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IN AN ACOUSTIC MEDIUM.

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DYNAMIC STABILITY OF A CYLINDRICAL SHELL
IN AN ACOUSTIC MEDIUM

DISSERTATION

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the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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The Ohio State University
1971

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INTRODUCTION

The interaction of fluids and solids in a dynamic environment has received considerable attention, lately. The equations governing the fluid and solid are coupled by the boundary condition that the fluid may not penetrate the solid. For an ideal fluid this is equivalent to equating the components of velocity of the fluid and solid normal to their interface. Since the equations of motion of the fluid and solid must be solved simultaneously, these problems are usually rather complicated.

Studies have considered the response of a submerged elastic structure to an acoustic wave. Under the action of acoustic loading, the elastic structure is forced to oscillate. The characteristics of the acoustic wave are changed due to not only the existence of the structure that causes scattering of the acoustic wave but also the motion of the elastic structure.

A second problem of interest is the vibration of an elastic structure in an acoustic medium and the associated radiation of sound. The dynamic characteristics, such as natural frequency and amplitude of forced vibration, of elastic structures submerged in a fluid may differ considerably from their characteristics in vacuo, see e. g. [1]-[21]^{*}. The radiation of sound

* Numbers in brackets refer to References.

created by the oscillation of a structure has received much attention since it was introduced by Stokes [3] .

Also, the dynamical behavior of shells containing a fluid have been considered. These studies have been concerned with either the dynamics and stability of rigid bodies containing a fluid, or the vibrations of a body containing a fluid [4] . Most of the investigators treated the shells as rigid bodies, however, a few investigators, e.g. Poincaré [5] , took into account the elasticity of the shell.

Apparently, the stability of the forced oscillatory motion of elastic structures submerged in a fluid has not been considered. Nevertheless the primary motion of these structures may become unstable. To determine the stability of this motion the system is perturbed. If the perturbation grows temporally the motion is unstable, otherwise it is stable. Under certain conditions the perturbed equations of motion may be reduced to a second-order differential equation with periodic coefficients, the Mathieu-Hill equation. The external loading occurs as a parameter in the perturbed equation and is referred to as parametric loading. It can be shown that for certain relationships between the frequency of the external loading and the natural frequency of vibration, parametric resonance may occur and the structure may become dynamically unstable. This resonance due to nonlinear interactions is not to be confused with the usual forced vibrational resonance. Also, it is to be expected that the resonance conditions will be different when the structures are vibrating in an acoustic medium and in vacuo.

Parametric excitation of the structure submerged in an acoustic medium is different from another type of solid-fluid interaction stability problem, the so called " flutter " in aeroelastic instability. " Flutter " of the airfoil is the self-excited oscillation of the external skin when exposed to an airflow on one side. The characteristic of this problem is that the elastic matrix of the governing differential equations is asymmetric, hence the problem is circulatory [6] . When the oscillation of the airfoil is near the critical point, it continuously absorbs energy from the airflow so that the amplitude of the oscillation increases. Similar effects can be found in suspension bridges ; the collapse of the Tacoma Narrow Bridge in 1940 due to flutter being a classical example.

In the present study, we consider a thin, elastic, infinitely long cylindrical shell, freely suspended in an infinite fluid medium, excited by harmonically varying internal pressure. Ordinarily only a radial or breathing motion would occur. However, at certain frequencies of the pressure, a slight disturbance of the shell may cause the amplitudes of the nonaxisymmetric modes to increase to large values. In other words, the radial motion becomes unstable. The development of nonaxisymmetric motion is due to the interaction of the circumferential stress with the flexural curvature.

Woinowsky-Krieger [7] , Dzhanelidze and Radtsig [8] , and Yao [9] have investigated the dynamic stability of forced oscillation of cylindrical shells and rings in vacuo. The present study is undertaken to examine the stability criteria for a shell in a fluid and compare it with that in vacuo.

The solution by taking into account the coupling effect of extension and bending of the shell and the solution by neglecting the extension of the shell were developed. In addition a study of the response, taking into consideration the kinematic nonlinearities, of the shell, after it was parametrically excited, has also been investigated.

CHAPTER I

MATHEMATICAL FORMULATION OF THE PROBLEM

In this chapter we derive the differential equations governing the dynamic behavior of an infinitely long, cylindrical shell in a fluid. The shell is subjected to a temporally harmonic pressure. The analysis is based on the following assumptions :

- a. The shell material is elastic, isotropic, and homogeneous ;
- b. The stress component normal to the middle surface is small compared with the other stress components and may be neglected in the stress-strain relations ;
- c. The normals to the unstrained middle surface deform into the normals to the deformed middle surface and undergo no extension ;
- d. The loading is normal to the midsurface and does not vary along the axial direction, i.e. the analysis is restricted to plane strain ;
- e. The fluid medium is homogeneous, nonviscous ;
- f. The domain of the fluid is infinite so that there is no boundary to reflect the radiating acoustic wave.

*

1. Strain-Displacement Relations

By the assumption "d", any cross section normal to the generator of the shell can represent the overall configuration of the deformed shell.

* The derivation of this section follows closely Popelar [10] .

A typical undeformed cross section, the coordinate system and shell geometry are shown in Fig. 1.

Let X_1 and X_2 be the Cartesian coordinates of a point P lying on a curve of the undeformed shell (see Fig. 2). The elemental arc length dS is expressed as

$$dS^2 = dX_1^2 + dX_2^2 = A dr^2 + 2B dr d\theta + C d\theta^2 \quad (1)$$

where

$$A = 1, \quad B = 0, \quad C = r^2$$

After the shell is deformed, point P is displaced to P' . The Cartesian coordinates $\bar{X}_1$, $\bar{X}_2$ of P' are

$$\bar{X}_1 = (r - \bar{w}) \sin \theta + \bar{v} \cos \theta \quad (2)$$

$$\bar{X}_2 = (r - \bar{w}) \cos \theta - \bar{v} \sin \theta$$

where $\bar{w}$ and $\bar{v}$ are radial and tangential displacements, respectively, and they are functions of polar coordinates r and θ and time t . The elemental arc length for the curve on the deformed shell is therefore expressed as

$$d\bar{S}^2 = d\bar{X}_1^2 + d\bar{X}_2^2 = \bar{A} dr^2 + 2\bar{B} dr d\theta + \bar{C} d\theta^2 \quad (3)$$

Taking differentials of (2), and substituting into (3), we obtain

$$\bar{A} = \left(1 - \frac{\partial \bar{w}}{\partial r}\right)^2 + \left(\frac{\partial \bar{v}}{\partial r}\right)^2$$

$$\begin{aligned}\bar{B} &= -(1 - \frac{\partial \bar{W}}{\partial r}) \frac{\partial \bar{W}}{\partial \theta} + \frac{\partial \bar{V}}{\partial r} \frac{\partial \bar{V}}{\partial \theta} - \bar{V} (1 - \frac{\partial \bar{W}}{\partial r}) + (r - \bar{W}) \frac{\partial \bar{V}}{\partial r} \\ \bar{C} &= (\frac{\partial \bar{W}}{\partial \theta})^2 + (r - \bar{W})^2 + (\frac{\partial \bar{V}}{\partial \theta})^2 + \bar{V}^2 + 2 \bar{V} \frac{\partial \bar{W}}{\partial \theta} + 2(r - \bar{W}) \frac{\partial \bar{V}}{\partial \theta}\end{aligned}\quad (4)$$

The radial strain ϵ_r , the shearing strain $\epsilon_{r\theta}$ and tangential strain ϵ_θ are expressed as

$$\begin{aligned}\epsilon_r &= \frac{1}{2} (\bar{A} - A) \\ \epsilon_{r\theta} &= \frac{1}{2r} (\bar{B} - B) \\ \epsilon_\theta &= \frac{1}{2r^2} (\bar{C} - C)\end{aligned}\quad (5)$$

Substituting (1) and (4) into (5), we obtain the nonlinear strain-displacement relations

$$\begin{aligned}\epsilon_r &= -\frac{\partial \bar{W}}{\partial r} + \frac{1}{2} \left[\left(\frac{\partial \bar{W}}{\partial r} \right)^2 + \left(\frac{\partial \bar{V}}{\partial r} \right)^2 \right] \\ \epsilon_{r\theta} &= \frac{1}{2r} \left(-\frac{\partial \bar{W}}{\partial \theta} - \bar{V} + r \frac{\partial \bar{V}}{\partial r} + \frac{\partial \bar{W}}{\partial r} \frac{\partial \bar{W}}{\partial \theta} + \frac{\partial \bar{V}}{\partial r} \frac{\partial \bar{V}}{\partial \theta} + \bar{V} \frac{\partial \bar{W}}{\partial r} - \bar{W} \frac{\partial \bar{V}}{\partial r} \right) \\ \epsilon_\theta &= \frac{1}{r} \left(\frac{\partial \bar{V}}{\partial \theta} - \bar{W} \right) \left[1 + \frac{1}{2r} \left(\frac{\partial \bar{V}}{\partial \theta} - \bar{W} \right) \right] + \frac{1}{2r^2} \left(\frac{\partial \bar{W}}{\partial \theta} + \bar{V} \right)^2\end{aligned}\quad (6)$$

Neglecting the nonlinear terms in ϵ_r and $\epsilon_{r\theta}$, then

$$\begin{aligned}\epsilon_r &= -\frac{\partial \bar{W}}{\partial r} \\ \epsilon_{r\theta} &= \frac{1}{2r} \left(-\frac{\partial \bar{W}}{\partial \theta} - \bar{V} + r \frac{\partial \bar{V}}{\partial r} \right)\end{aligned}$$

By assumption "c", ϵ_r and $\epsilon_{r\theta}$ must vanish,

$$\epsilon_r = -\frac{\partial \bar{w}}{\partial r} = 0$$

$$\epsilon_{r\theta} = \frac{1}{2} \left[-\frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{\bar{v}}{r} \right) \right] = 0 \quad (7)$$

Integrating (7a), we obtain

$$\bar{w} = w(\theta, t) \quad (8)$$

where w is radial displacement of the midsurface. Substituting (8) into (7b) and integrating with respect to r , then $\bar{v}$ can be expressed as

$$\bar{v} = -\frac{\partial \bar{w}}{\partial \theta} + r \bar{f}(\theta, t) \quad (9)$$

where $\bar{f}$ is an arbitrary function of θ and t . If we let

$$v = v(\theta, t) = \bar{v}(a, \theta, t)$$

be the midsurface tangential displacement, then it follows from (8) and (9) that

$$\bar{f}(\theta, t) = \frac{1}{a} \left(v + \frac{\partial w}{\partial \theta} \right)$$

Consequently, the tangential displacement $\bar{v}$ for a general point can be expressed as

$$\bar{v} = -\frac{\partial w}{\partial \theta} + \frac{r}{a} \left(v + \frac{\partial w}{\partial \theta} \right) \quad (10)$$

The quantity

$$\frac{1}{r} \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right)$$

is the tangential strain associated with infinitesimal deformation and may be neglected compared to unity. Thus, the expression for ϵ_θ simplifies to

$$\epsilon_\theta = \frac{1}{r} \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) + \frac{1}{2r^2} \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} \right)^2$$

Substituting (8), (10) into the above equation gives

$$\epsilon_\theta = \frac{1}{a} \frac{\partial v}{\partial \theta} - \frac{w}{r} + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 + \frac{1}{a} \left(1 - \frac{a}{r} \right) \frac{\partial^2 w}{\partial \theta^2} \quad (11)$$

It is convenient to introduce in (11) the change of variables

$$r = a \left(1 - \frac{z}{a} \right)$$

where z is a coordinate measured radially inward from the midsurface.

Expanding $1/(1 - \frac{z}{a})$ and retaining terms through second degree in z , we write the strain-displacement relation as

$$\epsilon_\theta = \frac{1}{a} \left(\frac{\partial v}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 - \frac{z}{a^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) - \frac{z^2}{a^3} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \quad (12)$$

2. Derivation of Governing Differential Equations

To derive the governing differential equations, we apply the generalized Hamilton's principle, in which the nonconservative forces are included. The extended principle appears in variational form as

$$\delta \int_{t_1}^{t_2} (T - V) dt + \int_{t_1}^{t_2} \sum_{j=1}^n Q_j \delta q_j dt = 0 \quad (13)$$

where t_1 , t_2 are end points of time t , T is the kinetic energy, V is the potential energy of conservative forces, Q_j is the generalized force associated with nonconservative forces, q_j is the generalized coordinates and n is the number of independent coordinates.

The kinetic energy of the shell is expressed as

$$T = \frac{\rho_s}{2} \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\left(\frac{\partial \bar{v}}{\partial t} \right)^2 + \left(\frac{\partial \bar{w}}{\partial t} \right)^2 \right] (a - z) dz d\theta \quad (14)$$

where ρ_s is the density per unit volume of the shell. Substituting (8) and (10) into (14) and neglecting the rotary inertia, we can write the kinetic energy as

$$T = \frac{\rho_s a h}{2} \int_0^{2\pi} (\dot{v}^2 + \dot{w}^2) d\theta \quad (15)$$

The dot denotes differentiation with respect to time.

We consider a submerged shell subjected to a uniform external pressure $-p$ and a harmonically varying component $f \cos \omega t$. A particular form

of motion for this loading is a uniform radial motion. If this motion is slightly disturbed, flexural motion of the shell will also occur. For sufficiently small initial radial motion ($w \ll a$) we may measure the additional displacements from the undeformed position. If we take the uniform radial motion as our datum, then the potential energy for conservative forces will consist of three portions

$$V = V_1 + V_2 - W, \quad (16)$$

The additional internal strain energy of deformation, the strain energy contributed from initial circumferential stress, the work done by external uniform distributed loadings are denoted by V_1 , V_2 and W , respectively.

If we neglect the shear deformation, then the internal strain energy is

$$V_1 = \frac{1}{2} \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta \epsilon_\theta (a - z) dz d\theta \quad (17)$$

With assumption "a", the tangential stress σ_θ is related to ϵ_θ by Hooke's law

$$\sigma_\theta = \frac{E}{1 - \nu^2} \epsilon_\theta \quad (18)$$

where E is the modulus of elasticity and ν is Poisson's ratio. If (12) and (18) are substituted into (17) and the integration with respect to z

performed, V_1 becomes

$$V_1 = \frac{E a h}{2(1-\nu^2)} \int_0^{2\pi} \left\{ \left[\frac{1}{a} \left(\frac{\partial V}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(\nu + \frac{\partial w}{\partial \theta} \right)^2 \right]^2 + \frac{h^2}{12a^4} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right)^2 \right\} d\theta \quad (19)$$

The strain energy due to the initial circumferential stress σ_θ^0 is

$$V_2 = \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta^0 \epsilon_\theta (a - z) dz d\theta \quad (20)$$

The stress is related to the uniform "equivalent external pressure" p , by membrane theory, as

$$\sigma_\theta^0 = - \frac{p a}{h} \quad (21)$$

In Appendix B, we obtained

$$p = -q + f (\beta^2 + r^2)^{\frac{1}{2}} \cos(wt - \phi) \quad (22)$$

Substituting (12) and (21) into (20) and integrating with respect to z yields

$$V_2 = -p a \int_0^{2\pi} \left[\left(\frac{\partial V}{\partial \theta} - w \right) + \frac{1}{2a} \left(\nu + \frac{\partial w}{\partial \theta} \right)^2 \right] d\theta \quad (23)$$

The work done by "equivalent external pressure" p during flexural motion of the shell is equal to the pressure times the change of volume of the shell. From Appendix C, the elemental change of volume per unit length of the shell is given as

$$dS = \frac{1}{2} \left[2aw - a \frac{\partial v}{\partial \theta} - w^2 + w \frac{\partial v}{\partial \theta} - v \frac{\partial w}{\partial \theta} - v^2 \right] d\theta$$

Therefore

$$W_1 = \int_0^{2\pi} pa \left[w + \frac{1}{2a} w \left(-w + \frac{\partial v}{\partial \theta} \right) - \frac{1}{2a} \left(\frac{\partial w}{\partial \theta} + v \right) v \right] d\theta \quad (24)$$

The nonaxisymmetric motion of the shell causes an additional fluid reaction on the shell. Such a loading is nonconservative and, therefore, must be treated differently from the above calculation. To evaluate the generalized forces associated with the fluid reaction, we further assume that the normal to the midsurface of the deformed shell differs little from the radial direction^{*}. From assumption "e", tangential displacement of the shell surface does not give rise to any work done by the fluid. Under these conditions no generalized forces are associated with the circumferential direction. If we let the generalized force associated with the variation δw be denoted by Q_w , then

* This is also equivalent to neglecting a small secondary thrust due to the perturbation compared to the primary thrust resulting from the initial purely radial motion.

$$\sum_{i=1}^n Q_i \delta q_i = \int_0^{2\pi} Q_w a \delta w d\theta \quad (25)$$

Substituting (15), (16) and (25) into (13), we obtain the equations of motion as

$$\begin{aligned} \rho_s a h \ddot{w} - \frac{E h}{1-\nu^2} \left[\frac{1}{a} \left(\frac{\partial v}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 \right] \\ - \frac{E h}{1-\nu^2} \frac{\partial}{\partial \theta} \left\{ \left[\frac{1}{a} \left(\frac{\partial v}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 \right] \frac{1}{a} \left(v + \frac{\partial w}{\partial \theta} \right) \right\} \\ + \frac{E h}{1-\nu^2} \frac{\alpha^2}{a} \left(\frac{\partial^4 w}{\partial \theta^4} + 2 \frac{\partial^2 w}{\partial \theta^2} + w \right) + p \left(w + \frac{\partial^2 w}{\partial \theta^2} \right) = -a Q_w \end{aligned}$$

$$\begin{aligned} \rho_s a h \ddot{v} - \frac{E h}{1-\nu} \frac{\partial}{\partial \theta} \left[\frac{1}{a} \left(\frac{\partial v}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 \right] \\ + \frac{E h}{1-\nu} \left[\frac{1}{a} \left(\frac{\partial v}{\partial \theta} - w \right) + \frac{1}{2a^2} \left(v + \frac{\partial w}{\partial \theta} \right)^2 \right] \frac{1}{a} \left(v + \frac{\partial w}{\partial \theta} \right) = 0 \end{aligned}$$

where $\alpha^2 = \frac{h^2}{12a^2} \quad (26)$

Discarding the tangential inertia and the external loading, we obtain the equations of motion for shallow cylindrical shell [10]. If all nonlinear terms in (26) are neglected, we have the linearized equations for small

motion

$$\rho_s a h \ddot{W} + \frac{Eh}{1-\nu^2} \frac{\alpha^2}{a} \left(\frac{\partial^4 W}{\partial \theta^4} + 2 \frac{\partial^2 W}{\partial \theta^2} + W \right) - \frac{Eh}{1-\nu^2} \frac{1}{a} \left(\frac{\partial V}{\partial \theta} - W \right) + p \left(W + \frac{\partial^2 W}{\partial \theta^2} \right) = -a Q_W$$

$$\rho_s a h \ddot{V} - \frac{Eh}{1-\nu^2} \frac{1}{a} \frac{\partial}{\partial \theta} \left(\frac{\partial V}{\partial \theta} - W \right) = 0 \quad (27)$$

Since the displacements, V and W , are the additional displacements beyond the breathing oscillation, then (26) and (27) may be viewed as the variational equations. The static and dynamic analogues of these equations have been obtained by different approaches by Timoshenko [12] and Bolotin [13], respectively.

For a sufficiently thin shell and suitably limited motion, we may use the linear equations. The resulting dynamic configuration of the shell can be expressed in terms of Fourier series

$$W = \sum_{n=2}^{\infty} a_n \cos n\theta + \sum_{n=2}^{\infty} C_n \sin n\theta$$

$$V = \sum_{n=2}^{\infty} b_n \sin n\theta + \sum_{n=2}^{\infty} d_n \cos n\theta$$

In the above series, $n=0, 1$ terms were not included, because they represent breathing mode and rigid body translation, respectively, which are not of interest in the present analysis. The first series and second

series represent, respectively, symmetric and asymmetric mode with respect to $\theta = 0$ axis. Without loss in generality we may consider the symmetric motion only and simply write

$$\begin{aligned} W &= \sum_{n=2}^{\infty} a_n(t) \cos n\theta \\ V &= \sum_{n=2}^{\infty} b_n(t) \sin n\theta \end{aligned} \quad (28)$$

The generalized force corresponding to the assumed configuration of the deformed shell is found in Appendix A to be

$$Q_{a_n} = -\rho c \sum_{n=2}^{\infty} \left[\Theta_n(K_m a) \dot{a}_n + \frac{1}{\omega_m} X_n(K_m a) \ddot{a}_n \right] \cos n\theta \quad (29)$$

where ω_m is the flexural frequency, K_m is the flexural wave numbers*, $X_n(K_m a)$ is the acoustic reactance and $\Theta_n(K_m a)$ is the acoustic resistance. Substituting (28) and (29) into (27), we obtain the temporal equations for the system in terms of generalized coordinates a_n and b_n

$$\begin{aligned} & \left[\rho_s a h + \frac{\rho a}{K_m} X_n(K_m a) \right] \ddot{a}_n + \rho c a \Theta_n(K_m a) \dot{a}_n \\ & + \left\{ \frac{E}{1-\nu^2} \frac{h}{a} [\alpha^2 (n^2-1)^2 + 1] + P(1-n^2) \right\} a_n - \frac{E}{1-\nu^2} \frac{h}{a} n b_n = 0 \end{aligned}$$

$$\rho_s a h \ddot{b}_n + \frac{E}{1-\nu^2} \frac{h}{a} [n^2 b_n - n a_n] = 0$$

$$(n = 2, 3, 4 \dots) \quad (30)$$

* The physical meaning of K_m will be discussed later.

In the above equations, the terms involving the acoustic reactance, $X_n(\kappa_n a)$, and the acoustic resistance, $\theta_n(\kappa_n a)$, play the roles of "added mass" and damping, respectively.

It is convenient to express the differential equations in nondimensional form by defining

$$\begin{aligned} f_n &= \frac{a_n}{a} \\ \psi_n &= \frac{b_n}{a} \\ t' &= \frac{c_s t}{a} \end{aligned} \quad (31)$$

where

$$c_s = \left[\frac{E}{(1-\nu^2)\rho_s} \right]^{\frac{1}{2}}$$

is a wave velocity of the solid media. Consequently, the pressure is expressed as

$$p = -\bar{p} + f(\beta^2 + \gamma^2)^{\frac{1}{2}} \cos(\Omega t' - \phi) \quad (32)$$

where

$$\Omega = \frac{\omega a}{c_s}$$

By making the change of variables

$$\Omega \tau = \Omega t' - \phi$$

we can eliminate the phase angle ϕ

$$p = -f + f(\beta^2 + r^2)^{\frac{1}{2}} \cos \Omega \tau \quad (33)$$

Substituting (31), (33) into (30), we obtain the governing differential equations

$$\begin{aligned} \ddot{f}_n + \bar{E}_n^m \dot{f}_n + \bar{G}_n^m (1 - 2\bar{\mu}_n \cos \Omega \tau) f_n - \bar{H}_n^m n \psi_n &= 0 \\ \ddot{\psi}_n + n^2 \psi_n - n f_n &= 0 \end{aligned} \quad (n = 2, 3, \dots) \quad (34)$$

where dot now denotes differentiation with respect to τ and

$$\begin{aligned} \bar{H}_n^m &= \frac{1}{1 + \frac{\rho}{\rho_s} \frac{1}{h k_m} X_n(k_m a)} \\ \bar{E}_n^m &= \frac{\frac{C}{C_s} \Theta_n(k_m a)}{\frac{\rho_s}{\rho} \frac{h}{a} \bar{H}_n^m} \end{aligned}$$

$$\bar{G}_n^m = \frac{[\alpha^2 (n^2 - 1)^2 + 1] - \frac{p}{p_n} (1 - n^2)^2 \alpha^2}{1 - \bar{I}_n^m}$$

$$\bar{U}_n = \frac{\frac{f}{p_n} \alpha^2 (n^2 - 1)^2 (\beta^2 + \gamma^2)^{\frac{1}{2}}}{2 \left\{ [\alpha^2 (n^2 - 1)^2 + 1] - \frac{p}{p_n} (1 - n^2)^2 \alpha^2 \right\}}$$

where

$$p_n = E \alpha^2 \frac{h}{a} \frac{n^2 - 1}{1 - \nu^2}$$

is the eigenbuckling load of an infinite cylindrical shell subject to an external pressure.

Equation (34) is the so called Mathieu equation with frequency dependent coefficients.

3. Natural Frequencies of a Cylindrical Shell in Vacuo

When the infinitely long cylindrical shell oscillates in vacuo, the acoustic loading is equal to zero

$$\theta_n(k_m a) = X_n(k_m a) = 0$$

Therefore, those coefficients in (34) simplify to

$$\bar{E}_n = 0$$

$$\bar{G}_n = \alpha^2 (n^2 - 1)^2 + 1$$

$$\bar{H}_n = 1$$

By setting the forcing function equal to zero, we then obtain the governing differential equations for the freely oscillating shell as

$$\ddot{f}_n + \bar{G}_n f_n - n \psi_n = 0 \quad (38)$$

$$\ddot{\psi}_n + n^2 \psi_n - n f_n = 0$$

$$(n = 2, 3, \dots)$$

To obtain the natural frequencies for the shell, we assume the solution for (38) as

$$f_n = A_n e^{i \Omega_n \tau} \quad (39)$$

$$\psi_n = B_n e^{i \Omega_n \tau}$$

Substituting (39) into (38), we obtain the characteristic equation in the form

$$(-\Omega_n^2 + \bar{G}_n)(-\Omega_n^2 + n^2) - n^2 = 0$$

The natural frequency is therefore obtained as

$$\begin{aligned}\Omega_{n1}^2 &= \frac{S_1 - (S_1^2 - 4S_2)^{\frac{1}{2}}}{2} \\ \Omega_{n2}^2 &= \frac{S_1 + (S_1^2 - 4S_2)^{\frac{1}{2}}}{2}\end{aligned}\tag{40}$$

where

$$S_1 = \alpha^2 (n^2 - 1)^2 + 1 + n^2$$

$$S_2 = n^2 \alpha^2 (n^2 - 1)^2$$

We thus obtain two frequency spectra : the higher frequency spectra , the Ω_{n2} 's , are extensional in character and are membrane modes, the lower frequency spectra, the Ω_{n1} 's , are associated with both extensional and bending behaviors, depending on n and a/h and are the composite modes.

CHAPTER II

DETERMINATION OF REGIONS OF DYNAMIC INSTABILITY *

From the Floquet theory we know that the solutions of the Mathieu equation are periodic and are either bounded or grow exponentially with time. The stability criterion is the condition for which these solutions cease to be bounded. In the present analysis, the Floquet theory is applied to determine the stability or instability of the trivial solution $f_n = \psi_n = 0$ of the Mathieu equation (34). Thus, the stability or instability of the breathing motion is determined.

In this chapter, we derive the equations of boundary frequencies which separate the unstable regions from the stable regions. Some numerical examples of the equations for the shell both in fluid and in vacuo are considered.

* The structure of the solutions of this chapter follows Bolotin [13] .

1. Derivation of the Boundary Frequency Equations

The system of equations (34) can be expressed in matrix form

$$\underline{\underline{E}} \ddot{\underline{\underline{f}}}_n + \underline{\underline{C}}_n^m \dot{\underline{\underline{f}}}_n + (\underline{\underline{A}}_n^m - \underline{\underline{B}}_n^m \cos \Omega \tau) \underline{\underline{f}}_n = 0 \quad (n=2, 3, \dots)$$

where

(41)

$$\begin{aligned} \underline{\underline{f}}_n &= \begin{bmatrix} f_n \\ \psi_n \end{bmatrix}, & \underline{\underline{E}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \underline{\underline{C}}_n^m &= \begin{bmatrix} \bar{C}_n^m & 0 \\ 0 & 0 \end{bmatrix}, \\ \underline{\underline{A}}_n^m &= \begin{bmatrix} \bar{G}_n^m & -n \bar{H}_n^m \\ -n & n^2 \end{bmatrix}, \\ \underline{\underline{B}}_n^m &= \begin{bmatrix} 2 \bar{\mu}_n \bar{G}_n^m & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

where n is integer which defines the mode shape of the shell and m is the integer which defines the Fourier components of the solution of the mode shape with respect to time.

Equation (41) is the vector form of Mathieu differential equation with damping term. Following Bolotin, the introduction of the following solution

$$\underline{\underline{f}}_n = e^{-\frac{\bar{C}_n^m}{2} \tau} \underline{\underline{u}}_n(\tau)$$

to (41) will lead to the Mathieu differential equation

$$\ddot{\underline{U}}_n + \left[-\frac{1}{4} (\underline{\epsilon}_n^m)^2 + \underline{A}_n^m - \underline{B}_n^m \cos \Omega \tau \right] \underline{U}_n = 0$$

The Floquet theory [13] * shows that the solutions for each component,

$\underline{U}_{nK}(\tau)$, of $\underline{U}_n(\tau)$ of the above differential equation have the following relations

$$\underline{U}_{nK}^1(\tau + T) = \rho_{nK}^1 \underline{U}_{nK}^1(\tau)$$

$$\underline{U}_{nK}^2(\tau + T) = \rho_{nK}^2 \underline{U}_{nK}^2(\tau)$$

where T is the period of the forcing function, ρ_{nK}^1 and ρ_{nK}^2 are the roots of the following characteristic equation

$$(\rho_{nK})^2 - 2 A_{nK} \rho_{nK} + 1 = 0$$

where $A_{nK} = \frac{1}{2} [\underline{U}_{nK}^1(T) + \underline{U}_{nK}^{2'}(T)]$. Real roots correspond to unstable solutions, while complex roots correspond to stable solutions. Multiple roots

$$\rho_{nK}^1 = \rho_{nK}^2 = 1$$

or

$$\rho_{nK}^1 = \rho_{nK}^2 = -1$$

separate the stable and unstable solutions. The first case corresponds to

* Chapter Fourteen, Sections 55-57.

the periodic solution with a period $T = 2\pi/\Omega$, in the second case the period is $2T$. It can be shown for the Mathieu equation that solutions of same periods bound the regions of instability and solutions of different periods bound the regions of stability.

We seek the periodic solution with a period $2T$ in the form

$$\underline{f}_n(\tau) = \sum_{m=1,3,5}^{\infty} \left(\underline{a}_n^m \sin \frac{m\Omega\tau}{2} + \underline{b}_n^m \cos \frac{m\Omega\tau}{2} \right) \quad (42)$$

where $\underline{a}_n^m$ and $\underline{b}_n^m$ are vectors with constant coefficients.

In equation (42), the value, $m\Omega/2$, is the nondimensionalized frequency of the flexural motion of the shell. The oscillation of the fluid particle will be at this same frequency. Consequently, the coefficients in (41) depend upon this frequency, instead of the frequency of the forcing function. Therefore, the wave number, k_m , shown in previous equations is represented, by the aid of equation (31c), as

$$k_m = \frac{m\Omega}{2a c/c_s} \quad (m = 1, 3, 5, \dots) \quad (43)$$

* Stevens and Evan-Iwanowski also found the frequency dependency of the coefficients in Mathieu equation for the analysis of dynamic stability of viscoelastic columns [14].

In substituting (42) into (41), one must remember to associate the appropriate frequency dependent coefficients of (41) with each of the terms in (42). For example, all coefficients in (41) associated with the m th terms in (42) should be evaluated at the frequency $m\Omega/2$. Applying the procedure outlined above and equating the coefficients of $\sin \frac{m\Omega\tau}{2}$ and $\cos \frac{m\Omega\tau}{2}$ to zero, we obtain the system of equations

$$\left(\underline{A}'_n + \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} \right) \underline{q}'_n - \frac{\Omega}{2} \underline{\epsilon}'_n \underline{b}'_n - \frac{1}{2} \underline{B}_n^3 \underline{q}_n^3 = 0$$

$$\left(\underline{A}'_n - \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} \right) \underline{b}'_n + \frac{\Omega}{2} \underline{\epsilon}'_n \underline{q}'_n - \frac{1}{2} \underline{B}_n^3 \underline{b}_n^3 = 0$$

$$\left(\underline{A}^m_n - \frac{m^2}{4} \Omega^2 \underline{E} \right) \underline{q}^m_n - \frac{m\Omega}{2} \underline{\epsilon}^m_n \underline{b}^m_n - \frac{1}{2} (\underline{B}^{m-2}_n \underline{q}^{m-2}_n + \underline{B}^{m+2}_n \underline{q}^{m+2}_n) = 0$$

$$\left(\underline{A}^m_n - \frac{m^2}{4} \Omega^2 \underline{E} \right) \underline{b}^m_n + \frac{m\Omega}{2} \underline{\epsilon}^m_n \underline{q}^m_n - \frac{1}{2} (\underline{B}^{m-2}_n \underline{b}^{m-2}_n + \underline{B}^{m+2}_n \underline{b}^{m+2}_n) = 0$$

$$(n = 2, 3, 4, \dots)$$

$$(m = 3, 5, 7, \dots)$$

(44)

The condition for the existence of nontrivial solutions of (44) is that the determinant of the coefficients vanishes, i.e.

$$\begin{vmatrix} \underline{A}_n^3 - \frac{9}{4} \Omega^2 \underline{E} & -\frac{1}{2} \underline{B}'_n & 0 & -\frac{3}{2} \Omega \underline{\epsilon}_n^3 \\ \frac{1}{2} \underline{B}_n^3 & \left[\underline{A}'_n + \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} \right] & -\frac{\Omega}{2} \underline{\epsilon}'_n & 0 \\ 0 & \frac{\Omega}{2} \underline{\epsilon}'_n & \left[\underline{A}'_n - \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} \right] & -\frac{1}{2} \underline{B}_n^3 \\ \frac{3}{2} \Omega \underline{\epsilon}_n^3 & 0 & -\frac{1}{2} \underline{B}'_n & \underline{A}_n^3 - \frac{9}{4} \Omega^2 \underline{E} \end{vmatrix} = 0 \quad (45)$$

To obtain the first approximation to the principal region of instability, we retain the central elements in the determinant (45)

$$\begin{vmatrix} \underline{A}'_n + \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} & -\frac{\Omega}{2} \underline{E}'_n \\ \frac{\Omega}{2} \underline{E}'_n & \underline{A}'_n - \frac{1}{2} \underline{B}'_n - \frac{1}{4} \Omega^2 \underline{E} \end{vmatrix} = 0 \quad (46)$$

After expanding (46), we obtain

$$\frac{f}{p_n} = \frac{2}{\alpha^2} \left\{ \frac{[\alpha^2(n^2-1)^2+1] - \beta(1-n^2)^2\alpha^2/p_n}{(n^2-1)^2(\beta^2+r^2)^{\frac{1}{2}}} \right\} \times \left[\frac{(-\frac{\Omega^2}{4}+n^2)^2(-\frac{\Omega^2}{4}+\bar{G}'_n)^2 - 2n^2(-\frac{\Omega^2}{4}+\bar{G}'_n)(-\frac{\Omega^2}{4}+n^2)\bar{H}'_n + (\bar{E}'_n)^2\frac{\Omega^2}{4}(-\frac{\Omega^2}{4}+n^2)^2 + n^4(\bar{H}'_n)^2}{(\bar{G}'_n)^2(-\frac{\Omega^2}{4}+n^2)^2} \right]^{\frac{1}{2}} \quad (47)$$

To determine the regions of instability bounded by the periodic solutions with a period T , we substitute the series

$$\underline{f}_n(\tau) = \frac{1}{2} \underline{b}_n^0 + \sum_{m=2,4,\dots}^{\infty} \left(\underline{a}_n^m \sin \frac{m\Omega\tau}{2} + \underline{b}_n^m \cos \frac{m\Omega\tau}{2} \right) \quad (48)$$

into (41).

Proceeding in an analogous manner, we obtain the first approximation to the second region of instability as

$$\frac{f}{p_n} = \frac{2}{\alpha^2} \times \left[\frac{\alpha^2(n^2-1)^2+1 - \beta(1-n^2)^2\alpha^2/p_n}{(n^2-1)^2(\beta^2+r^2)^{\frac{1}{2}}} \right] \times \left\{ \left[\frac{-\bar{H}_n^0 + \bar{G}_n^0}{\bar{G}_n^0} \right] \times \left[\frac{n^4(\bar{H}_n^0)^2 - 2n^2\bar{H}_n^0(-\Omega^2+n^2)(-\Omega^2+\bar{G}_n^0) + (-\Omega^2+n^2)(-\Omega^2+\bar{G}_n^0)^2 + \Omega^2(-\Omega^2+n^2)(\bar{E}_n^0)^2}{2\bar{G}_n^0[(-\Omega^2+\bar{G}_n^0)(-\Omega^2+n^2)^2 - n^2\bar{H}_n^0(-\Omega^2+n^2)]} \right] \right\}^{\frac{1}{2}} \quad (49)$$

The equivalent equations for the dynamic stability of the shell vibrating in vacuo can be obtained by simply putting $X_o^m = \theta_o^m = X_n^m = \theta_n^m = 0$, namely, the acoustic loadings vanish, in equations (47) and (49).

2. Numerical Examples and Discussions

Numerical computations of (47) and (49) were made in order to assess the effect of acoustic media on the instability regions. The computations were carried out for various parameters, and the results were plotted in Figs. 5-9. In all cases $\dot{f} = 0$, ω is the frequency of external loading and ω_n is the natural frequency of the shell in vacuo. For a better understanding of the following explanations, the numerical data at which the principal parametric resonance occurs are shown in Table 1.

Fig. 5 compares the first two instability regions for the shell in both fluid and vacuo. For the sake of clarity of presentation only the results for $n = 2$ are shown. The figure indicates the acoustic medium has a significant influence on the instability regions. Due to acoustic reactance, the "added mass", the instability regions for the shell immersed in water are shifted toward the lower frequencies. The acoustic resistance, the "damping" effect, of the water causes the instability regions to be raised slightly.

For different mode numbers, the appearances of the unstable regions are changed significantly. This is illustrated in Figs. 6-9. It is mainly due to the fact that we have taken the inertia force of the breathing mode

Table I. Numerical Data at Principal Parametric Resonance

ρ_s/ρ	a/h	n	ω_n (CPS)	ω_r^+ (CPS)	$\omega_s^* = \frac{\omega_r}{2}$ (CPS)	$K_s = \frac{\omega_s^{**}}{C}$ (in ⁻¹)	$K_s a^{***}$
7.7	10	2	1641	2680	1340	223×10^{-4}	0.223
7.7	10	3	4641	7830	3915	653×10^{-4}	0.653
7.7	10	4	8899	15450	7725	1290×10^{-4}	1.290
7.7	10	5	14391	25300	12650	2100×10^{-4}	2.100
7.7	10	6	21111	37500	18750	3125×10^{-4}	3.120
7.7	100	2	16.4	13.1	6.55	1.09×10^{-4}	0.0109
7.7	100	6	211.0	239.0	119.5	19.9×10^{-4}	0.199
7.7	100	10	602.4	798.0	399.0	66.5×10^{-4}	0.665
7.7	100	14	1189.0	1720.0	860.0	143×10^{-4}	1.43
2.7	100	2	16.4	8.2	4.1	0.685×10^{-4}	0.00685
2.7	100	6	211.5	161.0	80.5	13.4×10^{-4}	0.134
2.7	100	10	603.4	555.0	277.5	46.3×10^{-4}	0.463
2.7	100	14	1191.4	1250.0	625.0	104.0×10^{-4}	1.04

⁺ ω_r is the parametric resonance frequency of the forcing function.

^{*} ω_s is the parametric resonance frequency of the submerged shell.

^{**} $C = 6 \times 10^4$ in./sec.

^{***} $h = 1$ in.

oscillation into consideration during the derivations of variational equations. When the frequencies of the forcing function approach the natural frequencies of the breathing mode of the shell, resonances of the shell occur. Therefore, the amplitudes of the forcing function must approach zero in order to keep the system stable. For example, this is illustrated by $n = 4, 5, 6$ in Fig. 6a. However, if the shell is immersed in a fluid, at the frequencies which this kind of resonance develops, the acoustic resistance of the breathing mode is so large that it limits the amplitude of the breathing mode at linear resonance. Therefore, the amplitude of the forcing function can be finite instead of zero at this point, e. g. $n = 4, 5, 6$ in Fig 6b.

The " added mass " depends upon the frequency of excitation. The magnitude of the added mass is relatively large for low frequencies and relatively small for the higher frequencies. Junger [1] offers the following explanation for these: " The low frequency vibration tends to oscillate large quantities of fluid, while the high frequency vibration compresses a small amount of fluid ". Since parametric excitation, for example at the principal resonance, occurs at twice the natural frequency of a mode of vibration, then the " added mass " is larger for the lower modes than for higher modes. For example, this is the reason why a greater shift of the boundary occurs for $n = 2$ than $n = 14$.

The greater the ratio a/h , the greater is the shift of the boundaries. For two shells, identical except for thickness, the " added mass " of each

is of the same order of magnitude. However, the mass of the shell is obviously less for the thinner one. Therefore, the contribution of the " added mass " for the thinner shell is relatively greater and a larger shift of the boundaries occur.

By comparing Figs. 7 and 8 for steel shell and aluminum shell, respectively, we observed an even greater shift of the boundaries for the lighter aluminum shell. Since the speed of sound in aluminum and steel is approximately the same, the " added mass " of similar shells of steel and aluminum is nearly the same. However, since the mass of the aluminum shell is less, the effect of the " added mass " is greater and a larger shift of the boundaries is witnessed. A similar argument holds for other metallic shells.

In Fig. 9, the boundaries of the second regions of instability of steel shell were plotted for different modes and different geometrical parameter, a/h . The percentages of the shift of the unstable regions on the $\omega/2\omega_n$ coordinate are almost the same for the principal and secondary resonances.

The results presented are for the composite modes only. The numerical computations show that if the shell is vibrating in the vacuo the unstable regions for membrane modes, with relatively high natural frequencies, are so narrow that they can be neglected. Because a relatively high excitation frequency, almost 40 times that of composite mode, is required to excite a membrane mode the acoustic resistance of the shell

submerged in water is large (see Fig.4). Therefore, these narrow regions of instability are reduced and raised upward significantly, so that unreasonably large amplitudes of forcing function are required in order to cause the system to become unstable.

On the otherhand, we see from the Table 1 and Fig. 4 that when the parametric resonance for a composite mode occurs, the acoustic resistance is so small that the damping effect is little. Therefore, the boundary is not raised substantially in this case.

CHAPTER III

EXCITATION OF INEXTENSIONAL MODES

The foregoing analysis takes into account coupling between extensional and inextensional deformations, the latter being essentially flexural. An approximate solution may be obtained by neglecting extension entirely. As we have seen previously even the first approximation to the instability region is very complicated. The condition of inextensionality will simplify these equations considerably. Furthermore, this simplification now makes the calculation of higher order approximation of the principal instability region and other regions feasible.

In this chapter, we derive the governing differential equations for the inextensional oscillation of the immersed shell by applying Lagrange equation. Again, we derive the equations of boundary frequencies. Furthermore, the behavior of the shell after the occurrences of parametric resonance is investigated.

Except for the first section, the method of analysis follows closely that of Bolotin [13].

1. Derivation of Governing Equations

The expression that the middle surface undergoes no extension is

$$\frac{\partial V}{\partial \theta} - W = 0 \quad (61)$$

If we assume the form for the radial displacement W as

$$W = \sum_{n=2}^{\infty} a_n \cos n \theta \quad (62)$$

then, to satisfy the constraint (61), we need

$$V = \sum_{n=2}^{\infty} \frac{1}{n} a_n \sin n \theta \quad (63)$$

Substituting (62) and (63) into (15), we obtain the expression for kinetic energy as

$$T = \frac{\rho_s a h}{2} \pi \left(1 + \frac{1}{n^2} \right) \dot{a}_n^2 \quad (64)$$

Similarly, we obtain the expression for potential energy V as

$$V = -\frac{E a h}{2(1-\nu^2)} \left[\frac{1}{4 a^4} (n^2 - 1)^4 a_n^4 \frac{3\pi}{4} + \frac{\alpha^2}{a^2} (1 - n^2)^2 a_n^2 \pi \right] + \frac{P}{2} \pi a_n^2 (n^2 - 1) \quad (65)$$

The generalized coordinate a_n must satisfy the Lagrange equation of the system,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{a}_n} - \frac{\partial T}{\partial a_n} + \frac{\partial V}{\partial a_n} = Q_{a_n} \quad (66)$$

where Q_{a_n} is given by (29).

Substituting (29), (64) and (65) into (66), we obtain the temporal

differential equation

$$\left[\rho_s a h \left(1 + \frac{1}{n^2} \right) + \frac{\rho a}{K_m} X_n(K_m a) \right] \ddot{a}_n + \rho c a \Theta_n(K_m a) \dot{a}_n + \frac{E h}{a(1-\nu^2)} \alpha^2 (1-n^2)^2 a_n + \rho (1-n^2) a_n + \frac{E h}{1-\nu^2} \frac{3}{8 a^3} (n^{-1} - n)^4 a_n^3 = 0 \quad (67)$$

By introducing the nondimensional quantities (31), we write (67) as

$$\ddot{f}_n + \epsilon_n^m \dot{f}_n + G_n^m (1 - 2\mu_n \cos \Omega \tau) f_n + N_n^m f_n^3 = 0 \quad (68)$$

where

$$\epsilon_n^m = \frac{\frac{c}{c_s} \Theta_n(K_m a)}{\left[\frac{\rho_s}{\rho} \frac{h}{a} \left(1 + \frac{1}{n^2} \right) + \frac{1}{a K_m} X_n(K_m a) \right]}$$

$$G_n^m = \frac{\alpha^2 (n^2 - 1)^2 \left(1 - \frac{\rho}{\rho_n} \right)}{\left(1 + \frac{1}{n^2} \right) + \frac{\rho}{\rho_s} \frac{1}{h K_m} X_n(K_m a)}$$

$$\mu_n = \frac{\frac{f}{\rho_n} (\beta^2 + \gamma^2)^{\frac{1}{2}}}{2 \left(1 - \frac{\rho}{\rho_n} \right)}$$

$$N_n^m = \frac{\frac{3}{8} (n^{-1} - n)^4}{\left(1 + \frac{1}{n^2} \right) + \frac{\rho}{\rho_s} \frac{1}{h K_m} X_n(K_m a)}$$

2. Derivation of the Boundary Frequency Equation

To determine the stability criteria for the present problem, it is sufficient to work with the linearized differential equation of (68) in the analysis, because initially $\dot{f}_n$ is assumed small. The linearized differential equation,

$$\ddot{f}_n + \epsilon_n^m \dot{f}_n + G_n^m (1 - 2\mu_n \cos \Omega \tau) f_n = 0, \quad (69)$$

is a Mathieu equation with damping.

We seek the periodic solution with a period 2τ in the form

$$f_n = \sum_{m=1,3,\dots}^{\infty} \left(a_n^m \sin \frac{m\Omega\tau}{2} + b_n^m \cos \frac{m\Omega\tau}{2} \right) \quad (70)$$

In substituting (70) into (69), one must remember again to associate the appropriate frequency dependent coefficients of (69) with each term in (70). For example, all coefficients in (69) associated with the m th terms in (70) should be evaluated at the frequency $\frac{m\Omega}{2}$. We apply the procedure outlined above, introduce appropriate trigonometric identities, and equate the coefficients of $\sin \frac{m\Omega\tau}{2}$ and $\cos \frac{m\Omega\tau}{2}$ to zero, respectively. As a result, we obtain the system of linear algebraic equations

$$\left(-\frac{\Omega^2}{4} + G_n' + G_n' \mu_n \right) a_n' - \frac{\Omega}{2} \epsilon_n' b_n' - G_n^3 \mu_n a_n^3 = 0$$

$$\left(-\frac{\Omega^2}{4} + G_n' - G_n' \mu_n \right) b_n' + \frac{\Omega}{2} \epsilon_n' a_n' - G_n^3 \mu_n b_n^3 = 0$$

$$\left(-\frac{m^2 \Omega^2}{4} + G_n^m\right) a_n^m - G_n^{m-2} \mu_n a_n^{m-2} - G_n^{m+2} \mu_n a_n^{m+2} - \frac{m \Omega}{2} \epsilon_n^m b_n^m = 0$$

$$\left(-\frac{m^2 \Omega^2}{4} + G_n^m\right) b_n^m - G_n^{m-2} \mu_n b_n^{m-2} - G_n^{m+2} \mu_n b_n^{m+2} + \frac{m \Omega}{2} \epsilon_n^m a_n^m = 0$$

$$(n = 2, 3, 4 \dots) \quad (m = 3, 5, \dots)$$

Equating the determinant of the coefficients of the homogeneous system of equations to zero, we obtain the equation for the boundary frequency :

$$\begin{vmatrix} -\frac{\Omega^2}{4} + G_n^3 & -G_n' \mu_n & 0 & -\frac{3\Omega}{2} \epsilon_n^3 \\ -G_n^3 \mu_n & \begin{vmatrix} -\frac{\Omega^2}{4} + G_n'(1 + \mu_n) & -\frac{\Omega}{2} \epsilon_n' \\ \frac{\Omega}{2} \epsilon_n' & -\frac{\Omega^2}{4} + G_n'(1 - \mu_n) \end{vmatrix} & 0 & -G_n^3 \mu_n \\ 0 & 0 & -G_n' \mu_n & -\frac{3\Omega}{2} \epsilon_n^3 \\ \frac{3\Omega}{2} \epsilon_n^3 & 0 & -G_n' \mu_n & -\frac{\Omega^2}{4} + G_n^3 \end{vmatrix} = 0 \quad (71)$$

To obtain the first approximation to the boundaries of the principal instability regions, we retain the central elements in the determinant (71)

$$\begin{vmatrix} -\frac{\Omega^2}{4} + G_n'(1 + \mu_n) & -\frac{\Omega}{2} \epsilon_n' \\ \frac{\Omega}{2} \epsilon_n' & -\frac{\Omega^2}{4} + G_n'(1 - \mu_n) \end{vmatrix} = 0 \quad (72)$$

Solving (72) for the exciting amplitude, we obtain the approximate formula

$$\frac{f}{p_n} = \frac{2(1 - f/p_n)}{(\beta^2 + r^2)^{\frac{1}{2}}} \frac{\left[\left(-\frac{\Omega^2}{4} + G_n' \right)^2 + \frac{\Omega^2}{4} (\epsilon_n')^2 \right]^{\frac{1}{2}}}{G_n'} \quad (73)$$

To increase the accuracy of (73), we will consider the second approximation

by retaining the 4×4 determinant in (71). After expanding the determinant, we then find

$$\frac{f}{p_n} = \frac{2(1 - f/p_n)}{(\beta^2 + \gamma^2)^{\frac{1}{2}}} \left[\frac{-U_2 - (U_2^2 - 4U_1U_3)}{2U_1} \right]^{\frac{1}{2}} \quad (74)$$

where

$$U_1 = (G_n' G_n^3)^2$$

$$U_2 = -2G_n' G_n^3 \left(-\frac{\Omega^2}{4} + G_n'\right) \left(-\frac{\Omega^2}{4} + G_n^3\right) - (G_n')^2 \left(-\frac{\Omega^2}{4} + G_n^3\right)^2 \\ - \frac{\Omega^2}{4} (G_n^3 G_n')^2 + \frac{3\Omega^2}{2} G_n' G_n^3 G_n' G_n^3$$

$$U_3 = \left[\left(-\frac{\Omega^2}{4} + G_n'\right)^2 + \frac{\Omega^2}{4} (G_n')^2 \right] \left[\left(-\frac{\Omega^2}{4} + G_n^3\right)^2 + \frac{\Omega^2}{4} (G_n^3)^2 \right]$$

The second solution with period T is obtained by taking a solution in the form of the series

$$f_n = b_n^0 + \sum_{m=2,4,\dots}^{\infty} \left(a_n^m \sin \frac{m\Omega\tau}{2} + b_n^m \cos \frac{m\Omega\tau}{2} \right) \quad (75)$$

Once more, proceeding in an analogous manner, we obtain the first approximation for the boundaries of the second instability regions as

$$\frac{f}{p_n} = \frac{2(1 - f/p_n)}{(\beta^2 + \gamma^2)^{\frac{1}{2}}} \left[\frac{(-\Omega^2 + G_n^2)^2 + \Omega^2 (G_n^2)^2}{2 G_n^2 (-\Omega^2 + G_n^2)} \right]^{\frac{1}{2}} \quad (76)$$

and the second approximation for the second instability regions as

$$\frac{f}{p_n} = \frac{2(1 - \beta/p_n)}{(\beta^2 + \gamma^2)^{\frac{1}{2}}} \left[\frac{-X_2 - (X_2^2 - 4X_1X_3)^{\frac{1}{2}}}{2X_1} \right]^{\frac{1}{2}} \quad (77)$$

where

$$X_1 = (G_n^2)^2 G_n^4 [2(-4\Omega^2 + G_n^4) + G_n^4]$$

$$X_2 = G_n^2 \left[-2(-\Omega^2 + G_n^2)(-4\Omega^2 + G_n^4)^2 - 8\Omega^2(\epsilon_n^4)^2(-\Omega^2 + G_n^2) \right. \\ \left. - 2G_n^4(-\Omega^2 + G_n^2)(-4\Omega^2 + G_n^4) + 4\Omega^2\epsilon_n^4\epsilon_n^4 G_n^4 \right]$$

$$X_3 = [(-\Omega^2 + G_n^2)^2 + \Omega^2(\epsilon_n^4)^2] [(-4\Omega^2 + G_n^4)^2 + 4\Omega^2(\epsilon_n^4)^2]$$

By discarding the acoustic loading from (73) and (76), we can obtain the equations for the boundaries frequencies of the principal and secondary regions, respectively, of a shell in vacuo. In this case, (68) simplify to

$$\epsilon_n^m = \gamma = 0$$

$$G_n^m = \frac{\alpha^2(n^2 - 1)^2}{1 + \frac{1}{n^2}} \quad (78)$$

$$\beta = \frac{1}{1 + \frac{-\Omega^2}{1 - \nu^2}}$$

where G_n^m can be related to the natural frequencies Ω_{n1} of the shell in vacuo

$$\Omega_{n1} = (G_n^m)^{\frac{1}{2}} \quad (79)$$

Numerical Examples :

The instability regions for $n = 2$, based on the second approximation (74) and (77) are shown in Fig. 10. As can be seen, the first approximation gives good results, and is sufficiently accurate for most purposes. Fig. 10 should be compared with Fig. 5. One would expect the greatest difference to occur for $n = 2$; the difference being smaller for larger n .

Based on the first approximations as given by (73) and (76), the instability regions were plotted for the same parameters in Figs. 5-9. The results indicate that the approximate solutions obtained by neglecting the extension of the shell results in no appreciable difference from the more accurate solutions of Chapter II.

3. Amplitude of Vibration at the Principal Resonance

According to the linear theory, one expects the vibration amplitudes in the regions of dynamic instability to increase exponentially with time. However, with increasing amplitudes, the influence of nonlinear factors becomes more important and the linear analysis is no longer valid. The nonlinear factors tend to yield bounded amplitudes.

In the present analysis, the nonlinear terms considered in (68) are derived for small finite deformation of the shell. All nonlinearities associated with the behavior of the fluid are expressly neglected. It is conjectured here that the important nonlinearities occur in the shell.

The principal objective here is only to evaluate these nonlinearities.

In order to determine the amplitudes of the vibrations for the principal resonance, we assume the solution for (68) as

$$f_n = a_n'(\tau) \sin \frac{\Omega \tau}{2} + b_n'(\tau) \cos \frac{\Omega \tau}{2} \quad (80)$$

where $a_n'(\tau)$ and $b_n'(\tau)$ are functions of time.

Substituting (80) into (68), introducing appropriate trigonometric identities, and equating the coefficients of $\sin \frac{\Omega \tau}{2}$ and $\cos \frac{\Omega \tau}{2}$ to zero, we obtain two equations for the functions $a_n'(\tau)$ and $b_n'(\tau)$ as

$$\begin{aligned} \ddot{a}_n' + \epsilon_n' \dot{a}_n' - \Omega \dot{b}_n' + \left(-\frac{\Omega^2}{4} + G_n' + G_n' \mu_n\right) a_n' - \epsilon_n' \frac{\Omega}{2} b_n' + \frac{3}{4} N_n' (A_n')^2 a_n' &= 0 \\ \ddot{b}_n' + \epsilon_n' \dot{b}_n' + \Omega \dot{a}_n' + \left(-\frac{\Omega^2}{4} + G_n' - G_n' \mu_n\right) b_n' + \epsilon_n' \frac{\Omega}{2} a_n' + \frac{3}{4} N_n' (A_n')^2 b_n' &= 0 \end{aligned} \quad (81)$$

The equations for steady-state oscillation of the flexural modes of the shell can be obtained by discarding those terms involving the differentiation with respect to time, τ . As a result, we obtain

$$\left(-\frac{\Omega^2}{4} + G_n' + G_n' \mu_n\right) a_n' - \epsilon_n' \frac{\Omega}{2} b_n' + \frac{3}{4} N_n' (A_n')^2 a_n' = 0 \quad (82)$$

$$\left(-\frac{\Omega^2}{4} + G_n' - G_n' \mu_n\right) b_n' + \epsilon_n' \frac{\Omega}{2} a_n' + \frac{3}{4} N_n' (A_n')^2 b_n' = 0$$

where $(A_n')^2 = (a_n')^2 + (b_n')^2$

The nontrivial solutions for a_n' and b_n' for (82) lead to

$$A_n' = \left\{ \frac{\frac{\Omega^2}{4} - \epsilon_n' \pm \left[(\epsilon_n')^2 - (\epsilon_n')^2 \frac{\Omega^2}{4} \right]^{\frac{1}{2}}}{\frac{3}{4} N_n'} \right\}^{\frac{1}{2}} \quad (83)$$

The diagrams of the dependence of the amplitudes on the frequency are shown in Figs. 11-12. For a given mode, each sign in (83) yields a solution. The solutions represented by the dashed line, solution with minus sign in (83), are unstable*. Due to the fact that $N_n'' > 0$, the shell behaves like a hardening spring and the response curves are bent toward increasing excitation frequency. Due to the presence of very slight damping the solid and dash curves meet at a large amplitude. In all likelihood other forms of damping are more important here than that due to acoustic resistance.

Fig. 11 is obtained by keeping $f/p_n = 0.05$ in Fig. 7. The curves corresponding to $n = 6, 10, 14$ for the shell in vacuo are not plotted in order to keep the diagram uncluttered. Fig. 12 is obtained by keeping $f/p_n = 0.5$ in Fig. 7.

4. Stability of Vibrations

By assuming that $a_n'(\tau)$ and $b_n'(\tau)$ are slowly changing functions of time, namely

$$\begin{aligned} \frac{\dot{a}_n}{a_n} \frac{4\pi}{\Omega} &<< 1 \\ \frac{\ddot{a}_n}{\dot{a}_n} \frac{4\pi}{\Omega} &<< 1 \end{aligned} \quad a_n = a_n', \quad b_n = b_n' \quad (84)$$

* The details of the stability of each solution will be investigated in next section.

we can simplify equation (81) as

$$\begin{aligned}\Omega \dot{b}_n' &= \left(-\frac{\Omega^2}{4} + G_n' + G_n' u_n\right) a_n' - \frac{\Omega}{2} \epsilon_n' b_n' + \frac{3}{4} N_n' (A_n')^2 a_n' \\ \Omega \dot{a}_n' &= \left(\frac{\Omega^2}{4} - G_n' + G_n' u_n\right) b_n' - \frac{\Omega}{2} \epsilon_n' a_n' - \frac{3}{4} N_n' (A_n')^2 b_n'\end{aligned}\quad (85)$$

The steady-state solutions for a_n' and b_n' of the previous section are denoted as a_{n0} , b_{n0} , respectively. To investigate the stability of those solutions, we disturb the system. Then, we have

$$\begin{aligned}a_n' &= a_{n0} + \delta_n' \\ b_n' &= b_{n0} + K_n'\end{aligned}\quad (86)$$

The variational equations are obtained by substituting (86) into (85)

$$\begin{aligned}\Omega \frac{d\delta_n'}{d\tau} &= a_{11} \delta_n' + a_{12} K_n' \\ \Omega \frac{dK_n'}{d\tau} &= a_{21} \delta_n' + a_{22} K_n'\end{aligned}\quad (87)$$

where

$$\begin{aligned}a_{11} &= -\epsilon_n' \frac{\Omega}{2} - \frac{3}{2} N_n' a_{n0} b_{n0} \\ a_{12} &= \frac{\Omega^2}{4} - G_n' (1 - u_n) - N_n' \left(\frac{9}{4} b_{n0}^2 + \frac{3}{4} a_{n0}^2\right) \\ a_{21} &= -\frac{\Omega^2}{4} + G_n' (1 + u_n) + N_n' \left(\frac{9}{4} a_{n0}^2 + \frac{3}{4} b_{n0}^2\right) \\ a_{22} &= -\epsilon_n' \frac{\Omega}{2} + \frac{3}{2} N_n' a_{n0} b_{n0}\end{aligned}$$

After checking the solutions of variational equations (87) for each steady-state solution of (82), following Bolotin [13], we come to the following conclusions :

a. The trivial solution

$$a_{n0} = b_{n0} = 0$$

is stable everywhere except in the principal resonance regions ;

b. The nontrivial solution

$$a_{n0} = \left\{ \frac{\frac{\Omega^2}{4} - G_n' - \left[(G_n' \mu_n)^2 - (\epsilon_n' \frac{\Omega}{2})^2 \right]^{\frac{1}{2}}}{\frac{3}{4} N_n'} \right\}^{\frac{1}{2}}$$

$$b_{n0} = 0$$

is unstable ;

c. The nontrivial solution

$$a_{n0} = 0$$

$$b_{n0} = \left\{ \frac{\frac{\Omega^2}{4} - G_n' + \left[(G_n' \mu_n)^2 - (\epsilon_n' \frac{\Omega}{2})^2 \right]^{\frac{1}{2}}}{\frac{3}{4} N_n'} \right\}^{\frac{1}{2}}$$

is stable.

The existence of two stable solutions at same frequency leads to the phenomenon of jump. It can be illustrated by the following diagrams.

If the frequency of the external loading is gradually increased from small value, Fig. A, the amplitude of the flexural oscillation follows along the direction A B D F G. Namely, before the frequency reaches the parametric resonance range, B C, the shell undergoes no flexural oscillation. When the parametric range is reached, the amplitude of the flexural oscillation is determined by the branch B J. As the frequency is increased further, the amplitude is determined by the branch B J. Due to the presence of damping the amplitude will not become unbounded as the frequency is increased, but a sudden decrease (jump) in amplitude will occur. Such a jump is depicted graphically by D F. For frequencies beyond F the amplitude is zero.

In the other case, if the frequency of excitation is gradually decreased from a value beyond the range of parametric resonance, B C, Fig. B, the amplitude of the flexural oscillation follows along the direction G C H B A, with a jump occurring at C.

The unstable solution C K plays an important role in the determina-

tion of whether the shell will oscillate with the amplitude H J or C G. If the magnitude of the amplitude of the disturbance is greater than the unstable solution C K, for frequencies greater than C, then the shell will have flexural oscillation with amplitude determined by the branch H J. On the other hand if the magnitude is smaller than C K, the flexural oscillation will not occur.

Inside the parametric resonance region, B C, if we impart a strong disturbance to the shell, the flexural oscillation of the shell with amplitude given by branch B H will take place. In addition to the steady-state flexural vibration, beat phenomena can occur, if it is emanated from a small disturbance. This will be discussed later.

5. The Buildup of Vibrations

In the neighborhood of principal resonance, if we impart a small perturbation to the shell, then at the initial stage the amplitude is so small that we can neglect the nonlinear terms in (85) and obtain the linear differential equations

$$\begin{aligned} \Omega \frac{d b_n'}{d \tau} &= \left[-\frac{\Omega^2}{4} + G_n' (1 + \mu_n) \right] a_n' - \epsilon_n' \frac{\Omega}{2} b_n' \\ \Omega \frac{d a_n'}{d \tau} &= \left[\frac{\Omega^2}{4} - G_n' (1 - \mu_n) \right] b_n' - \epsilon_n' \frac{\Omega}{2} a_n' \end{aligned} \quad (88)$$

By assuming the solution of the above differential equation in the form

$$a'_n = a_{n_0} e^{\lambda \tau}$$

$$b'_n = b_{n_0} e^{\lambda \tau} \quad (89)$$

we obtain the eigen value λ as

$$\lambda = \frac{1}{\Omega} \left\{ \left[\left(G'_n \mu_n \right)^2 - \left(\frac{\Omega^2}{4} - G'_n \right)^2 \right]^{\frac{1}{2}} - \frac{1}{2} G'_n \right\} \quad (90)$$

In general, it is necessary to use numerical integration methods to solve equation (85). In the present analysis, we apply the Subroutine HPCG in IBM 360 Scientific Subroutine Package. The Hamming's modified predictor-corrector method is used in this program.

Numerical examples of the solution of (85) are shown in Figs. 13-14, with the numerical parameters corresponding to A, B, respectively, of Fig. 7. In Fig. 13, the dashed curves correspond to the exponential growth of the amplitude, equation (89). It is seen that the simplified exponential solution gives a good accuracy at the initial stage of excitation.

The results of the numerical integration show that there exists a subharmonic response, or beat phenomena, arising from initially small disturbance. An approximate analysis following Bolotin [13] ^{*} shows the maximum amplitude of the beat is

$$A_{max} = \left[\frac{\frac{\Omega^2}{4} - G'_n + G'_n \mu_n}{\frac{9}{16} N'_n} \right]^{\frac{1}{2}} \quad (91)$$

* Chapter Six, Section 25.

Boston [15] used the method of harmonic balance to obtain the maximum amplitude of the beat which for the present problem is predicted by

$$A_{max} = \left[\frac{\frac{\Omega^2}{4} - G_n' + G_n' u_n}{\frac{3}{8} N_n'} \right]^{\frac{1}{2}} \quad (92)$$

The difference between the two is the numerical factor in the denominator. By comparing the values given by (91) and (92) with that of computer simulation presented by Boston, we find that Bolotin's estimation is lower while Boston's analysis is higher. The difference is about eight percent for each case.

Fig. 15 shows the maximum of amplitude of the beat v.s. $\omega/2\omega_n$ for $f/p_n = 0.085$, for both equation (91) and (92).

A comment should be made about the frequencies that determine the values for the frequency dependent coefficients of (88). In the numerical integration of (88), we have assumed that these coefficients are determined by $m\Omega/2$ and that the oscillations of a_n' and b_n' do not influence these coefficients. The numerical results show that the beat frequency is much smaller than the exciting frequency. This justifies the assumption.

6. Amplitude of Vibration at the Secondary Resonance

In order to determine the amplitude of the steady-state vibration for the secondary resonance, we assume the solution for (68) as

$$f_n = b_n^0 + a_n^2 \sin \Omega \tau + b_n^2 \cos \Omega \tau \quad (93)$$

Substituting (93) into (68) gives

$$G_n^2 b_n^0 - G_n^2 \mu_n b_n^2 + N_n^2 b_n^0 \left[(b_n^0)^2 + \frac{3}{2} (A_n^2)^2 \right] = 0$$

$$(-\Omega^2 + G_n^2) a_n^2 - \epsilon_n^2 \Omega b_n^2 + N_n^2 a_n^2 \left[\frac{3}{4} (A_n^2)^2 + 3(b_n^0)^2 \right] = 0$$

$$-2 G_n^2 \mu_n b_n^0 + \epsilon_n^2 \Omega a_n^2 + (-\Omega^2 + G_n^2) b_n^2 + N_n^2 b_n^2 \left[\frac{3}{4} (A_n^2)^2 + 3(b_n^0)^2 \right] = 0$$

where $(A_n^2)^2 = (a_n^2)^2 + (b_n^2)^2$ (94)

The previous numerical computation reveals that the damping coefficient

ϵ_n^2 is so small that it can be neglected. In this case, (94) can be satisfied by

$$b_n^0 = b_n^2 = 0, \quad a_n^2 = A \quad (\text{1st solution})$$

or

$$a_n^2 = 0, \quad b_n^0 \neq 0, \quad b_n^2 = A \quad (\text{2nd solution})$$

The amplitude of the first solution is determined from (94b)

$$A_n^2 = \left[\frac{\Omega^2 - G_n^2}{\frac{3}{4} N_n^2} \right]^{\frac{1}{2}} \quad (95)$$

The second solution leads to the system of equations

$$\begin{aligned} G_n^2 b_n^0 - G_n^2 \mu_n A_n^2 + N_n^2 b_n^0 \left[(b_n^0)^2 + \frac{3}{2} (A_n^2)^2 \right] &= 0 \\ -2 G_n^2 \mu_n b_n^0 + (-\Omega^2 + G_n^2) A_n^2 + N_n^2 A_n^2 \left[\frac{3}{4} (A_n^2)^2 + 3 (b_n^0)^2 \right] &= 0 \end{aligned} \quad (96)$$

The linear approximation for b_n^0 from (96a) is

$$(b_n^0)_1 \approx \mu_n (A_n^2)_0 \quad (97)$$

Substituting (97) into (96b), we obtain

$$(A_n^2)_1 \approx \left[\frac{\Omega^2 - G_n^2 + 2 G_n^2 \mu_n^2}{N_n^2 \left(\frac{3}{4} + 3 \mu_n^2 \right)} \right]^{\frac{1}{2}} \quad (98)$$

$(b_n^0)_1$ and $(A_n^2)_1$ calculated after (97) and (98), respectively, constitutes the first approximation. Equation (96) can be rewritten as

$$\begin{aligned} (b_n^0)_m &= \frac{1}{G_n^2} \left\{ G_n^2 \mu_n (A_n^2)_{m-1} - N_n^2 (b_n^0)_{m-1} \left[(b_n^0)_{m-1}^2 + \frac{3}{2} (A_n^2)_{m-1}^2 \right] \right\} \\ (A_n^2)_m &= 2 \left\{ \frac{(\Omega^2 - G_n^2) (A_n^2)_{m-1} + 2 G_n^2 \mu_n (b_n^0)_m}{3 N_n^2 (A_n^2)_{m-1}} - 3 (b_n^0)_m^2 \right\}^{\frac{1}{2}} \end{aligned} \quad (99)$$

where m denotes the order of the approximation. Substituting the first approximate solutions, $(b_n^0)_1$ and $(A_n^2)_1$, into the right-hand-side of (99a), we get the second approximation, $(b_n^0)_2$. Then insert $(b_n^0)_2$ and $(A_n^2)_1$ into the right-hand-side of (99b), we obtain the second approximate value $(A_n^2)_2$. Continuing the same procedure we can obtain more accurate value for A_n^2 .

The numerical examples shown in Fig. 16 are obtained by setting $\frac{f}{p_n} = 0.05, 0.5$ in Fig. 9 with $\frac{a}{h} = 100$. The solid lines correspond to equation (95) and the dashed lines correspond to equation (96). Comparing with the appearance of amplitudes of principal resonance, we expect, from previous analysis, that equation (95) is the stable solution and (96) is the unstable solution. Qualitatively, Fig. 16 is similar to Figs. 11-12, however, the differences between the solid and dashed curves are much smaller than that for similar curves for the principal regions.

CONCLUSIONS AND FURTHER WORK

The effect of the acoustic media is primarily to cause the shift of the unstable regions to the lower parametric frequencies. This shift may be of the order of 50 percent . It is due mainly to the " added mass ", provided by the acoustic reactance. In the neighborhood of parametric resonance frequencies, the relative damping provided by the resistive impedance is usually very small. Therefore, the unstable regions are not raised appreciably. Undoubtly, other forms of damping are more important in parametric resonance of the composite modes. However, for the extensional modes the damping is so great that parametric excitation of these modes is virtually nonexistent. The relative damping for the breathing mode is so large that the forced resonance does not exist practically.

The approximate inextensional solution showed a good agreement with that of the more accurate analysis by taking into account coupling between extensional and inextensional deformation. The condition of inextensionality may be used very effectively to reduce the computational effort.

At a given frequency the amplitude of the parametric excited motion is bounded by the nonlinear terms. The response curves for the shell are similar to those of the hardening spring. From the concept of " stability in the large ", a steady-state finite amplitudes can be obtained. In addition to the steady-state flexural vibration, beats can also occur, if it

emanates from small perturbations. The maximum amplitude for the beats exceeds the steady-state amplitude by approximately 15 percent .

The next obvious extension of this work is to take into consideration perturbations in the axial direction. A further extension of the present investigation, which may lead to results of practical interest, is the study of more complicated shells, such as a stiffened cylindrical shell and a spherical shell. An extension of the present investigation can also be made by considering the cylindrical shell submerged in a semiinfinite fluid medium. Recently, Chang and DiMaggio [18] have investigated the vibration of such a system. Another possible extension of the present analysis is the study of the dynamic stability of the submerged shell under impulsive loading. This has practical importance, since underwater explosions may cause a serious damage to the structure.

APPENDIX A

ACOUSTICAL LOADINGS ON A CYLINDRICAL SHELL ^{*}

The wave equation for the polar coordinates (r , θ) is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} \quad (a)$$

where Φ is the velocity potential and c is the speed of sound in the fluid. The solution of (a) for outgoing wave only is

$$\Phi = \sum_{n=0,1}^{\infty} e^{j\omega t} (A_n \cos n\theta + B_n \sin n\theta) H_n^{(2)}(kr) \quad (b)$$

where $j = (-1)^{\frac{1}{2}}$ and $H_n^{(2)}(kr)$ is the Hankel function of the second kind, [16]. The undetermined coefficients A_n and B_n can be determined in terms of the generalized coordinates a_n and b_n of (28) by the boundary condition at the interface of the fluid and solid. The condition is that the radial displacement of the shell is equal to the radial displacement of the fluid, i.e.

$$-\dot{w} = \left(\frac{\partial \Phi}{\partial r} \right)_{r=a} \quad (c)$$

Therefore

$$\begin{aligned} 0 &= B_n \\ -\dot{a}_n &= e^{j\omega t} A_n \left[\frac{d H_n^{(2)}(kr)}{dr} \right]_{r=a} \end{aligned} \quad (n = 2, 3, \dots) \quad (d)$$

* The derivation of this Appendix follows Junger [1] , [17].

From (d), we obtain

$$\ddot{a}_n = j \omega \dot{a}_n \quad (e)$$

Substituting (d) into (b), the velocity potential Φ can be expressed as

$$\Phi = \sum_{n=2}^{\infty} - \frac{\dot{a}_n}{\left[\frac{d H_n^{(2)}(kr)}{dr} \right]_{r=a}} H_n^{(2)}(kr) \cos n\theta \quad (f)$$

The acoustic impedance ratio $f_n(kr)$ is defined by

$$f_n(kr) = - \frac{j H_n^{(2)}(kr)}{\left[\frac{d}{dr} H_n^{(2)}(kr) \right]_{r=a}} \quad (g)$$

For convenience, we separate the complex quantities into real and imaginary parts, respectively, acoustic resistance and reactance ratio

$$f_n(kr) = \theta_n(kr) + j X_n(kr) \quad (h)$$

$X_n(kr)$ and $\theta_n(kr)$ are plotted in Fig. 3 and Fig. 4, respectively, for value of n from 0 to 12 and for the range from $kr=0$ to $kr=11$.

Substituting (h) into (f), we obtain

$$\Phi = \sum_{n=2}^{\infty} \frac{1}{j} \dot{a}_n [\theta_n(kr) + j X_n(kr)] \cos n\theta \quad (i)$$

The pressure in the fluid is given by

$$Q_{a_n} = -\rho \frac{\partial \Phi}{\partial t} \quad (j)$$

Substituting (e) and (i) into (j), we obtain

$$Q_{a_n} = -\rho c \sum_{n=2}^{\infty} \left[\frac{1}{\omega} X_n(ka) \ddot{a}_n + \theta_n(ka) \dot{a}_n \right] \cos n\theta \quad (k)$$

APPENDIX B

BREATHING MODE OSCILLATION OF A CYLINDRICAL SHELL

Under the uniformly distributed simple harmonic force $-g + f e^{j\omega t}$ acting on the cylindrical shell, the shell is vibrating in the breathing mode. Consider the half shell with all the loadings acting on it as the following figure

The pressure P includes uniformly distributed pressure $-g + f e^{j\omega t}$, acoustic loading $Q_w^{(0)}$, and inertia force $\rho_s h \frac{d^2 w}{dt^2}$, and N is the hoop force at any cross section of the ring. For dynamical equilibrium of the system

$$N = (-g + f e^{j\omega t} + Q_w^{(0)} - \rho_s h \frac{d^2 w}{dt^2}) a \quad (a)$$

Assuming the stress is uniformly distributed across the cross section, then

$$N = \frac{WEh}{a}$$

Therefore

$$\rho_s a h \frac{d^2 W}{dt^2} - Q_w^{(0)} a + \frac{Eh}{a} W = (-g + f e^{j\omega t}) a$$

From Appendix A, the acoustic loading for breathing mode is

$$Q_w^{(0)} = \left[-\rho X_o(ka) \ddot{W} - \rho \omega \theta_o(ka) \dot{W} \right] \frac{1}{k}$$

The equation of motion can then be expressed as

$$\left[\rho_s a h + \frac{1}{k} \rho a X_o(ka) \right] \ddot{W} + \rho c a \theta_o(ka) \dot{W} + \frac{Eh}{a} W = (-g + f e^{j\omega t}) a \quad (b)$$

Assuming the solution of (b) as

$$W = A + B e^{j\omega t}$$

then

$$A = \frac{-g a^2}{E h}$$

$$B = \frac{a f}{-\omega^2 \left[\rho_s a h + \frac{1}{k} \rho a X_o(ka) \right] + j \omega \rho c a \theta_o(ka) + \frac{Eh}{a}}$$

For convenience, we express B as the sum of its real and imaginary part

$$B = f B_R + j f B_I \quad (c)$$

The acoustic loading $Q_W^{(o)}$ and the inertia force F_I can be expressed as

$$\begin{aligned} Q_W^{(o)} &= B \omega^2 \rho \left[X_o(ka) - j \theta_o(ka) \right] \frac{1}{K} e^{j\omega t} \\ F_I &= -\rho_s h \frac{d^2 w}{dt^2} \\ &= \rho h \omega^2 B e^{j\omega t} \end{aligned} \quad (d)$$

If instead of complex forcing function $-g + f e^{j\omega t}$, only simple harmonic force, say $-g + f \cos \omega t$, acting on the shell, then the total pressure will be the real part of P , namely

$$P = \text{Re} (-g + f e^{j\omega t} + Q_W^{(o)} + F_I)$$

After substituting (d), (e) into the above expression, P can be expressed as

$$P = -g + f (\beta \cos \omega t + \gamma \sin \omega t) \quad (f)$$

where

$$\beta = 1 + \frac{\rho \omega^2}{K} [B_R X_o(ka) + B_I \theta_o(ka)] + \rho_s h \omega^2 B_R \quad (j)$$

$$\gamma = -\frac{\rho \omega^2}{K} [-B_R \theta_o(ka) + B_I X_o(ka)] - \rho_s h \omega^2 B_I$$

From (f), we observe there exists phase angle between the forcing function and the total pressure acting on the vibrating shell. We may write P as

$$P = -g + f (\beta^2 + \gamma^2)^{\frac{1}{2}} \cos (\omega t - \phi) \quad (h)$$

where $\phi = \tan^{-1} \gamma / \beta$

APPENDIX C

CHANGE OF THE ELEMENTAL VOLUME OF A CYLINDRICAL SHELL

Two points P_1 and P_2 on the midsurface of the cylindrical shell are subtended by the arc $d\theta$. After the shell is deformed P_1 is displaced to P'_1 and P_2 to P'_2 .

Let $\vec{R}$ represents the position vector for P'_1 , then $\vec{R} + d\vec{R}$ is that for P'_2 . Expressing $\vec{R}$ by its components, we have

$$\vec{R} = [(a-w)\sin\theta + v\cos\theta] \hat{i} + [(a-w)\cos\theta - v\sin\theta] \hat{j} \quad (a)$$

where $\hat{i}$ and $\hat{j}$ are unit vectors on X_1 and X_2 coordinate respectively. Then

$$\begin{aligned} d\vec{R} = & \left[-\frac{\partial w}{\partial \theta} \sin\theta + (a-w)\cos\theta + \frac{\partial v}{\partial \theta} \cos\theta - v\sin\theta \right] d\theta \hat{i} \\ & + \left[-\frac{\partial w}{\partial \theta} \cos\theta - (a-w)\sin\theta - \frac{\partial v}{\partial \theta} \sin\theta - v\cos\theta \right] d\theta \hat{j} \end{aligned}$$

(b)

The area inside $\Delta OP_1P_2'$ is shown as

$$dA' = \frac{1}{2} | (\vec{R} + d\vec{R}) \times \vec{R} | \quad (c)$$

After substituting (a) and (b) into (c) and rearranging, we obtain

$$dA' = \frac{1}{2} \left[a^2 - 2aw + w^2 + a \frac{\partial v}{\partial \theta} - w \frac{\partial v}{\partial \theta} + v \frac{\partial w}{\partial \theta} + v^2 \right] d\theta$$

The area enclosed by ΔOP_1P_2 is

$$dA = \frac{1}{2} a^2 d\theta$$

Therefore, the change in area is

$$\begin{aligned} dS &= dA - dA' \\ &= \frac{1}{2} \left[2aw - a \frac{\partial v}{\partial \theta} - w^2 + w \frac{\partial v}{\partial \theta} - v \frac{\partial w}{\partial \theta} - v^2 \right] d\theta \end{aligned}$$

(d)

Figure 1. Coordinate System and Shell Geometry.

Figure 2. Location of a Point Before and After Deformation.

Figure 3. Acoustic Reactance Ratios on a Cylindrical Shell.

Figure 4. Acoustic Resistance Ratios on a Cylindrical Shell.

Figure 5. Influence of Acoustic Media on the Principal and Secondary Instability Regions.

Figure 6. Principal Instability Diagram for Various Modes of a Steel Cylindrical Shell, $a/h = 10$. 8

(A) In Vacuo

Figure 6. Principal Instability Diagram for Various Modes of a Steel Cylindrical Shell, $a/h = 10$.

(B) In Water

Figure 7. Principal Instability Diagram for Various Modes of a Steel Cylindrical Shell,

$a/h = 100.$

Figure 8. Principal Instability Diagram for Various Modes of an Aluminum Cylindrical Shell,
 $a / h = 100$.

Figure 9. Secondary Instability Diagrams for Various Modes of a Steel Cylindrical Shell.

Figure 10. First Approximate and Second Approximate Instability Regions, $a/h = 10$.

Figure 11. Steady-State Amplitudes for Various Modes at the Principal Parametric Resonance of a Steel Cylindrical Shell, $a/h = 100$, $f/p_n = 0.05$.

Figure 12. Steady-State Amplitudes for Various Modes at the Principal Parametric Resonance of a Steel Cylindrical Shell in Water, $a/h = 100$, $f/p_n = 0.5$.

Figure 13. Buildup of Nonsteady-State Oscillation at Point A of Figure 7.

Figure 14. Nonsteady-State Oscillation at Point B of Figure 7.

Figure 15. Maximum Amplitudes of Beats by Setting $f/p_n = 0.085$ in Fig. 7.

Figure 16. Steady-State Amplitudes for Various Modes at the Secondary Parametric Resonance of a Steel Cylindrical Shell, $a/h = 100$.

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