76-9930

ARMBRUST, Wayne Thomas, 1943-
A CLUSTER MODEL SATISFYING LIMITED
CHARGE EXCHANGE.

The Ohio State University, Ph.D., 1975
Physics, elementary particles & high
energy

Xerox University Microfilms, Ann Arbor, Michigan 48106

A CLUSTER MODEL SATISFYING LIMITED CHARGE EXCHANGE

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

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* * * * *

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1975

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ACKNOWLEDGMENTS

The author would like to thank his adviser, Prof. K. Tanaka for understanding and support over the years. Special words of thanks go to Prof. Stephen Pinsky, whose many helpful discussions were greatly appreciated, to the Experimental High Energy Physics Group for the use of their computing facilities, and to Mr. John Heimaster for assistance in programming.

The author is very pleased to acknowledge the debt of thanks owed Prof. David M. Scott. It is difficult to imagine how this dissertation could have been completed without his advice, assistance and counsel.

The author also wishes to thank Mrs. Arleen Danford for typing the final draft of this dissertation.

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PUBLICATION

David M. Scott and W.T. Armbrust, "A Cluster Model Satisfying Limited Charge Exchange" Phys. Rev D (To be published).

FIELDS OF STUDY

Major Field: Physics

Studies in High Energy Physics, Strong Interaction, Professors
K. Tanaka and D.M. Scott.

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INTRODUCTION

The idea of multiparticle production proceeding through intermediate states or 'clusters' is a very old one in high energy hadron physics. It dates back at least 17 years to the original 'fireball' model of Cioek et al.¹ In this model the intermediate state consisted of two leading particles plus two clusters which decayed isotropically in their rest frames into the final state products. These isotropically decaying objects were termed 'fireballs.' The fireball idea was a very popular one and it provided much of the basis for succeeding models of multiparticle production. In fact, most models for multiparticle production were generalized into ones which incorporated these intermediate states.²

It is possible to classify cluster models into three broad areas according to the proposed method of production. These are:

- (i) thermodynamical models,
- (ii) multiperipheral cluster models,
- (iii) independent cluster production models.

The first extensive study of thermodynamical multiparticle production models seems to be the one by Hagedorn³ in 1965. In this model all hadronic collisions above a certain minimum center of mass (c.m.) energy are assumed to produce a large 'fireball' or 'cluster' in thermodynamic equilibrium in some volume $\sim 1/m_n^3$. These 'fireballs' are assumed to be independent of the incident particles, and composed of lesser 'fireballs.' This distribution is characterized by a maximum temperature T_0 , also

independent of energy and incident particles. The mass density of the secondary clusters $\rho(m)$ has an exponential distribution. No distinctions are made between resonances and 'fireballs' ('fireballs' are just considered massive resonances). This model has been expanded upon by a large number of authors and has enjoyed considerable success in the interpretation of experimental results. The ideas that 'fireballs' are composed of lesser 'fireballs' leads to a selfconsistency condition or so-called 'statistical bootstrap', the general solution to which is very tedious as it contains many internal quantum numbers and conservation laws. A good source giving the present status of this model is the set of papers by Mohring.⁴

A very complete review of multiperipheral cluster models is given by Dremin and Dunaevskii⁵ starting from basic multiperipheral theory. In the basic multiperipheral model the total cross section $\sigma(s, p^2)$ is given by the multiperipheral equation $\sigma(s, p^2) = \bar{\sigma}(s, p^2) + \frac{1}{16\pi} \int d^2k ds_1 ds_2 \times D^2(k^2) \bar{\sigma}(s_1, p, k^2) H$ (I-1)

with H a term involving dynamical quantities and $D(k^2)$ the propagator of an exchanged meson. The quantity $\bar{\sigma}(s, p^2, k^2)$ is chosen as the sum of the total cross sections of all known resonances. The multiperipheral cluster model is formed by adding to σ the elastic diffraction cross section at high energy, $\bar{\sigma}_d$, and a term $\bar{\sigma}_b$, which is taken to be the cross section for inelastic non-one-pion-exchange processes. It is the term $\bar{\sigma}_b$ which is related to cluster production. Although it is claimed⁵ that this multiperipheral cluster model gives a framework in which all high energy multiparticle hadronic processes can be incorporated, detailed calculations are difficult if the kernel of Equation (I-1) is

obtained from a subset of Feynman graphs. The usual approach in calculating with this model is to choose the kernel phenomenologically.

Independent cluster models were suggested by studies of two particle rapidity correlations (for details, see for example the review by Whitmore⁶) which showed that particles are uncorrelated when separated by large values of rapidity but have strong positive correlations for small values. In the most popular versions of this model⁷, isoscalar clusters are independently produced with uniform spacing in rapidity. The number of clusters is allowed to increase with $\ln s$ to match the apparent⁶ energy increase of $\langle n \rangle$, the average number of negative pions. This is an attractive hypothesis because, since the total available rapidity also varies with $\ln s$, the spacing of clusters in rapidity is constant with energy. This model has several shortcomings which we will discuss below.

Until recently, although the cluster idea was a very popular one incorporated into most phenomenological and many theoretical models, direct evidence for their existence was lacking. The evidence offered in their support had been indirect, namely through comparisons of experimental multiplicity distributions, two particle correlations, and charge transfer between c.m. hemispheres to model dependent phenomenological and theoretical predictions.⁸ Recent experimental analysis of the rapidity difference between produced particles has, however, provided a new facet in these comparisons.^{9,10} These new data show that for large rapidity gaps the probability density of rapidity gap length, $P(\Delta y)$, as a function of rapidity gap length Δy , falls exponentially. This dependence indicates that final state particles separated by these large rapidity gaps are produced independently. For smaller rapidity gaps however,

an interdependence of production is indicated since the density is above the exponential decay line as illustrated in Figure 1. This is easily explained in a cluster model since particles produced with large rapidity gaps would come from separate clusters with essentially independent production, while those separated by small gaps come from the same cluster. On this basis it is claimed that the actual production of clusters is exhibited.¹⁰

This recent experimental analysis has also provided data on the charge carried between clusters by an exchange mechanism.¹¹ The non-zero value of this exchanged charge ΔQ shows that cluster production models must include charged clusters in addition to the isoscalar ones considered in the past.⁷ Also, the suppression of $|\Delta Q| \geq 2$ for large rapidity gaps indicates that the exchanged charge is limited to the values $Q = 0, \pm 1$. This is illustrated in Figure 2, where the probability density for rapidity gap lengths of specific charge $P_{\Delta Q}(\Delta y)$ is plotted as a function of Δy .

In this work, we consider the production of isoscalar clusters and isovector clusters satisfying this limited charge exchange (LCEX) constraint. The subsequent decay of these clusters is assumed to result only in final state pions and we consider different size clusters. The size of a cluster is specified by the resulting number of pions and we look at clusters decaying into from 1 to 4 pions.

Following Levy,¹² we note that there are three possible schemes within a cluster model to account for the energy dependence of the number of final state particles produced.

- (i) the average cluster size is constant but the average number of clusters increases with energy,

- (ii) the average number of clusters is constant but the average cluster size increases,
- (iii) both the average number of clusters and average cluster size varies.

We have investigated both of the first two possibilities and find that the experimental two particle correlation data seems inconsistent with a constant cluster size but is consistent with a varying cluster size. We also investigated the admixture of isoscalar and isovector clusters using the ratio of the number of charged links to the number of neutral links in the multiperipheral chain. The primary result of this dissertation is the development of the limited charge exchange distribution and the resulting analytical expressions for the average number of charged and neutral particles produced and the two particle correlation coefficients.

The distribution function for describing cluster production and the subsequent decay into final state particles and analytic expressions for the moments are developed in Chapter I. In Chapter II various comparisons to data are used to limit the generality of the model; Chapter III contains the conclusions of the paper.

CHAPTER I. THE LIMITED CHARGE EXCHANGE MODEL

In this section we shall develop an expression giving the probability of producing various final states according to a statistical model which satisfies the LCEX constraint. The model assumes that isoscalar and isovector clusters are produced independently (a Poisson distribution). A binomial distribution is used to describe the relative production of isoscalar and isovector clusters, the LCEX constraint is imposed on the isovector part and multinomial distributions are used to separate each of these into different sizes of clusters. Finally the possible decays of each type of cluster into final state pions via different decay modes are again described statistically with the binomial (or, where needed, the multinomial) distribution.

In this investigation we consider the production of intermediate clusters which exchange only neutral or singly charged objects (ie. particles or Reggeons) between them. Since we consider only isoscalar and isovector cluster production, each cluster is limited to being neutral or singly charged except for end clusters. The ends of the cluster chain are complicated by the need to consider the possible transfer of the incident particle's quantum numbers along the chain. We have circumvented this complication in our study by not considering charge exchange processes so that each incident particle emerges undisturbed as a leading particle. However, a cluster may occur in conjunction with

either incident particle and is termed an end cluster. Since we do not consider charge exchange processes, an end cluster will decay into the incident particle plus pions and the charge of the incident particle must be added to that of the meson cluster. This is illustrated at the top end of the chain of clusters in Figure 3(a) which shows an incident proton forming a doubly charged end cluster; the mesons in this cluster have a net positive charge.

The first stage of our statistical development gives the probability of producing k independent clusters of which k_v are isovector and k_s are isoscalar:

$$P(k_v, k_s) = e^{-\lambda} \frac{\lambda^k}{k!} \binom{k}{k_v} V^{k_v} S^{k_s} \quad (1.1)$$

with $k = k_v + k_s$ and $V + S = 1$. The admixture of isovector and isoscalar clusters must be determined from experiment in order to fix V and S . The average number of clusters produced is the sum of the average number of isovector clusters λ_v and isoscalar clusters λ_s :

$$\lambda = \lambda_v + \lambda_s \quad (1.2)$$

where

$$\lambda_v = V\lambda \text{ and } \lambda_s = S\lambda. \quad (1.3)$$

Equation (1.1) can thus be written as the product of two Poissons in the form

$$P(k_v, k_s) = e^{-\lambda_v} \frac{\lambda_v^{k_v}}{k_v!} \cdot e^{-\lambda_s} \frac{\lambda_s^{k_s}}{k_s!} \quad (1.4)$$

The LCEX constraint which affects only isovector cluster production prevents certain sequences of charged clusters along the

multiperipheral chain. Thus the combinatorial factor will be different⁸ from that of a multinomial in the number of positive, negative, and neutral clusters. In deriving the effects of LCEX, we first note that the suppression of $|\Delta Q| \geq 2$ exchanges requires all positive and negative clusters to be arranged in pairs along the chain.

When u pairs of charged clusters are produced, each pair can be arranged in two ways; thus the combinatorial factor will include a term 2^u . Also there are $2u + 1$ "bins" between and at the ends of the $2u$ charged clusters in which neutral clusters can occur. Since there is no known constraint on the placement of these neutral clusters, there is no a priori reason for not assigning equal probability to each of the possible arrangements of a total of v neutral clusters into the possible "bins" and the number of distinguishable distributions is¹³ the binomial coefficient $\binom{2u+v}{v}$. Thus the overall probability that k_v clusters are independently produced and that these consist of u pairs of charged clusters and v neutral clusters is

$$P(u,v) = \frac{1}{N} \frac{\lambda_v^{k_v}}{k_v!} \binom{2u+v}{v} 2^u a^{2u} b^v \quad (1.5)$$

where a and b are the relative probabilities that a cluster is charged or neutral so that

$$2a + b = 1 \quad (1.6)$$

and

$$2u + v = k_v. \quad (1.7)$$

The factor N is required to normalize the distribution function.

$$P(u,v) = \frac{1}{N} e^{-2a\lambda_v} \frac{(\sqrt{2}a\lambda_v)^{2u}}{(2u)!} e^{-b\lambda_v} \frac{(b\lambda_v)^v}{v!} = P(u) P(v). \quad (1.8)$$

N is now given by

$$N = e^{-2a\lambda_v} \sum_{u=0}^{\infty} \frac{(\sqrt{2}a\lambda_v)^{2u}}{(2u)!} = e^{-2a\lambda_v} \cosh(\sqrt{2}a\lambda_v) \quad (1.9)$$

and we have

$$P(u,v) = \frac{e^{-b\lambda_v}}{\cosh(\sqrt{2}a\lambda_v)} \frac{(\sqrt{2}a\lambda_v)^{2u}}{(2u)!} \frac{(b\lambda_v)^v}{v!} \quad (1.10)$$

To determine a and b, we first note that by not considering charge exchange reactions, the experimental⁶ result $2\sigma_{\pi^0} = \sigma_{\pi^+} + \sigma_{\pi^-}$ reduces to equal production of negative and neutral pions. Further, it is easy to show that for a given cluster size, equal numbers of negative and neutral clusters are required to produce equal numbers of negative and neutral pions. We also assume that the size distributions of charged and neutral clusters are the same thus assuring equal numbers of π^- s and π^0 s on average. From Equation (1.8), since each factor is normalized to one, we obtain for the average number of negative (or positive) and neutral clusters respectively,

$$\langle u \rangle = \frac{a\lambda_v}{\sqrt{2}} \tanh(\sqrt{2}a\lambda_v) \quad (1.11a)$$

$$\langle v \rangle = b\lambda_v \quad (1.11b)$$

where the symbol ' $\langle \rangle$ ' indicates the average of a quantity. The equality of these two quantities together with Equation (1.6) allows a to be found as the solution of the transcendental equation

$$1 - 2a = \frac{a}{\sqrt{2}} \tanh(\sqrt{2}a\lambda_v) \quad (1.12)$$

If λ_v is large enough so that $\tanh(\sqrt{2}a\lambda_v) \approx 1$,

$$a \approx \frac{\sqrt{2}}{2\sqrt{2} + 1} \quad (1.13a)$$

$$b \approx \frac{1}{2\sqrt{2} + 1} \quad (1.13b)$$

$$\text{We note that } 2 \langle u \rangle + \langle v \rangle \approx \frac{3\lambda_v}{2\sqrt{2} + 1}, \quad (1.14)$$

not λ_v as before the imposition of LCEX. By letting $\lambda_v \rightarrow \frac{(2\sqrt{2} + 1)\lambda_v}{3}$

we have $b\lambda_v \rightarrow \lambda_v/3$ and $\sqrt{2}a\lambda_v \rightarrow \frac{2\lambda_v}{3}$. Equation (1.10) then becomes in

this approximation

$$P(u, v) = \frac{e^{-\lambda_v/3}}{\cosh(2\lambda_v/3)} \frac{(2\lambda_v/3)^{2u}}{(2u)!} \frac{(\lambda_v/3)^v}{v!}. \quad (1.15)$$

For $\lambda_v \gtrsim 3$ we have $\langle u \rangle \approx \langle v \rangle$ and $2 \langle u \rangle + \langle v \rangle = \lambda_v$ to within a few percent. We can write the probability of producing u negative isovector clusters, v neutral isovector clusters, and k_s isoscalar clusters as

$$P(u, v, k_s) = \frac{e^{-\lambda_s \lambda_s k_s}}{k_s!} \frac{e^{-\lambda_v/3}}{\cosh(2\lambda_v/3)} \frac{(2\lambda_v/3)^{2u}}{(2u)!} \frac{(\lambda_v/3)^v}{v!}. \quad (1.16)$$

Each type of cluster that we have considered can, in general, decay into different numbers of pions (ie. have different sizes). The number of positive and negative isovector clusters decaying into i pions is denoted by u_{i+} and the number of neutral isovector clusters of size i is v_i . Likewise, the number of i -particle isoscalar clusters is k_{si} ($i = 1$ is not permitted because a 1-particle isoscalar cluster cannot decay into only pions). The relative

probability for production of each size of isovector cluster is d_1 and that for isoscalar clusters is e_1 . The distribution of the various sizes of clusters produced is described by a multinomial distribution. Thus, the probability for producing u_1^+ positive 1-particle isovector clusters, etc. is:

$$P(u_1^+, \dots, u_4^+, u_1^-, \dots, u_4^-, v_1, \dots, v_4, k_{s2}, \dots, k_{s4}) = P(u, v, k_s) \times \frac{u_1! d_1^{u_1} d_2^{u_2} d_3^{u_3} d_4^{u_4}}{u_1^+! u_2^+! u_3^+! u_4^+!} \frac{u_1! d_1^{u_1} d_2^{u_2} d_3^{u_3} d_4^{u_4}}{u_1^-! u_2^-! u_3^-! u_4^-!} \frac{v_1! d_1^{v_1} d_2^{v_2} d_3^{v_3} d_4^{v_4}}{v_1! v_2! v_3! v_4!} \times \frac{k_{s2}! e_2^{k_{s2}} e_3^{k_{s3}} e_4^{k_{s4}}}{k_{s2}! k_{s3}! k_{s4}!} \quad (1.17)$$

where $P(u, v, k_s)$ is given by Equation (1.16).

Finally we describe the decay of each type and size of cluster into final state pions. The possible decays of these clusters along with the relative probabilities of decay into each final state and the variable describing the number of events for each final state are given in

Table 1. When more than one final state is available for the decay of a cluster, the actual number of decays is predicted with a binomial distribution. We illustrate this procedure with a 3-particle positive isovector cluster. From Table 1, we see that the probability for producing

$n_{31} \pi^+ \pi^+ \pi^-$ events and $n_{32} \pi^+ \pi^0 \pi^0$ events from u_3^+ clusters is

$$P(n_{31} | u_3^+) = \frac{u_3^+}{n_{31}! n_{32}!} p_3^{n_{31}} q_3^{n_{32}} = \binom{u_3^+}{n_{31}} p_3^{n_{31}} (1-p_3)^{u_3^+ - n_{31}}. \quad (1.18)$$

This decay proceeds through three possible intermediate isospin states: isotensor, isovector, and isoscalar. For each intermediate state, the values of p_3 and q_3 are combinations of Clebsch-Gordon coefficients. The respective values of these relative probabilities for each intermediate isospin state for the different clusters considered are listed in Table 2. We note that for isovector clusters (as expected from the Wigner-Eckardt Theorem) we have $2p_1 + r_1 = 2$. Thus only one independent variable describes the number of final state events for these clusters. Also from Table 2, we find limits on these relative probabilities:

$$\begin{aligned} 1/2 &\leq p_3 \leq 2/3 \\ 3/5 &\leq p_4 \leq 1 \\ 1/3 &\leq \bar{r}_4 \leq 4/9 \\ 4/9 &\leq \bar{s}_4 \leq 2/3 \\ 0 &\leq \bar{t}_4 \leq 1/9. \end{aligned}$$

The various probability functions are now assembled to yield the distribution function which gives the probability for producing each possible final state:

$$\begin{aligned} P(u_1^+, \dots, u_4^+, u_1^-, \dots, u_4^-, v_1, \dots, v_4, k_{s2}, \dots, k_{s4}, n_{31}, n_{33}, n_{35}, n_{41}, n_{43}, n_{45}, \\ \bar{n}_{21}, \bar{n}_{41}, \bar{n}_{42}) = P(u_1^+, \dots, k_{s4}) P(n_{31} | u_3^+) P(n_{33} | u_3^-) P(n_{35} | v_3) P(n_{41} | u_4^+) \\ \times P(n_{43} | u_4^-) P(n_{45} | v_4) P(\bar{n}_{21} | k_{s2}) P(\bar{n}_{41}, \bar{n}_{42} | k_{s4}). \end{aligned} \quad (1.19)$$

The functions $P(u_1^+, \dots, k_{s4})$ and $P(n_{31} | u_3^+)$ are given in Equations (1.17) and (1.18) respectively.

The goal of this development is, of course, the distribution function giving the probability for producing n negative pions and m neutral pions. This function is obtained by summing Equation (1.19) over all values of final state events subject to the constraints that the

proper numbers of negative and neutral pions are produced. By considering the decays listed in Table 1, we see that these numbers are given by

$$n = u + v_2 + v_4 + n_{31} + n_{33} + n_{35} + n_{41} + n_{43} + n_{45} + \bar{n}_{21} + 2\bar{n}_{41} + \bar{n}_{42}, \quad (1.20a)$$

$$m = u_2^+ + u_2^- + 2(u_3^+ + u_3^-) + 3(u_4^+ + u_4^-) + v_1 + 3v_3 + 2v_4 - 2(n_{31} + n_{33} + n_{35} + n_{41} + n_{43} + n_{45}) + 2k_{s2} + k_{s3} + 4k_{s4} - 2\bar{n}_{21} - 4\bar{n}_{41} - 2\bar{n}_{42}. \quad (1.20b)$$

Thus, the probability for producing n negative and m neutral pions is

$$P(n,m) = \sum_{\text{final states}} P(u_1^+, \dots, \bar{n}_{42}) \Delta \quad (1.21)$$

where Δ is a product of Kronecker deltas

$$\Delta = \delta_{n, u + v_2 + \dots + 2\bar{n}_{41} + \bar{n}_{42}} \delta_{m, u_2^+ + u_2^- + \dots + 4\bar{n}_{41} - 2\bar{n}_{42}} \delta_{u, \sum u_i^+} \times \delta_{u, \sum u_i^-} \delta_{v, \sum v_i} \delta_{k, k_{s2} + k_{s3} + k_{s4}}. \quad (1.22)$$

From the distribution function (1.19) and the relations (1.20a) and (1.20b), we have derived analytic expressions for the average numbers of negative and neutral pions produced and the two particle correlation coefficients. These general expressions are listed in Table 3 and will be used for data comparisons in the next section.

CHAPTER II. COMPARISON TO DATA

The limited charge exchange model developed in Section II is quite general and has a large number of parameters, each of which may have an energy dependence. In this section, we consider various submodels which are defined by assuming a particular energy dependence for certain parameters. We then make crude comparisons to data in the sense that we do not adjust parameters to achieve cosmetic fits but instead consider the general trends of the data. We observe in Part A that experimental data on charge transferred between clusters suggests a strong suppression of isoscalar relative to isovector cluster production. Thus we look only at isovector clusters for the remainder of Chapter II. In Part B, we consider the model which has a fixed average cluster size (ie. the d_1 , the parameters for the probability of producing clusters of various sizes, defined in Chapter I, are assumed constant) so that results of varying the average number of clusters are developed. The reverse situation is in Part C where we report results of an energy dependent cluster size and a constant average number of clusters. The more general (and probably more realistic) case where both cluster size and average number of clusters have an energy dependence is discussed in Part D.

A. Isospin of Clusters

The general expression for producing n negative pions and m neutral pions (Equation (1.21)) contains contributions from both

isoscalar and isovector clusters. In order to determine the relative production of each type of cluster, we have investigated the behavior of a parameter introduced by Pirila, Thomas, and Quigg¹¹ in their work with LCEX models. They identify the ratio R of the number of charged exchanges or links between clusters to the number of neutral exchanges as being an important parameter for exchange models. To calculate this ratio in our model, we note that intra-charged pair exchanges are charged while inter-charged pair exchanges are neutral (see Figure 3(a)). Also, each neutral cluster that is added to a link causes another link of the same charge. Thus the number of charged links is equal to the number of pairs of charged clusters plus the number of neutral clusters located within a pair of charged ones. Since the probability that a neutral cluster falls inside a charged pair is $u/(2u+1)$, the average number of such neutral clusters produced is $(v+k_s)u/(2u+1)$ and the average number of charged exchanges is

$$\ell_{ch} = u + \frac{u(v+k_s)}{2u+1} \quad (2.1)$$

The total number of exchanges¹³ is $2u+v+k_s - 1$, therefore the number of neutral exchanges is

$$\ell_{neut} = u + (v+k_s)\left(1 - \frac{u}{2u+1}\right) - 1. \quad (2.2)$$

The average value of the ratio $R = \ell_{ch}/\ell_{neut}$ was calculated as a function of the average number λ of clusters produced and the ratio of the average number of isoscalar clusters to isovector clusters λ_s/λ_v . The results which are plotted in Figure 4 show that for a small number of clusters (ie. $\lambda < 3$), R is quite sensitive to both the total number

of clusters produced and to their isospin admixture. Pirila, et al.¹¹ ¹⁶ observed that the 205 GeV/c p-p data suggest approximately twice as many charged exchanges occur as neutral ones; that is $R \approx 2$. Therefore, unless the total number of clusters produced is very small ($\lambda \lesssim 1.5$), the ratio of isoscalar to isovector cluster production must be approximately zero. Such a small number of clusters can only occur if they become unreasonably large. For example, the production of an average number of 1.5 clusters in a 405 GeV/c p-p interaction would require an average cluster size of about 7 particles. Thus, we conclude that isoscalar cluster production is strongly suppressed and for the rest of our data comparisons, we assume it is zero.

B. Fixed Cluster Size

In order to fix the cluster size and type, we take the general expression for the probability of producing n negative pions and m neutral pions (Equation (1.21)) and set the probability for producing each size and type of cluster equal to one for the desired clusters and equal to zero for all others. Since we are assuming that no isoscalar clusters are produced, we set $\lambda_s = 0$. Then studying clusters decaying into three pions, for example, we set $d_3 = 1$ and all other d_i equal to zero. The resulting expressions for the average numbers of negative and neutral pions produced and the two particle integrated correlation coefficients for each cluster size considered are plotted as a function of λ_v in Figure 5.

Looking first at the average number of negative particles produced, we note that for large values of the average number of isovector clusters λ_v (ie. for λ_v such that $\tanh(\sqrt{2}\alpha\lambda_v) \approx 1$); $\langle n \rangle$ depends linearly on λ_v . Thus by giving λ_v a $\ln s$ dependence, we can match the apparent

experimental energy dependence of $\langle n \rangle$. Difficulties are encountered when we consider the integrated two particle correlation coefficients f_2^{--} , f_2^{-0} , and f_2^{00} defined as:

$$f_2^{ij} = \langle n_i, n_j \rangle - \langle n_i \rangle \langle n_j \rangle, \quad i = j \quad (2.3a)$$

$$= \langle (n_i(n_i-1)) \rangle - \langle n_i \rangle^2, \quad i = j. \quad (2.3b)$$

(Higher order correlations can be defined in a similar way.) Since the f_2 's also depend linearly on λ_v (again for large λ_v) we are led to a contradiction with experiment since a $\ln^2 s$ term is apparently needed^{6,15} to explain the experimental f_2^{--} data. The necessity of a $\ln^2 s$ term is clearly demonstrated in Figure 6 in which Whitmore's empirical expression⁶:

$$f_2^{--} = (2.47 \pm 0.19) - (1.51 \pm 0.09) \ln s + (0.21 \pm 0.01) \ln^2 s \quad (2.4)$$

obviously fits the data better than an expression containing only a $\ln s$ term would.

Although we have only investigated clusters that decay into fewer than five particles, this linearity in the f_2 's at large values of λ_v is quite general with a fixed cluster size (FCS)LCEX model. There does not appear to be any mechanism in this scheme to generate terms quadratic in λ_v for large values of λ_v . To see why this occurs, we refer to Table 3. It shows that while the expression for f_2^{--} contains a term quadratic in λ_v , there is the factor $(1 - \tanh(2\lambda_v/3))^2$ which goes to zero at large values of λ_v . The reason for this factor is that in the calculation of $f_2^{--} = \langle n^2 \rangle - \langle n \rangle^2 = \langle n^2 \rangle - \langle n \rangle$, $\langle n^2 \rangle$ contributes the λ_v^2 while $\langle n \rangle$ has a $\lambda_v \tanh(2\lambda_v/3)$ factor so that at high values of λ_v , the

cancellation occurs between the $\langle n^2 \rangle$ and $\langle n \rangle^2$ terms. It is easy to show that this is a general feature and will occur for all larger sized clusters.

In addition to the above difficulty, we have also found that this FCS-LCEX model contradicts the apparent KNO scaling¹⁶ of topological cross section data¹⁷ that has been observed for p-p interactions over the energy range of 50-405 GeV/c incident momenta. This behavior of the experimental data requires a phenomenological model to at least mimic KNO scaling for this energy range even though the theoretical basis of KNO scaling is controversial. We have checked on this requirement for the FCS-LCEX model by calculating the topological cross section predictions for 1,2,3, and 4-particle isovector clusters at different energies. Each energy was determined by fixing the average number of clusters produced so that the resulting value of $\langle n \rangle$ would correspond to the experimental values for the various incident momenta. A typical prediction is the case of 3-particle clusters at 102 and 405 GeV/c which is plotted in Figure 7 using the KNO variables $\Psi = \langle n \rangle \sigma_n / \sigma_{\text{inel}}$ and $x = n / \langle n \rangle$. Since KNO scaling predicts that topological cross section data displayed in this fashion are energy independent, the predictions at the two energies should lie on top of one another. The strong shift between them clearly demonstrates the lack of KNO scaling over this energy range.

Finally we consider a more general model in which only the average cluster is constant. To achieve this, we choose energy independent values for the d_i 's; thus more than one size of cluster is produced but the relative amounts of each size are constant. This generalization fails, however, because the failure of the FCS-LCEX model detailed above does not depend on cluster size. Therefore any energy independent combination

of different sized clusters will also fail.

C. Fixed Average Number of Clusters

The second way suggested by Levy¹² to account for the energy dependence of particle production is to have a varying cluster size. To investigate this situation, we have set λ_v equal to a constant and $\lambda_s = 0$ so that we are looking at an isovector cluster model having a fixed average number of clusters with limited charge exchange (FANC-LCEX).

The increase of $\langle n \rangle$ with energy must now be achieved by increasing the cluster size. To illustrate this procedure, we refer to Figure 3 and note that if $\lambda_v = 3$ then $\langle n \rangle$ is approximately equal to the size of the cluster. This particularly simple case occurs since λ_v is large enough so that $\tanh(2\lambda_v/3) \approx 1$. Under this condition,

$$\langle n \rangle = (d_1 + 2d_2 + 3d_3 + 4d_4) \lambda_v / 3 \quad (2.5)$$

and we see that small values of $\langle n \rangle$ occur when d_1 dominates over the other d_i . Then as the incident energy increases, first d_2 and then d_3 and d_4 dominate over the other d_i yielding an increase in $\langle n \rangle$.

Since $\langle n \rangle$ has a logarithmic energy increase, the dominance of the d_i s must occur at approximately equal ratios of the energy. As a simple example: if $d_1 = 1$ at $s = s_1$ (all other $d_i = 0$) and $d_2 = 1$ at $s = s_2 = \alpha s_1$, then the energy at which $d_3 = 1$ is given by $s_3 = \alpha s_2$. Noting that experimentally¹⁸

$$\langle n \rangle = 1.03 \ln s - 3.23 \quad (2.6)$$

for s in GeV^2 , we have

$$\alpha = e^{\lambda_v / 3.09} \quad (2.7)$$

In order to learn something about the average number of clusters produced, we compare the predictions of the model with Slattery's KNO curve¹⁷ representing the experimental data. By referring to Figure 5, we find that the experimental value of $\langle n \rangle \approx 3.5$ at 405 GeV/c requires λ_v , the average number of isovector clusters, to be 10.5, 5.25, 3.5, or 2.6 for the case where only 1-particle, 2-particle, 3-particle, or 4-particle clusters are produced. Each of these cases is plotted in Figure 8. Although the conclusion is not decisive, we take the 3-particle clusters as forming the best fit to the data and thus arrive at a value of $\lambda_v \approx 3.5$. It follows from (2.7) that $\alpha \approx 3.10$ and the values of the s_1 where the respective d_1 equal unity²⁰ are given in Table 4. The table also contains calculated values of the two particle correlation coefficient for negative particles, f_2^{--} , at these energies assuming that the respective d_1 equals one. For this crude model we are encouraged by the comparison of this correlation coefficient to the experimental values listed for nearby energies since the two values are changing in the same direction at roughly the same rate, showing that it is possible to obtain the necessary faster than linear rise with $\ln s$ of f_2^{--} in the FANC-LCEX model. We also note that this model, unlike independent cluster models, allows negative values for f_2^{--} at low energies (≤ 50 GeV/c) as required by data.⁶ We also see, of course, a definite need for a more sophisticated energy dependence of the d_1 . We show in Figure 9 the model's predictions for the average number of neutral pions produced for a given number of negative pions $\langle m \rangle_n$ and experimental values at approximately the same energy. We note that the general trends of each are comparable.

To gain further insight into this model, we have investigated an oversimplified choice for the energy dependence of the d_1 in which no more than two different sizes of clusters are produced at one time. The specific energy dependence of the d_1 which is shown in Figure 10 gives complete dominance of the production of each size cluster at the energies listed in Table 4 and causes each type to disappear at the energy at which the next one dominates. This particular choice of the d_1 gives a logarithmic energy dependence to $\langle n \rangle$ under the conditions leading to Equation (2.5) and allows the calculation of the energy dependence of the two particle correlation coefficients. These are compared to experimental values in Figure 11. Again, we note that the trends of the data and predictions are comparable.

D. Variation of Both Number and Size of Clusters

We have shown that the FCS-LCEX model fails on comparison with experiment in two ways, whereas these data can be explained by the FANC-LCEX model. However, there are suggestions within the model that the more general case of a variable number of clusters with a variable size is required. Although this case may prove to be the one closest to describing nature, the increase in the number of parameters causes difficulty in determining them uniquely. Consequently, in this section, we look at two reasons for considering this case but do not consider it in a quantitative fashion.

We look first at the probability for producing no negative or neutral pions which is $P(0,0)$ in Equation (1.21). Using Equations (1.17) and (1.16) with $\lambda_s = 0$, we find

$$P(0,0) = e^{\frac{-\lambda_v/3}{\cosh(2\lambda_v/3)}} \approx 0.06 \quad (2.8)$$

for $\lambda_v = 3.5$. In our model, this is the probability for an inelastic process such as the charge exchange reaction $p + p \rightarrow \Delta^{++} + n$. Dao, et al.²³ have measured the inclusive Δ^{++} production cross section at 303 GeV/c and have found $\sigma(p+p \rightarrow \Delta^{++} + \text{anything})/\sigma_{\text{inel}} \approx 0.14$. Since $P(0,0)$ is the probability for exclusive Δ^{++} production, there is no serious disagreement with this experimental value. However, Regge theory says that this reaction requires Reggeon exchange; thus asymptotically, the cross section should fall as a power of s . In the FANC-LCEX model, $P(0,0) = \text{const.}$ Therefore, λ_v must grow logarithmically with s in order to obtain agreement with this prediction of Regge theory. This model already contained a variation in the cluster size, so we now have both the size and number of clusters growing with energy.

A second indication of the need for this more general case comes from a closer look at the apparent KNO scaling of topological cross sections in the 50-405 GeV/c p-p data. We have used 3-particle clusters in Figure 7 to illustrate the general tendency for the KNO plots of cluster produced pions to become narrower as the average number of clusters (λ) increases. Figure 12 on the other hand, shows the effects of constant λ and varying cluster size on the KNO plot. We point out that an increasing cluster size causes a slight broadening of the plot. Thus, increasing both the size and number of clusters produces effects which can cancel and thereby satisfy to a greater degree the apparent scaling.

These two indications for a more general model show the need for determining the energy dependence of λ . We point out that such dependence can be resolved experimentally by investigating the energy

dependence of the ratio R discussed in Part A. For a constant value of λ_s/λ_v -- which is not unreasonable -- there is a unique relationship between R and λ . Further, the conclusion of Part A that isocalar production is suppressed serves to enhance the sensitivity of R on λ .

CHAPTER III. CONCLUSION

From this investigation, it is easily seen that many experimental parameters are needed to describe completely a LCEX cluster model. The accurate determination of $\langle n \rangle$, the average number of negative pions produced, tells very little about either the cluster production process or subsequent cluster decays. For example, for the sizes of clusters studied, $\langle n \rangle$ is essentially independent of the isospin characteristics of produced clusters. The two particle correlation coefficients, however, depend in a very sensitive fashion on cluster size, the limited charge constraint, and cluster production and decay parameters as seen in Table 3 and Figures 5 and 11. Even though present experimental data do not provide enough information to evaluate uniquely all the parameters of the model, the apparent experimental $\ln^2$ s dependence of f_2^{--} does preclude a fixed cluster size model such as the one considered in Section II B. The parameter R introduced by Pirila et al.¹¹ is found to be extremely important since it is independent of cluster size and subsequent cluster decay and affords a direct means of determining both the average number and isoscalar-isovector admixture of clusters produced.

A model is said to exhibit long range order unless for all orders k the correlation coefficients, defined in Section II B, are of the form

$$f_k = a_k \langle n \rangle + b_k; \quad a_k, b_k \text{ const.} \quad (3-1)$$

In a limited charge exchange model such as we have developed, long range

order enters in two ways. The first is via charge conservation and transfer of charge between clusters. This long range order disappears when large numbers of clusters are produced. The second mechanism for introducing long range order is through an energy dependent cluster size. This mechanism can be seen as resulting from a need for information on the total energy to be passed to each cluster along the multiperipheral chain. The FCS-LCEX model of Section II B has only the first type of long range order while the FANC-LCEX model of Section II C and the more general model of Section II D have both types. The failure of the FCS-LCEX model and the agreement of the FANC-LCEX model on comparison to experiment show the need for a variation of cluster size with energy.

Table 1

Decay of Clusters

<u>Cluster</u>	<u>Final State</u>	<u>No. of Events</u>	<u>Relative Probability</u>
<u>Isoscalar Clusters</u>			
$\bar{C}_2^0$	$\pi^+ \pi^-$	$\bar{n}_{11}$	2/3
	$\pi^0 \pi^0$	$\bar{n}_{12}$	1/3
$\bar{C}_3^0$	$\pi^+ \pi^- \pi^0$		1
$\bar{C}_4^0$	$\pi^+ \pi^+ \pi^- \pi^-$	$\bar{n}_{41}$	$\bar{r}_4$
	$\pi^+ \pi^- \pi^0 \pi^0$	$\bar{n}_{42}$	$\bar{s}_4$
	$\pi^0 \pi^0 \pi^0 \pi^0$	$\bar{n}_{43}$	$\bar{t}_4$
<u>Isovector Clusters</u>			
C_1^+	π^+	u_1^+	1
C_1^0	π^0	v_1	1
C_1^-	π^-	u_1^-	1
C_2^+	$\pi^+ \pi^0$	u_2^+	1
C_2^0	$\pi^+ \pi^-$	v_2	1
C_2^-	$\pi^- \pi^0$	u_2^-	1
C_3^+	$\pi^+ \pi^+ \pi^-$	n_{31}	p_3
C_3^0	$\pi^+ \pi^0 \pi^0$	n_{32}	q_3
	$\pi^+ \pi^- \pi^0$	n_{35}	r_3
C_3^-	$\pi^0 \pi^0 \pi^0$	n_{36}	s_3
	$\pi^+ \pi^- \pi^-$	n_{33}	p_3
	$\pi^- \pi^0 \pi^0$	n_{34}	q_3

Table 1 Con't.

<u>Cluster</u>	<u>Final State</u>	<u>No. of Events</u>	<u>Relative Probability</u>
<u>Isovector Clusters</u>			
C_4^+	$\pi^+ \pi^+ \pi^- \pi^0$	n_{41}	p_4
C_4^0	$\pi^+ \pi^0 \pi^0 \pi^0$	n_{42}	q_4
C_4^-	$\pi^+ \pi^+ \pi^- \pi^-$	n_{45}	r_4
	$\pi^+ \pi^- \pi^0 \pi^0$	n_{46}	s_4
	$\pi^+ \pi^- \pi^- \pi^0$	n_{43}	p_4
	$\pi^- \pi^0 \pi^0 \pi^0$	n_{44}	q_4
	$\pi^- \pi^- \pi^0 \pi^0$		

Dependence of Relative Decay Probabilities
on Intermediate Isospin States

4-Particle Isoscalar Clusters

Relative Probability	Intermediate Isospin States		
	<u>2 + 2</u>	<u>1 + 1</u>	<u>0 + 0</u>
$\bar{r}_4$	19/45	1/3	4/9
$\bar{s}_4$	22/45	2/3	4/9
$\bar{t}_4$	4/45	0	1/9

3-Particle Isovector Clusters

Relative Probability	Intermediate Isospin State		
	<u>2</u>	<u>1</u>	<u>0</u>
p_3	19/30	1/2	2/3
q_3	11/30	1/2	1/3
r_3	11/15	1	2/3
s_3	4/15	0	1/3

4-Particle Isovector Clusters

Relative Probability	Intermediate Isospin States			
	<u>2+2</u>	<u>2+1 & 1+2</u>	<u>1+1</u>	<u>1+0 & 0+1</u>
p_4	3/5	14/15	1	2/3
q_4	2/5	1/15	0	1/3
r_4	4/5	2/15	0	2/3
s_4	1/5	13/15	1	1/3

Table 3

Moments of Multiplicity Distributions in the Limited Charge Exchange Model

$$\langle n \rangle = ((1 + 2d_3p_3 + 2d_4p_4) \tanh(2\lambda_v/3) + d_2 + d_3r_3 + d_4(1 + r_4)) \lambda_v/3 + (2e_2 + 3e_3 + 4e_4) \lambda_s/3$$

$$\langle m \rangle = ((2d_2 + 4d_3(1-p_3) + 2d_4(3-2p_4) \tanh(2\lambda_v/3) + d_1 + d_3(3-2r_3) + 2d_4(1-r_4)) \lambda_v/3 + (2e_2 + 3e_3 + 4e_4) \lambda_s/3$$

$$f_2^{--} = (1 + 2(d_3p_3 + d_4p_4))^2 (1 - \tanh^2(2\lambda_v/3)) \lambda_v^2/9 + ((4(d_3p_3 + d_4p_4) - 1) \tanh(2\lambda_v/3) + 4d_4r_4) \lambda_v/6 + 2e_4 \bar{r}_4 \lambda_s$$

$$f_2^{oo} = (d_2 + 2d_3(1-p_3) + d_4(3-2p_4))^2 (1 - \tanh^2(2\lambda_v/3)) \frac{4\lambda_v^2}{9} + ((4d_3(1-p_3) + 12d_4(1-p_4)) \tanh(2\lambda_v/3)$$

$$+ 6d_3(1-r_3) + 2d_4(1-r_4)) \lambda_v/3 + (2e_2 + 6e_4(\bar{s}_4 + 6\bar{t}_4)) \lambda_s/3$$

$$f_2^{o-} = ((1 + 2d_3p_3 + 2d_4p_4)(d_2 + d_3(1-p_3) + d_4(3-2p_4))) (1 - \tanh^2(2\lambda_v/3)) \frac{2\lambda_v^2}{9} + ((d_2 + 2d_3(1-p_3) + 3d_4)$$

$$\times \tanh(2\lambda_v/3) + d_3r_3 + 2d_4) \lambda_v/3 + (e_3 + 2e_4\bar{s}_4) \lambda_s$$

Effects of the Dominance of Different Sizes
of Isovector Clusters at Different Energies

No. of pions in cluster	Value of s where cluster dominates (GeV ²)	Predicted Value of f_2	Exp. Value of f_2	Value of s at which Exp. performed (GeV ²)
1	79.2	-0.52	-0.04+0.09	95.8
2	245.9	-0.52	0.28+0.07	193.5
3	763.2	0.77-1.23	2.15+ .17	763.2
4	2368.9	2.16-2.91	---	---

Figure 1. Distribution of rapidity gaps between produced particles in pp collisions at Fermi-lab energies (from Ref. 1)

Figure 2. Distribution of rapidity gaps carrying specific charge in 205 GeV/c pp collisions (from Ref. 1)

Figure 3. Multiperipheral chains of clusters which are allowed and forbidden by the limited charge exchange constraint. Charges of clusters and of links between clusters are indicated by $+$, $-$, 0 , for positive, negative, and neutral charges respectively.

Figure 4. The ratio of charged to neutral exchanges between clusters as a function of the average number of clusters produced.

Figure 5. Topological cross section moments for the fixed cluster size LCEX model. The solid lines are for isovector cluster production with $p_3 = 0.67$ or $p_4 = 1$. The dashed lines are for isovector cluster production with $p_3 = 0.5$ or $p_4 = 0.6$.

Figure 6. The experimental energy dependence of f_2^{--} . The line is Whitmore's⁶ fit to the experimental data (see Equation (2.4)).

Figure 7. A KNO plot of topological cross section predictions of the fixed cluster size LCEX model for 3-particle isovector clusters. The curves are straight lines connecting the points for the various n which were calculated with $p_3 = 0.67$.

Figure 8. Comparison of predictions for different sizes of clusters with experimental data. The line is Slattery's empirical fit to experimental data.

Figure 9. The average number of neutral particles produced for a given number of negative pions, $\langle m \rangle_n$, as a function of the number of negative pions for various sizes of isovector clusters. The experimental data are taken from Ref. ²¹

Figure 10. The functional dependence on s of the probability for producing each size of isovector cluster which is used in Section II C.

Figure 11. The dependence of the integrated two particle correlation coefficients on energy for the model of Section II C. The experimental data are from Ref. 22.

Figure 12. A KNO plot of topological cross section predictions showing the small deviation from scaling of the model with the average number of fixed at $\lambda_v = 4$.

REFERENCES

- 1) P. Ciok et al., Nuovo Cimento 8, 166(1958); ibid 10, 741(1958).
- 2) A. Bialas, Proceedings of the IVth International Symposium on Multiparticle Hadrodynamics, Pavia, 1973.
- 3) R. Hagedorn, Nuovo Cimento Supp. 3, 147(1965).
- 4) H.J. M["]ehring et al., Nuc. Phys. B85, 221 (1975); ibid. B87, 509(1975).
- 5) I.M. Dremin and A.M. Dunaevskii, Phys. Repts. 18C, 159(1975).
- 6) J. Whitmore, Phys. Repts. 10C, 274(1974).
- 7) See, e.g., F. Hayot and A. Morel, Nucl. Phys. B59, 493(1973);
A. Arneodo and J. Kubar-Andre, Lett. al Nuovo Cimento 12 1(1975);
D. Drijard and S. Pokorski, Phys. Letts. 43B, 509(1973).
- 8) A.W. Chao and C. Quigg, Phys. Rev. D 9, 2016(1974); S. Pokorski and L. Van Hove, Acta. Phys. Polon. B5, 229(1974).
- 9) ANL-Fermilab-Stony Brook Collaboration, to be published; Michigan-Rochester Collaboration, to be published.
- 10) C. Quigg, P. Pirila, and G.H. Thomas, Phys. Rev. Lett. 34, 290(1975).
- 11) P. Pirila, G.H. Thomas, and C. Quigg, Fermilab Report No. Pub-75/16-THY, 1974(unpublished).
- 12) D. Levy, Nucl. Phys. B59, 583(1973).
- 13) See, e.g., W. Feller, An Introduction to Probability Theory and Its Applications, 3rd Edit. (John Wiley & Sons, New York, 1968), p.38.
- 14) The relations for the total number of exchanges and the number of neutral exchanges are invalid for the case $u=v=0$. When no clusters are produced, both are zero.
- 15) T. Ferbel, Univ. of Rochester Report No. UR-462, 1973 (unpublished).
- 16) Z. Koba, H.B. Nielsen, and P. Olesen, Nucl. Phys. B40, 317(1972).

- 17) P. Slattery, Phys. Rev. D 7, 2073(1973).
- 18) From a fit to the p - p experimental data at 102, 205, 303, and 405 GeV/c - see Ref. ¹⁸
- 19) $P_{lab} = 102$ GeV/c: J. Chapman et al., Phys. Rev. Lett. 29, 1686(1972) and C. Bromberg et al., ibid. 31, 1563(1973); $P_{lab} = 205$ GeV/c: G. Charlton et al., ibid. 29, 515(1972) and S. Barish et al., Phys. Rev. D 9, 2689(1974); $P_{lab} = 303$ GeV/c: F.T. Dao et al., Phys. Rev. Lett. 29, 1627(1972); $P_{lab} = 405$ GeV/c: C. Bromberg, et al., opus cit.
- 20) These are calculated using the value of α from Equation (2.7) which is based on the simplified form for $\langle n \rangle$ in Equation (2.5). This, in turn, was derived by setting $\tanh(2\lambda/3)=1$. This approximation introduces less than a 2% change in the results.
- 21) $P_{lab} = 19$ GeV/c: H. Bøggild, et al., Nucl. Phys. B27, 285(1971); $P_{lab} = 102$ GeV/c: J.W. Chapman, et al., Phys. Lett. 47B, 465(1973); $P_{lab} = 303$ GeV/c: F.T. Dao, et al., Phys. Rev. Lett. 30, 1151(1973); $P_{lab} \approx 1500$ GeV/c: G. Flugge, et al., CERN report submitted to the XVI International Conference on High Energy Physics, Batavia, Illinois (1972), (unpublished).
- 22) $P_{lab} = 24$ GeV/c: V. Blobel, et al., DESY Report No.73/36, 1973 (unpublished); $P_{lab} = 24.12$ GeV/c: D.B. Smith, et al., Phys. Rev. Lett. 23, 1064(1969); $P_{lab} = 50$ and 69 GeV/c: V.V. Ammosov, et al., Phys. Lett. 42B, 519(1972); $P_{lab} = 69$ GeV/c: H. Blumenfeld, et al., Phys. Lett. 45B, 528(1973); $P_{lab} = 200$ and 300 GeV/c: W. Ko, in Proceedings of the XVII International Conference on High Energy Physics, London, 1974, edited by J.R. Smith (The Science Research Council, Rutherford Laboratory, Chilton, Didcot, England, 1974), P. I-76; $P_{lab} = 205$ GeV/c: K. Jaeger, et al., Phys. Rev. D 11, 2405(1975); $P_{lab} = 300$ GeV/c: A. Sheng, et al., Phys. Rev. D 11, 1733(1975); and Ref. 18 & 20.
- 23) F.T. Dao, et al., Phys. Rev. Lett. 30, 34(1973).