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THE NORMAL CONTACT OF ARBITRARILY SHAPED
MULTILAYERED ELASTIC BODIES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of
The Ohio State University

by

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1973

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LIST OF SYMBOLS

a	radius of region about a node
$[a_k^t]$	matrix of layer characteristics
a_{ikj}	influence coefficient, deflection of i^{th} body at k^{th} node due to unit force at j^{th} node
a_L	Hertzian contact major radius
$A_{km}^t, B_{km}^t, C_{km}^t$	constants of integration
$D_{km}^t, E_{km}^t, F_{km}^t$	
$\{e\}$	unit diagonal matrix
f_{ik}, f_{2k}	equations of surface of contacting bodies
F_k	force at k^{th} node
$F_k(\eta)$	transformed function, equation 52
g_1, g_2, g_3, g_4	transformed functions, equation 58
$[I]$	unit column matrix
k	node number
t	layer number
$[M_k]$	transformed matrix, equation 50
N_o	number of nodes in model
N_L	nondimensional load
N	total number of layers
P	total normal load on bodies
$\{P_{km}^t\}$	matrix of transformed displacements and stresses, equation 36

$\{Q_{km}^l\}$	transformed matrix of integration
r, θ, z	coefficients, equation 36, cylindrical coordinated directions, radial, hoop, and normal to the half-space, respectively
R	sphere radius and contact region
$[T_k^l]$	transformed matrix of layer functions, equation 37
$U_{km}^l, V_{km}^l, W_{km}^l$ $R_{km}^l, S_{km}^l, Z_{km}^l$	transformed displacement and stresses, equation 34
$U_k, V_k, W_k,$ W_{1k}, W_{2k}	displacements beneath k^{th} node surface displacement at k^{th} node of bodies 1 and 2, respectively
Y_k	slack variable
α	relative approach of two bodies
η	dummy variable
Δ	dilatation
ϵ_{ij}	strain in i coordinate plane in j coordinate direction
ϵ_k	sum of surface functions, equation 4
τ, μ, ν_l	Lamé material constants, Poisson's ratio l^{th} layer
Γ	artificial variable
φ_k, γ_k	stress functions for k^{th} node
σ_{ijk}	stress on i^{th} coordinate plane in the j^{th} coordinate direction at the k^{th} node
σ_k	uniform normal stress about k^{th} node

σ_L

maximum Hertzian contact stress

$\nabla^2 \nabla^4$

harmonic, biharmonic operators

I. INTRODUCTION

When two bodies are placed in contact, the bodies touch over a compression or contact area. While the initial contact is determined by the geometric features of the bodies, the final contact area is determined by the amount the bodies are deformed by the applied forces. The stresses developed during the contact of elastic bodies has long been of great concern in the design of mechanical systems. The study of the mechanics of contacting elastic bodies is one of the oldest and most basic problems in the theory of elasticity. Questions associated with the contact of elastic bodies have received considerable attention in the literature from 1831 onward. The vast majority of this work has been directed toward homogeneous media. In recent years, a class of problems dealing with the contact of layered elastic bodies has become increasingly important. In modern engineering practice, one frequently meets with stratified structures, laminated composites or solid components coated with surface layers. For example, these problems stem from areas of design of airport runways, roadways, foundations, plated or clad rollers, tracks, ball bearings, electrical contacts, as well as layered shells. In biomedical engineering, for example, problems stem from the study of joints in the skeletal system, as well as design of equipment to measure the intraocular pressure within the eye.

In this thesis, a method was developed that enables the determination of contact area, normal displacements, relative approach and normal contact stress developed during the normal contact of arbitrarily shaped multilayered, isotropic elastic bodies.

The method centers about a programming procedure utilizing a simplex-type algorithm similar to that presented by Conry [44]. The method is restricted to normal surface loading applied to smooth surfaces having continuous first derivatives. The deformations are assumed small and the bodies are assumed to obey the laws of linear elasticity.

II. HISTORICAL BACKGROUND

The first results on the topic of contacting elastic bodies in the theory of elasticity were obtained in the last century. The first calculations directed toward determining the stresses during the contact of two bodies were obtained by Lamé and Clapeyron [1] in 1831. The general theory of contact of elastic bodies was formulated later (1881) by Heinrich Hertz [2]. Hertz [2], [3] assumed the bodies to be "flat" enough in the neighborhood of the area of contact to be treated by analytical methods of potential theory available for semi-infinite half spaces. He assumed that the actual shape was given by a general second-degree curve. His solutions, arrived at by means of the semi-inverse method, have long served as the bases of research into contact problems.

The next significant contribution to the subject was made in 1885 by Boussinesq[4]. Boussinesq solved a number of important cases including the classic problem of a normal concentrated load acting on the surface of a semi-infinite solid. This was followed in 1902 by Lamb's solution [5] of the case of a symmetrically distributed load on the surface of a semi-infinite elastic solid. Boussinesq also considered the contact of an absolutely rigid body (a punch) with a circular base against an elastic semi-infinite body. This treatment served as the basis of considerable work that followed in later literature dealing with problems of the indentation of an

elastic solid by a rigid stamp or punch. In the Russian literature, the elastic contact problem is often approached from this viewpoint, or simply as a problem of surface loading of a half space.

The group of problems made up by (1) displacement boundary conditions problems (punch problems) and (2) force boundary condition problems, make up the majority of the problems referred to as contact problems. The advancements in these areas will not be covered here except where they concern themselves with multilayered media. Two books in Russian are entirely devoted to the field of contact problems, Galin [6], and Shtaerman [7], and a third Russian book on elasticity theory treats them at length, Lurye [8]. The books by Galin and Lurye are available in English translation. These, in addition to a recent paper by Abramyan [9], provided an excellent survey of work done in the area of the punch and force boundary value problems through 1969. Also, such standard references as Timoshenko [10], Love [11], and Seely and Smith [12], can be used for additional details.

The first extensive evaluation of the stress field arising in general Hertz contact is due to Belyayev [13-17], (1917-1929). Other papers concerning the analysis of the state of stress due to normal contact which came later include the works of Dinnik [18] (1906), Way [19] (1935), Karas [20] (1940), Radzinovsky [21] (1953), Palmgren and Sundberg [22] (1919), Thomas and Hoersch [23] (1930), Weibull [24] (1919), Lundberg and Odqvist [25] (1932), and Fessler and Ollerton [26] (1957). The latter work of Fessler and Ollerton verifies solutions of the former investigators as well as putting forth

a theory of failure due to rolling contact as was first introduced in [21]. This theory remains the most reliable method in predicting fatigue of rolling contact. More recently, Hamilton and Goodman in 1966 [27], examined effects of tangential loadings on Hertzian contacts. Their results have been experimentally verified by Lawn (1967) [28].

Probably the most significant extension of the basic work of the Hertz contact theory consists of the addition of surface tractions to the bodies. This problem was first treated by Cattaneo in 1938 [29] and later in 1949 independently by Mindlin [30]. By an argument based upon considerations of symmetry, Mindlin demonstrates that the tangential force problem can be solved without appreciably affecting the normal pressure distributions of the Hertz problem. It was found that the magnitude of traction at actual interface rises from one-half the average at the center of the contact surface to infinity at the edge. Due to this theoretically infinite traction, there is an annulus of slip. These results were later verified experimentally by Johnson (1961) [31, 32, 33] and are also discussed at length by Lubkin (1962) [34], Mindlin, et al (1951) [35], and Mow et al (1967) [36].

Outside of the work of Mindlin, little has been done of a fundamental nature in the study of contact of elastic bodies. Some work however, has been done in the study of contact of anisotropic bodies. Willis (1966) [37], points out that complete solutions for general anisotropy cannot be found analytically, however, the functional form can be found explicitly. By using Fourier transform methods, he presents some explicit formula for transversely isotropic media.

Planer contact problems have been treated by Galin [6] and Green and Zerna [38]. It was brought out by Chen (1969) [39] that in certain situations, the stress functions for the isotropic and anisotropic materials are of the same form. Additional work in this area was written by Conway (1955-1956) [40-42] and Cooper (1969) [43].

Perhaps the most significant contributions to the topic of contact problems was made by Conry (1970) [44]. Realizing the intractable nature of problems involving bodies with more complex geometries than the ellipsoids of Hertz, Conry formulated a programming procedure utilizing a simplex-type algorithm. The procedure makes possible approximate solutions to the contact of homogeneous bodies of arbitrary geometry never before possible.

The first treatment of layered media was performed by Hertz in 1884 [3]. Hertz analyzed a force boundary value problem as opposed to a contact problem. No further treatments can be found until 1923 when Westergaard [45] began an exhaustive series of papers that was to stretch over 15 years. Also during this period 1933 to 1938, Marguerre [45], Biot [47], Pickett [48], and Passer [49] developed formulas for stresses and displacements for the one layer system, resting on a rigid base.

In 1938, Hogg [50] and, independently, Hall [51] solved the problem of a thin slab of infinite size supported by a semi-infinite elastic solid. Their solutions were restricted to symmetrically loaded areas. Solutions for a non-symmetrical loading were not developed until 1947 by Volterra [52]. The approach taken to the contact problem of layered media by all investigators from Westergaard (1923)

to Volterra (1947) is to model the problems by plates on an elastic foundation (also see Pickett [53]). This model, however, has inherent deficiencies. The foundation reaction (per unit area) at any point is assumed proportional to the layer deflection at that point, and independent of layer deflections at other points. However, the actual expression for substrate reaction for a layered elastic solid is an integral equation.

Burmister (1943) [54] was the first to consider each layer as an elastic solid. He considered a two-layer system subjected to an axially symmetric loading with a zero-th order Bessel function variation in the radial direction. The free surface was assumed traction free and continuity at the layer's interface was enforced. In 1945, he extended his theory to the three-layered system [55].

In 1948, Fox [56] developed explicit solutions for the two-layer case in the form of integrals for the computation of stresses on the axis of loading utilizing Burmister's general equations for the two-layer system. The integrals were evaluated by mechanical quadrature. In 1951, Acum and Fox [57] attempted to extend this work to calculate the stresses for the three-layered problem with little success. Bufler (1959) [58, 59] and Sundara Raja Lyengar (1961) [60] using the Fourier transform method performed an analysis of a two-layered elastic solid like Burmister. No numerical results were given; however, Lemcoe (1960) [61] gave limited numerical results due to sinusoidal loading of a two-layered system. Barovich et al (1964) [62] calculated the stresses within a two-layer system due to normal and tangential loading in the form of Hertzian stress distributions. Various com-

binations of layer and substrate modulus and Poisson's ratios were examined. The stresses developed in the two-layer system due to surface tractions were calculated by Westmann (1963) [63], also using Fourier transform methods.

In 1969, Kuo [64] by using a matrix method frequently, used in elastodynamics obtained a generalized solution to the problem of a multilayered medium under static loads on the surface of the medium.

Other work in the area of layered media includes the writings of Dundurs (1970) [65] and Keer (1973) [66] who while studying receding contacts considered contact between an elastic layer and a half-space when the two bodies are pressed together. Paralleling this work from slightly different approach are the works of Westmann (1969) [67] who approached it from the standpoint of a thin plate and Pu and Hussian (1970) [68] who used a variational method in obtaining the solution.

Much recent literature (within the last 5 years) has been generated concerning layered media under the action of punch loads. The normal procedure in such problems is to formulate a mixed boundary value problem as a dual integral equation and reduce it into a Fredholm integral equation which can be then solved approximately. Some early work in two dimensional problems of a two-layered system was performed by Wu and Chui [71]. Some earlier work performed by Low (1964) [69-70] solved the problem of a flat punch applied to a layer resting on a rigid half-space.

A rigid stratum supporting an elastic layer is a mathematical fiction that simplified the analysis. Some recent investigators

have considered indentation of an elastic layer bonded to an elastic half-space of a different material. Dhaliwal (1970) [72] was able to reduce the axisymmetrical problem of a flat ended cylindrical punch on a layered elastic medium to a Fredholm integral equation which he solved approximately. Subsequently Dhaliwal and Rau (1970) [73] extended the analysis of Dhaliwal to punches of arbitrary profile, but did not present any numerical results. Independently Abbas and Kuipers (1969-1971) [74-76] gave approximate numerical results for the two-dimensional problem of a rectangular and axisymmetric cylindrical stamp on an elastic layer overlying a rigid substrate.

Rau and Dhaliwal presented numerical results (1972) [77] to the analytical solutions developed earlier [72-73] for cylindrical, conical, paraboloidal and spherical punches. Some of this had been presented independently by Tolkacheva (1970) [78]. Yefimov, Malyi, Tolkacheva (1969) [79] have followed up some of this earlier work of contact of a body having a thin layer with a rigid stamp or die. Recently, Chen and Engle (1972) [80] also obtained solutions to Dhaliwal's flat and spherical punch for two and three-layer problems.

More recently, work in layered media has been done by Johns (1973) [81] who studied the effect of layer thickness and nearness of adjacent contact spots on the stresses within a layer attached to a rigid foundation. Further, Gupta and Walowit (1973) [82] developed a plane strain solution to the contact of a two-layered elastic half space and a cylindrical elastic indenter.

In this work is presented a method of determining the contact area and normal contact stress developed during the normal contact of

arbitrary shaped, layered elastic bodies. Contact stress distributions resulting from the contact of layered spheres is presented in detail. Also represented is the problem of a flat circular punch on a layered half space. The method assumes that the proposed zone of contact is divided into a regular array of finite areas over which acts a uniform pressure of unknown magnitude. Equations describing the displacements within each of the areas are solved by incorporating a programming procedure utilizing a simplex-type algorithm. The displacements are related to the pressure across each area by influence coefficients. The influence coefficients are determined for multilayered media using methods of analysis of axisymmetric problem of elasticity by Haskell transforms.

III. MATHEMATICAL FORMULATION

A. Formulation of General Equations

Consider two bodies pressed together; let Points (x, y, z_1) and (x, y, z_2) on the surfaces of body 1 and body 2 (Figure 1), respectively come in contact within the compressed area. Then for points within the compressed area

$$W_1 + W_2 = \alpha - (F_1 + f_2) . \quad (1)$$

In that points outside the contact area are separated it must be true that

$$W_1 + W_2 > \alpha - (F_1 + f_2) \quad (2)$$

where

α = relative approach of the two bodies

W_1, W_2 = normal displacement of surface points of
bodies 1 and 2 respectively, with respect
to undeformed (but moving) surfaces

f_1, f_2 = equations of the surfaces of bodies 1 and 2,
respectively, before deformation.

It will be assumed that the contacting bodies are isotropic and linearly elastic and that representative dimensions of the contact area are small compared to the radii of curvature of the undeformed bodies in the vicinity of the contact interface.

The latter assumption led Hertz [2] to make the further simpli-

Figure 1. Contact of Two Arbitrarily Shaped Elastic Bodies

fication that near the contact zone it is sufficiently accurate to represent the actual shape of the two nearly flat bodies by general surfaces of second degree, i.e.

$$f_1 = A_1 x^2 + B_1 y^2 + H_1 xy$$

and

$$f_2 = A_2 x^2 + B_2 y^2 + H_2 xy.$$

By meeting displacement boundary conditions of equations (1) and (2) in addition to the stress conditions on the plane $z_1 = z_2 = 0$, that the tractions are zero, i.e.

$$1\sigma_{zx} = 1\sigma_{zy} = 2\sigma_{zx} = 2\sigma_{zy} = 0$$

and the equilibrium condition that

$$\int_R 1\sigma_{zz} dA = \int_R 2\sigma_{zz} dA = P \quad (3)$$

where

R = the contact region

P = the normal load forcing the bodies together.

A solution for the contact problem may be obtained for simple geometries involving homogeneous bodies.

Let the functions describing the surface geometry be general but have continuous first derivatives. And, like the surfaces of Hertz let the radii of curvature of the bodies be sufficiently large (i.e., the bodies sufficiently flat) that in the neighborhood of the contact surface the analytical methods available for semi-infinite

solids may be employed.

Assume that a proposed zone of contact is divided into a regular array of finite square areas (Figure 2). At the center of each area is a node. If a particular node lies within the area of contact, the sum of the elastic deformations and any initial separations must be equal to the rigid body approach of the bodies, or for the k^{th} node

$$W_{1k} + W_{2k} + \epsilon_k - \alpha = 0 \quad (4)$$

where

W_{1k}, W_{2k} = the normal deformations of bodies 1 and 2 respectively at the k^{th} node

$\epsilon_k = f_{1k} + f_{2k}$ = the initial separation at the k^{th} node.

Outside the contact area

$$W_{1k} + W_{2k} + \epsilon_k - \alpha > 0. \quad (5)$$

Equations (4) and (5) can be combined into a single equation by introducing new variables, called "slack variables", which are either positive or zero. Equation (5) can then be written as

$$W_{1k} + W_{2k} + \epsilon_k - \alpha - Y_k = 0 \quad (6)$$

where

$$Y_k \geq 0 \quad (7)$$

is the slack variable.

Let F_j denote concentrated forces acting at each of N_0 possible contact nodes. For equilibrium the sum of all F_j must equal the total normal force, P , forcing the bodies together, i.e.,

Figure 2. Proposed Zone of Contact Showing Regular Array of Nodes Representing Potential Contact Spots

$$\sum_{j=1}^{N_0} F_j = P \quad (8)$$

The contact problem is characterized by equations (6), (7), and (8) in addition to the criterion that inside the contact region, R ,

$$Y_k = 0 \text{ and } F_k \geq 0 \text{ in } R \quad (9)$$

outside the contact region

$$Y_k \geq 0 \text{ and } F_k = 0 \text{ outside } R. \quad (10)$$

A solution to a discrete contact problem is, therefore, a set of forces which satisfies the boundary and continuity conditions [6], [7], [8], [9], and [10], and the field equations of elasticity of the two contacting bodies.

The continuous pressure distribution is approximated by a set of uniform pressures acting over small areas about each of the nodes. The product of the pressure and the area is F_k . Since both bodies are assumed to follow laws of linear elasticity the elastic deflection at a point k is the superposition of the influence of all the forces F_j ($j = 1, 2, \dots, N_0$). This is expressed by

$$W_{ik} = \sum_{j=1}^{N_0} a_{ikj} F_j \quad (11)$$

where a_{ikj} is the deflection of the i^{th} body at point k due to the force F_j distributed over the area about the j^{th} node, i.e., a_{ikj} is an influence coefficient describing the deflection at a node k due to

a unit load applied at node j . The a 's will be determined later.

The contact problem is then defined in matrix notation by the following. Find vectors $\{F\}$, $\{Y\}$ and the scalar α such that (from equations (6) and (8))

$$\begin{aligned} -[G] \{F\} + \alpha \{e\} + [I] \{Y\} &= \{e\} \\ \{e\}^T \{F\} &= P \end{aligned} \quad (12)$$

where

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} = 0 \quad \begin{pmatrix} F_1 \\ \vdots \\ F_k \end{pmatrix} > 0 \quad \text{nodes } (1, 2, \dots, k) \text{ in } R$$

$$\begin{pmatrix} Y_{k+1} \\ \vdots \\ Y_n \end{pmatrix} > 0 \quad \begin{pmatrix} F_{k+1} \\ \vdots \\ F_N \end{pmatrix} = 0 \quad \text{nodes } (k+1, k+2, \dots, N_o) \text{ outside } R$$

and where

$$\bar{G}_{kj} = \sum_{i=1,2} a_{ikj} \quad (13)$$

and where $\{e\}$ and $[I]$ are the unit column and diagonal matrices, respectively.

The solution to equation (12) is then two column vectors (F, Y) and a scalar (α) subject to the conditions that either $F_k = 0$ or $Y_k = 0$ and such that e and P are ≥ 0 . These solutions are called a basis and represent a unique solution to the contact problem [44].

One method of solution of equations (12) is that of the method of enumeration where all the possible combinations of bases are listed as sets of linear equations. After solving them all one set of (F, α, Y) which satisfy equation (12) are chosen. Even with the aid of modern high speed computers this method is still economically impractical.

An efficient method for sorting through the various combinations of bases to find a unique feasible solution is that of a modified Eximplex Algorithm. The Simplex method [84] of linear programming consists of finding values for a set of non-negative variables that satisfies a system of linear equations and which minimizes some functional.

The original problem may be replaced with one involving artificial variables, $\Gamma_1, \Gamma_2, \dots, \Gamma_{N+1}$, and an objective function Γ_0 to be minimized, where the objective function is given by

$$\Gamma_0 = \sum_{j=1}^{N_0+1} \Gamma_j \quad (14)$$

and

$$-[G] \{F\} + \alpha \{e\} + [I] \{Y\} + [I] \{\Gamma\} = [e] \\ \{e\}^T \{F\} + \Gamma_{N_0+1} + P \quad (15)$$

with either

$$F_k = 0$$

or

$$Y_k = 0$$

and

$$\begin{aligned} F_k \geq 0 \quad Y_k \geq 0 \quad \alpha \geq 0 \quad k = 1, 2, \dots N_0 \\ \Gamma_k \geq 0 \quad k = 1, 2, \dots N_0 + 1 \end{aligned} \quad (16)$$

The Simplex Algorithm is always initiated with a program whose equations are in canonical form [83]. To put equations (15) into canonical form subtract the first $N_0 + 1$ equations from the $N_0 + 2$ equation. The result is shown in tableau form in Table 1. The Simplex Algorithm for linear programming can be used by a modification of the entry rules like those by Wolf [84].

To reduce the value of Γ_0 , an element is picked, say C_s from the $N_0 + 2$ row of Table 1 which is the minimum negative of all the C_k . If the corresponding variable X_s is increased to any positive value (the other non-basic variables remaining zero) it is evident that the value of Γ_0 would be reduced by $\Gamma_0 + C_s X_s$. It is best, therefore, to try to make X_s as large as possible. The value of X_s cannot be increased indefinitely while the other non-basic variables remain zero, because the corresponding values of the basic variables are governed by the constraints of the problem; i.e., the other elements of the basis. To keep the values of the other members of the basis non-negative the value of X_s is taken such that at least one of these members has a value of zero and all the remaining members are positive. Therefore, if a_{is} is an element in the i^{th} row and the s^{th} column of the tableau and b_i is the i^{th} element of the right hand side of the tableau, then the greatest value of X_s permissible will be given by

$$\begin{array}{cccccccccccccccccccc}
F_1 & F_2 & \cdots & F_{N_0} & a & Y_1 & Y_2 & \cdots & Y_{N_0} & \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{N_0} & \Gamma_{N_0+1} \\
-\bar{G}_{11} & -\bar{G}_{12} & \cdots & -\bar{G}_{1N_0} & 1 & & & & & & & & & & = \epsilon_1 \\
-\bar{G}_{21} & -\bar{G}_{22} & \cdots & -\bar{G}_{2N_0} & & 1 & & & & & & & & & = \epsilon_2 \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & & & & & & & & \vdots \\
-\bar{G}_{N_0 1} & -\bar{G}_{N_0 2} & \cdots & -\bar{G}_{N_0 N_0} & & & & & 1 & & & & & & = \epsilon_{N_0} \\
1 & 1 & \cdots & 1 & & & & & & & & & 1 & & = P \\
C_1 & C_2 & \cdots & C_{N_0} & C_{N_0+1} & C_{N_0+2} & C_{N_0+3} & \cdots & C_{2N_0+1} & & & & & & = \Gamma_0 - \Gamma_1
\end{array}$$

Where

$$\Gamma_1 = P + \sum_{j=1}^{N_0} \epsilon_j$$

$$C_k = \sum_{j=1}^{N_0} \bar{G}_{jk} - 1 \quad k=1,2,\dots,N_0$$

$$C_{N_0+1} = -N_0$$

$$C_k = -1 \quad k=N_0+2, N_0+3, \dots, 2N_0+1$$

TABLE 1. CANONICAL FORM OF EQUATION 15 IN TABLEAU FORM

the minimum value of $\frac{b_i}{a_{is}}$ or, if this value is in the r^{th} row, $\frac{b_r}{a_{rs}}$.

In the above X_s is the entering variable and is said to be entering the basis. When some F_k , Y_k , or α is chosen to enter, the variable leaving the basis corresponds to the unit vector with a value 1 in its r^{th} element.

The conditions placed on the entering variable (therefore the modification to the Simplex Algorithm of Wolf [84]) are (1) suppose the entering variable was chosen as F_s . Then if Y_s corresponding to F_s is not already in the basis then F_s is free to enter. (2) If Y_s does not correspond to row r , F_s may not enter the basis and a new entering variable must be chosen. The new entering variable corresponds to the minimum of all the C_k except C_s . The process is repeated until a variable can enter the basis.

Replacement of the variables is accomplished by an operation called pivoting. The pivoting operation replaces the system of equations by an equivalent system in which a specific variable has a coefficient of unity in one equation and zero elsewhere. A flow diagram of the modified Simplex Algorithm is shown in Figure 3.

The Simplex Algorithm will converge to the unique feasible point in at most $\frac{3}{2} (N_0 + 1)$ cycles with the majority of cases converging in $N_0 + 1$.

B. Formulation of Influence Coefficients

The mathematical formulation of this method of treating contacting layered bodies is completed by formulating the influence coefficients (a_{ikj}) equation (11).

Figure 3. Flow Diagram for the Simplex-Type Algorithm

We begin by replacing the array of uniformly loaded squares shown in Figure 2 by an array of uniformly loaded circles, Figure 4. The stress on each circular area (say at node k) is the static equivalent of the associated nodal force, (F_k) . Or

$$F_k = \pi a^2 \left[\sigma_{zz} \right]_{\text{at } k} . \quad (17)$$

An error is introduced by this approach in the influence coefficients. The error is greatest (12.4 percent) in the coefficient associated with the deflection directly beneath the nodal force (a_{ikk}). The error, however, quickly decreases for the influence coefficients of the neighboring nodes (3.4 percent) and even less for distant nodes due to Saint-Venant's Principle. In that the solution to the contact problem (the set of nodal forces F_k , $k = 1, 2, \dots, N_o$) is dependent upon a large number of nodes, the error introduced in the determined contact stress will be shown to be very small (~ 1.4 percent, Figure 6).

The influence coefficients are determined from the expression for the deflection at some node k due to a uniform pressure applied to a single circular region about node j . The response of a multi-layered medium under normal or inclined surface loads was first examined by Kuo [64]. In his work Kuo followed the method of analysis of Sneddon and Harding [85] of three-dimensional axisymmetric and asymmetric problems of elasticity by Hankel Transforms. Through the use of the matrix method of Thomson-Haskell [86], the coefficients of the stress functions were obtained in integral form and were adaptable to numerical evaluation.

To determine the general influence coefficient (a_{ikj}) then consider the following approach. Assume that each layer of the multilayered medium is elastic, homogeneous, and isotropic and that the interfaces of the layers are perfectly welded. Let the origin of the cylindrical coordinates coincide with the top of the l^{th} layer of the multilayered medium and with the z axis directed into the layer below (Figure 5).

The equations of equilibrium for any layer, l , under no body forces in cylindrical coordinated (r, θ, z) are*

$$\begin{aligned} \frac{\partial \sigma_{rrk}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta k}}{\partial \theta} + \frac{\partial \tau_{rzk}}{\partial z} + \frac{\sigma_{rrk} \sigma_{\theta\theta k}}{r} &= 0 \\ \frac{\partial \tau_{r\theta k}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta k}}{\partial \theta} + \frac{\partial \tau_{\theta zk}}{\partial z} + 2 \frac{\tau_{r\theta k}}{r} &= 0 \\ \frac{\partial \tau_{rzk}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta zk}}{\partial \theta} + \frac{\partial \sigma_{zzk}}{\partial z} + \frac{\tau_{rzk}}{r} &= 0 \end{aligned} \quad (18)$$

The equations of equilibrium in terms of displacements u_k , v_k , w_k in the r , θ , and z directions, respectively, are

$$\begin{aligned} \nabla^2 u_k + \frac{1}{1-2\nu} \frac{\partial \Delta}{\partial r} - \frac{1}{r} \left(2 \frac{\partial v_k}{r \partial \theta} + \frac{u_k}{r} \right) &= 0 \\ \nabla^2 v_k + \frac{1}{1-2\nu} \frac{\partial \Delta}{r \partial \theta} - \frac{1}{r} \left(\frac{v_k}{r} - 2 \frac{\partial u_k}{r \partial \theta} \right) &= 0 \end{aligned} \quad (19)$$

* Here the coordinates, r , θ , and z , should be taken from the k^{th} node. For simplicity no subscripts will be carried on the coordinate, but it should be kept in mind that we are considering a typical node.

Figure 5. Multilayered Half-Space

$$\nabla^2 w_k + \frac{1}{1-2\nu} \frac{\partial \Delta}{\partial z} = 0, \quad (19 \text{ continued})$$

where ∇^2 and ν denote the harmonic operator and Poisson's ratio, respectively, and Δ denotes the dilatation, which is given by

$$\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = \frac{\partial u_k}{\partial r} + \frac{u_k}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial w_k}{\partial z}. \quad (20)$$

It is verified by direct substitution that the equations of equilibrium (19) are satisfied identically if stress functions φ_k and γ_k are chosen such that

$$\begin{aligned} u_k(r, \theta, z) &= -\frac{\partial^2 \varphi_k}{\partial r \partial z} + \frac{2}{r} \frac{\partial \gamma_k}{\partial \theta} \\ v_k(r, \theta, z) &= -\frac{\partial^2 \varphi_k}{r \partial \theta \partial z} - 2 \frac{\partial \gamma_k}{\partial r} \\ w_k(r, \theta, z) &= 2(1 - \nu) \nabla^2 \varphi_k - \frac{\partial^2 \varphi_k}{\partial z^2} \end{aligned} \quad (21)$$

where

$$\nabla^4 \varphi_k = 0$$

and

$$\nabla^2 \gamma_k = 0.$$

The stress field corresponding to the displacement field given in (21) are found by substituting the displacements of equation (21) into the strain displacement relations

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial U_k}{\partial r} \\
\epsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial V_k}{\partial \theta} + \frac{U_k}{r} \\
\epsilon_{zz} &= \frac{\partial W_k}{\partial z} \\
\epsilon_{r\theta} &= \frac{\partial V_k}{\partial r} - \frac{V_k}{r} + \frac{1}{r} \frac{\partial U_k}{\partial \theta} \\
\epsilon_{rz} &= \frac{\partial U_k}{\partial z} + \frac{\partial W_k}{\partial r} \\
\epsilon_{\theta z} &= \frac{1}{r} \frac{\partial W_k}{\partial \theta} + \frac{\partial V_k}{\partial z}
\end{aligned} \tag{23}$$

and into the isotropic Hooke's Law expressions

$$\begin{aligned}
\sigma_{rr_k} &= \lambda \Delta + 2\mu \epsilon_{rr} \\
\sigma_{\theta\theta_k} &= \lambda \Delta + 2\mu \epsilon_{\theta\theta} \\
\sigma_{zz_k} &= \lambda \Delta + 2\mu \epsilon_{zz} \\
\tau_{r\theta k} &= \tau_{\theta r k} = \mu \epsilon_r \\
\tau_{rz k} &= \tau_{zr k} = \mu \epsilon_{rz} \\
\tau_{\theta z k} &= \tau_{z\theta k} = \mu \epsilon_{\theta z}
\end{aligned} \tag{24}$$

where λ and μ are Lamé's constants. In this way stress field is found in terms of the stress functions φ_k and γ_k to be

$$\begin{aligned}
\frac{\sigma_{rrk}}{2\mu} &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \varphi_k - \frac{\partial^2 \varphi_k}{\partial r^2} \right) + \frac{2}{r} \frac{\partial^2 \gamma_k}{\partial \theta \partial r} - \frac{2}{r} \frac{\partial \gamma_k}{\partial \theta} \\
\frac{\sigma_{\theta\theta k}}{2\mu} &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \varphi_k - \frac{1}{r} \frac{\partial \varphi_k}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \varphi_k}{\partial \theta^2} - \frac{2}{r} \frac{\partial^2 \gamma_k}{\partial \theta \partial r} + \frac{2}{r^2} + \frac{\partial \gamma_k}{\partial \theta} \\
\frac{\sigma_{zzk}}{2\mu} &= \frac{\partial}{\partial z} \left[(2-\nu) \nabla^2 \varphi_k - \frac{\partial^2 \varphi_k}{\partial z^2} \right] \\
\frac{\tau_{\theta zk}}{2\mu} &= \frac{\partial}{r \partial \theta} \left[(1-\nu) \nabla^2 \varphi_k - \frac{\partial^2 \varphi_k}{\partial z^2} \right] - \frac{\partial^2 \gamma_k}{\partial r \partial z} \\
\frac{\tau_{zrk}}{2\mu} &= \frac{\partial}{\partial r} \left[(1-\nu) \nabla^2 \varphi_k - \frac{\partial^2 \varphi_k}{\partial z^2} \right] + \frac{\partial^2 \gamma_k}{r \partial \theta \partial z} \\
\frac{\tau_{r\theta k}}{2\mu} &= \frac{\partial^2}{r \partial \theta \partial z} \left[\frac{\varphi_k}{r} - \frac{\partial \varphi_k}{\partial r} \right] - 2 \frac{\partial^2 \gamma_k}{\partial r^2} - \frac{\partial^2 \gamma_k}{\partial z^2}
\end{aligned} \tag{25}$$

The biharmonic function φ_k and the harmonic function γ_k may be written in series form as

$$\begin{aligned}
\varphi_k(r, \theta, z) &= \sum_{m=0}^{\infty} \left[\varphi_{km}(r, z) \cos m \theta - \bar{\varphi}_{km}(r, z) \sin m \theta \right] \\
\gamma_k(r, \theta, z) &= \sum_{m=0}^{\infty} \left[\gamma_{km}(r, z) \sin m \theta + \bar{\gamma}_{km}(r, z) \cos m \theta \right]
\end{aligned} \tag{26}$$

Assuming $\bar{\varphi}_m = \bar{\gamma}_m = 0$ and considering only a single value of m in (27) can be done without loss of generality. This is true as long as the y axis is an axis of symmetry. It follows upon substituting (26) into (24) that φ_{km} and γ_{km} are solutions to

$$\begin{aligned}
\nabla_m^4 \varphi_{km} &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} \right)^2 \varphi_{km} = 0 \\
\nabla_m^2 \gamma_{km} &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2} \right) \gamma_{km} = 0
\end{aligned} \tag{27}$$

Using the formulae for Bessel functions [87] it can be shown that

$$\begin{aligned}
\int_0^\infty r \nabla_m^4 \varphi_{km} J_m(\eta r) dr &= \left(\frac{d^2}{dz^2} - \eta^2 \right)^2 \int_0^\infty r \varphi_{km} J_m(\eta r) dr = 0 \\
\int_0^\infty r \nabla_m^2 \gamma_{km} J_m(\eta r) dr &= \left(\frac{d^2}{dz^2} - \eta^2 \right) \int_0^\infty r \gamma_{km} J_m(\eta r) dr = 0
\end{aligned} \tag{28}$$

By defining

$$\begin{aligned}
G_{km}(\eta, z) &= \int_0^\infty r \varphi_{km} J_m(\eta, r) dr \\
H_{km}(\eta, z) &= \int_0^\infty r \gamma_{km} J_m(\eta, r) dr
\end{aligned} \tag{29}$$

that is, the Hankel functions, it follows that for φ_{km} and γ_{km} to be solutions to equations (27) then the Hankel functions $G_{km}(\eta, z)$ and $H_{km}(\eta, z)$, in light of (28), must be solutions to the ordinary differential equations

$$\begin{aligned}
&\left(\frac{d^2}{dz^2} - \eta^2 \right)^2 G_{km}(\eta, z) = 0 \\
&\left(\frac{d^2}{dz^2} - \eta^2 \right) H_{km}(\eta, z) = 0
\end{aligned} \tag{30}$$

The general solutions to (30) for the ℓ^{th} layer are

$$\begin{aligned} G_{km}^{\ell}(\eta, z) &= \left[(A_{km}^{\ell} + z B_{km}^{\ell}) e^{\eta z} + (C_{km}^{\ell} + z D_{km}^{\ell}) e^{-\eta z} \right] \\ H_{km}^{\ell}(\eta, z) &= E_{km}^{\ell} e^{\eta z} + F_{km}^{\ell} e^{-\eta z} \end{aligned} \quad (31)$$

or in terms of the hyperbolic functions

$$\begin{aligned} G_{km}^{\ell}(\eta, z) &= \left[(A_{km}^{\ell} + C_{km}^{\ell}) + z (B_{km}^{\ell} + D_{km}^{\ell}) \right] \cosh \eta z \\ &+ \left[(A_{km}^{\ell} - C_{km}^{\ell}) + z (B_{km}^{\ell} - D_{km}^{\ell}) \right] \sinh \eta z \end{aligned} \quad (32)$$

$$H_{km}^{\ell}(\eta, z) = (E_{km}^{\ell} + F_{km}^{\ell}) \cosh \eta z + (E_{km}^{\ell} - F_{km}^{\ell}) \sinh \eta z$$

where the arbitrary constants $A_{km}^{\ell} \dots F_{km}^{\ell}$ will be determined by the boundary conditions. Once $G_{km}^{\ell}(\eta, z)$ and $H_{km}^{\ell}(\eta, z)$ have been determined, the expressions for φ_{km}^{ℓ} and γ_{km}^{ℓ} are obtained by means of the Hankel inversion theorem

$$\begin{aligned} \varphi_{km}^{\ell}(r, z) &= \int_0^{\infty} \eta G_{km}^{\ell}(\eta, z) J_m(\eta, r) d\eta \\ \gamma_{km}^{\ell}(r, z) &= \int_0^{\infty} \eta H_{km}^{\ell}(\eta, z) J_m(\eta, r) d\eta \end{aligned}$$

The components of displacements and stresses for a multilayered medium for the ℓ^{th} layer may be obtained by substituting (26) into (21) and (25) which results in

$$U_{km}^{\ell}(r, \theta, z) = \sum_{m=0}^{\infty} \int_0^{\infty} \left[-U_{km}^{\ell} J_m'(\eta r) + \frac{2m}{\eta r} V_{km}^{\ell} J_m(\eta r) \right] \eta d\eta \cos m \theta \quad (a)$$

$$V_{km}^{\ell}(r, \theta, z) = \sum_{m=0}^{\infty} \int_0^{\infty} \left[\frac{m}{\eta r} U_{km}^{\ell} J_m(\eta r) + 2 V_{km}^{\ell} J_m'(\eta r) \right] \eta d\eta \sin m \theta \quad (b)$$

$$W_k^{\ell}(r, \theta, z) = \sum_{m=0}^{\infty} \int_0^{\infty} \left[W_{km}^{\ell} J_m(\eta r) \right] \eta d\eta \cos m \theta \quad (c)$$

$$\tau_{zr_k}^{\ell}(r, \theta, z) = - \sum_{m=0}^{\infty} \int_0^{\infty} \left[R_{km}^{\ell} J_m'(\eta r) - \frac{m}{\eta r} S_{km}^{\ell} J_m(\eta r) \right] \eta d\eta \cos m \theta \quad (d)$$

$$\tau_{z\theta_{km}}^{\ell}(r, \theta, z) = \sum_{m=0}^{\infty} \int_0^{\infty} \left[\frac{m}{\eta r} R_{km}^{\ell} J_m(\eta r) + S_{km}^{\ell} J_m'(\eta r) \right] \eta d\eta \sin m \theta \quad (e)$$

$$\sigma_{zz_k}^{\ell}(r, \theta, z) = \sum_{m=0}^{\infty} \int_0^{\infty} Z_{km}^{\ell} J_m(\eta r) \eta d\eta \cos m \theta \quad (f)$$

where*

$$U_{km}^{\ell} = \eta^2 \cosh \eta z (A_{km}^{\ell} - C_{km}^{\ell}) + \eta^2 \sinh \eta z (A_{km}^{\ell} + C_{km}^{\ell}) \quad (a)$$

$$+ \eta (\sinh \eta z \cosh \eta z) (B_{km}^{\ell} - D_{km}^{\ell}) \quad (34)$$

$$+ \eta (\cosh \eta z \sinh \eta z) (B_{km}^{\ell} + D_{km}^{\ell})$$

$$V_{km}^{\ell} = \eta \sinh \eta z (E_{km}^{\ell} - F_{km}^{\ell}) + \eta \cosh \eta z (E_{km}^{\ell} + F_{km}^{\ell}) \quad (b)$$

* The constants of integration carry superscripts to indicate that they correspond to the ℓ^{th} layer and subscripts to indicate that they correspond to the k^{th} node and the m^{th} Fourier coefficient of the potential function expansion.

$$\begin{aligned}
W_{km}^{\ell} &= \eta^2 \sinh \eta z (A_{km}^{\ell} - C_{km}^{\ell}) + \eta^2 \cosh \eta z (A_{km}^{\ell} + C_{km}^{\ell}) \\
&+ \eta [-2 (1-2\nu_{\ell}) \cosh \eta z + \eta z \sinh \eta z] (B_{km}^{\ell} + D_{km}^{\ell}) \\
&+ \eta [-2 (1-2\nu_{\ell}) \sinh \eta z + \eta z \cosh \eta z] (B_{km}^{\ell} - D_{km}^{\ell}).
\end{aligned} \tag{c}$$

$$\begin{aligned}
Z_{km}^{\ell} &= \mu_{\ell} -\eta^3 \cosh \eta z (A_{km}^{\ell} - C_{km}^{\ell}) - \eta^3 \sinh \eta z (A_{km}^{\ell} + C_{km}^{\ell}) \\
&+ \eta [(1-2\nu_{\ell}) \eta \sinh \eta z - \eta^2 z \cosh \eta z] (B_{km}^{\ell} - D_{km}^{\ell}) \\
&+ \eta [(1-2\nu_{\ell}) \eta \cosh \eta z - \eta^2 z \sinh \eta z] (B_{km}^{\ell} + D_{km}^{\ell}).
\end{aligned} \tag{d}$$

$$\begin{aligned}
R_{km}^{\ell} &= \mu_{\ell} \eta^3 \sinh \eta z (A_{km}^{\ell} - C_{km}^{\ell}) + \eta^3 \cosh \eta z (A_{km}^{\ell} + C_{km}^{\ell}) \\
&+ \eta [2\nu_{\ell} \eta \cosh \eta z + \eta^2 z \sinh \eta z] (B_{km}^{\ell} - D_{km}^{\ell}) \\
&+ \eta [2\nu_{\ell} \eta \sinh \eta z + \eta^2 z \cosh \eta z] (B_{km}^{\ell} + D_{km}^{\ell}).
\end{aligned} \tag{e}$$

$$S_{km}^{\ell} = \mu_{\ell} [\eta^2 \cosh \eta z (E_{km}^{\ell} - F_{km}^{\ell}) + \eta^2 \sinh \eta z (E_{km}^{\ell} + F_{km}^{\ell})] \tag{f}$$

where $J_m'(\eta r)$ is evaluated from the convenient recursion formula

$$J_m'(\eta r) = \frac{1}{2} [J_{m-1}(\eta r) - J_{m+1}(\eta r)].$$

For a multilayered medium, the boundary conditions at the interface require that the displacements u , v , w , and the stresses σ_{zz} , τ_{rz} , and $\tau_{\theta z}$ be continuous. Since this must be true for all (r) then it must be true that the coefficients of $J_m'(\eta r)$ and $J_m(\eta r)$ (U_{km}^{ℓ} , V_{km}^{ℓ} , W_{km}^{ℓ} , R_{km}^{ℓ} , Z_{km}^{ℓ} and S_{km}^{ℓ}) be continuous. This satisfies continuity of stresses and displacement.

It was pointed out by Kuo [64] that, by using a matrix formulation approach similar to that used by Haskell [86, 88], the cumbersome nature of insuring continuity between layers can be made easier.

Define

$$\left\{ P_{km}^{\ell} \right\} = \begin{Bmatrix} U_{km}^{\ell} \\ V_{km}^{\ell} \\ W_{km}^{\ell} \\ Z_{km}^{\ell} \\ R_{km}^{\ell} \\ S_{km}^{\ell} \end{Bmatrix} \quad \left\{ Q_{km}^{\ell} \right\} = \begin{Bmatrix} A_{km}^{\ell} - C_{km}^{\ell} \\ E_{km}^{\ell} - F_{km}^{\ell} \\ B_{km}^{\ell} - D_{km}^{\ell} \\ B_{km}^{\ell} - D_{km}^{\ell} \\ A_{km}^{\ell} - C_{km}^{\ell} \\ E_{km}^{\ell} - F_{km}^{\ell} \end{Bmatrix} \quad (36)$$

where the superscript ℓ again denotes the layer number and in the case of P_{km}^{ℓ} the column vector is evaluated at the bottom of the ℓ^{th} layer. Let $T_{km}^{\ell}(z)$ be defined as

$$\left[T_{km}^{\ell}(z) \right] = \begin{bmatrix} t_{11} & t_{12} & \cdot & \cdot & \cdot & t_{16} \\ t_{21} & t_{22} & \cdot & \cdot & \cdot & t_{26} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ t_{61} & t_{62} & & & & t_{66} \end{bmatrix} \quad (37)$$

where

$$t_{12} = t_{21} = t_{23} = t_{32} = t_{24} = t_{42} = t_{25} = t_{52} = t_{16} = t_{61} = 0$$

$$t_{36} = t_{63} = t_{46} = t_{64} = t_{56} = t_{65} = 0$$

and

$$t_{11} = \eta^2 \cosh \eta z$$

$$t_{13} = \eta (\sinh \eta z + \cosh \eta z)$$

$$t_{14} = \eta (\cosh \eta z + \eta z \sinh \eta z)$$

$$t_{15} = \eta^2 \sinh \eta z$$

$$t_{22} = \eta \cosh \eta z$$

$$t_{26} = \eta \sinh \eta z$$

$$t_{31} = \eta^2 \sinh \eta z \quad (37)$$

$$t_{33} = \eta [-2(1-2\nu_\ell) \cosh \eta z + \eta z \sinh \eta z]$$

$$t_{34} = \eta [-2(1-2\nu_\ell) \sinh \eta z + \eta z \cosh \eta z]$$

$$t_{35} = \eta^2 \cosh \eta z$$

$$t_{41} = \mu_\ell \eta^3 \cosh \eta z$$

$$t_{43} = \mu_\ell \eta [(1-2\nu_\ell) \eta \sinh \eta z - \eta^2 z \cosh \eta z]$$

$$t_{44} = \mu_\ell \eta [(1-2\nu_\ell) \eta \cosh \eta z - \eta^2 z \sinh \eta z]$$

$$t_{45} = -\mu_\ell \eta^3 \sinh \eta z$$

$$t_{51} = \mu_\ell \eta^3 \sinh \eta z$$

$$t_{53} = \mu_\ell \eta [2\nu_\ell \eta \cosh \eta z + \eta^2 z \sinh \eta z]$$

$$\begin{aligned}
t_{54} &= \mu_l \eta [2\nu_l \eta \sinh \eta z + \eta^2 z \cosh \eta z] \\
t_{55} &= \mu_l \eta^3 \cosh \eta z \\
t_{62} &= \mu_l \eta^2 \sinh \eta z \\
t_{63} &= \mu_l \eta^2 \cosh \eta z .
\end{aligned} \tag{37}$$

Then by taking $z = 0$ at the top of the l^{th} layer we have

$$\left\{ P_{km}^{l-1} \right\} = \left[T_{km}^l (0) \right] \left\{ Q_{km}^l \right\} \tag{38}$$

where

$$\left[T_{km}^l (0) \right] = \left[T_{km}^l (z) \right] \text{ at } z = 0. \tag{39}$$

By letting $z = d_l$ (the thickness of the l^{th} layer)

$$\left\{ P_{km}^l \right\} = \left[T_{km}^l (d_l) \right] \left\{ Q_{km}^l \right\} . \tag{40}$$

Then from equation (38)

$$\left\{ Q_{km}^l \right\} = \left[T_{km}^l (0) \right]^{-1} \left\{ P_{km}^{l-1} \right\} \tag{41}$$

where $\left[T_k^l (0) \right]$ is the matrix equal to

$$\begin{bmatrix}
 \frac{1}{\eta^2} - \frac{1}{2\eta^2(1-\nu_\ell)} & 0 & 0 & \frac{1}{2\mu_\ell \eta^3(1-\nu_\ell)} & 0 & 0 \\
 0 & \frac{1}{\eta} & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{1}{2\eta(\nu_\ell-1)} & 0 & \frac{1}{2\mu_\ell \eta^3(1-\nu_\ell)} & 0 \\
 \frac{1}{2\eta(1-\nu_\ell)} & 0 & 0 & \frac{1}{2\mu_\ell \eta^2(1-\nu_\ell)} & 0 & 0 \\
 0 & 0 & \frac{\nu_\ell}{\eta^2(1-\nu_\ell)} & 0 & \frac{1-2\nu_\ell}{\mu_\ell \eta^3(1-\nu_\ell)} & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{1}{\eta^3 \mu_\ell}
 \end{bmatrix} \quad (42)$$

Substitution into equation (40) yields

$$\left\{ P_{km}^\ell \right\} = \begin{bmatrix} a_k^\ell \end{bmatrix} \left\{ P_{km}^{\ell-1} \right\} \quad (43)$$

where

$$\begin{bmatrix} a_k^\ell \end{bmatrix} = \begin{bmatrix} T_k^\ell(d_\ell) \end{bmatrix} \begin{bmatrix} T_k^\ell(o) \end{bmatrix}^{-1} \quad (44)$$

For the $(\ell-1)^{\text{th}}$ layer $\left\{ P_{km}^{\ell-2} \right\}$ is the value of the matrix P_{km}

evaluated at the top of the layer $(\ell-1)$ and

$$\left\{ P_{km}^\ell \right\} = \begin{bmatrix} a_k^\ell \end{bmatrix} \begin{bmatrix} a_k^{\ell-1} \end{bmatrix} \left\{ P_{km}^{\ell-2} \right\} \quad (45)$$

By iterating it follows that

$$\left\{ P_k^{\ell-1} \right\} = \begin{bmatrix} a_k^{\ell-1} \end{bmatrix} \begin{bmatrix} a_k^{\ell-2} \end{bmatrix} \cdots \begin{bmatrix} a_k^1 \end{bmatrix} \left\{ P_k^o \right\} \quad (46)$$

Finally from equation (41) the coefficients of the l^{th} layer ($A_{km}^l - C_{km}^l$), ($E_{km}^l + F_{km}^l$) . . . are related to the displacements and stresses on the surface; i.e., U_{km}^o , V_{km}^o , W_{km}^o , Z_{km}^o , R_{km}^o , S_{km}^o

$$\left\{ Q_{km}^l \right\} = \left[T_{km}^l(o) \right]^{-1} \left[a_{km}^{l-1} \right] \left[a_{km}^{l-2} \right] \dots \left[a_{km}^1 \right] \left\{ P_{km}^o \right\} \quad (47)$$

Through the boundary conditions on the surface and the requirement that the stresses in the last layer (the N^{th} layer of the half space) must remain finite, the unknown coefficients, Q_{km}^l , can be determined.

At infinity the displacements and stresses within the N^{th} layer must be finite. The coefficients A_{km}^N , B_{km}^N , and E_{km}^N , must then be zero. Let the surface load functions be expressed as a column vector, i.e.

$$\left\{ P_{km}^o \right\} = \begin{Bmatrix} U_{km}^o \\ V_{km}^o \\ W_{km}^o \\ Z_{km}^o \\ R_{km}^o \\ S_{km}^o \end{Bmatrix} = \begin{Bmatrix} l_{km1} \\ l_{km2} \\ l_{km3} \\ l_{km4} \\ l_{km5} \\ l_{km6} \end{Bmatrix} \quad (48)$$

The coefficients of the last layer, N are then

$$\left\{ Q_{km}^N \right\} = \left[M_{km} \right] \left\{ P_{km}^o \right\} = \left[M_k \right] \left\{ l_{km} \right\} \quad (49)$$

where

$$\begin{bmatrix} M_k \end{bmatrix} = \begin{bmatrix} T_k^N(o) \end{bmatrix}^{-1} \begin{bmatrix} a_k^{N-1} \end{bmatrix} \begin{bmatrix} a_k^{N-2} \end{bmatrix} \dots \begin{bmatrix} a_k^1 \end{bmatrix} \quad (50)$$

The non-zero coefficients of the last layer C_{km}^N , D_{km}^N , F_{km}^N can be determined once the boundary conditions (i.e., l_{km4} , l_{km5} , l_{km6}) are known. The Hankel transformed displacement functions at the free surface can now be expressed in terms of these three boundary conditions as

$$\begin{aligned} U_{km}^o &= \frac{\Delta_{U_m}}{F_k(\eta)} \\ V_{km}^o &= \frac{\Delta_{V_m}}{F_k(\eta)} \\ W_{km}^o &+ \frac{\Delta_{W_m}}{F_k(\eta)} \end{aligned} \quad (51)$$

where

$$F_k(\eta) = \begin{vmatrix} M_{11} + M_{51} & M_{12} + M_{52} & M_{13} + M_{53} \\ M_{31} + M_{41} & M_{32} + M_{42} & M_{33} + M_{43} \\ M_{21} + M_{61} & M_{22} + M_{62} & M_{23} + M_{63} \end{vmatrix} \quad (52)(a)$$

$$\Delta_{U_m} = \begin{vmatrix} N_1 + N_3 & M_{12} + M_{52} & M_{13} + M_{53} \\ N_3 + N_4 & M_{32} + M_{42} & M_{33} + M_{43} \\ N_2 + N_6 & M_{22} + M_{62} & M_{23} + M_{63} \end{vmatrix}$$

$$\Delta_{V_m} = \begin{vmatrix} M_{11} + M_{51} & N_1 + N_3 & M_{13} + M_{53} \\ M_{31} + M_{41} & N_3 + N_4 & M_{33} + M_{43} \\ M_{21} + M_{61} & N_2 + N_6 & M_{23} + M_{63} \end{vmatrix}$$

$$\Delta_{W_m} = \begin{vmatrix} M_{11} + M_{51} & M_{12} + M_{52} & N_1 + N_3 \\ M_{31} + M_{41} & M_{32} + M_{42} & N_3 + N_4 \\ M_{21} + M_{61} & M_{22} + M_{62} & N_2 + N_6 \end{vmatrix} \quad (53) (b)$$

where

$$N_i = \sum_{j=4, 5, 6} M_{kij} \ell_{kj}.$$

Assume that the multilayered medium is stressed by only a circular uniform normal load over a radius a . The boundary conditions on the free surface are then

$$\left. \begin{aligned} \sigma_{zzk}(r, \theta, 0) &= \sigma_k \\ \tau_{rzk}(r, \theta, 0) &= \tau_{\theta zk}(r, \theta, 0) = 0 \end{aligned} \right\} r \leq a \quad (54)$$

and

$$\sigma_{zzk}(r, \theta, 0) = \tau_{rzk}(r, \theta, 0) = \tau_{\theta zk}(r, \theta, 0) = 0 \text{ for } r > a.$$

For an N -layered medium under the axisymmetric ($m = 0$) circular load distribution described by equations (54) the stress function ϕ_k does not depend upon θ and the stress function γ_k is zero as are

V_{km}^l , S_{km}^l , E_{km}^l , and F_{km}^l .

It follows that $\{P_k^l\}$ and $\{Q_k^l\}$ are then 4 element column vectors where the m subscript can now be dropped. $[T_k^l(z)]$, $[a_k^l]$, $[M_k]$, are then 4 x 4 matrices.

To determine the values of l_{k4} , l_{k5} , and l_{k6} , we first rewrite the boundary conditions (54) as

$$\sigma_{zzk}(r, \theta, 0) = \sigma_{zz}^a \int_0^{\infty} J_1(\eta r) d\eta$$

and

$$\tau_{rzk}(r, \theta, 0) = \tau_{\theta zk}(r, \theta, 0) = 0.$$

Since the values l_{k4} , l_{k5} , and l_{k6} , are the Hankel transforms of these boundary stresses, they can be written by inspection as

$$l_{k4} = \sigma_{zz}^a J_1(\eta a) \quad (55)$$

$$l_{k5} = l_{k6} = 0.$$

The non-zero displacements of the free surface are then obtained by substituting equations (51) into equations (33a) and (33c), or simply

$$u_k(r, \theta, 0) = \int_0^{\infty} \frac{u_k^0}{2} J_1(\eta r) \eta d\eta$$

$$w_k(r, \theta, 0) = \int_0^{\infty} \frac{w_k^0}{2} J_0(\eta r) \eta d\eta.$$

Letting $M_{2i} = M_{6i} = 0$ ($i = 1, 2, \dots, 6$) and simplifying, the displacements become expressed as

$$u_k(r, \theta, 0) = \int_0^{\infty} \frac{\sigma_k^a}{F_k(\eta)} J_1(\eta r) (g_1 - g_2) d\eta \quad (56)$$

$$w_k(r, \theta, 0) = \int_0^{\infty} \frac{\sigma_k^a}{F_k(\eta)} J_1(\eta a) J_0(\eta r) (g_3 - g_4) d\eta \quad (57)$$

where

$$\begin{aligned} g_1 &= (M_{14} + M_{54}) (M_{33} + M_{43}) \\ g_2 &= (M_{13} + M_{53}) (M_{34} + M_{44}) \\ g_3 &= (M_{11} + M_{51}) (M_{34} + M_{44}) \\ g_4 &= (M_{14} + M_{54}) (M_{31} + M_{41}) \end{aligned} \quad (58)$$

and

$$F_k(\eta) = (M_{11} + M_{51}) (M_{33} + M_{43}) - (M_{13} + M_{53}) (M_{31} + M_{41}).$$

Equation (57) provides the basis for establishing the influence coefficients for contacting layered bodies. The displacement at the k^{th} node of the i^{th} body due to the combined affect of N_o nodal forces, F_j (distributed uniformly over a circular region about the j^{th} node) is

$$w_{ik} = \sum_{j=1}^{N_o} \frac{F_j}{\pi a} \int_0^{\infty} \frac{(g_3 - g_4)}{F_k(\eta)} J_1(\eta a) J_0(\eta r_{kj}) d\eta \quad (59)$$

where $F_j = a^2 \sigma_j$, and r_{kj} is the distance between the k^{th} and j^{th} nodes. The influence coefficient for the i^{th} body is then simply

$$a_{ikj} = \frac{1}{a} \int_0^{\infty} \frac{(g_3 - g_4)}{F(\eta)} J_1(\eta a) J_0(\eta r_{kj}) d\eta. \quad (60)$$

IV. NUMERICAL RESULTS AND APPLICATIONS

The Simplex method shown in Figure 3 was programmed to solve the set of equations shown in canonical form in Table 1. The geometrical shapes considered in this section were for the most part kept simple in order to attempt to understand the effects of layering upon the contact stress phenomenon. The program was first used to solve for the normal contact stress distribution resulting from the normal contact of two layered spheres. The spheres were identical in every way with a 1.0-inch radius. The elastic properties of the layers, however, were first made identical to those of the substrate material to allow the results to be compared to known closed form solutions; i.e., the Hertz solution. The results are shown in Figure 6. The model, made using only $7 \times 7 = 49$ modes, demonstrates reasonable accuracy differing only by approximately 1.37 percent from the Hertz solution in the maximum contact stress and 1.44 percent in the relative approach of the spheres. The stress is seen to be nondimensionalized with respect to σ_L the maximum Hertz stress and the radius is nondimensionalized with respect to a_L the Hertz radius. (From here on the subscript "L" as in σ_L or a_L refers to a characteristic of the layer.) Note that 7 points are plotted in Figure 6 across the radius whereas with a 7×7 node model only 4 points can lie on any one radius. The additional 3 points came from points other than those along the radius.

Figure 6. Stress Distribution for Classical Hertzian Formulation and Approximate Method. $7 \times 7 = 49$ Node Model. Load $P = 100$ b. and the radius of the spheres is 1.0 inch.

Let us consider some problems of practical usefulness to demonstrate the useage of the method. Often in the design of mechanical or electrical components the engineer is confronted with determining contact stresses for plated bodies. A common combination of materials is that of copper plated steel $\frac{E_1}{E_2} = 0.5$. Another is nickle plated aluminum $\frac{E_1}{E_2} = 2.0$.

Figures 7 and 8 show the contact stress distribution for the normal contact of two layered spheres as before identical in every way with a radius of 1.0 inch. The problem was modeled using $11 \times 11 = 121$ nodes. The spheres are layered having a ratio of layer modulus to substrate modulus $\frac{E_1}{E_2} = 2.0$ in Figure 7 and $\frac{E_1}{E_2} = 0.5$ in Figure 8 and for various thickness of layers to radius of sphere ratios, $\frac{t}{R}$. The load P was taken as 100 lb. for each case. In these figures the stress is nondimensionalized with respect to maximum contact stress that would be obtained if the bodies were made completely out of the layer material, σ_L . The radius is nondimensionalized with respect to, a_L , the radius that would be obtained if the spheres were made completely of the layer material. From the figures it is seen that the stress distribution does not appear to change significantly from the ellipsoidal or Hertzian distribution.

By plotting the maximum normal contact stress in Figures 7 and 8 versus layer thickness, Figure 9, it is seen that for this case that when $0.1 < \frac{t}{R}$ the effect upon the contact stress distribution of the substrate material is small and when $\frac{t}{R} < 10^{-4}$ the effect of the layer material upon the contact stress distribution is small.

Figure 7. Contact Stress Distribution for Layered Spheres Having Various Thickness $\frac{E_1}{E_2} = 2.0$, $\nu_1 = \nu_2 = 0.3$, Load $11 \times 11 = 121$ Node Model $P = 100$ lb. and the radius of the spheres is 1.0 inch.

Figure 8. Contact Stress Distribution for Layered Spheres Having Various Thickness, $\frac{E_1}{E_2} = 0.5$, $\nu_1 = \nu_2 = 0.3$, Load $11 \times 11 = 121$ Node Model. $P = 100$ lb. and the radius of the spheres is 1.0 inch.

Figure 9. Maximum Normal Contact Stress Versus Layer Thickness, $\frac{E_1}{E_2} = 0.5$ and 2.0 of Layered Spheres, Load $P = 100$ lb. and the spheres have 1.0-inch radius.

We next choose again the problem of two spheres identical in every way, however, having relatively thin layers $\frac{t}{R} = 0.01$. Figure 10 shows the normal contact stress distribution resulting from the normal contact of the above two spheres of radius 1.0 inch for various ratios of $\frac{E_1}{E_2}$. This problem as before was modeled using $11 \times 11 = 121$ nodes. The normal load P was again 100 lb. The figure shows that as $\frac{E_1}{E_2}$ becomes large the stress distribution approaches that of contacting spherical shells [89]. At $\frac{E_1}{E_2} = 10$ the contact stress is seen to be uniform over much of the contact area. But as $\frac{E_1}{E_2}$ becomes larger the location of maximum contact stress moves to or concentrates near the periphery of the contact region.

By holding the ratio of layer modulus to substrate modulus, constant $\frac{E_1}{E_2} = 20$ for again two identical spheres and varying the layer thickness it is seen, Figure 11, that for thick layers the distribution is approximately ellipsoidal as is expected. However, as the layer becomes thinner the distribution becomes as in Figure 10 with the maximum stress at the periphery of the contact region. As the layer thickness continues to decrease the distribution begins to become again ellipsoidal in the central region. In Figure 11 with $\frac{t}{R} = 0.005$ the distribution still retains a high stress at the periphery of the contact region but in the central region the distribution has begun to become again ellipsoidal. The effect being seen in the central region is that the layer is having a lesser effect on the stress distribution as it becomes thin and the substrate begins to govern.

Figure 10. Contact Distribution for Layered Spheres and Various Layer Modulus,
Load $11 \times 11 = 121$ Node Model, $t/R = 0.01$, $P = 100$ lb, and
spheres have 1.0-inch radius.

Figure 11. Contact Stress Distribution for Layer Spheres $\frac{E_1}{E_2} = 20$ and Various Layer Thicknesses, $11 \times 11 = 121$ Node Model, Load $P = 100$ lb, and the spheres have radius 1.0 inch.

In Figure 12, the normal contact of two identical ellipsoidal layered solids is considered whose ratio of major R_1 to minor R'_1 radius of curvature is $\frac{R_1}{R'_1} = 2.0$ where $R_1 = 1.0$ inch and $R'_1 = 1.0$ inch. The model was constructed using $13 \times 9 = 127$ nodes and the normal load was 100 lb. The thickness considered was $\frac{t}{R_1} = 0.01$. Similar distributions are seen as in the case of layered spheres; i.e., the stress distribution becomes nonellipsoidal when $\frac{E_1}{E_2}$ becomes large. However, a most important fact is too, that the stress distribution in the minor plane of the ellipsoidal solids departs from its approximate ellipsoidal functional distribution and becomes more uniform.

One interesting area of contacting layered bodies is in the contact of a flat circular rigid punch with a layered half-space. Only recently have Chen and Engel [80] solved the axisymmetric problem assuming the punch rigidly adhered to the layered half-space (thus allowing tensile stresses at the contact interface). Figure 13 shows the result of Boussinesq for the same punch problem of the nonhomogeneous half-space and the results of the author's method. Since only $11 \times 11 = 121$ nodes were used, near the edges of the punch as to be expected the approximate method is at its worst. The rest of the contact stress distribution is very close to the closed form solution, however. Comparing the results of this theory to that of Chen and Engel for the ratio of twice the layer thickness to the radius of the punch $\frac{2h}{a} = 0.3$ shows that the allowance of tensile stresses at the contact interface has a great effect upon the stress distribution. It is also interesting to note that the stress at the

Figure 12. Contact Stress Distribution for Layered Ellipsoidal Solids $\frac{t}{R_1} =$

0.01 Various Layer Modulae Load = 1.6, $13 \times 9 = 117$ Nodes

$P = 100$ lb. $R_1 = 2.0$ inch, $R_1' = 1.0$ inch

Figure 13. Contact Stress Distribution for a Flat Circular Punch on a Layered Half-Space 11 x 11 = 121 Node Model

center of the contact region is 1.7 times the stress found to be at the center of the contact region by Chen and Engle.

The problems presented in this section are by no means a complete coverage of layered body problems. They do, however, demonstrate the strength of the method presented in this work. The method can be used to solve most of the problems involving normal contact of layered bodies.

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