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DIFFERENCE PROPERTIES FOR
BANACH-VALUED FUNCTIONS

DISSERTATION

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By

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* * * * *

The Ohio State University
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INTRODUCTION

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are groups and Q is a subset of $\mathcal{X}_2^{\mathcal{X}_1}$ (the set of functions from $\mathcal{X}_1$ to $\mathcal{X}_2$) one can pose the following question: If f is a function from $\mathcal{X}_1$ to $\mathcal{X}_2$ such that the differences of f given by: ${}_h\Delta f(x) = f(hx) - f(x)$ and $\Delta_h f(x) = f(xh) - f(x)$ are in Q for every $h \in \mathcal{X}_1$, what can be said about f ? The work that has been done on this and related problems treats the case when $\mathcal{X}_1$ is a topological group and $\mathcal{X}_2$ is the real numbers or the complex numbers.

In 1951, N. G. de Bruijn [1] (numbers in square brackets refer to numbers in the bibliography) showed that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are the real numbers and Q is the class of continuous functions from $\mathcal{X}_1$ to $\mathcal{X}_2$ and f is as prescribed above then $f = g + \gamma$, where $g \in Q$ and γ satisfies the equation: $\gamma(x) + \gamma(y) = \gamma(x + y)$. In this case Q is said to have the difference property. Among many other results, de Bruijn in [1] and [2] also showed that the class of polynomials has the difference property and that the class of Riemann integrable functions has the difference property.

J. H. B. Kemperman in [13] and [14] defined related properties, the local difference property and the weak

difference property. He showed among other results that the classes of continuous functions, polynomials and exponential polynomials on Euclidean n -space possess the former property while the class of functions bounded on every compact subset of Euclidean n -space possess the latter property.

Carroll in [3], [4], [5] and [6] extended many of the above results to locally compact abelian groups and compact groups. In [7] he took up the question posed in our first paragraph when Q is the class of complex-valued periodic L_p -functions, $L_p[0,1]$, and obtained the following: If f is a complex-valued function on $[0,1]$ such that for each h , $f(x+h) - f(x)$ is in $L_p[0,1]$, where $1 \leq p \leq \infty$, then $f = g + \gamma + S$, where g is in $L_p[0,1]$, γ satisfies the equation: $\gamma(x) + \gamma(y) = \gamma(x+y)$ and for each h , $S(x+h) - S(x) = 0$ almost everywhere. The proof works equally well for compact monothetic groups.

In this dissertation $\mathcal{G}_1$ will be either a compact abelian group, a compact second axiom group or, in one case, a locally compact abelian group. $\mathcal{G}_2$ will be a Banach space, which in some cases is the complex numbers. The main results concern the case when $\mathcal{G}_1$ need not be monothetic or when $\mathcal{G}_2$ need not be the complex numbers. Q will be the continuous functions, the Riemann integrable functions or the L_p -functions. Chapter I is given

over to notations used throughout the dissertation and statements of results which either appear in textbooks or are easy modifications of such results. Chapter II contains major lemmas. Chapter III gives the results when $\mathcal{S}_1$ is a compact abelian group and $\mathcal{S}_2$ is a Banach space. A large portion of Chapter III is an extension of results which appeared in [6], in which $\mathcal{S}_2$ was the complex numbers. In Chapter IV uniformly distributed sequences are introduced and groups possessing them are considered as well as compact second axiom groups. The generalized Cantor group and its dual group are considered in Chapter V and the L_p -functions on the former are shown to have a difference property. Chapter VI contains miscellaneous results.

CHAPTER I

PRELIMINARIES

I.1 Introduction to Chapter I and General Information

This chapter is a summary of theorems which can be found in the literature and which are referred to in this dissertation. Any such reference will be of the form (I.x.y) where x and y are decimal numbers. Any reference enclosed in parentheses in which the left-most character is not 'I' is a reference to a formula number. Formula number (X.y.z) will be found in section X.y. Other references will be preceded by the appropriate name, for example, Lemma III.2.1, which is found in section III.2.

$\mathbb{C}$ is the symbol for the complex numbers and if $a \in \mathbb{C}$, then $\bar{a}$ is the complex conjugate of a . We shall be dealing with a function from a topological group, $\mathcal{G}$, to a Banach space, B . The norm in B is written $|\cdot|$. A topological group, $\mathcal{G}$, is a Hausdorff space in which a group operation, $\cdot$, is defined such that $(x,y) \rightarrow x \cdot y^{-1}$ is continuous. We shall write the group operation as $+$ only when the group is abelian, in which case 0 denotes the identity. Otherwise, the operation will be represented by juxtaposition and the identity by I . Any locally compact topological group, $\mathcal{G}$, has a (left) Haar measure, that is, a measure, m , defined on the Borel sets,

Σ , which is positive on any open set, finite on any compact set and such that $m(E) = m(xE)$ for all $x \in \mathcal{G}$ and $E \in \Sigma$ [10, Chapter XI]. We also observe that a topological group is a uniform space since the neighborhood system at the identity, $\mathcal{N}(I)$, when left (or right) translated to x , becomes the neighborhood system at x , $\mathcal{N}(x)$.

If f is a function from $\mathcal{G}$ to B the left h -translate, the right h -translate, the left h -difference and the right h -difference are denoted by $S_h f$, $T_h f$, ${}_h \Delta f$ and $\Delta_h f$, respectively, and their values at a point, $x \in \mathcal{G}$ are given by $f(hx)$, $f(xh)$, $f(hx) - f(x)$ and $f(xh) - f(x)$, respectively. f is a class function if $f(xy) = f(yx)$ for all $x, y \in \mathcal{G}$ and f is an additive function if $f(xy) = f(x) + f(y)$ for all $x, y \in \mathcal{G}$.

1.2 Preliminaries from Measure Theory and Linear Spaces

The notation and results are taken for the most part from [8]. If μ is a measure, $|\mu|$ denotes the total variation of μ and μ^* denotes the outer measure associated with μ . We shall omit arguments from the notation $L_p(\mathcal{G}, \Sigma, m, B)$ when such omission causes no ambiguity. $[f]$ is the equivalence class of f within L_1 ; if $[f] \in L_p$ we write " f is an L_p -function." The integral of f over $\mathcal{G}$ is denoted $B \int f(x) dm(x)$ or $B \int f$. The integral over the subset, E , is denoted $B \int_E f(x) dm(x)$ or

$B \int_E f$. If B is a subset of the complex numbers then B will be omitted from the integral sign. χ_E will be the characteristic function of a set, E . $\|\cdot\|_p$ will denote the p -norm of an L_p -function.

I.2.1 (Egoroff's Theorem), [8, p. 149]. If $(\mathcal{X}, \Sigma, m)$ is a finite measure space, then a sequence $\{f_n\}$ of measurable functions defined on $\mathcal{X}$ with values in B is m -uniformly convergent to a function, f , if and only if $\{f_n(x)\}$ converges to $f(x)$ m -almost everywhere.

I.2.2 [8, p. 150]. If $(\mathcal{X}, \Sigma, m)$ is a finite measure space, an m -almost everywhere convergent sequence of measurable functions is convergent in m -measure.

I.2.3 (Lebesgue Dominated Convergence Theorem), [8, p. 151]. Let $1 \leq p < \infty$, let $(\mathcal{X}, \Sigma, m)$ be a measure space and let $\{f_n\}$ be a sequence of $L_p(\mathcal{X}, \Sigma, m, B)$ -functions converging m -almost everywhere to a function, f . Suppose there exists an L_p -function, g , such that $|f_n(x)| \leq |g(x)|$ m -almost everywhere for each n . Then f is an L_p -function and $\|f_n - f\|_p$ converges to zero.

I.2.4 [8, p. 147]. If $(\mathcal{X}, \Sigma, m)$ is a measure space and f is m -integrable then f vanishes m -almost everywhere if and only if $B \int_E f = 0$ for every $E \in \Sigma$.

I.2.5 (Fubini's Theorem), [8, p. 90]. Let $(\mathcal{X}, \Sigma, m)$ be a positive sigma-finite measure space. Let $(\mathcal{X} \times \mathcal{X}, \Sigma \times \Sigma, m \times m) = (\mathcal{X}, \Sigma, m) \times (\mathcal{X}, \Sigma, m)$, and let $[f] \in$

$L_1(\mathcal{X} \times \mathcal{X}, \Sigma \times \Sigma, m \times m)$. Then for m -almost all $x \in \mathcal{X}$, $f(x, \cdot)$ is an L_1 -function. Moreover $B \int f(\cdot, y) dm(y)$ is an L_1 -function and

$$\begin{aligned} B \int \left\{ B \int f(x, y) dm(y) \right\} dm(x) &= \\ B \int \left\{ B \int f(x, y) dm(x) \right\} dm(y) &= \\ B \int_{\mathcal{X} \times \mathcal{X}} f(z) d(m \times m)(z) & \end{aligned}$$

I.2.6 [18, p. 258]. For any normed linear space, X , the unit ball, S^* , of X^* is a compact Hausdorff space in the weak* topology of X^* .

I.2.7 (Mean Ergodic Theorem), [8, p. 668]. Let $(\mathcal{X}, \Sigma, m)$ be a finite positive measure space and let ϕ be a mapping of $\mathcal{X}$ into itself with the property that $m(e \Delta \phi^{-1}e) = 0$ implies that either e or its complement is null (metric transitivity). Then for every p , $1 \leq p < \infty$, the transformation, T , defined by $(Tf)(x) = f(\phi(x))$ is a continuous linear mapping in $L_p(\mathcal{X}, \Sigma, m)$ and the averages: $A(n)f = \frac{1}{n} \sum_{m=0}^{n-1} T^m f$ are convergent in the p -norm to $B \int f / m(\mathcal{X})$.

I.2.8. Let $(\mathcal{X}, \Sigma, m)$ be a finite positive measure space. The weak* limit of a sequence of L_p -functions is an L_p -function if $1 < p < \infty$, a finite regular measure if $p = 1$.

Proof. This is an application of (I.2.6). If $1 < p < \infty$, put $1/p + 1/q = 1$. Then $L_p = L_q^*$ and the result follows. For $p = 1$, we observe that the space of finite regular measures is the dual space of the space of bounded continuous functions on $\mathcal{X}$. Since L_1 is a subspace of the space of finite regular measures on $\mathcal{X}$, the weak* limit of a sequence of L_1 -functions is a finite regular measure.

I.3 Topological Preliminaries

The usual symbols will be used for union and intersection. $C_X E$, $c_X E$, $\text{Int}_X E$ and $b_X E$ are the symbols for complement, closure, interior and boundary, respectively, of a set, E , with respect to the space X . If it is clear what X is, X will be omitted from the symbols. Δ is the notation for symmetric difference and $\sim$ is the notation for difference, that is, $E \sim F = E \cap C F$ while $E \Delta F = (E \cup F) \sim (E \cap F)$. $\mathcal{N}(x)$ denotes the neighborhood system of x . If it is necessary to display the topology on a space, X , we follow Kelley [11] and write $(X, \mathcal{T})$ where X is the space and $\mathcal{T}$ is the topology.

I.3.1 [11, p. 72, 86]. If X is a second axiom topological space and Y is a topological space and $f: X \rightarrow Y$ then f is continuous at a point $x_0 \in X$ if and only if $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ whenever $\lim_{n \rightarrow \infty} x_n = x_0$.

I.3.2 (Baire Category Theorem), [11, p. 200]. Let X

be either a complete pseudo-metric space or a locally compact regular space. Then the intersection of a countable family of open dense subsets of X is itself dense in X . Conversely, the union of a countable family of closed nowhere dense sets cannot be all of X .

I.3.3. Let $\mathcal{X}$ be a compact metric space and let $C(\mathcal{X})$ be the space of continuous complex-valued functions on $\mathcal{X}$. $C(\mathcal{X})$ with the sup topology is separable.

Proof. Denote the metric on $\mathcal{X}$ by d . For each pair of distinct points, x and y in $\mathcal{X}$, there are disjoint open sets, $U_{x,y}$ and $V_{x,y}$, containing x and y , respectively, and a continuous real-valued function, $f_{x,y}$, such that if $s \in U_{x,y}$ and $t \in V_{x,y}$ then $f_{x,y}(s) \neq f_{x,y}(t)$. $u = \{U_{x,y} \times V_{x,y} : x \neq y\}$ is an open cover of the compact set: $K_n = \{(x,y) : d(x,y) \geq \frac{1}{n}\}$. Thus a finite subcollection of u covers K_n for each n , and a countable subcollection of u covers $\{(x,y) : x \neq y\}$. The associated functions separate points and the algebra generated by them together with the function identically 1 over the rationals is countable and by the Stone-Weierstrass Theorem [18, p. 250] dense in $C(\mathcal{X})$.

I.3.4 [9, p. 278]. A limit of a sequence of continuous functions from a compact space to a metric space is continuous on a residual set.

I.3.5. If X is a topological space and E and F are subsets of X , then: $b(E \cap F) \subset b(E) \cup b(F)$,
 $b(E \cup F) \subset b(E) \cup b(F)$ and $bcE \subset bE$.

I.4 Preliminaries for Topological Groups

These results and notations are taken for the most part from [18]. $\mathcal{G}$ will always be a locally compact topological group and m will be the Haar measure on $\mathcal{G}$. If $f: \mathcal{G} \rightarrow B$ and $g: \mathcal{G} \rightarrow C$ then the convolution of f

and g , $f * g$, is given by $f * g(x) = \int_B f(y)g(y^{-1}x)dm(y)$.

When $\mathcal{G}$ is abelian, $\hat{\mathcal{G}}$ will be the dual group of $\mathcal{G}$ and if $g \in \hat{\mathcal{G}}$, and $x \in \mathcal{G}$, we write $g(x) = \langle x, g \rangle = \langle g, x \rangle$. If f is a function on $\mathcal{G}$ and $\gamma \in \hat{\mathcal{G}}$, the γ -Fourier

coefficient of f is $\hat{f}(\gamma) = \int_B f(x)\langle -x, \gamma \rangle dm(x) =$

$f * \gamma(0)$. If $\mathcal{G}$ is compact, then $m(\mathcal{G}) = 1$ and $m(E) = m(E^{-1})$. S' is the topological subgroup generated by S , that is, the smallest closed subgroup containing S .

I.4.1 [17, p. 60]. If $\mathcal{G}$ is a compact topological group and $\mathcal{H}$ is a normal closed subgroup then $\mathcal{G}/\mathcal{H}$ is compact (the topology on $\mathcal{G}/\mathcal{H}$ is the quotient topology).

I.4.2. If $\mathcal{G}$ is a topological group, if $\mathcal{H}$ is a normal closed subgroup and if f is a continuous function from $\mathcal{G}$ to a Banach space, B , which is constant on the cosets of $\mathcal{H}$ and $f': \mathcal{G}/\mathcal{H} \rightarrow B$ is defined as follows: $f'(x\mathcal{H}) = f(x)$, then f' is continuous.

Proof. If U is open in B , then $f'^{-1}(U) = \{x\mathcal{H}: f'(x) \in U\}$. If ϕ denotes the natural mapping from $\mathcal{G}$ into $\mathcal{G}/\mathcal{H}$, then $f'^{-1}(U) = \phi(f^{-1}(U))$ and is thus open.

I.4.3. If $\mathcal{G}$ is a topological group, if $\mathcal{H}$ is a normal closed subgroup and if γ is an additive function from $\mathcal{G}/\mathcal{H}$ to a Banach space, B , then γ may be extended to an additive function on all of $\mathcal{G}$.

Proof. Put $\lambda(x) = \gamma(x\mathcal{H})$. Then $\lambda(xy) = \gamma(xy\mathcal{H}) = \gamma(x\mathcal{H}y\mathcal{H}) = \gamma(x\mathcal{H}) + \gamma(y\mathcal{H}) = \lambda(x) + \lambda(y)$.

I.4.4. If $\mathcal{G}$ is a topological group, if $\mathcal{H}$ is a normal closed subgroup and if f is a continuous function from $\mathcal{G}/\mathcal{H}$ to a Banach space, B , then f may be extended to a continuous function on all of $\mathcal{G}$.

Proof. Put $g(x) = f(x\mathcal{H})$. If ϕ is the natural mapping from $\mathcal{G}$ to $\mathcal{G}/\mathcal{H}$ and U is an open set in B , then $g^{-1}(U) = \phi^{-1}(f^{-1}(U))$ which is open.

I.4.5 (Parseval's Formula), [18, p. 26, 27]. If $\mathcal{G}$ is a locally compact abelian topological group, if n is the Haar measure on $\hat{\mathcal{G}}$ and if f and g are complex-valued L_2 -functions then:

$$\int_{\mathcal{G}} f(x)\overline{g(x)} \, dm(x) = \int_{\hat{\mathcal{G}}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)} \, dn(\gamma).$$

I.4.6. If S is a countable subset of the discrete group T then S' is countable.

Proof. Since T is discrete, S' is just the set of polynomials over the integers in the elements of S , which is countable.

I.4.7 [18, p. 9]. If $\mathcal{S}$ is compact and abelian then $\hat{\mathcal{S}}$ is discrete.

I.4.8 (Pontrjagin Duality Theorem), [18, p. 27]. If $\mathcal{S}$ is a locally compact abelian group then $\mathcal{S}$ is homeomorphically isomorphic to $\hat{\hat{\mathcal{S}}}$.

I.4.9 [18, p. 36]. If $\mathcal{S}$ is a locally compact abelian group and $\mathcal{H}$ is a closed subgroup of $\mathcal{S}$, every continuous character on $\mathcal{H}$ can be extended to a continuous character on $\mathcal{S}$.

I.4.10. If $\mathcal{S}$ is compact, if B is a Banach space and if f is an integrable function from $\mathcal{S}$ to B , then

$$\begin{aligned} B \int f(xh) dm(x) &= B \int f(hx) dm(x) \\ &= B \int f(x) dm(x), \end{aligned}$$

for each $h \in \mathcal{S}$.

Proof. If B is the complex numbers, this is well known. The point is that in a compact group the left Haar measure is also the right Haar measure [10, p. 261], and the proof for the complex case works in this case also.

I.4.11. If $\mathcal{S}$ is compact, if $1 \leq p < \infty$ and if f is an L_p -function then $\|f\|_1 \leq \|f\|_p$.

Proof. This follows from [8, p. 121] by taking g

to be the function identically equal to 1.

I.4.12. Let $\mathcal{G}$ be compact and let B be a Banach space. $C(\mathcal{G}, B)$ (the continuous functions from $\mathcal{G}$ to B) is dense (in the p -norm) in $L_p(\mathcal{G}, m, B)$, where $1 \leq p < \infty$.

Proof. The simple functions are dense in L_p and since m is regular and $\mathcal{G}$ is normal the continuous functions are dense in the simple functions.

I.4.13. Let $\mathcal{G}$ be a compact topological group, let $1 \leq p < \infty$ and let f be an $L_p(\mathcal{G}, m, B)$ -function. The mappings $T_h(f)$ and $S_k(f)$ from $\mathcal{G}$ to L_p are continuous and their composition, $S_k(T_h(f))$ is continuous from $\mathcal{G} \times \mathcal{G}$ to L_p .

Proof. Let $\epsilon > 0$. By (I.4.12) there exists a continuous function, g , such that $\|f - g\|_p < \epsilon/3$. Now g is uniformly continuous so there exists $U \in \mathcal{N}(I)$ such that $yz^{-1} \in U$ implies that $\|g(y) - g(z)\| < \epsilon/3$.

There exists, furthermore, $V \in \mathcal{N}(I)$ such that $h \in V$ and $k \in V$ implies that $kxhx^{-1} \in U$ for all $x \in \mathcal{G}$, because multiplication is jointly uniformly continuous.

Therefore, for $h \in V$ and $k \in V$ we have $\|g - S_k T_h g\|_p < \epsilon/3$. Combining the above inequalities we get:

$$\|f - S_k T_h(f)\|_p \leq$$

$$\|f - g\|_p + \|g - S_k T_h(g)\|_p + \|S_k T_h g - S_k T_h f\|_p < \epsilon,$$

the last summand above being less than $\epsilon/3$ on account of the left and right invariance of the integral (I.4.10).

I.4.14 [18, p. 24]. Let $\mathcal{G}$ be a compact abelian topological group and let $1 \leq p < \infty$. The trigonometric polynomials are dense in $L_p(\mathcal{G}, m, \mathbb{C})$.

I.4.15. Let $\mathcal{G}$ be a compact abelian group and let f be a complex-valued L_1 -function such that $\hat{f}(\gamma) = 0$ for all $\gamma \in \hat{\mathcal{G}}$. Then $[f] = 0$.

Proof. Let $\epsilon > 0$. By [18, p. 51] there exists an L_1 -function v such that $\|f * v - f\|_1 < \epsilon/2$. There is a trigonometric polynomial $p(x) = \sum_{i=1}^n a_i \gamma_i(x)$ such that $\|v - p\|_1 < \epsilon/(2\|f\|_1)$. Thus we have:

$$\begin{aligned} \|f - f * p\|_1 &\leq \|f - f * v\|_1 + \|f * (v - p)\|_1 \\ &< \epsilon. \end{aligned}$$

But $f * p = 0$ so that $\|f\|_1 < \epsilon$.

I.4.16. [18, p. 37]. If $\mathcal{G}$ is the complete direct sum of a family of compact abelian groups $\{\mathcal{G}_\alpha\}$ then $\hat{\mathcal{G}}$ is the direct sum of their dual groups $\{\hat{\mathcal{G}}_\alpha\}$.

I.4.17 (Orthogonality Relations), [18, p. 10]. If $\mathcal{G}$ is compact and abelian and $\gamma \in \hat{\mathcal{G}}$, then:

$$\int \langle x, \gamma \rangle dm(x) = \begin{cases} 1 & \text{if } \gamma \text{ is identically } 1 \\ 0 & \text{otherwise} \end{cases}$$

CHAPTER II

BASIC LEMMAS

II.1 Introduction

This chapter contains generalizations of results occurring in the literature, with one exception. Lemma II.2.1 appears in [3] and we permit the range space to be a Banach space and do not require $\mathcal{S}$ to be abelian. Lemma II.2.2 seems obvious and is related to [8, p. 198, Theorem 17] but we were unable to find it in the literature. In lemma II.2.3 and corollary II.2.4 we allow the range space to be a Banach space and the group to be non-abelian. They appear in [7]. Lemma II.2.5 originally was proved in [9] and an improvement was proved in [6]. Since [6] is generally unavailable we reproduce that proof. These results are thought to be of general interest and do not involve specialized definitions. Lemmas which are peculiar to the material in a later chapter are presented in that chapter.

II.2 The Lemmas

II.2.1. Lemma. Let $\mathcal{S}$ be compact and let $\{\phi_\lambda\}$ be a collection of measurable functions on $\mathcal{S}$ taking values in the Banach space, B , such that for each h in $\mathcal{S}$, the values $|\Delta_h \phi_\lambda(y)|$ are uniformly bounded in λ and y .

Then $|\phi_\lambda(y) - \phi_\lambda(I)|$ is uniformly bounded in λ and y .

Proof. For each positive integer, M , let

$$S_M = \{h : |\phi_\lambda(yh) - \phi_\lambda(y)| \leq M \text{ for all } \lambda \text{ and } y\}.$$

Then by hypothesis, $\mathcal{S} = \bigcup_{M=1}^{\infty} S_M$, and it follows that

$m^*(S_M) > 1/2$ for all M sufficiently large. Fix such an M and fix λ , and let $S = \{k : |\phi_\lambda(k) - \phi_\lambda(I)| \leq M\}$.

S is measurable and since $S_M \subset S$, $m(S) > 1/2$. Now if A is any measurable subset of $\mathcal{S}$ and Q is any subset of $\mathcal{S}$, then $m^*(Q \sim A) \leq 1 - m(A)$, since CA covers $Q \sim A$. Also $m^*(Q) \leq m^*(A \cap Q) + m^*(Q \sim A)$, so that:

$$m^*(A \cap Q) \geq m(A) + m^*(Q) - 1.$$

Now if $y \in \mathcal{S}$, then $m(S^{-1}y) > 1/2$ so that applying the previously displayed formula to $S^{-1}y$ and S_M gives $m(S^{-1}y \cap S_M) > 0$. Let h be a point in $S^{-1}y \cap S_M$. Then $h = k^{-1}y$ or $y = kh$ with $k \in S$ and $h \in S_M$. We now have: $|\phi_\lambda(y) - \phi_\lambda(I)| \leq |\phi_\lambda(kh) - \phi_\lambda(k)| + |\phi_\lambda(k) - \phi_\lambda(I)| \leq 2M$.

II.2.2. Lemma. Let $\mathcal{S}$ be compact, let $1 \leq p < \infty$ and let $[f] \in L_p(\mathcal{S}, m, B)$. Then

$$\begin{aligned} L_p \int T_x[f] \, dm(x) &= [B \int f] \\ &= L_p \int S_x[f] \, dm(x). \end{aligned}$$

Proof. Let $\epsilon > 0$. Let $E_1, E_2, \dots, E_n$ be a par-

tition of $\mathcal{G}$ and $x_j \in E_j$ such that $x' \in E_j$ implies $|T_{x',f} - T_{x_j,f}|_p < \epsilon/2$ and $|S_{x',f} - S_{x_j,f}|_p < \epsilon/2$. This can be done because of (I.4.13). $[\sum_{j=1}^n m(E_j) T_{x_j} f]$ is an approximating sum for $L_p \int T_x[f] dm(x)$, in fact

$$[\sum_{j=1}^n m(E_j) T_{x_j} f] = L_p \int [\sum_{j=1}^n \chi_{E_j}(x) T_{x_j} f] dm(x), \text{ and}$$

$$\left| L_p \int [T_x f - \sum_{j=1}^n \chi_{E_j}(x) T_{x_j} f] dm(x) \right|_p =$$

$$\left| L_p \int \sum_{j=1}^n [\chi_{E_j}(x) (T_x f - T_{x_j} f)] dm(x) \right|_p \leq$$

$$\sum_{j=1}^n \int_{E_j} |T_x f - T_{x_j} f|_p dm(x) <$$

$$(\epsilon/2) \sum_{j=1}^n m(E_j) = \epsilon/2.$$

Likewise $[\sum_{j=1}^n m(E_j) S_{x_j} f]$ is an approximating sum for $L_p \int S_x[f] dm(x)$ and their p-norm difference is less than

$$\epsilon/2. \text{ Now } B \int f(z) dm(z) = B \int f(xz) dm(z) = B \int f(zx) dm(z)$$

for any $x \in \mathcal{G}$ (I.4.10). We have:

$$\int \left| B \int f(xz) dm(z) - \sum_{j=1}^n m(E_j) (T_{x_j} f)(x) \right| dm(x) =$$

$$\int \left| B \int (f(xz) - \sum_{j=1}^n \chi_{E_j}(z) (T_{x_j} f)(x)) dm(z) \right| dm(x) =$$

$$\int \left| B \int \left(\sum_{j=1}^n \chi_{E_j}(z) (f(xz) - f(xx_j)) \right) dm(z) \right| dm(x) \leq \\ \iint \sum_{j=1}^n \chi_{E_j}(z) |f(xz) - f(xx_j)| dm(z) dm(x).$$

This means that:

$$\left| \left[B \int f(z) dm(z) \right] - \left[\sum_{j=1}^n m(E_j) T_{x_j} f \right] \right|_1 \leq \\ \iint \sum_{j=1}^n \chi_{E_j}(z) |f(xz) - f(xx_j)| dm(z) dm(x) = \\ \iint \sum_{j=1}^n \chi_{E_j}(z) |f(xz) - f(xx_j)| dm(x) dm(z) = \\ \sum_{j=1}^n \int_{E_j} \int |f(xz) - f(xx_j)| dm(x) dm(z) = \\ \sum_{j=1}^n \int_{E_j} |T_z f - T_{x_j} f|_1 dm(z) < \\ \epsilon / 2 \sum_{j=1}^n m(E_j) = \epsilon / 2.$$

The interchange of the order of integration was justified by Fubini's Theorem and the fact that $f(xy)$ is jointly measurable in x and y . As one expects, we also have:

$$\int \left| B \int f(zx) dm(z) - \sum_{j=1}^n m(E_j) (S_{x_j} f)(x) \right| dm(x) \leq \\ \iint \sum_{j=1}^n \chi_{E_j}(z) |f(zx) - f(x_j x)| dm(z) dm(x),$$

so that $\left| [B \int f(z) dm(z)] - \left[\sum_{j=1}^n m(E_j) S_{x_j} f \right] \right|_1 < \epsilon/2$.

The following four inequalities have now been shown:

$$\left| [B \int f(z) dm(z)] - \left[\sum_{j=1}^n m(E_j) T_{x_j} f \right] \right|_1 < \epsilon/2,$$

$$\left| [B \int f(z) dm(z)] - \left[\sum_{j=1}^n m(E_j) S_{x_j} f \right] \right|_1 < \epsilon/2,$$

$$\left| L_p \int T_x[f] dm(x) - \left[\sum_{j=1}^n m(E_j) T_{x_j} f \right] \right|_1 < \epsilon/2 \text{ and}$$

$$\left| L_p \int S_x[f] dm(x) - \left[\sum_{j=1}^n m(E_j) S_{x_j} f \right] \right|_1 < \epsilon/2.$$

Combining these we have:

$$\left| [B \int f] - L_p \int T_x[f] dm(x) \right|_1 < \epsilon \text{ and}$$

$$\left| [B \int f] - L_p \int S_x[f] dm(x) \right|_1 < \epsilon.$$

Since ϵ is arbitrary, we are finished with the proof.

II.2.3. Lemma. Let $\mathcal{X}$ be compact, let $1 \leq p < \infty$ and let B be a Banach space. Suppose f is a function from $\mathcal{X}$ to B such that for each h in $\mathcal{X}$, $\Delta_h f$ and $h \Delta f$ are L_p -functions. Then there exists M such that

$$|h \Delta_k f|_p \leq M \text{ for all } h \text{ and } k \text{ in } \mathcal{X}.$$

Proof. Let $A_n = \bigcap_{k \in \mathcal{X}} \{h: |h \Delta_k f|_p \leq n\}$, for

each positive integer n . For each fixed k ,

$\{h: |h \Delta_k f|_p \leq n\}$ is closed because of (I.4.13) so that

A_n is closed. Moreover, $\mathcal{S} = \bigcup_n A_n$, since if $x \in CA_n$ for all n , there would be $\{k_n\}$ such that $|x \Delta_{k_n} f|_p > n$ for each n . If k_0 is a limit point of $\{k_n\}$ then we would have by (I.4.13) $|x \Delta_{k_0} f|_p > n$ for each n . Now according to (I.3.2) one of the A_N contains an open set, Vh_0 , where V is a neighborhood of the identity. Let $Vh_0, Vh_1, \dots, Vh_q$ cover $\mathcal{S}$ and let $L = \max \{|h_0^{-1}h_i \Delta f|_p : 1 \leq i \leq q\}$. Let h and k be arbitrary elements of $\mathcal{S}$. Then $h = vh_i$ for some $i, 1 \leq i \leq q$ and $v \in V$, and we have: $h \Delta_k f = vh_0 h_0^{-1} h_i \Delta_k f$ with $v \in V$. Now if it shown that

$$(II.1.1) \quad |ab \Delta g|_p \leq |a \Delta g|_p + |b \Delta g|_p$$

we are finished, for

$$\begin{aligned} |h \Delta_k f|_p &\leq |vh_0 \Delta_k f|_p + |h_0^{-1} h_i \Delta_k f|_p \\ &= |vh_0 \Delta_k f|_p + |\Delta_k (h_0^{-1} h_i \Delta f)|_p \\ &\leq N + 2L. \end{aligned}$$

Now as for (II.1.1), the left-hand side is equal to:

$$\left\{ \int |g(abx) - g(x)|^p dm(x) \right\}^{(1/p)} =$$

$$\begin{aligned}
& \left\{ \int |g(ax) - g(bx) + g(bx) - g(x)|^{p_{dm}(x)} dx \right\}^{(1/p)} \leq \\
& \left\{ \int |g(ax) - g(bx)|^{p_{dm}(x)} dx \right\}^{(1/p)} + \\
& \left\{ \int |g(bx) - g(x)|^{p_{dm}(x)} dx \right\}^{(1/p)} = \\
& |a^\Delta g|_p + |b^\Delta g|_p.
\end{aligned}$$

II.2.4. Corollary. We may allow $p = \infty$ in lemma II.2.3 and the inequality holds uniformly for $1 \leq p \leq \infty$.

Proof. Define $A_n = \bigcap_{1 \leq p < \infty} \bigcap_{k \in \mathcal{H}} \{h: |h^\Delta_k f|_p \leq n\}$ and $L = \max \{ |h_0^{-1} h_i \Delta f|_\infty : 1 \leq i \leq q \}$ and the proof of lemma II.2.3 will suffice.

II.2.5. Lemma. Let X_1 , X_2 and X_3 be topological spaces. Suppose X_2 is second axiom and X_3 is second axiom and regular (thus, metrizable). Let $g(x, y)$ be a function from $X_1 \times X_2$ into X_3 which is continuous in each variable separately. Then there is a residual set $Q \subset X_1$ such that $g(x, y)$ is continuous on $Q \times X_2$.

Proof. Let $\{V_m\}$ and $\{W_n\}$ be countable bases for the topologies of X_2 and X_3 , respectively. Let D_{nm} denote the set of points, x , in X_1 such that each neighborhood of x contains a pair of points x' , x'' with $g(x', V_m) \subset cW_n$ and $g(x'', V_m) \not\subset cW_n$. Clearly D_{nm} is closed. Let $x \notin D_{nm}$, let U be any neighborhood of x and let $x'' \in U$ be such that $g(x'', V_m) \not\subset cW_n$. There

exists $y_0 \in V_m$ such that $g(x'', y_0) \notin cW_n$. There is, then, a neighborhood $U_1 \subset U$ of x'' such that $g(x', y_0) \notin cW_n$ for each $x' \in U_1$, showing that $g(x', V_m) \not\subset cW_n$ and, consequently, that $x'' \notin D_{nm}$. This means that D_{nm} is nowhere dense. Let Q be the complement of the union of the D_{nm} . Q is residual. Let $(x_0, y_0) \in Q \times X_2$ and let W be an open subset of X_3 containing $g(x_0, y_0)$. Let n be such that $g(x_0, y_0) \in W_n \subset cW_n \subset W$. There exists m such that $g(x_0, V_m) \subset W_n$. Since $x_0 \notin D_{nm}$ there exists a neighborhood U of x_0 such that, for each $x \in U$, $g(x, V_m) \subset cW_n$, which is to say that $g(U, V_m) \subset W$. This shows that g is continuous at (x_0, y_0) .

CHAPTER III

FOR COMPACT ABELIAN GROUPS

III.1 Introduction and Statement of the Main Results

Lemma III.2.1, lemma III.2.2 and theorem III.3.1 may be regarded as structure theorems for abelian groups. They appeared in [6] and so we present their proofs. Theorem III.1.1 is an improvement of a result in [6] in that the range space may be a Banach space. This point is essential for theorem III.1.2, the proof of which uses theorem III.1.1 with the range space, $L_p(\mathcal{G}, m, B)$. There are two main results, as follows:

III.1.1. Theorem. Let $\mathcal{G}$ be a compact, abelian topological group and let B be a Banach space. Let F be a function on $\mathcal{G}$ with values in B and suppose that for every h in $\mathcal{G}$ the function $\Delta_h F$ is continuous on $\mathcal{G}$. Then

$$(III.1.1) \quad F = G + \Gamma,$$

where G is continuous on $\mathcal{G}$ and Γ is an additive function on $\mathcal{G}$ given by:

$$(III.1.2) \quad \Gamma(x) = \int B (F(x + y) - F(y)) dm(y).$$

III.1.2. Theorem. Let $\mathcal{G}$ be a compact, abelian topological group and let B be a Banach space. Let $1 \leq p \leq \infty$ and let f be a function on $\mathcal{G}$ with values in B such that for every $h \in \mathcal{G}$, $\Delta_h f$ is an L_p -function. Then:

$$(III.1.3) \quad f = g + \gamma + S,$$

where g is an L_p -function, γ is additive on $\mathcal{G}$ and each difference of S is null.

III.2 Lemmas

III.2.1. Lemma. Let F be an uncountable discrete abelian group and let S be an uncountable subset of F . Then there exists a character on F which assumes values different from 1 on an uncountable subset of S .

Proof. Assume the contrary. Then the following is true: (*) if H is a subgroup of F and C a coset of H different from H , then $S \cap C$ is countable. For by the Pontrjagin Duality Theorem (I.4.8) there is always a character on F/H assuming a value different from 1 on the canonical image of C in F/H . That character defines a character on F , constant on cosets of H and assuming a value different from 1 at each point of C . If $S \cap C$ were uncountable, then the conclusion of the lemma would be true.

Some terminology at this point will make the proof easier to give. By a multiple of an element f of F we

mean an expression nf where n is an integer. If $n = 0$, then nf is the identity of F and is called the zero-multiple. If $n > 0$, $nf = f + f + \dots + f$, the expression on the right having n terms. If $n < 0$, then $nf = -((-n)f)$. The latter two cases are termed non-zero-multiples, while a multiple unequal to the identity of F is called a non-zero multiple. Let T be the torsion subgroup of F . There are two cases.

CASE A. $S \sim T$ is uncountable. By (*) S has a non-empty intersection with uncountably many cosets of T . Hence, S^* , the image of S under the natural mapping is uncountable. Well-order S^* , and, using transfinite induction, form the set S^{**} as follows: $s \in S^{**}$ if no non-zero-multiple of s is a sum of multiples of elements preceding s . S^{**} is uncountable since each element of F/T is of infinite order. Again using transfinite induction, extend S^{**} to a set, Q , having the two properties:

- 1) a sum of multiples of elements of Q is the identity only if each multiple is the zero-multiple,
- 2) each element of F/T has a non-zero-multiple equal to a sum of multiples of elements of Q .

Such a set is called a Hamel basis for F/T . The reader is referred to [19, Chapter VI] and [3, p. 288] for a discussion of similar notions. Now define an additive function Γ on F/T by putting $\Gamma(b) = 1/2$ for b in Q ,

and for $x \in F/T$ and $Nx = n_1 b_1 + \dots + n_p b_p$, $N \neq 0$, $b_i \in Q$, let $\Gamma(x) = 1/(2N) \sum_{i=1}^p n_i$. The uniqueness of the expression for Nx shows that Γ is well-defined and one can verify that Γ is additive on F/T . Γ defines an additive function (I.4.3) on F , denoted hereafter by Γ , which is constant on cosets of T . The character given by $\exp(2\pi i \Gamma)$ is unequal to 1 at each point of the canonical preimage of S^* , which contains uncountably many points of S . Thus if we deny the conclusion of the lemma CASE A cannot occur. There remains:

CASE B. $S \sim T$ is countable. Thus $S \cap T$ is uncountable. Let $n > 1$ be the smallest integer with the property that $T_n \cap S$ is uncountable, where $f \in T_n$ if nf is the identity of F . Well-order $T_n \cap S$ and inductively define the subset, S^* , of $T_n \cap S$ as follows: $s \in S^*$ if no non-zero multiple of s equals a sum of multiples of elements of S^* which precede s . We claim that S^* is uncountable. If S^* were countable, then the set of sums of multiples of elements of S^* would also be countable. Now because of the definition of S^* , there would exist an m ($0 < m < n$) such that the m^{th} multiples of uncountably many elements of $T_n \cap S$ would be equal. This means that some coset of T_m intersects S in an uncountable set. The coset could not be T_m because of the definition of n and in view of (*) no other coset can intersect S in

an uncountable set. Thus S^* is uncountable. Let $S^{*'}$ be the subgroup generated by S^* . In view of the definition of S^* each element of $S^{*'}$ is associated with a unique $(\text{mod}(n))$ integer-valued function on S^* taking non-zero values on at most a finite subset of S^* . If for f' in $S^{*'}$, $s(f')$ denotes the sum of those values then $\exp(2\pi i s(f')/n)$ is a character on $S^{*'}$ which can be extended to a character on F (I.4.9). If $f' \notin S^*$ then $s(f') \neq 1$ and the value of the above character is unequal to 1. This contradicts the denial assumed at the beginning of the proof, and so the denial is false and the proof is complete.

III.2.2. Lemma. Let $\mathcal{G}$ be an arbitrary abelian group, let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ and let Γ_1 be an additive function on $\mathcal{H}$. Then there exists an additive function Γ on $\mathcal{G}$ such that $\Gamma_1(x) = \Gamma(x)$ for $x \in \mathcal{H}$.

Proof. We form a Hamel basis, Q_1 , for $\mathcal{H}$ as described in the proof of lemma III.2.1, CASE A. Now well-order $\mathcal{G}$ in such a way that each element of Q_1 precedes any element of $\mathcal{G} - Q_1$. We then form a Hamel basis, Q , of $\mathcal{G}$ using the well-ordering obtained above. We note that each element of Q_1 is in Q , so that Q is a superset of Q_1 . Now since an additive function is given essentially on a Hamel basis, we define $\Gamma(b) = \Gamma_1(b)$ if b is in Q_1 and let $\Gamma(b)$ be arbitrary if $b \in Q - Q_1$.

III.3 The Proof of Theorem III.1.1

We shall assume for now that $\mathcal{X}$ is second axiom and remove this hypothesis later. If $\phi: \mathcal{X} \times \mathcal{X} \rightarrow B$ is defined by $\phi(x, y) = F(x + y) - F(x) - F(y)$ then ϕ is continuous in each variable separately and lemma II.1.1 shows that there is an M such that $|\phi(x, y)| \leq M$ for all x and y in $\mathcal{X}$. If Γ is defined by (III.1.2) then it is elementary that Γ is additive; moreover, if G is defined by (III.1.1) we see that for all $x \in \mathcal{X}$:

$$(III.3.1) \quad G(x) = -B \int \phi(x, y) \, dm(y).$$

Now let x_0 be any point of $\mathcal{X}$ and let $\{x_n\}$ be a sequence in $\mathcal{X}$ converging to x_0 . For each fixed y , $\{\phi(x_n, y)\}$ converges to $\phi(x_0, y)$ because of the separate continuity of ϕ . Since $\mathcal{X}$ has finite measure, Egoroff's Theorem (I.2.1) may be applied in the form (I.2.2) to show that $\{\phi(x_n, \cdot)\}$ converges in measure to $\phi(x_0, \cdot)$; moreover, since $|\phi(x_n, y)| \leq M$ for all n and y , the Dominated Convergence Theorem (I.2.3) shows that $B \int \phi(x_n, y) \, dm(y)$ converges to $B \int \phi(x_0, y) \, dm(y)$. In view of (I.3.1) and (III.3.1) we see that G is continuous.

We can now remove the hypothesis that $\mathcal{X}$ be second axiom by using the following theorem the proof of which will be deferred:

III.3.1. Theorem. Let $\mathcal{X}$ and F be as in theorem III.1.1. Then there exist an additive function Γ_1 on $\mathcal{X}$ and a closed subgroup $\mathcal{H}$ of $\mathcal{X}$ such that $\mathcal{X}/\mathcal{H}$ is second axiom and $F - \Gamma_1$ is constant on cosets of $\mathcal{H}$.

The proof of theorem III.1.1 is now easy, for if $\mathcal{X}_1 = \mathcal{X}/\mathcal{H}$ and $F_1 = F - \Gamma_1$, then $\mathcal{X}_1$ is second axiom, compact (I.4.1) and abelian and F_1 may be considered to be a function on $\mathcal{X}_1$, having continuous differences (I.4.2). Thus $F_1 = G + \Gamma_2$, where Γ_2 is additive on $\mathcal{X}/\mathcal{H}$ and G is continuous on $\mathcal{X}/\mathcal{H}$. Γ_2 and G may be used to define functions on $\mathcal{X}$ (I.4.3, I.4.4) which we now denote by Γ_2 and G , respectively, which are additive on $\mathcal{X}$ and continuous on $\mathcal{X}$, respectively. We now have : $F = G + (\Gamma_1 + \Gamma_2) = G + \Gamma$, and

$$\Gamma(x) = B \int (F(x+y) - F(y)) \, dm(y).$$

One sees that if (III.1.1) holds then Γ must be given by (III.1.2).

We now prove theorem III.3.1. For each h in $\mathcal{X}$ and $\hat{x}$ in $\hat{\mathcal{X}}$, let

$$(III.3.2) \quad a(h, \hat{x}) = B \int \Delta_h F(x) \langle -x, \hat{x} \rangle \, dm(x).$$

We claim that for fixed h , $a(h, \hat{x}) = 0$ for all but countably many $\hat{x}$. If we suppose the contrary then there exist $\alpha > 0$ and an uncountable subset $S_1(h)$ of $\hat{\mathcal{X}}$

such that for each $\hat{x} \in S_1(h)$

$$(III.3.3) \quad |a(h, \hat{x})| > \alpha.$$

Since $\Delta_h F$ is continuous there exist a finite collection of measurable subsets of $\mathcal{X}$, say, $A_1, \dots, A_n$, and non-zero elements $e_1, \dots, e_n$, of B such that

$$(III.3.4) \quad |\Delta_h F(x) - \sum_{i=1}^n \chi_{A_i}(x) e_i| < \alpha/3$$

for all x . (III.3.2), (III.3.3) and (III.3.4) imply that

$$(III.3.5) \quad \left| B \int \sum_{i=1}^n \chi_{A_i}(x) e_i \langle -x, \hat{x} \rangle dm(x) \right| > 2\alpha/3$$

for all $\hat{x} \in S_1(h)$. Since the left-hand side of (III.3.5)

is less than or equal to $\sum_{i=1}^n \left| B \int \chi_{A_i}(x) e_i \langle -x, \hat{x} \rangle dm(x) \right|$ we

have that for each $\hat{x} \in S_1(h)$ there exists j such that

$$(III.3.6) \quad \left| B \int \chi_{A_j}(x) e_j \langle -x, \hat{x} \rangle dm(x) \right| > 2\alpha/(3n).$$

Thus for some j , (III.3.6) must hold for uncountably many $\hat{x}$ in $S_1(h)$. This would mean that $2\alpha/(3n|e_j|) <$

$\left| \int \chi_{A_j}(x) \langle -x, \hat{x} \rangle dm(x) \right|$ for uncountably many $\hat{x}$, contra-

dicting Parseval's Formula (I.4.5). The assertion following (III.3.2) is now proved. For each h let $S_h =$

$\{\hat{x} \in \mathcal{X} : a(h, \hat{x}) \neq 0\}$. S_h is countable for each h .

Let S be the union of the S_h and let S^* be the sub-

group of $\hat{\mathcal{G}}$ generated by S . We shall show that S^* is at most denumerable. Since $\Delta_h, \Delta_{h'} F = \Delta_h \Delta_{h'} F$ for all h, h' in $\mathcal{G}$ and since

$$(III.3.7) \quad B \int \Delta_h, \Delta_{h'} F(x) \langle -x, \hat{x} \rangle dm(x) = a(h, \hat{x}) (\langle -h', \hat{x} \rangle - 1)$$

we have for all h, h' in $\mathcal{G}$, $\hat{x}$ in $\hat{\mathcal{G}}$ that

$$(III.3.8) \quad a(h, \hat{x}) (\langle h', \hat{x} \rangle - 1) = a(h', \hat{x}) (\langle h, \hat{x} \rangle - 1).$$

If for some h , $S_h = S$, then S is countable and so is S^* (I.4.6), so we assume that for each h , $S \sim S_h \neq \emptyset$.

If h is fixed (III.3.8) shows that

$$(III.3.9) \quad \langle h, \hat{x} \rangle = 1$$

for all $\hat{x} \in S \sim S_h$. If S were uncountable then by lemma III.2.1 and (I.4.7) there would be a character on S^* taking values different from 1 on an uncountable subset of S . This character could be extended to a character on all of $\hat{\mathcal{G}}$ (I.4.9) so that by the Pontrjagin Duality Theorem (I.4.8) there would be an element h of $\mathcal{G}$ such that $\langle h, \hat{x} \rangle \neq 1$ for uncountably many $\hat{x}$. But any such $\hat{x}$ would belong to S_h in view of (III.3.9), and we have shown that S_h is countable. This contradiction shows that S and therefore S^* is countable (I.4.6).

Let $\mathcal{H} = \{x \in \mathcal{G} : \langle x, \hat{x} \rangle = 1 \text{ for all } \hat{x} \in S^*\}$. $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$, S^* is the character group of $\mathcal{G} / \mathcal{H}$ [18, p. 35] and since S^* is countable, $\mathcal{G} / \mathcal{H}$

is second axiom [18, p. 35, 38]. Now fix $h \in \mathcal{H}$. If $h' \in \mathcal{H}$ and $\hat{x} \in S_h$, then $\langle h', \hat{x} \rangle = 1$, and, because of (III.3.7),

$$(III.3.10) \quad B \int \Delta_h \Delta_{h'} F(x) \langle -x, \hat{x} \rangle dm(x) = 0.$$

If $\hat{x} \notin S_h$ then $a(h, \hat{x}) = 0$ and again (III.3.10) holds. Thus for $h' \in \mathcal{H}$ and $h \in \mathcal{H}$, (III.3.10) holds for all $\hat{x} \in \mathcal{H}$. Now since the trigonometric polynomials are dense in L_1 (I.4.14) and since $\Delta_h \Delta_{h'} F$ is bounded on $\mathcal{H}$, (III.3.10) shows that for any measurable set, E ,

$$B \int_E \Delta_h \Delta_{h'} F(x) dm(x) = 0. \quad \text{By (I.2.4) the integrand in the}$$

above expression is null and being continuous, vanishes identically. We have now shown that $\Delta_h F$ is constant on the cosets of $\mathcal{H}$ for every h . For $h \in \mathcal{H}$ define $\Gamma_1(h)$ to be the value of $\Delta_h F$ on $\mathcal{H}$. For x, h_1 and h_2 in $\mathcal{H}$ we have

$$\begin{aligned} \Gamma_1(h_1 + h_2) &= F(x + h_1 + h_2) - F(x) \\ &= F(x + h_1 + h_2) - F(x + h_1) \\ &\quad + F(x + h_1) - F(x) \\ &= \Gamma_1(h_2) + \Gamma_1(h_1), \end{aligned}$$

so that Γ_1 is an additive function on $\mathcal{H}$. By lemma III.2.2, Γ_1 can be extended to an additive function, denoted hereafter by Γ_1 , on all of $\mathcal{H}$. Now for h in $\mathcal{H}$, $\Delta_h(F - \Gamma_1)$ is constant on $\mathcal{H}$; in particular, for h in

$\mathcal{H}$, the constant value is 0. Thus $F - \Gamma_1$ is constant on $\mathcal{H}$, and, consequently, on each coset of $\mathcal{H}$.

This concludes the proofs of theorem III.3.1 and theorem III.1.1.

III.4 The Proof of Theorem III.1.2

We treat the case $1 \leq p < \infty$ first and the case $p = \infty$ will follow as a corollary. Let $F(h) = [\Delta_h f]$ for h in $\mathcal{H}$. Then the functional equation

$$(III.4.1) \quad F(x + h) - F(x) = T_x F(h)$$

shows that $\Delta_h F$ is continuous for each h in $\mathcal{H}$. Hence by theorem III.1.1 we have $F(x) = G(x) + \Gamma(x)$ where G is continuous and Γ is additive and given by

$$(III.4.2) \quad \Gamma(x) = L_p \int (F(x + h) - F(h)) \, dm(h).$$

In view of (III.4.1) and lemma II.2.2 we see that $\Gamma(x) = [B \int \Delta_x f]$ for x in $\mathcal{H}$. Define $\gamma(x) = B \int \Delta_x f$ and $[g] = -L_p \int G(h) dm(h)$. Now if $S = f - g - \gamma$ and k is in $\mathcal{H}$, then

$$\begin{aligned} [\Delta_k S] &= [\Delta_k f] + T_k L_p \int G(h) \, dm(h) \\ &\quad - L_p \int G(h) \, dm(h) - [\gamma(k)] \\ &= G(k) + L_p \int (T_k G(h) - G(h)) \, dm(h) \end{aligned}$$

$$\begin{aligned}
&= G(k) + L_p \int (T_k F(h) - F(h)) dm(h) \\
&= G(k) + L_p \int (F(k+h) - F(k) - F(h)) dm(h) \\
&= G(k) + \Gamma(k) - F(k) \\
&= 0.
\end{aligned}$$

Now, for the case $p = \infty$, corollary II.2.4 shows that $|\Delta_h \Delta_k f|_p \leq M$ for some M , for all h and k in $\mathcal{X}$ and for all p , $1 \leq p < \infty$. We apply the theorem as proved so far for $1 \leq p < \infty$ and obtain $F = G + \Gamma$ where G is continuous and Γ is additive. Now since

$$G(h) = L_p \int (F(h+k) - F(h) - F(k)) dm(k),$$

we have that $|G(h)|_p \leq M$ for all h and p , $1 \leq p < \infty$.

Moreover, since $[g] = -L_p \int G(h) dm(h)$, we have

$$(III.4.3) \quad |[g]|_p \leq M$$

for all p , $1 \leq p < \infty$. Now if $|g(x)| > M + 1$ on a set, E , of positive measure, α , choosing p such that $\alpha^{(1/p)} > M/(M + 1)$ gives:

$$\begin{aligned}
|g|_p &\geq \left\{ \int_E |g(x)|^p dm(x) \right\}^{(1/p)} \\
&> \alpha^{(1/p)} (M + 1) \\
&> M,
\end{aligned}$$

a contradiction of (III.4.3). Thus $[g] \in L_\infty$.

CHAPTER IV
 COMPACT SECOND AXIOM GROUPS, UNIFORMLY
 DISTRIBUTED SEQUENCES AND PROPERTY C

IV.1 Introduction and Definitions

In weakening the algebraic requirements on the group, $\mathcal{G}$, namely, permitting $\mathcal{G}$ to be non-abelian, we have had to impose stronger topological or analytic conditions to get results. If one tries to rework the proofs of the previous chapter to accommodate a non-abelian group, he cannot get far. The concept of Hamel basis is worthless and the nice properties of the dual group are unavailable. These properties are algebraic in nature and are compensated for by the topological properties: "second axiom" and "C" and the property of having a uniformly distributed sequence. These properties are all used to show that a certain function is continuous at at least one point.

IV.1.1. Definition. A bounded function, f , from a compact metric space, X , with measure, m , to a Banach space, B , will be called m -Riemann integrable provided there exists $b \in B$ such that for every sequence, $\{\pi_i\}$, of measurable partitions, $\pi_i = \{A_{i,j} : j = 1, \dots, m_i\}$, of X satisfying the two conditions:

$$1) \quad \lim_{i \rightarrow \infty} \sup_{j=1, \dots, m_i} d(A_{i,j}) = 0, \text{ and}$$

2) $i_1 < i_2$ and $1 \leq j_2 \leq m_{i_2}$ implies there exists

j_1 , $1 \leq j_1 \leq m_{i_1}$, such that $A_{i_2, j_2} \subset A_{i_1, j_1}$,

and every sequence, $\{t_i\}$, of m_i -tuples, $t_i = (z_{i, 1}, \dots, z_{i, m_i})$, such that $z_{i, j} \in A_{i, j}$; $i = 1, 2, \dots$; $1 \leq j$

$\leq m_i$, we have $\lim_{i \rightarrow \infty} \sum_{j=1}^{m_i} f(z_{i, j}) m(A_{i, j}) = b$.

IV.1.2. Definition. A sequence $\{\lambda_i\}$ in a compact topological group, $\mathcal{X}$, will be called uniformly distrib-

uted provided $m(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi_A(\lambda_i)$ for all

closed sets, A , with null boundary.

IV.1.3. Definition. A pair $(\mathcal{X}, B)$ where $\mathcal{X}$ and B are topological spaces is said to have property C if any function from $\mathcal{X} \times \mathcal{X}$ to B which is continuous in each variable separately is continuous at some point.

IV.2 Lemmas

IV.2.1. Lemma. Let $\mathcal{X}$ be a compact second axiom group and let B be a Banach space. Let g be a function from $\mathcal{X}$ to B such that for all $h \in \mathcal{X}$:

i) $|g(\cdot)|$ is a bounded function on $\mathcal{X}$,

ii) $B \int_h \Delta g(x) dm(x) = 0$,

iii) $\Delta_h g(\cdot)$ is a continuous function on $\mathcal{X}$ and

iv) $g(I) = 0$.

Then g is continuous at I .

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{X}$ converging to I. Let $|g(x)| \leq M$ for each $x \in \mathcal{X}$. For each n and $z \in \mathcal{X}$ define:

$$\begin{aligned} h_n(z) &= \Delta_z g(x_n) - g(z) \\ &= g(x_n z) - g(x_n) - g(z). \end{aligned}$$

For each $z \in \mathcal{X}$, $\Delta_z g(\cdot)$ is a continuous function on $\mathcal{X}$, so that $\lim_{n \rightarrow \infty} h_n(z) = \Delta_z g(I) - g(z) = 0$. That is, the

sequence $\{h_n(\cdot)\}$ converges pointwise to the function which is identically zero on $\mathcal{X}$. We now see that $\{h_n\}$ converges in measure to the function identically zero (I.2.2), and since $|h_n(z)| \leq 3M$ for all n and z we

have $\lim_{n \rightarrow \infty} \int h_n(z) \, d\mu(z) = 0$, by using the Dominated

Convergence Theorem (I.2.3). But

$$\begin{aligned} \int h_n(z) \, d\mu(z) &= \int (g(x_n z) - g(x_n) - g(z)) \, d\mu(z) \\ &= -g(x_n) + \int x_n \Delta g(z) \, d\mu(z) \\ &= -g(x_n). \end{aligned}$$

Thus we see that $\lim_{n \rightarrow \infty} g(x_n) = 0$. Since the sequence

$\{x_n\}$ is arbitrary, save that it converge to I, we have shown that g is continuous at I (I.3.1).

IV.2.2. Lemma [3]. Let X be a compact metric space with a fixed finite regular Borel measure, μ , and let Y be a finite measure space with measure ν . Let $\phi(x, y)$ be a function on $X \times Y$ taking values in the

Banach space, B , such that $|\phi(x, y)| \leq M$ for all $(x, y) \in X \times Y$. Suppose that for each fixed $x \in X$, $\phi(x, \cdot)$ is a measurable function and that for each fixed $y \in Y$, $\phi(\cdot, y)$ is a u -Riemann integrable function. Then the function, g , given by:

$$(IV.2.1) \quad g(x) = B \int \phi(x, y) dv(y)$$

is a u -Riemann integrable function of x .

Proof. Let the $A_{i,j}$ and $z_{i,j}$ be as specified in definition IV.1.1. The corresponding sequence of Riemann sums for the function g is:

$$(IV.2.2) \quad \left\{ \sum_{j=1}^{m_i} B \int \phi(z_{i,j}, y) dv(y) u(A_{i,j}) : i=1, 2, \dots \right\} = \\ \left\{ B \int \sum_{j=1}^{m_i} \phi(z_{i,j}, y) u(A_{i,j}) dv(y) : i=1, 2, \dots \right\}.$$

The integrand in the above expression is bounded by

$$\sum_{j=1}^{m_i} |\phi(z_{i,j}, y)| |u(A_{i,j})| \leq \\ M \sum_{j=1}^{m_i} |u(A_{i,j})| \leq MN$$

where $N = |u|(X)$.

The sequence of integrands for $i = 1, 2, \dots$ converges pointwise to the function, h , the value of which at a point, y , of Y is $B \int \phi(x, y) du(x)$. By (I.2.2) the

sequence of integrands converges in measure to the function h . The Dominated Convergence Theorem (I.2.3) then shows that the sequence (IV.2.2) converges. The limit is independent of the choice of $A_{i,j}$ and $z_{i,j}$ since the limit of the sequence of integrands is. It is clear that g is bounded and so g is u-Riemann integrable.

IV.2.3. Lemma. Let $\mathcal{X}$ be a compact topological group having a uniformly distributed sequence $\{\lambda_i\}$. Let B be a Banach space and let f be a continuous function from $\mathcal{X}$ to B . Then

$$(IV.2.3) \quad B \int f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(\lambda_i).$$

Proof. If the space, B , is the complex numbers, this is a well-known result. A function, g , from $\mathcal{X}$ into B will be called strongly simple provided:

$$g(x) = \sum_{i=1}^N c_i \chi_{A_i}(x),$$

where

- i) A_i is closed, $1 \leq i \leq N$,
- ii) $m(bA_i) = 0$, $1 \leq i \leq N$,
- iii) $m(A_i \cap A_j) = 0$, $1 \leq i < j \leq N$, and
- iv) $c_i \in B$, $1 \leq i \leq N$.

The reader is referred to the relations (I.3.5). It is

clear that (IV.2.3) holds for a strongly simple function, f . Let M be a fixed positive integer. For each point, y_0 , in $\mathcal{X}$ the function $|f(\cdot) - f(y_0)|$ is continuous and so $\{x : |f(x) - f(y_0)| < 1/M\}$ is an open neighborhood of y_0 . For each r , $0 < r < 1/M$, let $S_r = \{x : |f(x) - f(y_0)| < r\}$ and $D_r = \{x : |f(x) - f(y_0)| = r\}$. We claim that $bS_r \subset D_r$. Since S_r is open, $bS_r = cS_r \sim S_r$. If $z \in bS_r$, then there is a net $T: D \rightarrow S_r$ converging to z . Since $|f(\cdot) - f(y_0)|$ is continuous we must have $|f(z) - f(y_0)| \leq r$. But $z \notin S_r$ so $|f(z) - f(y_0)| = r$ and $z \in D_r$. Now the sets D_r cannot all have positive measure, so there is an open set O_{y_0} such that:

$$i) \quad y_0 \in O_{y_0},$$

$$ii) \quad m(bO_{y_0}) = 0, \text{ and}$$

$$iii) \quad x \in O_{y_0} \text{ implies } |f(x) - f(y_0)| < r < 1/M.$$

In addition we remark that $x \in cO_{y_0}$ implies $1/M > |f(x) - f(y_0)|$. Such an O_y exists for each $y \in \mathcal{X}$, and a finite number of them covers $\mathcal{X}$, say $\{O_{y_i} : 1 \leq i \leq N\}$. Put $O_i = O_{y_i}$, $1 \leq i \leq N$. Now put

$$A_i = c(O_i \sim c \bigcup_{j=1}^{i-1} O_j)$$

for $1 \leq i \leq N$, and

$$g_M(x) = \sum_{i=1}^N f(y_i) \chi_{A_i}(x).$$

We claim that $\mathcal{G}_M$ is strongly simple.

i) A_i is closed by definition, $1 \leq i \leq N$,

$$\begin{aligned}
 \text{ii) } bA_i &= bc(0_i \cap cc \bigcup_{j=1}^{i-1} 0_j) \\
 &= bc(0_i \cap \bigcap_{j=1}^{i-1} cc0_j) \\
 &\subset b(0_i \cap \bigcap_{j=1}^{i-1} cc0_j) \\
 &\subset b0_i \cup \bigcup_{j=1}^{i-1} bcc0_j \\
 &= b0_i \cup \bigcup_{j=1}^{i-1} bc0_j \\
 &\subset \bigcup_{j=1}^i b0_j,
 \end{aligned}$$

so that $m(bA_i) = 0$, $1 \leq i \leq N$,

iii) If $1 \leq i < j \leq N$, then

$$\begin{aligned}
 A_i \cap A_j &= c(0_i \cap c \bigcup_{m=1}^{i-1} 0_m) \cap c(0_j \cap c \bigcup_{m=1}^{j-1} 0_m) \\
 &= c(0_i \cap c \bigcup_{m=1}^{i-1} 0_m) \cap c(0_j \cap c \bigcup_{m=1}^{j-1} 0_m) \\
 &\subset ccc \bigcup_{m=1}^{j-1} 0_m \cap c0_i \\
 &= c \bigcap_{m=1}^{j-1} cc0_m \cap c0_i
 \end{aligned}$$

$$\subset c c c O_i \cap c O_i$$

$$= b c O_i$$

$$\subset b O_i,$$

consequently, $m(A_i \cap A_j) = 0$,

iv) $f(y_i) \in B$ for $1 \leq i \leq N$.

Now let Q_M be the set of all points of $\mathcal{S}$ which are members of more than one A_i . Q_M is a closed set of measure zero. The above construction can be carried out for each positive integer M . The result is: for each positive integer m there exists a strongly simple function g_m and a closed null set Q_m such that:

$$x \notin Q_m \text{ implies } |g_m(x) - f(x)| < 1/m.$$

Thus $\left| B \int f - \lim_n \frac{1}{n} \sum_{i=1}^n g_m(\lambda_i) \right| < 1/m$. Now let $\epsilon > 0$

be given. We claim there is an N such that $n > N$ im-

plies $\left| B \int f - \frac{1}{n} \sum_{i=1}^n f(\lambda_i) \right| < \epsilon$. The verification of

this last statement will establish the lemma.

Let M be such that $1/M < \epsilon/4$. Let R be the maximum value assumed on $\mathcal{S}$ by $|g_M(\cdot) - f(\cdot)|$. Let N be such that $n > N$ implies

- i) $\left| Bf - \frac{1}{n} \sum_{i=1}^n g_M(\lambda_i) \right| < 2/M$ and
- ii) $\frac{1}{n} \sum_{i=1}^n \chi_{Q_M}(\lambda_i) < \epsilon/(4R).$

Then for $n > N$

$$\begin{aligned} & \left| Bf - \frac{1}{n} \sum_{i=1}^n f(\lambda_i) \right| \leq \\ & \left| Bf - \frac{1}{n} \sum_{i=1}^n g_M(\lambda_i) \right| + \left| \frac{1}{n} \sum_{i=1}^n (\chi_{Q_M}(\lambda_i) + \right. \\ & \quad \left. \chi_{C_{Q_M}}(\lambda_i)) (g_M(\lambda_i) - f(\lambda_i)) \right| \leq \\ & \epsilon/2 + \left(\frac{1}{n} \sum_{i=1}^n \chi_{Q_M}(\lambda_i) \right) R + \frac{1}{n} \sum_{i=1}^n \frac{1}{M} < \epsilon. \end{aligned}$$

IV.3 The Continuous Case

IV.3.1. Theorem. Let $\mathcal{G}$ be a compact, second axiom group, let B be a Banach space and let f be a function from $\mathcal{G}$ to B such that for each $h \in \mathcal{G}$,

- i) ${}_h\Delta f$ is measurable and
- ii) $\Delta_h f$ is continuous.

Then $f = g + \Gamma$ where g is continuous on $\mathcal{G}$ and Γ is additive on $\mathcal{G}$.

Proof. Put $f_1 = f - f(I)$. The result of the theorem will be proved for f_1 , and the theorem will follow from that proof. The hypotheses i) and ii) both hold for

f_1 . For each $k \in \mathcal{H}$ we wish to define:

$$(IV.3.1) \quad g(k) = B \int (f_1(k) + f_1(h) - f_1(kh)) \, dm(h).$$

The prospective integrand above is measurable, being just $f_1(k) - {}_k\Delta f_1(h)$. We will resort to lemma II.2.1 to show that the integrand is uniformly bounded in h and k , thus justifying (IV.3.1) and showing that $|g(\cdot)|$ is bounded. In the notation of lemma II.2.1, let

$$\begin{aligned} \phi_\lambda(x) &= f_1(x) + f_1(\lambda) - f_1(\lambda x) \\ &= {}_\lambda\Delta f_1(x) + f_1(\lambda) \end{aligned}$$

for each $\lambda \in \mathcal{H}$ and $x \in \mathcal{H}$. $\phi_\lambda(\cdot)$ is measurable. Fix $h \in \mathcal{H}$. Then for each $\lambda \in \mathcal{H}$ and $x \in \mathcal{H}$,

$$\begin{aligned} \Delta_h \phi_\lambda(x) &= f_1(xh) + f_1(\lambda) - f_1(\lambda xh) \\ &\quad - f_1(x) - f_1(\lambda) + f_1(\lambda x) \\ &= -f_1(\lambda xh) + f_1(\lambda x) + f_1(xh) - f_1(x) \\ &= -\Delta_h f_1(\lambda x) + \Delta_h f_1(x) \end{aligned}$$

Now $|\Delta_h f_1|$ is continuous on the compact group $\mathcal{H}$ and is therefore bounded, say by M . For each $y \in \mathcal{H}$ and $x \in \mathcal{H}$, therefore we have $|\Delta_h \phi_\lambda(x)| \leq 2M$. Now by lemma IV.2.1, $|\phi_\lambda(x) - \phi_\lambda(I)|$ is bounded uniformly in λ and x . This means that $|f_1(k) + f_1(h) - f_1(kh)|$ is uniformly bounded for h and k in $\mathcal{H}$, say by N . Now for

each $k \in \mathcal{K}$ we wish to define:

$$(IV.3.2) \quad \Gamma(k) = B \int (f_1(kh) - f_1(h)) dm(h).$$

Again the prospective integrand is measurable, and we must verify that it is bounded as a function of h for each fixed k . But for fixed k ,

$$\begin{aligned} |f_1(kh) - f_1(h)| &\leq |f_1(k) + f_1(h) - f_1(kh)| \\ &\quad + |f_1(k)| \\ &\leq N + |f_1(k)|. \end{aligned}$$

The plan is now to use theorem IV.2.1 to show that g is continuous at I , and hence, because of the definition of g , continuous everywhere, and to show that Γ is additive. It is clear that $f_1 = g + \Gamma$. For $x \in \mathcal{K}$ and $y \in \mathcal{K}$ we have:

$$\begin{aligned} \Gamma(xy) &= B \int (f_1(xyh) - f_1(h)) dm(h) \\ &= B \int (f_1(xyh) - f_1(yh)) dm(h) \\ &\quad + B \int (f_1(yh) - f_1(h)) dm(h) \\ &= \Gamma(x) + \Gamma(y), \end{aligned}$$

so that Γ is additive. The next paragraphs verify the hypotheses of theorem IV.2.1.

We have already remarked that $|g(\cdot)|$ is bounded.

For $k \in \mathcal{K}$ and $x \in \mathcal{X}$ we have:

$$\begin{aligned} {}_k\Delta g(x) &= (f_1(kx) - \Gamma(kx)) - (f_1(x) - \Gamma(x)) \\ &= f_1(kx) - f_1(x) - \Gamma(k), \end{aligned}$$

consequently, for each $k \in \mathcal{K}$,

$$\begin{aligned} B \int {}_k\Delta g(x) dm(x) &= -\Gamma(k) + B \int (f_1(kx) \\ &\quad - f_1(x)) dm(x) \\ &= -\Gamma(k) + \Gamma(k) \\ &= 0. \end{aligned}$$

For any $h \in \mathcal{H}$,

$$\begin{aligned} \Delta_h g(x) &= (f_1(xh) - \Gamma(xh)) - (f_1(x) - \Gamma(x)) \\ &= \Delta_h f_1(x) - \Gamma(h), \end{aligned}$$

so that $\Delta_h g(\cdot)$ is a continuous function on $\mathcal{X}$.

It is easy to see that $g(I) = 0$.

Theorem IV.2.1 can now be used and we have that g is continuous at I . The fact that g is continuous at x_0 is equivalent to the continuity of the function q_{x_0} at I , where q_{x_0} is given by: $q_{x_0}(h) = {}_h\Delta g(x_0)$.

We already know that q_I is continuous at I , and since

$$q_{x_0}(h) = q_I(h) + \Delta_{x_0} f_1(h) - f_1(x_0),$$

we see that q_{x_0} is continuous at I . This shows that g is continuous and the result of the theorem is proved for f_1 . Now $f = (g + f(I)) + \Gamma$ and the theorem is proved.

IV.4 The Riemann Integrable Case

IV.4.1. Theorem. Let $\mathcal{X}$ be a compact second axiom group, let B be a Banach space and let f be a function from $\mathcal{X}$ to B such that for each $h \in \mathcal{X}$,

- i) ${}_h\Delta f(\cdot)$ is measurable, and
- ii) $\Delta_h f(\cdot)$ is m -Riemann integrable.

Then $f = g + \Gamma$ where g is m -Riemann integrable and Γ is additive.

Proof. Let g and Γ be tentatively defined by (IV.3.1) and (IV.3.2), respectively. We again must show that the integrand in (IV.3.1) is bounded in h and k . The proof goes just as before by appeal to theorem II.2.1. Now lemma IV.2.2 is used to show that (IV.3.1) defines an m -Riemann integrable function.

IV.4.2. Remark. We were unable to prove the familiar result that f is Riemann integrable if and only if f is bounded and continuous almost everywhere when B is permitted to be an arbitrary Banach space. The usual technique for proving this cannot be used since upper and lower sums cannot be defined and we could find no way to circumvent their use.

IV.5 The L_p Case

Theorem IV.5.5 and theorem IV.5.7 are the main results of this section. Theorem IV.5.1 is a specific instance of theorem IV.5.5. A group with a uniformly distributed sequence (theorem IV.5.7) is separable since each non-empty open set contains a closed set with null boundary and non-empty interior (consult the proof of lemma IV.2.3) and, having positive measure, contains infinitely many points of the sequence. Here we begin to encounter algebraic difficulties for the first time and the reader will perceive that the results are not so strong as those for compact abelian groups.

IV.5.1. Theorem. Let $\mathcal{G}$ be a compact metric group. Let $1 \leq p \leq \infty$. Let f be a complex-valued function on $\mathcal{G}$ such that for each $h \in \mathcal{G}$, ${}_h\Delta f$ and $\Delta_h f$ are L_p -functions. Then $f = g + \gamma + S$ where g is an L_p -function, γ is additive and S has either null right differences or null left differences.

Proof. At first we let $1 \leq p < \infty$. For $h \in \mathcal{G}$ and $k \in \mathcal{G}$ define $F(h, k) = [{}_k\Delta \Delta_h f]$. We will see that:

$$\begin{aligned} F(h, k) &= [S_k \Delta_h f] - [\Delta_h f] \\ &= [\Delta_h {}_k\Delta f] \\ &= [T_h {}_k\Delta f] - [{}_k\Delta f]. \end{aligned}$$

Only the first part of this calculation will be carried

out. Let $x \in \mathcal{X}$. Then for any h and k we have:

$$\begin{aligned} ({}_k\Delta\Delta_h f)(x) &= f(kxh) - f(xh) - f(kx) + f(x) \\ &= f(kxh) - f(kx) - (f(xh) - f(x)) \\ &= (S_k \Delta_h f)(x) - (\Delta_h f)(x). \end{aligned}$$

By (I.4.13) we see that F is continuous in each variable separately. By (I.3.3) and (I.4.12) we have that L_p is separable. The hypotheses of lemma II.2.5 are satisfied (recall that a separable metric space is second axiom) as regards F and so F is continuous at some point, say (a_0, b_0) . We will show that F is continuous. Let (a_1, b_1) be any point of $\mathcal{X} \times \mathcal{X}$. Then $(a_1, b_1) = (a_0 x, y b_0)$ for some $(x, y) \in \mathcal{X} \times \mathcal{X}$. $x = a_0^{-1} a_1$ and $y = b_1 b_0^{-1}$.

Consider the function, Q , of h and k given by

$$|(F(ha_0, b_0 k) - F(a_0, b_0)) - (F(ha_1, b_1 k) - F(a_1, b_1))|_p.$$

The function $q(h, k)(\cdot) = b_0 k \Delta \Delta_{ha_0} f(\cdot) - b_0 \Delta \Delta_{a_0} f(\cdot) -$

$b_1 k \Delta \Delta_{ha_1} f(\cdot) + b_1 \Delta \Delta_{a_1} f(\cdot)$ is a representative of the

function, the p -norm of which is $Q(h, k)$, and

$$\begin{aligned} &f(b_0 k g h a_0) - f(b_0 k g) - f(g h a_0) + f(g) \\ &- f(b_0 g a_0) + f(b_0 g) + f(g a_0) - f(g) \\ &- f(y b_0 k g h a_0 x) + f(y b_0 k g) + f(g h a_0 x) - f(g) \\ &+ f(y b_0 g a_0 x) - f(y b_0 g) - f(g a_0 x) + f(g) \end{aligned}$$

is the value of $q(h, k)$ at a point $g \in \mathcal{G}$. The above expression is seen to be:

$$\begin{aligned}
 & T_{ha_0} \Delta_x f(g) + S_{b_0 k} ({}_y \Delta f)(g) \\
 & - T_{ha_0 x} S_{b_0 k} ({}_y \Delta f)(g) - T_{ha_0} S_{b_0 k} \Delta_x f(g) \\
 & + T_{a_0 x} S_{b_0} ({}_y \Delta f)(g) + T_{a_0} S_{b_0} \Delta_x f(g) \\
 & - T_{a_0} \Delta_x f(g) - S_{b_0} ({}_y \Delta f)(g).
 \end{aligned}$$

Since $Q(h, k) = \left(\int |q(h, k)(g)|^p dm(g) \right)^{\frac{1}{p}}$, we have:

$$\begin{aligned}
 Q(h, k) \leq & \left| [T_{ha_0} \Delta_x f] - [T_{a_0} \Delta_x f] \right|_p + \\
 & \left| [S_{b_0 k} ({}_y \Delta f)] - [S_{b_0} ({}_y \Delta f)] \right|_p + \\
 & \left| [T_{ha_0 x} S_{b_0 k} ({}_y \Delta f)] - [T_{a_0 x} S_{b_0} ({}_y \Delta f)] \right|_p + \\
 & \left| [T_{ha_0} S_{b_0 k} \Delta_x f] - [T_{a_0} S_{b_0} \Delta_x f] \right|_p.
 \end{aligned}$$

The first term of the expression bounding $Q(h, k)$ is a continuous function of h and the second is a continuous function of k ; the third and fourth terms are jointly continuous in h and k (I.4.13). Furthermore $Q(I, I) = 0$. If $\epsilon > 0$ is given there is a neighborhood $U \times V$

of (I, I) such that $Q(h, k) < \epsilon/2$ and $\|F(ha_0, b_0k) - F(a_0, b_0)\|_p < \epsilon/2$ for $(h, k) \in U \times V$. In view of the definition of Q we have for $(h, k) \in U \times V$:

$$\|F(ha_1, b_1k) - F(a_1, b_1)\|_p < \epsilon.$$

This shows that F is continuous on $\mathcal{S} \times \mathcal{S}$, and Fubini's Theorem (I.2.5) may be applied to F . Define, for $h \in \mathcal{S}$:

$$(IV.5.1) \quad G(h) = L_p \int F(h, k) \, dm(k).$$

$G(h)$, therefore, is an element of $L_p(\mathcal{S}, m, L_p(\mathcal{S}, m, C))$ for almost all $h \in \mathcal{S}$, and is an integrable function of h . Now define:

$$(IV.5.2) \quad [g] = L_p \int G(h) \, dm(h),$$

$$(IV.5.3) \quad \gamma(k) = \int \Delta_k f(x) \, dm(x), \text{ for all } k \in \mathcal{S}, \text{ and}$$

$$(IV.5.4) \quad S = f - g - \gamma.$$

g is an L_p -function and if h and k are in $\mathcal{S}$, then

$$\begin{aligned} \gamma(hk) &= \int (f(xhk) - f(xh) + f(xh) - f(x)) dm(x) \\ &= \int (f(xhk) - f(xh)) dm(x) \\ &\quad + \int (f(xh) - f(x)) dm(x) \\ &= \gamma(k) + \gamma(h), \end{aligned}$$

so that γ is an additive function, so we must now show

that S has null right differences. For notational convenience, denote the element of L_p which has as a representative the constant function whose value is $\gamma(h)$ by $\Gamma(h)$. That is, $\Gamma(h) \in L_p$, and $\Gamma(h) = [\gamma(h)]$, for each $h \in \mathcal{H}$. Then Γ is also additive. We shall now compute the right differences of S .

$$\begin{aligned}
 \text{(IV.5.5)} \quad [\Delta_q S] &= [\Delta_q f] - [\Delta_q g] - [\Delta_q \gamma] \\
 &= [\Delta_q f] - \Gamma(q) + L_p \int G(h) \, dm(h) \\
 &\quad - T_q \left(L_p \int G(h) \, dm(h) \right).
 \end{aligned}$$

Now by the definition of G (IV.5.1) and lemma II.2.2,

$$\begin{aligned}
 G(h) &= L_p \int ([S_k \Delta_h f] - [\Delta_h f]) \, dm(k) \\
 &= [\gamma(h)] - [\Delta_h f] \\
 &= \Gamma(h) - [\Delta_h f],
 \end{aligned}$$

so that (IV.5.5) gives:

$$\begin{aligned}
 [\Delta_q S] &= -G(q) - L_p \int T_q G(h) \, dm(h) + L_p \int G(h) \, dm(h) \\
 &= L_p \int (-T_q G(h) + G(h) - G(q)) \, dm(h).
 \end{aligned}$$

Again using the definition of G we obtain:

$$\begin{aligned}
 [\Delta_q S] &= L_p \int L_p \int (-T_q F(h, k) + F(h, k) \\
 &\quad - F(q, k)) \, dm(k) \, dm(h)
 \end{aligned}$$

The value of a representative of the integrand above at the point x of $\mathcal{S}$ is $-f(kxqh) + f(kxq) + f(xqh) - f(xq) + f(kxh) - f(kx) - f(xh) + f(x) - f(kxq) + f(kx) + f(xq) - f(x) = -T_{qh}({}_k\Delta f)(x) + T_h({}_k\Delta f)(x)$, so that

$$\begin{aligned} [\Delta_q S] &= L_p \int L_p \int ([T_h({}_k\Delta f)] \\ &\quad - [T_{qh}({}_k\Delta f)]) dm(k) dm(h) \\ &= L_p \int L_p \int ([T_h({}_k\Delta f)] \\ &\quad - [T_{qh}({}_k\Delta f)]) dm(h) dm(k) \\ &= L_p \int ([\int {}_k\Delta f] - [\int {}_k\Delta f]) dm(k) \\ &= 0. \end{aligned}$$

This completes the proof for $1 \leq p < \infty$. Now for the case $p = \infty$, apply the theorem as proved so far for each p , $1 \leq p < \infty$. We obtain

$$f = g_p + \gamma_p + S_p$$

where g_p is an L_p -function, γ_p is additive and S_p has null right differences. We argue that the apparent dependence of g_p , γ_p and S_p upon p is illusory. γ_p is defined in every case by (IV.5.3) which is independent of p . g_p is defined by (IV.5.2). Since the 1-norm is not greater than the p -norm (I.4.11), a determining sequence for $G(h)$ in the space L_p is a determining

sequence in L_1 as well with the consequence that

$$L_1 \int G(h) dm(h) = L_p \int G(h) dm(h).$$

We see then that g_p is also independent of p and so, therefore, is S_p (IV.5.4).

Now corollary II.2.4 shows that there exists M such that $|F(h, k)|_p \leq M$ for all h and k in $\mathcal{X}$ and for all p , $1 \leq p < \infty$. Then $|G(h)|_p \leq M$ for all h in $\mathcal{X}$ and p , $1 \leq p < \infty$, and finally $|g|_p \leq M$ for all p . This is sufficient to show that $[g] \in L_\infty$ (confer the proof of theorem III.1.2).

IV.5.2. Remark. The reader has no doubt observed that while the hypotheses of the last theorem were algebraically symmetric, the result was not. We know of no example showing that the left differences of S are not null, however the following computation of $[{}_q \Delta S]$ shows that that may indeed be the case.

$$\begin{aligned} [{}_q \Delta S] &= [{}_q \Delta f] - [{}_q \Delta g] - [{}_q \Delta \gamma] \\ &= -L_p \int (S_q(\Gamma(h) - [\Delta_h f]) \\ &\quad - \Gamma(h) + [\Delta_h f]) dm(h) \\ &\quad + [{}_q \Delta f] - [\Delta_q f] - G(q) \\ &= \int (S_q[\Delta_h f] - [\Delta_h f]) dm(h) \\ &\quad + [{}_q \Delta f] - [\Delta_q f] - G(q) \end{aligned}$$

$$\begin{aligned}
&= G(q) + [{}_q\Delta f] - [\Delta_q f] - G(q) \\
&= [{}_q\Delta f] - [\Delta_q f] \\
&= [S_q f - T_q f].
\end{aligned}$$

IV.5.3. Remark. If we choose to define $G(k) = L_p \int F(h, k) dm(h)$ then it is the left differences of S which are null while $[\Delta_q S] = [T_q f - S_q f]$.

IV.5.4. Remark. We have, in the proof of the last theorem, shown, in effect, that $(\mathcal{S}, L_p(\mathcal{S}, m, C))$ has property C. This fact is the rationale for the comment made at the beginning of section IV.5 concerning the last theorem and the following theorem.

IV.5.5. Theorem. Suppose $\mathcal{S}$ is a compact topological group, let $1 \leq p < \infty$ and let B be a Banach space such that $(\mathcal{S}, L_p(\mathcal{S}, m, B))$ has property C. If f is a function from $\mathcal{S}$ to B having L_p left and right differences, then $f = g + \gamma + S$ where g is an L_p -function, γ is additive and S has either null left differences or null right differences. If, in addition, $(\mathcal{S}, L_p)$ has property C for all p , $1 \leq p < \infty$, and f has L_∞ differences then the above decomposition holds with g an L_∞ -function.

Proof. The proof is that of theorem IV.5.1 save only the demonstration that F is continuous at some point,

which in this theorem is a direct consequence of the hypothesis that $(\mathcal{G}, L_p)$ have property C.

IV.5.6. Remark. We have now shown that we may decompose a function, f , from $\mathcal{G}$ to B , having L_p differences if:

- i) $\mathcal{G}$ is compact and abelian, or
- ii) $\mathcal{G}$ is compact and $(\mathcal{G}, L_p)$ has property C.

We now obtain a decomposition if $\mathcal{G}$ has a uniformly distributed sequences and f is a class function (definition I.1).

IV.5.7. Theorem. Let $\mathcal{G}$ be a compact topological group having a uniformly distributed sequence $\{\lambda_i\}$, let $1 \leq p \leq \infty$ and let f be a class function from $\mathcal{G}$ to the Banach space, B , having L_p differences. Then $f = g + \gamma + S$ where $[g] \in L_p$, γ is additive and $[\Delta_h S] = 0$ for $h \in \mathcal{G}$.

Proof. We treat the case $1 \leq p < \infty$ first. Let $F(h) = [\Delta_h f]$ for $h \in \mathcal{G}$. Since $(\Delta_{hk} f - \Delta_h f)(z) = f(zhk) - f(zh) = (T_h \Delta_k f)(z)$ it follows that for all h and k in $\mathcal{G}$ we have:

$$F(hk) - F(h) = T_h F(k).$$

Similarly, $(\Delta_{kh} f - \Delta_h f)(z) = f(zkh) - f(zh) = f(hzk) - f(hz)$ so that for all h and k in $\mathcal{G}$ we have:

$$F(kh) - F(h) = S_h F(k).$$

This means that the left and right differences of F are

continuous. Now for $h \in \mathcal{S}$, put

$$\Gamma(h) = L_p \int (F(kh) - F(k)) \, dm(k), \text{ and}$$

$$G(h) = F(h) - \Gamma(h).$$

$\Gamma(\cdot)$ is clearly additive. We now want to demonstrate that G is continuous. According to lemma IV.2.3

$$\begin{aligned} G(h) &= L_p \int (F(h) + F(k) - F(kh)) \, dm(k) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (F(h) + F(\lambda_i) - F(\lambda_i h)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (F(\lambda_i) - S_h F(\lambda_i)) \end{aligned}$$

For each n the sum above is a continuous function of h and so by lemma I.3.4 G is continuous somewhere, say at a_0 . Let $\epsilon > 0$ be given. Let U be a neighborhood of I such that $h \in U$ implies $|G(ha_0) - G(a_0)|_p < \epsilon/2$. Let k_0 be fixed in $\mathcal{S}$. Let V be a neighborhood of I such that $h \in V$ implies $|S_{ha_0} F(k_0) - S_{a_0} F(k_0)|_p < \epsilon/2$.

Then for $h \in U \cap V$ we have:

$$\begin{aligned} |(G(ha_0 k_0) - G(a_0 k_0)) - (G(ha_0) - G(a_0))|_p &= \\ |(F(ha_0 k_0) - F(a_0 k_0)) - (F(ha_0) - F(a_0))|_p &= \\ |(F(ha_0 k_0) - F(ha_0)) - (F(a_0 k_0) - F(a_0))|_p &= \\ |S_{ha_0} F(k_0) - S_{a_0} F(k_0)|_p &< \epsilon/2. \end{aligned}$$

Therefore $\|G(ha_0k_0) - G(a_0k_0)\|_p < \varepsilon$ and since ε and k_0 are arbitrary, G is continuous. We have observed in a previous proof that for h in $\mathcal{H}$,

$$\Gamma(h) = [B \int \Delta_h f(x) \, dm(x)].$$

We now make the definitions:

$$[g] = L_p \int G(h) \, dm(h),$$

$$\gamma(h) = B \int \Delta_h f(x) \, dm(x) \quad \text{for } h \in \mathcal{H}, \text{ and}$$

$$f = g + \gamma + S.$$

g is an L_p -function and γ is additive. It remains only to show that the differences of S are null.

$$\begin{aligned} [\Delta_h S] &= F(h) + T_h \left(L_p \int G(k) \, dm(k) \right) \\ &\quad - L_p \int G(k) \, dm(k) - [\gamma(h)] \\ &= F(h) + L_p \int T_h G(k) \, dm(k) \\ &\quad - L_p \int G(k) \, dm(k) - \Gamma(h) \\ &= F(h) + L_p \int (T_h F(k) - T_h \Gamma(k) - F(k) \\ &\quad \quad \quad + \Gamma(k)) \, dm(k) - \Gamma(h) \\ &= L_p \int (F(hk) - F(h) - F(k)) \, dm(k) \\ &\quad \quad \quad + F(h) - \Gamma(h) \end{aligned}$$

$$\begin{aligned}
&= L_p \int S_k F(h) \, dm(k) - \Gamma(h) \\
&= L_p \int S_k [\Delta_h f] \, dm(k) - \Gamma(h) \\
&= 0.
\end{aligned}$$

A similar calculation shows that $[\Delta_h S] = 0$. This completes the proof when $p < \infty$. For the case $p = \infty$, the same technique as that used at the end of the proof of theorem IV.5.1 (p. 53) is used.

IV.5.8. Remark. Among the groups known to have uniformly distributed sequences are compact second axiom groups. We refer the reader to [11] for a discussion of which groups possess uniformly distributed sequences.

CHAPTER V

ALMOST UNIFORMLY DISTRIBUTED SEQUENCES AND THE GENERALIZED CANTOR GROUP

V.1 Introduction

The results of this chapter are those obtained by the author at the outset of his research. They are presented mainly for historic purposes and are for the most part contained in the results proved thus far. This chapter does give an indication of another method for solving the difference problem for L_p , namely the use of Fourier coefficients. An investigation of the generalized Cantor group and its dual group is carried out and some difference properties are proved for their L_p spaces. Almost uniformly distributed sequences may be regarded as an artifice to facilitate the description of the generalized Cantor group and we claim no further significance for them. Notation concerning the generalized Cantor group will be given in section V.3. For now $\mathcal{G}$ will be a locally compact abelian group, $\hat{\mathcal{G}}$ will be its dual group and I denotes the identity of $\hat{\mathcal{G}}$. Functions will be complex-valued. Lemma V.2.2 gives an indication of the method of proof used in this chapter. We verify the hypotheses of that lemma and then use a completeness

property of the characters to show that $\Delta_n(f - F)$ is null. Lemma V.2.3 shows what kind of groups we can tackle with this method. Carroll in [7] used a similar method for monothetic groups, where translation is metrically transitive. We shall deal with a group having an almost uniformly distributed sequence each element of which is of finite order.

V.1.1. Definition. A sequence $\{h_i\}$ in $\mathcal{G}$ will be called almost uniformly distributed if there is a subsequence $\{m_j\}$ of the positive integers such that

$$\lim_{j \rightarrow \infty} m_j^{-1} \sum_{i=1}^{m_j} \chi(h_i) = 0$$

for all χ in $\hat{\mathcal{G}}$, $\chi \neq I$.

Lemma V.2.1 shows that if $\mathcal{G}$ is compact and $m_j = j$ then the sequence is uniformly distributed (definition IV.1.2). If a group has been stipulated to have an almost uniformly distributed sequence then subsequent reference to m_j is a reference to the subsequence showing the sequence to be almost uniformly distributed.

V.2 Lemmas

V.2.1. Lemma. Let $\mathcal{G}$ be a compact topological group having the almost uniformly distributed sequence $\{h_i\}$. Then

$$(V.2.1) \quad B \int f = \lim_{j \rightarrow \infty} m_j^{-1} \sum_{i=1}^{m_j} f(h_i)$$

for any continuous function, f , from $\mathcal{G}$ to the Banach

space, B , and for any closed set, A , with null boundary:

$$(V.2.2) \quad m(A) = \lim_{j \rightarrow \infty} m_j^{-1} \sum_{i=1}^{m_j} \chi_A(h_i).$$

Proof. (V.2.1) holds for any character and therefore for any trigonometric polynomial. Since the trigonometric polynomials are dense in the space of complex-valued continuous functions on $\mathcal{S}$ [18, p. 24], (V.2.1) holds for any continuous complex-valued function, f . Now let A be a closed set with null boundary and let $\epsilon > 0$. We may cover bA with an open set, O , of measure less than $\epsilon/2$. Furthermore, we may construct continuous functions, f and g , from $\mathcal{S}$ to $[0, 1]$ such that:

- i) $f(x) = 1$ and $g(x) = 1$ for $x \in A \cap CO$,
- ii) $1 - (\epsilon/2) \leq f(x) \leq 1$ and $(\epsilon/2) \leq g(x) \leq 1$ for $x \notin A$,
- iii) $f(x) = 1 - (\epsilon/2)$ and $g(x) = \epsilon/2$ for $x \in bA$,
- iv) $0 \leq f(x) \leq 1 - (\epsilon/2)$ and $0 \leq g(x) \leq \epsilon/2$ for $x \in CA$ and
- v) $f(x) = 0$ and $g(x) = 0$ for $x \in CA \cap CO$.

Now f and g coincide with χ_A except on O and $m(O) < \epsilon/2$, consequently

$$\left| \int (\chi_A(x) - h(x)) dm(x) \right| < \epsilon/2$$

where we may put f or g for h . Consequently there exists J such that $j > J$ implies:

$$m(A) - \epsilon/2 < m_j^{-1} \sum_{i=1}^{m_j} h(h_i) < m(A) + \epsilon/2$$

where again h may be f or g . Now since f and g vanish on $CO \cap CA$ and are 1 on $A \cap CO$ we have:

$$\begin{aligned} (V.2.3) \quad m(A) - \epsilon/2 &< m_j^{-1} \sum_{i=1}^{m_j} \chi_{A \cap CO}(h_i) + \\ &m_j^{-1} \sum_{i=1}^{m_j} \chi_{O \cap A}(h_i) h(h_i) + \\ &m_j^{-1} \sum_{i=1}^{m_j} \chi_{O \cap CA}(h_i) h(h_i) \\ &< m(A) + \epsilon/2 \end{aligned}$$

where h may be f or g . Now using the fact that $f(x) \geq 1 - \epsilon/2$ for $x \in A$ and $f(x) \geq 0$ for $x \in CA$ in (V.2.3) we get:

$$\begin{aligned} m(A) + \epsilon/2 &> m_j^{-1} \sum_{i=1}^{m_j} \chi_{A \cap CO}(h_i) \\ &+ m_j^{-1} \sum_{i=1}^{m_j} \chi_{O \cap A}(h_i) (1 - \epsilon/2) \\ &= m_j^{-1} \sum_{i=1}^{m_j} \chi_{A \cap CO}(h_i) \\ &- (\epsilon/2) m_j^{-1} \sum_{i=1}^{m_j} \chi_{O \cap A}(h_i) \\ &\geq m_j^{-1} \sum_{i=1}^{m_j} \chi_{A \cap CO}(h_i) - \epsilon/2. \end{aligned}$$

Consequently $m_j^{-1} \sum_{i=1}^{m_j} \chi_{A \cap CO}(h_i) < m(A) + \epsilon$ for $j > J$.

Likewise we obtain $m_j^{-1} \sum_{i=1}^{m_j} \chi_A(h_i) > m(A) - \epsilon$ for $j > J$ as follows: $g(x) \leq 1$ for $x \in A$ and $g(x) \leq \epsilon/2$ for $x \in CA$ and using these inequalities in (V.2.3) we get:

$$\begin{aligned}
 m(A) - \epsilon/2 &< m_j^{-1} \sum_{i=1}^{m_j} \chi_A \cap CO(h_i) \\
 &\quad + m_j^{-1} \sum_{i=1}^{m_j} \chi_O \cap A(h_i) \\
 &\quad + m_j^{-1} \sum_{i=1}^{m_j} \chi_O \cap CA(h_i) (\epsilon/2) \\
 &= m_j^{-1} \sum_{i=1}^{m_j} \chi_A(h_i) \\
 &\quad + (\epsilon/2) m_j^{-1} \sum_{i=1}^{m_j} \chi_O \cap CA(h_i) \\
 &\leq m_j^{-1} \sum_{i=1}^{m_j} \chi_A(h_i) + \epsilon/2.
 \end{aligned}$$

This establishes (V.2.2). Now if the reader will examine the proof of lemma IV.2.3 he will see that the proof works mutatis mutandis when the subsequence $\{m_j\}$ is used in place of the sequence $\{n\}$ and that proves the first conclusion of the lemma.

V.2.2. Lemma. Let $\mathcal{G}$ be a compact group with an almost uniformly distributed sequence $\{h_i\}$. Let $1 < p < \infty$ and let f be a function on $\mathcal{G}$ such that the differences of f are L_p -functions. Put $\Gamma(x) =$

$\int \Delta_x f(t) dm(t)$ and suppose the functions F_j given by:

$$F_j(x) = -m_j^{-1} \sum_{i=1}^{m_j} (\Delta_{h_i} f(x) - \Gamma(h_i))$$

have a subsequence which is uniformly bounded in p-norm.

Then there exists F an L_p -function such that for any h in $\mathcal{G}$ and any character, γ , unequal to I , the γ -Fourier coefficients of $\Delta_h F$ and $\Delta_h f$ are identical.

Proof. The formula $m_j^{-1} \sum_{i=1}^{m_j} \overline{\gamma(x - h_i)} = \overline{\gamma(x)}$

$m_j^{-1} \sum_{i=1}^{m_j} \gamma(h_i)$ shows that for $\gamma \neq I$, the sequence given by

the left hand side of the above equation converges uniformly to the function identically 0. The last hypothesis implies that there is a subsequence of $\{m_j\}$ which can be taken to be $\{m_j\}$ such that the sequence $\{F_j\}$ has a weak* limit F which is an L_p -function (I.2.6, I.2.8). Now we compute the Fourier coefficients of $\Delta_h F$. Let $\gamma \neq I$ and $h \in \mathcal{G}$.

$$\int \Delta_h F(x) \overline{\gamma(x)} dm(x) =$$

$$\lim_{j \rightarrow \infty} \int \Delta_h (-m_j^{-1} \sum_{i=1}^{m_j} (\Delta_{h_i} f(x) - \Gamma(h_i))) \overline{\gamma(x)} dm(x) =$$

$$\lim_{j \rightarrow \infty} -m_j^{-1} \sum_{i=1}^{m_j} \int (\Delta_h f(x+h_i) - \Delta_h f(x)) \overline{\gamma(x)} dm(x) =$$

$$\int \Delta_h f(x) \overline{\gamma(x)} dm(x) - \lim_{j \rightarrow \infty} m_j^{-1} \int \Delta_h f(x)$$

$$\overline{\sum_{i=1}^{m_j} \gamma(x - h_i)} \, dm(x) =$$

$$\int \Delta_h f(x) \overline{\gamma(x)} \, dm(x).$$

V.2.3. Lemma. Let $\mathcal{G}$ be a compact group with an almost uniformly distributed sequence $\{h_i\}$. Let $1 < p < \infty$ and let f be a function on $\mathcal{G}$ such that the differences of f are L_p -functions. Suppose that for each i , either translation by h_i is metrically transitive [8, p. 667] or h_i is of finite order. Then there exists an L_p -function, F , such that for any $h \in \mathcal{G}$ and any character, $\gamma \neq I$, the γ -Fourier coefficients of $\Delta_h F$ and $\Delta_h f$ are identical.

Proof. Let F_j be as in lemma V.2.2. To use lemma V.2.2 we must show that the F_j are bounded uniformly for $j = 1, 2, \dots$. By lemma II.2.3 there exists M such that $|\Delta_h \Delta_k f|_p \leq M$ for all h, k in $\mathcal{G}$. We argue that $\Delta_{h_i} f(\cdot) - \Gamma(h_i)$ is bounded in p -norm by M .

If h_i is of finite order, then $\Gamma(h_i) = 0$ and the fact that if $nh = 0$ then $\sum_{j=1}^n \Delta_{jh} \Delta_h f(x) = -n \Delta_h f(x)$

shows that $|\Delta_{h_i} f(\cdot) - \Gamma(h_i)|_p \leq M$.

If translation by h_i , T_{h_i} , is metrically transitive we have by the Mean Ergodic Theorem (I.2.7):

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{j=0}^{n-1} \Delta_{h_i} T_{h_i}^j [f] = [\Gamma(h_i)].$$

and therefore when T_{h_i} is metrically transitive we have:

$$\|\Delta_{h_i} f(\cdot) - \Gamma(h_i)\|_p =$$

$$\lim_{n \rightarrow \infty} n^{-1} \left\| n \Delta_{h_i} f(\cdot) - \sum_{j=0}^{n-1} \Delta_{h_i} (T_{h_i}^j f)(\cdot) \right\|_p =$$

$$\lim_{n \rightarrow \infty} n^{-1} \left\| \sum_{j=0}^{n-1} \Delta_{(n-j)h_i} \Delta_{h_i} f(\cdot) \right\|_p \leq M.$$

Thus we have $\|F_j\|_p \leq M$ for all j and lemma V.2.3 may be applied directly.

V.2.4. Lemma. We may allow $p = 1$ in lemma V.2.2 and lemma V.2.3.

Proof. We shall adopt the following convention: if u is a measure and C is u -integrable, then $u(C) =$

$\int C(x) du(x)$. As in lemma V.2.2 the functions F_j have

a weak* limit which in this case is a finite regular measure (I.2.8). We will show that the weak* limit of the F_j , denoted hereafter by u , is absolutely continuous with respect to m . For any continuous function, C , on $\mathcal{X}$:

$$u(C) = \lim_{j \rightarrow \infty} \int F_j(x) C(x) dm(x).$$

We shall establish the following for $h \in \mathcal{X}$ and $\delta \neq I$:

$$u(T_{-h} \gamma - \gamma) = \int \Delta_h f(x) \gamma(x) dm(x).$$

Let h be in $\mathcal{H}$ and let $\gamma \neq I$ be in $\hat{\mathcal{H}}$. Then

$$u(T_{-h} \gamma - \gamma) =$$

$$\lim_{j \rightarrow \infty} -m_j^{-1} \sum_{i=1}^{m_j} \left(\int \Delta_{h_i} f(x) \gamma(x-h) dm(x) \right. \\ \left. - \int \Delta_{h_i} f(x) \gamma(x) dm(x) \right) =$$

$$\lim_{j \rightarrow \infty} -m_j^{-1} \sum_{i=1}^{m_j} \int (\Delta_{h_i} f(x+h) - \Delta_{h_i} f(x)) \gamma(x) dm(x) =$$

$$\lim_{j \rightarrow \infty} -m_j^{-1} \int \sum_{i=1}^{m_j} \Delta_h \Delta_{h_i} f(x) \gamma(x) dm(x) =$$

$$\lim_{j \rightarrow \infty} -m_j^{-1} \sum_{i=1}^{m_j} \int (\Delta_h f(x+h_i) - \Delta_h f(x)) \gamma(x) dm(x) =$$

$$\lim_{j \rightarrow \infty} - \int \Delta_h f(x) (m_j^{-1} \sum_{i=1}^{m_j} \gamma(x - h_i)) dm(x)$$

$$+ \int \Delta_h f(x) \gamma(x) dm(x) =$$

$$\int \Delta_h f(x) \gamma(x) dm(x).$$

We will now prove for a continuous function, C , such that $m(C) = 0$ what we have just shown for $\gamma \neq I$. Let C be a continuous function such that $m(C) = 0$, let $h \in \mathcal{H}$ and

let $\epsilon > 0$. There is a finite linear combination $\sum_{i=1}^r a_i \gamma_i$

not involving I and differing from C in absolute value by no more than $\epsilon' = \epsilon / (3(|u|(\mathcal{X}) + |\Delta_h f|_1))$ on $\mathcal{X}$.

Therefore we have:

$$\begin{aligned}
 & |u(T_{-h}C - C) - \int \Delta_h f(x) C(x) dm(x)| \leq \\
 & |u(T_{-h}C - C) - u(\sum_{i=1}^r a_i (T_{-h} \gamma_i - \gamma_i))| + \\
 & |u(\sum_{i=1}^r a_i (T_{-h} \gamma_i - \gamma_i)) - \int \Delta_h f(x) C(x) dm(x)| \leq \\
 & |u(T_{-h}C - C) - u(\sum_{i=1}^r a_i (T_{-h} \gamma_i - \gamma_i))| + \\
 & \quad | \int \Delta_h f(x) \sum_{i=1}^r a_i \gamma_i(x) dm(x) \\
 & \quad - \int \Delta_h f(x) C(x) dm(x) | \leq \\
 & |u(T_{-h}(C - \sum_{i=1}^r a_i \gamma_i))| + \\
 & \quad |u(C - \sum_{i=1}^r a_i \gamma_i)| + \epsilon' |\Delta_h f|_1 \leq \\
 & 2 \sup_{x \in \mathcal{X}} (|C(x) - \sum_{i=1}^r a_i \gamma_i(x)|) \lim_{j \rightarrow \infty} \\
 & \quad \int |F_j(x)| dm(x) + \epsilon' |\Delta_h f|_1 \leq \\
 & \epsilon' (2|u|(\mathcal{X}) + |\Delta_h f|_1) < \epsilon.
 \end{aligned}$$

Since ϵ and h are arbitrary we have:

$$u(T_{-h}C - C) = \int \Delta_h f(x) C(x) dm(x).$$

We see in general that for the continuous function, C :

$$u(T_{-h}C - C) = \int \Delta_h f(x) C(x) dm(x) - m(C)m(\Delta_h f).$$

Now let $u = u_c + u_s$ be the Lebesgue decomposition of u with respect to m . We have for all continuous C :

$$u_c(C) = \int F(x)C(x)dm(x),$$

where F is a fixed L_1 -function. Now let $m(S) = 0$, let $h \in \mathcal{A}$ and let $\epsilon > 0$. Since u_s is regular there exist K_1 and K_2 , compact subsets of S and O_1 and O_2 , open supersets of S such that:

- i) $|u_s|(O_1 - K_1) < \epsilon/4$,
- ii) $|u_s|(O_2 + h - K_2 + h) < \epsilon/4$,
- iii) $m(O_1) < \alpha$ and
- iv) $m(O_2) < \alpha$

where α is the minimum of $\epsilon/(4|m(\Delta_h f)|)$ and the modulus of continuity for $\epsilon/4$ for the set function λ :

$$\lambda(E) = \int_E |\Delta_h(f - F)(x)| dm(x). \text{ Let } O = O_1 \cap O_2 \text{ and let}$$

$K = K_1 \cup K_2$. There is a continuous function, g , from $\mathcal{A}$ into $[0, 1]$ such that $g(x) = 0$ for $x \notin O$ and $g(x) = 1$ for $x \in K$. The following are then clear:

- i) $K \subset S \subset O$ and $K + h \subset S + h \subset O + h$,
- ii) $m(O) < \alpha$,

$$\text{iii)} \quad |u_s|(0 \sim K) < \epsilon/4,$$

$$\text{iv)} \quad |u_s|(0 + h \sim K + h) < \epsilon/4,$$

$$\text{v)} \quad \chi_S - g \text{ is } 0 \text{ except on } 0 \sim K \text{ and}$$

$$\text{vi)} \quad T_{-h}(\chi_S - g) \text{ is } 0 \text{ except on } 0 + h \sim K + h.$$

$$\begin{aligned} \text{Now } u_s(S + h) - u_s(S) &= u_s(\chi_{S+h} - \chi_S) = u_s(T_{-h}\chi_S - \chi_S) \\ &= u_s(T_{-h}g - g) + u_s(T_{-h}(\chi_S - g)) - u_s(\chi_S - g) \text{ so that:} \end{aligned}$$

$$|u_s(S + h) - u_s(S)| =$$

$$\left| \int \Delta_h(f - F)(x)g(x)dm(x) - m(g)m(\Delta_h f) \right.$$

$$\left. + \int_{0+h \sim K+h} (\chi_S(x-h) - g(x-h))du_s(x) \right.$$

$$\left. - \int_{0 \sim K} (\chi_S(x) - g(x))du_s(x) \right| \leq$$

$$\int |\Delta_h(f - F)(x)\chi_0(x)|dm(x) + m(g)|m(\Delta_h f)|$$

$$+ \int_{0+h \sim K+h} |\chi_S(x-h) - g(x-h)|d|u_s|(x)$$

$$+ \int_{0 \sim K} |\chi_S(x) - g(x)|d|u_s|(x) < \epsilon.$$

Since ϵ and h are arbitrary, we have $u_s(S + h) = u_s(S)$ for all S such that $m(S) = 0$. Now let $u_s = u_1 - u_2 + iu_3 - iu_4$ be the Jordan decomposition of u_s and let v be any of the u_i , say u_1 . v is concentrated on a set, A , of Haar measure zero and is non-negative. We note that u_1 and u_2 are concentrated on dis-

joint sets. Now let B be any measurable set and let h be arbitrary in $\mathcal{S}$. Then $v(B) = v(B \cap A) = (u_1 - u_2)(B \cap A) = (u_1 - u_2)((B \cap A) + h) \leq v((B \cap A) + h) \leq v(B + h)$. We also have, therefore, $v(B + h) \leq v((B + h) + (-h)) = v(B)$. Consequently $v(B + h) = v(B)$ for any measurable set, B , and any h in $\mathcal{S}$. v is therefore a multiple of m , and, being singular with respect to m , must be identically 0. Similar calculations show that v is identically 0 for any choice of v and thus u_s is identically 0. We have now shown that for any continuous function, C : $u(C) = \int F(x)C(x)dm(x)$. Now the computation of the Fourier coefficients used in the proof of lemma V.2.2 finishes the proof.

V.2.5. Lemma. We may allow $p = \infty$ in lemma V.2.3 and lemma V.2.4.

Proof. The hypotheses imply that there exists M such that $\|F_j\|_p \leq M$ for all p , $1 \leq p \leq \infty$ and all j . By using lemma V.2.2 or lemma V.2.3 we obtain the L_1 -function, F , which is the weak* limit of a subsequence of $\{F_j\}$, say $\{F_{j'}\}$. If $1 < q < \infty$, then $\|F_{j'}\|_q \leq M$ for each j' and there is a subsequence $\{F_{j''}\}$ of $\{F_{j'}\}$ converging weak*-ly to an L_q -function, F_q . But $\{F_{j''}\}$ also converges weak*-ly to F and so the Fourier coefficients are identical for F and F_q . By (I.4.15), $[F_q] = [F]$ and F is in fact an L_q -function. This is

true for every q , $1 \leq q < \infty$ and furthermore, $|F|_q \leq M$ for every q , $1 \leq q < \infty$. As we have observed before, this shows that F is an L_∞ -function.

V.3 The Generalized Cantor Group

We now look into some of the detail of the groups which are the main concern of this chapter, the generalized Cantor group and its dual group. The notation established here will be fixed throughout the remainder of the chapter. The generalized Cantor group, $\mathcal{G}$, is the complete direct sum of countably many cyclic groups, $\mathcal{G}_i$, of finite order, r_i . $\mathcal{L} = \widehat{\mathcal{G}}$ and $\mathcal{L}_i = \widehat{\mathcal{G}_i}$. Elementary calculation shows that $\mathcal{L}_i = \mathcal{G}_i$. h_i will be a generating element of $\mathcal{G}_i$. $\oplus$ is the symbol for direct sum of two groups, $\sum$ is the symbol for an infinite direct sum and $\bigoplus$ is the symbol for complete direct sum. $\cong$ denotes algebraic isomorphism and topological homeomorphism when dealing with topological groups. I will be the identity of $\mathcal{G}$ and I^i will be the identity of $\mathcal{G}_i$. Accordingly, $\mathcal{G} = \bigoplus_{i=1}^{\infty} \mathcal{G}_i$ and $\mathcal{L} = \sum_{i=1}^{\infty} \mathcal{L}_i$. An element, $g \in \mathcal{G}$ has an expansion $g: (g^1, g^2, \dots, g^n, \dots)$, $g^i \in \mathcal{G}_i$ and g will be identified with its expansion.

We will introduce an indexing of the elements of $\mathcal{G}$ which correspond to elements of $\mathcal{L}$. Put $O_1 = \mathcal{G}_1$ and $\Theta_1 = \mathcal{L}_1$. For $n > 1$ put $O_n = O_{n-1} \oplus \mathcal{G}_n$ and $\Theta_n = \Theta_{n-1} \oplus \mathcal{L}_n$. O_n is a closed subgroup of $\mathcal{G}$ and the dual

group, θ_n , of O_n is a closed subgroup of $\mathcal{L}$. We see

that $O_n = \{g : g^i = I^i \text{ for } i > n\}$ and that $\mathcal{L} = \bigcup_{n=1}^{\infty} \theta_n$.

For $1 \leq j \leq r_1$, put $g_j = (jh_1, I^2, I^3, \dots)$. This indexes the elements of O_1 ($= \theta_1$). Suppose we have indexed the elements of O_k ($= \theta_k$) to agree with the indexing of the elements of O_{k-1} ($= \theta_{k-1}$) considered as a subset of O_k . An element, g , of O_{k+1} can be written uniquely in the form:

$$g = g' + (I^1, I^2, \dots, (r_{k+1} - r)h_{k+1}, I^{k+2}, \dots)$$

where $g' \in O_k$ and $0 \leq r < r_{k+1}$. The element g' is already indexed as an element of O_k , say $g' = g_j$. We

give g the index $j + r \left(\sum_{i=1}^k r_i \right)$. Thus the new index

agrees with the index already assigned to g if $g \in O_k$, for in that case $r = 0$. Also every element of O_{k+1} is

indexed. By induction on k each element of $\bigcup_{k=1}^{\infty} O_k$ is

indexed. The elements of $\mathcal{L}$ which have been indexed have expansions the entries of which are the identities after a certain point in the expansion. An element of $\mathcal{L}$ has precisely that form of expansion and will be given the same index as the element of $\mathcal{L}$ with the identical expansion. The element in $\mathcal{L}$ with index i is denoted g_i and the element in $\mathcal{L}$ with index i is denoted l_i .

Let $\mathcal{A}$ be the set of non-empty finite subsets of the positive integers. For $F \in \mathcal{A}$ let $U_F = \{g : g^i = I^i \text{ for } i \in F\}$. From the definition of the product topology we see that $\{U_F : F \in \mathcal{A}\}$ is a base for $\mathcal{N}(I)$. For $F \in \mathcal{A}$ let $P_F = \{g : g^i = I^i \text{ for } i \notin F\}$. Evidently

$P_F \cong \sum_{i \notin F} \mathcal{L}_i$, and P_F is thus its own dual group. Let

$u_F(x) = \sum_{l \in P_F} l(x)$ for each $F \in \mathcal{A}$. We shall say more

about u_F after we see how to compute $\langle g, l \rangle$. Let $g \in \mathcal{G}$ and $l \in \mathcal{L}$. Recall that h_i is a generating element of $\mathcal{L}_i$ and $\mathcal{L}_i$. Let $g^i = p_i h_i$ and $l^i = q_i h_i$ where $0 \leq p_i < r_i$, $0 \leq q_i < r_i$ and $i = 1, 2, \dots$. Then

$$\langle g, l \rangle = \prod_{i=1}^{\infty} \rho_i^{p_i q_i},$$

where ρ_i is a primitive r_i^{th} root of unity. The product is meaningful since $q_i = 0$ for all but finitely many i . We see that $\langle g, l \rangle$ may be computed from those entries which are different from the identities in g and l . Therefore if $l \in P_F$, $\langle g, l \rangle$ can be computed from the entries $\{g^i : i \in F\}$. If $g \in \mathcal{G}$ and $F \in \mathcal{A}$ let g_F be the element of P_F the i^{th} entry of which is g^i for each i in F . For $l \in P_F$, then we have $\langle g, l \rangle = \langle g_F, l \rangle$. It is easy to see now that u_F vanishes outside of U_F and is non-negative and that $m(u_F) = 1$.

V.3.1. Theorem. $\mathcal{G}$ has an almost uniformly distributed sequence each element of which is of finite order.

Proof. The sequence is, of course, $\{g_i\}$ and the required subsequence is given by: $m_j = \prod_{i=1}^j r_i$. We note that $\{g_i : i = 1, 2, \dots, m_j\} = O_j$ and that the dual group of O_j is θ_j . Let $l \in \mathcal{L}$, $l \neq I$. There exists N such that $l \notin \theta_n$ for $n > N$. For $j > N$, we have:

$$\begin{aligned} \sum_{i=1}^{m_j} l(g_i) &= \int_{O_j} \langle g, l \rangle dm(g) \\ &= 0, \end{aligned}$$

by the orthogonality relations (I.4.17). Therefore

$$\lim_{j \rightarrow \infty} m_j^{-1} \sum_{i=1}^{m_j} l(g) = 0.$$

V.3.2. Theorem. $\mathcal{L}$ has an almost uniformly distributed sequence each element of which is of finite order.

Proof. The sequence is $\{l_i\}$ and the required subsequence is given by: $m_j = \prod_{i=1}^j r_i$. A character on $\mathcal{L}$ is an element of $\mathcal{G}$; accordingly let $g \neq I$ be in $\mathcal{G}$. there exists $l \in \mathcal{L}$ such that $\langle l, g \rangle \neq 1$ (the real number, one). Since $\theta_j = \{l_i : i = 1, 2, \dots, m_j\}$ we have for $l \in \theta_j$:

$$\sum_{i=1}^{m_j} \langle l_i, g \rangle = \langle l, g \rangle \sum_{i=1}^{m_j} \langle l_i, g \rangle,$$

which shows that $\sum_{i=1}^{m_j} g(l_i) = \sum_{i=1}^{m_j} \langle l_i, g \rangle = 0$. Therefore

$$\lim_{j \rightarrow \infty} m_j^{-1} \sum_{i=1}^{m_j} g(l_i) = 0 \quad \text{for all } g \in \widehat{\mathcal{L}}, g \neq I.$$

V.2.3. Theorem. If $[f] \in L_1(\mathcal{L})$ and if for every $l \in \mathcal{L}$, $\int f(x) \overline{l(x)} dm(x) = 0$, then $[f] = 0$.

Proof. This is the meaning of I.4.15 but we give a more specific proof to further illustrate the detail of $\mathcal{L}$. For any $V \in \mathcal{N}(I)$ there exists $F \in \mathcal{O}$ such that $U_F \subset V$. Put $u_V = u_F$. Then the function u_V is non-negative, vanishes outside of V and has an integral over $\mathcal{L}$ equal to 1, which means $\{u_V : V \in \mathcal{N}(I)\}$ is an approximate identity [15, p. 124-125] so that:

$$\|f * u_V\|_1 \xrightarrow{V \in \mathcal{N}(I)} \|f\|_1.$$

But for any $y \in \mathcal{L}$,

$$\begin{aligned} (f * u_F)(y) &= \int f(x) u_F(y - x) dm(x) \\ &= \sum_{l \in P_F} \int f(x) l(y - x) dm(x) \\ &= \left(\sum_{l \in P_F} l(y) \right) \left(\int f(x) \overline{l(x)} dm(x) \right) \\ &= 0. \end{aligned}$$

Thus $\|f\|_1 = 0$ and the proof is complete.

V.3.4. Theorem. If $[f] \in L_1(\mathcal{L})$ and for every $g \in \mathcal{L}$, $\int f(x) \overline{g(x)} dm(x) = 0$, then $[f] = 0$.

Proof. Since $[f] \in L_1(\mathcal{L})$, $[|f|] \in L_1(\mathcal{L})$. Given $\epsilon > 0$ there exists N such that $n > N$ implies:

$$\sum_{i=n}^{\infty} |f(l_i)| < \epsilon.$$

Let $m_j > N$ and $1 \leq i^* \leq m_j$. Then we have:

$$\begin{aligned} 0 &= \overline{g_k(-l_{i^*})} \sum_{i=1}^{\infty} f(l_i) \overline{g_k(l_i)} \\ &= \sum_{i=1}^{\infty} f(l_i) \overline{g_k(l_i - l_{i^*})}, \end{aligned}$$

for all k . We shall establish the following:

$$\begin{aligned} \text{(V.3.1)} \quad 0 &= \sum_{i=1}^{\infty} f(l_i) \left(m_j^{-1} \sum_{k=1}^{m_j} \overline{g_k(l_i - l_{i^*})} \right) \\ &= f(l_{i^*}) + \sum_{i=m_j+1}^{\infty} f(l_i), \end{aligned}$$

where $\sum'$ indicates that certain of the indices are omitted from the summation. Two things must be verified to

substantiate (V.3.1). That $m_j^{-1} \sum_{k=1}^{m_j} \left(\sum_{i=1}^{\infty} f(l_i) \overline{g_k(l_i - l_{i^*})} \right)$

$= \sum_{i=1}^{\infty} f(l_i) \left(m_j^{-1} \sum_{k=1}^{m_j} \overline{g_k(l_i - l_{i^*})} \right)$ holds is a consequence of

the convergence of the series and the finiteness of the outside summation in the left hand side of the above for-

mula. Next we must compute the function (of l_i) given by:

$$h(l_i) = m_j^{-1} \sum_{k=1}^{m_j} \overline{\varepsilon_k(l_i - l_{i*})}.$$

$h(l_i)$ will be determined according to certain conditions satisfied by the first j entries in the expansion of l_i .

CASE A. $(l_i - l_{i*})^n = I^n$ for $1 \leq n \leq j$. In this case $h(l_i) = 1$. The condition means that the expansion of l_i agrees with that of l_{i*} at least to the j^{th} term. One such element is l_{i*} and this accounts for the presence of $f(l_{i*})$ in (V.3.1). The remaining indices for which l_i is an instance of CASE A are all larger than m_j and it is they which are included in Σ' .

CASE B. $(l_i - l_{i*})^n \neq I^n$ for some n , $1 \leq n \leq j$. Then there exists k^* such that $1 \leq k^* \leq m_j$ and $\varepsilon_{k^*}(l_i - l_{i*}) \neq 1$. For example k^* could be $m_{n-1} + 1$. Now $\varepsilon_k : k = 1, 2, \dots, m_j$ is the group O_j so we get:

$$h(l_i) = \overline{\varepsilon_{k^*}(l_i - l_{i*})} h(l_i).$$

This means $h(l_i) = 0$. The indices considered here are those less than m_j with the exception of i^* and are those omitted from Σ' . These considerations establish (V.3.1). Therefore, for $1 \leq i^* \leq m_j$ and $m_j > N$ we have

$$|f(l_{i*})| \leq \left| \sum_{i=m_j+1}^{\infty} f(l_i) \right| < \epsilon. \text{ Since } \epsilon \text{ is arbitrary, } f(i^*) = 0 \text{ and the proof is complete.}$$

V.3.5. Theorem. Let $1 \leq p \leq \infty$. Suppose f is a function on $\mathcal{X}$ such that $\Delta_h f$ is an L_p -function for each h in $\mathcal{H}$. Then $f = F + \Gamma + S$ where F is an L_p -function, Γ is additive and S has null differences.

Proof. $\mathcal{X}$ has an almost uniformly distributed sequence each element of which is of finite order. We invoke lemma V.2.2, lemma V.2.3 or lemma 2.4 depending on the value of p and obtain the L_p -function, F . Γ is defined in the statement of lemma V.2.2. Put $f = F + \Gamma + S$. Let $h \in \mathcal{H}$ and let $l \in \mathcal{L}$. Then if $l \neq I$

$$\begin{aligned}\Delta_h S(l) &= \int \Delta_h(f - F)(x) \overline{l(x)} dm(x) \\ &\quad - \Gamma(h) \int \overline{l(x)} dm(x) \\ &= 0.\end{aligned}$$

If $l = I$ then

$$\begin{aligned}\Delta_h S(l) &= \int \Delta_h f(x) dm(x) - \int \Delta_h F(x) dm(x) - \Gamma(h) \\ &= 0.\end{aligned}$$

Thus the Fourier coefficients of $\Delta_h S$ are all zero and $[\Delta_h S] = 0$.

V.3.6. Theorem. Suppose that f is a function on $\mathcal{L}$ and that there exists M such that for all $l \in \mathcal{L}$, $\Delta_1 f$ is an L_1 -function and $|\Delta_1 f|_1 \leq M$. Then there is an

L_1 -function, F , and a constant, Z , such that $f = F + Z$.

Proof. Again let $F_j = -m_j^{-1} \sum_{i=1}^{m_j} \Delta_{1_i} f$. We have

$|F_j|_1 \leq M$ and if $m_j \geq n$ we have:

$$\begin{aligned} (V.3.2) \quad F_j(1_n) &= -m_j^{-1} \left(\sum_{i=1}^{m_j} f(1_i + 1_n) \right) + f(1_n) \\ &= -m_j^{-1} \left(\sum_{i=1}^{m_j} f(1_i) \right) + f(1_n). \end{aligned}$$

Now $|F_j(1_1)| \leq M$ for all j so that $|m_j^{-1} \sum_{i=1}^{m_j} f(1_i)| \leq$

$|F_j(1_1)| + |f(1_1)| \leq M + |f(1_1)|$ for all j . Thus

there exist a subsequence $\{m_{j'}\}$ of $\{m_j\}$ and a complex number Z such that

$$\lim_{j' \rightarrow \infty} m_{j'}^{-1} \sum_{i=1}^{m_{j'}} f(1_i) = Z.$$

Now let n and $\epsilon > 0$ be given. Choose j such that $m_{j'} > n$ and

$$(V.3.3) \quad \left| m_{j'}^{-1} \sum_{i=1}^{m_{j'}} f(1_i) - Z \right| < \epsilon/n.$$

We have $\sum_{i=1}^n |F_{j'}(1_i)| \leq |F_{j'}|_1 \leq M$ and

$$\begin{aligned} \sum_{i=1}^n |f(1_i) - Z| &\leq \sum_{i=1}^n |f(1_i) - m_{j'}^{-1} \sum_{k=1}^{m_{j'}} f(1_k)| \\ &\quad + \sum_{i=1}^n |m_{j'}^{-1} \sum_{k=1}^{m_{j'}} f(1_k) - Z| \end{aligned}$$

$$\leq \sum_{i=1}^n |F_j(l_i)| + n(\epsilon/n)$$

$$\leq M + \epsilon.$$

The penultimate inequality above follows from (V.3.2) and (V.3.3). Since n and ϵ are arbitrary we have $f - Z$ is an L_1 -function with $\|f - Z\|_1 \leq M$.

V.3.7. Remark. It is not possible to drop the hypothesis that $\|\Delta_1 f\|_1$ be bounded. For $l \in \mathcal{L}$ let $f(l)$ be the index of the last non-identity entry in the expansion of l . f is not bounded and so $f - Z$ is not integrable for any value of Z , yet $\Delta_1 f$ is zero except for a finite number of elements of $\mathcal{L}$.

CHAPTER VI

ADDITIONAL RESULTS CONCERNING THE DIFFERENCE PROPERTY

VI.1 Introduction

This chapter contains results which are either incomplete or do not fit nicely into any previous chapter. One may state the problem of the difference property for measurable functions thus: if $\mathcal{G}$ is a topological group and f is a function from $\mathcal{G}$ to a Banach space, B , such that $\Delta_h f$ is measurable for each $h \in \mathcal{G}$ and ${}_h \Delta f$ is measurable for each $h \in \mathcal{G}$ then what can be said about f ? Can one prove for instance that $f = g + \Gamma + S$ where g is measurable, Γ is additive and S has null differences? We obtain an answer when f is a certain type of step function.

We also show that the commutator subgroup of the group SU_2 is the entire group, SU_2 so that an additive function is identically zero on this group.

VI.2 A Certain Step Function

VI.2.1. Theorem. Let $\mathcal{G}$ be locally compact with a right Haar measure, m . If $f = \chi_E$ where $E \subset \mathcal{G}$ and $\Delta_h f$ is measurable for each $h \in \mathcal{G}$, then there exists $Q \subset \mathcal{G}$ such that $f - \chi_Q$ is measurable and χ_Q has null

right differences.

Proof. Let F be a measurable kernel of E [10, p. 59], that is: F is measurable, $F \subset E$ and if S is measurable and $S \subset E \sim F$ then $m(S) = 0$. Let $Q = E \sim F$. By hypothesis $(\Delta_h - 1)f(\cdot) = \chi_{E_h}(\cdot) - \chi_E(\cdot)$ is measurable and thus $E \sim E_h = \{x : (\Delta_h - 1)f(x) > 0\}$ and $E_h \sim E = \{x : (\Delta_h - 1)f(x) < 0\}$ are measurable and so is $E_h \Delta E$. Now F_h is a measurable kernel of E_h so that if S is measurable and $S \subset Q_h$ then $m(S) = 0$. Since $Q \Delta Q_h = (E_h \Delta E) \Delta (F_h \Delta F)$, $Q \Delta Q_h$ is measurable.

Let us put:

- i) $S_1 = (Q \Delta Q_h) \cap F$,
- ii) $S_2 = (Q \Delta Q_h) \cap F_h$,
- iii) $S_3 = (Q \Delta Q_h) \cap (E \cup CE_h)$ and
- iv) $S_4 = (Q \Delta Q_h) \cap (E_h \cup CE)$.

Each of the S_i is measurable and we see that:

- i) $S_1 = (Q \cap CQ_h \cap F) \cup (Q_h \cap CQ_h \cap F) \subset Q_h$,
- ii) $S_2 = (Q_h \cap CQ \cap F_h) \cup (Q \cap CQ_h \cap F_h) \subset Q$,
- iii) $S_3 = (Q_h \cap CQ \cap E \cap CE_h) \cup (Q \cap CQ_h \cap E \cap CE_h) \subset Q$ and
- iv) $S_4 = (Q \cap CQ_h \cap E_h \cap CE) \cup (Q_h \cap CQ \cap E_h \cap CE) \subset Q_h$.

These relations are easy to see, for the left hand side of the middle term of each of them is the empty set. Thus

$m(S_i) = 0$, $i=1,2,3,4$. On the other hand $Q \sim Q_h = (Q \cap CQ_h \cap E \cap CE_h) \cup (Q \cap CQ_h \cap E_h) \subset (E \cap CE_h) \cup F_h$

and $Qh \sim Q \subset (Eh \cap CE) \cup F$ so that $Q \Delta Qh \subset F \cup Fh \cup (E \cap CEh) \cup (Eh \cap CE)$ and finally $Q \Delta Qh = S_1 \cup S_2 \cup S_3 \cup S_4$. Hence $m(Q \Delta Qh) = 0$. Now $f - \chi_Q = \chi_F$ and is therefore measurable and $0 = m(Q \Delta Qh) =$

$$\int |(\Delta_h - 1)\chi_Q(x)| dm(x).$$

VI.2.2. Theorem. Let $\mathcal{X}$ be locally compact with a right Haar measure, m . Let $f = c_1 \chi_{E_1} + c_2 \chi_{E_2}$ have measurable right differences, where $c_1 + c_2 \neq 0$ and $E_1 \cap E_2 = \emptyset$. Then there exists S , having null right differences such that $f - S$ is measurable.

Proof. Since $(\Delta_h - 1)f(x) = c_1 \chi_{E_1 h}(x) + c_2 \chi_{E_2 h}(x) - c_1 \chi_{E_1}(x) - c_2 \chi_{E_2}(x)$, the following sets are measurable:

- i) $\{x: (\Delta_h - 1)f(x) = c_1\} = E_1 h \sim (E_1 \cup E_2)$
 $= E_1 h \cap CE_1 \cap CE_2,$
- ii) $\{x: (\Delta_h - 1)f(x) = c_2\} = E_2 h \sim (E_1 \cup E_2)$
 $= E_2 h \cap CE_1 \cap CE_2,$
- iii) $\{x: (\Delta_h - 1)f(x) = c_2 - c_1\} = E_2 h \cap E_1,$
- iv) $\{x: (\Delta_h - 1)f(x) = c_1 - c_2\} = E_1 h \cap E_2,$
- v) $\{x: (\Delta_h - 1)f(x) = -c_1\} = E_1 \sim (E_1 h \cup E_2 h)$
 $= E_1 \cap CE_1 h \cap CE_2 h$ and
- vi) $\{x: (\Delta_h - 1)f(x) = -c_2\} = E_2 \sim (E_1 h \cup E_2 h)$
 $= E_2 \cap CE_1 h \cap CE_2 h.$

Denote these six sets in order by S_1, S_2, S_3, S_4, S_5 and

S_6 . Then the four sets:

$$S_1 \cup S_4 = E_1 h \cap CE_1,$$

$$S_3 \cup S_5 = E_1 \cap CE_1 h,$$

$$S_2 \cup S_3 = E_2 h \cap CE_2 \quad \text{and}$$

$$S_4 \cup S_6 = E_2 \cap CE_2 h$$

are measurable and the two functions:

$$\chi_{S_1 \cup S_4}(\cdot) - \chi_{S_3 \cup S_5}(\cdot) = \chi_{E_1 h}(\cdot) - \chi_{E_1}(\cdot) \quad \text{and}$$

$$\chi_{S_2 \cup S_3}(\cdot) - \chi_{S_4 \cup S_6}(\cdot) = \chi_{E_2 h}(\cdot) - \chi_{E_2}(\cdot)$$

are measurable. Thus theorem VI.2.1 may be applied to the functions χ_{E_1} and χ_{E_2} . There exist Q_1 and Q_2 such

that: i) $Q_1 \subset E_1$ and $Q_2 \subset E_2$, ii) $E_1 \sim Q_1 = F_1$ and $E_2 \sim Q_2 = F_2$ are measurable and iii) $m(Q_1 h \Delta Q_1) = m(Q_2 h \Delta Q_2) = 0$ for all $h \in \mathcal{H}$. Let $S = c_1 \chi_{Q_1} +$

$c_2 \chi_{Q_2}$. Then $f - S = c_1 \chi_{F_1} + c_2 \chi_{F_2}$ is measurable and

$$|(\Delta_h - 1)S(x)| \leq |c_1| \chi_{Q_1 h \Delta Q_1}(x) + |c_2| \chi_{Q_2 h \Delta Q_2}(x) \quad \text{which}$$

is zero almost everywhere. Thus $(\Delta_h - 1)S$ is null.

VI.3 The Group SU_2

VI.3.1. Theorem. The commutator subgroup of SU_2 is SU_2 .

Proof. Consider $A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$; $a\bar{a} + b\bar{b} = 1$. The

characteristic values, λ , of A are given by $(a - \lambda)$

$$(\bar{a} - \bar{\lambda}) + b\bar{b} = 0, \text{ or } \lambda^2 - (a + \bar{a})\lambda + 1 = 0. \text{ Putting}$$

$p = \operatorname{Re}(a) = (a + \bar{a})/2$ we get $\lambda = p \pm \sqrt{p^2 - 1}$. Denote by λ_1 the value obtained when the $+$ sign is used and by λ_2 the value obtained when the $-$ sign is used. A , being unitary is unitarily similar to $\operatorname{DIAG}(\lambda_1, \lambda_2)$ (the notation DIAG means the square matrix of appropriate order having the diagonal elements which are listed and zero entries off the diagonal) [16, p. 194-195]. Thus there exists Y , unitary, such that $YAY^{-1} = \operatorname{DIAG}(\lambda_1, \lambda_2)$. If

$\Delta = \det(Y)$, then Y is of the form:
$$\begin{bmatrix} c & d \\ -\bar{d}\Delta & \bar{c}\Delta \end{bmatrix};$$

$c\bar{c} + d\bar{d} = 1$, $\Delta\bar{\Delta} = 1$, and we see that $Z = \sqrt{\Delta} Y$ is special unitary. We have $ZAZ^{-1} = YAY^{-1}$ and we now consider the commutator $ZAZ^{-1}A^{-1}$.

$$ZAZ^{-1}A^{-1} = \begin{bmatrix} \lambda_1 \bar{a} & -\lambda_1 b \\ \lambda_2 \bar{b} & \lambda_2 a \end{bmatrix}$$

We now show that the matrix $ZAZ^{-1}A^{-1}$ is a typical element of SU_2 . To this end let:

$$Y = \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}; \quad c\bar{c} + d\bar{d} = 1.$$

If $c\bar{c} = 1$, then $d = 0$ and we choose as a a square root of $\bar{c}$ which has a non-positive imaginary part. Then we have $\lambda_1 = \bar{a}$ and $ZAZ^{-1}A^{-1} = \operatorname{DIAG}(\bar{a}^2, a^2) = Y$.

If $c\bar{c} = r^2 < 1$, then any candidate for a must be of modulus r . If for instance $a = r \exp(i\theta)$, then $\lambda_1 = r \cos \theta + i \sqrt{1 - r^2 \cos^2 \theta} = \exp(i \cos^{-1}(r \cos \theta))$. Therefore $\lambda_1 \bar{a} = r \exp(i (-\theta + \cos^{-1}(r \cos \theta)))$. The function $\cos^{-1}(r \cos \theta) - \theta$ is continuous and maps the interval $[0, 2\pi]$ onto the interval $[\cos^{-1}r - 2\pi, \cos^{-1}r]$. c has an argument in the interval just mentioned and there exists then θ in the interval $[0, 2\pi]$ such that the argument of $\lambda_1 \bar{a}$ equals that of c . Putting $a = r \exp(i \theta)$ and $b = -d/\lambda_1$ shows that $Y = ZAZ^{-1}A^{-1}$ and is thus a commutator.

APPENDIX

We will here discuss some questions the answers to which would improve the state of knowledge concerning the question posed in our opening paragraph. The result in chapter VI is, so far as the author is aware, the only result that has been proved when Q is the class of measurable functions. We admit that that result is meager and the question of whether the class of measurable functions has a difference property may be considered open. On a non-abelian group we may ask whether membership in Q of all left (right) differences implies membership in Q of all right (left) differences. We did not attempt to solve this and the question is apparently open. Results of the type we have obtained might be possible if we suppose that the h -differences of f be in Q for h in a subset of $\mathcal{H}$, say, possibly a dense subset or a second category subset.

We must also be concerned with finding which groups, $\mathcal{H}$, and which Banach spaces, B , are such that $(\mathcal{H}, L_p)$ has property C. If such a group be compact, theorem IV.5.5 may be invoked forthwith. The reader will have noticed that in theorem IV.5.5 the function F need not be continuous; it need only be integrable. We could, there-

fore make the appropriate change in the hypothesis concerning $(\mathcal{X}, L_p)$. The resulting theorem would be a theorem without application, since we could find no example for which the hypothesis could be proved, and therefore we have not included it.

We mention, finally, that the absence of examples and counterexamples in a place where they would seem appropriate should be taken to mean that we could find none to illustrate the point in question (as in remark IV.5.2).

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