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Positive definite unimodular lattices with trivial automorphism groups

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The Ohio State University, 1988

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**POSITIVE DEFINITE UNIMODULAR LATTICES
WITH TRIVIAL AUTOMORPHISM GROUPS**

DISSERTATION

**Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University**

by

Etsuko Bannai, M.S.

*** * * * ***

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1988

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A handwritten signature in dark ink, appearing to read "John S. Hsia", is written over a horizontal line.

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ACKNOWLEDGMENTS

I would like to express my appreciation to Professor John S. Hsia for supervising this thesis, in particular for suggesting this research problem and valuable discussions. I also would like to thank Professor P. Ponomarev and Professor D. B. Shapiro in the reading committee of this thesis for their careful reading of this manuscript. I would like to thank Professor A. G. Earnest for making the thesis of Rehmann available to me. I would like to thank Ms. Sue Hunter for her excellent typing within a short period. I would like to thank my parents, my husband's parents and many of my friends for their encouragement. Finally I would like to thank my husband Eiichi and four sons, Kenichi, Hideo, Shinzo and Koshiro for their patience and understanding.

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2. Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2, *J. Math. Soc. Japan*, Vol. 28, No. 3 (1976), pp. 447-454.
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INTRODUCTION

The main objective of this thesis is to prove the following two theorems.

Theorem 1. If m is sufficiently large, then there exists a lattice with the trivial automorphism group (i.e. $\{\pm 1\}$) in every genus of positive definite unimodular integral lattices of rank m . More precisely, (i) the assertion holds if $m \geq 43$ for odd unimodular lattices, (ii) the assertion holds if $m \geq 144$ for even unimodular lattices.

Theorem 1 is obtained from the following stronger theorem.

Theorem 2. Let ω be the mass of the given genus of positive definite unimodular lattices of rank m and ω' be the mass of all the classes in the genus with nontrivial automorphisms. Then the ratio of the mass ω'/ω is bounded above by $33(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ for odd unimodular lattices of dimension $m \geq 43$ and by $2^{m+1}(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ for even unimodular lattices of dimension $m \geq 144$. In particular, this ratio ω'/ω approaches 0 very rapidly as m increases.

Here is a brief historical background concerning this problem. The existence of lattices with the trivial automorphism group is known. O'Meara [9, 1975] gave an algorithm to construct such a lattice starting from any given lattice. In this process the discriminants of the lattices increase in each step. Biermann [1, 1981] proved the existence of a lattice with the trivial automorphism group in every genus of positive definite integral lattices of any dimension with sufficiently large discriminant. In his proof the

fact that the discriminant is very large is crucial. We are instead interested in lattices with small discriminant. It seems that the existence of any unimodular lattice with the trivial automorphism group has not been known. This was, however, anticipated in [6]. For a treatment over localizations of polynomial rings, see [13].

On the other hand Watson [17, 1979] has shown the existence of an indecomposable lattices in every genus of positive definite integral lattices if the dimension $m \geq 14$. Clearly a lattice having the trivial automorphism group is indecomposable, but not vice-versa. But his idea of estimating the mass of decomposable lattices in the given genus is, however, very useful for our study. (Contrarily to Watson, Biermann estimated the number of the classes in the genus with non-trivial automorphism groups).

Our Theorems 1 and 2 show that there are abundance of lattices with the trivial automorphism group in every genus of positive definite unimodular integral lattices if the dimension is not so small, and furthermore the ratio

$$\frac{\text{mass of classes in } G_L \text{ with trivial automorphism group}}{\text{mass of } G_L}$$

approaches to 1 as the dimension increases.

It is important to note, however, that the *explicit construction* of a lattice with the trivial automorphism group is still an open problem!

Now we will give an outline of the proof of Theorems 1 and 2.

Let L be a positive definite unimodular integral lattice of rank m with a bilinear form $B(-, -)$. Let G_L be the genus of L . Define $\omega(L) = \sum_{cls \ M \subseteq G_L} \frac{1}{|O(M)|}$ and $\omega'(L) = \sum_{\substack{cls \ M \subseteq G_L \\ |O(M)| > 2}} \frac{1}{|O(M)|}$. The basic strategy of the proof is to estimate this ratio $\omega'(L)/\omega(L)$.

Let M be a lattice in G_L with non-trivial automorphisms. If the order of the automorphism group of M is divisible by some odd prime number q , then there exists an isometry g of order q . Since $g^q = 1$, the minimal polynomial of g must divide $x^q - 1$. Therefore the minimal polynomial of g is $x^q - 1$ (reducible) or $x^{q-1} + \dots + x + 1$ (irreducible). If the minimal polynomial is $x^q - 1$ then $q \leq m$ and if the minimal polynomial is $x^{q-1} + \dots + x + 1$ then $q - 1 \leq m$. Next, consider the case when order of the automorphism group is divisible by a power of 2. If M has an isometry g of order 2 which is not trivial then the minimal polynomial is $x^2 - 1$. If every isometry of M of order 2 is trivial and $4 \mid |O(M)|$, then any isometry of order 4 has minimal polynomial is $x^2 + 1$.

Thus we have seen that the lattices with nontrivial automorphisms is either (or both) of the following two types.

Type $R(q)$. Lattice with an isometry of order a prime number $q \leq m$ (q is odd or 2) whose minimal polynomial is reducible, i.e., $x^q - 1$.

Type $IR(q)$. Lattice with an isometry of order $q \leq m + 1$ (where q is an odd prime number or 4) whose minimal polynomial is irreducible i.e., if q is an odd prime then $x^{q-1} + \dots + x + 1$ and if $q = 4$ then $x^2 + 1$.

Let

$$\omega_{R(q)} = \sum_{\substack{cls\ M \subseteq G_L \\ type\ R(q)}} \frac{1}{|O(M)|} \text{ and } \omega_{IR(q)} = \sum_{\substack{cls\ M \subseteq G_L \\ type\ IR(q)}} \frac{1}{|O(M)|}.$$

Then we have

$$\omega'(L) \leq \sum_{\substack{q\ prime \\ 2 \leq q \leq m}} \omega_{R(q)} + \sum_{\substack{q\ odd\ prime \\ (q-1) \mid m\ or \\ q=4}} \omega_{IR(q)}$$

Remark. If G_L contains a lattice of type $IR(q)$ then the rank m has to be an even number. Therefore if m is an odd number then we only need to consider the type $R(q)$.

We are going to estimate $\omega_{R(q)}/\omega(L)$ and $\omega_{IR(q)}/\omega(L)$ separately. The idea of dividing lattice in the given genus into those two types is due to Biermann [1]. He introduced hermitian structures to the lattices of type $IR(q)$. In this thesis we go further and introduce also hermitian structures to the lattices of type $R(q)$.

Type $R(q)$. Let M be a lattice of type $R(q)$ in G_L . Let $G = \langle g \rangle$ be the cyclic group of order q generated by $g \in O(M)$, $\mathbb{Q}G$ be the group algebra and $\Lambda = \mathbb{Z}G$ be the group ring. Let ζ be a primitive q -th root of unity. Then $\mathbb{Q}G$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}(\zeta)$ under the mapping which corresponds g to $(1, \zeta)$. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}[\zeta] \subset \mathbb{Q} \times \mathbb{Q}(\zeta)$. Then $\Lambda = \mathbb{Z}G$ is injected into Γ by above isomorphism. We often identify $\mathbb{Q}G$ and $\mathbb{Q} \times \mathbb{Q}(\zeta)$. (Note that if $q = 2$ then $\zeta = -1$ and $\mathbb{Q}(\zeta) = \mathbb{Q}$.)

The following arguments are shown in Quebbemann [11, §2] in more general way but we only need the rational number and rational integer case, so we give them here again.

Let $W = \mathbb{Q}M$, $W_0 = (\sum_{i=0}^{q-1} g^i)W$ and $W_1 = (g - 1)W$. Then $W = W_0 \perp W_1$ (with respect to the bilinear form B). The group algebra $\mathbb{Q}G$ acts on W , and M is a Λ -module. Let $h(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x, y)g^i$ for $x, y \in W$. Then h is an hermitian form with respect to the involution sending g to g^{-1} (or equivalently $(1, \zeta)$ to $(1, \bar{\zeta})$ where $\bar{\zeta}$ is the complex conjugate of ζ). Clearly $h(x, y) \in \Lambda$ for any $x, y \in M$ and M is an hermitian Λ -lattice. Let us denote the hermitian structure of W and M by $\tilde{W}$ and $\tilde{M}$ respectively. Let $\Gamma\tilde{M} = M_0 \times M_1$ where $M_0 = (\mathbb{Z} \times \{0\})\tilde{M}$ and $M_1 = (\{0\} \times \mathbb{Z}[\zeta])\tilde{M}$. Let B_0 be the restriction of h to M_0 and h_1 to be the restriction of h to M_1 . We will

see in §3 that (M_0, B_0) is a positive definite integral $\mathbb{Z}$ -lattice with the bilinear form B_0 and (M_1, h_1) is a totally positive definite hermitian $\mathbb{Z}[\zeta]$ -lattice with the hermitian form h_1 . Let us denote the structure (M_1, h_1) by $(\mathcal{M}_1, h_1)$. Thus while the original $\mathbb{Z}$ -lattice (M, B) may be orthogonally indecomposable the Γ -lattice $(\Gamma\tilde{M}, h)$ which is slightly bigger, has important orthogonal (with h) decomposition. This is the central reason for dealing with Γ . We are going to estimate $\omega_{R(q)}$ through Siegel's mass formula for integral $\mathbb{Z}$ -lattices M_0 and the hermitian $\mathbb{Z}[\zeta]$ -lattices $\mathcal{M}_1$. (Note that if $q = 2$ then $\mathcal{M}_1$ is also a positive definite integral $\mathbb{Z}$ -lattice with bilinear form h_1 . Only this special case we shall denote $\mathcal{M}_1$ by M_1 and h_1 by B_1 .)

Let λ be a prime element in $\mathbb{Z}[\zeta]$ above q . Let $I = q\Lambda + (1 - g)\Lambda \subset \mathbb{Z}G$, then I corresponds to $q\mathbb{Z} \times \lambda\mathbb{Z}[\zeta]$ in $\Gamma = \mathbb{Z} \times \mathbb{Z}[\zeta]$.

Let $V_0(M_0) = M_0/qM_0$, $V_1(\mathcal{M}_1) = \mathcal{M}_1/\lambda\mathcal{M}_1$ be the vector spaces over the finite field $\mathbb{F}_q$ of q elements. Let b_0 and b_1 be the bilinear form induced by $B_0(\text{mod } q\mathbb{Z})$ and $h_1(\text{mod } \lambda\mathbb{Z}[\zeta])$ respectively. Let $V'_0(M_0) = V_0(M_0)/V_0(M_0)^\perp$ and $V'_1(\mathcal{M}_1) = V_1(\mathcal{M}_1)/V_1(\mathcal{M}_1)^\perp$ be the vector spaces over the field $\mathbb{F}_q$ with the nonsingular bilinear-form b'_0 and b'_1 induced by b_0 and b_1 . (Here a bilinear form b on a vector space V is said to be non-singular if $b(x, y) = 0$ for any $x \in V$ implies $y = 0$.) Then it is shown in Quebbemann [11], Satz 2.1, that $(V'_0(M_0), b'_0) \cong (V'_1(\mathcal{M}_1), b'_1)$. See also Proposition 3 in §3 below.

Let $q \leq m$ be a prime number, and let $r \geq 1$ and $\rho \geq 0$ be integers satisfying $r(q - 1) \leq m - 1$ and $\rho \leq \min(r, m_0)$, where $m_0 = m - r(q - 1)$ (if $q = 2$ assume $r \leq m_0$, i.e., $r \leq \lfloor \frac{m}{2} \rfloor$). For each such q, r and ρ define $L(q, r, \rho)$ to be the set of.

all the lattices M in G_L such that M is of type $R(q)$, $\text{rank}_{\mathbb{Z}[\zeta]} M_1 = r$ (therefore $\text{rank}_{\mathbb{Z}} M_0 = m_0$), and $\dim_{\mathbb{F}_q} V'_0(M_0) = \dim_{\mathbb{F}_q} V'_1(M_1) = \rho$.

Define $G(q, r, \rho)$ to be the set of all the pairs of genera $(G_{N_0}, G_{\mathcal{N}_1})$ satisfying the following conditions:

If $q = 2$ then

- (i) N_0 and $\mathcal{N}_1 = N_1$ are positive definite integral $\mathbb{Z}$ -lattices of rank m_0 and r respectively.
- (ii) $dN_0 = 2^{m_0-\rho}$, $dN_1 = 2^{r-\rho}$.
- (iii) Let y be in the vector space $\mathbb{Q}N_i$, $i = 0, 1$, then $B_i(x, y) \in 2\mathbb{Z}$ for all $x \in N_i$ implies $y \in N_i$. Where B_i is the bilinear form of N_i .
- (iv) $(V'_0(N_0), b'_0)$ is isometric to $(V'_1(N_1), b'_1)$.
- (v) $\dim_{\mathbb{F}_q} V'_i(N_i) = \rho$, $i = 0, 1$.

If $q \neq 2$ then

- (i) N_0 is a positive definite integral $\mathbb{Z}$ -lattice of rank m_0 , and $\mathcal{N}_1$ is a totally positive definite hermitian $\mathbb{Z}[\zeta]$ -lattice of rank r .
- (ii) $dN_0 = q^{m_0-\rho}$, $N_E \mathbb{Q}(\delta \mathcal{N}_1) = q^{r-\rho}$.
- (iii) Let $y \in \mathbb{Q}N_0$, then $B_0(x, y) \in q\mathbb{Z}$ for all $x \in N_0$ implies $y \in N_0$. Let $y \in \mathbb{Q}(\zeta)\mathcal{N}_1$, then $h_1(x, y) \in \lambda\mathbb{Z}[\zeta]$ for all $x \in \mathcal{N}_1$ implies $y \in \mathcal{N}_1$. Here B_0 is the bilinear form of N_0 and h_1 is the hermitian form of $\mathcal{N}_1$.
- (iv) $(V'_0(N_0), b'_0)$ is isometric to $(V'_1(\mathcal{N}_1), b'_1)$.
- (v) $\dim_{\mathbb{F}_q} V'_0(N_0) = \dim_{\mathbb{F}_q} V'_1(\mathcal{N}_1) = \rho$.

(vi) If L is even then $\mathcal{N}(N_0) \subseteq 2\mathbb{Z}$.

Then we show in §3 that for any lattices M_0 and M_1 constructed from a lattice $M \in L(q, r, \rho)$, the pair (G_{M_0}, G_{M_1}) is contained in $G(q, r, \rho)$ and

$$\sum_{\substack{\text{cls } K \subseteq L(q, r, \rho) \\ \Gamma K \cong \Gamma M}} \frac{1}{|O(K)|} \leq \frac{|I(V'_0(M_0), V'_1(M_1))|}{|Aut M_0| |Aut M_1|}.$$

Therefore

$$\sum_{\text{cls } K \subseteq L(q, r, \rho)} \frac{1}{|O(K)|} \leq \sum_{(G_{N_0}, G_{M_1}) \in G(q, r, \rho)} |I(V'_0(N_0), V'_1(M_1))| \omega(N_0) \omega(M_1)$$

where $Aut M_0$ is the orthogonal group of M_0 with respect to B_0 , $Aut M_1$ is the unitary group of M_1 with respect to h_1 (if $q = 2$ then $M_1 = M_1$, $h_1 = B_1$, and $Aut M_1$ is the orthogonal group of M_1 with respect to B_1), $\omega(N_0)$ is the mass of N_0 , $\omega(M_1)$ is the mass of M_1 , and $I(V'_0(N_0), V'_1(M_1))$ is the set of all the isometries from $(V'_0(N_0), b'_0)$ to $(V'_1(M_1), b'_1)$ which depends only on the pair (G_{N_0}, G_{M_1}) (if $q = 2$ then $M_1 = N_1$ is an integral $\mathbb{Z}$ -lattice). The order of the orthogonal groups over the finite fields are explicitly known (see e.g. [4]). Finally we will show that

$$\omega_{R(q)} < \sum_{r=1}^{\lfloor \frac{m-1}{q-1} \rfloor} \sum_{\rho=0}^{\min(r, m_0)} \sum_{(G_{N_0}, G_{M_1}) \in G(q, r, \rho)} |I(V'_0(N_0), V'_1(M_1))| \omega(N_0) \omega(M_1).$$

Using the classification of quadratic lattices over q -adic integers by O'Meara and the classification of hermitian lattices over local fields by Jacobowitz and Shimura we show in Chapter III that the number of the pairs of genera in $G(q, r, \rho)$ is bounded by a constant (independent of m, q, r and ρ).

Type $IR(q)$. Let M be a lattice in G_L of type $IR(q)$ with the isometry g . The following argument for the lattices of Type $IR(q)$ is based on Biermann [1]. He did not define hermitian form explicitly. Here we will give a explicit definition.

Let M be a lattice in G_L of type $IR(q)$, g the isometry of M . Let $W = \mathbb{Q}M$ and define the action of $\mathbb{Q}(\zeta)$ on W through $\zeta \cdot x = g(x)$ for all $x \in W$. Then W is a $\mathbb{Q}(\zeta)$ vector space and M is a $\mathbb{Z}[\zeta]$ -module. For $q \neq 4$ define $h(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x, y)\zeta^i$ and for $q = 4$, $h(x, y) = \frac{1}{2} \sum_{i=0}^3 B((q^{-i}x, y)\zeta^i$. Then h is a totally positive definite hermitian form. Clearly $h(x, y) \in \mathbb{Z}[\zeta]$ for $x, y \in M$. We show in §4 that with this hermitian form M has the $\lambda\mathbb{Z}[\zeta]$ -modular hermitian structure if $q \neq 4$, and unimodular hermitian lattice structure if $q = 4$.

Let us denote the hermitian $\mathbb{Z}[\zeta]$ -lattice structure of M by $\mathcal{M}$. Then we will see in §4 the following inequalities:

$$\begin{aligned} \omega_{IR(q)} &\leq \omega(\mathcal{M}) \quad \text{for } q \neq 4 \\ \omega_{IR(4)} &\leq \sum_{\mathcal{N}} \omega(\mathcal{N}) \text{ for } q = 4 \end{aligned}$$

where the summation in the second inequality is over the set of genera of totally positive definite unimodular hermitian $\mathbb{Z}[\sqrt{-1}]$ -lattices $\mathcal{N}$ of rank $r = \frac{m}{2}$.

The classification of hermitian lattices over local fields by Jacobowitz and Shimura shows that there is exactly one genus of totally positive definite $\lambda\mathbb{Z}[\zeta]$ -modular hermitian $\mathbb{Z}[\zeta]$ lattices of rank $r = \frac{m}{q-1}$ with $q \neq 4$, and exactly two genera of totally positive definite unimodular hermitian $\mathbb{Z}[\sqrt{-1}]$ -lattices of rank $r = \frac{m}{2}$ of norm $\mathbb{Z}[\sqrt{-1}]$ and exactly one genus of such lattices of norm $2\mathbb{Z}[\sqrt{-1}]$.

In Chapter IV we give the actual evaluations of $\omega_{R(q)}/\omega(L)$ and $\omega_{IR(q)}/\omega(L)$. In Chapter V we complete the proof of the theorems.

In Chapter III we give some formulas of local densities of quadratic or hermitian lattices. Formulas for the quadratic cases are already studied extensively by Siegel [15], Watson [16] and others (e.g. Pall [10]). We present part of their results here for the reader's convenience. For the hermitian case, it seems that it has been so far calculated only for the unimodular lattices at unramified primes (see Braun [3], Rehmann [12]). Here we obtain formulas for certain lattices at ramified primes.

In Theorem 2, the conditions on rank ($m \geq 43$ for odd unimodular case and $m \geq 144$ for even unimodular case) seems to be the best possible from the approach taken by our method. The ratio $\omega_{R(2)}/\omega(L)$ has the same order as that of the function $(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ in the odd case and $2^m(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ in the even case. The ratio $\omega_{R(2)}/\omega(L)$ turned out to be the dominant term in $\omega'(L)/\omega(L)$ by calculation. If $m \leq 42$ (resp. $m \leq 136$) then our method can not give a good estimation.

Also using the proof of Theorem 2 we can show the following Theorem 3.

Theorem 3. If $m \geq 43$, then there exists a lattice whose full automorphism group has order a power of 2 in every genus of positive definite unimodular lattices of rank m .

CHAPTER I

PRELIMINARIES

In this chapter we will give the definitions and the basic concepts we need in the proof of the theorems.

Let W be a vector space over a number field E . Let S be the ring of algebraic integers of E . We call a S -submodule M of W a lattice in W if there is a base $x_1, \dots, x_m$ for W such that $M \subseteq Sx_1 + \dots + Sx_m$. We say that M is a lattice on W if M generates W over E , and in such a case m is the rank $\text{rank}_S(M)$ of M . Unexplained notations and terminology may be found in O'Meara's book [8].

§1. Quadratic lattices and their mass formula.

In this section our field E is the rational number field $\mathbb{Q}$.

Let W be a quadratic space of dimension m . Let Q be the quadratic form on W and B be the associated bilinear form on $W \times W$, i.e., $Q(x + y) = Q(x) + Q(y) + 2B(x, y)$. Let L be a lattice on W . Then there is a base $x_1, \dots, x_m$ of W such that $L = \mathbb{Z}x_1 + \dots + \mathbb{Z}x_m$. Define the discriminant dL of L to be the determinant of the matrix $(B(x_i, x_j))$. Then dL is independent of the choice of the base. Scale sL of L is the $\mathbb{Z}$ -module generated by the subset $B(L, L)$ of $\mathbb{Q}$. Define the norm nL to be the $\mathbb{Z}$ -module generated by $Q(L) \subseteq B(L, L)$. We say L is unimodular if $dL = \pm 1$ and $sL = \mathbb{Z}$. We define a unimodular lattice to be odd if $nL = \mathbb{Z}$ and even if $nL = 2\mathbb{Z}$.

Let p be a prime spot (prime number in $\mathbb{Z}$ or infinite prime). Let $\mathbb{Q}_p$ be the completion of the field $\mathbb{Q}$ at p . For a finite prime p , $\mathbb{Z}_p$ denotes the ring of p -adic integers. Let $W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then the bilinear form B induces a bilinear form on $W_p \times W_p$. We define the genus G_L of the lattice L to be the set of all lattices M on W with the following property: for each finite prime p there exists an isometry σ_p in $O(W_p)$ such that $M_p = \sigma_p L_p$. We say a lattice M on W is in the class of L , $cls L$, if there exists an isometry σ in $O(W)$ such that $M = \sigma L$. It is known that the number of the isometric classes in a genus is finite (see O'Meara [8], §103).

If B is a positive definite bilinear form then the orthogonal group $O(M)$ of any lattice M on W is a finite group.

Now we assume that the quadratic space W has positive definite bilinear form B and the lattice L on W is integral, i.e., $B(L, L) \subset \mathbb{Z}$. Define the mass of the lattice L by

$$w(L) = \sum_{cls M \subseteq G_L} \frac{1}{|O(M)|}.$$

Let μ be a positive integer. Let $A_{p^\mu}(L, L)$ be the number of $\mathbb{Z}$ linear maps $\sigma : L \rightarrow L$ which are distinct modulo $p^\mu L$ and satisfy $B(\sigma x, \sigma y) \equiv B(x, y) \pmod{p^\mu}$ for every x and y in L . Then it is known that the limit

$$\lim_{\mu \rightarrow \infty} p^{-\frac{1}{2}\mu m(m-1)} A_{p^\mu}(L, L)$$

exists and stabilizes for μ sufficiently large (see Siegel [15], Hilfssatz 13). We define local density of L at p by

$$\alpha_p(L) = \lim_{\mu \rightarrow \infty} \frac{1}{2} p^{-\frac{1}{2}\mu m(m-1)} A_{p^\mu}(L, L).$$

Then we have the following Siegel's mass formula (see [15], §10) for $m > 1$.

$$(1) \quad w(L) = \frac{2 \prod_{i=1}^m \Gamma(\frac{i}{2}) (dL)^{\frac{m+1}{2}}}{\pi^{\frac{m(m+1)}{4}} \prod_{p \text{ finite}} \alpha_p(L)}.$$

where $\Gamma(x)$ denotes the gamma function.

§2. Hermitian lattices and their mass formula.

In this section our field E is $\mathbb{Q}(\zeta)$ where ζ is a primitive q -th root of unity with $q = 4$ or an odd prime number. Let K be the maximal real sub-field in E (i.e., $K = E \cap \mathbb{R}$). Let $\mathcal{W}$ be a hermitian space over E of dimension r with respect to the complex conjugation of E . Let h be the hermitian form on $\mathcal{W}$. We assume $h : \mathcal{W} \times \mathcal{W} \rightarrow E$ is linear in the first component, $h(x, y) = \overline{h(y, x)}$, and $h(x, \alpha y) = \overline{\alpha} h(x, y)$ for any $\alpha \in E$, where $\overline{\alpha}$ is the complex conjugate of α . Let $\mathcal{M}$ be a lattice on $\mathcal{W}$. Then there exists a base $x_1, \dots, x_r$ of $\mathcal{W}$ and fractional ideals $\mathcal{A}_1, \dots, \mathcal{A}_r$ of E such that

$$\mathcal{M} = \mathcal{A}_1 x_1 + \dots + \mathcal{A}_r x_r.$$

Let $d_h(x_1, \dots, x_r)$ be the determinant of the matrix $(h(x_i, x_j))$. Define the discriminant ideal $\delta \mathcal{M}$ by

$$\delta \mathcal{M} = (\mathcal{A}_1 \overline{\mathcal{A}}_1) \cdots (\mathcal{A}_r \overline{\mathcal{A}}_r) d_h(x_1, \dots, x_r).$$

Scale $s\mathcal{M}$ of $\mathcal{M}$ is a S -module generated by $h(\mathcal{M}, \mathcal{M})$. Define the norm $n\mathcal{M}$ of $\mathcal{M}$ to be the S -module generated by the subset $\{h(x, x) \mid x \in \mathcal{M}\}$ of $E = \mathbb{Q}(\zeta)$. We say $\mathcal{M}$ is $\mathcal{A}$ -modular if $s\mathcal{M} = \mathcal{A}$ and $\delta \mathcal{M} = \mathcal{A}^r$. Particularly, $\mathcal{M}$ is unimodular if $s\mathcal{M} = S$ and $\delta \mathcal{M} = S$.

Let R be the ring of algebraic integers and $\mathcal{P}$ a prime ideal in K or an infinite prime spot of K . Let $K_{\mathcal{P}}$ be a completion of K at $\mathcal{P}$ and $R_{\mathcal{P}}$ be the ring of integers of $K_{\mathcal{P}}$. We define the following:

$$\begin{aligned} E_{\mathcal{P}} &= E \otimes_K K_{\mathcal{P}}, \quad S_{\mathcal{P}} = S \otimes_R R_{\mathcal{P}}, \\ \mathcal{W}_{\mathcal{P}} &= \mathcal{W} \otimes_E E_{\mathcal{P}} \text{ and } \mathcal{M}_{\mathcal{P}} = \mathcal{M} \otimes_S S_{\mathcal{P}} \end{aligned}$$

The complex conjugation induces an involution of $E_{\mathcal{P}}$ which leaves the elements of $K_{\mathcal{P}}$ invariant. There is a unique extension of h on $\mathcal{W}_{\mathcal{P}} \times \mathcal{W}_{\mathcal{P}} \longrightarrow E_{\mathcal{P}}$ at each $\mathcal{P}$. We defined the genus $G_{\mathcal{M}}$ of the lattice $\mathcal{M}$ to be the set of all lattices $\mathcal{N}$ on $\mathcal{W}$ with the following property: for each finite prime $\mathcal{P}$ there exists an isometry $\sigma_{\mathcal{P}}$ in the local unitary group $U(\mathcal{W}_{\mathcal{P}})$ such that $\mathcal{N}_{\mathcal{P}} = \sigma_{\mathcal{P}} \mathcal{M}_{\mathcal{P}}$. We say a lattice $\mathcal{N}$ on $\mathcal{W}$ is in the class $[M]$ of $\mathcal{M}$ if there exists an isometry σ in $U(\mathcal{M})$ such that $\mathcal{N} = \sigma \mathcal{M}$. It is known that the number of the isometric classes in a genus is finite (see Rehmann [12], Satz 3, page 37).

We say the hermitian form h is totally positive definite if h is positive definite at every infinite prime spot $\mathcal{P}$ of K .

Henceforth we assume h is a totally positive definite hermitian form. Then for any lattice $\mathcal{N}$ in the genus $G_{\mathcal{M}}$ of $\mathcal{M}$ the unitary group $U(\mathcal{M})$ is a finite group. Define the mass of the hermitian S -lattice $\mathcal{M}$ on $\mathcal{W}$ by

$$w(\mathcal{M}) = \sum_{[\mathcal{N}] \in G_{\mathcal{M}}} \frac{1}{|U(\mathcal{N})|}$$

Now we assume that h is integral, i.e., $h(\mathcal{M}, \mathcal{M}) \subset S$. Let $\mathcal{P}$ be a finite prime ideal in K , and μ be a positive integer. Let $A_{\mathcal{P}^{\mu}}(\mathcal{M}, \mathcal{M})$ be the number of S -linear maps $\sigma : \mathcal{M} \longrightarrow \mathcal{M}$ which are distinct modulo $\mathcal{P}^{\mu} \mathcal{M}$ and satisfy $h(\sigma x, \sigma y) \equiv h(x, y)$

mod $\mathcal{P}^\mu S$ for every x and y in $\mathcal{M}$. Let p be the prime number in $\mathbb{Z}$ such that $\mathcal{P} \mid p$ and f_p be the residual degree $[R/\mathcal{P} : \mathbb{Z}/p]$. Then it is known that the limit

$$\lim_{\mu \rightarrow \infty} p^{-f_p r^2 \mu} A_{\mathcal{P}^\mu}(\mathcal{M}, \mathcal{M})$$

exists (see Rehmann [12], Hilfssatz 5.3).

Define the hermitian local density $\beta_{\mathcal{P}}(\mathcal{M})$ of $\mathcal{M}$ by

$$\beta_{\mathcal{P}}(\mathcal{M}) = \lim_{\mu \rightarrow \infty} p^{-f_p r^2 \mu} A_{\mathcal{P}^\mu}(\mathcal{M}, \mathcal{M}).$$

Then we have the following mass formula of totally positive definite hermitian S -lattice $\mathcal{M}$ on $\mathcal{W}$ (see Rehmann [12], (4.5), also Braun [3, Satz VI] and Böge [2, pp. 112]).

$$(2) \quad w(\mathcal{M}) = 2N_{K/\mathbb{Q}}(D(E/K))^{\frac{r(r+1)}{4}} \left(\prod_{j=1}^r \frac{(j-1)!}{(2\pi)^j} \right)^{\frac{\varphi(q)}{2}} \\ \cdot D(K/\mathbb{Q})^{\frac{r^2}{2}} N_{E/\mathbb{Q}}(\delta \mathcal{M})^r \prod_{\mathcal{P} \text{ finite}} \beta_{\mathcal{P}}(\mathcal{M})^{-1}.$$

Where $D(E/K)$ is the discriminant of the field E over K , $D(K/\mathbb{Q})$ is the absolute discriminant of the field K , $N_{K/\mathbb{Q}}(D(E/K)) = |R/D(E/K)|$, $N_{E/\mathbb{Q}}(\delta \mathcal{M}) = |S/\delta \mathcal{M}|$, and $\varphi(q)$ is the Euler number of q .

CHAPTER II

GENERAL THEORY

§3. Lattices with nontrivial automorphisms whose minimal polynomials are reducible.

Let M be a lattice in the genus G_L of L of type $R(q)$ with the isometry g of order prime $q \geq 2$. Let $G = \langle g \rangle$ be the cyclic group of order q generated by g . Let $\mathbb{Q}G$ be the group algebra and $\mathbb{Z}G$ be the group ring. Let ζ be a primitive q -th root of unity. Then the group algebra $\mathbb{Q}G$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}(\zeta)$ under mapping which corresponds g to $(1, \zeta)$. Let $\Gamma = \mathbb{Z} \times \mathbb{Z}[\zeta] \subset \mathbb{Q} \times \mathbb{Q}(\zeta)$. Then $\mathbb{Z}G$ is injected into Γ by above isomorphism. We often identify $\mathbb{Q}G$ and $\mathbb{Q} \times \mathbb{Q}(\zeta)$ in the following arguments. The following arguments are based on Quebbemann [11], §2.

Let $W = \mathbb{Q}M$, $W_0 = (\sum_{i=0}^{q-1} g^i)W$ and $W_1 = (g - 1)W$. Then we can easily see that $W = W_0 \perp W_1$, $gx = x$ for every $x \in W_0$ and $(\sum_{i=0}^{q-1} g^i)x = 0$ for every $x \in W_1$. The group algebra $\mathbb{Q}G$ acts on W . With this action M is a $\mathbb{Z}G$ -module. Let $h(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x, y)g^i$ for x, y in W . Then h is an hermitian form with respect to the involution sending g to g^{-1} . Clearly $h(x, y) \in \mathbb{Z}G$ for any x and y in M . Let us denote the hermitian $\mathbb{Q}G$ structure of W (resp. $\mathbb{Z}G$ structure of M) by $\tilde{W}$ (resp. by $\tilde{M}$).

Proposition 3.1. Let $y \in \tilde{W}$, then $h(x, y) \in \mathbb{Z}G$ for any $x \in \tilde{M}$ if and only if $y \in \tilde{M}$.

Proof. Assume $h(x, y) \in \mathbb{Z}G$ for any $x \in \tilde{M}$. Then by the definition of $h(x, y)$ we have $B(x, y) \in \mathbb{Z}$ for any $x \in M$. Therefore, by 82 : 14b in [8] we have $y \in M = \tilde{M}$. Conversely, if $y \in \tilde{M} = M$ then $B(g^{-i}x, y) \in \mathbb{Z}$ for any $x \in M$ because $g \in O(M)$. Hence we have $h(x, y) \in \mathbb{Z}G$. $\square$

For any $x, y \in \tilde{W}$, let $h(x, y) = (B_0(x, y), h_1(x, y)) \in \mathbb{Q} \times \mathbb{Q}(\zeta)$.

Proposition 3.2. Let $x, y \in W$.

$$(i) \quad \begin{aligned} B_0(x, y) &= \sum_{i=0}^{q-1} B(g^{-i}x, y) \in \mathbb{Q}, \\ h_1(x, y) &= \sum_{i=0}^{q-1} B(g^{-i}x, y)\zeta^i \in \mathbb{Q}(\zeta). \end{aligned}$$

(ii) Let $x = x_0 + x_1$, $y = y_0 + y_1$ where $x_i, y_i \in W_i$ for $i = 0, 1$. Then $h(x, y) = (B_0(x_0, y_0), h_1(x_1, y_1))$.

Proof. (i) is clear because $g^i \in \mathbb{Q}G$ corresponds to $(1, \zeta^i) \in \mathbb{Q} \times \mathbb{Q}(\zeta)$. As for (ii), we have from (i)

$$\begin{aligned} B_0(x, y) &= \sum_{i=0}^{q-1} B(g^{-i}x, y) = \sum_{i=0}^{q-1} B(g^{-i}(x_0 + x_1), y_0 + y_1) \\ &= B\left(\sum_{i=0}^{q-1} g^{-i}x_0 + \sum_{i=0}^{q-1} g^{-i}x_1, y_0 + y_1\right). \end{aligned}$$

Since $x_0 \in W_0 = \left(\sum_{i=0}^{q-1} g^i\right)W$, we have $g^{-i}x_0 = x_0$ for $i = 0, \dots, q-1$ and therefore $\left(\sum_{i=0}^{q-1} g^{-i}\right)x_0 = qx_0$. Since $x_1 \in W_1 = (g-1)W$ we have $\left(\sum_{i=0}^{q-1} g^{-i}\right)x_1 = 0$.

Therefore we have $B_0(x, y) = B(qx_0, y_0 + y_1) = B(qx_0, y_0) = B\left(\sum_{i=0}^{q-1} g^{-i}x_0, y_0\right) = B_0(x_0, y_0)$.

Similarly $h_1(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x, y)\zeta^i$

$$\begin{aligned}
 &= \sum_{i=0}^{q-1} B(g^{-i}x_0 + g^{-i}x_1, y_0 + y_1)\zeta^i \\
 &= \sum_{i=0}^{q-1} B(g^{-i}x_0, y_0 + y_1)\zeta^i + \sum_{i=0}^{q-1} B(g^{-i}x_1, y_0 + y_1)\zeta^i \\
 &= \sum_{i=0}^{q-1} B(x_0, y_0 + y_1)\zeta^i + \sum_{i=0}^{q-1} B(g^{-i}x_1, y_0 + y_1)\zeta^i
 \end{aligned}$$

Since $\sum_{i=0}^{q-1} \zeta^i = 0$ and $g^{-i}x_1 \in W_1$ we have $h_1(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x_1, y_1)\zeta^i = h_1(x_1, y_1)$.

□

The isomorphism between $\mathbb{Q}G$ and $\mathbb{Q} \times \mathbb{Q}(\zeta)$ corresponds $\frac{1}{q} \sum_{i=0}^{q-1} g^i$ to $(1, 0)$ and $g - \frac{1}{q}(\sum_{i=0}^{q-1} g^i)$ to $(0, \zeta)$. Moreover for any $x \in W_0$ we have $(\frac{1}{q} \sum_{i=0}^{q-1} g^i)x = x$ and for any $x \in W_1$, $[g - \frac{1}{q}(\sum_{i=0}^{q-1} g^i)]x = gx$.

Therefore we can consider W_0 as a vector space over $\mathbb{Q}$ and W_1 as a vector space over $\mathbb{Q}(\zeta)$. Where for any $x \in W_1$, $\zeta \cdot x = gx$.

Proposition 3.3. (i) Restrict B_0 to W_0 , then (W_0, B_0) is a positive definite quadratic space over $\mathbb{Q}$. Moreover $B_0(x, y) = qB(x, y)$ for any $x, y \in W_0$

(ii) Restrict h_1 to W_1 . If $q \neq 2$ then (W_1, h_1) is a totally positive definite hermitian space over $\mathbb{Q}(\zeta)$ with respect to the complex conjugation. Moreover we have $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(h_1(x, y)) = qB(x, y)$ for any $x, y \in W_1$. If $q = 2$, then (W_1, h_1) is also a positive definite quadratic space and $h_1(x, y) = 2B(x, y)$ for any $x, y \in W_1$.

Proof (i) As we showed in the proof of Proposition 3.2 $B_0(x, y) = qB(x, y)$ for all $x, y \in W_0$, (i) is clear.

$$\begin{aligned}
 \text{(ii) We have } \overline{h_1(y, x)} &= \sum_{i=0}^{q-1} \overline{B(g^{-i}y, x)\zeta^i} \\
 &= \sum_{i=0}^{q-1} B(g^{-i}y, x)\bar{\zeta}^i = \sum_{i=0}^{q-1} B(y, g^i x)\zeta^{-i} \\
 &= \sum_{i=0}^{q-1} B(g^i x, y)\zeta^{-i} = \sum_{i=0}^{q-1} B(g^{-i}x, y)\zeta^i \\
 &= h_1(x, y)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } h_1(x, \zeta \cdot y) &= \sum_{i=0}^{q-1} B(g^{-i}x, gy)\zeta^i \\
 &= \sum_{i=0}^{q-1} B(g^{-(i-1)}x, y)\zeta^i = \sum_{i=0}^{q-1} B(g^{-(i+1)}x, y)\zeta^{i+1-1} \\
 &= \zeta^{-1} \sum_{i=0}^{q-1} B(g^{-(i+1)}x, y)\zeta^{i+1} = \zeta^{-1} h_1(x, y) \\
 &= \bar{\zeta} h_1(x, y).
 \end{aligned}$$

Therefore $h_1(x, y)$ is a hermitian form with respect to the complex conjugation.

Moreover we have $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(h_1(x, y))$

$$\begin{aligned}
 &= \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\left(\sum_{i=0}^{q-1} B(g^{-i}x, y)\zeta^i\right) = (q-1)B(x, y) \\
 &\quad - \sum_{i=1}^{q-1} B(g^{-i}x, y) = qB(x, y) - \sum_{i=0}^{q-1} B(g^{-i}x, y) \\
 &= qB(x, y) - B\left(\sum_{i=0}^{q-1} g^{-i}x, y\right) = qB(x, y).
 \end{aligned}$$

Since (W_1, B) is a positive definite quadratic space over $\mathbb{Q}$, (W_1, h_1) is a totally positive definite hermitian space (see Biermann ([1], pp. 47, Folgerung 2) also Feit ([5], Theorem 6.1) □

Let $\Gamma\tilde{M} = M_0 \times M_1$ where $M_0 = (\mathbb{Z} \times \{0\}) \times \tilde{M}$ and $M_1 = (\{0\} \times \mathbb{Z}[\zeta])\tilde{M}$. Then M_i is a lattice on W_i for $i = 0, 1$. For every x and y in $\Gamma\tilde{M}$, $h(x, y) \in \Gamma$. Therefore $B_0(x, y) \in \mathbb{Z}$ and $h_1(x, y) \in \mathbb{Z}[\zeta]$ for any $x, y \in \Gamma\tilde{M}$. Hence (M_0, B_0) is a positive definite integral $\mathbb{Z}$ -lattice on W_0 and (M_1, h_1) is a totally positive definite hermitian $\mathbb{Z}[\zeta]$ -lattice on W_1 . Let us denote (M_1, h_1) by (M_1, h_1) . If $q = 2$ then M_1 is also a positive definite integral $\mathbb{Z}$ -lattice. In this case we denote (M_1, h_1) by (M_1, B_1) .

Proposition 3.4. (i) Let $y \in W_0$ and $B_0(x, y) \in q\mathbb{Z}$ for any x in M_0 , then $y \in M_0$.
(ii) Let $y \in W_1$ and λ be a prime element in $\mathbb{Z}[\zeta]$ above q . Assume $h_1(x, y) \in \lambda\mathbb{Z}[\zeta]$ for every $x \in M_1$, then $y \in M_1$. Here if $q = 2$, then $\zeta = -1$ and $\lambda = 2$.

Proof (i) Since x and y are in W_0 , $B_0(x, y) = qB(x, y)$. Therefore we have $B(x, y) \in \mathbb{Z}$ for all $x \in M_0$. Since W_1 and W_0 are orthogonal to each other with respect to B , $B(x, y) \in \mathbb{Z}$ for all $x \in \Gamma\tilde{M}$ and hence for all $x \in M$. Therefore $y \in M \cap W_0 \subset M_0$.

(ii) Since $Tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\lambda) = \pm q$, we have $qB(x, y) = Tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}(h_1(x, y)) \in Tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\lambda\mathbb{Z}[\zeta]) \subset q\mathbb{Z}$. Hence we have $B(x, y) \in \mathbb{Z}$ for all $x \in M_1$. Similar argument as in (i) gives $y \in M_1$. $\square$

Thus we constructed an integral lattice M_0 and a hermitian $\mathbb{Z}[\zeta]$ -lattice M_1 . The construction depends on the choice of g . Therefore we can possibly construct more than one pair (M_0, M_1) of lattices from one class in G_L and also from not isometric two lattices we may have same pair (M_0, M_1) . However Lemma 3.10 and 3.13 in this section will show it does not matter so much.

Let $\Lambda = \mathbb{Z}G$ and $I = q\Lambda + (1 - g)\Lambda \subset \mathbb{Z}G$. Then I corresponds to $q\mathbb{Z} \times \lambda\mathbb{Z}[\zeta]$ in $\Gamma = \mathbb{Z} \times \mathbb{Z}[\zeta]$. Let $V = \Gamma\tilde{M}/I\tilde{M}$, $V_0 = M_0/qM_0$ and $V_1 = M_1/\lambda M_1$. Then $V = V_0 \times V_1$. Define bilinear forms b, b_0 and b_1 on V, V_0 and V_1 induced by h, B_0 and h_1 respectively, i.e. $b(\bar{x}, \bar{y}) = h(x, y) \bmod I$, $b_0(\bar{x}_0, \bar{y}_0) = B_0(x_0, y_0) \bmod q\mathbb{Z}$ and $b_1(\bar{x}_1, \bar{y}_1) = h_1(x_1, y_1) \bmod \lambda\mathbb{Z}[\zeta]$ where x and y are in $\Gamma\tilde{M}$, x_0 and y_0 are in M_0 , x_1 and y_1 are in M_1 , and $\bar{x}$ is the equivalent class of x . The quotient $\Lambda/I \cong \mathbb{F}_q$ (finite field of q elements) is injected onto the diagonal Δ in $\Gamma/I \cong \mathbb{F}_q \times \mathbb{F}_q$.

Let $\mathcal{L}_{\tilde{M}}$ be the set of all hermitian Λ -lattices $\tilde{K}$ in $\Gamma\tilde{M}$ which satisfy the following two conditions (i) and (ii).

(i) $\Gamma\tilde{K} = \Gamma\tilde{M}$.

(ii) For any given $y \in \Gamma\tilde{M}$ the necessary and sufficient condition that $h(x, y) \in \Lambda$ for all $x \in \tilde{K}$ is $y \in \tilde{K}$. Here $\tilde{K}$ may not necessarily be constructed from a lattice in the genus G_L of L .

Let $\tilde{K}$ be in $\mathcal{L}_{\tilde{M}}$, $U_{\tilde{K}} = \tilde{K}/I\tilde{K}$, $U_{\tilde{K}}^{(0)} = U_{\tilde{K}} \cap (V_0 \times \{0\})$ and $U_{\tilde{K}}^{(1)} = U_{\tilde{K}} \cap (\{0\} \times V_1)$. Then we have the following Proposition 3.5.

This Proposition 3.5 is very crucial in our whole argument and is proved by Quebbmann (Lemma 2.2 in [11]) but for the purpose of making this thesis self contained and easier to be read we will give the proof here.

Proposition 3.5 (Quebbemann [11, Lemma (2.2)])

- (i) $U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$ is a maximal Γ/I submodule of V contained in $U_{\tilde{K}}$.
- (ii) $V^\perp = U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$.

(iii) Let $V/V^\perp = V'_0 \times V'_1$, then there exists an isometry $\tau_{\tilde{K}} : V'_0 \cong V'_1$ such that $U_{\tilde{K}}/V^\perp = \{(v_0, \tau_{\tilde{K}}(v_0)) \mid v_0 \in V'_0\}$. Therefore $U_{\tilde{K}} = \{(v_0, \tau_{\tilde{K}}(v_0)) \mid v_0 \in V'_0\} \oplus V^\perp$ because $V'_0 \times V'_1$ can be considered as a direct summand of V .

Proof (i) Let $u = (u_0, 0)$ and $v = (v_0, 0)$ be in $U_{\tilde{K}}^{(0)}$. Since $\tilde{K}$ is a Λ -lattice $u + v \in U_{\tilde{K}}$. Therefore $u + v \in U_{\tilde{K}}^{(0)}$. Let $a = (a_0, a_1) \in \mathbb{F}_q \times \mathbb{F}_q$, then $au = (a_0u_0, 0) = (a_0, a_0)(u_0, 0)$. Since $(a_0, a_0) \in \Lambda/I$ we have $au \in U_{\tilde{K}}^{(0)}$. Therefore $U_{\tilde{K}}^{(0)}$ is an $\mathbb{F}_q \times \mathbb{F}_q$ module. Similar argument shows $U_{\tilde{K}}^{(1)}$ is also an $\mathbb{F}_q \times \mathbb{F}_q$ module. Let U' be an $\mathbb{F}_q \times \mathbb{F}_q$ module with $U_{\tilde{K}}^{(0)} + U_{\tilde{K}}^{(1)} \subseteq U' \subseteq U_{\tilde{K}}$. Let $u = (u_0, u_1) \in U'$. Since U' is an $\mathbb{F}_q \times \mathbb{F}_q$ -submodule of V , $(1, 0)u$ and $(0, 1)u$ are in U' . Therefore $(1, 0)u \in U_{\tilde{K}}^{(0)}$ and $(0, 1)u \in U_{\tilde{K}}^{(1)}$ and hence we have $u = (1, 0)u + (0, 1)u \in U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$. This shows $U' = U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$.

(ii) Since $\Gamma\tilde{K} = \Gamma\tilde{M}$, $U_{\tilde{K}}$ generates V over Γ/I . Therefore it is sufficient to show that $U_{\tilde{K}}^\perp = U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$. Let $u \in U_{\tilde{K}}^\perp$, then $b(v, u) = 0$ for all $v \in U_{\tilde{K}}$, i.e., $b(v, u) \in \Lambda/I$ for all v in $U_{\tilde{K}} = \tilde{K}/I\tilde{M}$. Since $\tilde{K}$ is in $\mathcal{L}_{\tilde{M}}$, this implies $u \in U_{\tilde{K}}$, that is, $U_{\tilde{K}}^\perp \subseteq U_{\tilde{K}}$. Hence by (i) of this proposition it is enough to show $U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)} \subseteq U_{\tilde{K}}^\perp$. Let $u = (u_0, 0) \in U_{\tilde{K}}^{(0)}$ and $v = (v_0, v_1) \in U_{\tilde{K}}$ then $b(u, v) = (b_0(u_0, v_0), b_1(0, v_1)) = (b_0(u_0, v_0), 0)$. On the other hand we have $b(u, v) \in \Delta$. Therefore $b_0(u_0, v_0) = 0$. This means $b(u, v) = 0$, i.e., $U_{\tilde{K}}^{(0)} \subseteq U_{\tilde{K}}^\perp$. Similarly we can show $U_{\tilde{K}}^{(1)} \subseteq U_{\tilde{K}}^\perp$. This gives $U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)} \subseteq U_{\tilde{K}}^\perp$.

(iii) Let $u = (u_0, u_1)$ and $v = (v_0, v_1) \in U_{\tilde{K}}$. If $(u_0 - v_0, 0) \in U_{\tilde{K}}^{(0)}$, then $(1, 0)(u - v) = (u_0 - v_0, 0) \in U_{\tilde{K}}^{(0)} \subseteq U_{\tilde{K}}$. Since $U_{\tilde{K}}$ is a Λ/I module, we have $(1, 1)(u - v) \in U_{\tilde{K}}$. Therefore we have $(0, u_1 - v_1) = (0, 1)(u - v) = (1, 1)(u - v) - (1, 0)(u - v) \in U_{\tilde{K}}$. Hence $(0, u_1 - v_1) \in U_{\tilde{K}}^{(1)}$. This means for any u and v in $U_{\tilde{K}}$ if $u_0 \equiv v_0 \pmod{U_{\tilde{K}}^{(0)}}$

then $u \equiv v \pmod{U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}}$, that is the second component of $U_{\tilde{K}} \pmod{U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}}$ corresponds to the first component uniquely. Let us denote this correspondence with $\tau_{\tilde{K}}$. Let us assume that $\bar{u}_0 \in V'_0$ and $(\bar{u}_0, \tau_{\tilde{K}}(\bar{u}_0))$ is in $U_{\tilde{K}}/U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$. Since $(a, a)(\bar{u}_0, \tau_{\tilde{K}}(\bar{u}_0)) = (a\bar{u}_0, a\tau_{\tilde{K}}(\bar{u}_0)) \in U_{\tilde{K}}/U_{\tilde{K}}^{(0)} \oplus U_{\tilde{K}}^{(1)}$, we have $\tau_{\tilde{K}}(a\bar{u}_0) = a\tau_{\tilde{K}}(\bar{u}_0)$. Since $U_{\tilde{K}}$ generates V over Γ/I we can extend $\tau_{\tilde{K}}$ linearly to V'_0 as follows. For any $v_0 \in V_0$ let $(v_0, 0) = \sum_i (a_{i0}, a_{i1})(u_{i0}, u_{i1})$ where $a_{i0}, a_{i1} \in \mathbb{F}_q$ and $(u_{i0}, u_{i1}) \in U_{\tilde{K}}$. Since $\sum_i (a_{i0}, a_{i1})(u_{i0}, u_{i1}) = (\sum_i a_{i0}u_{i0}, \sum_i a_{i1}u_{i1})$ we have $\sum_i a_{i1}u_{i1} = 0$. Hence we have $(v_0, 0) = (\sum_i a_{i0}u_{i0}, 0)$ for any $v_0 \in V_0$. Define $\tau_{\tilde{K}}(\bar{v}_0) = \sum_i a_{i0}\tau_{\tilde{K}}(\bar{u}_{i0})$. Clearly $\tau_{\tilde{K}}$ is an injection from V'_0 to V'_1 . We will show that $\tau_{\tilde{K}}$ is surjective. Let $v_1 \in V_1$. Then $(0, \bar{v}_1) = \sum_i (a_{i0}, a_{i1})(\bar{u}_{i0}, \bar{u}_{i1}) = (\sum_i a_{i0}\bar{u}_{i0}, \sum_i a_{i1}\bar{u}_{i1}) = (0, \sum_i a_{i1}\bar{u}_{i1})$ with some $(a_{i0}, a_{i1}) \in \mathbb{F}_q \times \mathbb{F}_q$ and $(u_{i0}, u_{i1}) \in U_{\tilde{K}}$. Then $(\bar{u}_{i0}, \bar{u}_{i1}) = (\bar{u}_{i0}, \tau_{\tilde{K}}(\bar{u}_{i0}))$. Therefore we have $\bar{v}_1 = \sum_i a_{i1}\bar{u}_{i1} = \sum_i a_{i1}\tau_{\tilde{K}}(\bar{u}_{i0}) = \tau_{\tilde{K}}(\sum_i a_{i1}\bar{u}_{i0})$. Since for any u and v in $U_{\tilde{K}}$, $b(u, v) \in \Lambda I = \Delta$, we have $b_0(u_0, v_0) = b_1(u_1, v_1)$. This shows that $\tau_{\tilde{K}}$ is an isometry.

Remark. By Proposition 3.1 we have $\tilde{M} \in \mathcal{L}_{\tilde{M}}$, therefore V'_0 is isometric to V'_1 . Let $I(V'_0, V'_1)$ be the set of all the isometries from V'_0 to V'_1 . Then we have the following proposition.

Proposition 3.6. Let τ be an element in $I(V'_0, V'_1)$. Let π be the projection $\Gamma\tilde{M} \rightarrow \Gamma\tilde{M}/I\tilde{M}$, and $U = \{(v_0, \tau(v_0)) \mid v_0 \in V'_0\} \oplus V^\perp$. Then $\tilde{K} = \pi^{-1}(U)$ is in $\mathcal{L}_{\tilde{M}}$.

Proof. Since V_0 is a finite $\mathbb{F}_q$ -vector space, and $v = (v_0, v_1)$ in $V'_0 \times V'_1$ can be written as $v_0 = \sum_i a_i u_i$ and $\sum_i b_i \tau(u_i)$ with some $a_i, b_i \in \mathbb{F}_q$ and $u_i \in V'_0$. Therefore $v = (\sum_i a_i u_i, \sum_i b_i \tau(u_i)) = \sum_i (a_i, b_i)(u_i, \tau(u_i)) \in (\Gamma/I)U$. This proves $\Gamma\tilde{K} = \Gamma\tilde{M}$. Next we will show that for any $y \in \Gamma\tilde{M}$ the necessary and sufficient condition that

$h(x, y) \in \Lambda$ for all $x \in \tilde{K}$ is $y \in \tilde{K}$. Suppose $h(x, y) \in \Lambda$ for any x in $\tilde{K}$. Let $\bar{x} = (\bar{x}_0, \tau(\bar{x}_0)) \oplus \bar{x}'$ and $\bar{y} = (\bar{y}_0, \bar{y}_1) \oplus \bar{y}'$ where $\bar{x}_0 \in V'_0$, $(\bar{y}_0, \bar{y}_1) \in V'_0 \times V'_1$ and $\bar{x}', \bar{y}' \in V^\perp$. Then $b(\bar{x}, \bar{y}) = (b_0(\bar{x}_0, \bar{y}_0), b_1(\tau(\bar{x}_0), \bar{y}_1)) \in \Lambda$. Therefore $b_0(\bar{x}_0, \bar{y}_0) = b_1(\tau(\bar{x}_0), \bar{y}_1)$. On the other hand, we have $b_1(\tau(\bar{x}_0), \tau(\bar{y}_0)) = b_0(\bar{x}_0, \bar{y}_0)$. Therefore $b_1(\tau(\bar{x}_0), \bar{y}_1 - \tau(\bar{y}_0)) = 0$ for all $\bar{x}_0 \in V'_0$ and hence we have $\bar{y}_1 = \tau(\bar{y}_0)$. This shows $y \in \tilde{K}$. $\square$

Proposition 3.7. There is a one to one correspondence between $\mathcal{L}_{\tilde{M}}$ and $I(V'_0, V'_1)$.

Proof. This is clear from Propositions 3.5 and 3.6. $\square$

Proposition 3.8. Let $\tilde{K}$ and $\tilde{N}$ be in $\mathcal{L}_{\tilde{M}}$. Assume that there exists an isometry $\sigma = (\sigma_0, \sigma_1)$ in $\text{Aut } M_0 \times \text{Aut } M_1$ which induces the isometry from $U_{\tilde{K}} = \tilde{K}/I\tilde{M}$ to $U_{\tilde{N}} = \tilde{N}/I\tilde{M}$. Then $\tilde{K}$ and $\tilde{N}$ are isometric as hermitian Λ -lattices.

Proof. This is clear because $\sigma(\tilde{K}) = \sigma(\tilde{K} + I\tilde{K}) = \sigma(\tilde{K} + I\tilde{M}) = \tilde{N} + I\tilde{M} = \tilde{N} + I\tilde{N} = \tilde{N}$.

Proposition 3.9.

$$\sum_{\text{Class } \tilde{K} \in \mathcal{L}_{\tilde{M}}} \frac{1}{|\text{Aut}(\tilde{K})|} \leq \frac{|I(V'_0, V'_1)|}{|\text{Aut } M_0| |\text{Aut } M_1|}$$

Here $\text{Aut } M_0$ is the orthogonal group of M_0 and $\text{Aut } M_1$ is the unitary group of M_1 . If $q = 2$ then $\text{Aut } M_1$ is also the orthogonal group of $M_1 = M_1$.

Proof. Let $\tilde{K} \in \mathcal{L}_{\tilde{M}}$ and $\sigma = \sigma_0 \times \sigma_1 \in \text{Aut } M_0 \times \text{Aut } M_1$. Clearly we have $\sigma \tilde{K} \in \mathcal{L}_{\tilde{M}}$. Let $H = \text{Aut } M_0 \times \text{Aut } M_1$. Let $\mathcal{L}_{\tilde{M}} = H\tilde{K}_1 \cup \dots \cup H\tilde{K}_s$ be the decomposition of $\mathcal{L}_{\tilde{M}}$ into the orbits given by the action of H on $\mathcal{L}_{\tilde{M}}$. Let $H_i = \{\sigma \in H \mid \sigma \tilde{K}_i = \tilde{K}_i\}$ and ℓ_i be the length of the orbit of $\tilde{K}_i$ that is $\ell_i = |\{\tilde{K} \in \mathcal{L}_{\tilde{M}} \mid \tilde{K} = \sigma(\tilde{K}_i) \text{ with some}$

$\sigma \in H\}$. Then for any $i = 1, \dots, s$ we have $|H| = \ell_i |H_i|$. Therefore we have

$$\sum_{i=1}^s \frac{1}{|H_i|} = \sum_{i=1}^s \frac{\ell_i}{|H|} = \frac{|\mathcal{L}_{\tilde{M}}|}{|H|} = \frac{|I(V'_0, V'_1)|}{|H|}.$$

By Proposition 3.8, the Λ -lattices in the orbit $H\tilde{K}_i$ are isometric for each $i = 1, \dots, s$.

On the other hand H_i is a subgroup of $\text{Aut}(\tilde{K}_i)$. Therefore we have

$$\sum_{\text{Class } \tilde{K} \text{ in } \mathcal{L}_{\tilde{M}}} \frac{1}{|\text{Aut } \tilde{K}|} \leq \sum_{i=1}^s \frac{1}{|\text{Aut } \tilde{K}_i|} \leq \sum_{i=1}^s \frac{1}{|H_i|}.$$

This gives the Proposition 3.9. $\square$

Lemma 3.10. Let M be in the genus G_L of L and of type $R(q)$. Then we have the following inequality.

$$\sum_{\substack{\text{cls } K \subseteq G_L \\ \text{of type } R(q) \\ \text{with } \Gamma \tilde{K} \cong \Gamma \tilde{M}}} \frac{1}{|O(K)|} \leq \frac{|I(V'_0, V'_1)|}{|\text{Aut } M_0| |\text{Aut } M_1|}$$

Proof. Let K and N be the classes in G_L of type $R(q)$ with $\Gamma \tilde{K} \cong \Gamma \tilde{N} \cong \Gamma \tilde{M}$.

Then it is clear from the construction of $\tilde{K}$ and $\tilde{N}$ that an Λ -isometry $\sigma : (\tilde{K}, h_K) \longrightarrow (\tilde{N}, h_N)$ induces an $\mathbb{Z}$ -isometry $\sigma : (K, B) \longrightarrow (N, B)$. Therefore we have

$$\sum_{\substack{\text{cls } K \subseteq G_L \\ \text{of type } R(q) \\ \text{with } \Gamma \tilde{K} \cong \Gamma \tilde{M}}} \frac{1}{|O(K)|} \leq \sum_{\substack{\text{class } \tilde{K} \\ \text{constructed} \\ \text{from } K \in G_L \\ \text{with } \Gamma \tilde{K} \cong \Gamma \tilde{M}}} \frac{1}{|\text{Aut}(\tilde{K})|} \leq \sum_{\text{Class } \tilde{K} \text{ in } \mathcal{L}_{\tilde{M}}} \frac{1}{|\text{Aut}(\tilde{K})|}$$

Hence by Proposition 3.9 we have the proof for Lemma 3.10. $\square$

Let $L(q, r, \rho)$ and $G(q, r, \rho)$ be the set defined in the introduction. Then there are finitely many representatives of lattices $M^{(1)}, \dots, M^{(k)}$ in $L(q, r, \rho)$ with the following conditions.

- (i) $(\Gamma \tilde{M}^{(i)}, h^{(i)})$ is not isometric to $(\Gamma \tilde{M}^{(j)}, h^{(j)})$ for $i \neq j$ as Γ -lattices.
- (ii) For every $K \in L(q, r, \rho)$ there exists some $M^{(i)}$ such that $(\Gamma \tilde{K}, h_K) \cong (\Gamma \tilde{M}^{(i)}, h^{(i)})$. Then Lemma 3.10 gives the following proposition.

Proposition 3.11.

$$\sum_{cls K \subseteq L(q, r, \rho)} \frac{1}{|O(K)|} \leq \sum_{i=1}^k \frac{|I(V'_0(M_0^{(i)}), V'_1(M_1^{(i)}))|}{|Aut M_0^{(i)}| |Aut M_1^{(i)}|}$$

Proposition 3.12. Let M be a lattice in $L(q, r, \rho)$ then the pair of genera (G_{M_0}, G_{M_1}) of the lattices M_0 and M_1 which are constructed from M is contained in $G(q, r, \rho)$.

Proof. Since $(1, 0)$ corresponds to $\frac{1}{q} \sum_{i=0}^{q-1} g^i$ in $\mathbb{Q}G$ and $(0, 1)$ corresponds to $1 - \frac{1}{q} (\sum_{i=0}^{q-1} g^i)$ in $\mathbb{Q}G$, we have $M_0 = (\frac{1}{q} \sum_{i=0}^{q-1} g^i)M$ and $M_1 = (1 - \frac{1}{q} (\sum_{i=0}^{q-1} g^i))M$. Then M_0 and M_1 are lattice on W_0 and W_1 respectively. Therefore we have $q(M_0 \perp M_1) \subseteq M \subseteq M_0 \perp M_1$ with respect to the bilinear form B . This shows that the discriminant of $qM_i, i = 0, 1$, with respect to the bilinear form B is power of q . On the other hand by Proposition 3.3 we have $B_0(x, y) = qB(x, y)$ for any $x, y \in M_0 = M_0$ (if $q = 2$ $B_1(x, y) = 2B(x, y)$ for any $x, y \in M_1 = M_1$). Hence discriminant of M_0 with respect to the bilinear form B_0 is also a power of q (if $q = 2$ the discriminant of M_1 with respect to the bilinear form B_1 is also power of 2).

Proposition 3.4 shows that $(M_0)_p$ is unimodular at the prime $p \neq q$ and $(M_0)_q$ has a Jordan splitting (see O'Meara [8] pp. 243) $J_0 \perp J_1$ with some unimodular $\mathbb{Z}_q$ lattice J_0 and $q\mathbb{Z}_q$ -modular $\mathbb{Z}_q$ -lattice J_1 ($(M_0)_q$ could be unimodular or $q\mathbb{Z}_q$ -modular). If $q = 2$ then this is also true for $(M_1)_2$. Since $\dim_{\mathbb{F}_q} V'_0(M_0) = \dim_{\mathbb{F}_q} V'_1(M_1) = \rho$, the rank of J_0 has to be ρ . This shows that $d_{B_0} M_0 = q^{m_0 - \rho}$ (if $q = 2$ then $d_{B_1} M_1 = q^{r - \rho}$). Similarly Proposition 3.4 shows that $(M_1)_p$ is unimodular at the prime $p \mid q$ and

$(M_1)_P$ has a Jordan splitting $J_0 \perp J_1$ with some unimodular hermitian S_P -lattice J_0 and λS_P -modular hermitian S_P -lattice J_1 (see Jacobowitz [7] page 448, 449). Since $\dim_{\mathbb{F}_q} V'_1(M_1) = \rho$, the rank of J_0 has to be ρ . This shows that $N_{E/\mathbb{Q}}(\delta M_1) = q^{r-\rho}$. If $q = 2$ and a lattice M in G_L of type $R(2)$ has an isometry g which gives $\text{rank}_{\mathbb{Z}} M_0 < r$ then taking $-g$ we can consider $M \in L(q, r, \rho)$. Other conditions in the definition of $G(q, r, \rho)$ is clearly obtained from the conditions in the definition of $L(q, r, \rho)$.

Lemma 3.13.

- (i) $\sum_{cls K \subseteq L(q, r, \rho)} \frac{1}{|O(K)|} \leq \sum_{(G_{N_0}, G_{M_1}) \in G(q, r, \rho)} |I(V'_0(N_0), V'_1(M_1))| \omega(N_0) \omega(M_1).$
- (ii) $\omega_{R(q)} < \sum_{r=1}^{\lfloor \frac{m-1}{q-1} \rfloor} \sum_{\rho=0}^{\min(r, m_0)} \sum_{(G_{N_0}, G_{M_1}) \in G(q, r, \rho)} |I(V'_0(N_0), V'_1(M_1))| \omega(N_0) \omega(M_1).$

Proof. This is clear from the definition of mass, Propositions 3.11, and 3.12.

§4. Lattices with nontrivial automorphisms whose minimal polynomials are irreducible.

Case $q = \text{odd prime}$

Let M be a lattice in the genus G_L of type $IR(q)$ with the isometry g of order odd prime number q . Let $W = \mathbb{Q}M$ and define the action of $\mathbb{Q}(\zeta)$ on W through $\zeta x = g(x)$ for $x \in W$. Then M is a $\mathbb{Z}[\zeta]$ -lattice with this action. Let $h(x, y) = \sum_{i=0}^{q-1} B(g^{-i}x, y) \zeta^i$ for $x, y \in W$. Then $h(x, y)$ is a totally positive definite hermitian form on W with respect to the complex conjugation. Clearly $h(x, y) \in \mathbb{Z}[\zeta]$ for any $x, y \in M$. Thus W has a structure of a hermitian vector space over $\mathbb{Q}(\zeta)$ and M has a hermitian $\mathbb{Z}[\zeta]$ -lattice structure. Let us denote them by $\mathcal{W}$ and $\mathcal{M}$ respectively.

Proposition 4.1. Let $y \in \mathcal{W}$. Then $h(x, y) \in \lambda\mathbb{Z}[\zeta]$ for every $x \in \mathcal{M}$ if and only if $y \in \mathcal{M}$.

Proof. First assume $h(x, y) \in \lambda\mathbb{Z}[\zeta]$ for every $x \in \mathcal{M}$. Then by Proposition 3.3 (ii) we have $qB(x, y) = \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(h(x, y))$. Therefore by the assumption $qB(x, y) \in \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\lambda\mathbb{Z}[\zeta]) \subseteq q\mathbb{Z}$. Hence we have $B(x, y) \in \mathbb{Z}$ for every $x \in \mathcal{M} = M$. Since M is a unimodular lattice with respect to B , we have $y \in M = \mathcal{M}$. Conversely if $y \in \mathcal{M} = M$ then $B(x, y) \in \mathbb{Z}$ for all $x \in M = \mathcal{M}$. Therefore $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(h(x, y)) = qB(x, y) \in q\mathbb{Z}$. This implies $h(x, y) \in \lambda\mathbb{Z}[\zeta]$. $\square$

Proposition 4.2. Let M and N be lattices in G_L of type $IR(q)$. Let $\mathcal{M}$ and $\mathcal{N}$ be the hermitian $\mathbb{Z}[\zeta]$ -lattices constructed from M and N , h_M and h_N be the hermitian form of $\mathcal{M}$ and $\mathcal{N}$ respectively. Suppose (M, h_M) and (N, h_N) are isometric as hermitian $\mathbb{Z}[\zeta]$ -lattices then (M, B) and (N, B) are isometric as $\mathbb{Z}$ -lattices.

Proof. Let σ be the isometry from $\mathcal{M}$ to $\mathcal{N}$. Let g_M and g_N be the isometries of M and N which determine the structure $\mathcal{M}$ and $\mathcal{N}$ respectively. Since $\sigma(\zeta x) = \zeta \sigma(x)$, $\zeta \cdot x = g_M(x)$ for $x \in M$ and $\zeta \cdot u = g_N(u)$ for $u \in N$, we have $\sigma g_M = g_N \sigma$. Hence we have $h_N(\sigma x, \sigma y) = \sum_{i=0}^{q-1} B(g_N^{-i} \sigma x, \sigma y) \zeta^i$

$$\begin{aligned} &= \sum_{i=0}^{q-1} B(g_N^{-i} \sigma x, \sigma y) \zeta^i - \sum_{i=0}^{q-1} B(g_N^{-(q-1)} \sigma x, \sigma y) \zeta^i \\ &= \sum_{i=0}^{q-2} \{B(g_N^{-i} \sigma x, \sigma y) - B(g_N^{-(q-1)} \sigma x, \sigma y)\} \zeta^i \\ &= \sum_{i=0}^{q-2} B((g_N^{-i} - g_N) \sigma x, \sigma y) \zeta^i = \sum_{i=0}^{q-2} B(\sigma(g_M^{-i} - g_M) x, \sigma y) \zeta^i. \end{aligned}$$

On the other hand, we have $h_M(x, y) = \sum_{i=0}^{q-1} B(g_M^{-i}x, y)\zeta^i$

$$= \sum_{i=0}^{q-2} B((g_M^{-i} - g_M^{-(q-1)})x, y)\zeta^i = \sum_{i=0}^{q-2} B((g_M^{-i} - g_M)x, y)\zeta^i.$$

Therefore we have $B(\sigma(1 - g_M)x, \sigma y) = B((1 - g_M)x, y)$ for all $x, y \in M = M$.

Since the minimal polynomial of g_M is $x^{q-1} + \dots + x + 1$, $1 - g_M$ is an automorphism of $W = \mathbb{Q}M$ as a vector space. This implies that $B(\sigma x, \sigma y) = B(x, y)$ for all $x, y \in W$. Hence σ is an isometry as $\mathbb{Z}$ -lattices. $\square$

Corollary 4.3.

$$|U(M)| \leq |O(M)|$$

Proof. Take $N = M$ in the proof of Proposition 4.2. $\square$

Lemma 4.4. There is exactly one genus G_M of totally positive definite $\lambda\mathbb{Z}[\zeta]$ -modular hermitian $\mathbb{Z}[\zeta]$ -lattices of rank $\frac{m}{q-1}$ and we have

$$\omega_{IR(q)} \leq \omega(M).$$

Proof. Proposition 4.1 and similar argument as in [8], §82 F and §82G show that a hermitian $\mathbb{Z}[\zeta]$ -lattice M which is constructed from a integral $\mathbb{Z}$ -lattice M in G_L of type $IR(q)$ is $\lambda\mathbb{Z}[\zeta]$ -modular. Proposition 3.2 in [14], Theorem 7.1 and Proposition 8.1 in [7] shows that there is exactly one genus of totally positive definite $\lambda\mathbb{Z}[\zeta]$ -modular hermitian $\mathbb{Z}[\zeta]$ -lattice of rank $\frac{m}{q-1}$. Therefore by Proposition 4.1 and Corollary 4.3, we have this lemma. $\square$

Case $q = 4$.

Let M be a lattice in G_L of type $IR(4)$ with the isometry g of order 4 whose minimal polynomial is $x^2 + 1$. Let $\zeta = \sqrt{-1}$, $W = \mathbb{Q}M$. Let $\mathbb{Q}(\zeta)$ act on M through $\zeta \cdot x = g(x)$. Define $h(x, y) = \frac{1}{2} \sum_{i=0}^3 B(g^{-i}x, y)\zeta^i$. Then $h(x, y)$ is a totally positive definite hermitian form with respect to complex conjugation.

Proposition 4.5. $h(x, y) = B(x, y) - B(gx, y)\zeta$.

Proof.

$$\begin{aligned} h(x, y) &= \frac{1}{2} \{B(x, y) + B(g^{-1}x, y)\zeta + B(-x, y)(-1) + B(gx, y)(-\zeta)\} \\ &= \frac{1}{2} \{2B(x, y) + B((g^{-1} - g)x, y)\zeta\} \\ &= B(x, y) + \frac{1}{2} B(g(g^{-2} - 1)x, y)\zeta \\ &= B(x, y) - B(gx, y)\zeta. \end{aligned} \quad \square$$

By Proposition 4.5, $h(x, y) \in \mathbb{Z}[\zeta]$ for any x and $y \in M$. Thus M has a hermitian $\mathbb{Z}[\zeta]$ lattice structure and W has a hermitian vector space structure. Let us denote them by $\mathcal{M}$ and $\mathcal{W}$ respectively.

Proposition 4.6. Let $y \in \mathcal{W}$. Then $h(x, y) \in \mathbb{Z}[\zeta]$ for every $x \in \mathcal{M}$ if and only if $y \in \mathcal{M}$.

Proof. First assume that $h(x, y) \in \mathbb{Z}[\zeta]$ for every $x \in \mathcal{M} = M$. Then by Proposition 4.5, $B(x, y) \in \mathbb{Z}$ for every $x \in \mathcal{M} = M$. Since M is unimodular with respect to B , we have $y \in M = \mathcal{M}$. Conversely, it is clear that $h(x, y) \in \mathbb{Z}[\zeta]$ for any x and $y \in \mathcal{M}$. $\square$

Proposition 4.7. Let M and N be lattice in G_L of type $IR(4)$. Let $\mathcal{M}$ and $\mathcal{N}$ be the hermitian $\mathbb{Z}[\zeta]$ -lattices constructed from M and N , h_M and h_N be the hermitian form

of $\mathcal{M}$ and $\mathcal{N}$ respectively. Suppose $(\mathcal{M}, h_{\mathcal{M}})$ is isometric to $(\mathcal{N}, h_{\mathcal{N}})$ as a hermitian $\mathbb{Z}[\zeta]$ -lattices. Then (M, B) is isometric to (N, B) as quadratic lattices.

Proof. The proof is similar to that of Proposition 4.2. $\square$

Corollary 4.8.

$$|U(\mathcal{M})| \leq |O(\mathcal{M})|$$

Proof. In Proposition 4.7 take $N = M$. $\square$

Lemma 4.9. There is exactly one genus $G_{\mathcal{M}}$ of totally positive definite unimodular hermitian $\mathbb{Z}[\sqrt{-1}]$ lattice with norm $n(\mathcal{M}) = 2\mathbb{Z}[\sqrt{-1}]$ and of rank $\frac{m}{2}$. Moreover if L is an even unimodular lattice then we have $\omega_{IR(4)} \leq \omega(\mathcal{M})$.

Proof. Proposition 3.2 in [14], Propositions 10.3, 9.2, 10.4 and Theorem 7.1 in [7] show that there is exactly one genus of such lattices. By Proposition 4.6 and similar arguments as in [8], §82 F and §82 G, show that every lattice $\mathcal{N}$ constructed from a lattice N in the genus G_L of type $IR(4)$ is unimodular. Since $B(g_N x, x) = B(g_N^2 x, g_N x) = B(-x, g_N x)$, we have $h(x, x) = B(x, x)$ for any x in N . Therefore if L is even unimodular then by Proposition 4.5, $n(\mathcal{N}) \subset 2\mathbb{Z}[\sqrt{-1}]$. Therefore $\mathcal{N} \in G_{\mathcal{M}}$. By Corollary 4.8 we have $\omega_{IR(4)} \leq \omega(\mathcal{M})$. $\square$

Lemma 4.10. There are exactly two genera $G_{\mathcal{M}}$ and $G_{\mathcal{N}}$ of totally positive definite unimodular hermitian $\mathbb{Z}[\sqrt{-1}]$ -lattice of rank $r = \frac{m}{2}$ with norm $\mathbb{Z}[\sqrt{-1}]$. Moreover if L is odd unimodular lattice then we have

$$\omega_{IR(4)} \leq \omega(\mathcal{M}) + \omega(\mathcal{N})$$

Proof. The same reason as in the proof of Lemma 4.9 gives Lemma 4.10. $\square$

CHAPTER III

LOCAL DENSITIES

§5. Local densities of hermitian lattices.

In this section we use the notation given in §2. First we introduce some more definitions and notation (see also §2).

Let M be a hermitian S_P -lattice on $\mathcal{W}$. Define dual lattice $M^\#$ of M to be the set $\{y \in \mathcal{W} \mid h(y, M) \subseteq S_P\}$ (clearly this set is a lattice on $\mathcal{W}$). If P does not split in E then define $ord_P(\mathcal{A})$ to be the order of the generator of $\mathcal{A}$ with respect to P for any fractional ideal in E_P . If P splits in E then $ord_P(\mathcal{A} \otimes R_P)$ is defined to be the order of the generator of $\mathcal{A}_{\mathbb{P}}$ with respect to P for any fractional ideal $\mathcal{A}$ in E , where $\mathbb{P}$ is a prime ideal in S such that $\mathbb{P} \mid P$. Let Δ_P be the discriminant of S_P over R_P . Let N be a hermitian S_P -lattice. Let r be the rank of M and s be the rank of N respectively, where $r \geq s$. Define $A_{P^\mu}(M, N)$ to be the number of S_P -linear map $\sigma : N \rightarrow M$ which are distinct modulo $P^\mu M$ and satisfy $h_M(\sigma x, \sigma y) \equiv h_N(x, y) \pmod{P^\mu S_P}$ for every x and y in N . Then we have the next two propositions by Rehmann.

Proposition 5.1 (Hilfssatz 5.3 [12])

(i) If $P \mid 2$ or P splits in E then

$$A_{P^{\mu+1}}(M, N) = p^{f_P s(2r-s)} A_{P^\mu}(M, N)$$

for any $\mu \geq 2\alpha + 1$ with $\alpha \geq -\text{ord}_p(sN\#)$, where p is a rational prime number such that $\mathcal{P} \mid p$.

(ii) If $\mathcal{P} \mid 2$ and $\mathcal{P}$ does not split in E then $A_{\mathcal{P}\mu+1}(\mathcal{M}, \mathcal{N}) = p^{f_{\mathcal{P}}s(2r-s)} A_{\mathcal{P}\mu}(\mathcal{M}, \mathcal{N})$ for any $\mu \geq 2\alpha + 1$ with $\alpha \geq \text{ord}_p(\sqrt{\Delta_{\mathcal{P}}}(sN\#)^{-1})$.

Proposition 5.2 (Hilfssatz 6.1 [12]). Let $\mathcal{P}$ be an unramified prime ideal in K . Then local density $\beta_{\mathcal{P}}(\mathcal{M})$ of a unimodular hermitian $S_{\mathcal{P}}$ -lattice $\mathcal{M}$ of rank r is given by the following:

$$\beta_{\mathcal{P}}(\mathcal{M}) = \prod_{i=1}^r (1 - p^{f_{\mathcal{P}}i}) \text{ for } \mathcal{P} \text{ which splits in } K.$$

$$\beta_{\mathcal{P}}(\mathcal{M}) = \prod_{i=1}^r (1 - (-1)^i p^{-f_{\mathcal{P}}i}) \text{ for } \mathcal{P} \text{ which remains prime in } K.$$

Rehmann also gave some estimation for the upper bound of the local densities of unimodular lattices and $\lambda S_{\mathcal{P}}$ -modular lattices at the ramified primes (See Hilfssatz 6.2, 6.3 in [12]). It seems that there is some error in Hilfsatz 6.3 [12]. Hilfssatz 6.3 shows that the local density of a $\lambda S_{\mathcal{P}}$ -modular hermitian $S_{\mathcal{P}}$ -lattice at the prime $\mathcal{P}$ above q is bounded by $q^{\frac{1}{2}r(r-1)}$ from above. This apparently contradicts our Proposition 5.8.

In the following we will give local densities of some hermitian $S_{\mathcal{P}}$ -lattices at the ramified prime ideal $\mathcal{P}$ in K . (That is $\mathcal{P} \mid q$ if q is odd prime and $\mathcal{P} \mid 2$ if $q = 4$.) We assume scales of the lattices in this section are ideals in $S_{\mathcal{P}}$.

Proposition 5.3. (Analog of 82:15 in [8].) Let $\mathcal{M}$ be a lattice on a hermitian space $\mathcal{W}$ over $E_{\mathcal{P}}$. Let J be a modular sublattice of $\mathcal{M}$ such that $h(\mathcal{M}, J) \subseteq sJ$. Then there exists a sublattice $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{M} = J \perp \mathcal{N}$.

Proof. Let $\mathcal{U}$ be the orthogonal compliment of $E_{\mathcal{P}}J$ in $\mathcal{W}$. Let $x \in \mathcal{M}$ then $x = y + z$ with some $y \in E_{\mathcal{P}}J$ and $z \in \mathcal{U}$. Then $h(y, J) = h(x - z, J) = h(x, J) \subseteq h(\mathcal{M}, J) \subseteq$

sJ . Since J is a modular lattice, $y \in J \subset M$ and therefore $z = x - y \in M \cap U$. Take $N = M \cap U$. $\square$

Proposition 5.4. Let r and ℓ be integers $r \geq \ell \geq 1$. Let M and N be hermitian lattice over S_p of rank r and ℓ respectively. Let H_M and H_N be the hermitian matrices over S_p with respect to certain basis of M and N respectively. Assume that there exists an integer μ_0 such that for any hermitian (ℓ, ℓ) -matrix C and any integer $\mu \geq \mu_0$, $C \equiv N$ modulo $P^\mu S_p$ implies that the lattice defined by C is isometric to N as S_p -lattices. Then $q^{2\ell r} A_{P^\mu}(M, N) = q^{\ell^2} A_{P^{\mu+1}}(M, N)$ for any $\mu \geq \mu_0$.

Proof. Let $X_j, j = 1, \dots, A_{P^\mu}(M, N)$ be all the mod $P^\mu S_p$ distinct (r, ℓ) -matrix such that ${}^t \bar{X}_j H_M X_j \equiv H_N \pmod{P^\mu S_p}$. For each j let $Y_i^j, i = 1, \dots, q^{2\ell r}$ be all the mod $P^{\mu+1} S_p$ distinct (r, ℓ) -matrices such that $Y_i^j \equiv X_j \pmod{P^\mu S_p}$. Then ${}^t \bar{Y}_i^j H_M Y_i^j \equiv H_N \pmod{P^\mu S_p}$. Let $B_t, t = 1, \dots, q^{\ell^2}$ be all the mod $P^{\mu+1} S_p$ distinct (ℓ, ℓ) -hermitian matrix over S_p such that $B_t \equiv H_N \pmod{P^\mu S_p}$. Then ${}^t \bar{Y}_i^j H_M Y_i^j \equiv B_t \pmod{P^{\mu+1} S_p}$ with some t . Therefore we have $q^{2\ell r} A_{P^\mu}(M, N) \leq \sum_{i=1}^{q^{\ell^2}} A_{P^{\mu+1}}(M, B_i)$ (here we denote the S_p -lattice defined by the hermitian matrix B_i by B_i also). On the other hand any matrix such that ${}^t \bar{Y} H_M Y \equiv B_i \pmod{P^{\mu+1} S_p}$ gives ${}^t \bar{Y} H_M Y \equiv H_N \pmod{P^\mu S_p}$. Therefore $Y \equiv Y_i^j \pmod{P^{\mu+1} S_p}$ with some Y_i^j . This shows that $q^{2\ell r} A_{P^\mu}(M, N) = \sum_{i=1}^{q^{\ell^2}} A_{P^{\mu+1}}(M, B_i)$. By assumption B_i is isometric to N therefore $A_{P^{\mu+1}}(M, B_i) = A_{P^{\mu+1}}(M, N)$ for $i = 1, \dots, q^{\ell^2}$. This completes the proof. $\square$

Case $q = \text{odd prime}$ and $P \mid q$

Proposition 5.5. Let $M = J \perp N$ with some $\lambda^s S_p$ -modular lattice J ($s = 0$ or 1) such that $sM = sJ$. Then $A_{P^\mu}(M, M) = q^{st(r-t)} A_{P^\mu}(M, J) A_{P^\mu}(N, N)$, where $\text{rank}_{S_p} M = r$, $\text{rank}_{S_p} J = t$ and μ is any integer ≥ 1 .

Proof. Let H_M, H_J and H_N be the hermitian matrices of M, J and N corresponding to certain basis of M respectively. Therefore $H_M = \begin{pmatrix} H_J & 0 \\ 0 & H_N \end{pmatrix}$. Let $C = (c_{ij})$ be a (r, t) -matrix over S_P such that ${}^t\overline{C}H_MC \equiv H_J(\text{mod } P^\mu S_P)$. Then ${}^t\overline{C}H_MC$ corresponds to a sublattice J_1 of M . Since ${}^t\overline{C}H_MC \equiv H_J(\text{mod } P^\mu S_P)$, J_1 is isometric to J by Theorem 8.2 in [7]. Therefore by Proposition 5.3 J_1 splits M . That is there exists a (r, r) -matrix $A = (a_{ij})$ such that $a_{ij} = c_{ij}$ for $i = 1, \dots, r, j = 1, \dots, t$ and ${}^t\overline{A}H_MA \equiv H_M(\text{mod } P^\mu S_P)$. Let $X = (x_{ij})$ be a (r, r) -matrix such that $x_{ij} \equiv c_{ij}(\text{mod } P^\mu S_P), i = 1, \dots, r, j = 1, \dots, t$, and ${}^t\overline{X}H_MX \equiv H_M(\text{mod } P^\mu S_P)$. Then $A^{-1}X \equiv \begin{pmatrix} 1 & & 0 & \\ & \ddots & & B \\ 0 & & 1 & \\ & 0 & & X_1 \end{pmatrix} (\text{mod } P^\mu S_P)$ with some (t, n) -matrix B and (n, n) -matrix X_1 , where $n = r - t$. Then we have ${}^t\overline{X}H_MX \equiv \begin{pmatrix} H_J & H_J B \\ {}^t\overline{B}H_J & {}^t\overline{X}_1 H_N X_1 \end{pmatrix} \equiv \begin{pmatrix} H_J & 0 \\ 0 & H_N \end{pmatrix} (\text{mod } P^\mu S_P)$. Hence we have $H_J B \equiv 0(\text{mod } P^\mu S_P)$. By Proposition 8.1 in [7] we may assume $H_J = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & dJ \end{pmatrix}$ if $s = 0$ ($dJ = \text{a unit in } K_P$) and $H_J = \begin{pmatrix} 0 & \lambda & & 0 \\ -\lambda & 0 & & \\ & & \ddots & \\ 0 & & & 0 & \lambda \\ & & & -\lambda & 0 \end{pmatrix}$ if $s = 1$. Therefore $B \equiv 0(\text{mod } P^\mu S_P)$ if $s = 0$, and $B \equiv 0(\text{mod } \lambda^{2\mu-1} S_P)$ if $s = 1$. Therefore the number of choices for B is $q^{st(r-t)}$. This completes the proof. $\square$

Proposition 5.6. Let $M = \langle \epsilon \rangle \perp \langle 1 \rangle \perp \dots \perp \langle 1 \rangle$ of $\text{rank}_{S_P} M = r$. Then we have

- (i) if $r = \text{even}$ then $A_P(M, \langle \epsilon \rangle) = q^{2r-1} (1 - (\frac{(-1)^{\frac{r}{2}} \epsilon}{p}) q^{-\frac{r}{2}})$
- (ii) if $r = \text{odd}$ then $A_P(M, \langle \epsilon \rangle) = q^{2r-1} (1 + (\frac{(-1)^{\frac{r-1}{2}}}{p}) q^{-\frac{r-1}{2}})$.

Proof. The number $A_P(M, \langle \epsilon \rangle)$ is the same as the number of modulo $P S_P$ distinct

vectors $\mathbf{a} = {}^t(a_1, \dots, a_r)$ such that

$$(3) \quad {}^t\bar{\mathbf{a}}H\mathbf{a} \equiv \epsilon \pmod{\mathcal{P}S_{\mathcal{P}}}.$$

Since $S_{\mathcal{P}}$ is generated by 1 and λ over $R_{\mathcal{P}}$, $a_i = \xi_i + \lambda\eta_i$ with some $\xi_i, \eta_i \in R_{\mathcal{P}}$.

Then the equation (3) is equivalent to

$$(4) \quad \epsilon\xi_i^2 + \sum_{i=2}^r \xi_i^2 \equiv \epsilon \pmod{\mathcal{P}R_{\mathcal{P}}}$$

with η_i chosen arbitrarily $\pmod{\mathcal{P}S_{\mathcal{P}}}$. The number of the solutions of (4) is well-known (see Hilfssatz 12 in [15]) and is $q^{r-1}(1 - (\frac{(-1)^{\frac{r}{2}}\epsilon}{p})q^{-\frac{r}{2}})$ if r = even, and $q^{r-1}(1 + (\frac{(-1)^{\frac{r-1}{2}}}{p})q^{-\frac{r-1}{2}})$ if r = odd. This gives the Proposition 5.6. $\square$

Proposition 5.7. Let M be a unimodular hermitian $S_{\mathcal{P}}$ -lattice, $\text{rank}_{S_{\mathcal{P}}} M = r$ and $dM = \epsilon$. Then the local density of M is given by the following,

(i) if r = even then

$$\beta_{\mathcal{P}}(M) = 2(1 - (\frac{(-1)^{\frac{r}{2}}\epsilon}{p})q^{-\frac{r}{2}}) \prod_{i=1}^{\frac{r-2}{2}} (1 - q^{-2i}),$$

if r = odd then

$$\beta_{\mathcal{P}}(M) = 2 \prod_{i=1}^{\frac{r-1}{2}} (1 - q^{-2i}).$$

Proof. By Proposition 8.1 in [7], $M \cong \langle \epsilon \rangle \perp \langle 1 \rangle \perp \dots \perp \langle 1 \rangle$. By Proposition 5.1, $\beta_{\mathcal{P}}(M) = q^{-r^2} A_{\mathcal{P}}(M, M)$. By Proposition 5.5, $A_{\mathcal{P}}(M, M) = A_{\mathcal{P}}(M, \langle \epsilon \rangle) A_{\mathcal{P}}(N, N)$, where $N = \langle 1 \rangle \perp \dots \perp \langle 1 \rangle$ and $\text{rank}_{S_{\mathcal{P}}} N = r - 1$. Therefore by Proposition 5.6, if r = even then

$$A_{\mathcal{P}}(M, M) = q^{2r-1} (1 - (\frac{(-1)^{\frac{r}{2}}\epsilon}{p})q^{-\frac{r}{2}}) A_{\mathcal{P}}(N, N),$$

and if $r = \text{odd}$ then

$$A_P(\mathcal{M}, \mathcal{M}) = q^{2r-1} \left(1 + \left(\frac{(-1)^{\frac{r-1}{2}}}{p} \right) q^{-\frac{r-1}{2}} \right) A_P(\mathcal{N}, \mathcal{N}).$$

Inductive computations give the Proposition 5.7. $\square$

Proposition 5.8 Let $\mathcal{M}$ be a λS_P -modular hermitian lattice of $\text{rank}_{S_P} \mathcal{M} = r$ then the local density at P is given by

$$\beta_P(\mathcal{M}) = q^{\frac{1}{2}r(r+1)} \prod_{i=1}^{\frac{r}{2}} (1 - q^{-2i})$$

Proof. By Proposition 8.1 in [7], r has to be even and $\mathcal{M}$ is isometric to $\begin{pmatrix} -\lambda & \lambda \end{pmatrix} \perp$

$$\begin{pmatrix} -\lambda & \lambda \\ 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} -\lambda & \lambda \end{pmatrix}. \text{ Let } H_r \text{ be the } (r, r)\text{-matrix}$$

$$\begin{pmatrix} 0 & \lambda & & & \\ -\lambda & 0 & & & \\ & & 0 & \lambda & \\ & & -\lambda & 0 & \\ & & & \ddots & \\ 0 & & & & 0 & \lambda \\ & & & & -\lambda & 0 \end{pmatrix}.$$

Let C be a $(r, 2)$ -matrix $\begin{pmatrix} a_{11}b_{11} & \cdots & a_{\frac{r}{2},1}b_{\frac{r}{2},1} \\ a_{12}b_{12} & \cdots & a_{\frac{r}{2},2}b_{\frac{r}{2},2} \end{pmatrix}$ such that

$$(5) \quad {}^t \overline{C} H_r C \equiv H_2 \pmod{P S_P}.$$

Then number of the $\text{mod } P S_P$ distinct solutions of (5) is the same as the number of the solutions which satisfy the following three equations (6), (7) and (8).

$$(6) \quad \lambda \sum_{i=1}^{\frac{r}{2}} (\overline{a}_{i1} b_{i1} - a_{i1} \overline{b}_{i1}) \equiv 0 \pmod{P S_P}.$$

$$(7) \quad \lambda \sum_{i=1}^{\frac{r}{2}} (\overline{a}_{i2} b_{i2} - a_{i2} \overline{b}_{i2}) \equiv 0 \pmod{P S_P}.$$

$$(8) \quad \lambda \sum_{i=1}^{\frac{r}{2}} (\overline{a}_{i1} b_{i2} - \overline{b}_{i1} a_{i2}) \equiv \lambda \pmod{P S_P}.$$

Let $a_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}\lambda$, $b_{ij} = b_{ij}^{(0)} + b_{ij}^{(1)}\lambda$ with $a_{ij}^{(\ell)}, b_{ij}^{(\ell)} \in R_P$. Then the number of solutions which satisfy (6), (7) and (8) is equal to the number of solutions which satisfy the following three equations.

$$(9) \quad \lambda^2 \sum_{i=1}^{\frac{r}{2}} (a_{i1}^{(0)} b_{i1}^{(1)} - a_{i1}^{(1)} b_{i1}^{(0)}) \equiv 0 \pmod{P R_P},$$

$$(10) \quad \lambda^2 \sum_{i=1}^{\frac{r}{2}} (a_{i2}^{(0)} b_{i2}^{(1)} - a_{i2}^{(1)} b_{i2}^{(0)}) \equiv 0 \pmod{P R_P},$$

$$(11) \quad \lambda \sum_{i=1}^{\frac{r}{2}} (a_{i1}^{(0)} b_{i2}^{(0)} - b_{i1}^{(0)} a_{i2}^{(0)}) + \lambda^3 \sum_{\lambda=1}^{\frac{r}{2}} (b_{i1}^{(1)} a_{i2}^{(1)} - a_{i1}^{(1)} b_{i2}^{(1)}) \\ + \lambda^2 \sum_{i=1}^{\frac{r}{2}} (a_{i1}^{(0)} b_{i2}^{(1)} - a_{i1}^{(1)} b_{i2}^{(0)} - b_{i1}^{(0)} a_{i2}^{(1)} + b_{i1}^{(1)} a_{i2}^{(0)}) \equiv \lambda \pmod{P R_P}$$

Since $\lambda^2 R_P = P$ equations (9) and (10) are satisfied automatically. Equation (11) is equivalent to

$$(12) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i1}^{(0)} b_{i2}^{(0)} - b_{i1}^{(0)} a_{i2}^{(0)}) \equiv 1 \pmod{P R_P}.$$

Then by (12) $a_{ik}, i = 1, \dots, \frac{r}{2}, k = 1, 2$, are not in $P R_P$ at the same time. For any such $\{a_{ik}^{(0)}\}_{i=1, \dots, \frac{r}{2}, k=1, 2}$ fixed, the number of the solution of (12) is q^{r-1} . The number of such $\{a_{ik}^{(0)}\}_{i=1, \dots, \frac{r}{2}, k=1, 2}$ is $q^r - 1$. Therefore we have $A_P(\mathcal{M}, H_2) = q^{2r} q^{r-1} (q^r - 1) = q^{4r-1} (1 - q^{-r})$. By Proposition 5.5 $A_P(\mathcal{M}, \mathcal{M}) = q^{2(r-2)} A_P(\mathcal{M}, H_2) = q^{6r-5} (1 - q^{-r}) A_P(H_{r-2}, H_{r-2})$, where H_ℓ denotes the lattice defined by the hermitian matrix H_ℓ also. By inductive computation we have $A_P(\mathcal{M}, \mathcal{M}) = q^{\frac{3}{2}r^2 + \frac{1}{2}r} \prod_{i=1}^{\frac{r}{2}} (1 - q^{-2i})$. Therefore $\beta_P(\mathcal{M}) = q^{-r^2} A_P(\mathcal{M}, \mathcal{M}) = q^{\frac{1}{2}r(r+1)} \prod_{i=1}^{\frac{r}{2}} (1 - q^{-2i})$. $\square$

Proposition 5.9. Let $\mathcal{M} \cong \langle \epsilon \rangle \perp \langle 1 \rangle \perp \dots \perp \langle 1 \rangle \perp H_{r-\rho}$, where $\text{rank}_{S_P} \mathcal{M} = r$, $r - \rho$ is even and $0 \leq \rho \leq r$. Then the local density of $\mathcal{M}$ is given by the following equations.

(i) If $\rho = \text{even}$ then

$$\beta_{\mathcal{P}}(\mathcal{M}) = \frac{2q^{\frac{1}{2}(r-\rho)(r-\rho+1)} \prod_{i=1}^{\frac{\rho}{2}} (1 - q^{-2i}) \prod_{i=1}^{\frac{r-\rho}{2}} (1 - q^{-2i})}{1 + \left(\frac{(-1)^{\frac{\rho}{2}} \epsilon}{q}\right) q^{-\frac{\rho}{2}}}$$

(ii) If $\rho = \text{odd}$ then

$$\beta_{\mathcal{P}}(\mathcal{M}) = 2q^{\frac{1}{2}(r-\rho)(r-\rho+1)} \prod_{i=1}^{\frac{\rho-1}{2}} (1 - q^{-2i}) \prod_{i=1}^{\frac{r-\rho}{2}} (1 - q^{-2i}).$$

Proof. If $\rho = 0$ then it is already shown in Proposition 5.8. Therefore we may assume $\rho \geq 1$. Let $\mathbf{a} = {}^t(a_1, \dots, a_r)$ be a vector over $S_{\mathcal{P}}$ such that

$$(13) \quad {}^t \mathbf{a} \begin{pmatrix} \epsilon & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & 0 \\ & & & & 0 & \lambda & & \\ & & & & -\lambda & 0 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & 0 & \lambda \\ & & & & & & & -\lambda & 0 \end{pmatrix} \mathbf{a}$$

$$\equiv \begin{pmatrix} \epsilon & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & 0 \\ & & & & 0 & \lambda & & \\ & & & & -\lambda & 0 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & 0 & \lambda \\ & & & & & & & -\lambda & 0 \end{pmatrix} \pmod{\mathcal{P} S_{\mathcal{P}}}.$$

Let $a_i = a_{i0} + \lambda a_{i1}$ with $a_{i0}, a_{i1} \in R_{\mathcal{P}}$ for $i = 1, \dots, r$. Then (13) is equivalent to

$$(14) \quad \epsilon a_{10}^2 + \sum_{i=2}^{\rho} a_{i0}^2 \equiv \epsilon \pmod{\mathcal{P} R_{\mathcal{P}}}.$$

The number of the solutions of (14) is well known (see Hilfssatz 12 in [15]) and is $q^{\rho-1}(1 - (\frac{(-1)^{\frac{\rho}{2}} \epsilon}{p})q^{-\frac{\rho}{2}})$ if $\rho = \text{even}$, and $q^{\rho-1}(1 + (\frac{(-1)^{\frac{\rho-1}{2}}}{p})q^{-\frac{\rho}{2}})$ if $\rho = \text{odd}$.

Therefore $A_{\mathcal{P}}(\mathcal{M}, \langle \epsilon \rangle) = q^{2r-1}(1 - (\frac{(-1)^{\frac{r}{2}}\epsilon}{p})q^{-\frac{r}{2}})$ if $\rho = \text{even}$, and $A_{\mathcal{P}}(\mathcal{M}, \langle \epsilon \rangle) = q^{2r-1}(1 + (\frac{(-1)^{\frac{r-1}{2}}}{p})q^{-\frac{r-1}{2}})$ if $\rho = \text{odd}$. On the other hand if $a \equiv \epsilon(\mathcal{P})$ and $a \in R_{\mathcal{P}}$ then $\langle a \rangle \cong \langle \epsilon \rangle$ as hermitian $S_{\mathcal{P}}$ -lattice. Therefore by Proposition 5.4 we have $q^{2r}A_{\mathcal{P}\mu}(\mathcal{M}, \langle \epsilon \rangle) = qA_{\mathcal{P}\mu+1}(\mathcal{M}, \langle \epsilon \rangle)$ for $\mu \geq 1$. Therefore $A_{\mathcal{P}3}(\mathcal{M}, \langle \epsilon \rangle) = q^{4r-2}A_{\mathcal{P}}(\mathcal{M}, \langle \epsilon \rangle)$.

Hence we have by Proposition 5.5

$$\begin{aligned} A_{\mathcal{P}3}(\mathcal{M}, \mathcal{M}) &= A_{\mathcal{P}3}(\mathcal{M}, \langle \epsilon \rangle)A_{\mathcal{P}3}(\mathcal{N}, \mathcal{N}) \\ &= q^{6r-3}(1 - (\frac{(-1)^{\frac{r}{2}}\epsilon}{q})q^{-\frac{r}{2}})A_{\mathcal{P}3}(\mathcal{N}, \mathcal{N}) \text{ if } \rho \text{ even} \\ \text{and} \\ &= q^{6r-3}(1 + (\frac{(-1)^{\frac{r-1}{2}}}{q})q^{-\frac{r-1}{2}})A_{\mathcal{P}3}(\mathcal{N}, \mathcal{N}) \text{ if } \rho \text{ odd,} \end{aligned}$$

where $\mathcal{N} = \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle \perp H_{r-\rho}$ and $\text{rank}_{S_{\mathcal{P}}} \mathcal{N} = r - 1$. Then by inductive computation and Proposition 5.8 we can complete the proof. $\square$

Case $q = 4$ (dyadic ramified case) and $\mathcal{P} \mid 2$.

In this case $E = \mathbb{Q}(\sqrt{-1})$, $K = \mathbb{Q}$, $K_2 = \mathbb{Q}_2$, $R_2 = \mathbb{Z}_2$, $E_2 = \mathbb{Q}_2(\sqrt{-1})$ and $S_2 = \mathbb{Z}_2[\sqrt{-1}]$.

Let H_i be the uniodular hermitian S_2 -lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ where $\text{rank}_{S_2} H_i = i$. We also denote the (i, i) -hermitian matrix $\begin{pmatrix} 0 & 1 & & 0 \\ 1 & 0 & & \\ & & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}$ by H_i .

Proposition 5.10. Let C be a hermitian (r, r) -matrix over S_2 such that $C \equiv H_r \pmod{2S_2}$, then the lattice defined by C is isometric to H_i .

Proof. Let $\mathcal{M}$ be the lattice defined by C . Clearly $\mathcal{M}$ is unimodular and $n(\mathcal{M}) \subseteq 2S_2$. By Proposition 10.3 in [7], $\mathcal{M} = J \perp H_{r-2}$ with some unimodular lattice J of rank 2 and $n(J) = n(\mathcal{M})$. By Proposition 9.1 in [7], $n(J) \supseteq n(H_2) = 2S_2$. Therefore by Proposition 9.2 in [7], $J \cong H_2$. $\square$

Proposition 5.11. Let $\mathcal{M} = H_2 \perp \mathcal{N}$, where $\mathcal{N}$ is a unimodular hermitian S_2 -lattice of rank $r - 2$, then we have $A_{2^\mu}(\mathcal{M}, \mathcal{M}) = A_{2^\mu}(\mathcal{M}, H_2)A_{2^\mu}(\mathcal{N}, \mathcal{N})$ for any $\mu \geq 1$.

Proof. Let $H_{\mathcal{M}}$ and $H_{\mathcal{N}}$ be the hermitian matrices of the lattices $\mathcal{M}$ and $\mathcal{N}$ with respect to certain base of $\mathcal{M}$ respectively. Then we may assume $H_{\mathcal{M}} = \begin{pmatrix} H_2 & 0 \\ 0 & H_{\mathcal{N}} \end{pmatrix}$. Let C be a $(r, 2)$ -matrix such that ${}^t\overline{C}H_{\mathcal{M}}C \equiv H_2 \pmod{2^\mu S_2}$. Then ${}^t\overline{C}HC$ defines a sublattice J of $\mathcal{M}$. By Proposition 5.10 $J \cong H_2$. By Proposition 5.3, J splits $\mathcal{M}$, i.e., $\mathcal{M} = J \perp \mathcal{K}$ with some unimodular sublattice $\mathcal{K}$ of $\mathcal{M}$. By Proposition 9.3 in [7], $\mathcal{N} \cong \mathcal{K}$. Therefore there exists a (r, r) -matrix $A = (a_{ij})$ such that $a_{ij} = c_{ij}$, $i = 1, \dots, r, j = 1, 2$, and ${}^t\overline{A}HA \equiv H_{\mathcal{M}} \pmod{2^\mu S_2}$.

Let $X = (x_{ij})$ be a (r, r) -matrix over S_2 such that $x_{ij} \equiv c_{ij} \pmod{2^\mu S_2}$, $i = 1, \dots, r, j = 1, 2$, and ${}^t\overline{X}H_{\mathcal{M}}X \equiv H_{\mathcal{M}} \pmod{2^\mu S_2}$, then $A^{-1}X \equiv \begin{pmatrix} H_2 & B \\ 0 & X_1 \end{pmatrix} \pmod{2^\mu S_2}$ with some $(2, r - 2)$ -matrix B over S_2 . Then we have ${}^t\overline{X}H_{\mathcal{M}}X \equiv \begin{pmatrix} H_2 & H_2 B \\ {}^t\overline{B}H_2 & {}^t\overline{X}_1 H_{\mathcal{N}} X_1 \end{pmatrix}$. Therefore we have $B \equiv 0 \pmod{2^\mu S_2}$ and we have the Proposition. $\square$

Proposition 5.12. Let $\mathcal{M} \cong H_r$. Then

$$\beta_2(\mathcal{M}) = 2^r \prod_{i=1}^{\frac{r}{2}} (1 - 2^{-2i}).$$

Proof. By Propositions 5.10 and 5.4 we have $\beta_2(\mathcal{M}) = 2^{-r^2} A_2(H_r, H_r)$. By Proposition

5.11 we have $A_2(H_r, H_r) = A_2(H_r, H_2)A_2(H_{r-2}, H_{r-2})$. Let

$$C = \begin{pmatrix} a_{11}b_{11} & \cdots & a_{\frac{r}{2},1}b_{\frac{r}{2},1} \\ a_{12}b_{12} & \cdots & a_{\frac{r}{2},2}b_{\frac{r}{2},2} \end{pmatrix}$$

be a $(r, 2)$ -matrix over S_2 such that

$$(15) \quad {}^t\overline{C}H_rC \equiv H_2 \pmod{2S_2}.$$

Then (15) is equivalent to the following equations.

$$(16) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i1}\overline{b}_{i1} + \overline{a}_{i1}b_{i1}) \equiv 0 \pmod{2S_2}$$

$$(17) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i2}\overline{b}_{i2} + \overline{a}_{i2}b_{i2}) \equiv 0 \pmod{2S_2}$$

$$(18) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i2}\overline{b}_{i1} + \overline{a}_{i1}b_{i2}) \equiv 1 \pmod{2S_2}$$

Let

$$\begin{aligned} a_{ij} &= a_{ij}^{(0)} + a_{ij}^{(1)}\sqrt{-1} \\ b_{ij} &= b_{ij}^{(0)} + b_{ij}^{(1)}\sqrt{-1} \end{aligned}$$

where $a_{ij}^{(k)}, b_{ij}^{(k)} \in \mathbb{Z}_2$ $i = 1, \dots, \frac{r}{2}$; $j = 1, 2$; $k = 0, 1$.

Then we easily see that (16) and (17) are satisfied for any $a_{ij}^{(k)}, b_{ij}^{(k)}$ and that (18) is equivalent to

$$\begin{aligned} (19) \quad & \sum_{i=1}^{\frac{r}{2}} (a_{i2}^{(0)}b_{i1}^{(0)} + a_{i1}^{(0)}b_{i2}^{(0)} + a_{i2}^{(1)}b_{i1}^{(1)} + a_{i1}^{(1)}b_{i2}^{(1)}) \\ & + \sqrt{-1} \sum_{i=1}^{\frac{r}{2}} (a_{i2}^{(1)}b_{i1}^{(0)} - a_{i2}^{(0)}b_{i1}^{(1)} - a_{i1}^{(1)}b_{i2}^{(0)} + a_{i1}^{(0)}b_{i2}^{(1)}) \\ & \equiv 1 \pmod{2\mathbb{Z}_2} \end{aligned}$$

and hence it is equivalent to the following two equations.

$$(20) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i2}^{(0)} b_{i1}^{(0)} + a_{i1}^{(0)} b_{i2}^{(0)} + a_{i2}^{(1)} b_{i1}^{(1)} + a_{i1}^{(1)} b_{i2}^{(1)}) \equiv 1 \pmod{2\mathbb{Z}_2}$$

$$(21) \quad \sum_{i=1}^{\frac{r}{2}} (a_{i2}^{(1)} b_{i1}^{(0)} - a_{i1}^{(1)} b_{i2}^{(0)} - a_{i2}^{(0)} b_{i1}^{(1)} + a_{i1}^{(0)} b_{i2}^{(1)}) \equiv 0 \pmod{2\mathbb{Z}_2}$$

Let $A = \begin{pmatrix} \cdots & a_{i2}^{(0)} & a_{i1}^{(0)} & a_{i2}^{(1)} & a_{i1}^{(1)} & \cdots \\ \cdots & a_{i2}^{(1)} & a_{i1}^{(1)} & a_{i2}^{(0)} & a_{i1}^{(0)} & \cdots \end{pmatrix}$ and $\mathbf{b} = {}^t (\cdots b_{i1}^{(0)} b_{i2}^{(0)} b_{i1}^{(1)} b_{i2}^{(1)} \cdots)$

Then (20) and (21) are equivalent to

$$(22) \quad A\mathbf{b} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod{2\mathbb{Z}_2}.$$

Therefore rank of A has to be 2. Hence, for a fixed such A the number of solutions of $\pmod{2\mathbb{Z}_2}$ is 2^{2r-2} . It is easy to see that the number of such matrices A of rank 2 is $2^{2r} - 2^r$. Therefore we have $A_2(H_r, H_2) = 2^{4r-2}(1 - 2^{-r})$ and $A_2(H_r, H_r) = 2^{4r-2}(1 - 2^{-r})A_2(H_{r-2}, H_{r-2})$. By inductive computation we have

$$A_2(H_r, H_r) = 2^{r^2+r} \prod_{i=1}^{\frac{r}{2}} (1 - 2^{-2i}).$$

This completes the proof. □

Proposition 5.13. Let $\mathcal{M} = \langle \epsilon \rangle \perp H_{r-1}$ with some odd integer $r \geq 1$ and a unit ϵ in $\mathbb{Z}_2$. Then we have

$$\beta_2(\mathcal{M}) = 2 \prod_{i=1}^{\frac{r-1}{2}} (1 - 2^{-2i}).$$

Proof. If $r = 1$, then by Proposition 5.1

$$\beta_2(\langle \epsilon \rangle) = 2^{-3} A_{2^3}(\langle \epsilon \rangle, \langle \epsilon \rangle).$$

It is easy to see $A_{2^3}(\langle \epsilon \rangle, \langle \epsilon \rangle) = 2^4$, i.e., $\beta_2(\langle \epsilon \rangle) = 2$. Therefore we may assume $r \geq 3$.

Let $\mathcal{N} = \langle \epsilon \rangle \perp H_{r-3}$. Then Proposition 5.11 gives

$$A_{2^\mu}(\mathcal{M}, \mathcal{M}) = A_{2^\mu}(\mathcal{M}, H_2)A_{2^\mu}(\mathcal{N}, \mathcal{N})$$

for any $\mu \geq 1$. First we evaluate $A_2(\mathcal{M}, H_2)$. Let

$C = {}^t \begin{pmatrix} a_1 a_{11} b_{11} & \cdots & a_{\frac{r-1}{2}, 1} b_{\frac{r-1}{2}, 1} \\ a_2 a_{12} b_{12} & \cdots & a_{\frac{r-1}{2}, 2} b_{\frac{r-1}{2}, 2} \end{pmatrix}$ be a $(r, 2)$ -matrix over S_2 such that

$$(23) \quad {}^t \overline{C} H C \equiv H_2 \pmod{2S_2}.$$

Then (23) is equivalent to the following three equations.

$$(24) \quad \epsilon a_1 \bar{a}_1 + \sum_{i=1}^{\frac{r-1}{2}} (a_{i1} \bar{b}_{i1} + \bar{a}_{i1} b_{i1}) \equiv 0 \pmod{2S_2}$$

$$(25) \quad \epsilon a_2 \bar{a}_2 + \sum_{i=2}^{\frac{r-1}{2}} (a_{i2} \bar{b}_{i2} + \bar{a}_{i2} b_{i2}) \equiv 0 \pmod{2S_2}$$

$$(26) \quad \epsilon \bar{a}_1 a_2 + \sum_{i=1}^{\frac{r-1}{2}} (a_{i2} \bar{b}_{i1} + \bar{a}_{i1} b_{i2}) \equiv 1 \pmod{2S_2}$$

Put

$$\begin{aligned} a_i &= a_i^{(0)} + a_i^{(1)} \sqrt{-1} \quad i = 1, 2, \\ a_{ij} &= a_{ij}^{(0)} + a_{ij}^{(1)} \sqrt{-1} \quad i = 1, \dots, \frac{r-1}{2}; j = 1, 2, \\ b_{ij} &= b_{ij}^{(0)} + b_{ij}^{(1)} \sqrt{-1} \quad i = 1, \dots, \frac{r-1}{2}; j = 1, 2, \end{aligned}$$

where $a_i^{(j)}$, $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$ are in $\mathbb{Z}_2$. Then equations (24), (25) and (26) are equivalent to the following (27), (28) and (29).

$$(27) \quad (a_1^{(0)})^2 + (a_1^{(1)})^2 \equiv 0 \pmod{2\mathbb{Z}_2},$$

$$(28) \quad (a_2^{(0)})^2 + (a_2^{(1)})^2 \equiv 0 \pmod{2\mathbb{Z}_2},$$

$$\begin{aligned}
(29) \quad & \epsilon(a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)}) + \epsilon(a_1^{(0)}a_2^{(1)} - a_1^{(1)}a_2^{(0)})\sqrt{-1} \\
& + \sum_{i=1}^{\frac{r-1}{2}} (a_{i2}^{(0)}b_{i1}^{(0)} + a_{i1}^{(0)}b_{i2}^{(0)} + a_{i2}^{(1)}b_{i1}^{(1)} + a_{i1}^{(1)}b_{i2}^{(1)}) \\
& + \sqrt{-1} \sum_{i=1}^{\frac{r-1}{2}} (a_{i2}^{(1)}b_{i1}^{(0)} - a_{i1}^{(1)}b_{i2}^{(0)} - a_{i2}^{(0)}b_{i1}^{(1)} + a_{i1}^{(0)}b_{i2}^{(1)}) \equiv 1 \pmod{2\mathbb{Z}_2}.
\end{aligned}$$

The equation (29) is equivalent to the following two equations (30) and (31).

$$\begin{aligned}
(30) \quad & \sum_{i=1}^{\frac{r-1}{2}} (a_{i2}^{(0)}b_{i1}^{(0)} + a_{i1}^{(0)}b_{i2}^{(0)} + a_{i2}^{(1)}b_{i1}^{(1)} + a_{i1}^{(1)}b_{i2}^{(1)}) \\
& \equiv 1 + (a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)}) \pmod{2\mathbb{Z}_2},
\end{aligned}$$

$$\begin{aligned}
(31) \quad & \sum_{i=2}^{\frac{r-1}{2}} (a_{i2}^{(1)}b_{i1}^{(0)} + a_{i1}^{(1)}b_{i2}^{(0)} + a_{i2}^{(0)}b_{i1}^{(1)} + a_{i1}^{(0)}b_{i2}^{(1)}) \\
& \equiv a_1^{(0)}a_2^{(1)} + a_1^{(1)}a_2^{(0)} \pmod{2\mathbb{Z}_2}.
\end{aligned}$$

For any $a_i^{(j)}$ satisfying (27) and (28), we have $a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)} \equiv a_1^{(0)}a_2^{(1)} + a_1^{(1)}a_2^{(0)}$.

Let

$$A = \begin{pmatrix} \cdots & a_{i2}^{(0)}, a_{i1}^{(0)}, a_{i2}^{(1)}, a_{i1}^{(1)}, & \cdots \\ \cdots & a_{i2}^{(1)}, a_{i1}^{(1)}, a_{i2}^{(0)}, a_{i1}^{(0)}, & \cdots \end{pmatrix}$$

be a $(2, 2(r-1))$ -matrix. Then (30) and (31) show that the rank of A is 2. Therefore for fixed $a_i^{(j)}, a_{ij}^{(k)}$ the number of the solutions of (30) and (31) is $2^{2(r-1)-2} = 2^{2r-4}$. On the other hand, the number of such matrices A of rank 2 is $2^{2(r-1)} - 2^{r-1}$, and the number of solutions satisfying (27) and (28) is 4.

Therefore we have $A_2(\mathcal{M}, H_2) = 2^{4(r-1)}(1 - 2^{-(r-1)})$. By Propositions 5.4 and 5.10

$$\begin{aligned}
A_{23}(\mathcal{M}, H_2) &= 2^{4(r-1)}A_{22}(\mathcal{M}, H_2) = 2^{8(r-1)}A_2(\mathcal{M}, H_2) \\
&= 2^{12(r-1)}(1 - 2^{-(r-1)}).
\end{aligned}$$

Hence we have by

$$\begin{aligned} A_{2^3}(\mathcal{M}, \mathcal{M}) &= A_{2^3}(\mathcal{M}, H_2)A_{2^3}(\mathcal{N}, \mathcal{N}) \\ &= 2^{12(r-1)}(1 - 2^{-(r-1)})A_{2^3}(\mathcal{N}, \mathcal{N}) \end{aligned}$$

Therefore inductive computation gives

$$A_{2^3}(\mathcal{M}, \mathcal{M}) = 2^{\sum_{i=1}^{\frac{r-1}{2}} 24i} \prod_{i=1}^{\frac{r-1}{2}} (1 - 2^{-2i}) A_{2^3}(\langle \epsilon \rangle, \langle \epsilon \rangle)$$

It is easy to see $A_{2^3}(\langle \epsilon \rangle, \langle \epsilon \rangle) = 2^4$. Thus we have $A_{2^3}(\mathcal{M}, \mathcal{M}) = 2^{3r^2+1} \prod_{i=1}^{\frac{r-1}{2}} (1 - 2^{-2i})$.

Since, by Proposition 5.1, $\beta_2(\mathcal{M}) = 2^{-3r^2} A_{2^3}(\mathcal{M}, \mathcal{M})$, we have

$$\beta_2(\mathcal{M}) = 2^{\sum_{i=1}^{\frac{r-1}{2}} (1 - 2^{-2i})}.$$

□

Proposition 5.14. Let $\mathcal{M} = \langle \epsilon \rangle \perp \langle 1 \rangle \perp H_{r-2}$ with some even integer $r \geq 2$ and a unit ϵ in $\mathbb{Z}_2$. Then we have $\beta_2(\mathcal{M}) = 2^{\sum_{i=1}^{\frac{r-2}{2}} (1 - 2^{-2i})}$.

Proof. First we estimate $A_2(\mathcal{M}, H_2)$. Let

$$C = {}^t \begin{pmatrix} a_1 b_1 a_{11} b_{11} & \cdots & a_{\frac{r-2}{2}, 1} b_{\frac{r-2}{2}, 1} \\ a_2 b_2 a_{12} b_{12} & \cdots & a_{\frac{r-2}{2}, 2} b_{\frac{r-2}{2}, 2} \end{pmatrix}$$

be a $(r, 2)$ -matrix over S_2 such that

$$(32) \quad {}^t \overline{C} H_{\mathcal{M}} C \equiv H_2 \pmod{2S_2},$$

where $H_{\mathcal{M}} = \begin{pmatrix} \epsilon & & 0 \\ & 1 & \\ 0 & & H_{r-2} \end{pmatrix}$. Let $a_i = a_i^{(0)} + a_i^{(1)}\sqrt{-1}$, $b_i = b_i^{(0)} + b_i^{(1)}\sqrt{-1}$, $a_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}\sqrt{-1}$ and $b_{ij} = b_{ij}^{(0)} + b_{ij}^{(1)}\sqrt{-1}$, where $a_i^{(j)}$, $a_{ij}^{(k)}$, $b_{ij}^{(k)}$ are in $\mathbb{Z}_2$. Then,

equation (32) is equivalent to the following equations.

$$(33) \quad (a_1^{(0)})^2 + (a_1^{(1)})^2 + (b_1^{(0)})^2 + (b_1^{(1)})^2 \equiv 0 \pmod{2\mathbb{Z}_2}$$

$$(34) \quad (a_2^{(0)})^2 + (a_2^{(1)})^2 + (b_2^{(0)})^2 + (b_2^{(1)})^2 \equiv 0 \pmod{2\mathbb{Z}_2}$$

$$(35) \quad \sum_{i=1}^{\frac{r-2}{2}} (a_{i2}^{(0)}b_{i1}^{(0)} + a_{i1}^{(0)}b_{i2}^{(0)} + a_{i2}^{(1)}b_{i1}^{(1)} + a_{i1}^{(1)}b_{i2}^{(1)}) \\ \equiv 1 + a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)} + b_1^{(0)}b_2^{(0)} + b_1^{(1)}b_2^{(1)} \pmod{2\mathbb{Z}_2}$$

$$(36) \quad \sum_{i=1}^{\frac{r-2}{2}} (a_{i2}^{(1)}b_{i1}^{(0)} + a_{i1}^{(1)}b_{i2}^{(0)} + a_{i2}^{(0)}b_{i1}^{(1)} + a_{i1}^{(0)}b_{i2}^{(1)}) \\ \equiv a_1^{(0)}a_2^{(1)} + a_1^{(1)}a_2^{(0)} + b_1^{(0)}b_2^{(1)} + b_1^{(1)}b_2^{(0)} \pmod{2\mathbb{Z}_2}.$$

For any $a_i^{(j)}$ and $b_i^{(j)}$, $i = 1, 2; j = 0, 1$, satisfying (33) and (34) we have $a_1^{(0)}a_2^{(1)} + a_1^{(1)}a_2^{(0)} + b_1^{(0)}b_2^{(1)} + b_1^{(1)}b_2^{(0)} \equiv a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)} + b_1^{(0)}b_2^{(0)} + b_1^{(1)}b_2^{(1)} \pmod{2\mathbb{Z}_2}$.

$$\prod a_1^{(1)}a_2^{(0)} + b_1^{(0)}b_2^{(1)} + b_1^{(1)}b_2^{(0)} \equiv a_1^{(0)}a_2^{(0)} + a_1^{(1)}a_2^{(1)} + b_1^{(0)}b_2^{(0)} + b_1^{(1)}b_2^{(1)} \pmod{2\mathbb{Z}_2}.$$

rank 2 Therefore, the $(2, 2(r-2))$ -matrix $A = \begin{pmatrix} \cdots & a_{i2}^{(0)} & a_{i1}^{(0)} & a_{i2}^{(1)} & a_{i1}^{(1)} & \cdots \\ \cdots & a_{i2}^{(1)} & a_{i1}^{(1)} & a_{i2}^{(0)} & a_{i1}^{(0)} & \cdots \end{pmatrix}$ is of

rank 2. For a fixed A , and $a_i^{(j)}, b_i^{(j)}$ satisfying (33) and (34), the number of solutions satisfying (35) and (36) is 2^{2r-6} . Number of such matrices A of rank 2 is $2^{2(r-2)} - 2^{r-2}$ and number of the solutions satisfying (33) and (34) is 2^6 . Hence we have $A_2(\mathcal{M}, H_2) = 2^{4r-4}(1 - 2^{-(r-2)})$ from which $A_{23}(\mathcal{M}, H_2) = 2^{12(r-1)}(1 - 2^{-(r-2)})$ follows by Propositions 5.4 and 5.10.

Let $\mathcal{N} = \langle \epsilon \rangle \perp \langle 1 \rangle \perp H_{r-4}$. Then Proposition 5.11 gives

$$A_{23}(\mathcal{M}, \mathcal{M}) = A_{23}(\mathcal{M}, H_2)A_{23}(\mathcal{N}, \mathcal{N}) \\ = 2^{12(r-1)}(1 - 2^{-(r-2)})A_{23}(\mathcal{N}, \mathcal{N}).$$

By inductive computation we have

$$A_{2^3}(\mathcal{M}, \mathcal{M}) = 2^{3r^2-12} \sum_{i=1}^{\frac{r-2}{2}} (1 - 2^{-2i}) A_{2^3}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Let $C = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in S_2$, such that

$$(37) \quad {}^t\overline{C} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} C \equiv \epsilon \pmod{2^3 S_2}.$$

Similar argument as above in page 40 shows that there exist a $(2, 2)$ -matrix

$A = \begin{pmatrix} a & * \\ b & * \end{pmatrix}$ satisfying ${}^t\overline{A} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} A \equiv \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \pmod{2^3 S_2}$. Let $X = (x_{ij})$ be a $(2, 2)$ -matrix such that $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \pmod{2^3 S_2}$ and ${}^t\overline{X} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} X \equiv \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \pmod{2^3 S_2}$. Then we have $X \equiv A \begin{pmatrix} 1 & c \\ 0 & x \end{pmatrix}$ with some $c, x \in S_2$. Since ${}^t\overline{X} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} X \equiv \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$, we have $c \equiv 0 \pmod{2^3 S_2}$.

Therefore

$$A_{2^3}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right) = A_{2^3}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \langle \epsilon \rangle\right) A_{2^3}(\langle 1 \rangle, \langle 1 \rangle) = 2^4 A_{2^3}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \langle \epsilon \rangle\right).$$

Now we estimate $A_{2^3}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \langle \epsilon \rangle\right)$. Let $a = a_0 + a_1\sqrt{-1}$ and $b = b_0 + b_1\sqrt{-1}$ with $a_i, b_i \in \mathbb{Z}_2$. Equation (37) is equivalent to

$$(38) \quad \epsilon a_0^2 + \epsilon a_1^2 + b_0^2 + b_1^2 \equiv \epsilon \pmod{2^3 \mathbb{Z}_2}$$

Let $T(a)$ be the number of the solutions of $x^2 + y^2 \equiv a \pmod{2^3 \mathbb{Z}_2}$. We have $T(0) = T(4) = 2^3$, $T(1) = T(2) = T(5) = 2^4$ and $T(3) = T(7) = T(6) = 0$.

Then the number of the solutions of (38) is $2(T(0)T(1) + T(2)T(7) + T(3)T(6) + T(4)T(5)) = 2^9$ if $\epsilon = 1$, and $T(0)T(1) + T(1)T(6) + T(2)T(3) + T(3)T(0) + T(4)T(5) + T(5)T(2) + T(6)T(7) + T(7)T(4) = 2^9$ if $\epsilon = 3$. Hence $A_{2^3}(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \langle \epsilon \rangle) = 2^9$ and thus, $A_{2^3}(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}) = 2^{13}$ and $A_{2^3}(\mathcal{M}, \mathcal{M}) = 2^{3r^2+1} \prod_{i=1}^{\frac{r-2}{2}} (1 - 2^{-2i})$. Since $\beta_2(\mathcal{M}) = 2^{-3r^2} A_{2^3}(\mathcal{M}, \mathcal{M})$ by Proposition 5.1, we have the proposition. $\square$

§6. List of quadratic lattices over 2-adic integers and their local densities.

Classification of quadratic lattices over 2-adic integers is completely done by O'Meara (see [8]). Local densities of quadratic lattices at the prime 2 are obtained by Watson (see [16]).

In this section we give the local densities of quadratic $\mathbb{Z}_2$ -lattices N satisfying the following conditions.

- (i) $\text{rank}_{\mathbb{Z}_2} N = n$ and $dN = 2^{n-\rho}$ with some integer ρ such that $0 \leq \rho \leq n$
- (ii) $N = N_u \perp N_t$ with some sublattices N_u and N_t of N such that N_u is unimodular of rank ρ and N_t is $2\mathbb{Z}_2$ -modular of rank $n - \rho$.

Notation:

$$\begin{aligned}
 H_i &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ of rank } i \geq 0, \\
 A_i &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ of rank } i \geq 2, \\
 P_2(k) &= \prod_{i=1}^k (1 - 2^{-2i}), P_2(0) = 1,
 \end{aligned}$$

the small letters a, b, c and d denote units in $\mathbb{Z}_2$, and $2H_i, 2\langle a \rangle$ etc., mean lattice $H_i, \langle a \rangle$ etc., scaled by 2 respectively.

Let N be a lattice over $\mathbb{Z}_2$ satisfying the conditons (i) and (ii) given above, then N is one of the following 14 types of lattices

- 1) $n = \text{even}$, $\rho = \text{odd}$, $\langle a \rangle \perp H_{\rho-1} \perp 2\langle b \rangle \perp 2H_{n-\rho-1}$, with $ab \equiv (-1)^{\frac{n-2}{2}}$ modulo square. The number of classes of this type is 4.
- 2) $n \equiv 0 \pmod{4}$, $\rho = \text{even} \geq 0$, $H_\rho \perp 2H_{n-\rho}$. There is exactly one class of this type.
- 3) $n \equiv 0 \pmod{4}$, $\rho = \text{even} \geq 2$, $n - \rho \geq 2$, $A_\rho \perp 2A_{n-\rho}$. There is exactly one class of this type.
- 4) $n = \text{even}$, $\rho = \text{even}$, $n - \rho \geq 2$, $H_\rho \perp 2\{\langle a \rangle \perp \langle b \rangle \perp H_{n-\rho-2}\}$, with $a \equiv b \pmod{4}$ for $n \equiv 2 \pmod{4}$, and $a \not\equiv b \pmod{4}$ for $n \equiv 0 \pmod{4}$. The number of classes of this type is 2.
- 5) $n \equiv 0 \pmod{4}$, $\rho = \text{even}$, $n - \rho \geq 4$, $H_\rho \perp 2\{\langle a \rangle \perp \langle b \rangle \perp H_{n-\rho-2}\}$ with $a \not\equiv b \pmod{4}$. The number of classes of this type is 2 respectively.
- 6) $n = \text{even}$, $\rho = \text{even} \geq 2$, $\langle a \rangle \perp \langle b \rangle \perp H_{\rho-2} \perp 2H_{n-\rho}$, with $ab \equiv -1$ modulo square if $n \equiv 0 \pmod{4}$ and $ab \equiv 1$ modulo square if $n \equiv 2 \pmod{4}$.
- 7) $n \equiv 0 \pmod{4}$, $\rho = \text{even} \geq 4$, $\langle a \rangle \perp \langle b \rangle \perp A_{\rho-2} \perp 2H_{n-\rho}$, with $a \not\equiv b \pmod{4}$. The number of classes of this type is 2.
- 8) $n = \text{even}$, $\rho \geq 2$, $n - \rho \geq 2$, $\langle a \rangle \perp \langle b \rangle \perp H_{\rho-2} \perp 2\{\langle c \rangle \perp \langle d \rangle \perp H_{n-\rho-2}\}$, with $abcd \equiv (-1)^{\frac{n}{2}}$ modulo square. The number of classes of this type is 16.
- 9) $n = \text{odd}$, $\rho = \text{odd} \geq 1$, $\langle a \rangle \perp H_{\rho-1} \perp 2H_{n-\rho}$, with $a \equiv (-1)^{\frac{n-1}{2}}$ modulo square. There is exactly one class of this type.
- 10) $n = \text{odd}$, $\rho = \text{odd} \geq 3$, $\langle a \rangle \perp A_{\rho-1} \perp 2H_{n-\rho}$, with $3a \equiv (-1)^{\frac{n-3}{2}}$ modulo square. There is exactly one class of this type.

11) $n = \text{odd}$, $\rho = \text{odd} \geq 1$, $\langle a \rangle \perp H_{\rho-1} \perp 2\{\langle b \rangle \perp \langle c \rangle \perp H_{n-\rho-2}\}$, with $abc \equiv (-1)^{\frac{n-3}{2}}$ modulo square. The number of classes of this type is 8.

12) $n = \text{odd}$, $\rho = \text{even} \leq n-1$, $H_{\rho} \perp 2\{\langle a \rangle \perp H_{n-\rho-1}\}$, with $a \equiv (-1)^{\frac{n-1}{2}}$ modulo square. There is exactly one class of this type.

13) $n = \text{odd}$, $\rho = \text{even} \leq n-3$, $H_{\rho} \perp 2\{\langle a \rangle \perp A_{n-\rho-1}\}$, with $a \equiv 3(-1)^{\frac{n-3}{2}}$ modulo square. There is exactly one class of this type.

14) $n = \text{odd}$, $\rho = \text{even} \geq 2$, $\rho \leq n-1$, $\langle a \rangle \perp \langle b \rangle \perp H_{\rho-2} \perp 2\{\langle c \rangle \perp H_{n-\rho-1}\}$ with $abc \equiv (-1)^{\frac{n-3}{2}}$ modulo square. The number of classes of this type is 8.

Local densities of the lattices given above are as follows:

$N = \text{type 1)}$ then

$$\alpha_2(N) = 2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 - \rho) + 2} P_2\left(\frac{\rho-1}{2}\right) P_2\left(\frac{n-\rho-1}{2}\right).$$

$N = \text{type 2) or 3)}$ then

$$\alpha_2(N) = \frac{2^{\frac{1}{2}n^2 + \frac{3}{2}n - n\rho + \frac{1}{2}(\rho^2 - \rho) + 1} P_2\left(\frac{\rho}{2}\right) P_2\left(\frac{n-\rho}{2}\right)}{(1 + \delta 2^{-\frac{\rho}{2}})(1 + \delta 2^{-\frac{1}{2}(n-\rho)})},$$

where $\delta = 1$ for type 1) and $\delta = -1$ for type 2).

$N = \text{type 4) or 5)}$ then

$$\alpha_2(N) = \frac{2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 + \rho) - 2} P_2\left(\frac{\rho}{2}\right) P_2\left(\frac{n-\rho-2}{2}\right)}{1 + \delta 2^{-\frac{1}{2}(n-\rho-2)}},$$

where $\delta = 1$ for type 4) and $\delta = -1$ for type 5).

$N = \text{type 6) or 7)}$ then

$$\alpha_2(N) = \frac{2^{\frac{1}{2}n^2 + \frac{3}{2}n - n\rho + \frac{1}{2}(\rho^2 - 3\rho) + 2} P_2\left(\frac{\rho-2}{2}\right) P_2\left(\frac{n-\rho}{2}\right)}{1 + \delta 2^{-\frac{1}{2}(\rho-2)}},$$

where $\delta = 1$ for type 6) and $\delta = -1$ for type 7).

$N = \text{type 8) then}$

$$\alpha_2(N) = 2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 + \rho)} P_2\left(\frac{\rho - 2}{2}\right) P_2\left(\frac{n - \rho - 2}{2}\right).$$

$N = \text{type 9) or 10) then}$

$$\alpha_2(N) = \frac{2^{\frac{1}{2}n^2 + \frac{3}{2}n - n\rho + \frac{1}{2}(\rho^2 - 3\rho) + 2} P_2\left(\frac{\rho - 1}{2}\right) P_2\left(\frac{n - \rho}{2}\right)}{1 + \delta 2^{-\frac{1}{2}(\rho - 1)}},$$

where $\delta = 1$ for type 9) and $\delta = -1$ for type 10).

$N = \text{type 11) then}$

$$\alpha_2(N) = 2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 - \rho) + 2} P_2\left(\frac{\rho - 1}{2}\right) P_2\left(\frac{n - \rho - 2}{2}\right).$$

$N = \text{type 12) or 13) then}$

$$\alpha_2(N) = \frac{2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 + \rho) + 2} P_2\left(\frac{\rho}{2}\right) P_2\left(\frac{n - \rho - 1}{2}\right)}{1 + \delta 2^{-\frac{1}{2}(n - \rho - 1)}},$$

where $\delta = 1$ for type 12) and $\delta = -1$ for type 13).

$N = \text{type 14) then}$

$$\alpha_2(N) = 2^{\frac{1}{2}n^2 + \frac{1}{2}n - n\rho + \frac{1}{2}(\rho^2 - \rho) + 2} P_2\left(\frac{\rho - 2}{2}\right) P_2\left(\frac{n - \rho - 1}{2}\right).$$

For fixed n and ρ we have the following table which shows the number of the classes of lattices satisfying the condition (i) and (ii) given at the beginning of this section.

n	ρ	number of classes
$n = \text{even}$	$\rho = \text{odd}$	4
$n = 4$	$\rho = 0$	5
	$\rho = 2$	22
	$\rho = 4$	5
$n \equiv 0 \pmod{4}, n \geq 8$	$\rho = 0$	5
	$\rho = 2$	24
	$4 \leq \rho \leq n - 4, \rho = \text{even}$	26
	$\rho = n - 2$	24
	$\rho = n$	5
$n \equiv 2 \pmod{4}$	$\rho = 0$	2
	$2 \leq \rho \leq n - 2, \rho = \text{even}$	20
	$\rho = n$	2
$n = 1$	$\rho = 0$	1
	$\rho = 1$	1
$n = \text{odd}, n \geq 3$	$\rho = 0$	2
	$\rho = 1$	9
	$2 \leq \rho \leq n - 2$	10
	$\rho = n - 1$	9
	$\rho = n$	2

CHAPTER IV

ESTIMATIONS

In this chapter we estimate $\omega_{R(2)}/\omega(L)$ in §7, $\omega_{R(q)}/\omega(L)$ (for $q \neq 2$) in §8 and $\omega_{IR(q)}/\omega(L)$ in §9. First we give some notation and some estimations of functions we need in this chapter.

Notation:

$$X(N_0, N_1) = |I(V'_0(N_0), V'_1(N_1))| \omega(N_0) \omega(N_1) / \omega(L) \text{ (if } q = 2 \text{ then } N_1 = N_0).$$

$$P_p(k) = \prod_{i=1}^k (1 - p^{-2i}) \text{ for any prime } p.$$

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_{p=\text{prime}} (1 - p^{-k})^{-1}.$$

$$(39) \quad Y(m, r, 2) = \frac{2^{\frac{1}{2}r(m-r-1)} \pi^{\frac{1}{2}r(m-r)} \prod_{i=1}^{m_0} \Gamma(\frac{i}{2}) \prod_{i=1}^r \Gamma(\frac{i}{2})}{\prod_{i=1}^m \Gamma(\frac{i}{2})}.$$

$$(40) \quad Y(m, r, q) = 2^{\frac{1}{4}r(q-1)(2m-rq-3)} \pi^{\frac{1}{4}r(q-1)(2m-rq-1)} \\ \cdot q^{\frac{1}{4}r(2m-rq-1)} \left(\prod_{j=1}^r (j-1)! \right)^{\frac{q-1}{2}} \{m_0!(m_0+2)! \cdots (m-2)!\}^{-1},$$

where $m_0 = m - r(q-1)$.

Local densities at nondyadic primes.

Let N be an integral $\mathbb{Z}$ -lattice of rank n then

$$(41) \quad \alpha_p(N) = P_p\left(\frac{n-1}{2}\right) \text{ for } n = \text{odd and } p \nmid 2dN,$$

$$(42) \quad \alpha_p(N) = \left(1 - \left(\frac{(-1)^{\frac{n}{2}} dN}{p}\right) p^{-\frac{n}{2}}\right) P_p\left(\frac{n-2}{2}\right) \text{ for } n = \text{even}$$

and $p \nmid 2dN$ (see Hilfssatz 12 in [14]).

If

$$N_q \cong \underbrace{\langle 1 \rangle \perp \cdots \perp \langle 1 \rangle \perp \langle \epsilon \rangle}_{\rho \text{ copies}} \perp \langle q \rangle \perp \cdots \perp \langle q\epsilon \rangle$$

then we have the following (43) - (46) (see 6.4 in [16]).

$$(43) \quad \alpha_q(N) = \frac{2q^{\frac{1}{2}(n-\rho)(n-\rho+1)} P_q\left(\frac{\rho-1}{2}\right) P_q\left(\frac{n-\rho}{2}\right)}{1 + \delta q^{-\frac{1}{2}(n-\rho)}}$$

for $n = \text{odd}$ and $\rho = \text{odd}$, where $\delta = \left(\frac{(-1)^{\frac{n-\rho}{2}} \epsilon}{q}\right)$.

$$(44) \quad \alpha_q(N) = \frac{2q^{\frac{1}{2}(n-\rho)(n-\rho+1)} P_q\left(\frac{\rho}{2}\right) P_q\left(\frac{n-\rho-1}{2}\right)}{1 + \delta q^{-\frac{1}{2}(n-\rho-1)}}$$

for $n = \text{odd}$ and $\rho = \text{even}$, where

$$\delta = \left(\frac{(-1)^{\frac{n-\rho-1}{2}} \epsilon}{q}\right).$$

$$(45) \quad \alpha_q(N) = 2q^{\frac{1}{2}(n-\rho)(n-\rho+1)} P_q\left(\frac{\rho-1}{2}\right) P_q\left(\frac{n-\rho-1}{2}\right)$$

for $n = \text{even}$ and $\rho = \text{odd}$.

$$(46) \quad \alpha_q(N) = \frac{2^{\frac{1}{2}(n-\rho)(n-\rho+1)} P_q\left(\frac{\rho}{2}\right) P_q\left(\frac{n-\rho}{2}\right)}{(1 + \delta_1 q^{-\frac{\rho}{2}})(1 + \delta_2 q^{-\frac{n-\rho}{2}})}$$

for $n = \text{even}$ and $\rho = \text{even}$, where

$$\delta_1 = \left(\frac{(-1)^{\frac{\rho}{2}}\epsilon}{q}\right) \text{ and } \delta_2 = \left(\frac{(-1)^{\frac{n-\rho}{2}}\epsilon}{q}\right).$$

Gamma function

$$(47) \quad \Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\pi^{\frac{1}{2}}\Gamma(2x),$$

$$(48) \quad \Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} e^{\eta(x)}, \text{ where } 0 < \eta(x) < \frac{1}{8x},$$

$$(49) \quad \Gamma(n) = (n-1)! \text{ for integer } n \geq 1,$$

$$(50) \quad \zeta(2) = \frac{\pi^2}{6}, \zeta(k) < 1.01 \text{ if } k \geq 7,$$

$$(51) \quad \prod_p P_p(k)^{-1} < \zeta(2)e^{\frac{1}{6}} \text{ for any integer } k \geq 2$$

$$(52) \quad \prod_{p \neq 2} (1 - (\frac{-1}{p})p^{-1})^{-1} = \frac{\pi}{4}, \prod_{p \neq 2} (1 - (\frac{-2}{p})p^{-1}) = \frac{\pi}{2\sqrt{2}},$$

$$(53) \quad \prod_p \prod_{i=3}^r (1 - p^{-i})^{-1} < e^{\frac{1}{2}} \text{ for any integer } r \geq 3,$$

$$(54) \quad \prod_{p \neq 2} (1 - (\frac{-1}{p})p^{-i}) < (1 - 3^{-i})^{-1},$$

$$(55) \quad \prod_{\substack{p \neq 2 \\ p \neq q}} (1 - (\frac{-q}{p})p^{-1})^{-1} < 0.36q \text{ for } q \geq 3$$

(see 5.7 in [16]).

(see 5.7 in [16]).

Order of $I(V'_0(N_0), V'_1(N_1))$. (See [4], pages xi and xii)

If $q = 2$ and $V'_0(N_0) \cong V'_1(N_1) \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then

$$(56) \quad |I(V'_0(N_0), V'_1(N_1))| = 2^{\frac{1}{2}\rho(\rho+1)} \prod_{i=1}^{\frac{\rho}{2}} (1 - 2^{-2i}).$$

If $q = 2$ and $V'_0(N_0) \cong V'_1(N_1) \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 1 \rangle$ then

$$(57) \quad |I(V'_0(N_0), V'_1(N_1))| = 2^{\frac{1}{2}\rho(\rho-1)} \prod_{i=1}^{\frac{\rho-1}{2}} (1 - 2^{-2i}).$$

If $q = 2$ and $V'_0(N_0) \cong V'_1(N_1) \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 1 \rangle \perp \langle 1 \rangle$ then

$$(58) \quad |I(V'_0(N_0), V'_1(N_1))| = 2^{\frac{1}{2}\rho(\rho-1)} \prod_{i=1}^{\frac{\rho-2}{2}} (1 - 2^{-2i}).$$

If $q =$ an odd prime then

$$(59) \quad |I(V'_0(N_0), V'_1(N_1))| = 2q^{\frac{1}{2}\rho(\rho-1)} P_q\left(\frac{\rho-1}{2}\right)$$

for $\rho =$ odd and

$$(60) \quad |I(V'_0(N_0), V'_1(N_1))| = 2q^{\frac{1}{2}\rho(\rho-1)} P_q\left(\frac{\rho}{2}\right) / (1 + \delta q^{-\frac{\rho}{2}})$$

for $\rho =$ even, where $\delta = \left(\frac{(-1)^{\frac{\rho}{2}} dV'_0(N_0)}{q}\right)$.

§7. Estimation of $\omega_{R(2)}/\omega(L)$.

In this section we give an upper bound of $\omega_{R(2)}/\omega(L)$. Let N_0 and N_1 be the lattices such that the pair of genera (G_{N_0}, G_{N_1}) is in $G(2, r, \rho)$, where $1 \leq r \leq [\frac{m}{2}]$ and $0 \leq \rho \leq r$. First we evaluate

$$|I(V'_0(N_0), V'_1(N_1))| \omega(N_0) \omega(N_1) / \omega(L) = X(N_0, N_1).$$

Proposition 7.1. Let $r = 1$ and $m \geq 30$ then we have the followings.

i) If L is odd unimodular then

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{m}{2}-2} \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}.$$

ii) If L is even unimodular then

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{3}{2}m-4} \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}.$$

Proof. Since $r = 1$ we have $\rho = 0$ or 1 , $\omega(N_1) = \frac{1}{2}$, and $|I(V'_0(N_0), V'_1(N_1))| = 1$.

Therefore by the mass formula (1) we have

$$X(N_0, N_1) = \frac{1}{2} \cdot \frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) \cdot \prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)} \cdot \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}.$$

If $m = \text{odd}$ then by (41) and (42)

$$\begin{aligned} \prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)} &= \prod_{p \neq 2} \frac{P_p(\frac{m-1}{2})}{(1 - \delta p^{-\frac{m-1}{2}}) P_p(\frac{m-3}{2})} \\ &= \prod_{p \neq 2} (1 - \delta p^{-\frac{m-1}{2}})^{-1} (1 - p^{-(m-1)}), \end{aligned}$$

where $\delta = (\frac{(-1)^{\frac{m-1}{2}} dN_0}{p})$.

Therefore we have $\prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)} < \prod_{p \neq 2} ((1 - p^{-\frac{m-1}{2}})^{-1} = (1 - 2^{-\frac{m-1}{2}})^{-1} \zeta(\frac{m-1}{2})$. We have also $\alpha_2(L) = 2^2 P_2(\frac{m-1}{2}) / (1 + \delta 2^{-\frac{m-1}{2}})$, where $\delta = \pm 1$ (see §6 type 9) or 10) with $\rho = n = m$).

a) If $\rho = 0$ then $(N_0)_2$ is of type 2), 4) or 5) in §6, and $dN_0 = 2^{m-1}$. Therefore if N_0 is of type 2) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{-m+3} (1 + 2^{-\frac{m-1}{2}}) / (1 + \delta 2^{-\frac{m-1}{2}}),$$

if N_0 is of type 4) or 5) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^4 (1 + \delta_0 2^{-\frac{m-3}{2}}) (1 - 2^{-(m-1)}) / (1 + \delta 2^{-\frac{m-1}{2}}),$$

where $\delta_0 = 0$ or ± 1 and $\delta = \pm 1$. Hence we have if N_0 is of type 2) then

$$X(N_0, N_1) < 1.01 \cdot 2^{-m+2} \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}$$

if N_0 is of type 4) or 5) then

$$X(N_0, N_1) < 1.01 \cdot 2^3 \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}$$

b) If $\rho = 1$ then $(N_0)_2$ is of type 1) and

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{\frac{m}{2}-1} (1 - 2^{-(m-1)}) / (1 + \delta 2^{-\frac{m-1}{2}})$$

Hence we have

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{m}{2}-2} \pi^{\frac{m}{2}} \Gamma(\frac{m}{2})^{-1}$$

If $m = \text{even}$ then by (41) and (42)

$$\prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)} = \prod_{p \neq 2} \frac{(1 - \delta p^{-\frac{m}{2}}) P_p(\frac{m-2}{2})}{P_p(\frac{m-2}{2})} = \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}),$$

where $\delta = \left(\frac{(-1)^{\frac{m}{2}}}{p}\right)$. Hence by (54) $\prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)} < (1 - 3^{-\frac{m}{2}})^{-1}$.

Also we have the followings.

If L is odd then

$$\alpha_2(L) = 2^2 P_2\left(\frac{m-2}{2}\right) / (1 + \delta 2^{-\frac{m-2}{2}}),$$

where $\delta = 0, \pm 1$ (see §6 type 6) or 7) with $\rho = n = m$). If L is even then

$$\alpha_2(L) = 2^m P_2\left(\frac{m}{2}\right) / (1 + 2^{-\frac{m}{2}}),$$

(see §6 type 2) with $\rho = n = m$).

a) If $\rho = 0$ then $(N_0)_2$ is of type 12) or 13) in §6 and $dN_0 = 2^{m-1}$. Hence we have if L is odd then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = \frac{1 + \delta_0 1^{-\frac{m-2}{2}}}{1 + \delta 2^{-\frac{m-2}{2}}},$$

where $\delta_0 = \pm 1$ and $\delta = 0, \pm 1$, and if L is even then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = \frac{2^{m-2}(1 + \delta_0 2^{-\frac{m-2}{2}})(1 - 2^{-m})}{1 + 2^{-\frac{m}{2}}},$$

where $\delta_0 = \pm 1$.

Hence we have

$$X(N_0, N_1) < \frac{1}{2} \cdot 1.01 \cdot \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1}$$

for L odd and

$$X(N_0, N_1) < 1.01 \cdot 2^{m-3} \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1}$$

for L even.

b) If $\rho = 1$ then $(N_0)_2$ is of type 9), 11) and $dN_0 = 2^{m-2}$. Hence we have if L is odd and $(N_0)_2$ is of type 9) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{-\frac{m}{2}+2} / (1 + \delta 2^{-\frac{m-2}{2}}),$$

and if L is odd and $(N_0)_2$ is of type 11) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{\frac{m}{2}-1} (1 - 2^{-(m-2)}) (1 + \delta 2^{-\frac{m-2}{2}}),$$

if L is even and $(N_0)_2$ is of type 9) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{\frac{m}{2}} (1 - 2^{-m}) / (1 + 2^{-\frac{m}{2}}),$$

and if L is even and $(N_0)_2$ is of type 11) then

$$\frac{dN_0^{\frac{m}{2}}}{\alpha_2(N_0)} \cdot \alpha_2(L) = 2^{\frac{3}{2}m-3} (1 - 2^{-m}) (1 - 2^{-(m-2)}) / (1 + 2^{-\frac{m}{2}}),$$

Hence if L is odd and $(N_0)_2$ is of type 9) then

$$X(N_0, N_1) < 1.01 \cdot 2^{-\frac{m}{2}+1} \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1},$$

if L is odd and $(N_0)_2$ is of type 11) then

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{m}{2}-2} \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1},$$

if L is even and $(N_0)_2$ is of type 9) then

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{m}{2}-1} \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1},$$

if L is even and $(N_0)_2$ is of type 11) then

$$X(N_0, N_1) < 1.01 \cdot 2^{\frac{3}{2}m-4} \pi^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)^{-1}.$$

□

Now we assume $r \geq 2$. Then we have

$$(61) \quad X(N_0, N_1) = \frac{2|I(V'_0(N_0), V'_1(N_1))| dN_0^{\frac{m_0+1}{2}} dN_1^{\frac{r+1}{2}}}{\alpha_2(N_0)\alpha_2(N_1)} \cdot \alpha_2(L) \\ \cdot \prod_{p \neq 2} \frac{\alpha_p(L)}{\alpha_p(N_0)\alpha_p(N_1)} \cdot \pi^{\frac{1}{2}r(m-r)} \cdot \frac{\prod_{i=1}^{m_0} \Gamma(\frac{i}{2}) \prod_{i=1}^r \Gamma(\frac{i}{2})}{\prod_{i=1}^m \Gamma(\frac{i}{2})}.$$

Define $F(\rho) = \frac{2|I(V'_0(N_0), V'_1(N_1))| dN_0^{\frac{m_0+1}{2}} dN_1^{\frac{r+1}{2}}}{\alpha_2(N_0)\alpha_2(N_1)}$. Then other factors in (61) do not depend on ρ .

The following table is the list of possible pairs of N_0 and N_1 such that $(G_{N_0}, G_{N_1}) \in G(2, r, \rho)$ with $r \geq 2$, and the value $F(\rho)$. In this table $\delta_i \in \{0, 1, -1\}$ depends on the type of $N_i, i = 0, 1$ (see §6). Number of the pairs for fixed m, r and ρ is also given.

$m = \text{odd}, r = \text{odd} (\geq 3) \text{ and } \rho = \text{odd}$

i) N_0 is type 1) and N_1 is type 9) or 10),

$$F(\rho) = \frac{2^{-r-4+\frac{1}{2}\rho(m-\rho+1)}(1 + \delta_1 2^{-\frac{\rho-1}{2}})}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho}{2})}.$$

There are 8 possible pairs in this case.

ii) N_0 is type 1) and N_1 is type 11),

$$F(\rho) = \frac{2^{-4+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 32 possible pairs in this case.

$m = \text{odd}, r = \text{odd} (\geq 3) \text{ and } \rho = \text{even}$

iii) N_0 is type 2) or 3), and N_1 is type 12) or 13),

$$F(\rho) = \frac{2^{-m+r-3+\frac{1}{2}\rho(m-\rho-3)}(1+\delta_0 2^{-\frac{\rho}{2}})(1+\delta_0 2^{-\frac{m_0-\rho}{2}})(1+\delta_1 2^{-\frac{r-\rho-1}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 4 possible pairs in this case.

iv) N_0 is type 4) or 5), and N_1 is type 12) or 13),

$$F(\rho) = \frac{2^{\frac{1}{2}\rho(m-\rho-5)}(1+\delta_0 2^{-\frac{m_0-\rho-2}{2}})(1+\delta_1 2^{-\frac{r-\rho-1}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 8 possible pairs in this case.

v) N_0 is type 6) or 7) , and N_1 is type 14),

$$F(\rho) = \frac{2^{-m+r-4+\frac{1}{2}\rho(m-\rho+1)}(1+\delta_0 2^{-\frac{\rho-2}{2}})}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 32 possible pairs in this case.

vi) N_0 is type 8) and N_1 is type 14),

$$F(\rho) = \frac{2^{-2+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 128 possible pairs in this case.

$m = \text{odd}, r = \text{even} (\geq 2) \text{ and } \rho = \text{odd}.$

vii) N_0 is type 9) or 10), and N_1 is type 1),

$$F(\rho) = \frac{2^{-m+r-4+\frac{1}{2}\rho(m-\rho-1)}(1+\delta_0 2^{-\frac{\rho-1}{2}})}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 8 possible pairs in this case.

viii) N_0 is type 11) and N_1 is type 1),

$$F(\rho) = \frac{2^{-4+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 32 possible pairs in this case.

$m = \text{odd}, r = \text{even} (\geq 2) \text{ and } \rho = \text{even}$

ix) N_0 is type 12) or 13), and N_1 is type 2) or 3),

$$F(\rho) = \frac{2^{-r-3+\frac{1}{2}\rho(m-\rho-3)}(1+\delta_0 2^{-\frac{m_0-\rho-1}{2}})(1+\delta_1 2^{-\frac{\rho}{2}})(1+\delta_1 2^{-\frac{r-\rho}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho}{2})}.$$

There are 4 possible pairs in this case.

x) N_0 is type 12) or 13), and N_1 is type 4) or 5),

$$F(\rho) = \frac{2^{\frac{1}{2}\rho(m-\rho-5)}(1+\delta_0 2^{-\frac{m_0-\rho-1}{2}})(1+2^{-\frac{r-\rho-2}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 8 possible pairs in this case.

xi) N_0 is type 14) and N_1 is type 6) or 7),

$$F(\rho) = \frac{2^{-r-4+\frac{1}{2}\rho(m-\rho+1)}(1+\delta_1 2^{-\frac{\rho-2}{2}})}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho}{2})}.$$

There are 32 possible pairs in this case.

xii) N_0 is type 14) and N_1 is type 8),

$$F(\rho) = \frac{2^{-2+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 128 possible pairs in this case.

$m = \text{even}, r = \text{even} (\geq 2) \text{ and } \rho = \text{odd}$

xiii) N_0 and N_1 are type 1),

$$F(\rho) = \frac{2^{-4+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 16 possible pairs in this case.

$m = \text{even}, r = \text{even} (\geq 2) \text{ and } \rho = \text{even}$

xiv) N_0 and N_1 are type 2) or 3),

$$F(\rho) = \frac{2^{-m-2+\frac{1}{2}\rho(m-\rho-1)}(1+\delta_0 2^{-\frac{\rho}{2}})(1+\delta_0 2^{-\frac{m_0-\rho}{2}})(1+\delta_1 2^{-\frac{\rho}{2}})(1+\delta_1 2^{-\frac{r-\rho}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho}{2})}.$$

There are 4 possible pairs in this case.

xv) N_0 is type 2) or 3), and N_1 is type 4) or 5),

$$F(\rho) = \frac{2^{-(m-r)+1+\frac{1}{2}\rho(m-\rho-3)}(1+\delta_0 2^{-\frac{\rho}{2}})(1+\delta_0 2^{-\frac{m_0-\rho}{2}})(1+\delta_1 2^{-\frac{r-\rho-2}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 8 possible pairs in this case.

xvi) N_0 is type 4) or 5), and N_1 is type 2) or 3),

$$F(\rho) = \frac{2^{-r+1+\frac{1}{2}\rho(m-\rho-3)}(1+\delta_0 2^{-\frac{m_0-\rho-2}{2}})(1+\delta_1 2^{-\frac{\rho}{2}})(1+\delta_1 2^{-\frac{r-\rho}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 8 possible pairs in this case.

xvii) N_0 and N_1 are type 4) or 5),

$$F(\rho) = \frac{2^{4+\frac{1}{2}\rho(m-\rho-5)}(1+\delta_0 2^{-\frac{m_0-\rho-2}{2}})(1+\delta_1 2^{-\frac{r-\rho-2}{2}})}{P_2(\frac{\rho}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 16 possible pairs in this case.

xviii) N_0 and N_1 are type 6) or 7),

$$F(\rho) = \frac{2^{-m-4+\frac{1}{2}\rho(m-\rho+3)}(1+\delta_0 2^{-\frac{\rho-2}{2}})(1+\delta_1 2^{-\frac{\rho-2}{2}})}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho}{2})}.$$

There are 16 possible pairs in this case.

xix) N_0 is type 6) or 7), and N_1 is type 8),

$$F(\rho) = \frac{2^{-(m-r)-2+\frac{1}{2}\rho(m-\rho-1)}(1+\delta_0 2^{-\frac{\rho-2}{2}})}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 48 possible pairs in this case.

xx) N_0 is type 8), and N_1 is type 6) or 7),

$$F(\rho) = \frac{2^{-r-2+\frac{1}{2}\rho(m-\rho-1)}(1+\delta_1 2^{-\frac{\rho-2}{2}})}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho}{2})}.$$

There are 48 possible pairs in this case.

xxi) N_0 and N_1 are type 8),

$$F(\rho) = \frac{2^{\frac{1}{2}\rho(m-\rho-5)}}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 256 possible pairs in this case.

$m = \text{even}, r = \text{odd} (\geq 3) \text{ and } \rho = \text{odd}$

xxii) N_0 is type 9) or 10) and N_1 is type 9) or 10),

$$F(\rho) = \frac{2^{-m-4+\frac{1}{2}\rho(m-\rho+3)}(1+\delta_0 2^{-\frac{\rho-1}{2}})(1+\delta_1 2^{-\frac{\rho-1}{2}})}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho}{2})}.$$

There are 4 possible pairs in this case.

xxiii) N_0 is type 9) or 10), and N_1 is type 11)

$$F(\rho) = \frac{2^{-(m-r)-4+\frac{1}{2}\rho(m-\rho-1)}(1+\delta_0 2^{-\frac{\rho-1}{2}})}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 16 possible pairs in this case.

xxiv) N_0 is type 11) and N_1 is type 9) or 10),

$$F(\rho) = \frac{2^{-r-4+\frac{1}{2}\rho(m-\rho+1)}(1+\delta_1 2^{-\frac{\rho-1}{2}})}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho}{2})}.$$

There are 16 possible pairs in this case.

xxv) N_0 and N_1 are type 11)

$$F(\rho) = \frac{2^{-4+\frac{1}{2}\rho(m-\rho-1)}}{P_2(\frac{\rho-1}{2})P_2(\frac{m_0-\rho-2}{2})P_2(\frac{r-\rho-2}{2})}.$$

There are 64 possible pairs in this case.

$m = \text{even}, r = \text{odd} (\geq 3) \text{ and } \rho = \text{even}$

xxvi) N_0 and N_1 are type 12) or 13),

$$F(\rho) = 2^{-4+\frac{1}{2}\rho(m-\rho-5)}(1+\delta_0 2^{-\frac{m_0-\rho-1}{2}})(1+\delta_1 2^{-\frac{r-\rho-1}{2}}),$$

There are 4 possible pairs in this case.

xxvii) N_0 and N_1 are type 14),

$$F(\rho) = \frac{2^{-4+\frac{1}{2}\rho(m-\rho-3)}}{P_2(\frac{\rho-2}{2})P_2(\frac{m_0-\rho-1}{2})P_2(\frac{r-\rho-1}{2})}.$$

There are 64 possible pairs in this case.

Upper bound of $F(\rho)$ for fixed $m(\geq 30)$ and $r(\geq 2)$.

$m = \text{odd}$ and $r = \text{odd}$ (i) – vi) of the table).

In cases i), ii), iii), v) and vi), $F(\rho)$ increases as ρ increases. In case iv) $F(\rho) \leq F(\rho+2) \leq \dots \leq F(r-3) \leq 2^2 F(r-1)$. Therefore we have the following upper bound for each case.

- i) $2^{-\frac{r}{2}-3+\frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r-1}{2})$,
- ii) $2^{-m+\frac{3}{2}r-5+\frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r+1}{2})$,
- iii) $2^{-m+\frac{1}{2}r+1+\frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r+1}{2})$,
- iv) $2^{-\frac{m}{2}-\frac{3}{2}r+6+\frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r+1}{2})$,
- v) $2^{-\frac{3}{2}m+\frac{5}{2}r-3+\frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r+1}{2})$,
- vi) $2^{-\frac{m}{2}+\frac{r}{2}-2+\frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r+1}{2})$.

Since $r \leq \frac{1}{2}(m-1)$ we have

$$(62) \quad F(\rho) \leq 2^{\frac{1}{2}r(m-r-1)-3} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r+1}{2})$$

for any case i) - vi).

$m = \text{odd}$ and $r = \text{even}$ (vii) – xii) of the table)

In case vii) viii), x), xi) and xii), $F(\rho)$ increases as ρ increases. In case ix), $F(\rho) \leq F(r-2) \leq 2^3 F(r)$. Therefore we have the following upper bound for each case .

- vii) $2^{-\frac{3}{2}m+\frac{3}{2}r-3+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r+1}{2})$,
- viii) $2^{-\frac{m}{2}-4+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r-1}{2})$,
- ix) $2^{-\frac{5}{2}r+3+\frac{1}{2}r(m-r)} / P_2(\frac{r}{2}) P_2(\frac{m_0-r-1}{2})$,

$$\text{x)} \quad 2^{-m-\frac{r}{2}+5+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r+1}{2}),$$

$$\text{xi)} \quad 2^{-\frac{r}{2}-3+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r-1}{2}),$$

$$\text{xii)} \quad 2^{-m+\frac{3}{2}r-3+\frac{1}{2}r(m-r)} / P_2(\frac{r-4}{2}) P_2(\frac{m_0-r+1}{2}).$$

Since $r \leq \frac{1}{2}(m-1)$ we have the following (63) and (64) in any case vii) - xii).

$$(63) \quad F(\rho) \leq 2^{\frac{1}{2}r(m-r-1)-3} / P_2(\frac{r}{2}) P_2(\frac{m_0-r+1}{2}) \text{ for } r \geq 4.$$

and

$$(64) \quad \begin{aligned} F(\rho) &\leq 2^{\frac{1}{2}r(m-r-1)-1} / P_2(\frac{r}{2}) P_2(\frac{m_0-r+1}{2}) \\ &= 2^{m-1} / P_2(1) P_2(\frac{m_0-1}{2}) \text{ for } r = 2. \end{aligned}$$

$m = \text{even and } r = \text{even (xiii) - xxi in the table}$

We have the following upper bound of $F(\rho)$ in each case.

$$\text{xiii)} \quad 2^{-\frac{m}{2}+\frac{r}{2}-4+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xiv)} \quad 2^{-m-\frac{r}{2}+9+\frac{1}{2}r(m-r)} / P_2(\frac{r}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xv)} \quad 2^{-2m+\frac{3}{2}r+7+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r+2}{2}),$$

$$\text{xvi)} \quad 2^{-\frac{5}{2}r+8+\frac{1}{2}r(m-r)} / P_2(\frac{r}{2}) P_2(\frac{m_0-r-2}{2}) \text{ for } r \leq \frac{m-2}{2},$$

$$2^{-\frac{5}{2}r+8+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(1) \text{ for } r = \frac{m}{2},$$

$$\text{xvii)} \quad 2^{-m-\frac{r}{2}+10+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xviii)} \quad 2^{-m+\frac{3}{2}r-1+\frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xiv)} \quad 2^{-2m+\frac{5}{2}r-2+\frac{1}{2}r(m-r)} / P_2(\frac{r-4}{2}) P_2(\frac{m_0-r+2}{2}),$$

$$\text{xx)} \quad 2^{-\frac{3}{2}r + \frac{1}{2}r(m-r)} / P_2(\frac{r-2}{2}) P_2(\frac{m_0-r-2}{2}) \text{ for } r \leq \frac{m-2}{2},$$

$$2^{-\frac{3}{2}r - 2 + \frac{1}{2}r(m-r)} / P_2(\frac{r-4}{2}) P_2(1) \text{ for } r = \frac{m}{2},$$

$$\text{xxi)} \quad 2^{-m - \frac{r}{2} + 4 + \frac{1}{2}r(m-r)} / P_2(\frac{r-4}{2}) P_2(\frac{m_0-r}{2}).$$

Since $r \leq \frac{m}{2}$ we have the following (65) and (66) in any case xiii) - xxi).

$$(65) \quad F(\rho) \leq 2^{\frac{1}{2}r(m-r-1)} / P_2(\frac{r}{2}) P_2(\frac{m_0-r+2}{2}) \text{ for } r \geq 4$$

and

$$(66) \quad \begin{aligned} F(\rho) &\leq 2^{4 + \frac{1}{2}r(m-r-1)} / P_2(\frac{r}{2}) P_2(\frac{m_0-r+2}{2}) \\ &= 2^{m+1} / P_2(1) P_2(\frac{m_0}{2}) \text{ for } r = 2. \end{aligned}$$

$m = \text{even and } r = \text{odd (xxii) - xxvii) in the table}$

We have the following upper bound of $F(\rho)$ in each case

$$\text{xxii)} \quad 2^{-m + \frac{3}{2}r - 1 + \frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xxiii)} \quad 2^{-2m + \frac{7}{2}r - 6 + \frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r+2}{2}),$$

$$\text{xxiv)} \quad 2^{-\frac{r}{2} - 3 + \frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r-2}{2}) \text{ for } r \leq \frac{m-2}{2},$$

$$2^{-\frac{r}{2} - 6 + \frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(1) \text{ for } r = \frac{m}{2},$$

$$\text{xxv)} \quad 2^{-m + \frac{3}{2}r - 5 + \frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r}{2}),$$

$$\text{xxvi)} \quad 2^{-\frac{m}{2} - \frac{3}{2} + 2 + \frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r}{2}) \text{ for } r \leq \frac{m-2}{2},$$

$$2^{-\frac{m}{2} - \frac{3}{2}r + 1 + \frac{1}{2}r(m-r)} / P_2(\frac{r-1}{2}) \text{ for } r = \frac{m}{2},$$

$$\text{xxvii)} \quad 2^{-\frac{m}{2}-\frac{3}{2}r-2+\frac{1}{2}r(m-r)} / P_2(\frac{r-3}{2}) P_2(\frac{m_0-r}{2}).$$

Since $r \leq \frac{m}{2}$ we have the following (67) in any case xxii) - xxvii).

$$(67) \quad F(\rho) \leq 2^{-1+\frac{1}{2}r(m-r-1)} / P_2(\frac{r-1}{2}) P_2(\frac{m_0-r+2}{2})$$

Upperbound of $X(N_0, N_1)$ for fixed $m(\geq 30)$ and $r(\geq 2)$

$m = \text{odd and } r = \text{odd}$

In this case L is an odd lattice (see §6, type 9) and 10) for $\alpha_2(L)$).

By (41) and (42) we have

$$\begin{aligned} & \prod_{p \neq 2} \alpha_p(L) \alpha_p(N_0)^{-1} \alpha_p(N_1)^{-1} \\ &= \prod_{p \neq 2} P_p(\frac{m-1}{2}) P_p(\frac{m_0-2}{2})^{-1} P_p(\frac{r-1}{2})^{-1} (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} \\ &< \prod_{p \neq 2} (1 - p^{-\frac{m_0}{2}})^{-1} P_p(\frac{r-1}{2})^{-1}, \end{aligned}$$

where $\delta_0 = (\frac{(-1)^{\frac{m_0}{2}} dN_0}{p})$.

Therefore by (39), (61) and (62) we have

$$X(N_0, N_1) < \prod_p (1 - p^{-\frac{m_0}{2}})^{-1} P_p(\frac{r-1}{2})^{-1} \cdot Y(m, r, 2)$$

Since $r \geq 3$ and $m_0 \geq \frac{1}{2}(m+1) \geq 16$, we have (by (50) and (51))

$$X(N_0, N_1) < 2 \cdot Y(m, r, 2).$$

$m = \text{odd and } r = \text{even}$

In this case L is an odd lattice. For $\alpha_2(L)$ see §6, type 9) and 10).

By (41) and (42) we have

$$\begin{aligned} & \prod_{p \neq 2} \alpha_p(L) \alpha_p(N_0)^{-1} \alpha_p(N_1)^{-1} \\ &= \prod_{p \neq 2} P_p\left(\frac{m-1}{2}\right) P_p\left(\frac{m_0-1}{2}\right)^{-1} P_p\left(\frac{r-2}{2}\right)^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1} \\ &< \prod_{p \neq 2} P_p\left(\frac{r-2}{2}\right)^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1}, \end{aligned}$$

where $\delta_1 = \left(\frac{(-1)^{\frac{r}{2}} dN_1}{2}\right)$.

Therefore if $r \geq 4$ we have (by (61) and (63))

$$\begin{aligned} X(N_0, N_1) &= P_2\left(\frac{r}{2}\right)^{-1} (1 + \delta 2^{-\frac{m-1}{2}})^{-1} \prod_{p \neq 2} P_p\left(\frac{r-2}{2}\right)^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1} Y(m, r, 2) \\ &< \frac{3}{4} (1 - 2^{-\frac{m-1}{2}})^{-1} \zeta(2) \prod_p P_p\left(\frac{r}{2}\right)^{-1} \cdot Y(m, r, 2) \\ &< 2.40 Y(m, r, 2), \\ &\text{(by (50) and (51))} \end{aligned}$$

where $\delta = \pm 1$.

If $r = 2$ then we have by (61) and (64)

$$\begin{aligned} X(N_0, N_1) &= P_2(1)^{-1} (1 + \delta 2^{-\frac{m-1}{2}}) \prod_{p \neq 2} \left(1 - \left(\frac{-dN_1}{p}\right) p^{-1}\right)^{-1} \cdot 2^2 Y(m, 2, 2) \\ &< 1.49 \cdot 2^2 Y(m, 2, 2). \\ &\text{(by (52))} \end{aligned}$$

$m = \text{even and } r = \text{even}$

by (41) and (42) we have

$$\begin{aligned}
 & \prod_{p \neq 2} \alpha_p(L) \alpha_p(N_0)^{-1} \alpha_p(N_1)^{-1} \\
 &= \prod_{p \neq 2} P_p\left(\frac{m-2}{2}\right) P_p\left(\frac{m_0-2}{2}\right)^{-1} P_p\left(\frac{r-2}{2}\right)^{-1} (1 - \delta p^{-\frac{m}{2}}) \\
 & \quad \cdot (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1} \\
 &< \prod_{p \neq 2} P_p\left(\frac{r-2}{2}\right)^{-1} (1 - \delta p^{-\frac{m}{2}}) (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1},
 \end{aligned}$$

where $\delta = \left(\frac{(-1)^{\frac{m}{2}}}{p}\right)$, $\delta_0 = \left(\frac{(-1)^{\frac{m_0}{2}} dN_0}{p}\right)$, and $\delta_1 = \left(\frac{(-1)^{\frac{r}{2}} dN_1}{p}\right)$.

Therefore if L is odd and $r \geq 4$ we have (by (39), (61), (65), §6 type 6) and 7))

$$\begin{aligned}
 X(N_0, N_1) &< (1 - 2^{-\frac{m-2}{2}}) \prod_p P_p\left(\frac{r}{2}\right)^{-1} \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}) \\
 & \quad (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1} \cdot 2^3 Y(m, r, 2) \\
 &< 2.43 \cdot 2^3 Y(m, r, 2). \\
 & \text{(by (50), (51) and (54))}
 \end{aligned}$$

If L is odd and $r = 2$ then we have (by (39), (61), (66) and §6 type 6) and 7))

$$\begin{aligned}
 X(N_0, N_1) &< (1 - 2^{-\frac{m-2}{2}})^{-1} (1 - 2^{-2})^{-1} \\
 & \quad \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}) (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} (1 - \delta_1 p^{-1})^{-1} \cdot 2^7 Y(m, 2, 2) \\
 &< 1.50 \cdot 2^7 Y(m, 2, 2). \\
 & \text{(by (50), (52))}
 \end{aligned}$$

If L is even and $r \geq 4$ then we have (by (39), (61), (65) and §6 type 2))

$$\begin{aligned} X(N_0, N_1) &< \prod_p P_p\left(\frac{r}{2}\right)^{-1} \prod_{p \neq 2} (1 - \delta_0 p^{-\frac{m_0}{2}})^{-1} (1 - \delta_1 p^{-\frac{r}{2}})^{-1} \\ &\quad \cdot 2^{m+1} Y(m, r, 2) \\ &< 2.43 \cdot 2^{m+1} Y(m, r, 2). \end{aligned}$$

(by (50), (51))

If L is even and $r = 2$ then we have (by (39), (61), (66) and §6 type 2))

$$\begin{aligned} X(N_0, N_1) &< (1 - 2^{-2})^{-1} (1 + 2^{-\frac{m}{2}})^{-1} \zeta\left(\frac{m_0}{2}\right) \prod_{p \neq 2} (1 - \delta_1 p^{-1})^{-1} \\ &\quad \cdot 2^{m+5} Y(m, 2, 2) \\ &< 1.50 \cdot 2^{m+5} Y(m, 2, 2). \end{aligned}$$

(by (50) and (52))

$m = \text{even and } r = \text{odd}$

By (41) and (42) we have

$$\begin{aligned} &\prod_{p \neq 2} \alpha_p(L) \alpha_p(N_0)^{-1} \alpha_p(N_1)^{-1} \\ &= \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}) P_p\left(\frac{m-2}{2}\right) P_p\left(\frac{m_0-1}{2}\right)^{-1} P_p\left(\frac{r-1}{2}\right)^{-1} \\ &< \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}) P_p\left(\frac{r-1}{2}\right)^{-1}, \end{aligned}$$

where $\delta = \left(\frac{(-1)^{\frac{m}{2}}}{p}\right)$.

Therefore if L is odd we have (by (39), (61), (67) and §6 type 6) and 7))

$$\begin{aligned} X(N_0, N_1) &< (1 - 2^{-\frac{m-2}{2}})^{-1} \prod_{p \neq 2} (1 - \delta p^{-\frac{m}{2}}) \prod_p P_p\left(\frac{r-1}{2}\right)^{-1} \\ &\quad \cdot 2^2 \cdot Y(m, r, 2) \\ &< 1.95 \cdot 2^2 \cdot Y(m, r, 2). \end{aligned}$$

(by (50), (51) and (54))

If L is even then we have (by (39), (61), (67) and §6 Type 2))

$$\begin{aligned} X(N_0, N_1) &< \prod_p P_p\left(\frac{r-1}{2}\right)^{-1} 2^m Y(m, r, 2) \\ &< 1.95 \cdot 2^m Y(m, r, 2). \end{aligned}$$

(by (51))

Next proposition is the summary of above computations and Proposition 7.1.

Proposition 7.2. i) If L is odd then

$$X(N_0, N_1) < 2.43 \cdot 2^3 Y(m, r, 2) \quad \text{for } r \geq 3,$$

$$X(N_0, N_1) < 1.50 \cdot 2^7 Y(m, 2, 2) \quad \text{for } r = 2,$$

$$X(N_0, N_1) < 1.01 \cdot 2^{-1} Y(m, 1, 2) \quad \text{for } r = 1.$$

ii) If L is even then

$$X(N_0, N_1) < 2.43 \cdot 2^{m+1} Y(m, r, 2) \quad \text{for } r \geq 3,$$

$$X(N_0, N_1) < 1.50 \cdot 2^{m+5} Y(m, 2, 2) \quad \text{for } r = 2,$$

$$X(N_0, N_1) < 1.01 \cdot 2^{m-3} Y(m, 1, 2) \quad \text{for } r = 1.$$

Next we examine the function $Y(m, r, 2)$.

Proposition 7.3. Assume $m \geq 38$. Then $Y(m, r, 2) \leq Y(m, 3, 2)$ for any r such that $3 \leq r \leq \frac{m}{2}$.

Proof. Consider the natural logarithm $\ln(Y(m, r, 2)/Y(m, r-1, 2))$ for any r with $3 \leq r-1 < r \leq \frac{m}{2}$. By Stirling's formula (48) we have $\ln(Y(m, r, 2))/Y(m, r-1, 2) < f_m(r)$, where $f_m(r) = \frac{1}{2}(m-2r)\ln 2 + \frac{1}{2}(m-2r+1)\ln \pi - \frac{r}{2} + \frac{1}{2}(r-1)\ln \frac{r}{2} + \frac{1}{2}(m-r+1) - \frac{1}{2}(m-r)\ln \frac{1}{2}(m-r+1) + \frac{1}{8}$. Since $r \leq \frac{m}{2}$, $f_m''(r) = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{m-r+1}\right) + \frac{1}{2}\left(\frac{1}{r^2} - \frac{1}{(m-r+1)^2}\right) > 0$ for any such r . Computations shows $f_m'(6) > 0$. Therefore $f_m(r)$ is increasing on $[6, \frac{m}{2}]$. On the other hand we have $f_m(\frac{m}{2}) = \frac{1}{2}\ln \pi + \frac{1}{2} + \frac{m}{4}(\ln m - \ln(m+2)) - \frac{1}{2}\ln \frac{m}{4} + \frac{1}{8} < 0$ for any $m \geq 38$.

Therefore we have $f_m(r) < 0$ for any r such that $6 \leq r \leq \frac{m}{2}$ and for any $m \geq 38$.

Next we consider $f_m(5)$ and $f_m(4)$ as a function of m and evaluate them. We have

$$\begin{aligned} f_m(5) &= \frac{1}{2}(m-10)\ln 2 + \frac{1}{2}(m-9)\ln \pi + 2\ln \frac{5}{2} + \frac{1}{2}(m-4) \\ &\quad - \frac{1}{2}(m-5)\ln \frac{1}{2}(m-4) - 5 + \frac{1}{8}, \\ \frac{df_m(5)}{dm} &= \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln \frac{1}{2}(m-4) + \frac{1}{2(m-4)}, \\ f_m(4) &= \frac{1}{2}(m-8)\ln 2 + \frac{1}{2}(m-7)\ln \pi + \frac{3}{2}\ln 2 + \frac{1}{2}(m-3) \\ &\quad - \frac{1}{2}(m-4)\ln \frac{1}{2}(m-3) - 4 + \frac{1}{8}, \\ \frac{df_m(4)}{dm} &= \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln \frac{1}{2}(m-3) + \frac{1}{2(m-3)}. \end{aligned}$$

Computations show $\frac{df_m(5)}{dm} < 0$, $f_{38}(5) < 0$, $\frac{df_m(4)}{dm} < 0$ and $f_{38}(4) < 0$. Therefore we have $f_m(5) < 0$ and $f_m(4)$ for all $m \geq 38$. This completes the proof. $\square$

Propositon 7.4.

- (i) $Y(m, 3, 2) < Y(m, 1, 2)m^{-3.2}$ for $m \geq 43$
- (ii) $Y(m, 2, 2) < Y(m, 1, 2)m^{-1.57}$ for $m \geq 43$.

Proof.

- (i) $Y(m, 3, 2)Y(m, 1, 2)^{-1} = 2^{2m-9}\pi^{m-4}/(m-3)!$
 $< m^{-3.2}$ for $m \geq 43$.
- (ii) $Y(m, 2, 2)Y(m, 1, 2)^{-1} = 2^{\frac{1}{2}(m-4)}\pi^{\frac{1}{2}(m-3)}/\Gamma(\frac{m-1}{2})$
 $< m^{-1.57}$ for $m \geq 43$.

$\square$

We are now ready to show the following Lemma 7.5.

Lemma 7.5. i) Let L be odd unimodular then

$$\omega_{R(2)}/\omega(L) < 64.81Y(m, 1, 2) \quad \text{for } m \geq 43.$$

ii) Let L be even unimodular then

$$\omega_{R(2)}/\omega(L) < 3.95 \cdot 2^m Y(m, 1, 2) \quad \text{for } m \geq 144,$$

where $Y(m, 1, 2) = (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2})$.

Proof. i) Since $(N_0)_q$ and $(N_1)_q$ are determined uniquely by the discriminants (92:1 in [8]) the genera G_{N_0} and G_{N_1} are determined by $(N_0)_2$ and $(N_1)_2$. From the table in §6 we can show that if $r = 1$ then there are at most 11 pairs of genera (adding the numbers for $\rho = 0$ and $\rho = 1$ together), if $r = 2$ then there are at most 74 pairs of genera (adding the numbers for $\rho = 0, 1$ and 2 together), and if $r \geq 3$ then there are at most 26^2 pairs of genera for fixed ρ, r and m . Hence we have

$$\begin{aligned} \omega_{R(2)}/\omega(L) &< 11 \cdot 1.01 \cdot 2^{-1} Y(m, 1, 2) + 74 \cdot 1.50 \cdot 2^7 (m, 2, 2) \\ &\quad + 26^2 \cdot 2.43 \cdot 2^3 Y(m, 3, 2) \prod_{r=1}^{\lfloor \frac{m}{2} \rfloor} (r+1) \\ &< (5.56 + 74 \cdot 1.50 \cdot 2^7 m^{-1.57} \\ &\quad + 26^2 \cdot 2.43 \cdot 2^3 (\frac{m^2}{8} + \frac{3}{4}m) m^{-3.2}) Y(m, 1, 2) \\ &< 64.81 Y(m, 1, 2) \quad \text{for } m \geq 43. \end{aligned}$$

ii) Similarly we have

$$\begin{aligned} \omega_{R(2)}/\omega(L) &< 11 \cdot 1.01 \cdot 2^{m-3} Y(m, 1, 2) \\ &\quad + 74 \cdot 1.50 \cdot 2^{m+5} Y(m, 2, 2) + 26^2 \cdot 2.43 \cdot 2^{m+1} Y(m, 3, 2) (\frac{m^2}{8} + \frac{3m}{4}) \\ &< 2^m (1.39 + 1.46 + 1.10) = 3.95 \cdot 2^m Y(m, 1, 2) \quad \text{for } m \geq 144. \quad \square \end{aligned}$$

Remark. If $m \leq 41$ then we can show that our upper bound for $\omega_{R(2)}/\omega(L)$ is greater than 1. If $m = 42$ then $Y(m, 1, 2)m^{-2.97} < Y(m, 3, 2) < Y(m, 1, 2)m^{-2.98}$. With this estimation the similar argument as in Lemma 7.5 cannot give good result. Careful computations might give us a good upper bound but we would rather avoid complicated computations and use restriction that $m \geq 43$.

§8. Estimation of $\omega_{R(q)}/\omega(L)$, $q \neq 2$.

In this section we give an upper bound of $\omega_{R(q)}/\omega(L)$, $q \neq 2$. Let N_0 and N_1 be the lattices such that the pair of genera (G_{N_0}, G_{N_1}) is in $G(q, r, \rho)$, where $1 \leq r \leq [\frac{m-1}{q-1}]$ and $0 \leq \rho \leq \min(r, m_0)$ with $m_0 = m - r(q-1)$. First we evaluate $|I(V'_0(N_0), V'_1(N_1))|\omega(N_0)\omega(N_1)/\omega(L) = X(N_0, N_1)$.

Proposition 8.1.

- (i) $\prod_{p|q} \beta_p(N_1) = 2^{\frac{1}{2}(q-3)} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)}$ for $r = 1$.
- (ii) $\prod_{p|q} \beta_p(N_1) \leq 2^{\frac{1}{2}(q-3)} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} \prod_{p \neq q} \prod_{i=2}^r (1 - \frac{1}{p^i})^{-\frac{q-1}{2}}$ for $r \geq 2$.

Proof. Let M be a unimodular hermitian S -lattice of rank 1. Then $w(M) = \frac{1}{2q}$.

Therefore by Mass formula (2) and Proposition 5.2 we have

$$\begin{aligned} \frac{1}{2q} &= 2 \cdot q^{\frac{1}{2}(q^{\frac{q-1}{2}}-1)} (2\pi)^{-\frac{q-1}{2}} \prod_p \beta_p(M)^{-1} \\ &= (2\pi)^{-\frac{q-1}{2}} q^{\frac{q-1}{4}} \prod_{\substack{p \neq q \\ p|p \\ p \text{ remains} \\ \text{prime in } E}} (1 + p^{-fp})^{-tp} \prod_{\substack{p \neq q \\ p|p \\ p \text{ splits} \\ \text{in } E}} (1 - p^{-fp})^{-tp}, \end{aligned}$$

where $t_p f_p = \frac{1}{2}(q-1)$. Hence we have

$$\prod_{p \nmid q} \beta_p(\mathcal{M}) = \prod_{\substack{p \nmid q \\ p|p \\ p \text{ remains} \\ \text{prime in } E}} (1 + p^{-f_p})^{t_p} \prod_{\substack{p \nmid q \\ p|p \\ p \text{ splits} \\ \text{in } E}} (1 - p^{-t_p})^{-t_p} = 2^{\frac{q-3}{2}} \pi^{\frac{q-1}{2}} q^{-\frac{q+3}{4}}.$$

Since $(\mathcal{M}_1)_p$ is unimodular at $p \nmid q$ we have $\beta_p(\mathcal{M}_1) = \beta_p(\mathcal{M})$ for $r = 1$ at the prime ideal $p \nmid q$. This gives the proof of i). As for ii) by Proposition 5.2 we have

$$\begin{aligned} \prod_{p \nmid q} \beta_p(\mathcal{M}_1) &= \prod_{\substack{p \nmid q \\ p|p \\ p \text{ remains} \\ \text{prime in } E}} \prod_{i=1}^r (1 - (-1)^i p^{-f_p i})^{-t_p} \cdot \prod_{\substack{p \nmid q \\ p|p \\ p \text{ splits} \\ \text{in } E}} \prod_{i=1}^r (1 - p^{-f_p i})^{-t_p} \\ &< 2^{\frac{1}{2}(q-3)} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} \prod_{p \nmid q} \prod_{i=2}^r (1 - p^{-i})^{-\frac{q-1}{2}} \quad \text{for } r \geq 2. \end{aligned}$$

□

If $m_0 \geq 2$ then by mass formulas (1) and (2) we have

$$\begin{aligned} X(N_0, \mathcal{M}_1) &= N_{K/\mathbb{Q}}(D(E/K))^{\frac{r(r+1)}{4}} D(K/\mathbb{Q})^{\frac{r^2}{2}} \\ &\quad \cdot \frac{|I(V'_0(N_0), V'_1(\mathcal{M}_1))| (dN_0)^{\frac{m_0+1}{2}} N_{E/\mathbb{Q}}(\delta \mathcal{M}_1)^r}{\alpha_q(N_1) \beta_q(\mathcal{M}_1)} \cdot \alpha_q(L) \\ &\quad \pi^{\frac{1}{4}r(q-1)(2m-rq)} \cdot 2^{-\frac{1}{4}r(r+1)(q-1)} \\ &\quad \prod_{p \nmid q} \frac{\alpha_p(L)}{\alpha_p(N_0) \beta_p(\mathcal{M}_1)} \cdot \frac{(\prod_{j=1}^r (j-1)!)^{\frac{q-1}{2}} \prod_{i=1}^{m_0} \Gamma(\frac{i}{2})}{\prod_{i=1}^m \Gamma(\frac{i}{2})}, \end{aligned}$$

where $\beta_p(\mathcal{N}) = \prod_{p|p} \beta_p(\mathcal{M}_1)$ for any p .

$$\text{Let } F_q(\rho) = \frac{|I(V'_0(N_0), V'_1(\mathcal{M}_1))| (dN_0)^{\frac{m_0+1}{2}} N_r(\delta \mathcal{M}_1)^r}{\alpha_q(N_0) \beta_q(\mathcal{M}_1)}.$$

Upper bound of $F_q(\rho)$

$m = \text{odd and } r = \text{odd}$

Since m and r are odd, m_0 and ρ are also odd. By (43), Proposition 5.9 and (59) we have

$$F_q(\rho) = \frac{1}{2}(1 + \delta_0 q^{-\frac{m_0-\rho}{2}}) q^{\frac{1}{2}(m_0\rho+r\rho-\rho^2-r)} P_q\left(\frac{m_0-\rho}{2}\right)^{-1} P_q\left(\frac{\rho-1}{2}\right)^{-1} P_q\left(\frac{r-\rho}{2}\right)^{-1},$$

where $\delta_0 = \pm 1$ (if $\rho = m_0$ then $\delta_0 = 1$).

Let $1 \leq \rho < \rho + 2 \leq \min(m_0, r)$, then we can easily check $F_q(\rho) \leq F_q(\rho + 2)$.

Therefore we have

$$(68) \quad F_q(\rho) \leq \frac{1}{2}(1 + \delta'_0 q^{-\frac{m_0-r}{2}}) q^{\frac{1}{2}r(m_0-1)} P_q\left(\frac{m_0-r}{2}\right)^{-1} P_q\left(\frac{r-1}{2}\right)^{-1} \text{ for } r \leq m_0$$

and

$$(69) \quad F_q(\rho) \leq q^{\frac{1}{2}(m_0-1)} P_q\left(\frac{m_0-1}{2}\right)^{-1} P_q\left(\frac{r-m_0}{2}\right)^{-1} \text{ for } r \geq m_0,$$

where $\delta'_0 = \pm 1$ (if $r = m_0$ then $\delta'_0 = 1$).

$m = \text{odd and } r = \text{even}$

Since m is odd and r is even, m_0 is odd and ρ is even. Proposition 5.9, (44) and (60) give

$$F_q(\rho) \leq \frac{1}{2}(1 + \delta q^{-\frac{\rho}{2}}) q^{\frac{1}{2}(m_0\rho+r\rho-\rho^2-r)} P_q\left(\frac{m_0-\rho-1}{2}\right)^{-1} P_q\left(\frac{r-\rho}{2}\right)^{-1} P_q\left(\frac{\rho}{2}\right)^{-1},$$

where $\delta = \pm 1$ (if $\rho = 0$ then $\delta = 1$).

It is easy to check $F_q(\rho) \leq F_q(\rho + 2)$ for ρ such that $0 \leq \rho < \rho + 2 \leq \min(r, m_0)$.

Hence we have

$$(70) \quad F_q(\rho) \leq F_q(m_0 - 1) = \frac{1}{2}(1 + \delta_0 q^{-\frac{m_0-1}{2}}) q^{\frac{1}{2}(rm_0+m_0-2r-1)}$$

$$\cdot P_q\left(\frac{r-m_0+1}{2}\right)^{-1} P_q\left(\frac{m_0-1}{2}\right)^{-1} \quad \text{for } r > m_0$$

and

$$(71) \quad F_q(\rho) \leq F(r) = \frac{1}{2}(1 + \delta_1 q^{-\frac{r}{2}}) q^{\frac{1}{2}r(m_0-1)} \\ \cdot P_q\left(\frac{m_0-r-1}{2}\right)^{-1} P_q\left(\frac{r}{2}\right)^{-1} \quad \text{for } r < m_0,$$

where $\delta_0 = \pm 1$ and $\delta_1 = \pm 1$ (if $r = 0$ then $\delta_1 = 1$).

$m = \text{even and } r = \text{odd}$

Since m is even and r is odd, m_0 is even and ρ is odd. We have

$$F_q(\rho) = \frac{1}{2} q^{\frac{1}{2}(m_0\rho+r\rho-\rho^2-r)} P_q\left(\frac{\rho-1}{2}\right)^{-1} P_q\left(\frac{m_0-\rho-1}{2}\right)^{-1} P_q\left(\frac{r-\rho}{2}\right)^{-1}$$

Then we have $F_q(\rho) \leq F_q(\rho+2)$ for any ρ such that $1 \leq \rho < \rho+2 \leq \min(m_0, r)$.

Therefore we have

$$(72) \quad F_q(\rho) \leq q^{\frac{1}{2}r(m_0-1)} P_q\left(\frac{r-1}{2}\right)^{-1} P_q\left(\frac{m_0-r-1}{2}\right)^{-1} \quad \text{for } r < m_0$$

and

$$(73) \quad F_q(\rho) \leq \frac{1}{2} q^{\frac{1}{2}(m_0r+m_0-2-1)} P_q\left(\frac{m_0-2}{2}\right)^{-1} P_q\left(\frac{r-m_0+1}{2}\right)^{-1} \quad \text{for } r > m_0.$$

$m = \text{even and } r = \text{even}$

Since m and r are even, m_0 and ρ are also even. Then we have

$$F_q(\rho) = \frac{1}{2} q^{\frac{1}{2}(m_0\rho+r\rho-\rho^2-r)} (1 + \delta_0 q^{-\frac{m_0-\rho}{2}}) (1 + \delta q^{-\frac{\rho}{2}}) \\ \cdot P_q\left(\frac{\rho}{2}\right)^{-1} P_q\left(\frac{m_0-\rho}{2}\right)^{-1} P_q\left(\frac{r-\rho}{2}\right)^{-1},$$

where $\delta_0 = \pm 1$ and $\delta = \pm 1$ (if $\rho = m_0$ then $\delta_0 = 1$ and if $\rho = 0$ then $\delta = 1$).

We can easily show that $F(\rho) \leq F(\rho + 2)$ for any ρ such that $0 \leq \rho < \rho + 2 \leq \min(r, m_0)$.

Hence we have

$$(74) \quad F_q(\rho) \leq (1 + \delta'_0 q^{-\frac{m_0}{2}}) q^{\frac{1}{2}r(m_0-1)} P_q\left(\frac{m_0}{2}\right)^{-1} P_q\left(\frac{r-m_0}{2}\right)^{-1}$$

for $r \geq m_0$ and

$$(75) \quad F_q(\rho) \leq \frac{1}{2} (1 + \delta''_0 q^{-\frac{m_0-r}{2}}) (1 + \delta_1 q^{-\frac{r}{2}}) P_q\left(\frac{r}{2}\right)^{-1} P_q\left(\frac{m_0-r}{2}\right)^{-1}$$

for $r \leq m_0$, where $\delta'_0 = \pm 1$, $\delta''_0 = \pm 1$ (if $r = m_0$ then $\delta''_0 = 1$) and $\delta_1 = \pm 1$.

Upper bound of $X(N_0, N_1)$ for $m \geq 43$.

Using the functional equation (47) we have

$$\begin{aligned} X(N_0, N_1) &= 2^{1+\frac{1}{4}r(q-1)(2m-rq-3)} \pi^{\frac{1}{4}r(q-1)(2m-rq-1)} \\ &\cdot q^{\frac{1}{4}r(rq-2r+1)} \cdot F_q(\rho) \alpha_q(L) \cdot \prod_{p \neq q} \frac{\alpha_p(L)}{\alpha_p(N_0) \beta_p(N_1)} \\ &\cdot \left(\prod_{j=1}^r (j-1)! \right)^{\frac{q-1}{2}} \cdot (m_0!(m_0+2)! \cdots (m-2)!)^{-1} \quad \text{for } m_0 = m - r(q-1) \geq 2. \end{aligned}$$

$m = \text{odd}$ and $r = \text{odd}$

If $m_0 = 1$ then $\rho = 1$, $\omega(N_0) = \frac{1}{2}$ and $|I(V'_0(N_0), V'_1(N_1))| = 2$. Therefore $X(N_0, N_1) = \omega(N_1)/\omega(L)$. Since m is odd, L is odd lattice. Therefore by §6 type 9) and 10) (for $\alpha_2(L)$), (41) (for $\alpha_p(L)$, $p \neq 2$), Proposition 5.7 and Proposition 8.1 we have the following.

$$X(N_0, \mathcal{N}_1) < 2.01 \cdot 2^{\frac{q-3}{2}} \pi^{\frac{q-1}{2}} q^{-\frac{q+3}{4}} Y(m, 1, q) \text{ for } r = 1 \text{ (therefore } q = m),$$

and

$$X(N_0, \mathcal{N}_1) < 2.01 \cdot 2^{-1} \cdot 3^{-\frac{q-1}{2}} e^{\frac{q-1}{4}} \pi^{\frac{3}{2}(q-1)} q^{-\frac{q+3}{4}} \cdot Y(m, r, q) \text{ for } r \geq 3$$

$$\text{with } m = r(q - 1) + 1.$$

If $m_0 \geq 3$ then using (68) and (69) we get

$$X(N_0, \mathcal{N}_1) < 1.51 \cdot 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 1, q) \text{ for } r = 1$$

and

$$X(N_0, \mathcal{N}_1) < 1.51 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \text{ for } r \geq 3.$$

$m = \text{odd and } r = \text{even}$

Since m is odd, m_0 is odd. If $m_0 = 1$ then $\rho = 0$, $|I(V'_0(N_0), V'_1(\mathcal{N}_1))| = 1$ and $\omega(N_0) = \frac{1}{2}$. Therefore we have $X(N_0, \mathcal{N}_1) = \frac{1}{2}\omega(\mathcal{N}_1)/\omega(L)$. Using the mass formulas (1) and (2) we have

$$X(N_0, \mathcal{N}_1) < 1.01 \cdot 3^{-\frac{1}{2}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, q) \text{ for } r = 2$$

(therefore $m = 2q - 1$),

and

$$X(N_0, \mathcal{N}_1) < 1.01 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \text{ for } r \geq 4.$$

If $2 \leq r < m_0$, then using (71) we have

$$X(N_0, \mathcal{N}_1) < 1.51 \cdot 2^{-1} \cdot 3^{-\frac{1}{2}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, q) \text{ for } r = 2,$$

and

$$X(N_0, \mathcal{N}_1) < 1.26 \cdot 2^{-1} \cdot 3^{-\frac{1}{2}(q-1)} \cdot \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \text{ for } r \geq 4.$$

If $3 \leq m_0 < r$, then using (70) we have

$$X(N_0, N_1) < 1.01 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q).$$

$m = \text{even and } r = \text{odd}$

Since m is even, m_0 is even. If $1 \leq r < m_0$, then using (72) we have the following: For odd lattice L

$$X(N_0, N_1) < 0.37 \cdot 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q)$$

for $r = 1$ and $m_0 = 2$.

$$X(N_0, N_1) < 1.51 \cdot 3^{-1} \cdot 2^{-2+\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q+1)} q^{-\frac{1}{4}(q+3)}$$

for $r = 1$ and $m_0 \geq 4$.

$$X(N_0, N_1) < 1.51 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-2} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q)$$

for $r \geq 3$.

For even lattice L ,

$$X(N_0, N_1) < 0.54 \cdot 2^{-1+\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q) \cdot 2^{(q-1)}$$

for $r = 1$ and $m_0 = 2$.

$$X(N_0, N_1) < 1.25 \cdot 3^{-1} \cdot 2^{-2+\frac{1}{2}(q-1)} \pi^{2+\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 1, q) \cdot 2^{(q-1)}$$

for $r = 1$ and $m_0 \geq 4$.

$$X(N_0, N_1) < 1.25 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-2} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \cdot 2^{r(q-1)}$$

If $2 \leq m_0 < r$ then using (73) we have the following:

For odd lattice L ,

$$\begin{aligned}
 X(N_0, N_1) &< 0.37 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} \\
 &\cdot q^{\frac{1}{2}(m-rq-1)} Y(m, r, q) \text{ for } m_0 = 2, \\
 X(N_0, N_1) &< 1.51 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-2} \pi^{2+\frac{3}{2}(q-1)} \\
 &\cdot q^{-\frac{1}{4}(q+3)+\frac{1}{2}(m-rq-1)} Y(m, r, q) \text{ for } m_0 \geq 4.
 \end{aligned}$$

For even lattice L ,

$$\begin{aligned}
 X(N_0, N_1) &< 0.54 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{\frac{3}{2}(q-1)} \\
 &\cdot q^{1-\frac{1}{4}(q+3)+\frac{1}{2}(m-rq-1)} Y(m, r, q) 2^{r(q-1)} \text{ for } m_0 = 2, \\
 X(N_0, N_1) &< 1.25 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-2} \pi^{2+\frac{3}{2}(q-1)} \\
 &\cdot q^{-\frac{1}{4}(q+3)+\frac{1}{2}(m-rq-1)} Y(m, r, q) 2^{r(q-1)} \text{ for } m_0 \geq 4.
 \end{aligned}$$

$m = \text{even and } r = \text{even}$

Since m is even, m_0 is also even. If $2 \leq r \leq m_0$ then by (75) we have the following:

For odd lattice L ,

$$\begin{aligned}
 X(N_0, N_1) &< 2.67 \cdot 3^{-\frac{1}{2}(q-1)} 2^{-2} \pi^{1+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, q) \\
 &\text{for } r = 2, m_0 = 2, \\
 X(N_0, N_1) &< 2.01 \cdot 3^{-\frac{1}{2}(q+1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, q) \\
 &\text{for } r = 2, m_0 \geq 4, \\
 X(N_0, N_1) &< 1.67 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \\
 &\text{for } r \geq 4
 \end{aligned}$$

For even lattice L ,

$$X(N_0, N_1) < 3^{-\frac{1}{2}(q-1)} 2^{-1} \pi^{1+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, 2) 2^{2(q-1)}$$

$$\text{for } r = 2, m_0 = 2,$$

$$X(N_0, N_1) < 1.67 \cdot 3^{-\frac{1}{2}(q+1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 2, m_0) 2^{2(q-1)}$$

$$\text{for } r = 2, m_0 \geq 4,$$

$$X(N_0, N_1) < 1.39 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)}$$

$$\text{for } r \geq 4.$$

If $2 \leq m_0 < r$ then by (74) we have the following:

For odd lattice L

$$X(N_0, N_1) < 2.67 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} 2^{-2} \pi^{1+\frac{3}{2}(q-1)}$$

$$\cdot q^{-\frac{1}{4}(q+3)} Y(m, r, q) \text{ for } m_0 = 2,$$

$$X(N_0, N_1) < 1.67 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)}$$

$$\cdot q^{-\frac{1}{4}(q+3)} Y(m, r, q) \text{ for } m_0 \geq 4.$$

For even lattice L

$$X(N_0, N_1) < 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{1+\frac{3}{2}(q-1)}$$

$$\cdot q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)} \text{ for } m_0 = 2,$$

$$X(N_0, N_1) < 1.39 \cdot 3^{-\frac{1}{2}(q+1)} e^{\frac{1}{4}(q-1)} 2^{-1} \pi^{2+\frac{3}{2}(q-1)}$$

$$\cdot q^{-\frac{1}{4}(q+3)} Y(m, r, q) \cdot 2^{r(q-1)} \text{ for } m_0 \geq 4.$$

Next proposition is the summary of above computations.

Propositon 8.2. i) Let L be odd unimodular then

$$X(N_0, N_1) < 2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q)$$

for $r \geq 2$,

$$X(N_0, N_1) < 0.37 \cdot 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q)$$

for $r = 1$ and $q \geq 5$,

$$X(N_0, N_1) < 1.51 \cdot 2\pi \cdot 3^{-\frac{3}{2}} Y(m, 1, 3)$$

for $r = 1$ and $q = 3$.

ii) Let L be even unimodular then

$$X(N_0, N_1) < 2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)}$$

for $r \geq 2$,

$$X(N_0, N_1) < 0.54 \cdot 2^{-1} \cdot 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q) 2^{q-1}$$

for $r = 1, q \geq 5$,

$$X(N_0, N_1) < 1.25 \cdot 3^{-1} 2^{-2} \pi^2 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, 1, q) 2^{q-1}$$

for $r = 1, q = 3$.

Proposition 8.3.

$Y(m, r, q) Y(m, 1, 2)^{-1} m^6 < Y(m-1, r, q) Y(m-1, 1, 2)^{-1} (m-1)^6$ for $m-1 \geq 43$ and $r(q-1) \leq m-1$.

Proof. We have

$$\begin{aligned} & \frac{Y(m, r, q) Y(m, 1, 2)^{-1} m^6}{Y(m-1, r, q) Y(m-1, 1, 2)^{-1} (m-1)^6} \\ &= \frac{(\sqrt{2\pi})^{r(q-1)}}{(m-r(q-1))(m-r(q-1)+2) \cdots (m-4)(m-2)} \cdot \frac{\Gamma(\frac{m}{2})}{\sqrt{2\pi} \Gamma(\frac{m-1}{2})} \cdot \frac{m^6}{(m-1)^6} \\ &\leq \frac{(\sqrt{2\pi})^x}{(m-x)(m-x+2) \cdots (m-2)} \cdot \left(\frac{m-1}{2}\right)^{\frac{1}{2}} = f(x) \quad (\text{by (47)}), \end{aligned}$$

where $x = r(q - 1)$. Then $f(x)/f(x - 2) = 2\pi/(m - x)$ shows $f(2) > f(4) > \dots < f(2[\frac{m-1}{2}])$. On the other hand we have

$$f(2[\frac{m-1}{2}]) = (\sqrt{2\pi})^{m-1} (\frac{m-1}{2})^{\frac{1}{2}} / 1 \cdot 3 \cdot \dots \cdot (m-2) \text{ for } m = \text{odd}$$

$$f(2[\frac{m-1}{2}]) = (\sqrt{2\pi})^{m-2} (\frac{m-1}{2})^{\frac{1}{2}} / 2 \cdot 3 \cdot \dots \cdot (m-2) \text{ for } m = \text{even}$$

and

$$f(2) = \frac{2\pi}{m-2} \sqrt{\frac{m-1}{2}}.$$

We can easily check $f(2) < 1$ and $f(2[\frac{m-1}{2}]) < 1$. Hence $f(x) < 1$ for all $x \leq m - 1$. □

Proposition 8.4 i) Let $m = r(q - 1)$ and $m \geq 44$ then we have

$$2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) \cdot 2^{r(q-1)} \\ < m^{-6} Y(m, 1, 2) \text{ for } r \geq 2.$$

ii) Let $m = q - 1$ and $m \geq 44$ then

$$2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q) \cdot 2^{(q-1)} \\ < m^{-6} Y(m, 1, 2).$$

Proof i) By Stirling's formula (48) we have

$$(76) \quad \sum_{i=1}^{\frac{m-2}{2}} \ln(2i)! > \frac{1}{2}(m-2) \ln \sqrt{2\pi} - \frac{1}{2}(m-2)m \\ + (\frac{m-2}{2})^2 \ln(m-2) - \frac{1}{2}(\frac{m-2}{2})^2 + \frac{1}{2}(\frac{m-2}{2}) \ln(m-2) \\ - \frac{1}{4}(m-2) + \ln 2 + 2 > \frac{m^2}{4} (\ln(m-2) - 1.64) \text{ for } m \geq 44.$$

Therefore $\ln 2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)}$ is bounded by the following function $g_m(r)$.

$$\begin{aligned} g_m(r) = & 2 \ln 2 - \frac{m}{2r} \ln 3 + \frac{m}{4r} + \frac{3m}{2r} \ln \pi - \left(\frac{m}{4r} + 1\right) \ln \left(\frac{m}{r} + 1\right) \\ & + \frac{1}{4} m(m-r+1) \ln 2 + \frac{1}{4} m(m-r-1) \ln \pi + \frac{1}{4} r(m-r+1) \ln \left(\frac{m}{r} + 1\right) \\ & + \frac{1}{2} m \ln \sqrt{2\pi} + \frac{m}{4} r \ln r - \frac{3}{8} mr + \frac{1}{4} m \ln r \\ & + \frac{1}{16} \frac{m \ln(r-1)}{r} + \frac{7m}{16r} - \frac{m}{2r} \ln \sqrt{2\pi} - \frac{m^2}{4} (\ln(m-2) - 1.64). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d g_m(r)}{dr} = & \left(\frac{m}{4r} - \frac{m \ln(r-1)}{16r^2} - \frac{7m}{16r^2}\right) + \left(\frac{1}{r} - \frac{1}{m+r} - \frac{m}{4r(m+r)}\right) \\ & + \frac{m \ln 3}{2r^2} + \frac{m}{4r^2} \ln \left(\frac{m}{r} + 1\right) - \frac{3m \ln \pi}{2r^2} - \frac{m \ln 2\pi}{4} - \frac{3}{8} m \\ & + \frac{m}{4} \ln r + \frac{1}{4}(r-1) + \frac{1}{4}(m-2r+1) \ln \left(\frac{m}{r} + 1\right) \\ & > 0 \text{ for } 4 \leq r \leq \frac{m}{13}. \end{aligned}$$

If $r > \frac{m}{13}$, then $r = \frac{m}{12}, \frac{m}{10}, \frac{m}{6}, \frac{m}{4}$ or $\frac{m}{2}$.

We can check $g_m(2), g_m(3), g_m(\frac{m}{13}), g_m(\frac{m}{12}), g_m(\frac{m}{10}), g_m(\frac{m}{6}), g_m(\frac{m}{4})$ are bounded by $-0.02m^2$ for $m \geq 44$. For example

$$\begin{aligned} g_m(2) = & 2 \ln 2 - \frac{m}{4} \ln 3 + \frac{m}{8} + \frac{3}{8} m \ln \left(\frac{m}{2} + 1\right) \\ & - \frac{3}{2} \ln \left(\frac{m}{2} + 1\right) - \frac{1}{4} m \ln 2 + \frac{1}{4} m^2 \ln 2\pi - \frac{1}{4} m^2 (\ln(m-2) - 1.64) \\ = & m^2 \left(\frac{2 \ln 2}{m^2} - \frac{\ln 3}{4m} + \frac{1}{8m} + \frac{3}{8m} \ln \left(\frac{m}{2} + 1\right) - \frac{3}{2m^2} \ln \left(\frac{m}{2} + 1\right)\right. \\ & \left. - \frac{1}{4m} \ln 2 + \frac{1}{4} \ln 2\pi - \frac{1}{4} \ln(m-2) + \frac{1.64}{4}\right) \end{aligned}$$

As a function of m ,

$$\begin{aligned} & \frac{2 \ln 2}{m^2} - \frac{\ln 3}{4m} + \frac{1}{8m} \\ & + \frac{3}{8m} \ln\left(\frac{m}{2} + 1\right) - \frac{3}{2m^2} \ln\left(\frac{m}{2} + 1\right) - \frac{1}{4m} \ln 2 + \frac{1}{4} \ln 2\pi \\ & - \frac{1}{4} \ln(m-2) + \frac{1.64}{4} \end{aligned}$$

is decreasing for $m \geq 44$ and at $m = 44$ less than -0.04 . This gives $g_m(2) \leq -0.04m^2$ for $m \geq 44$. Summarize above computations we have $g_m(r) \leq -0.02m^2$ for any $2 \leq r < \frac{m}{2}$.

On the other hand we can show that

$$-0.02m^2 < \ln(Y(m, 1, 2)m^{-6}).$$

This gives the proof for i), $r < \frac{m}{2}$.

For $r = \frac{m}{2}$, which gives $q = 3$, the left hand side of the inequality i) is

$$2^2 \cdot 3^{-\frac{5}{2}} e^{\frac{1}{2}} \pi^3 \cdot 2^{\frac{m^2}{8} + \frac{m}{4}} \pi^{\frac{m^2}{8} - \frac{m}{4}} \cdot 3^{\frac{m^2}{16} + \frac{m}{8}} \cdot \prod_{j=1}^{\frac{m}{2}} (j-1)! / 2!4! \dots (m-2)!.$$

We can show directly that this is bounded by $Y(m, 1, 2)m^{-6}$ for $m \geq 44$. (Note that $Y(m, 1, 2) = 2^{\frac{m}{2}-1} \pi^{\frac{m}{2}} / (\frac{m}{2} - 1)!$.)

(ii) By (76) we have

$$\begin{aligned}
& \ln 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q) 2^{q-1} \\
&= \frac{m}{2} \ln 2 + \frac{m}{2} \ln \pi - \frac{m}{4} \ln(m+1) + m \ln 2 \\
&+ \frac{1}{4} m(m-2) \ln 2 + \frac{1}{4} m^2 \ln \pi + \frac{m}{4} \ln(m+1) \\
&- \sum_{i=1}^{\frac{m-2}{2}} \ln(2i)! \\
&< \frac{1}{4} m^2 \ln 2\pi + m \ln 2 + \frac{m}{2} \ln \pi - \frac{1}{4} m^2 (\ln(m-2) - 1.64). \\
&< -0.03m^2 < Y(m, 1, 2)m^{-6} \text{ for } m \geq 44.
\end{aligned}$$

□

Direct computations give the following proposition.

Proposition 8.5.

- (i) $2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{q+3}{4}} 2^{r(q-1)} Y(43, r, q)$
 $< Y(43, 1, 2) 43^{-6}$ for $r \geq 2$ with $r(q-1) < 43$.
- (ii) $2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} 2^{q-1} Y(43, 1, q)$
 $< Y(43, 1, 2) 43^{-6}$ for $q-1 < 43$.

Proposition 8.6. Let $m \geq 43$

- (i) $2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)} < Y(m, 1, 2) m^{-6}$
for any $r \geq 2$ such that $r(q-1) \leq m$
- (ii) $2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} Y(m, 1, q) \cdot 2^{q-1} < Y(m, 1, 2) m^{-6}$
for any $q \leq m+1$,

where $Y(m, 1, 2) = (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2})$.

Proof. Proof is by induction on m . Propositions 8.3, 8.4 show if Proposition 8.6 is true for $m - 1$ then it is also true for m . Proposition 8.5 gives the first step of the induction.

□

Lemma 8.7. Let L be a unimodular lattice of rank $m \geq 43$ then

$$\omega_{R(q)}/\omega(L) < m^{-3}Y(m, 1, 2),$$

where $Y(m, 1, 2) = (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2})$.

Proof. The class of $(N_0)_p, p \neq q$ and $p \neq 2$, is determined by the discriminant of N_0 (see 92:1 in [8]) uniquely. Since $(N_1)_p, p \nmid q$, is unimodular $(N_1)_p, p \nmid q$, determined uniquely (see Proposition 3.2 in [14] and Theorem 7.1 in [7]). There are at most 2 possible classes for $(N_0)_q$ (see 92:2 in [8]). For each $(N_0)_q$ the class of $(N_1)_p, p \mid q$, determined uniquely by the definition (iv) of $G(q, r, \rho)$ and Theorem 8.2 in [7]. There are at most 2 possible classes for $(N_0)_2$ (see 93: 16 in [8]). Therefore the number of pairs (G_{N_0}, G_{N_1}) in $G(q, r, \rho)$ is at most 4.

By Proposition 8.2 we have

$$X(N_0, N_1) < 2^2 \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} q^{-\frac{1}{4}(q+3)} \\ \cdot Y(m, r, q) 2^{r(q-1)} \text{ for } r \geq 2$$

and

$$X(N_0, N_1) < 2^{\frac{1}{2}(q-1)} \pi^{\frac{1}{2}(q-1)} q^{1-\frac{1}{4}(q+3)} \\ \cdot Y(m, 1, q) 2^{q-1} \text{ for } r = 1$$

Therefore by Proposition 8.6 $X(N_0, N_1) < Y(m, 1, 2)m^{-6}$ for any r, q, ρ and $m \geq$

43 such that $\rho \leq \min(r, m_0)$, $m_0 = m - r(q - 1)$. Hence we have by Lemma 3.13

$$\begin{aligned} \omega_{R(q)}/\omega(L) &< 4 \sum_{r=1}^{\lfloor \frac{m-1}{q-1} \rfloor} (r+1) \cdot m^{-6} Y(m, 1, 2) \\ &< m^{-3} Y(m, 1, 2). \end{aligned}$$

□

§9. Estimation of $\omega_{IR(q)}/\omega(L)$

First consider the case $q \neq 4$.

Let $\mathcal{M}$ be a totally positive definite λS -modular hermitian S -lattice of rank $\frac{m}{q-1}$.

Mass formulas (1) and (2) give

$$\begin{aligned} \omega(\mathcal{M})/\omega(L) &= 2^{-\frac{1}{4}r(r+1)(q-1)} \pi^{\frac{1}{4}m(m+1) - \frac{1}{4}r(r+1)(q-1)} \\ &\cdot q^{\frac{1}{4}r(1+rq)} \cdot \prod_p \alpha_p(L) \cdot \prod_p \beta_p(\mathcal{M})^{-1} \cdot \left\{ \prod_{j=1}^r (j-1)! \right\}^{\frac{q-1}{2}} \\ &\cdot \left[\prod_{i=1}^m \Gamma\left(\frac{i}{2}\right) \right]^{-1}. \end{aligned}$$

By (42) we have $\alpha_p(L) = (1 - p^{-\frac{m}{2}}) P_p / \frac{m-2}{2}$ for $p \neq 2$ (note that $2 \mid r$ and $m = r(q-1)$ gives $4 \mid m$). By Proposition 5.8 we have $\beta_q(\mathcal{M}) = q^{\frac{1}{2}r(r+1)} P_q(\frac{r}{2})$. By Proposition 8.1, ii) we have $\prod_{p \neq q} \beta_p(\mathcal{M}) \leq 2^{\frac{q-3}{2}} \pi^{\frac{1}{2}(q-1)} q^{-\frac{1}{4}(q+3)} \prod_{p \neq q} \prod_{i=2}^r (1 - p^{-i})^{-\frac{1}{2}(q-1)}$.

Using these local densities and (53) we obtain

$$\begin{aligned} \omega(\mathcal{M})/\omega(L) &< \alpha_2(L) 2^{\frac{1}{4}m(m-r-3)} \pi^{\frac{1}{4}m(m-r-1)} \\ &\cdot q^{\frac{1}{4}r(rq-2r+1)} 2^{-1} 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} \\ &\cdot q^{-\frac{1}{4}(q+3)} \left[\prod_{j=1}^r (j-1)! \right]^{\frac{1}{2}(q-1)} (2!4! \dots (m-2)!)^{-1} \end{aligned}$$

If L is unimodular then $\alpha_2(L) = P_2(\frac{m-2}{2}) / (1 + \delta 2^{-\frac{m+2}{2}})$, $\delta = \pm 1$ and if L is even unimodular then $\alpha_2(L) = 2^m P_2(\frac{m}{2}) / (1 + 2^{-\frac{m}{2}})$ (see §6, types 6), 7) and 2). Then we

obtain (using the definition (40))

$$\omega(\mathcal{M})/\omega(L) < 1.01 \cdot 2^{-1} \cdot 3^{-\frac{1}{2}(q-1)} e^{\frac{1}{4}(q-1)} \pi^{\frac{3}{2}(q-1)} \\ q^{-\frac{1}{4}(q+3)} Y(m, r, q) 2^{r(q-1)}.$$

Therefore by Proposition 8.6, i) and Lemma 4.4 we obtain the following Lemma.

Lemma 9.1. Let L be a unimodular lattice then $\omega_{IR(q)}/\omega(L) < Y(m, 1, 2)m^{-6}$, where $Y(m, 1, 2) = (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2})$ for $q \neq 4$.

Next, consider the case $q = 4$.

(In this case $E = \mathbb{Q}(\zeta) = \mathbb{Q}(\sqrt{-1})$ and $K = \mathbb{Q}$.)

Proposition 9.2. Let $\mathcal{M}$ be a unimodular hermitian S lattice of rank $\frac{m}{2}$.

$$\prod_{p \neq 2} \beta_p(\mathcal{M})^{-1} < \frac{1}{32} \pi^3 e^{\frac{1}{2}} \prod_{i=3}^r (1 - 2^{-i})$$

Proof. By Proposition 5.2 and the fact that p splits in E for $p \equiv 1 \pmod{4}$ and remains prime in E for $p \equiv 3 \pmod{4}$, we have

$$\begin{aligned} \prod_{p \neq 2} \beta_p(\mathcal{M})^{-1} &= \prod_{p \equiv 1(4)} \prod_{i=1}^r (1 - p^{-i})^{-1} \prod_{p \equiv 3(4)} \prod_{i=1}^r (1 - (-1)^i p^{-i})^{-1} \\ &< \prod_{p \equiv 1(4)} (1 - p^{-1})^{-1} \prod_{p \equiv 3(4)} (1 + p^{-1})^{-1} \prod_{p \neq 2} \prod_{i=2}^r (1 - p^{-i})^{-1} \\ &< \frac{\pi^3 e^{\frac{1}{2}}}{32} \prod_{i=3}^r (1 - 2^{-i}) \\ &\text{(by (50), (52) and (53))} \end{aligned}$$

□

With the notation in Proposition 9.2, Mass formulas (1) and (2) give

$$\begin{aligned} \omega(\mathcal{M})/\omega(L) &< \frac{\pi^3 e^{\frac{1}{2}}}{32} \prod_{i=3}^r (1 - 2^{-i}) \cdot \frac{\alpha_2(L)}{\beta_2(\mathcal{M})} \cdot \prod_{p \neq 2} (1 - (\frac{(-1)^{\frac{m}{2}}}{p}) p^{-\frac{m}{2}}) P_2(\frac{m-2}{2}) \\ &\cdot \pi^{\frac{1}{4}m(m+1) - \frac{1}{2}r(r+1)} \prod_{j=1}^r (j-1)! [\prod_{i=1}^m \Gamma(\frac{i}{2})]^{-1}. \end{aligned}$$

If L is odd unimodular and $n(M) = S$, then by Propositions 5.13 and 5.14 we have

$$\prod_{i=3}^r (1 - 2^{-i}) \frac{\alpha_2(L)}{\beta_2(M)} = \frac{P_2(\frac{m-2}{2})}{1 + \delta_2 - \frac{m-2}{2}} \cdot \frac{1}{2P_2(\lfloor \frac{r-1}{2} \rfloor)} \prod_{i=3}^r (1 - 2^{-i}) < 0.51.$$

If L is even unimodular and $n(M) = 2S$, then by Proposition 5.12 we have

$$\prod_{i=3}^r (1 - 2^{-i}) \frac{\alpha_2(L)}{\beta_2(M)} = \frac{2^m P_2(\frac{m}{2})}{1 + 2^{-\frac{m}{2}}} \cdot \frac{1}{2^r P_2(\frac{r}{2})} \prod_{i=3}^r (1 - 2^{-i}) < 2^{\frac{m}{2}}.$$

Therefore by using (54) we have the following Proposition.

Proposition 9.3. Let M be a unimodular lattice of rank $\frac{m}{2}$. Assume that $nM = S$ if L is odd, and $n(M) = 2S$ if L is even then we have the following.

$$\begin{aligned} \omega(M)/\omega(L) &< 1.01 \cdot \frac{\pi^3 e^{\frac{1}{2}}}{32} \cdot 2^{\frac{m^2}{4}} \pi^{\frac{m^2}{8} - \frac{m}{4}} \prod_{j=1}^{\frac{m}{2}} (j-1)! \\ &\cdot [2!4! \cdots (m-2)!]^{-1}. \end{aligned}$$

Lemma 9.4. Let L be a unimodular lattice of rank $m \geq 44$ then

$$\omega_{IR(4)}/\omega(L) < Y(m, 1, 2)m^{-6},$$

where $Y(m, 1, 2) = (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2})$.

Proof. By Lemmas 4.9 and 4.10 and Proposition 9.3 we have

$$\begin{aligned} \omega_{IR(4)}/\omega(L) &< 2 \cdot 1.01 \cdot \frac{\pi^3 e^{\frac{1}{2}}}{32} \cdot 2^{\frac{m^2}{4}} \pi^{\frac{m^2}{8} - \frac{m}{4}} \prod_{j=1}^{\frac{m}{2}} (j-1)! \\ &\cdot [2!4! \cdots (m-2)!]^{-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\frac{\omega_{IR(4)}}{\omega(L)} \cdot Y(m, 1, 2)^{-1} \\ &< \frac{1.62 \cdot 2^{\frac{m^2}{4}} \pi^{\frac{m^2}{8} - \frac{m}{4}} \prod_{j=1}^{\frac{m}{2}} (j-1)!}{2!4! \cdots (m-2)!} \cdot \frac{2(\frac{m-2}{2})!}{(\sqrt{2\pi})^m}. \end{aligned}$$

Let us denote this upper bound by $g(m)$ then we obtain

$$\frac{g(m)}{g(m-2)} \cdot \frac{m^6}{(m-2)^6} = \frac{2^{m-3} \pi^{\frac{1}{2}m-2}}{(m-3)!} \left(\frac{m}{m-2} \right)^6 < 1.$$

Direct computations show that $g(44) \cdot 44^6 < 1$. Therefore we have this Lemma. $\square$

CHAPTER V

PROOF OF THE THEOREMS

§10. Proof of the theorems.

In this section we complete the proof of the Theorems 1, 2 and 3.

Since we have

$$\omega'(L) \leq \sum_{\substack{q \text{ prime} \\ 2 \leq q \leq m}} \omega_{R(q)} + \sum_{\substack{q \text{ odd prime} \\ (q-1) | m \text{ or} \\ q=4}} \omega_{IR(q)},$$

and there are at most $[\frac{m}{2}]$ prime numbers less than $m + 1$,

Lemmas 7.5, 8.7, 9.1 and 9.4 give the following.

If L is odd unimodular lattice then

$$\begin{aligned} \omega'(L)/\omega(L) &< 64.81Y(m, 1, 2) + \frac{m}{2} \cdot m^{-3}Y(m, 1, 2) \\ &\quad + \frac{m}{2} \cdot m^{-6}Y(m, 1, 2) + m^{-6}Y(m, 1, 2) \\ &< 65Y(m, 1, 2) = 65 \cdot (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2}) \\ &< 33(\sqrt{2\pi})^m / \Gamma(\frac{m}{2}) \text{ for } m \geq 43. \end{aligned}$$

If L is even unimodular lattice then

$$\begin{aligned} \omega'(L)/\omega(L) &< (3.95 \cdot 2^m + \frac{m}{2} \cdot m^{-3} + \frac{m}{2} \cdot m^{-6} + m^{-6})Y(m, 1, 2) \\ &< 4 \cdot 2^m Y(m, 1, 2) = 4 \cdot 2^m (\sqrt{2\pi})^m / 2\Gamma(\frac{m}{2}) \\ &< 2^{m+1} (\sqrt{2\pi})^m / \Gamma(\frac{m}{2}) \text{ for } m \geq 144. \end{aligned}$$

This gives Theorem 2. By estimation of the functions $33(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ and $2^{m+1}(\sqrt{2\pi})^m/\Gamma(\frac{m}{2})$ for $m \geq 43$ (resp. $m \geq 144$) we obtain Theorem 1.

Let $\omega''(L) = \sum_{cls K \subseteq G_L} \frac{1}{|O(K)|}$, where the summation goes over the classes of lattices K such that $q \mid |O(K)|$ with some odd prime number q . Then

$$\omega''(L) < \sum_{\substack{q \neq 2 \\ q \text{ prime} \\ q \leq m}} \omega_{R(q)} + \sum_{\substack{q \text{ odd prime} \\ (q-1) \mid m}} \omega_{IR(q)}.$$

Therefore Lemmas 8.7 and 9.2 give the proof of Theorem 3.

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