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The Ohio State University, Ph.D., 1973
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NUMERICAL SOLUTION OF A DEEP DRAWING
PROBLEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Eugene Irving Odell, B.S., M.S.

The Ohio State University
1973

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NOTATION

A_j	Coefficient in the $d\Gamma$ function
A	Constant in the stress-strain relationship
B	Constant in the stress-strain relationship
β	Angle between middle surface normal and the ram centerline for Zone 3
x	Current curvature in γ direction
x'	Previous curvature in γ direction
$d\epsilon$	Incremental effective strain
$d\epsilon_\gamma$ $d\epsilon_\theta$ $d\epsilon_z$	Incremental strains in the γ , θ , and z directions, respectively
$d\Gamma$	Unknown change in the γ position
dr	Incremental material movement in the radial direction — middle surface
$dr^{(2)}$	Incremental material movement in the radial direction — assuming no γ strains— middle surface
ϵ	Total effective strain for current increment
ϵ'	Total effective strain from previous increment
e	Engineering strain
ϵ_γ ϵ_θ ϵ_z	Total strains in the γ , θ , and z directions, respectively

$\left. \begin{matrix} \epsilon_l \\ \epsilon_w \\ \epsilon_t \end{matrix} \right\}$	True strains of the tensile test specimen in the length, width and thickness directions, respectively
F, G, H	Variable parameters in the plasticity equations
F_H	Total holding pressure (force)
F_R	Ram load
γ	Coordinate, measured along the middle surface for current increment (Fig. 5.1)
γ'	γ from the previous increment
i	Material position number (Fig. 5.1)
k	Material position number (Fig. 5.1)
λ	Parameter in the flow equations
M_γ, M_θ	Bending moments in the γ and θ directions, respectively
μ_1	Coefficient of friction between the inside of the shell and the tools
μ_2	Coefficient of friction between the outside of the shell and the tools
n	Constant in the stress-strain relationship
N_γ, N_θ	Inplane shell forces in the γ and θ directions, respectively
P	Tool pressure--normal to the middle surface
ϕ	Angle between the middle surface normal and the ram centerline
Q_γ	Shear force in the γ direction
R	R-value, see Chapter IV
R_P	Initial blank radius (Fig. 5.2)

r	Radius to shell middle surface
r_γ	Radius of curvature in γ -direction, current
r_γ'	Radius of curvature in γ -direction, previous
r_θ	Radius of curvature in θ -direction, current
ρ_j	Tool geometry (see Fig. 6.1)
S	Engineering stress
σ	Effective yield stress (true)
σ_γ	Stresses in the γ , θ , and z directions, respectively
σ_θ	
σ_z	
t	Current shell thickness
t'	Previous shell thickness
t^0	Initial blank thickness
T	Time
θ	Circumferential coordinate
V_o	Ram velocity
v_k	Volume of material to k (see Fig. 5.2)
z	Coordinate--inward normal (see Fig. 5.1)

CHAPTER I

INTRODUCTION

Description of Deep Drawing, Redrawing, and Ironing

Deep drawing is generally understood to be the process of forming sheet metal into a "cup shape" by the use of a ram and die (see Fig. 1.1). The ram, die, and material do not, necessarily, have to be of circular cross section, but are in this study.

Redrawing is the process of reducing the cup to a smaller diameter, using a smaller diameter ram and die (Fig. 1.2). Redrawing may be repeated many times to obtain parts with large length-diameter ratios.¹ Some cups may require intermediate annealing due to work hardening, depending upon the extent of redrawing required.

Ironing is the process of reducing the wall thickness, of a part, without reducing the inside diameter. This is accomplished by forcing a part, while on a ram, through an ironing ring (see Fig. 1.3).

The solution given in this work is limited to the basic cup-forming problem. Analyses of redrawing and ironing problems are not included in this treatise.

¹The patented Metal-Flo process utilizes telescopic rams and a unique hydraulic system to obtain parts of very large length-diameter ratios.

Figure 1.1 - Basic Cup Forming

Figure 1.2 - Redrawing

Figure 1.3 - Ironing

Problem Definition and Initial Assumptions

The problem analyzed here is defined as: finding the stress-strain distribution throughout the material, due to plastic deformation as the ram is incremented down, for the case of complete axial symmetry with respect to ram, die, and material (see Fig. 1.1).

Incremental plasticity theory employing modified¹ von-Mises yield criterion and flow equations, is used along with thin shell theory to obtain the solutions.

Other initial assumptions are: shape of middle surface is known² and is conical between the ram and die (see Fig. 6.1); material volume remains constant; elastic strains are neglected; principal axes of strain are parallel to, perpendicular to, and circumferential relative to the shell middle surface; thickness distribution from the prior increment is used to calculate γ_F (see Fig. 6.1); the holding

¹Modified to include effects of normal anisotropy.

²If a shape is assumed other than the true shape, the calculated normal pressure will be in error. Between the ram and the die the true normal pressure is zero. By assuming a conical shape in this region small residual pressures result, thereby implying the true shape is not conical. For the examples studied in this work, these residual pressures are less than 10% of those existing between the die and the material, thus leading to the conclusion that the errors in the stress-strain distribution are minimal. (Note--the assumption of conical shape greatly simplifies the analysis.)

pressure³ distribution is linear between E and F (Fig. 5.4) where

$$\gamma_E = \frac{\gamma_D + \gamma_F}{2} . \quad (1-1)$$

Solutions are given for both membrane and bending theories, where all numerical work was done with the aid of a digital computer using Fortran IV programs.

³Holding pressure is used to keep the material in contact with the flat region of the die to prevent wrinkling as the material is drawn inward. Generally, this pressure is applied through a flat holding pad via an adjustable hydraulic actuator or springs.

CHAPTER II

REVIEW OF LITERATURE

The mechanics of deep drawing received relatively little attention until about thirty years ago. Since then much fine work has been done, most notable is that of Professor Swift and colleagues at Sheffield University.

In 1960, Alexander (1)¹ presented a very complete review of the literature, studying in detail the work of Swift and others. One section of this review is the solution by Chung and Swift of the basic cup-forming problem; where many simplifying assumptions were made to facilitate integration of the incremental strain equations. Also lacking was a solution for the material in zones 1, 2, and 3 (see Fig. 5.4). This work, however, should be considered a very valuable pioneer effort for future analysis.

A practical study of Swift's work was given in 1954 by J. Willis (21). Woo (22), (23), (24) has provided the most complete solutions thus far. Using numerical integration, Woo has attained a solution to the basic cup forming problem, for membrane theory only, employing a much improved holding pressure model.

¹Numbers in parentheses in the text refer to items in the List of References.

Of much interest to people in industry is the "limiting draw ratio."¹ This was studied by Chiang and Kobayashi (6) and more recently by Chakrabarty and Mellor (5).

For other analyses of deep drawing, see References (2), (4), (8), (9), (10), (11), (12), (13), and (20).

A very fine solution to the redrawing problem was given by Fogg (7). Other references on redrawing are (1), (8), (9), (12), and (21).

Experimental work relative to sheet metal forming is found in References (4), (14), (16), (17), (18), (25), and (26).

¹The limiting draw ratio is defined as, the ratio of initial blank diameter to finished cup diameter.

CHAPTER III

MATERIAL DESCRIPTION

Stress-Strain Relationship

The use of the modified von-Mises yield stress criterion¹ and associated flow equations requires an empirical stress-strain relationship which, in this work, is assumed to be:

$$\sigma = A + B\epsilon^n \quad (3-1)$$

where σ is the true uniaxial stress, ϵ is the true axial strain, and A, B, and n are constants obtained via material tests.

True strain² ϵ is related to engineering strain e as follows: A true infinitesimal strain is defined as:

$$d\epsilon = \frac{dl}{l} \quad (3-2)$$

where l is the current length of the test specimen. By integrating Equation (3-2) from some initial length l_0 , to l , an equation for ϵ is obtained as:

$$\epsilon = \int_{l_0}^l \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right) = \ln(1 + e) \quad (3-3)$$

¹Developed in Chapter IV.

²True stress and strain are developed in most texts on theory of plasticity, such as Hill (9).

where

$$e = \frac{l - l_0}{l_0} \quad (3-4)$$

True stress¹ σ is defined as

$$\sigma = \frac{P}{A} \quad (3-5)$$

where P is the applied uniaxial load and A is the current cross-sectional area. A relationship between true stress σ and engineering stress S is obtained by using volumetric constancy; i.e.,

$$Al = A_0 l_0 \quad \text{or} \quad A = \frac{A_0 l_0}{l} \quad (3-6)$$

where A , l and A_0 , l_0 are the current and initial cross-sectional area and length, respectively, of the test specimen. Substituting Equation (3-6) into Equation (3-5), σ becomes:

$$\sigma = \frac{P}{A_0} \left(\frac{l}{l_0} \right) = S(1 + e) \quad (3-7)$$

where,

$$S = \frac{P}{A_0} \quad (3-8)$$

and e is from Equation (3-4).

The material used in this analysis was tested by Armco Steel Corporation of Middletown, Ohio. Armco ran 24 load-extension curves

¹Ibid.

for specimens cut from 4 production blanks. True stresses and strains were then calculated using Equations (3-3), (3-4), (3-7), and (3-8). A single stress-strain curve was developed by averaging the 24 data sets¹ (see Fig. 3.1). The stress-strain data are given in Appendix A, Table 1.

The constant A, in the empirical equation (3-1), is clearly the initial yield stress. The constants B and n were obtained from the logarithmic plot of $(\sigma - A)$ vs. ϵ , Fig. 3.2.

With A, B, and n known, Equation (3-1) becomes

$$\sigma = 21,000 + 92,000 (\epsilon)^{0.633} \quad (3-9)$$

Equation (3-9) is used throughout this analysis as the relationship between stress and strain.

R-Value

The anisotropic parameter R is defined as:

$$R = \frac{\epsilon_w}{\epsilon_t} \quad (3-10)$$

where ϵ_w and ϵ_t are the true width and thickness strains, respectively, for the uniaxial tension test.

¹Only data up to the point of maximum applied load have been used. The stress-strain distribution is non-uniform for greater strains as necking starts to occur. This is also known as the point of plastic instability.

FIGURE 3.1 STRESS-STRAIN

Figure 3.2 - Logarithmic Stress-Strain Plot

Since thickness strains are difficult to measure, the need for them can be eliminated by using volumetric constancy:

$$\epsilon_t = -\epsilon_w - \epsilon_l \quad (3-11)$$

to rewrite Equation (3-10) as

$$R = \frac{-\epsilon_w}{\epsilon_w + \epsilon_l} \quad (3-12)$$

where ϵ_l is the longitudinal strain.

Using the true strain equation (3-3), Equation (3-12) becomes

$$R = \frac{\ln \left(\frac{w_0}{w} \right)}{\ln \left(\frac{wl}{w_0 l_0} \right)} \quad (3-13)$$

where, w_0 = the initial width of the test specimen
 w = the current width of the test specimen
 l_0 = the initial length of the test specimen
 l = the current length of the test specimen

R is obtained by measuring w_0 and l_0 , then straining the material a sufficient amount¹ and measuring w and l . Equation (3-13)

¹The actual amount of straining is of minimal significance as R is relatively constant for a wide range of strains.

is then used to calculate R. R-values of the material tested are given in Appendix A, Table 2. An average value of $R = 1.51$ has been used in this analysis.

Recent work in R-value determination has been done by Lafferty, Schmidt and Lawley (14) and also Pearce and Woodthorpe.(17).

CHAPTER IV

PLASTICITY RELATIONSHIPS

Yield Criterion and Flow Equations

A general yield criterion for an anisotropic material possessing three mutually orthogonal planes of symmetry, as given by Hill (9), is

$$F(\sigma_{\theta} - \sigma_z)^2 + G(\sigma_{\gamma} - \sigma_z)^2 + H(\sigma_{\gamma} - \sigma_{\theta})^2 = 1 \quad (4-1)$$

where F, G, and H may vary throughout the material; and where the shear stress terms are zero, per initial assumptions given in Chapter I.

The incremental flow equations associated with this yield criterion, also from Hill (9), are

$$d\epsilon_{\gamma} = \lambda \left[H(\sigma_{\gamma} - \sigma_{\theta}) + G(\sigma_{\gamma} - \sigma_z) \right] \quad (4-2)$$

$$d\epsilon_{\theta} = \lambda \left[F(\sigma_{\theta} - \sigma_z) + H(\sigma_{\theta} - \sigma_{\gamma}) \right] \quad (4-3)$$

$$d\epsilon_z = \lambda \left[G(\sigma_z - \sigma_{\gamma}) + F(\sigma_z - \sigma_{\theta}) \right] \quad (4-4)$$

where λ is an unknown variable. F, G, H, and λ are found via the following considerations:

Inplane symmetry requires that,

$$F = G \quad (4-5)$$

For uniaxial loading with $\sigma_y = \sigma$ and $\sigma_\theta = \sigma_z = 0$,
Equations (4-3) and (4-4) become

$$d\epsilon_\theta = -\lambda H \sigma \quad (4-6)$$

$$d\epsilon_z = -\lambda G \sigma \quad (4-7)$$

Dividing Equation (4-6) by Equation (4-7) and using the definition of R-value,

$$\frac{d\epsilon_\theta}{d\epsilon_z} = \frac{H}{G} \quad \text{or } H = GR \quad (4-8)$$

Substituting Equations (4-5) and (4-8) into Equation (4-1), the yield criterion becomes

$$G \left[(\sigma_\theta - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + R(\sigma_y - \sigma_\theta)^2 \right] = 1 \quad (4-9)$$

G is found from uniaxial loading, where again $\sigma_y = \sigma$ and $\sigma_\theta = \sigma_z = 0$.
Yielding

$$G = \frac{1}{\sigma^2(1+R)} \quad (4-10)$$

The final yield stress equation is then,

$$(\sigma_\theta - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + R(\sigma_y - \sigma_\theta)^2 = (1 + R)\sigma^2 \quad (4-11)$$

where σ is the yield stress from Equation (3-1).

λ is determined by substituting Equations (4-2), (4-3), and (4-4) into Equation (4-11), to get

$$\lambda = (d\epsilon) \sigma \quad (4-12)$$

where

$$d\epsilon = \frac{(1+R)^{\frac{1}{2}}}{(1+2R)} \left[(d\epsilon_{\theta} - R d\epsilon_z)^2 + (d\epsilon_{\gamma} - R d\epsilon_z)^2 + R(d\epsilon_{\theta} - d\epsilon_{\gamma})^2 \right]^{\frac{1}{2}} \quad (4-13)$$

The final flow equations are now:

$$d\epsilon_{\gamma} = \frac{d\epsilon(1+R)}{\sigma} \left[R(\sigma_{\gamma} - \sigma_{\theta}) + (\sigma_{\gamma} - \sigma_z) \right] \quad (4-14)$$

$$d\epsilon_{\theta} = \frac{d\epsilon(1+R)}{\sigma} \left[(\sigma_{\theta} - \sigma_z) + R(\sigma_{\theta} - \sigma_{\gamma}) \right] \quad (4-15)$$

$$d\epsilon_z = \frac{d\epsilon(1+R)}{\sigma} \left[(\sigma_z - \sigma_{\gamma}) + (\sigma_z - \sigma_{\theta}) \right] \quad (4-16)$$

Stresses from Incremental Strains

In Chapters VI and VII it is required to calculate σ_{γ} and σ_{θ} using the flow equations. From Equations (4-14) and (4-15), equations for σ_{γ} and σ_{θ} are obtained.

$$\sigma_{\gamma} = \frac{1}{[(1+R)^2 - R^2]} \left\{ \frac{(1+R)\sigma}{d\epsilon} \left[(1+R)d\epsilon_{\gamma} + R d\epsilon_{\theta} \right] + (1+2R)\sigma_z \right\} \quad (4-17)$$

$$\sigma_{\theta} = \frac{1}{[(1+R)^2 - R^2]} \left\{ \frac{(1+R)\sigma}{d\epsilon} \left[(1+R)d\epsilon_{\theta} + R d\epsilon_{\gamma} \right] + (1+2R)\sigma_z \right\} \quad (4-18)$$

CHAPTER V

THIN SHELL EQUATIONS

Coordinates

A discrete number of material points are studied during deformation. Each point (i, k) has material coordinates (z_i, γ_k) . Where z_i is the distance from the middle surface and γ_k is the meridian distance from the cup centerline (see Fig. 5.1).

z_i is obtained from

$$z_i = \left(\frac{I-1}{I} - \frac{1}{2} \right) t'_k \quad (5-1)$$

where $I+1$ is the total number of i positions, and t'_k is the material thickness at k for the previous increment.¹

γ_k is determined from the volume integral

$$2\pi \int_0^{\gamma_k} r t' d\gamma = v_k \quad (5-2)$$

where v_k is the volume of material between k and the ram centerline. v_k is obtained from the initial calculation

$$v_k = \pi t^0 (r_k^0)^2 \quad (5-3)$$

¹ Primed variables are from the previous increment.

Figure 5.1 - Coordinates

Figure 5.2 - Initial Blank

where

$$r_k^0 = \frac{k R_F}{K} \quad (5-4)$$

with K the total number of k points and R_F the initial blank radius (see Fig. 5,2).

The radius r_k is to the middle surface for each position k .¹

Incremental Displacement--Incremental Strain

When the ram is incremented, each point² k on the middle surface will change position $d\Gamma_k$ and dr_k amounts. Where

$$d\Gamma_k = \gamma_k - \gamma_k' \quad (5-5)$$

and

$$dr_k = r_k - r_k' \quad (5-6)$$

dr_k is composed of two parts:³

$$dr_k = dr_k^{(1)} + dr_k^{(2)} \quad (5-7)$$

¹ r_k assumed the same for all i values, via thin material.

²Only points in the plastic region will be displaced.

³ $dr_k^{(1)}$ is the radial component of $d\Gamma_k$ and $dr_k^{(2)}$ is due to the change in shape of the middle surface.

where

$$dr_k^{(1)} = d r_k \cos \phi_k \quad (5-8)$$

and

$$dr_k^{(2)} = r_k^{(2)} - r_k' \quad (5-9)$$

where $r_k^{(2)}$ is the radius obtained by assuming $\gamma = \gamma'$.

The circumferential incremental strain is then,

$$d\epsilon_\theta = \frac{dr_k}{r_k} \quad (5-10)$$

Using Equations (5-7) and (5-8), this becomes

$$d\epsilon_\theta = \frac{1}{r_k} (d r_k \cos \phi_k + dr_k^{(2)}) \quad (5-11)$$

From Figure 5.3, the incremental meridian strain γ is clearly

$$d\epsilon_\gamma = \frac{L - L'}{L'} \quad (5-12)$$

where

$$L' = L_m' \left(1 - \frac{z}{r_\gamma'}\right) \quad (5-13)$$

and

$$L = L_m \left(1 - \frac{z}{r_\gamma}\right) \quad (5-14)$$

with L_m' and L_m the before and after lengths of the middle surface, respectively.

Figure 5.3 - Meridian Geometry

The incremental strain of the middle surface is,

$$d\epsilon_{\gamma}^{(m)} = \frac{d}{d\gamma} (d\Gamma) = \frac{L_m - L_m^i}{L_m^i} \quad (5-15)$$

Combining Equations (5-12), (5-13), (5-14), and (5-15), the final equation for $d\epsilon_{\gamma}$ is obtained:

$$d\epsilon_{\gamma} = \left[1 + \frac{d}{d\gamma} (d\Gamma_k) \right] \frac{(1 - x_k z_i)}{(1 - x_k^i z_i)} - 1 \quad (5-16)$$

where the curvature terms x_k and x_k^i are,¹

$$x_k = \pm \frac{1}{r_{\gamma}} ; \quad x_k^i = \pm \frac{1}{r_{\gamma}^i} \quad (5-17)$$

The incremental thickness strain $d\epsilon_z$ is obtained from volumetric constancy,

$$d\epsilon_z = -d\epsilon_{\theta} - d\epsilon_{\gamma}. \quad (5-18)$$

Inplane Forces and Bending Moments

The inplane forces and bending moments² are obtained by integrating the stresses across the material thickness t --see Ref. (19).

¹Positive curvatures are shown in Fig. 5.3; negative curvatures are when $L > L_m$ for positive z values.

²Positive N_{γ} and N_{θ} are from tension stresses. Positive M_{γ} and M_{θ} tend to decrease the curvatures x and $1/r_{\theta}$, respectively.

$$N_{\gamma} = \int_{-t/2}^{t/2} \sigma_{\gamma} \left(1 - \frac{z}{r_{\theta}}\right) dz \simeq \int_{-t/2}^{t/2} \sigma_{\gamma} dz \quad (5-19)$$

$$N_{\theta} = \int_{-t/2}^{t/2} \sigma_{\theta} \left(1 - \frac{z}{r_{\gamma}}\right) dz = \int_{-t/2}^{t/2} \sigma_{\theta} (1 - xz) dz \quad (5-20)$$

$$M_{\gamma} = \int_{-t/2}^{t/2} \sigma_{\gamma} z \left(1 - \frac{z}{r_{\theta}}\right) dz \simeq \int_{-t/2}^{t/2} \sigma_{\gamma} z dz \quad (5-21)$$

$$M_{\theta} = \int_{-t/2}^{t/2} \sigma_{\theta} z \left(1 - \frac{z}{r_{\gamma}}\right) dz = \int_{-t/2}^{t/2} \sigma_{\theta} z (1 - xz) dz \quad (5-22)$$

where the z/r_{θ} terms are very small and may be ignored.¹ The current curvature x is from Equation (5-17) and the thickness t is from

$$t = \int_{-t'/2}^{t'/2} (1 + d\epsilon_x) dz. \quad (5-23)$$

¹For the class of problems studied in this work, the plastic region does not extend into areas of small radius r . With $r_{\theta} \geq r$; z/r_{θ} will always be $\ll 1$. However, the xz terms must be retained and are as large as 0.125 for some geometries in current industrial production.

Equilibrium Equations

The three non-trivial equilibrium equations, as obtained from Timoshenko and Woinowsky-Krieger (19), are:¹

$$\frac{d}{d\gamma} (N_{\gamma} r) - N_{\theta} \cos \phi - x r Q_{\gamma} + r Y = 0 \quad (5-24)$$

$$N_{\gamma} x r + N_{\theta} \sin \phi + \frac{d}{d\gamma} (Q_{\gamma} r) + r Z = 0 \quad (5-25)$$

$$\frac{d}{d\gamma} (M_{\gamma} r) - M_{\theta} \cos \phi - Q_{\gamma} r = 0 \quad (5-26)$$

where Q_{γ} is the shear force, and where

$$Y = \pm \mu_i P \quad (5-27)$$

with μ_i the coefficient of friction² between the material and the tools, and P the normal tool pressure. Therefore,

$$Z = \pm P. \quad (5-28)$$

¹These equations are written for the independent variable γ instead of ϕ as found in Ref. (19). Also, the curvature x has been used, as defined in Equation (5-17).

² μ_1 and μ_2 are for the inside and outside of the shell, respectively.

Using Equations (5-27) and (5-28) and by omitting the $\frac{d}{dy}(Q_y r)$ term¹ from Equation (5-25); the three equilibrium equations can be combined into a single equation in one unknown, where Q_y has been eliminated and where all the other terms are derived from the $d\bar{r}$ function.

The remaining equilibrium equation for each zone is then (see Fig. 5.4):

$$\text{Zone 1:} \quad \frac{d}{dy}(N_y r) - N_\theta = 0 \quad (5-29)$$

¹Dropping this term will result in errors as high as 10-15% in the tool pressure P , as calculated from Equation (5-25). However, when P is multiplied by μ_i ($\mu_i < 0.1$) per Equation (5-27), the error reflected into Equation (5-24) will be less than 2% of N_θ , for the examples analyzed here.

Figure 5.4 - External Forces Applied to Material

Zone 2: $\frac{d}{d\gamma}(N_{\gamma r}) - N_{\theta} \cos \phi - xrQ_{\gamma} - r\mu_1 P = 0$ (5-30)

where

$$P = N_{\gamma x} + \frac{N_{\theta}}{r} \sin \phi$$
 (5-31)

and

$$Q_{\gamma} = \frac{1}{r} \left[\frac{d}{d\gamma}(M_{\gamma r}) - M_{\theta} \cos \phi \right]$$
 (5-32)

Zone 3: $\frac{d}{d\gamma}(N_{\gamma r}) - N_{\theta} \cos \beta = 0$ (5-33)

and

$$P = -\frac{N_{\theta}}{r} \sin \phi^1$$
 (5-34)

Zone 4: $\frac{d}{d\gamma}(N_{\gamma r}) - N_{\theta} \cos \phi - xrQ_{\gamma} + r\mu_2 P = 0$ (5-35)

where

$$P = -N_{\gamma x} - \frac{N_{\theta}}{r} \sin \phi$$
 (5-36)

and Q_{γ} is from Equation (5-32).

~~Zones 5 & 6:~~ $\frac{d}{d\gamma}(N_{\gamma r}) - N_{\theta} - (\mu_1 + \mu_2) r \sigma_z = 0$ (5-37)

¹See Footnote 2, page 3.

CHAPTER VI

MEMBRANE SOLUTION

Basic Equations

The thirteen basic equations used for the membrane solution are¹ (5-11), (5-16), (5-18), (4-13), (3-1), (4-17), (4-18), (5-23), (5-19), (5-20), (5-25), and (5-24), plus one² not given earlier.

Shown here for convenience they are:

$$d\epsilon_{\theta} = \frac{1}{r_k} \left(d\Gamma_k \cos \phi_k + dr_k^{(2)} \right) \quad (6-1)$$

$$d\epsilon_{\gamma} = \frac{d}{d\gamma} (d\Gamma_k) \quad (6-2)$$

$$d\epsilon_z = -d\epsilon_{\gamma} - d\epsilon_{\theta} \quad (6-3)$$

$$d\epsilon = \frac{(1+R)^{\frac{1}{2}}}{(1+2R)} \left[(d\epsilon_{\theta} - Rd\epsilon_z)^2 + (d\epsilon_{\gamma} - Rd\epsilon_z)^2 + R(d\epsilon_{\gamma} - d\epsilon_{\theta})^2 \right]^{\frac{1}{2}} \quad (6-4)$$

$$\epsilon = \epsilon' + d\epsilon \quad (6-5)$$

$$\sigma = A + B\epsilon^n \quad (6-6)$$

$$\sigma_{\gamma} = \frac{1}{(1+R)^2 - R^2} \left\{ \frac{(1+R)\sigma}{d\epsilon} \left[(1+R)d\epsilon_{\gamma} + Rd\epsilon_{\theta} \right] + (1+2R)\sigma_z \right\} \quad (6-7)$$

¹In Chapters III, IV, and V, the equations were given for bending theory; whereas, the equations here are the simplified membrane theory version (i.e., stresses and strains are constant through the shell thickness).

²Equation (6-5), not shown before, is used to calculate the new effective strain from the old effective strain ϵ' .

$$\sigma_{\theta} = \frac{1}{[(1+R)^2 - R^2]} \left\{ \frac{(1+R)\sigma}{d\epsilon} \left[(1+R)d\epsilon_{\theta} + Rd\epsilon_{\gamma} \right] + (1+2R)\sigma_z \right\} \quad (6-8)$$

$$t = t'(1+d\epsilon_z) \quad (6-9)$$

$$N_{\gamma} = \sigma_{\gamma} t \quad (6-10)$$

$$N_{\theta} = \sigma_{\theta} t \quad (6-11)$$

$$P = \pm(N_{\gamma} x + \frac{N_{\theta}}{r} \sin \phi) \quad (6-12)$$

$$\frac{d}{d\gamma}(N_{\gamma} r) - N_{\theta} \cos \phi \pm r\mu_i P = 0 \quad (6-13)$$

Where the thirteen unknowns are: $d\Gamma$, $d\epsilon_{\theta}$, $d\epsilon_{\gamma}$, $d\epsilon_z$, $d\epsilon$, ϵ , σ , σ_{γ} , σ_{θ} , t , N_{γ} , N_{θ} , and P .¹

Preliminary Calculations

Before proceeding with the solution, it is required to calculate numerical values for β , γ_k , ϕ_k , r_k , $dr_k^{(2)}$ and σ_z .

From Figure 6.1,

$$T_1 = \left[T_2^2 - (\rho_2 + \rho_3)^2 \right]^{\frac{1}{2}} \quad (6-14)$$

¹These thirteen equations can be combined into a single equation with one unknown, $d\Gamma$; i.e., if $d\Gamma$ were known, all the unknowns are dependent on it and are easily calculated using Equations (6-1) through (6-13), in order.

Figure 6.1 - Geometry

where,

$$T_2 = \left[(\rho_4 - \rho_1)^2 + (\rho_2 + \rho_3 - v_o T)^2 \right]^{\frac{1}{2}} \quad (6-15)$$

Then β is,

$$\beta = \arccos \left(\frac{T_1}{T_2} \right) - \arctan \left(\frac{\rho_3 + \rho_2 - v_o T}{\rho_4 - \rho_1} \right) \quad (6-16)$$

where $v_o T$, the ram advancement distance, is known.

Next γ_k , ϕ_k , r_k and $dr_k^{(2)}$ are calculated as follows:¹

$$\gamma_k = \gamma_{k-1} + \gamma_k' - \gamma_{k-1}' \quad (6-17)^2$$

where γ_{k-1} is known from the previous point.

Next, ϕ_k is approximated as,

$$\phi_k = \phi_{k-1} + (\gamma_k - \gamma_{k-1}) x_k \quad (6-18)$$

where x_k is the curvature for the zone, the material, k , is in.

$r_k^{(2)}$ is then:

$$r_k^{(2)} = r_{k-1}^{(2)} + (\gamma_k - \gamma_{k-1}) \cos \left(\frac{\phi_k + \phi_{k-1}}{2} \right) \quad (6-19)$$

¹These calculations are made working from the ram centerline outward to F...

²This assumes no straining of the middle surface and is used in obtaining $dr_k^{(2)}$.

and from Equation (5-9), $dr_k^{(2)}$ is

$$dr_k^{(2)} = r_k^{(2)} - r_k' \quad (6-20)$$

Using a numerical approximation to Equation (5-2), more accurate values for γ_k are:

$$\gamma_k = \gamma_{k-1} + \frac{v_k - v_{k-1}}{2\pi \left(\frac{r_k^{(2)} + r_{k-1}'}{2} \right) \left(\frac{t_k' + t_{k-1}'}{2} \right)} \quad (6-21)$$

where v_k and v_{k-1} are known from Equation (5-3).

Knowing γ_k , ϕ_k is improved from Equation (6-18), i.e.,

$$\phi_k = \phi_{k-1} + (\gamma_k - \gamma_{k-1}) x_k \quad (6-22)$$

and r_k is from,

$$r_k = r_{k-1} + (\gamma_k - \gamma_{k-1}) \cos \left(\frac{\phi_k + \phi_{k-1}}{2} \right) \quad (6-23)$$

σ_z , for each k -position, is obtained via the initial holding pressure distribution assumption,¹ where

$$-2\pi \int_{r_H}^{r_P} \sigma_z r dr = F_H \quad (6-24)$$

with F_H the total holding pressure load.²

¹See Chapter I.

²Obtained from hydraulic pressure measurements on production presses.

Rim Solution

Now that the coordinates of the rim¹ are known, ϵ_θ is clearly,

$$\epsilon_\theta = \ln \left(\frac{r_F}{R_F} \right) \quad (6-25)$$

Using the fact that each total strain is the summation of the incremental strains, $d\epsilon_\theta$ is,

$$d\epsilon_\theta = \epsilon_\theta - \epsilon_\theta' \quad (6-26)$$

The boundary condition at F is

$$\sigma_\gamma = 0. \quad (6-27)$$

With $d\epsilon_\theta$, σ_γ and σ_z known, Equations (6-3) through (6-8), with unknowns $d\epsilon_\gamma$, $d\epsilon_z$, $d\epsilon$, ϵ , σ , and σ_θ , form a basis for solution on the rim.

These six equations are easily solved using the Newton-Raphson method - see Ref.(15).

To obtain interior solutions, an incremental displacement field, $d\Gamma$, that yields $d\epsilon_\theta$ and $d\epsilon_\gamma$ from above and also satisfies equilibrium on the rim is required. The $d\Gamma$ field used is,²

$$d\Gamma = A_1\gamma + A_2\gamma^2 + A_3\gamma^3. \quad (6-28)$$

¹Position F is referred to as the rim. All the calculations in this section are for the rim only.

² $A_0 = 0$; in order that $d\Gamma(0) = 0$.

Using Equations (6-1) and (6-2) for the rim position,

$$\begin{bmatrix} \gamma_F & \gamma_F^2 \\ 1 & 2\gamma \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} d\epsilon_\theta r_F - dr_F^{(2)} \\ d\epsilon_\gamma \end{bmatrix} - \begin{bmatrix} \gamma_F^3 \\ 3\gamma_F^2 \end{bmatrix} A_3 \quad (6-29)$$

Solving this, A_1 and A_2 are obtained as functions of A_3 . A_3 is then found by satisfying the equilibrium equation (6-13).¹

Interior Solutions

The interior is defined as: all points γ_k , where $0 \leq \gamma_k < \gamma_F$. An interior solution is then defined as finding a $d\Gamma$ function that is continuous through second derivatives² and also satisfies equilibrium at each point k .

The solution procedure is to work inward, point by point; that is, find a $d\Gamma$ function that satisfies the continuity requirements at $k+1$ and also satisfies equilibrium at k .

The $d\Gamma$ function used is,

$$d\Gamma = A_1\gamma + A_2\gamma^2 + A_3\gamma^3 + A_4\gamma^4 \quad (6-30)$$

¹Equation (6-13) becomes a non-linear function of A_3 and is solved by the Newton Method - see Ref. (15).

²Requiring continuity through second derivatives implies: satisfaction of compatibility for all values of γ and continuity of the equilibrium equation terms - Equation (6-13).

The continuity requirements are:

$$\begin{bmatrix} \gamma_{k+1} & \gamma_{k+1}^2 & \gamma_{k+1}^3 \\ 1 & 2\gamma_{k+1} & 3\gamma_{k+1}^2 \\ 0 & 2 & 6\gamma_{k+1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} - \begin{Bmatrix} \gamma_{k+1}^4 \\ 4\gamma_{k+1}^3 \\ 12\gamma_{k+1}^2 \end{Bmatrix} A_4 \quad (6-31)$$

where

$$\begin{aligned} C_1 &= d \Gamma_{k+1} \\ C_2 &= \frac{d}{d\gamma} (d \Gamma_{k+1}) \\ C_3 &= \frac{d^2}{d\gamma^2} (d \Gamma_{k+1}) \end{aligned} \quad (6-32)$$

Solving Equations(6-31), A_1 , A_2 , and A_3 are determined as functions of A_4 . A_4 is then found by satisfying the equilibrium equation (6-13).¹

The point-by-point inward solution continues until $d\epsilon$ becomes ≤ 0 .² This is assumed to be the end of the plastic region.

With all stresses known, the ram load P_R can be calculated with:

$$P_R = 2 \pi r_k N_\gamma \sin \phi_k \quad (6-33)$$

where r_k , ϕ_k , and N_γ are the results for any point k in Zone 3.

¹Again, this is solved via the Newton method.

²This is predicted by extrapolating the $d\epsilon$ function inward to γ_{k-1} using a straight line approximation through γ_k and γ_{k+1} .

Results

Data for a steel cup¹ drawing operation has been obtained using the membrane theory procedure given in this chapter.

The tool geometry and the other parameters used are presented in Fig. 6.2. The maximum ram load and die tool pressure is also given in this figure. Stress-strain data has been plotted for the even-numbered² increments, for a ram increment distance of 0.400 inches.³ abscissa for each graph is material position number. This solution was obtained using 100 material positions; i.e., $K = 100$ in Equation (5-4).

The meridian distance y_k and radius r_k are given in Figures 6.3 and 6.4, respectively.

The total strains: ϵ , ϵ_y , ϵ_θ , and ϵ_z are shown in Figures 6.5, 6.6, 6.7, and 6.8, respectively. Note, there is no straining for material position numbers less than 53, that is, material adjacent to that flat end of the ram does not deform.

¹The material specifications are given in Appendix A. The stress-strain equation and R-value used are per Chapter III. The tool geometry is for an actual cup in current production at the Metal-Flo Corporation of Columbus, Ohio.

²Some curves are too close together if all increments are plotted.

³Accurate results have been obtained for ram increment distances as high as 8 times material thickness.

Stress curves for σ_r , σ_γ , and σ_θ are given in Figures 6.9, 6.10, and 6.11, respectively. The σ_γ and σ_θ curves start at the plastic region boundary and end at the rim.

Ram load-displacement curves for both membrane and bending theories are given in Figure 7.6, where a comparison of the two theories can be made.

Membrane theory results are also superimposed on the bending theory charts--Figures 7.3, 7.4, and 7.5.

Scale = 1/2 size

Incremental Distance	= 0.400 inches
Ram Load (max.)	= 56,360 lbs.
Blank Radius	= 8.375 inches
Blank Thickness	= 0.0635 inches
ρ_1	= 4.250 inches
ρ_2	= 0.5322 inches
ρ_3	= 0.5322 inches
ρ_4	= 5.321 inches
Die Pressure (max.)	= 3,895 psi
R-Value	= 1.51
$\mu_1 = \mu_2$	= 0.06

Figure 6.2—Tool and Other Specifications

MEMBRANE THEORY RESULTS

FIGURE 6.3--GAMMA DISTANCE

MEMBRANE THEORY RESULTS

FIGURE 6.4--RADIUS TO MIDDLE SURFACE

MEMBRANE THEORY RESULTS

FIGURE 6.5---TOTAL EFFECTIVE STRAIN

MEMBRANE THEORY RESULTS

FIGURE 6.6-----TOTAL GAMMA STRAIN

MEMBRANE THEORY RESULTS

FIGURE 6.7-----TOTAL THETA STRAIN

MEMBRANE THEORY RESULTS

FIGURE 6.8--TOTAL THICKNESS STRAIN

FIGURE 6.9---EFFECTIVE STRESS

MEMBRANE THEORY RESULTS

FIGURE 6.10-----GAMMA STRESS

MEMBRANE THEORY RESULTS

FIGURE 6.11---THETA STRESS

CHAPTER VII

BENDING SOLUTION

Basic Equations

The sixteen basic equations used for the bending solution are (5-11), (5-16), (5-18), (4-13), (6-5), (3-1), (4-17), (4-18), (5-23), (5-19), (5-20), (5-21), (5-22), (5-24), (5-25), and (5-26). Repeated here for convenience, they are:

$$d\epsilon_{\theta} = \frac{1}{r_k} (d\Gamma_k \cos \phi_k + dr_k^{(2)}) \quad (7-1)$$

$$d\epsilon_{\gamma} = \left[1 + \frac{d}{\gamma} (d\Gamma_k) \right] \frac{(1-x_k z_i)}{(1-x_k z_i)} - 1 \quad (7-2)$$

$$d\epsilon_z = -d\epsilon_{\theta} - d\epsilon_{\gamma} \quad (7-3)$$

$$d\epsilon = \frac{(1+R)^{\frac{1}{2}}}{(1+2R)} \left[(d\epsilon_{\theta} - Rd\epsilon_z)^2 + (d\epsilon_{\gamma} - Rd\epsilon_z)^2 + R(d\epsilon_{\gamma} - d\epsilon_{\theta})^2 \right]^{\frac{1}{2}} \quad (7-4)$$

$$\epsilon = \epsilon' + d\epsilon \quad (7-5)$$

$$\sigma = A + B\epsilon^n \quad (7-6)$$

$$\sigma_{\gamma} = \frac{1}{[(1+R)^2 - R^2]} \left\{ \frac{(1+R)}{d\epsilon} [(1+R)d\epsilon_{\gamma} + Rd\epsilon_{\theta}] + (1+2R)\sigma_z \right\} \quad (7-7)$$

$$\sigma_{\theta} = \frac{1}{[(1+R)^2 - R^2]} \left\{ \frac{(1+R)}{d\epsilon} [(1+R)d\epsilon_{\theta} + Rd\epsilon_{\gamma}] + (1+2R)\sigma_z \right\} \quad (7-8)$$

$$t = \int_{-t^*/2}^{t^*/2} (1+d\epsilon_z) dz \quad (7-9)$$

$$N_{\gamma} = \int_{-t/2}^{t/2} \sigma_{\gamma} dz \quad (7-10)$$

$$N_{\theta} = \int_{-t/2}^{t/2} \sigma_{\theta} (1-xz) dz \quad (7-11)$$

$$M_{\gamma} = \int_{-t/2}^{t/2} \sigma_{\gamma} z dz \quad (7-12)$$

$$M_{\theta} = \int_{-t/2}^{t/2} \sigma_{\theta} z (1-xz) dz \quad (7-13)$$

$$Q_{\gamma} = \frac{1}{r} \left[\frac{d}{dy} (M_{\gamma} r) - M_{\theta} \cos \phi \right] \quad (7-14)$$

$$P = \pm (N_{\gamma} x + \frac{N_{\theta}}{r} \sin \phi) \quad (7-15)$$

$$\frac{d}{dy} (N_{\gamma} r) - N_{\theta} \cos \phi - x r Q_{\gamma} \pm r \mu_i P = 0 \quad (7-16)$$

where the sixteen unknowns are: $d\Gamma$, $d\epsilon_{\theta}$, $d\epsilon_{\gamma}$, $d\epsilon_z$, $d\epsilon, \epsilon$, σ , σ_{γ} , σ_{θ} , t , N_{γ} , N_{θ} , M_{γ} , M_{θ} , Q_{γ} , and P .¹

¹Similar to the membrane equations, these sixteen equations can also be combined into one equation in one unknown, $d\Gamma$.

Preliminary Calculations

Calculations for values of β , γ_k , ϕ_k , r_k , $dr_k^{(2)}$, and σ_z are done exactly the same as for the membrane solution. (See Chapter VI, p. 29.)

Rim Solution

The boundary condition that should be used for the rim position is,

$$N_\gamma = 0 \quad (7-17)$$

where N_γ is obtained from Equation (7-10).

This boundary condition is replaced by

$$\sigma_\gamma(m, F) = 0 \quad (7-18)$$

where $\sigma_\gamma(m, F)$ is σ_γ for the middle surface at γ_F .

Equations (7-17) and (7-18) yield exactly the same results when $\gamma_F > \gamma_D$, and are nearly equal when F is on the die.¹

The rim solution now proceeds exactly as for the membrane solution, where now all the calculations apply to the middle surface only (see Chapter VI, p. 33).

¹When F is on the die, a fictitious bending moment $M_\gamma(F)$ must be admitted to maintain the assumed curvature, $-1/\rho_3$, at F .

The stresses and strains for positions not on the middle surface are then easily calculated using Equations (7-1) through (7-8). N_y , N_θ , M_y , M_θ , Q_y , and P are from Equations (7-9) through (7-15).¹

Interior Solutions

Discontinuities

There are two types of discontinuities occurring at a finite number of points. Type I--discontinuous curvatures κ at points A, B, C, and D (see Figures 7.1 and 7.2), showing expected $d\epsilon_y$ strains. Type II--material just bent(or just unbent) adjacent to material bent (or unbent) the prior increment. These are at points A', B', C', and D' (see Figure 7.2).

For both types, $d\epsilon_y$ is not continuous for all z , where $-t/2 \leq z \leq t/2$.

Solutions. No Discontinuities

When there is no discontinuity between y_k and y_{k+1} , the solution procedure is the same as for the membrane theory. That is, Equation (6-30) is used for the $d\Gamma$ function and is subjected to the continuity equations (6-31) to find A_1 , A_2 , and A_3 , as functions of A_4 . A_4 is then found by satisfying the equilibrium equation (7-16).

¹Numerical values for all the integrals in this chapter are obtained using the trapezoidal rule - see Ref. (15).

Figure 7.1 - After Increment No. 1

Figure 7.2 - After Increment No. 2

Solutions, Across Discontinuities

When $\gamma_k < \gamma_s \leq \gamma_{k+1}$, where at γ_s is either type of discontinuity, $d\epsilon_\gamma$ cannot be expected to be continuous. This implies that $\frac{d}{d\gamma}(d\Gamma_s)$ and higher order derivatives will not be continuous.

From physical considerations continuity of $d\Gamma$ and N_γ are required. Note that continuous $d\Gamma$ implies continuity of $d\epsilon_\theta$.

Starting with a solution at γ_{k+1} , the required continuity is attained by first extrapolating the $d\Gamma_{k+1}$ function from γ_{k+1} to γ_s^+ and calculating $d\Gamma_{k+1}^+$ and N_{γ^+} there.

Next, a new $d\Gamma_k$ function is sought that yields $d\Gamma_k^-$ and N_{γ^-} when extended to γ_s^- , such that

$$N_{\gamma^-} = N_{\gamma^+} \quad (7-19)$$

$$d\Gamma_k^- = d\Gamma_{k+1}^+ \quad (7-20)$$

and also satisfies the equilibrium equation (7-16) at γ_k . The $d\Gamma_k$ function used is, again,

$$d\Gamma_k = A_1\gamma + A_2\gamma^2 + A_3\gamma^3 \quad (7-21)$$

A_1 , A_2 , and A_3 are found by the Newton-Raphson method.

The solution continues inward, as before, until $d\epsilon$ for the middle surface becomes ≤ 0 .¹ This is assumed to be the end of the plastic region.²

¹See Footnote 2, page 35.

²The plastic region may extend a little further inward, as in the case of pure bending, where $d\epsilon$ for the middle surface is zero. This is not considered important and is ignored.

The ram load F_R is calculated from

$$F_R = 2 \pi r_k \left[N_\gamma \sin \phi_k + Q_\gamma \cos \phi_k \right] \quad (7-22)$$

where r_k , ϕ_k , N_γ , and Q_γ are for the γ_k position nearest the ram.

Results

Data for the same cup drawing operation analyzed using membrane theory, have been generated using the bending theory method presented in this chapter. The material description, tool geometry and all other parameters are given on pages 36 - 38. Eleven z_i points, evenly distributed through the thickness of the shell, were used as required in the bending theory equations; i.e., $I = 10$ in Equation (5-1).

Stress-strain curves for bending theory similar to the membrane theory curves are impractical to plot, due to the many discontinuities involved. Also many more curves would be required as the stresses and strains vary for each different value of z_i .

The method used to display the bending theory results is to plot the stresses and strains as functions of z , for three different material positions¹ plotted are for $k = 65$, 77 , and 88 , shown in Figures 7.3, 7.4, and 7.5, respectively.

¹The material positions and increment numbers plotted were chosen to show the most interesting and important results.

Column One, of these figures, gives the increment number for all plots in that row. Column Two shows the geometry, where the material position k is marked with a small circle. The next four columns are plots of ϵ , ϵ_y , ϵ_θ , and ϵ_z versus z , respectively. The last three columns are σ , σ_y , and σ_θ versus z , respectively. Note, the z -axis has been drawn at an angle such that it remains normal to the shell middle surface. Also the positive z and strain (or stress) directions are marked with arrow-heads. The stress and strain scales are calibrated along the bottom of each figure.

Results from the membrane theory are shown with dashed lines.

Ram load-displacement curves for both membrane and bending theories are given in Figure 7.6.

BENDING RESULTS

MEMBR

FIGURE 7.3

STRAIN SCALE ϵ_{LL}

MEMBRANE RESULTS - - - - -

MATERIAL NO. 65

BENDING RESULTS

MEMBRANE

FIGURE 7.4

STRAIN SCALE

MEMBRANE RESULTS - - - - -

MATERIAL NO. 77

SCALE 0 50%

STRESS SCALE 0 100 K

BENDING RESULTS

MEMBR

INC NO.	GEOMETRY 1/4 SIZE	EFFECTIVE STRAIN	GAMMA STRAIN	
01
08
09
10
11

FIGURE 7.5

STRAIN SCALE 0.1

MEMBRANE RESULTS - - - - -

MATERIAL NO. 88

N SCALE 0 50%

STRESS SCALE 0 100K

FIGURE 7.6--RAM LOAD VS. DISPLACEMENT

CHAPTER VIII

DISCUSSION

Without experimental work to compare the solutions with, some other method of verification should be attempted. One way to do this, is to examine the curves to see if all results look reasonable.

Consider first, the membrane theory curves: since the material is pulled in by positive N_y forces, positive ϵ_y and σ_y values would be expected. Also, the boundary condition, $\sigma_y = 0$ on the rim, should be noted. Figures 6.6 and 6.10 do show these expected results.

Since the radii of all material, on both the flat and curved portions of the die, are decreasing, negative circumferential stresses and strains are anticipated. Figures 6.7 and 6.11 show this to be true.

It is well known that the material thickness increases as it is drawn in. Also known is that the material thins slightly as it is stretched around the ram radius. Figure 6.8 does reflect these facts.

As the material is work-hardened, the effective stress and strain values should increase. Figures 6.5 and 6.9 do check relative to that fact.

Next, consider the bending theory results (Figures 7.3, 7.4, and 7.5). The first row of each of these charts shows the bending and membrane theory results are the same. This is expected as the material has not yet been bent.

The second row of each chart shows the material just after first being bent. ϵ_y and σ_y do show a bending type of distribution as would be expected. ϵ_θ is constant, per an initial assumption, and agrees very closely with the membrane results.

Row 3 shows the results while the material is on the curved portion of the die, where the strain increments are constant through the thickness. Therefore, ϵ_y and ϵ_θ are merely increased and decreased, respectively, as one would expect.

Row 4 shows the material just after being unbent. A bending type of stress distribution opposite of Row 2 is observed. This is as expected.

Row 5 shows the residual strains after the plastic deformation is complete for that material position. σ_y and σ_θ are shown to be zero at this point - actually they are not zero, but are indeterminate by the method used here. However, it should be noted that σ_y can be approximated from a consideration of equilibrium for any position in Zone 3. That is,

$$\sigma_y = \frac{F_R}{A \sin \beta} \quad (8-1)$$

where, F_R is the ram load as obtained from Equation (7-22) and A is the cross-sectional area of the cup wall. When the ram has

advanced to a position such that the cup walls are essentially vertical, σ_y becomes proportional to the ram load. For these solutions, σ_y (maximum) = 29,209 psi.

The final results obtained by the membrane and bending theories are very similar, as observed in the last row of Figures 7.3, 7.4, and 7.5. The effective stress-strain values are somewhat higher near the inner and outer surfaces of the shell, using bending theory, but are nearly equal elsewhere. These higher values are expected as the material has been work-hardened from bending.

Figure 7.6 shows the membrane theory ram loads to be higher than the bending theory loads. One explanation of this is, that the bending theory, being a more correct theory, will predict a more correct ram load, whereas the membrane theory being subjected to more constraints on the material, will require more energy to form a part and hence, result in greater ram loads. This seems reasonable relative to a consideration of upper-bound theory, used to solve many metal-working problems.

The basic conclusion is that membrane theory appears adequate for analyzing cups of moderate die radius/material thickness ratios, such as the one analyzed in this work.

Solution Limitations

The solution method presented here works well for geometries and material characteristics, such that the plastic region does not extend into the ram centerline.

One hard-to-analyze geometry is when the ram end is of hemispherical shape. The main difficulty is that the zone of material between the ram and the die is large and the assumption of conical shape leads to large errors in the stresses and strains obtained.

The averaging of the R-values and the other material characteristics also introduces errors, hence the method should not be applied to the drawing of materials where these values vary greatly.

Neglecting the shear stresses due to friction, also can lead to errors, especially when the tool and holding pressures are large. For thin materials, these errors are not sufficient, however.

Attempts to predict the limiting draw ratio also have been difficult due to the lack of an acceptable failure theory for sheet materials. The fact that the plastic region generally extends inward to the ram centerline for cups approaching maximum draw ratios, also adds to the problem.

This method does, however, work very well in analyzing cups for geometries and materials found in industrial production.

Further Studies

This method can very easily be applied to analyzing drawing and redrawing for rams and dies of many different geometries. When enough computer programs are available, studies can be made to improve tool geometries, and thus, lengthen life and reduce production costs.

An efficient method for analyzing geometries of larger zones of material between the ram and die should be sought.

An acceptable failure theory for the highly ductile deep drawing materials is required before prediction of the limiting draw ratio can be achieved.

In conclusion, much future work needs to be done.

APPENDIX A
MATERIAL DESCRIPTION

Basic description: CRS - .06 C Max. -DQ-Ak

Cold Rolled Sheet Steel

Drawing Quality - Aluminum killed

Typical Chemistry:

Carbon	.03%
Manganese	.30%
Phosphorus	.01%
Sulfur	.018%
Silicon	.02%

TABLE 1

STRESS-STRAIN DATA
FOR BLANK NO. 1

STRAIN (TRUE)	1	2	3	4	5	6
0.0000	20763.	20763.	21941.	20153.	21331.	20805.
0.0050	21788.	22095.	22969.	21788.	22663.	22446.
0.0099	24055.	23130.	24007.	23561.	23391.	23485.
0.0149	26034.	25786.	26415.	25848.	25549.	25216.
0.0198	28031.	27594.	28161.	28031.	27353.	27150.
0.0247	29670.	29295.	30361.	30108.	29049.	28914.
0.0296	30947.	30696.	32078.	31576.	30634.	30316.
0.0344	32362.	32109.	33432.	33183.	31982.	31730.
0.0392	33661.	33280.	34862.	34677.	33087.	32965.
0.0440	34780.	34525.	36112.	35929.	34392.	34146.
0.0488	35908.	35652.	37117.	37191.	35453.	35017.
0.0583	37804.	37545.	39344.	39358.	37341.	37036.
0.0677	39402.	39206.	41084.	41101.	38932.	38630.
0.0770	40892.	40760.	42653.	42606.	40349.	40114.
0.0862	42202.	41926.	43978.	43933.	41520.	41420.
0.0953	43194.	43059.	45186.	45075.	42772.	42540.
0.1044	44197.	44129.	46206.	46230.	43838.	43606.
0.1133	45142.	45005.	47236.	47194.	44642.	44410.
0.1222	45959.	45890.	48002.	48029.	45523.	45221.
0.1310	46714.	46714.	48844.	48802.	46273.	46110.
0.1398	47405.	47475.	49553.	49511.	47100.	46796.
0.1506	48346.	48275.	50446.	50405.	47895.	47660.
0.1613	49009.	49081.	51203.	51162.	48625.	48316.
0.1719	49748.	49748.	51965.	51851.	49359.	49121.
0.1823	50492.	50492.	52731.	52617.	50098.	49932.
0.1927	51166.	51166.	53280.	53165.	50620.	50526.
0.2029	51768.	51768.	53830.	53713.	51217.	51122.
0.2131	52372.	52372.	54379.	54261.	51890.	51719.
0.2231	52901.	52901.	54928.	54809.	52567.	52395.
0.2331	53507.	53430.	55478.	55357.	53092.	52919.
0.2429	54037.	53959.	55949.	55983.	53618.	53443.
0.2527	54645.	54488.	56498.	56532.	54144.	53967.
0.2624	55176.	55017.	57046.	57160.	54669.	54491.

TABLE 1

STRESS-STRAIN DATA
FOR BLANK NO. 2

STRAIN (TRUE)	1	2	STRESSES (TRUE)				6
			3	4	5		
0.0000	20615.	21027.	21495.	20965.	21111.	20275.	
0.0050	21646.	22357.	22714.	22309.	22139.	21481.	
0.0099	22251.	23391.	23571.	23354.	22867.	22205.	
0.0149	24048.	25363.	25745.	25035.	24968.	23864.	
0.0198	25924.	26980.	27689.	26919.	26776.	25414.	
0.0247	27754.	28736.	29776.	28884.	28726.	27104.	
0.0296	29220.	30132.	31756.	30422.	30379.	28620.	
0.0344	30636.	31477.	33117.	32229.	31793.	30402.	
0.0392	31936.	32833.	34555.	33539.	32965.	31883.	
0.0440	33054.	33883.	35812.	35182.	34019.	33121.	
0.0488	34246.	34941.	37015.	36451.	35017.	34241.	
0.0583	36138.	36824.	39125.	38759.	36842.	36315.	
0.0677	37730.	38345.	40875.	40708.	38564.	38030.	
0.0770	39345.	39888.	42450.	42354.	40114.	39573.	
0.0862	40649.	41121.	43915.	43755.	41420.	40871.	
0.0953	41766.	42236.	45128.	44970.	42540.	42119.	
0.1044	42897.	43297.	46220.	46063.	43606.	43180.	
0.1133	43835.	44164.	47187.	47100.	44547.	44184.	
0.1222	44644.	45041.	48094.	48008.	45429.	44993.	
0.1310	45460.	45856.	48870.	48784.	46180.	45878.	
0.1398	46142.	46609.	49651.	49567.	47007.	46492.	
0.1506	47215.	47470.	50548.	50464.	47802.	47423.	
0.1613	47868.	48195.	51308.	51296.	48604.	48220.	
0.1719	48596.	49070.	51927.	52062.	49339.	48950.	
0.1823	49477.	49733.	52768.	52684.	50005.	49759.	
0.1927	50067.	50398.	53392.	53307.	50749.	50351.	
0.2029	50658.	50993.	54018.	54008.	51347.	51020.	
0.2131	51251.	51588.	54645.	54635.	51947.	51616.	
0.2231	51923.	52186.	55197.	55187.	52471.	52214.	
0.2331	52442.	52708.	55749.	55739.	52996.	52736.	
0.2429	52962.	53307.	56301.	56291.	53521.	53258.	
0.2527	53481.	53830.	56774.	56843.	54045.	53780.	
0.2624	54000.	54352.	57325.	57395.	54570.	54302.	

TABLE 1

STRESS-STRAIN DATA
FOR BLANK NO. 3

STRAIN (TRUE)	1	2	STRESSES (TRUE)				5	6
			3	4				
0.0000	21417.	21252.	21819.	21679.	20308.		20308.	
0.0050	21893.	22401.	22357.	23138.	21522.		21337.	
0.0099	22249.	23500.	22776.	24117.	22748.		22002.	
0.0149	24285.	25166.	26353.	26158.	24735.		23798.	
0.0198	26214.	27034.	28410.	28280.	26677.		25861.	
0.0247	28099.	28794.	30361.	30234.	28511.		27565.	
0.0296	29686.	30129.	32015.	31954.	30108.		29220.	
0.0344	31160.	31477.	33559.	33562.	31591.		30700.	
0.0392	32519.	32708.	34925.	34931.	32768.		32128.	
0.0440	33635.	33823.	36303.	36120.	33954.		33311.	
0.0488	34567.	34754.	37245.	37191.	34957.		34375.	
0.0583	36518.	36639.	39344.	39228.	36855.		36268.	
0.0677	38172.	38161.	41019.	41036.	38586.		38059.	
0.0770	39652.	39638.	42653.	42606.	40076.		39678.	
0.0862	40953.	41004.	43912.	43866.	41454.		41051.	
0.0953	42203.	42052.	45119.	45142.	42578.		42240.	
0.1044	43131.	43044.	46138.	46162.	43580.		43307.	
0.1133	44136.	43911.	47168.	47125.	44524.		44318.	
0.1222	45014.	44786.	48071.	48029.	45339.		45130.	
0.1310	45761.	45530.	48844.	48802.	46231.		45951.	
0.1398	46514.	46281.	49623.	49582.	46849.		46708.	
0.1506	47447.	47139.	50375.	50405.	47788.		47573.	
0.1613	48245.	48005.	51131.	51090.	48518.		48374.	
0.1719	48976.	48660.	51892.	51851.	49254.		49108.	
0.1823	49638.	49319.	52585.	52617.	49920.		49846.	
0.1927	50378.	50055.	53280.	53313.	50664.		50589.	
0.2029	51047.	50795.	53830.	53863.	51262.		51262.	
0.2131	51644.	51387.	54379.	54412.	51861.		51785.	
0.2231	52165.	51908.	54928.	54962.	52462.		52308.	
0.2331	52764.	52427.	55478.	55511.	52986.		52831.	
0.2429	53287.	52947.	56027.	56061.	53511.		53354.	
0.2527	53809.	53466.	56576.	56611.	54035.		53877.	
0.2624	54332.	53985.	57125.	57160.	54560.		54400.	

TABLE 1

STRESS-STRAIN DATA
FOR BLANK NO. 4

STRAIN (TRUE)	1	2	STRESSES (TRUE)				6
			3	4	5		
0.0000	20722.	20763.	21514.	21331.	20348.		20112.
0.0050	22051.	22217.	22296.	22847.	21751.		21132.
0.0099	23084.	23315.	22899.	24007.	22669.		21976.
0.0149	24930.	25290.	25054.	26167.	24784.		23878.
0.0198	26856.	27096.	27104.	28161.	26667.		25985.
0.0247	28611.	28731.	29236.	30111.	28252.		27674.
0.0296	30069.	30192.	31136.	31890.	29850.		29253.
0.0344	31414.	31477.	32675.	33369.	31208.		30783.
0.0392	32706.	32645.	34101.	34735.	32641.		32073.
0.0440	33819.	33823.	35411.	35921.	33765.		33246.
0.0488	34749.	34754.	36541.	37053.	34833.		34365.
0.0583	36630.	36509.	38633.	39021.	36668.		36436.
0.0677	38215.	38030.	40497.	40823.	38399.		38019.
0.0770	39691.	39573.	42126.	42390.	39890.		39625.
0.0862	41055.	40871.	43579.	43712.	41268.		40922.
0.0953	42169.	41985.	44851.	44985.	42460.		42102.
0.1044	43229.	42975.	45867.	46070.	43531.		43161.
0.1133	44028.	43911.	46827.	47031.	44406.		44096.
0.1222	44903.	44717.	47727.	47865.	45291.		44903.
0.1310	45787.	45530.	48636.	48705.	46113.		45787.
0.1398	46469.	46211.	49272.	49413.	46801.		46399.
0.1506	47328.	47068.	50233.	50304.	47740.		47399.
0.1613	48124.	47861.	50988.	50988.	48471.		48124.
0.1719	48853.	48588.	51748.	51748.	49206.		48925.
0.1823	49659.	49319.	52512.	52512.	49946.		49659.
0.1927	50325.	49981.	53207.	53207.	50616.		50251.
0.2029	50918.	50721.	53755.	53830.	51289.		50843.
0.2131	51513.	51314.	54304.	54379.	51888.		51438.
0.2231	52110.	51908.	54852.	54928.	52413.		52033.
0.2331	52708.	52427.	55401.	55478.	52937.		52554.
0.2429	53229.	52947.	55949.	56027.	53461.		53074.
0.2527	53751.	53466.	56498.	56576.	53985.		53594.
0.2624	54273.	53985.	57046.	57125.	54509.		54115.

TABLE 1---AVERAGE STRESSES

ϵ	σ	$\sigma - A$
0.0000	20951.	0.
0.0050	22139.	1188.
0.0099	23109.	2158.
0.0149	25186.	4235.
0.0198	27099.	6148.
0.0247	28940.	7989.
0.0296	30512.	9561.
0.0344	31976.	11025.
0.0392	33297.	12346.
0.0440	34504.	13553.
0.0488	35568.	14617.
0.0583	37549.	16598.
0.0677	39234.	18284.
0.0770	40779.	19828.
0.0862	42098.	21148.
0.0953	43270.	22319.
0.1044	44319.	23369.
0.1133	45255.	24305.
0.1222	46109.	25158.
0.1310	46923.	25973.
0.1398	47643.	26692.
0.1506	48531.	27580.
0.1613	49283.	28333.
0.1719	50023.	29072.
0.1823	50763.	29812.
0.1927	51419.	30468.
0.2029	52036.	31086.
0.2131	52624.	31673.
0.2231	53194.	32243.
0.2331	53735.	32785.
0.2429	54271.	33320.
0.2527	54803.	33852.
0.2624	55338.	34387.

TABLE 2
R-VALUE DATA

Blank No.	Sample No.	<u>w</u>	<u>l</u>	<u>R</u>
1	1	.6782	.8787	1.75
	2	.6801	.8752	1.74
	3	.6886	.8800	1.15
	4	.6878	.8813	1.16
	5	.6750	.8794	1.96
	6	.6727	.8829	2.00
2	1	.6798	.8748	1.76
	2	.6802	.8736	1.78
	3	.6899	.8776	1.14
	4	.6888	.8776	1.18
	5	.6726	.8835	1.99
	6	.6726	.8825	2.03
3	1	.6791	.8773	1.73
	2	.6767	.8821	1.74
	3	.6888	.8806	1.13
	4	.6880	.8806	1.17
	5	.6703	.8888	1.97
	6	.6740	.8808	1.97
4	1	.6798	.8756	1.73
	2	.6767	.8829	1.70
	3	.6877	.8841	1.12
	4	.6895	.8770	1.17
	5	.6727	.8840	1.96
	6	.6745	.8806	1.95

$$w_o = l_o = 0.7500$$

$$R(\text{average}) = 1.51$$

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