed form using Gradshteyn [23, 6.574] and Abramowitz [3, 15.3.24, 15.3.30]. As a results, the surface current density distribution can be expressed as

$$\frac{J_z(\rho, \phi)}{J_{ave}} = \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A_m^n}{A_0^0} \cos m\phi \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \frac{\rho^m}{\sqrt{1-\rho^2}} P_n^{(m, -0.5)}(1-2\rho^2), \quad (4.91)$$

for $\rho \leq 1$. Here $\Gamma(\cdot)$ is the Gamma function and $P_n^{(\alpha, \beta)}$ a Jacobi polynomial of order n . On the surface of the second disk the current density is the mirror image of this distribution.

For $z > 0$, i.e. at points below the surface, the current density cannot be expressed in closed form. To evaluate the integral in (4.90) by numerical integration is a prohibitive task because of the variable period oscillatory integrand.

4.8.3 Interelectrode Impedance.

From the expression for I given in (4.88) it is easy to see that the interelectrode impedance for two electrodes with a potential difference of 2 volt is given by

$$Z = \frac{2}{I} = \frac{1}{2\sigma a A_0^0} . \quad (4.92)$$

This impedance is often referred to as the “spreading” impedance in order to distinguish it from the contact impedance which is strongly affected by the surface finish of the electrode and the impedance of a contact layer such as a layer of gel or a skin layer between the electrodes and the underlying tissues.

4.8.4 Monopolar Electrode Impedance.

The impedance of a monopolar disk electrode is a useful reference against which to compare the impedance between two closely spaced electrodes. Since a return electrode is required in the measurement of low frequency electrode impedances, the impedance of a single electrode is a theoretical quantity that cannot be directly measured. In practice it is closely approximated by one half of the interelectrode impedance between two identical electrodes when the electrodes are both on the surface of a uniform lossy halfspace, and the distance between the electrodes is large compared to the electrode diameters. In the case of a single electrode, one can also assume that the surface current density has rotational symmetry, which makes it possible to express the current density distribution and the electrode impedance in closed form. The impedance of a monopolar disk electrode

Figure 44: Normalized interelectrode impedance for two disk electrodes, each of radius a with center-to-center spacing p .

is given by the equation

$$Z_{\text{mon}} = \frac{1}{4\hat{\sigma}a}, \quad (4.93)$$

where a is the electrode radius and $\hat{\sigma}$ the complex body conductivity.

In the next section it will be shown that for an electrode spacing of more than $5a$, the interelectrode impedance for a bipolar electrode pair is within 10% of $2Z_{\text{mon}}$.

4.8.5 Impedance Computations.

Using the values of A_0^0 computed by Kobayashi, the interelectrode impedance was determined for values of the center-to-center electrode spacing p in the range $2.2 < p/a < 10$. The results are shown in Figure 44. Note that $p/a = 2.0$ corresponds to the electrodes just touching. For large values of p/a , the normalized interelectrode impedance, $Z\hat{\sigma}a$, approaches a value of 0.5, which is twice the monopolar

electrode impedance given by (4.93). At $p/a = 10$ the interelectrode impedance is within 6% of the value for very large spacing.

4.8.6 Discussion.

It can be shown that the Kobayashi solution is a special case of the MM solution, namely one in which the basis functions used for the expansion of the surface current density are the entire basis functions

$$\frac{\sin m\phi}{\cos m\phi} \frac{\rho^m}{\sqrt{1-\rho^2}} P_n^{(m,-0.5)}(1-2\rho^2) . \quad (4.94)$$

Elements from the same set of functions are used as weighting functions in what amounts to the Galerkin approach to the MM solution of an integral equation.

The Kobayashi solution converges faster than the MM solution and the number of basis functions required for the accurate representation of the surface current density decrease with increasing distance between the electrodes. The reason for this is that the Kobayashi solution is an extension of the exact solution for a monopolar disk electrode. Because it is expressed in terms of the eigenfunction solution of Laplace's equation in cylindrical coordinates it explicitly includes the edge singularity. Unfortunately, the difficulty of evaluating the expressions for the near-field current density distribution prohibits the use of the Kobayashi solution for computing the current density anywhere but on the surface of the electrode.

4.9 Two-dimensional Approximation.

This section presents the derivation of two expressions to evaluate the accuracy of approximating a three-dimensional (3D) electrode configuration with its two-dimensional (2D) equivalent. The reason for introducing this analysis is that in the literature 2D finite element models are often used to compute the current

Figure 45: Two-dimensional approximation of two disk electrodes on a half-space.

distribution of surface electrodes due to the prohibitive computational effort of 3D finite element models [44,45,46]. The following two expressions are presented here:

1. A closed form expression for the current density distribution of a 2D approximation of two surface electrodes. This is compared with the Kobayashi (3D) solution in order to demonstrate the difference between the 2D and 3D approximations of the surface current density distribution.
2. An expression for the far-field current density of a 2D dipole consisting of two infinite lines in a bipolar configuration. This is compared with an equivalent expression for a 3D dipole to show the limitations of the 2D approximation in the prediction of the far-field.

The basic assumption in the 2D approximation is that the field quantities are constant along one of the three cartesian axes. In the 2D approximation of two disk electrodes on a uniform half-space, this is equivalent to taking the electrodes to be infinitely long strips as shown in Figure 45.

4.9.1 Surface Current Density.

Sneddon [81, p. 267] gives a closed form expression for the electrostatic surface charge distribution due to two parallel electrified strips of opposite polarity, each of width $2a$ with center-to-center spacing p . The surface current density in the analogous quasistatic volume conductor problem is given by:

$$J_z(x) = \frac{\sigma d}{K(c/d)} \left\{ (x^2 - c^2) (d^2 - x^2) \right\}^{-1/2}, \quad (c < x < d), \quad (4.95)$$

where x is the distance from the point halfway between the strips. $K(\cdot)$ is the complete elliptic integral of the first kind and c and d are the positions of the edge of the electrodes as defined in Figure 45.

The total current per unit length in the y -direction is given by the integral of the surface current density over the width of one of the electrodes:

$$I = \int_0^1 dy \int_c^d dx J_z(x) = \frac{\sigma d}{K(c/d)} \frac{2}{(c+d)} K(2a/p). \quad (4.96)$$

This can be expressed in closed form by the use of Gradshteyn [23, 3.147(7)]. The average current density per unit length on the surface of the strip is given by

$$J_{\text{ave}} = \frac{I}{d-c}. \quad (4.97)$$

Combining (4.95) thru (4.97) allows one to express the normalized surface current density as

$$\frac{J_z(x)}{J_{\text{ave}}} = \frac{pa}{K(2a/p)} \left\{ (x^2 - c^2) (d^2 - x^2) \right\}^{-1/2}, \quad (c < x < d). \quad (4.98)$$

In order to compare this distribution with that of two disk electrodes, it is convenient to define a new variable $\rho = x + p/2$. This allows one to draw the normalised current density distribution of two disk electrodes as given in (4.91) and that of the strip electrodes on the same normalized axes.

Figure 46: Point electrodes in a dipole configuration.

4.9.2 Far Field Approximation.

To find the far-field approximation of the current density distribution in the 2D model the electrodes can be approximated by two infinite lines separated by a distance p . The current density distribution per unit length of each of the lines is given by

$$J_{ps}^{2D} = \frac{I}{\pi R}, \quad (4.99)$$

where R is the distance from the source in the plane perpendicular to the surface and the direction of the electrodes. The subscript (ps) is used as a reminder that this result pertains to the 2D approximation of a *point source*.

For a line source 'dipole' with dipole strength Ip the current density component parallel to the surface at a point halfway between the two electrodes is given by

$$J_{dip}^{2D} = 2 \frac{I}{\pi R} \sin \theta \approx \frac{Ip}{\pi R^2}, \quad (4.100)$$

where θ is the angle between the normal on the surface and the line connecting one of the electrodes with the observation point as defined in Figure 46. This result will be compared to that of two point electrodes separated by a distance p to show how the 2D approximation overestimates the magnitude of the far-field current density.

4.10 Point Sources and Point Dipoles.

This section presents the derivation of the expressions for the current density of point electrodes and point dipoles. A point dipole is defined as a dipole pair with currents $+I$ and $-I$ and an infinitesimal separation distance. In a later section the expressions derived here are used to compare the MM solution for the current density distribution of a monopolar disk electrode with the point dipole and with a bipolar pair of electrodes with varying spacing.

Monopolar Point Source.

For a monopolar point source with current I on the surface of a uniform half-space the current density vector is directed radially away from the point source. At a distance R from the source it has a magnitude

$$J_{\text{ps}}^{3D} = \frac{I}{2\pi R^2} \quad (4.101)$$

In order to compare this with the current density distribution of a disk electrode, I is taken to be equal to the current through a disk electrode with radius a and constant current density J_{ave} , i.e., $I = \pi a^2 J_{\text{ave}}$. The magnitude of the normalized current density is then given by

$$\frac{J_{\text{ps}}^{3D}}{J_{\text{ave}}} = \frac{1}{2(R/a)^2} . \quad (4.102)$$

Point Dipole.

Consider the current density at a depth z below the surface halfway between the source and sink of a bipolar pair of electrodes separated by distance p as shown in Figure 46.

The dipole moment of this dipole is Ip . At any point P in the plane perpendicular to the surface and to the line connecting the electrodes, the x -component of the current density is given by

$$J_x = \frac{I}{2\pi R_1^2} \sin \theta_1 + \frac{I}{2\pi R_2^2} \sin \theta_2 , \quad (4.103)$$

where R_1, R_2, θ_1 , and θ_2 are defined in Figure 46. At the point halfway between the source and sink, $R_1 = R_2 = R$, $\sin \theta_1 = \sin \theta_2 = p/2R$, and $|\mathbf{J}| = J_x$. As z becomes very large, $R \rightarrow z$ and $\sin \theta_1 \approx \sin \theta_2 \approx p/2R$. The same result is obtained if one takes the limit as $p \rightarrow 0$. The normalized current density at the point P is then given by

$$\frac{J_{\text{dip}}}{J_{\text{ave}}} = \frac{p/a}{2(z/a)^3} . \quad (4.104)$$

4.10.1 Comparison with Two-dimensional Approximations.

For the 2D approximation the far-field of a dipole source decays as $1/R^2$ as compared to the $1/R^3$ decay of a three-dimensional dipole source. From this comparison it can be inferred that computations based on two-dimensional models of surface electrode stimulation, even with a more complex geometry, would tend to overestimate the normalized current density at points far away from the electrodes.

4.10.2 Comparison of Point Source and Disk Electrode Fields.

Figure 47 shows the normalized current density distribution of a disk electrode and of a bipolar pair of disk electrodes compared to that of a point electrode and an

Figure 47: Comparison of monopolar and bipolar disk electrode fields with that of a point source and an infinitesimal dipole.

infinitesimal dipole respectively. At a depth equal to three times the disk electrode radius, the magnitude of the current density of a monopolar disk electrode is within 10% of that of a point source at located at the center of the disk electrode. Likewise, the electric field and current density due to any combination of disk electrodes can be well approximated by that of an array of point electrodes if one restricts the region of interest to being more three electrode radii away from any of the disk electrodes.

At a depth equal to three times the electrode spacing p of a bipolar pair of electrodes, the field is within 10% of that produced by an infinitesimal dipole of strength I_p located halfway between the two electrodes.

In the far-field the current density decays as $1/R^2$ for a monopolar and $1/R^3$ for a bipolar electrode pair, where R is the distance from the electrode(s). The accuracy of the point source approximation in the region beyond $R/a = 3$ also shows that the current distribution in this region is independent of the actual surface current distribution. This suggest that the computation of the electric field and current density in the far field ($R/a > 3$) can be significantly simplified by replacing the actual electrodes by point electrodes.

4.10.3 Comparison of 2D and 3D Surface Current Density.

Figure 49 and Figure 48 show comparative data for the same electrode spacings for the 2D and 3D approximations. These data also illustrate the extent of current density skewing associated with the “leading edge effect”. Note how the 2D approximation overemphasizes the extent of the leading edge effect.

Figure 48: Leading edge effect versus electrode spacing for two disk electrodes.

Figure 49: Leading edge effect versus electrode spacing for two infinite strip electrodes.

4.11 Numerical Results for the MM solution.

This section presents illustrative examples of current density distributions due to disk electrodes obtained with the Moment Method (MM) formulation. The first examples serve to validate the solution. The other examples show the differences in the current density fields produced by monopolar, bipolar and four-electrode configurations.

4.11.1 Validation of the MM Solution.

The accuracy of the MM solution is demonstrated by the following comparisons with numerical results derived by other methods:

- The surface and subsurface current density is compared with an exact solution for the case of a single disk electrode,
- The surface current density is compared with the solution derived from the Kobayashi formulation for two closely spaced disk electrodes.

Comparison with Exact Solution for a Monopolar Disk Electrode.

The exact solution for the current distribution due to a monopolar disk electrode can be derived from the electrostatic solution given by Smythe [80, p. 124] (See Appendix C). It is used here to check the accuracy of the MM solution. Figure 50 shows a comparison of the exact solution with the surface current density and the residual potential of the MM solution. The residual potential is defined as the difference between the (constant) potential specified on the surface of the electrode and the potential distribution computed from the MM solution. Here the potential at an arbitrary point on the patch is computed using the rectangular patch approximation. The expression for the potential at an arbitrary point

on the patch is derived in Appendix B. As expected, the residual is zero at the matching points which are chosen at the centers of the source patches. At all other points the residual potential is less than 10% of the applied potential over most of the radial range. The surface current density J_z differs markedly from the exact solution only at the edge of the electrode. A better approximation of the surface current density close to the edge can be obtained by explicitly including the edge singularity in the basis functions used for the MM solution as was done in Chapter 3. This approximately doubles the CPU time for computing the elements of the impedance matrix since the edge terms then have to be evaluated by numerical integration. For the present application the additional complexity this would incur should be weighed against the following considerations:

- The edge singularity in the exact solution for the surface current density is a consequence of the assumption of a zero thickness electrode. Any practical electrode will have finite thickness and the edge will be rounded to some extent, so that the current density near the edge would actually be closer to the pulse basis function approximation than to the exact solution.
- At points below the surface, the difference between the MM solution and the exact solution rapidly decreases with increasing depth as shown in Figure 51.

Comparison with the Kobayashi Solution.

Figure 52 shows 3D surface plots in which the height above the XY -plane is proportional to the normalized surface current density J_z/J_{ave} of one of two identical disk electrodes. In (a) the second electrode is at a normalized distance $p/a > 10$. At this distance the second electrode has negligible effect on the surface current distribution of the first electrode. In (b) the second electrode is to the

Figure 50: Comparison of the surface current density and the potential residual with the exact solution for a monopolar disk electrode.

Figure 51: Comparison of the subsurface surface current density with the exact solution for a monopolar disk electrode.

(a)

(b)

Figure 52: Surface current density distribution on one of two identical coplanar disk electrodes: (a) Electrode spacing large ($p/a > 10$), (b) Electrode spacing small ($p/a = 3$).

Figure 53: Moment method solution compared to Kobayashi solution for two disk electrodes of opposite polarity with center-to-center spacing $p/a=3$.

upper right of the one shown, at a distance $p/a = 3$. Notice the skewing of the surface current density on the leading edge side, that is toward the second electrode. The rise of J_z on the near side is more pronounced than the reduction on the far side of the electrode. The dark circumferential contour shows the average current density level. The other dark line marks the cut through the 3D current density distribution shown in Figure 48.

The accuracy of the MM solution can be seen in the close correspondence with the Kobayashi solution as shown in Figure 53.

4.11.2 Current Density Fields of Various Electrode Configurations.

Examples of the numerical results obtained with the MM formulation are shown in Figure 54 through Figure 61. The spatial coordinates of the current density field plots are scaled to the size of the electrodes, which are shown with each plot so that the results can be generalized to electrodes of arbitrary size. Likewise, the magnitude of the current density is scaled to the average current density on the surface of the electrode, so as to make the presented data independent of the actual electrode size or the amplitude of the applied current. Various two-dimensional display formats are used to emphasize different aspects of the field distribution. In several cases, a comparative field for point electrodes is shown together with the field due to the disk electrodes in order to illustrate the similarity between the field due to point electrodes and that of actual disk electrodes.

In the following discussions the *central plane* will refer to the plane through the centers of the two electrodes and perpendicular to the surface. The *neutral plane* is the plane perpendicular to the central plane halfway between the electrodes.

Figure 54 shows the potential and current density distribution due to two disk electrodes of opposite polarity on a uniform half-space. This field is computed

Figure 54: Potential and current density distribution in the central plane between two disk electrodes with a center-to-center spacing $p/a = 5.4$.

using a center-to-center spacing $p/a = 5.4$ and electrode potentials of +1 and -1 volt respectively. The neutral plane coincides with the zero potential equipotential. At all points in the neutral plane the current density vectors are directed parallel to the surface. Although this display format gives a qualitative representation of both the magnitude and the direction of the field in the central plane in a single plot, a separate display of the magnitude and direction of the field is more useful for showing the quantitative relation between the current density and the applied current. The plots showing contours of J/J_{ave} where J is the magnitude of the current density and $J_{\text{ave}} = I/\pi a^2$ the average current density on the surface of the electrode, will be referred to as *magnitude* plots. The plots showing the direction of the field will be referred to as *field line* plots. Notice that in the central plane, the magnitude of the current density is given by $J = |\mathbf{J}| = \sqrt{J_x^2 + J_z^2}$ and the direction by $\theta = \tan^{-1}(J_z/J_x)$. Except where noted otherwise, the computations are based on a subdivision of each electrode into 100 uniform current density patches, with $N_\rho = N_\phi = 10$.

For 2 electrodes and $N_\rho = N_\phi = 10$ the CPU time for computation of the elements of the impedance matrix and the inversion of the 200×200 matrix is about 30 seconds on a VAX 8550 minicomputer.

Field line and Magnitude Plots.

Figure 55 shows the field line and magnitude plots for the current density field of a monopolar disk electrode. The electrode is shown in the left half of the plot in order to facilitate comparison with the fields due to bipolar disk electrodes shown in subsequent plots. Notice that at a distance of about 1 electrode diameter from the center of the electrode, the contour lines almost become semicircles, suggesting that at this distance the field distribution approximates that of a point source at

(a)

(b)

Figure 55: (a) Field line and (b) magnitude plots of the current density field of a monopolar disk electrode.

the center of the disk electrode. A quantitative comparison between the fields of a disk electrode and that of a point source will be given in the next section.

Figures 56 and 57 show the field line and magnitude plots for two disk electrodes in a bipolar configuration with center-to-center spacing $p/a = 3$ compared to that of point electrodes with the same total current. For this electrode spacing, the magnitude of J is almost constant in the region between the electrode centers at any particular depth, for depths ranging from a to $3a$. Beyond a depth of $3a$ the current density field of a bipolar disk electrode pair closely resembles that of a pair of point electrodes.

Figure 58 shows the field line and magnitude plots for two electrodes with spacing $p/a = 6$. For this spacing there is a significant variation in J at a given depth, with the maximum occurring at the point closest to each electrode, rather than in the neutral plane as for $p/a = 3$. The field line plot shows that the field is almost parallel with the surface over a larger region between the electrodes than with $p/a = 3$.

Figure 59 shows the effect of increasing the electrode spacing on the current density at a given depth below the surface. Notice that the peak magnitude occurs right under the electrode. For points deeper into the body, computations show that the peaks are more prominent while for points closer to the surface, there is less variation in the magnitude of the field as a function of distance from the neutral plane. The data in Figure 59 compares very well with corresponding data derived from an electrolytic tank solution of the two-electrode problem by Gold and Schuster [26, Fig.6]. Notice that the normalization in the present case is with respect to J_{ave} rather than to the square root of the applied power as in the data presented by Gold and Shuster. This facilitates the use of Figure 59 to derive the

(a)

(b)

Figure 56: Field line plots of the current density field of two electrodes with $p/a = 3$; (a) Point electrodes, (b) Disk electrodes.

Figure 57: Magnitude plots of the current density field of two electrodes with $p/a = 3$; (a) Point electrodes, (b) Disk electrodes.

Figure 58: Field line plots of the current density field of two disk electrodes with $p/a = 6$.

Figure 59: Magnitude and angle of the current density field in the central plane at a depth of $z/a=1$ for electrode spacings.

current required to produce a given field strength at a particular point in the body.

Design Example.

As an example of the use of Figure 59 in designing a suitable electrode configuration consider the derivation of the current density and current required to produce a field strength of 50 V/m (100 mV over a 2mm internodal distance) in the neutral plane at a depth $z=2$ cm using two electrodes, each with radius $a=2$ cm, and with center-to-center spacing $p=4$ cm. The procedure is as follows:

- The normalized values of depth and spacing is $z/a=1$ and $p/a=4$ respectively. From the magnitude plot for $p/a=4$ one finds that at $x/a=0$, i.e. in the neutral plane,

$$J/J_{ave} = \frac{\sigma E}{J_{ave}} = 0.18 . \quad (4.105)$$

- Assuming a tissue conductivity of $\sigma = 0.2$ S/m (a typical value for muscle tissue), it follows that the average surface current density required to produce an electric field $E=50$ V/m at the required depth is given by

$$J_{ave} = \sigma E / 0.18 = 56 \text{ A/m}^2 = 5.6 \text{ mA/cm}^2 . \quad (4.106)$$

- The applied current would be $I = J_{ave} \pi a^2 = 17$ mA.

Since the average current density is about 1/6 of the threshold for thermal damage [10] it could be concluded that the electrodes are large enough to keep the surface current density within safe limits. If this were not the case one would repeat the procedure with a larger electrode size.

Fields of Four Coplanar Electrodes.

Figure 60 (A) and (B) illustrate the effect of two different excitation patterns on four electrodes in a row. In (A) the field of the two inner electrodes is enhanced

Figure 60: Field line plots for two excitation patterns of four identical disk electrodes in a row. The interelectrode (center-to-center) spacing is $4a$.

by that of the outer electrodes to produce a pattern similar to what could be obtained with only two electrodes spaced halfway between each of the (+) and (-) electrodes. In (B) the outer electrode pair produce a field that is oppositely directed to that of the inner pair. The result is that with increasing depth there is a reversal in the direction of the parallel component of the current density in the region between the inner pair. The parallel component (J_x) changes direction, and hence goes through zero, at the point marked (a). By switching between the excitation patterns (A) and (B) it might be possible to selectively stimulate nerves at different depths.

Figure 61 shows a surface plot where the perpendicular component of the E -field (E_z) for the electrode configuration of Figure 60 (B) is shown as a height above the XZ -plane for E_z inwardly directed and below the XZ -plane for E_z outwardly directed. The darkened contour corresponds to the point where E_z changes direction as it goes through zero.

A multiple electrode configuration consisting of two coplanar electrode pairs arranged as in Figure 62 would produce a field configuration that is the corrolary of that shown in Figure 60 (A). If such a configuration is used, the perpendicular components of the field due to each pair would add to produce a field similar to what could be obtained with only two electrodes spaced halfway between each of the (+) and (-) electrodes. The parallel components of the fields of the two electrode pairs would be orthogonal and have no effect on each other. The field in each of the (perpendicular) central planes of the two electrode pairs would be similar to that of a single bipolar pair.

The three configurations of four electrodes considered here are representative of what can be achieved in terms of field "shaping" with any number of coplanar

Figure 61: Surface plot of E_z for the electrode configuration shown in Figure 60 (B).

Figure 62: Four electrodes in a configuration where the parallel components of the field produced by each pair are orthogonal.

electrodes. Due to the $1/R^3$ decay in the field produced by any bipolar pair, it can be expected that widely spaced bipolar pairs which are simultaneously excited will have little effect on the local fields of the other pairs. If the spacing between the electrodes of any bipolar pair exceeds $3a$, where a is the electrode radius, the local field near each electrode would be well approximated by that of a monopolar electrode of the proper polarity.

4.12 Chapter Summary.

The Moment Method solution for the quasistatic current density field of multiple coplanar electrodes was presented as an efficient and accurate method to predict the relation between the field in both the near and far-field of the electrodes and the electrode configuration in a model where the body is approximated by a uniform resistive half-space. The accuracy of the solution was verified by comparison with other mathematical and physical models described in the literature as well as with asymptotic forms that apply to the far field.

The magnitude of the current density is always a maximum at the electrode surface irrespective of the electrode configuration [85]. Examples of the current density fields of several colinear electrode configurations were presented by means of field line and field magnitude plots in the central plane of the electrodes. The possibility of achieving a reversal in the field direction of both the parallel and perpendicular component of the field was demonstrated. Up to a depth of about one half an electrode radius, the distribution of the perpendicular component of current density appears as a smoothed version of the current distribution on the surface of the electrode.

By comparing the fields of disk electrodes with that of point electrodes it was found that the point electrode approximation is within 10% of that of disk

electrodes if the observation point is more than three electrode radii away from the nearest electrode. The results were normalized to the size of the electrode so as to be applicable to any specific electrode size. For other electrode shapes such as rectangular electrodes the same approximation would hold if the electrode radius is replaced by $1/2$ the maximum dimension of the (noncircular) electrode.

A comparison was made between two and three-dimensional models for the same electrode configuration. It was shown that in two-dimensional models the leading edge effect is overestimated as well as the magnitude of the far-field for a given input current.

A universal spreading resistance curve was derived. It shows that the interelectrode resistance comes within 10% of that of two monopolar electrodes if the separation distance between two coplanar electrodes is larger than 5 electrode radii.

Since the model makes no provision for nonlinearities, inhomogeneities, and anisotropy, or for the complex geometry of the body tissues, the results can at best be taken as qualitative guidelines of the relation between surface electrode configuration and the resulting field distribution. However, the model serves as a valuable test base for checking the accuracy of more complex models and should be viewed as a first step toward the development of a more comprehensive model for the prediction of the current density distribution.

The model could be improved by incorporating the edge conditions in the Moment Method solution. The benefit of such an improvement has been demonstrated for two-dimensional models by Wilton et al. [97]. However, such improvement would only be relevant in predicting the fields close to the edge of the electrode.

Inhomogeneities modelled as planar layers of different conductivity could be incorporated into the model by using an appropriately modified Green's function

in the integral equation formulation. More complex inhomogeneities and complex boundaries would require a finite element or finite difference model. The present model can be used to derive an approximate field distribution which could serve as a good starting point to accelerate the convergence of a more complex finite element or finite difference model.

CHAPTER V

SUMMARY AND CONCLUSIONS.

This dissertation has endeavoured to use mathematical modelling to extend the present knowledge of the factors that play a role in the use of surface electrodes for the electrical stimulation of nerves. To this end two new models were developed for the prediction of the current density distribution produced by surface electrode stimulation. The analysis was based on the solution of Laplace's equation which governs the electric field distribution in volume conductors such as the human body at frequencies where the wavelength is large compared to the dimensions of the part of the body that is stimulated.

Resistive Electrode Model.

In the first of these models the study focussed on some of the factors that affect the current density distribution at the electrode-body interface in the use of rectangular resistive electrodes since such electrodes are in common use. The surface current density was of particular interest since with any combination of surface electrodes applied to a volume conductor, the current density is always the highest at the surface. The surface-body contact is the point where harmful effects ranging from skin irritation to electrode burn is thus most likely to occur if the current density exceeds acceptable limits. In Functional Electrical Stimulation the use of resistive electrodes made out of conductive rubber has been motivated

primarily because of the improved contact to the complex geometry of the body afforded by the pliability of the electrodes. The exact current density distribution of such electrodes is not well known since no noninvasive methods are available for the direct measurement of the surface current density.

With respect to the resistive electrode model, the following was accomplished:

- 1.0 A general model was developed which is capable of representing a wide range of electrode shapes, feed configurations and electrode-to-body conductivity ratios.
- 2.0 The model implemented a novel combination of the finite difference technique and the integral equation technique for solving Laplace's equation.
- 3.0 The accuracy of the solution was checked against known solutions and measured data.
- 4.0 The model was implemented in a versatile computer code which permits interactive modification of the problem parameters and the graphical display of the computed current density and voltage distribution in the form of two-dimensional surface plots.
- 5.0 By the inclusion of the known edge condition in the three-dimensional moment method solution of the integral equation on the half-space representing the body, greater accuracy and faster convergence was obtained than was previously achieved with only ordinary pulse basis functions.
- 6.0 It was shown that the modification of the electrode thickness and the electrode conductivity can have equivalent effects on improving the surface current density distribution in situations where feed region peaks would otherwise occur.

7.0 An in vitro measurement technique was presented which produces a current density distribution image on a simulated volume conductor. Measurements performed by this technique demonstrated the possibility of a feed region peak for two types of commercially produced conductive rubber electrodes.

8.0 A good comparison between the predicted and measured voltage distribution was demonstrated for an non-invasive in vivo measurement technique whereby the surface current density distribution can be determined indirectly. This comparison was achieved using measured values of the electrode parameters and a value of body conductivity which produced an optimum match. The value of the body conductivity thus inferred is within the range of measured values of effective body conductivity. The current density distribution corresponding to the measured voltage distribution suggests that harmful feed region peaks are not likely to occur in vivo.

Multiple Electrode Model.

The second model endeavoured to address the following questions concerning the use of multiple electrodes: (a) How the current distribution on each electrode is affected by the presence of proximal electrodes; (b) How much current is required to achieve a given field strength at a particular depth below the surface and how it depends on the electrode configuration; (c) What is the difference between the fields of monopolar and that of bipolar electrodes; (d) Is it possible to control the magnitude and direction of a particular field component by the use of multiple electrodes; and (d) At what distance from the electrodes is it acceptable to replace the electrodes by point sources.

A simple model representing two or more disk electrodes was chosen in order to facilitate comparison with earlier results and results available in the literature. The

body was represented by a uniform conducting half-space. An integral formulation of Laplace's equation was solved by means of the moment method with pulse basis functions and collocation.

With this model the following was accomplished:

- 1.0 The model was set up to accommodate general configuration of coplanar disk electrodes on a resistive half-space. The computational cost increases roughly proportional to the number of electrodes.
- 2.0 The accuracy of the model was checked against known surface current density distributions for a single disk electrode and for two disk electrodes, using an alternative solution method proposed by Kobayashi.
- 3.0 The dependence of the leading edge effect on the proximity of two electrodes was demonstrated using the Kobayashi model and compared to the exact results for a two-dimensional equivalent model. It was shown that the two-dimensional approximation, which has been used in finite element models with more complex body representations, overestimates the extent of the leading edge effect.
- 4.0 The accuracy of the computed solution for two disk electrodes of the current density distribution at points not on the electrodes was verified by comparison with results in the literature which were obtained by means of a physical model.
- 5.0 A far-field criterion for volume conductor fields was established by showing at what point disk electrode and point source current density fields become equivalent. The use of this criterion can simplify the modelling of multiple electrodes if the region of interest is in the far-field.

- 6.0 Computed data was provided in the form of field line and magnitude plots for a number of collinear electrode configurations. The plots can be used to infer to first approximation the surface current to produce a given field strength in a given direction at a particular depth in the body.
- 7.0 A limited extent of control of the current density field by the use of multiple electrodes was demonstrated by showing the possibility of a reversal in the direction of both parallel and perpendicular components of the field at a particular direction.

Future Work.

The need for further research into several areas became apparent during the course of this study. For this study the working hypothesis was that it is the component of the electric field parallel to the nerve that produces electrical excitation if of sufficient strength and duration to cause depolarization. The exact mechanism by which an externally applied field produces an excitation of the nerve is not well understood. However, in order to specify a desired field distribution, this mechanism would have to be expressed in quantitative terms.

The present models can be improved in many ways. For example, the extent of the leading edge effect in resistive surface electrodes could be studied by including more than one electrode in the surface current density model. Alternative models for the body, e.g. a conductive slab or a solid or multilayer cylinder could be incorporated in the solution by using the appropriate Green's function for these geometries in the integral equation formulation. For multilayer models, whether planar or cylindrical, the Green's function is available in the form of a slowly converging series. Alternate representations of this series could make such models more feasible.

More complex models such as a finite element model of a specific limb which represents the complex geometry and distribution of electrical properties of the various tissues could be used if appropriate software and sufficient computing power is available. The lack of consistent and accurate data on the distribution of the electrical properties of human body tissues coincident with anatomical data could be reduced in order to make such complex modelling more meaningful.

The direct in vivo measurement of electric field distribution remains a very difficult task which might be accomplished by the use of miniature percutaneous or implanted field probes.

APPENDIX A

EQUATIONS FOR THE IMPEDANCE MATRIX FOR SINGULAR EDGE AND CORNER PATCHES.

This appendix presents a summary of the equations used in the computation of the elements of the impedance matrix that involve to the singular edge patches defined in Chapter III.

A.1 Background.

The normalized current density distribution of an infinite perfectly conducting strip electrode and a perfectly conducting circular disk electrode have the same

Figure 63: Geometry for comparing an infinite strip and a circular disk electrode.

form, i.e.,

$$\frac{J_z}{J_{ave}} = \frac{c}{\sqrt{(1+x/a)(1-x/a)}} , \quad (A.1)$$

where x/a is the normalized distance from the center of the electrode to the edge as shown in Figure 63. For the disk electrode $c=0.5$, while for the strip electrode $c=2/\pi$. By explicitly including the singular behaviour of this function in the basis functions used for the moment method solution of the surface current density, the convergence is accelerated and more accurate results are obtained than with ordinary constant current density basis functions. This was demonstrated for two-dimensional moment method solutions by Wilton et al. [97].

The general form for the elements of the impedance matrix is given by

$$Z_{m,n} = \frac{1}{2\pi\sigma_b A_n} \int_{x_n-\Delta x/2}^{x_n+\Delta x/2} dx' \int_{y_n-\Delta y/2}^{y_n+\Delta y/2} dy' \frac{P_n(x', y')}{R_m} , \quad (A.2)$$

where

$$\begin{aligned} \sigma_b &= \text{conductivity of the resistive half-space,} \\ A_n &= \text{surface area of the } n\text{-th patch,} \\ (x_n, y_n) &= \text{center of the } n\text{-th source patch,} \\ \Delta x, \Delta y &= \text{dimensions of the surface patches, and} \\ R_m &= \sqrt{(x' - x_m)^2 + (y' - y_m)^2} . \end{aligned}$$

The mutual impedance $Z_{m,n}$ is equal to the potential at the center of the m -th rectangular patch, i.e., at (x_m, y_m) due to a unit current through the n -th patch. The normalized current density distribution on the n -th patch is given by $P_n(\zeta, \eta)$, where (ζ, η) are rectangular coordinates in a patch centered coordinate system. For ordinary constant current patches, the current density distribution is given by $P_n^o = 1$.

The edge and corner patches are shown graphically in Figure 12. Here only square patches are considered for which $\Delta x = \Delta y = \Delta$.

For edge patches the current density distribution $P_n(\zeta, \eta)$, is taken as

$$P_n^e(\zeta, \eta) = \frac{1}{\sqrt{2}\sqrt{1 - 2\zeta/\Delta}}, \quad (\text{A.3})$$

where ζ is the perpendicular distance from the center of the patch to the singular edge and η the orthogonal dimension. This function is a good approximation of (A.1) at the edge and is simpler to integrate over the remainder of the patch. The factor $\sqrt{2}$ is a normalization factor to give a unit current through the patch.

For corner patches the current distribution on the rounded part of the corner patch is taken to be the same as that of a disk electrode with radius Δ . On the rest of the corner patch it is taken as the product of two edge distributions, of the type defined in (A.3), i.e.

$$P_n^c(\zeta, \eta) = \begin{cases} \frac{0.96}{\sqrt{2}\sqrt{1 - (2r/\Delta)^2}} & \text{for } r = \sqrt{\zeta^2 + \eta^2} \leq \Delta/2 \text{ and } \zeta, \eta > 0, \\ \frac{0.96}{\sqrt{2}\sqrt{1 - 2\zeta/\Delta} \sqrt{1 - 2\eta/\Delta}} & \text{otherwise.} \end{cases} \quad (\text{A.4})$$

Here the normalization factor is $0.96\sqrt{2}$. The normalization factors for both edge and corner patches will be derived in the next section. The edge and corner patches are collectively referred to as singular patches and the basis functions used for the singular patches are denoted by P_n^s . For the singular patches, the elements of the impedance matrix are computed using the closed form expressions for ordinary patches and then multiplied by a correction factor given by

$$E = \frac{Z_{m,n}^s}{Z_{m,n}^o}, \quad (\text{A.5})$$

where

$$Z_{m,n}^s = \iint_{B_n} \frac{P_n^s(\zeta, \eta) d\zeta d\eta}{\sqrt{\zeta^2 + \eta^2}}, \quad (\text{A.6})$$

$$Z_{m,n}^o = \iint_{B_n} \frac{P_n^o(\zeta, \eta) d\zeta d\eta}{\sqrt{\zeta^2 + \eta^2}}, \quad (\text{A.7})$$

$$(\text{A.8})$$

Figure 64: Geometry for singular edge and corner patches.

where B_n is the domain of the n -th source patch. The correction factor is independent of the size of the patches.

Three results are derived in this appendix:

- The normalization factor used in the definition of the singular patches,
- The correction factor for self-impedance terms involving singular patches,
- The correction factor for mutual impedance terms involving singular patches.

Without loss of generality one can consider only the singular patches shown in Figure 64.

A.2 Normalization Factor.

The ratio of the current density at the center of the singular patch to the average current density through the patch is designated as C_0 and is defined by the relation

$$1 = C_0 \frac{1}{A_n} \iint_{B_n} \bar{P}_n^s(\zeta, \eta) d\zeta d\eta, \quad (A.9)$$

where $\bar{P}_n^s$ is the un-normalized form of the singular patch distribution. The un-normalized form has unit amplitude at the center of the patch. Hence $P_n^s = C_0 \bar{P}_n^s$.

Corner Patch

Figure 65: Subdivision of singular corner patches for the computation of the normalization factor.

Edge Patches.

For edge patches the area of the patch is $A_n = \Delta^2$. The expression in (A.9) becomes

$$1 = C_0 \frac{1}{\Delta^2} \int_{-\Delta/2}^{\Delta/2} \int_{-\Delta/2}^{\Delta/2} \frac{1}{\sqrt{1 - 2\zeta/\Delta}} d\zeta d\eta . \quad (\text{A.10})$$

The integral can be expressed in closed form by the use of Dwight [16, 191.01].

The result is

$$C_0 = 1/\sqrt{2} . \quad (\text{A.11})$$

Corner Patches.

For corner patches it is convenient to divide the patch into four subregions as shown in Figure 65. The integral is then written as

$$\iint_{B_n} = D_0 + D_1 + D_2 + D_3 , \quad (\text{A.12})$$

where D_k is the integral over the k -th subregion, R_k . The integral over R_0 is best expressed in a polar coordinate system (r, ψ) , i.e.,

$$D_0 = \frac{C_0}{A_n} \int_0^{\pi/2} d\psi \int_0^{\Delta/2} \frac{r dr}{\sqrt{1 - (2r/\Delta)^2}} . \quad (\text{A.13})$$

By using the transformation $x = 2r/\Delta$ and Dwight [16, 321.01] it is easy to show that

$$D_0 = \frac{C_0}{A_n} \left(\frac{\Delta}{2} \right)^2 \frac{\pi}{2} = 0.393 \Delta^2 \frac{C_0}{A_n} . \quad (\text{A.14})$$

For the remaining subregions (A.9) is given by

$$D_k = \frac{C_0}{A_n} \iint_{R_k} \frac{d\zeta d\eta}{\sqrt{1 - 2\zeta/\Delta} \sqrt{1 - 2\eta/\Delta}} . \quad (\text{A.15})$$

By use of the transformation equations $x = 2\zeta/\Delta$ and $y = 2\eta/\Delta$ the integrals can be written in a form that is independent of the patch size. The result is

$$D_1 = D_3 = 0.414 \Delta^2 \frac{C_0}{A_n} , \quad D_2 = (0.414)^2 \Delta^2 \frac{C_0}{A_n} . \quad (\text{A.16})$$

Figure 66: Subdivision of singular edge patches for the computation of the self-impedance terms.

Since the surface area of the corner patch is given by

$$A_n = \frac{3}{4}\Delta^2 + \frac{1}{4}\pi\left(\frac{\Delta}{2}\right)^2 = 0.946 \Delta^2, \quad (\text{A.17})$$

the normalization factor for corner patches becomes

$$C_0 = \frac{0.946}{0.393 + 0.414 + 0.414 + 0.171} = 0.96/\sqrt{2}. \quad (\text{A.18})$$

A.3 Correction Factor for Self-Impedance Terms.

For the self-impedance terms it is required to evaluate the potential at the center of the patch. To remove the singularity from the integral, a transformation to a patch centered polar coordinate system is used. The transformation is defined by

$$\zeta = \rho \cos \psi, \quad \eta = \rho \sin \psi. \quad (\text{A.19})$$

With this transformation the integral in (A.2) can be written as

$$\iint_{B_n} \frac{P_n}{R_m} d\zeta d\eta = \int_0^{2\pi} d\psi \int_0^{r(\psi)} P_n(r, \psi) \rho dr. \quad (\text{A.20})$$

Corner Patch

Figure 67: Subdivision of singular corner patches for the computation of the self-impedance terms.

Edge Patches.

For edge patches it is convenient to divide the patch into four subregions as shown in Figure 66. The required integral in the k -th region is of the form

$$\int_{\psi_a}^{\psi_b} d\psi \int_0^{r(\psi)} \frac{1}{\sqrt{1 - (2\rho \cos \psi)/\Delta}} d\rho . \quad (\text{A.21})$$

In regions 1 and 3 it is possible to express the integral in closed form by the use of the transformation $x = (2\rho \cos \psi)/\Delta$. By using the symmetry about the ζ -axis the integral in region 1 can be written as

$$D_1 = \frac{1}{\sqrt{2}} 2 \int_0^{\pi/4} d\psi \frac{\Delta}{2 \cos \psi} \int_0^1 \frac{dx}{\sqrt{1-x}} \quad (\text{A.22})$$

$$= 2\Delta \ln(1 + \sqrt{2})/\sqrt{2} . \quad (\text{A.23})$$

The corresponding integral in region 3 is given by

$$D_3 = \frac{1}{\sqrt{2}} 2 \int_{\pi}^{5\pi/4} d\psi \frac{-\Delta}{2 \cos \psi} \int_0^{-1} \frac{dx}{\sqrt{1-x}} \quad (\text{A.24})$$

$$= \sqrt{2}(1 - \sqrt{2})\Delta \ln(1 + \sqrt{2}) \quad (\text{A.25})$$

$$= 0.8248\Delta \ln(1 + \sqrt{2})/\sqrt{2} . \quad (\text{A.26})$$

In region 2 the integral can be written as

$$D_2 = \frac{1}{\sqrt{2}} \int_{\pi/4}^{3\pi/4} d\psi \frac{\Delta}{2 \cos \psi} \int_0^{1/\tan \psi} \frac{dx}{\sqrt{1-x}} \quad (\text{A.27})$$

$$= \frac{1}{\sqrt{2}} \int_{\pi/4}^{3\pi/4} d\psi \frac{\Delta}{2 \cos \psi} \left[1 - \sqrt{1 - 1/\tan \psi} \right] . \quad (\text{A.28})$$

In this case the integral on ψ has to be done numerically. Due to symmetry the integrals in regions 2 and 4 are equal. The result is

$$D_2 = D_4 = 0.934 \Delta/\sqrt{2} = 1.059 \Delta \ln(1 + \sqrt{2})/\sqrt{2} . \quad (\text{A.29})$$

Note that for a constant current density, each region contributes exactly

$$\Delta \ln(1 + \sqrt{2}) \quad (\text{A.30})$$

to the integral.

The correction factor for singular edge patches is then given by

$$E = \frac{2 + 1.059 + 0.8248 + 1.059}{4\sqrt{2}} = 0.874 . \quad (\text{A.31})$$

Corner Patches.

For corner patches it is convenient to divide the patch into six subregions as shown in Figure 67. In the region R_0 the integral is given by

$$D_0 = \frac{0.96}{\sqrt{2}} \int_0^{\pi/2} d\psi \int_0^{\Delta/2} \frac{dr}{\sqrt{1 - (2r/\Delta)^2}} = 1.40c\Delta \ln(1 + \sqrt{2}) , \quad (\text{A.32})$$

where $c = \ln(1 + \sqrt{2})$. In each of the other regions the integral is evaluated using Gaussian quadrature with a product rule after the application of the transformation $x = 2\rho/\Delta$. The resulting equations and the numerical values are given by

$$D_1 = \frac{0.96}{\sqrt{2}} \frac{\Delta}{2} \int_{-\pi/4}^0 d\psi \int_0^{1/\cos\psi} dx \frac{1}{\sqrt{1 - x \cos\psi}} \frac{1}{\sqrt{1 - x \sin\psi}} \quad (\text{A.33})$$

$$= 0.574 c\Delta , \quad (\text{A.34})$$

$$D_3 = \frac{0.96}{\sqrt{2}} \frac{\Delta}{2} \int_{3\pi/4}^{5\pi/4} d\psi \int_0^{-1/\cos\psi} dx \frac{1}{\sqrt{1 - x \cos\psi}} \frac{1}{\sqrt{1 - x \sin\psi}} \quad (\text{A.35})$$

$$= 0.592 c\Delta . \quad (\text{A.36})$$

Due to symmetry $D_1 = D_2$ and $D_3 = D_4$. The correction factor for singular corner patches is then given by

$$E = \frac{0.95 + 2(0.574) + 2(0.592)}{4} = 0.821 . \quad (\text{A.37})$$

A.4 Correction Factor for Mutual Impedance Terms.

For mutual impedance terms, the distance between the source and field patches does not become singular. The correction factor in this case is computed by the numerical integration of (A.2).

Example.

Table A.4 gives the correction factor computed for terms close to an edge and a corner patch. The correction factor is independent of the size of the patches. Notice that with increasing distance from the edge, the correction factor goes to unity, as one would expect.

Table 15: Mutual impedance correction factor for singular edge and corner source patches.

Correction factor: $E = Z_{m,n}^s / Z_{m,n}^o$						
$Z_{m,n}^o$: n =ordinary source patch						
$Z_{m,n}^s$: n =singular source patch						
□ = Location of source patch						
EDGE PATCH						
1.01	1.00	0.99	0.99	0.99	0.99	0.99
1.01	1.00	0.99	0.98	0.99	0.99	0.99
1.01	0.97	0.97	0.98	0.98	0.98	0.99
0.98	0.93	0.95	0.97	0.97	0.98	0.98
□ 0.87	0.86	0.93	0.96	0.97	0.98	0.98
0.98	0.93	0.95	0.97	0.97	0.98	0.98
1.01	0.97	0.97	0.98	0.98	0.98	0.99
1.01	1.00	0.99	0.98	0.99	0.99	0.99
1.01	1.00	0.99	0.99	0.99	0.99	0.99
CORNER PATCH						
□ 0.82	0.95	0.98	1.00	1.00		
	0.95	0.94	0.97	0.99	1.00	
	0.98	0.97	0.98	0.99	1.00	
	1.00	0.99	0.99	1.00	1.00	
	1.00	1.00	1.00	1.00	1.00	

APPENDIX B

THE POTENTIAL AT AN ARBITRARY POINT ON A RECTANGULAR PATCH WITH AN ASSUMED CONSTANT CURRENT DENSITY.

This appendix presents a derivation of the equation for the potential at an arbitrary point on the surface of a rectangular patch on a uniform half-space with an assumed constant current density. The potential at the center of the patch is used in Chapter IV for the computation of the self-impedance terms in the impedance matrix for the moment method solution of the surface current density of a disk electrode when the rectangular patch approximation is employed. The potential at other points on the patch is used for the computation of the residual potential in Figure 44.

Consider a rectangular patch with side dimensions $2a \times 2b$ as shown in Figure 68. The patch is assumed to have a constant current density over its surface and to be applied to a uniform resistive half-space with conductivity σ . The potential at an arbitrary field point (x, y) is given by the integral

$$\Psi(x, y) = \frac{1}{2\pi\sigma} \iint_S dx' dy' \frac{1}{\sqrt{(x')^2 + (y')^2}}, \quad (B.1)$$

where the domain of integration, S , spans the surface of the patch. The singularity in the integrand which occurs when the source and field points coincide can be removed by use of a transformation to a polar coordinate system, (ξ, ψ) , centered

$$\theta_2 = \tan^{-1}\left(\frac{b-y}{a-x}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{b-y}{a+x}\right)$$

$$\theta_1 = \tan^{-1}\left(\frac{b+y}{a-x}\right)$$

$$\theta_4 = \tan^{-1}\left(\frac{b+y}{a+x}\right)$$

Figure 68: Geometry for the evaluation of the potential at an arbitrary point on a rectangular constant current density patch.

Table 16: Integrand and limits in the equation for the potential at an arbitrary point on a constant current density rectangular patch with dimensions $2a \times 2b$.

General Form:			
$\Psi(x, y) = \frac{1}{2\pi\sigma} \int_{\psi_a}^{\psi_b} r(\psi) d\psi$			
Region	ψ_a	ψ_b	$r(\psi)$
1	$-\theta_1$	θ_2	$\frac{a-x}{\cos \psi}$
2	θ_2	$\pi - \theta_3$	$\frac{b-y}{\sin \psi}$
3	$\pi - \theta_3$	$\pi + \theta_4$	$-\frac{a+x}{\cos \psi}$
4	$\pi + \theta_4$	$2\pi - \theta_1$	$-\frac{b+y}{\sin \psi}$

on the field point. This converts the integral to the form

$$\Psi(x, y) = \frac{1}{2\pi\sigma} \int_{\psi_a}^{\psi_b} d\psi \int_0^{r(\psi)} d\xi . \quad (B.2)$$

The integral on ξ can be done by inspection so that one only needs to determine the integral on ψ by numerical integration, i.e.,

$$\Psi(x, y) = \frac{1}{2\pi\sigma} \int_{\psi_a}^{\psi_b} r(\psi) d\psi . \quad (B.3)$$

The equations for ψ_a , ψ_b , and $r(\psi)$ in each of the regions defined in Figure 68 are given in Table 16.

APPENDIX C

AN ANALYTIC SOLUTION FOR THE CURRENT DENSITY DISTRIBUTION DUE TO A PERFECTLY CONDUCTING DISK ELECTRODE ON A UNIFORM RESISTIVE HALF-SPACE.

This appendix presents a derivation of the equations for the current density distribution of a perfectly conducting disk electrode applied to the surface of a resistive half-space. The derivation is based on the electrostatic solution for the equipotentials of a perfectly conducting circular disk given by Smythe [80, p. 124]. The potential distribution is given in the form of a closed form solution of Laplace's equation. The results are used to check the accuracy of the moment method solution for multiple coplanar disk electrodes in Chapter IV.

C.1 Electrostatic Solution.

For a perfectly conducting circular disk in an infinite dielectric medium, dielectric constant ϵ , charged to a specified potential V_0 , Smythe gives the following results for the potential distribution, $\Psi(\rho, \phi, z)$, the surface charge density, $q(\rho)$, and the capacitance, C :

$$\Psi(\rho, \phi, z) = \frac{2V_0}{\pi} \tan^{-1} \frac{a}{\sqrt{\theta}}, \quad (\text{C.1})$$

$$q = \frac{Q}{4\pi a^2 \sqrt{1 - (\rho/a)^2}}, \quad (\text{C.2})$$

$$C = 8\epsilon a, \quad (\text{C.3})$$

where

$$\begin{aligned}\theta &= \frac{1}{2} \left(p + \sqrt{p^2 + 4a^2 z^2} \right) , \\ p &= \rho^2 + z^2 - a^2 , \\ Q &= \text{total charge on the disk} , \\ a &= \text{radius of the disk} ,\end{aligned}$$

and (ρ, ϕ, z) is a cylindrical coordinate system with its origin at the center of the disk.

In (C.1) each value of θ corresponds to a different equipotential surface, $\theta=0$ describing the surface of the disk where $\Psi = V_0$. At all points not on the surface of the disk, $\theta > 0$. For points far away from the disk $\theta \rightarrow \infty$ and $\Psi \rightarrow 0$.

In the equivalent volume conductor problem of a circular disk electrode applied to the surface of a uniform resistive half-space, the charge on one side of the disk ($Q/2$), is replaced by the current I , and the surface charge density on one side of the disk, q , by the surface current density, J_z . The expression for the potential distribution is the same as for the electrostatic problem.

C.2 Electrode Resistance.

The resistance of the disk $R = V_0/I$ is obtained as follows:

At an infinite distance from the disk, the current density is directed radially away from the center of the disk and has a magnitude given by

$$J = \frac{I}{2\pi r^2} , \quad r = \sqrt{\rho^2 + z^2} . \quad (C.4)$$

The equipotential surfaces at a large distance become hemispheres of radius r and $\theta \rightarrow r^2$ so that

$$\frac{\partial \theta}{\partial r} \rightarrow 2r. \quad (C.5)$$

In spherical coordinates the radial component of the current density, J_r , is given by

$$J_r = -\sigma \frac{\partial \Psi}{\partial r} , \quad (C.6)$$

which can be applied to (C.1) to give

$$J_r \approx J \rightarrow -\sigma \frac{\partial \Psi}{\partial \theta} \frac{\partial \theta}{\partial r} = \frac{2\sigma a V_0}{\pi r^2} , \quad (C.7)$$

for $\theta \rightarrow r^2 \rightarrow \infty$. Equating (C.4) to (C.7) then gives

$$R = \frac{V_0}{I} = \frac{1}{4\sigma a} . \quad (C.8)$$

This result can also be obtained from (C.3) and the relation

$$RC = \frac{2\epsilon}{\sigma} , \quad (C.9)$$

which is a general result describing the analogy between the infinite dielectric electrostatic problem and the equivalent surface electrode volume conductor problem [80, p. 258].

In Chapter IV this result is used in the expression for the impedance of a monopolar disk electrode.

C.3 Current Density Distribution

The current density in the resistive half-space is symmetric about the z -axis and given by

$$\mathbf{J}(\rho, z) = -\sigma \nabla \Psi = -\sigma \frac{\partial \Psi}{\partial \theta} \nabla \theta = J_\rho \hat{\rho} + J_z \hat{z} \quad (C.10)$$

where $\hat{\rho}$ and $\hat{z}$ are unit vectors along the ρ - and z -axes respectively. From (C.1) with V_0 substituted from (C.8) one obtains

$$\frac{\partial \Psi}{\partial \theta} = -\frac{I}{4\pi\sigma} \cdot \frac{1}{(a^2 + \theta)\sqrt{\theta}} . \quad (C.11)$$

Let

$$M = \frac{\rho^2}{(a^2 + \theta)^2} + \frac{z^2}{\theta^2} = \frac{1}{(a^2 + \theta)^2} \left[\rho^2 + \left(\frac{a^2 + \theta}{\theta} \cdot z \right)^2 \right] \quad (C.12)$$

and

$$\zeta = \frac{a^2 + \theta}{\theta} \cdot z . \quad (C.13)$$

Then differentiating (C.12) with respect to ρ and z gives

$$\frac{2\rho}{a^2 + \theta} - M \frac{\partial \theta}{\partial \rho} = 0 \quad \text{and} \quad \frac{2z}{\theta} - M \frac{\partial \theta}{\partial z} = 0 \quad (C.14)$$

which implies

$$\frac{\partial \theta}{\partial \rho} = \frac{2\rho}{M(a^2 + \theta)} \quad \text{and} \quad \frac{\partial \theta}{\partial z} = \frac{2z}{M\theta} \quad (C.15)$$

so that

$$\nabla \theta = \hat{\rho} \frac{\partial \theta}{\partial \rho} + \hat{z} \frac{\partial \theta}{\partial z} = \frac{2}{M} \left\{ \frac{\rho}{a^2 + \theta} \hat{\rho} + \frac{z}{\theta} \hat{z} \right\} . \quad (C.16)$$

Substitution of (C.11) and (C.16) in (C.10) then yields the desired expression for the surface current density at points both on and below the surface

$$\mathbf{J}(\rho, z) = \frac{I}{2\pi a^2} \left\{ \left[\theta/a^2 \right]^{\frac{1}{2}} \left[(\rho/a)^2 + (\zeta/a)^2 \right] \right\}^{-1} \left\{ \frac{\rho}{a} \hat{\rho} + \frac{\zeta}{a} \hat{z} \right\} \quad (C.17)$$

The current density on the surface of the electrode, is obtained by setting $\theta = 0$ in (C.1) to obtain

$$\mathbf{J}(\rho) = \frac{I}{2\pi a^2} \left[1 - (\rho/a)^2 \right]^{-\frac{1}{2}} \hat{z} , = \frac{J_{\text{ave}}}{2\sqrt{1 - (\rho/a)^2}} \hat{z} , \quad (C.18)$$

for $0 \leq \rho \leq a$, where $J_{\text{ave}} = I/(\pi a^2)$ is the average current density for a disk electrode. This is a well-known result, first obtained by Weber in 1873 [93]. It is the same as that given by (C.2) with the appropriate substitution for Q .

The surface current density is normal to the surface of the electrode and reaches a maximum at the edge ($\rho = a$).

This completes the derivation of the equations for the resistance and current density of a circular disk electrode applied to a uniform resistive half-space.

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