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ESTIMATES FOR THE POINTS OF INTERSECTION
OF TWO POLYNOMIAL REGRESSIONS

DISSERTATION

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the Degree Doctor of Philosophy in the Graduate
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By

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To my adviser,
Professor D.R.Whitney,
who has made academic life a pleasure;

and to my family,
Emily, David, and Jeannie,
who make the rest of life a pleasure.

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ESTIMATES FOR THE POINTS OF INTERSECTION OF TWO POLYNOMIAL REGRESSIONS

CHAPTER I INTRODUCTION

1.1 Description of the Problem

Research workers in the experimental sciences are frequently confronted with experimental data in which a sudden and decisive change in trend is observed. It is of interest to know where or when this change takes place. The topic of this paper is the statistical estimation of these points of change in trend.

The first point of view is to regard the data as samples from two intersecting normal processes. The samples taken are assumed to be discrete but not necessarily evenly spaced with respect to any parameter. The problem at hand is to provide estimates and distribution theory for the point or points of intersection of the two estimated regression curve. This problem will be studied under various polynomial assumptions for the form of the regression curves. The solution obtained consists of providing maximum likelihood estimates and confidence regions when such exist, and of presenting alternative procedures otherwise.

A more general point of view is to regard the data as samples from two intersecting processes for which continuous sample records are available. The problem is to provide estimates and their distributions

for the points of intersection of the regressions of the processes. This problem will be studied for the case of simple linear regression from a Fundamental Random Process. Estimates of the intersection and corresponding confidence regions are obtained.

1.2 A Graphical Example

There are a variety of situations where the point of intersection, P , is of interest. The point P might correspond to some maximum or minimum of some physical process. The abscissa of P might correspond to a position of demarkation, to the left of which interpolation and extrapolation would be performed according to a first function, and to the right of which the same operations would be performed by a second function. In the case of missing interior data points, P might be used to establish a thread of continuity through the missing region. Finally, P may be of just plain academic interest.

As an explicit example consider a sample taken from telemetry data from an upper atmospheric research problem. Radiation belts are known to exist above the earth, but the radial distances of certain intensities varies from time to time. Suppose the experiment provides intensities and times as coordinate pairs. Sudden changes in trend of radiation as a function of time are known to occur. Knowledge of the exact time of this change may be of considerable interest (Figure 1). By visual inspection the point of change seems to be somewhere near t_* . It is desired to have a sound statistical estimate for the abscissa of this point.

Figure 1

1.3 Definitions and Notation

Usually we shall try to denote random variables by capital letters and fixed observations of random variables by lower case letters. However, we shall not forsake the convenience of denoting maximum likelihood estimates of a number a by $\hat{a}$ and other estimates of a by $\tilde{a}$.

A stochastic process is an indexed set of random variables, $\{X_t\}$. For our purposes t will be selected from some set of real numbers. It is convenient to regard t as time. For any particular value of t , say t_0 , the symbol X_{t_0} is a random variable. A process is considered known if for any finite set $(t_1, \dots, t_n)$ the joint distribution of $(X_{t_1}, \dots, X_{t_n})$ is known. When there is little chance of misinterpretation we shall shorten the notation $\{X_t\}$ to X_t .

A sample function from a process X_t is a function $x(t)$ composed of a single observation of X_t for each t in some subset of t 's. A sample function can be either discrete or continuous.

The mean value function for a process is the function $m(t)$ which gives the expectation $E X_t$ at any time t . It is usually called the regression function of X_t . The covariance function is the function $\sigma_{12} = E[X_{t_1} - m(t_1)][X_{t_2} - m(t_2)]$. The variance function is obtained from the covariance function when $t_1 = t_2 = t$. It is denoted σ_t^2 .

Consider any function $g(t)$ associated with a process X_t and consider also the corresponding function $g^*(t)$ associated with a second process X_t^* . The two processes are said to intersect if the associated functions g and g^* intersect in the usual geometric sense. In this paper we consider only processes whose regression functions intersect. This definition shall include the special case when a process X_t , $t \in T$, is considered as the union of two other processes $X_{t'}^1$, $t' \in T^1$, and $X_{t''}^2$, $t'' \in T^2$, with $T^1 + T^2 = T$, so that the associated functions g^1 and g^2 can be regarded as intersecting branches of a more inclusive $g(t)$.

Most of this paper will be concerned with discrete samples from two processes X_t and Y_t . But in Chapter III one continuous model will be studied, requiring the following definition (Mann, 1953).

A Fundamental Random Process (FRP) is a process X_t satisfying

$$1) \quad X_{t+\tau} - X_t = \epsilon_{t,\tau}$$

where $\epsilon_{t,\tau}$ is a random variable with zero mean and is independent of X_t and also of $\epsilon_{t',\tau'}$ if the

intervals $(t, t+\tau)$ and $(t', t'+\tau)$ do not overlap.

2) The distribution of $\epsilon_{t,\tau}$ depends only on τ .

3) The variance of $\epsilon_{t,\tau}$ exists and is given by

$$\sigma_{\epsilon_{t,\tau}}^2 = \sigma_{\tau}^2 = c\tau.$$

Here, c is a positive constant.

4) Large values of $\epsilon_{t,\tau}$ are unlikely; precisely, it is required that for sufficiently small τ and arbitrary $\rho > 0$ and $\eta > 0$, $\int_{|a| > \rho} a^2 dF_{\tau}(a) < \eta \sigma^2$, where $F_{\tau}(a)$ is the distribution function of $\epsilon_{t,\tau}$.

Thus an FRP has a single parameter, c . As a consequence of the definition it can be shown that $\epsilon_{t,\tau}$ is normally distributed with variance c .

We will often consider a related process $Y_t = f(t) + X_t$, where $f(t)$ is a non-random regression function, usually unknown, and where X_t is an FRP. Another related process is $Z_t = \int_0^t X_s ds$. If X_s is an FRP then Z_t is Gaussian with mean zero and covariance

$$\sigma_{t,t'} = \frac{c}{2} \max(t, t') [\min(t, t')]^2 - \frac{c}{6} [\min(t, t')]^3.$$

1.4 Maximum Likelihood Theory of Estimation

The principle of maximum likelihood (M.L.) states that to estimate parameters $(\theta_1, \dots, \theta_k)$ from a sample $(x_1, \dots, x_n)$ one should form the joint probability function of the sample and then find the values $(\hat{\theta}_1, \dots, \hat{\theta}_k)$ that maximize this function. This function is called the likelihood function.

In this paper we shall use the likelihood principle not only in the

above context but also in the case when the data are given by a continuous sample record. To proceed, suppose $y(t)$ is given on $[0, T]$. Form a subdivision of $(n + 1)$ points from $[0, T]$ and consider the discrete sample points $(y_0, y_1, \dots, y_n)$. The likelihood function for this sample can be formed and a solution $\hat{\theta}$ obtained. Now consider a sequence of such subdivisions as the modulus converges to zero. If $\hat{\theta}$ converges in some probability sense to a limiting estimator, we shall refer to that limiting random variable as the maximum likelihood estimate in the continuous case.

Mann(1954) has shown that for the multiple regression estimation problem associated with the FRP the limiting estimators exist as limits in the mean and are unbiased.

1.5 Multiple Regression Theory

In this section we review the standard results in multiple regression theory. Let Y denote an observable random variable and let $g_1, \dots, g_k$ denote k observable nonrandom parameters. Consider a sample consisting of n independent sets $(y_i, g_{1i}, \dots, g_{ki})$, where g_{ji} represents the i -th observation on the j -th parameter, $i = 1, \dots, n$. We assume

$$y_i = g_{1i}\beta_1 + \dots + g_{ki}\beta_k + \epsilon_i, \quad k \leq n$$

where the ϵ_i are independent normal variates with zero means and constant variances σ^2 ; the β_j are constants to be estimated. It is necessary that the rank $(g_{ji}) = k$.

The solution for the maximum likelihood estimates b_p of the β_p and $\hat{\sigma}^2$ of σ^2 is well known, (Mood, p. 301). Let

$$a_{pq} = \sum_{i=1}^n g_{pi} g_{qi} \quad \text{and} \quad c_q = \sum_{i=1}^n y_i g_{qi} \quad .$$

Then the normal equations are

$$\sum_{p=1}^k a_{pq} b_p = c_q \quad q = 1, \dots, k$$

and the solution is

$$b_p = \sum_{q=1}^k a^{pq} c_q, \quad \text{where} \quad (a^{pq}) = (a_{pq})^{-1} \quad .$$

Furthermore

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - g_{1i} b_1 - \dots - g_{ki} b_k)^2 \quad .$$

The b_p are unbiased estimates of the β_p . They are jointly normally distributed and the elements of the covariance matrix are given by

$$\sigma_{b_p b_q} = a^{pq} \sigma^2, \quad p, q = 1, \dots, k \quad .$$

It can be shown that $n\hat{\sigma}^2/\sigma^2$ has the chi square distribution with $(n - k)$ degrees of freedom and that it is independently distributed of the b_p . To obtain polynomial regressions in time, take $g_{i-1} = t^i$.

1.6 A Summary of the Results in This Paper

Chapter II deals with the intersections of two polynomial regressions in the context of a discrete sampling procedure. It is assumed that a first process X_t precedes a second process Y_t in time. Points P of intersection are denoted by (t_*, x_*) . Usually it can be assumed that some true t_* falls in a known interval I corresponding to the time lag after the first process and preceding the second. If the sample consists of values $(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2})$ at times $(s_1, \dots, s_{N_1}, t_1, \dots, t_{N_2})$ then I is the interval $[s_{N_1}, t_1]$.

The maximum likelihood estimates of P are obtained and are found to be the intersections of the sample M.L. regressions. But the M.L. estimates of P do not always exist because the two sample polynomials may not always intersect. The M.L. formulation is checked for correctness and the results are affirmative. Thus if practical difficulties arise, it is the fault of the sample and not of the general theory. For cases when the M.L. estimates do exist in I , confidence intervals are obtained for t_* . For cases where the M.L. estimators exist and lie outside I (but correspond to some t_* in I) a revised point estimator is found to be the endpoint of I nearest $\hat{t}_*$. Confidence limits for t_* are again found and the intersection of these with I is taken as an improved confidence interval for t_* , thus completing the distribution theory. Explicit examples for polynomials of degree one and two are presented.

For cases where the M.L. estimates clearly do not correspond

to a t_* in I , or where the sample regressions completely fail to intersect, three alternative procedures are considered. The preferred method is to find an optimal point $(\tilde{t}_*, \tilde{x}_*)$ which could have been included in the regression estimates and which would have minimized an associated likelihood-type function. The solution for such a point is numerically difficult because of the functional interdependence of all of the parameters. Another method is that of finding the abscissa in I where the sample regressions are closest. This method is easy, but has certain shortcomings. A third method is that of constructing confidence bands about each of the sample regressions and to consider the intersection of these bands in I as defining a confidence region for P . This method is not always possible. The common virtue of all three methods is that they all recover the M.L. estimate when $\tilde{t}_*$ is in I .

In many cases the interval I may not be known exactly. In such cases the likelihood is also a function of N_1 . The correct procedure is to try various divisions of the sample, corresponding to various choices of N_1 and N_2 . For each such choice calculate separate regression estimates and pick the choice which eventually maximizes the likelihood.

With I not known exactly, probabilities associated with selecting (N_1, N_2) are of interest. These probabilities are derived for the non regression case under conditions of both correct and incorrect selection. These results are of interest in the classical two-sample problem. When regression assumptions are added, the selection probabilities are difficult to calculate.

Some progress on this problem, in terms of approximations, is reported.

Chapter III deals with simple linear regressions for two fundamental random processes. Under a special assumption on the F.R.P. the distribution of t_* is obtained.

CHAPTER II

THE INTERSECTION OF SIMPLE POLYNOMIAL REGRESSIONS:

DISCRETE MODEL

2.1 Statement of the Problem

In a simple regression problem the regression is assumed to be a function of just one observable parameter, say t . Polynomial regressions are polynomials in the variable t . The parameter t will be called time.

Consider a process with unknown regression $\mu(t)$ having two intersecting branches X_s and Y_t such that

$$\mu(\cdot) = \begin{cases} E(X_s) = \mu_1(s) = \alpha_0 + \alpha_1 s + \dots + \alpha_k s^k & \text{if } s \leq t_* \\ E(Y_t) = \mu_2(t) = \beta_0 + \beta_1 t + \dots + \beta_q t^q & \text{if } t \geq t_* \end{cases} .$$

The variables s and t are measured on the same axis and in the same units ; different letters have been used simply for easy identification with X and Y . Let the variances, σ_1^2 and σ_2^2 , of X_s and Y_t be constant. For the discrete case no covariance structure is required. The point t_* is an unknown abscissa of intersection of the two branch regressions. The corresponding ordinate of intersection will be denoted by x_* . We assume that X_s precedes Y_t and that each is undefined in the other's domain. Thus we shall not consider the problem of intersecting regressions if X_s and Y_t overlap. Initially we shall assume that $\mu(t)$ is continuous at t_* .

For the case of a jump at the intersection see the comments in section 2.5 . Assume $k \geq q$; if this is not the case interchange the roles of k and q .

Given a discrete sample from each process, our problem is to estimate t_* and to determine the distribution of the estimator. This problem will be studied under the usual independence and normality assumptions. This means that discrete samples from X_s and Y_t at times s_i, t_j will be regarded as observations on independent normal random variables $X_i = X_{s_i}$ and $Y_j = Y_{t_j}$ having means $\mu_1(s_i)$ and $\mu_2(t_j)$ and constant variances σ_1^2 and σ_2^2 .

Our solution, when possible, consists of presenting the maximum likelihood point estimator for t_* and of showing a method for constructing confidence intervals about t_* . These results rely heavily on being able to solve certain sample polynomials for all real roots. When all roots are not real, certain alternative procedures are suggested.

2.2 Intersecting Polynomials of Arbitrary Degree: N_1 and N_2 Known

Let $(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2})$ be a given set of independent observations on X_s and Y_t at given ordered times $(s_1, \dots, s_{N_1}, t_1, \dots, t_{N_2})$. Let $N = N_1 + N_2$. Then the likelihood for this composite sample is a function of $(k + 1) + (q + 1) + 2$ continuous parameters and two discrete parameters:

$$L(\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_q, \sigma_1^2, \sigma_2^2; N_1, N_2) =$$

$$\prod_{i=1}^{N_1} \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{1}{2\sigma_1^2} [x_i - \mu_1(s_i)]^2} \cdot \prod_{i=1}^{N_2} \frac{1}{\sqrt{2\pi} \sigma_2} e^{-\frac{1}{2\sigma_2^2} [y_i - \mu_2(t_i)]^2}.$$

The α_j, β_j and σ^2 's are unknown parameters which will be estimated by maximizing $\log L$. For the remainder of this section regard N_1 and N_2 as known. See section 2.6 for the case when N_1 and N_2 are unknown. This formulation of the likelihood equation will later be referred to as the "original" formulation.

The required t_* is defined by equating μ_1 and μ_2 in the interval $I: [s_{N_1}, t_1]$,

$$\alpha_0 + \alpha_1 t_* + \dots + \alpha_k t_*^k = \beta_0 + \beta_1 t_* + \dots + \beta_q t_*^q.$$

Let

$$\gamma_j = \begin{cases} \alpha_j - \beta_j & \text{for } j = 0, 1, \dots, q \\ \alpha_j & \text{for } j > q. \end{cases}$$

Then t_* is any solution to the system

$$\gamma_0 + \gamma_1 t_* + \dots + \gamma_k t_*^k = 0, \quad s_{N_1} \leq t_* \leq t_1.$$

We would like to assume that t_* is unique in I since that is the problem under consideration. But since polynomial regressions may be only approximations to the true regression, and since in many cases I may be large relative to the size of the sampling intervals, it is more prudent to study all k abscissae defined by the equation

$$\mu_1(t_*) - \mu_2(t_*) = 0.$$

After finding a set of solutions for t_* we would then pick that t_* which falls in I . If there are more than one such t_* which falls in I we must establish some decision procedure. If no t_* falls in I we must establish alternative procedures.

2.2.1 The Maximum Likelihood Estimates for (t_*, x_*)

Since t_* is an unknown quantity which is to be estimated one might wonder why it does not occur explicitly in the likelihood function. The answer is that it could, but that the situation is easier if it doesn't. This conclusion follows from the following discussion. We will demonstrate that from the likelihood standpoint all information about t_* is obtainable from estimates of the α 's and β 's.

From the likelihood function consider the set of independent parameters $(\alpha_0, \alpha_1, \dots, \alpha_k, \beta_0, \dots, \beta_q)$. The equations defining γ_j define a 1-1 transformation from parameters $(\alpha_0, \dots, \beta_q)$ into parameters $(\gamma_0, \gamma_1, \dots, \gamma_k, \beta_0, \dots, \beta_q)$. If $t_*^{(1)}, \dots, t_*^{(k)}$ denote the k solutions for t_* then

$$\begin{aligned} \gamma_0 + \gamma_1 t_* + \dots + \gamma_k t_*^k &= \gamma_k (\delta_0 + \delta_1 t_* + \dots + \delta_{k-1} t_*^{k-1} + t_*^k) \\ &= \gamma_k (t_* - t_*^{(1)}) \dots (t_* - t_*^{(k)}) \end{aligned}$$

where $\delta_j = \gamma_j / \gamma_k$, $j = 0, 1, \dots, k-1$.

But from the theory of equations the δ_j are expressible in 1-1

fashion with the $t_*^{(j)}$:

$$\begin{aligned}\delta_{k-1} &= - (\text{sum of the roots, } t_*^{(j)}) \\ \delta_{k-2} &= + (\text{sum of the products of the roots taken 2 at a time}) \\ &\vdots \\ \delta_0 &= (-1)^k \cdot (\text{product of all the roots, } t_*^{(j)})\end{aligned}$$

Thus there exists a 1-1 transformation from the parameter set $(\alpha_0, \dots, \beta_q)$ into the parameter set $(t_*^{(1)}, \dots, t_*^{(k)}, \gamma_k, \beta_0, \dots, \beta_q)$.

Consider now the following theorem given by Anderson (p. 48).

Theorem. If on the basis of a given sample, $\hat{\theta}_1, \dots, \hat{\theta}_m$ are maximum likelihood estimates of parameters $\theta_1, \dots, \theta_m$ of a distribution, then $\hat{\phi}_1(\hat{\theta}_1, \dots, \hat{\theta}_m), \dots, \hat{\phi}_m(\hat{\theta}_1, \dots, \hat{\theta}_m)$ are maximum likelihood estimates of $\phi_1(\theta_1, \dots, \theta_m), \dots, \phi_m(\theta_1, \dots, \theta_m)$ if the transformation from $\theta_1, \dots, \theta_m$ to $\phi_1, \dots, \phi_m$ is one-to-one. If the estimates of $\theta_1, \dots, \theta_m$ are unique then the estimates of $\phi_1, \dots, \phi_m$ are unique.

Applying this result to the 1-1 correspondence demonstrated above, it is evident that the M.L. estimators of the t_* are obtainable from the M.L. estimates of the α 's and β 's. We shall soon see that this explicitly involves solving the polynomial equation obtained from equating the sample regressions $\hat{\mu}_1$ and $\hat{\mu}_2$. Thus it is not necessary that the t_* occur explicitly in the likelihood function since $(\alpha_0, \dots, \beta_q)$ and $(t_*^{(1)}, \dots, \beta_q)$ are equivalent parameter sets. The choice is made in favor of $(\alpha_0, \dots, \beta_q)$ because the other choice produces likelihood equations which are more difficult

to solve.

It would seem that the existence of the M.L. estimators depends on the existence of real solutions in I of the polynomial $\hat{\mu}_1(t_*) = \hat{\mu}_2(t_*)$. This observation prompted the author to consider alternative formulations of the likelihood equation, such as in terms of the parameter set just discussed, to see if there was a more favorable likelihood formulation. The answer is, of course, in the negative, but two of these formulations will be discussed in passing.

One possibility considered was that of maximizing L subject to the "condition" that $\mu_1(t_*) = \mu_2(t_*)$ by using Lagrange multipliers. But further study indicated that this was not truly a restriction on $(\alpha_0, \dots, \beta_q)$ but rather an equation defining alternative parameters $(t_*^{(1)}, \dots, t_*^{(k)})$. These could replace any k of the α 's or β 's.

Another possibility studied was to write each regression as a polynomial about some root, notationally designated as $t_*^{(1)}$, which lies in I :

$$\begin{aligned}\mu_1(t) &= \alpha_0^* + \alpha_1^*(t - t_*^{(1)}) + \dots + \alpha_k^*(t - t_*^{(1)})^k \\ \mu_2(t) &= \beta_0^* + \beta_1^*(t - t_*^{(1)}) + \dots + \beta_q^*(t - t_*^{(1)})^q.\end{aligned}$$

Then the $t_*^{(j)}$ are solutions to

$$\begin{aligned}P(t) &= (\alpha_0^* - \beta_0^*) + (\alpha_1^* - \beta_1^*)(t - t_*^{(1)}) + \dots \\ &\quad + (\alpha_k^* - \beta_k^*)(t - t_*^{(1)})^k = 0, \beta_j^* = 0 \text{ for } j > q.\end{aligned}$$

In particular, $P(t_*^{(1)}) = 0$ means that $\alpha_0^* = \beta_0^*$. Hence $(t_*^{(1)}, \alpha_1^*, \dots, \beta_q^*)$ is an alternative parameter set. Furthermore,

knowledge of this set and of the identities

$$\begin{aligned}\alpha_0 + \alpha_1 t + \dots + \alpha_k t^k &= \alpha_0^* + \alpha_1^* (t - t_*^{(1)}) + \dots + \alpha_k^* (t - t_*^{(1)})^k \\ \beta_0 + \beta_1 t + \dots + \beta_q t^q &= \beta_0^* + \beta_1^* (t - t_*^{(1)}) + \dots + \beta_q^* (t - t_*^{(1)})^q\end{aligned}$$

will recover $(\alpha_0, \dots, \beta_q)$ explicitly. Hence there exists a 1-1 transformation between $(\alpha_0, \dots, \beta_q)$ and $(t_*^{(1)}, \alpha_1^*, \dots, \beta_q^*)$. Since these parameter sets are equivalent we select $(\alpha_0, \dots, \beta_q)$ because of easier likelihood equations.

Thus it is argued that the original likelihood formulation given is correct and is the easiest such formulation to work with. This conclusion is of great importance when we see that the M.L. estimator of t_* doesn't always exist. We immediately wonder if our likelihood formulation is correct, and the previous argument shows that it is.

We now turn to the mechanics of implementing the preceding solution. We will obtain the M.L. estimator and then show a way of deriving confidence intervals for the t_* .

2.2.2 Computation Formulas for t_* and x_*

Taking logarithms of the likelihood function we obtain

$$\begin{aligned}\log L &= -\frac{N}{2} \log 2\pi - \frac{N_1}{2} \log \sigma_1^2 - \frac{N_2}{2} \log \sigma_2^2 \\ &\quad - \frac{1}{2\sigma_1^2} \sum_{i=1}^{N_1} [x_i - \mu_1(s_i)]^2 - \frac{1}{2\sigma_2^2} \sum_{i=1}^{N_2} [y_i - \mu_2(t_i)]^2.\end{aligned}$$

By maximizing $\log L$ we can separately obtain the usual (section 1.6) maximum likelihood estimates a_i of α_i and b_j of β_j . The estimates for the regressions and variances are:

$$\hat{\mu}_1 = m_1(s) = a_0 + a_1 s + \dots + a_k s^k \quad (\text{based on } x_1, \dots, x_{N_1})$$

$$\hat{\mu}_2 = m_2(t) = b_0 + b_1 t + \dots + b_q t^q \quad (\text{based on } y_1, \dots, y_{N_2})$$

$$\hat{\sigma}_1^2 = \frac{1}{N_1} \sum_{i=1}^{N_1} [x_i - m_1(s_i)]^2$$

$$\hat{\sigma}_2^2 = \frac{1}{N_2} \sum_{i=1}^{N_2} [y_i - m_2(t_i)]^2 \quad .$$

It is known that the $(a_0, \dots, a_r)$ are jointly normally distributed with means $(\alpha_0, \dots, \alpha_r)$ and covariances $\sigma_{a_i a_j} = r^{ij} \sigma_1^2$, $(i, j = 1 \text{ to } k)$, where each r^{ij} is a function of $(s_1, \dots, s_{N_1})$ only and is to be identified with a^{ij} in section 1.6. Likewise the $(b_0, \dots, b_s)$ are jointly normally distributed with means $(\beta_0, \dots, \beta_s)$ and covariances $\sigma_{b_i b_j} = s^{ij} \sigma_2^2$ $(i, j = 1, \dots, s)$, where each s^{ij} is a function of $(t_1, \dots, t_{N_2})$ only and is also to be identified with a^{ij} in section 1.6. The $(a_0, \dots, a_r)$ and $(b_0, \dots, b_s)$ are distributed independently of themselves and of $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$.

Equating the sample regressions $m_1(t)$ and $m_2(t)$ we obtain

$$a_0 + a_1 \hat{t}_* + \dots + a_k \hat{t}_*^k = b_0 + b_1 \hat{t}_* + \dots + b_q \hat{t}_*^q \quad .$$

Let

$$c_j = \begin{cases} a_j - b_j & \text{for } j = 0, \dots, q \\ a_j & \text{for } j > q \end{cases} .$$

We note that the c_j are jointly normal and are unbiased estimates of the γ_j . Then the required estimates $\hat{t}_*$ are solutions to the system

$$c_0 + c_1 \hat{t}_* + \dots + c_k \hat{t}_*^k = 0$$

$$s_{N_1} \leq \hat{t}_* \leq t_1 .$$

Furthermore $\hat{x}_* = \hat{\mu}_1(\hat{t}_*)$ for any $\hat{t}_*$ in I . The above polynomial, with coefficients c_j , will be called the t_* -estimator polynomial.

What should our M.L. estimator be if the sample regressions don't intersect? The answer, in view of the correctness of the likelihood formulation argued previously, is that the M.L. estimators simply don't exist. The maximization situation is completely determined. There are no free parameters to adjust (such as t_*) to determine other smooth maxima. There is no point to looking for a maximum other than by the use of partial derivatives since L , being quadratic in the α 's and β 's, has only smooth extrema. The estimates of α 's and β 's contain all likelihood information about t_* . We still may be able to find alternative estimators, but they won't be M.L. estimators. For fixed (k, q) and a fixed sample, we have gone as far as we can go with M.L. estimation theory. We conclude with the following theorem, which summarizes the discussions in the last several pages.

Theorem. Let $\hat{\mu}_1(t)$ and $\hat{\mu}_2(t)$ be maximum likelihood estimates of true polynomial regressions $\mu_1(t)$ and $\mu_2(t)$. Let the abscissae of intersection be defined by equating $\mu_1(t_*) = \mu_2(t_*)$. If the maximum likelihood estimators of t_* exist as real numbers, they are given by the roots of the sample equation $\hat{\mu}_1(\hat{t}_*) = \hat{\mu}_2(\hat{t}_*)$. The estimate of the ordinates, $\hat{x}_*$, of intersection are the common values of $\hat{\mu}_1(\hat{t}_*)$ and $\hat{\mu}_2(\hat{t}_*)$ for any real $\hat{t}_*$.

2.2.3 Confidence Intervals for t_*

We would like to find the distribution of $\hat{t}_*$. But since $\hat{t}_*$ is a solution to a polynomial equation there is in general no explicit formula for any $\hat{t}_*$ in terms of the equations's coefficients. This and other situations make the distribution problem virtually unsolvable. We surmount this difficulty by seeking confidence intervals for t_* directly. The intervals obtained contain $\hat{t}_*$ but are not necessarily symmetric about it.

Consider the random variable

$$u = c_0 + c_1 t_* + \dots + c_k t_*^k \quad .$$

This random variable is normally distributed because it is a linear combination of normal variates. Note that u contains the true t_* and not the M.L. estimator. The mean of u is zero since

$$E[u] = \gamma_0 + \gamma_1 t_* + \dots + \gamma_k t_*^k = 0 \quad .$$

The variance of u is given by

$$\sigma_u^2 = \sum_{i=0}^k \sum_{j=0}^q t_*^{i+j} \sigma_{c_i c_j} \quad .$$

But

$$\sigma_{c_i c_j} = \begin{cases} \sigma_{(a_i - b_i)(a_j - b_j)} & i = 1, \dots, q \quad j = 1, \dots, q \\ \sigma_{a_i a_j} & i = 1, \dots, k \quad j = q+1, \dots, k \end{cases}$$

$$= \begin{cases} \sigma_{a_i a_j} + \sigma_{b_i b_j} \\ \sigma_{a_i a_j} \end{cases}$$

$$= \begin{cases} r^{ij} \sigma_1^2 + s^{ij} \sigma_2^2 \\ r^{ij} \sigma_1^2 \end{cases}$$

where the r^{ij} and s^{ij} are the elements of the inverses of the matrices of coefficients obtained from the s 's and t 's.

Thus

$$\begin{aligned} \sigma_u^2 &= \left(\sum_{i=0}^q + \sum_{i=q+1}^k \right) \left(\sum_{j=0}^q + \sum_{j=q+1}^k \right) t_*^{i+j} \sigma_{c_i c_j} \\ &= \sum_{i=0}^q \sum_{j=0}^q t_*^{i+j} (r^{ij} \sigma_1^2 + s^{ij} \sigma_2^2) \\ &\quad + \sum_{i=0}^q \sum_{j=q+1}^k t_*^{i+j} r^{ij} \sigma_1^2 + \sum_{i=q+1}^k \sum_{j=0}^q t_*^{i+j} r^{ij} \sigma_1^2 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=q+1}^k \sum_{j=q+1}^k t_*^{i+j} r^{ij} \sigma_1^2 \quad . \\
& = \sigma_1^2 \sum_{i=0}^k \sum_{j=0}^k t_*^{i+j} r^{ij} + \sigma_2^2 \sum_{i=0}^q \sum_{j=0}^q t_*^{i+j} s^{ij} \\
& = \sigma_1^2 P_{2k}(t_*; s_1, \dots, s_{N_1}) + \sigma_2^2 Q_{2q}(t_*; t_1, \dots, t_{N_2}) \quad ,
\end{aligned}$$

where P and Q are polynomials in t_* of degrees indicated by the subscripts and whose coefficients are functions of the observed values of times s 's and t 's . We now construct a Student- t variate. Observe that u/σ_u is normally distributed with mean zero and unit variance and that

$$u = (a_0 + \dots + a_k t_*^k) - (b_0 + \dots + b_q t_*^q) \quad .$$

Let

$$t = \frac{\frac{u}{\sqrt{\sigma_1^2 P_{2k}(t_*) + \sigma_2^2 Q_{2q}(t_*)}}}{\sqrt{\frac{\sum_{i=1}^{N_1} [x_i - m_1(s_i)]^2}{\sigma_1^2} + \frac{\sum_{j=1}^{N_2} [y_j - m_2(t_j)]^2}{\sigma_2^2}}} \sqrt{N_1 + N_2 - k - q - 2} \quad .$$

Then t is a Student variate with $(N_1 + N_2 - k - q - 2)$ degrees of

freedom because the numerator is normal(0,1), the denominator is $\sqrt{\chi^2/d.f.}$, and the numerator and denominator are independent because the terms in the numerator are linear combinations of the a's and b's, which in turn are known to be independent of the sums of squares in the variance estimates.

The t variate constructed has three unknown parameters, t_* , σ_1^2 and σ_2^2 . This is similar to the situation in the problem of testing the difference between means of two normal populations. The only solution for this more elementary problem is obtained by assuming $\sigma_1^2 = \sigma_2^2$ or, more generally, $\sigma_1^2 = h\sigma_2^2$ for some known constant h . Making this latter assumption for our problem we have

$$t = \frac{\frac{u(t_*)}{\sqrt{h^2 P(t_*) + Q(t_*)}}}{\sqrt{\frac{\sum_{i=1}^{N_1} [x_i - m_1(s_i)]^2}{h^2} + \frac{\sum_{i=1}^{N_2} [y_i - m_2(t_i)]^2}{1}}} \sqrt{N_1 + N_2 - k - q - 2}.$$

This Student variate involves just one parameter t_* . Confidence limits at significance ϵ are

$$-t_{\epsilon/2} < t < t_{\epsilon/2}, \quad \text{or} \quad 0 \leq t^2 < t_{\epsilon/2}^2.$$

Let the denominator radical of t be denoted by H . Then by manipulating the inequality,

$$0 \leq u^2(t_*) < (h^2 P(t_*) + Q(t_*)) H^2 t_{\epsilon/2}^2 .$$

All t_* values satisfying the above would constitute the confidence region for all t_* values defined by the original polynomial equation. The limits of the region are obtained by solving the polynomial equation of degree $2k$

$$u^2(t_*) = [h^2 P(t_*) + Q(t_*)] H^2 t_{\epsilon/2}^2 .$$

This polynomial will be called the confidence interval polynomial.

If all $2k$ roots of the above equation are real, the M.L. estimates of t_* are contained in the above confidence set. To see this, note that the region defined by

$$- t_{\epsilon/2} < t(t_*) < t_{\epsilon/2}$$

has been replaced by an equivalent set of intervals about t_* .

Since $t(t_*) = 0$ only if $u(t_*) = 0$, then the solutions to $u(t_*) = 0$ must lie in the new equivalent sets.

In case σ_1^2 and σ_2^2 are not known-multiples of each other, it would theoretically be possible to establish a joint three dimensional confidence region for $(t_*, \sigma_1^2, \sigma_2^2)$. Consider the event

$$E : \left[-p < \frac{u(t_*)}{\sigma_u(t_*)} < p, q_1 < \frac{N_1 \hat{\sigma}_1^2}{\sigma_1^2} < q_2, q_3 < \frac{N_2 \hat{\sigma}_2^2}{\sigma_2^2} < q_4 \right] .$$

The three component events are independent; call them E_1, E_2, E_3 . Then $P(E) = P(E_1) \cdot P(E_2) \cdot P(E_3)$. If we require $P(E) = .95$ (say), then one way this can be realized is to take each $P(E_i) = (.95)^{1/3}$.

Such a specification of the three probabilities defines a three dimensional region in the space of $(t_*, \sigma_1^2, \sigma_2^2)$.

The principal result of this section may be summarized in the following theorem.

Theorem. Let $\mu_1(t)$ and $\mu_2(t)$ be unknown polynomial branch regressions with largest degree k , and let $\hat{\mu}_1(t)$ and $\hat{\mu}_2(t)$ be their maximum likelihood estimators. Let t_* be defined by equating $\mu_1(t_*) = \mu_2(t_*)$ and assume the unknown constant variances of the branches satisfy $\sigma_1^2 = h\sigma_2^2$ for some positive known h . Define $u(t_*) = \hat{\mu}_1(t_*) - \hat{\mu}_2(t_*)$ and consider the equation

$$u^2(t_*) = [h^2 P(t_*) + Q(t_*)] \cdot H^2 t_{1-\epsilon/2}^2,$$

where P and Q are polynomials in t_* , H^2 is a function of h , and $t_{1-\epsilon/2}^2$ is a student percentile. All of these quantities are previously defined explicitly. If the above equation has $2k$ real roots, these roots define the end points of k confidence intervals, with joint confidence $1-\epsilon$, for the k roots t_* .

2.3.1 Explicit Results for Polynomials of 1st Degree

Consider the assumptions that the variances are equal, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, and that both branch regressions are of first degree. Then t_* is defined by the intersection $\mu_1 = \mu_2$,

$$\alpha_0 + \alpha_1 t_* = \beta_0 + \beta_1 t_*.$$

By the theorem in section 2.2.2 the M.L. estimate $\hat{t}_*$ is obtained

by equating the sample regressions,

$$a_0 + a_1 \hat{t}_* = b_0 + b_1 \hat{t}_*$$

$$\hat{t}_* = \frac{b_0 - a_0}{a_1 - b_1} .$$

Then according to the previous section, confidence intervals with confidence ϵ for t_* are given by the solutions A to the second degree equation

$$u^2(A) = [P_2(A) + Q_2(A)] H^2 t_{1-\epsilon/2}^2$$

$$[(a_0 - b_0) + (a_1 - b_1)A]^2 = t_{1-\epsilon/2}^2 \hat{\sigma}^2 [r^{00} + 2r^{01}A + r^{11}A^2 + s^{00} + 2s^{01}A + s^{11}A^2] .$$

But in the solution for the a 's and b 's (sections 2.1 and 1.6)

$$(r_{ij}) = \begin{bmatrix} \sum 1 & \sum s_i \\ \sum s_i & \sum s_i^2 \end{bmatrix} \quad (\text{summations from 1 to } N_1)$$

$$(s_{ij}) = \begin{bmatrix} \sum 1 & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \quad (\text{summations from 1 to } N_2) .$$

Hence

$$(r^{ij}) = \frac{1}{\det(r_{ij})} \begin{bmatrix} \sum s_i^2 & -\sum s_i \\ -\sum s_i & N_1 \end{bmatrix}$$

$$(s^{ij}) = \frac{1}{\det(s_{ij})} \begin{bmatrix} \sum t_i^2 & -\sum t_i \\ -\sum t_i & N_2 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = (r^{ij}) \begin{bmatrix} \sum x_i \\ \sum x_i s_i \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = (s^{ij}) \begin{bmatrix} \sum y_i \\ \sum y_i t_i \end{bmatrix}$$

$$\hat{\sigma}^2 = \frac{1}{N_1 + N_2 - 4} [\sum (x_i - a_0 - a_1 s_i)^2 + \sum (y_i - b_0 - b_1 t_i)^2] .$$

We now present a numerical example.

Example. Consider two normal random variables X and Y having true branch regressions

$$\begin{aligned} E[X] = \mu_1 &= 2 - 2t & t \leq 0 \\ E[Y] = \mu_2 &= 2 + t & t \geq 0 \end{aligned} .$$

Obviously $t_* = 0$. Assume $\sigma_1^2 = \sigma_2^2$.

Using tables of normal(0,1) random numbers, samples of X and Y were constructed. See Table 1. The regression analysis was performed and t_* estimated. In the following results $N_1 = 11$ and $N_2 = 10$.

The sample regressions obtained were

$$\begin{aligned} m_1 &= 1.01 - 2.06 t \\ m_2 &= 2.50 + 0.98 t \end{aligned} .$$

Equating $m_1 = m_2$, the point estimate for the abscissa of intersection

was

$$\hat{t}_* = -0.49.$$

The sample variances were

$$\hat{\sigma}_1^2 = 1.1025$$

$$\hat{\sigma}_2^2 = 0.5041$$

$$\hat{\sigma}^2 = 0.8212 \quad (\text{pooled estimate from } \sigma_1^2 \text{ and } \sigma_2^2).$$

For $\epsilon = .05$, $t_{.975} = 2.11$. The equation for confidence limits A was

$$[(-1.49) + (-3.04)A]^2 = (2.11)^2(.8212)\left(\frac{385}{1210} + \frac{2(55)}{1210}A + \right.$$

$$\left. \frac{11}{1210}A^2 + \frac{385}{825} + \frac{2(-55)}{825}A + \frac{10}{825}A^2\right)$$

$$2.22 + 9.06A + 9.22A^2 = (4.46)(.8212)(.776 - .041A + .130A^2)$$

$$-.62 + 9.21A + 9.70A^2 = 0$$

$$A = -1.01 \text{ and } .07.$$

The numbers used in the calculations were obtained from Table 2.

Now to analyze the results. By taking $N_1 = 11$ and $N_2 = 10$ we are saying that we know t_* is in the interval $I : [0,1]$. For this choice of N_1 and N_2 the point estimate was $\hat{t}_* = -0.49$, while the 95% confidence intervals was $[-1.01 \leq t_* \leq 0.07]$.

Are these results contradictory or helpful?

The results are extremely helpful because they suggest a way to incorporate certain types of non-statistical information into an estimation procedure. In general (but using current notation) suppose

we want to estimate a constant t_* subject to the known condition that t_* is restricted to some interval I . Let $\hat{t}_*$ denote an estimator of t_* whose values may or may not lie in I . Note that I may be regarded as a 100% confidence interval. Let J denote a randomly determined confidence interval for t_* having confidence $(1 - \epsilon)100\%$. The events " I covers t_* " and " J covers t_* " are independent. Define $G = I \cap J$. Then

$$P(G \text{ covers } t_*) = (1.)(1 - \epsilon) = 1 - \epsilon.$$

If the interval G is non-empty it is an improved confidence interval with coefficient $1 - \epsilon$. If G is empty, the situation corresponds to the ϵ proportion of the samples for which a confidence interval fails to cover the true parameter. Another interpretation is that the a priori information that I covers t_* is better than the " J covers t_* " information obtained from the sample by the M.L. procedures.

A point estimator for t_* can now be associated with G . From elementary statistics recall that point estimators are usually contained in associated confidence intervals. In fact, if the confidence intervals were somehow derived first, one could define the point estimator as the limit of confidence intervals as their lengths approach zero. This is the logic to be used here, using the improved confidence intervals.

The number ϵ can be taken small enough so that $I \cap J$ is not empty. Then determine $\tilde{t}_*$ as the limiting abscissa of $I \cap J$ as $(1 - \epsilon)$ approaches zero. Since I is fixed, this value is the

end point of J closest to the intersection $\hat{t}_*$. The estimator for t_* is thus

$$\tilde{t}_* = \begin{cases} \hat{t}_* & \text{(the M.L. estimate) if } \hat{t}_* \text{ is in } I \\ \text{end point of } I \text{ closest to } \hat{t}_* & \text{if } \hat{t}_* \text{ is not in } I. \end{cases}$$

It should be noted that the idea of defining the point estimate as the limit of the improved confidence interval produces the M.L. estimate when $\hat{t}_*$ is in I .

To estimate x_* , minimize the weighted sum of squared residuals from $\tilde{x}_*$ to $\hat{\mu}_1(\tilde{t}_*)$ and $\hat{\mu}_2(\tilde{t}_*)$. Thus consider

$$R = \frac{1}{\sigma_1^2} [\tilde{x}_* - \hat{\mu}_1(\tilde{t}_*)]^2 + \frac{1}{\sigma_2^2} [\tilde{x}_* - \hat{\mu}_2(\tilde{t}_*)]^2.$$

Differentiating with respect to x_* , equating to zero and solving gives

$$\tilde{x}_* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \hat{\mu}_1(\tilde{t}_*) + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \hat{\mu}_2(\tilde{t}_*).$$

If $\hat{t}_*$ falls in I this estimate is the abscissa of intersection and hence agrees with M.L. estimator.

Returning to our example we obtain the improved, truncated estimate

$$\tilde{t}_* = 0.0$$

and the improved 95% confidence interval

$$I \cap J : 0 \leq t \leq 0.07.$$

We couldn't expect all results with improved confidence intervals and truncated estimators to be this good.

Sample Regression Data

<u>i</u>	<u>x_i, t_i</u>	<u>$\varepsilon_i, N(0,1)$</u>	Linear Case		Quadratic Case	
			k = 1		k = 2	
			<u>μ</u>	<u>X and Y</u>	<u>μ</u>	<u>X and Y</u>
1	-10	-2.052	22	19.948	122	119.948
2	- 9	-1.378	20	18.622	101	99.622
3	- 8	1.453	18	19.453	82	83.453
4	- 7	0.355	16	16.355	65	65.355
5	- 6	-0.636	14	13.364	50	49.364
6	- 5	-0.577	12	11.423	37	36.423
7	- 4	-1.026	10	8.974	26	24.974
8	- 3	0.124	8	8.124	17	17.124
9	- 2	-1.299	6	4.701	10	8.701
10	- 1	-1.094	4	2.906	5	3.906
11	0	-1.697	2	0.303	2	0.303
12	1	0.798	3	3.798	5	5.798
13	2	-0.331	4	3.669	12	11.669
14	3	0.446	5	5.446	23	23.446
15	4	0.759	6	6.759	38	38.759
16	5	1.673	7	8.673	57	58.673
17	6	-0.341	8	7.659	80	79.659
18	7	-0.534	9	8.466	107	106.466
19	8	0.637	10	10.637	138	138.637
20	9	0.054	11	11.054	173	173.054
21	10	0.661	12	12.661	212	212.661

Table 1

Calculations

For the a_i

$$N_1 = 11$$

$$\sum 1 = 11$$

$$\sum s_i = -55$$

$$\sum s_i^2 = 385$$

$$\sum s_i^3 = -3025$$

$$\sum s_i^4 = 25333$$

For the b_i

$$N_2 = 10$$

$$\sum 1 = 10$$

$$\sum t_i = 55$$

$$\sum t_i^2 = 385$$

$$\sum t_i^3 = 3025$$

$$\sum t_i^4 = 25333$$

Linear Case

$$\sum x_i = 124.173$$

$$\sum s_i x_i = -847.062$$

$$\sum s_i^2 x_i = 6554.658$$

$$\sum x_i^2 = 1876.856345$$

$$\sum y_i = 78.822$$

$$\sum t_i y_i = 514.283$$

$$\sum t_i^2 y_i = 3925.257$$

$$\sum y_i^2 = 704.421334$$

Quadratic Case

$$\sum x_i = 509.173$$

$$\sum s_i x_i = -3872.062$$

$$\sum s_i^2 x_i = 31877.658$$

$$\sum x_i^2 = 40319.172345$$

$$\sum y_i = 78.822$$

$$\sum t_i y_i = 6564.283$$

$$\sum t_i^2 y_i = 54591.257$$

$$\sum y_i^2 = 117,737.449334$$

Table 2

2.3.2 Explicit Results for Polynomials of 2nd Degree

Consider the assumptions that the variances are equal, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, and that both branch regressions are of second degree. Then t_* is defined as the intersection $\mu_1 = \mu_2$:

$$a_0 + a_1 t_* + a_2 t_*^2 = b_0 + b_1 t_* + b_2 t_*^2 \quad .$$

The point estimate is obtained by equating the sample regressions

$$a_0 + a_1 \hat{t}_* + a_2 \hat{t}_*^2 = b_0 + b_1 \hat{t}_* + b_2 \hat{t}_*^2 \quad .$$

Confidence limits with confidence $1-\alpha$ for both roots are obtained from the solutions A of the fourth degree polynomial equation (with $c_j = a_j - b_j$)

$$[c_0 + c_1 A + c_2 A^2]^2 = t_{1-\alpha/2}^2 \hat{\sigma}^2.$$

$$[(r^{00} + s^{00}) + (r^{11} + s^{11})A^2 + (r^{22} + s^{22})A^4 + 2(r^{01} + s^{01})A + 2(r^{02} + s^{02})A^2 + 2(r^{12} + s^{12})A^3] \quad .$$

In this case, with summations from 1 to N_1

$$(r_{ij}) = \begin{bmatrix} \sum 1 & \sum s_i & \sum s_i^2 \\ \sum s_i & \sum s_i^2 & \sum s_i^3 \\ \sum s_i^2 & \sum s_i^3 & \sum s_i^4 \end{bmatrix} \quad .$$

Similarly the matrix (s_{ij}) is formed from the observed t 's, where the summations would run from 1 to N_2 .

From the theory of determinants,

$$(r^{ij}) = \frac{1}{|(r_{ij})|} \begin{bmatrix} (r_{11}r_{22}-r_{12}r_{21}) & -(r_{01}r_{22}-r_{02}r_{21}) & (r_{01}r_{12}-r_{11}r_{02}) \\ -(r_{10}r_{22}-r_{20}r_{12}) & (r_{00}r_{22}-r_{20}r_{02}) & -(r_{00}r_{12}-r_{10}r_{02}) \\ (r_{10}r_{21}-r_{11}r_{20}) & -(r_{00}r_{21}-r_{20}r_{01}) & (r_{00}r_{11}-r_{10}r_{01}) \end{bmatrix}.$$

Similarly (s^{ij}) is obtained using the elements of (s_{ij}) .

Then

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [r^{ij}] \begin{bmatrix} \sum x_i \\ \sum x_i s_i \\ \sum x_i s_i^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = [s^{ij}] \begin{bmatrix} \sum y_i \\ \sum y_i t_i \\ \sum y_i t_i^2 \end{bmatrix}.$$

The unbiased variance estimate is given by

$$\hat{\sigma}^2 = \frac{1}{N_1 + N_2 - 6} \left\{ \sum_{i=1}^{N_1} [x_i - \mu_1(s_i)]^2 + \sum_{i=1}^{N_2} [y_i - \mu_2(t_i)]^2 \right\}.$$

Example. Let the true regressions for X_s and Y_t be

$$\begin{aligned} \mu_1 &= 2 - 2t + t^2 & t \leq 0 \\ \mu_2 &= 2 + t + 2t^2 & t > 0. \end{aligned}$$

Obviously $t_* = 0$. The true variance is one, but this is assumed unknown.

Consider the sample of size $N = 21$ given in Table 1. Then with $N_1 = 11$ (the s_i are $-10, -9, \dots, -1, 0$) and $N_2 = 10$:

$$(r_{ij}) = \begin{bmatrix} 11 & -55 & 385 \\ -55 & 385 & -3025 \\ 385 & -3025 & 25333 \end{bmatrix}.$$

Using the results in Table 2 we obtain

$$\det (r_{ij}) = 1038180$$

$$(r^{ij}) = \frac{1}{1038180} \begin{bmatrix} 602580 & 228690 & 18150 \\ 228690 & 130438 & 12100 \\ 18150 & 12100 & 1210 \end{bmatrix} .$$

In a similar manner

$$(s_{ij}) = \begin{bmatrix} 10 & 55 & 385 \\ 55 & 385 & 3025 \\ 385 & 3025 & 25333 \end{bmatrix}$$

$$\det (s_{ij}) = 435600$$

$$(s^{ij}) = \frac{1}{435600} \begin{bmatrix} 602580 & -228690 & 18150 \\ -228690 & 105105 & -9125 \\ 18150 & -9125 & 825 \end{bmatrix} .$$

From these results we obtain

$$\hat{\mu}_1(t) = -0.101 - 2.795t + 0.926t^2$$

$$\hat{\mu}_2(t) = 2.591 + 0.933t + 2.004t^2$$

$$\hat{\sigma}^2 = 0.9631 .$$

Equating the sample regressions and solving gives

$$t_*^{(1)} = -0.54 \quad \text{and} \quad t_*^{(2)} = -2.4 .$$

Let $\epsilon = .05$; then $t_{.975} = 2.11$. The equation defining

end points A for the confidence intervals is

$$[2.692 + 3.728A + 1.078A^2]^2 = (2.11)^2 (.9631) \cdot$$

$$[0.5804 + 0.4406A + 0.1256A^2 + 0.0234A^3 + 0.0012A^4 \\ + 1.3833 - 1.0500A + 0.2413A^2 - 0.0416A^3 + 0.0019A^4] ,$$

or

$$-1.1730 + 22.6846A + 17.5781A^2 + 8.1157A^3 + 1.1489A^4 = 0 .$$

The solutions obtained are

$$A = 0.06 , \quad -5.42 , \quad \text{and two complex roots.}$$

As in the linear example the situation arises where the M.L. estimator doesn't fall in the known interval I , which in this case is $[0,1]$. Nevertheless these results provide a great deal of valuable information. The graph of $u(A) = 0$ is of the nature given in figure 2. The graph of $u^2(A) = 0$ is therefore of the nature given in figure 3. By equating $u^2(A)$ to a positive polynomial, as our derivation requires, we obtain

$$u^2(A) = [h^2P + Q] \cdot H^2 t^2 .$$

This function is shown in figure 3. For the above example it is geometrically clear that this equation defines two complex roots and hence the real roots obtained must be a lower bound for the smallest root and an upper bound for the largest root. Thus the 95% confidence intervals are undefined as closed intervals. Instead we obtain two 97.5% one-sided confidence intervals: $(-5.4, \infty)$ for $t_{*}^{(2)}$ and $(-\infty, 0.06)$ for $t_{*}^{(1)}$. Using the concepts introduced

Figure 2

Figure 3

in section 2.3.1 we readily obtain the 97.5% improved confidence interval $(0.0, 0.06)$. The associated point estimate is $\tilde{t}_* = 0.0$.

2.3.3 General Theorem for Polynomials of k-th Degree

The results derived from the previous two examples can be summarized and extended to any degree polynomial assumptions in the following procedural theorem.

Theorem. Let $\hat{t}_*$ be a solution to the equation $\hat{\mu}_1(\hat{t}_*) = \hat{\mu}_2(\hat{t}_*)$ and suppose $\hat{t}_*$ can be identified with a true value t_* known to be in I , the interval between the sample of $(x_1, \dots, x_{N_1})$ and the sample of $(y_1, \dots, y_{N_2})$. For ϵ sufficiently small construct a confidence interval J which intersects I . Then $I \cap J$ is a $(1-\epsilon)100\%$ confidence interval for t_* , and the limit of $I \cap J$ as ϵ approaches 1 defines a point estimator $\tilde{t}_*$. Let

$$\tilde{x}_* = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2} \hat{\mu}_1(\tilde{t}_*) + \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2} \hat{\mu}_2(\tilde{t}_*).$$

If $\hat{t}_*$ falls in I then $(\tilde{t}_*, \tilde{x}_*)$ are the maximum likelihood estimators for (t_*, x_*) . If $\hat{t}_*$ does not fall in I then $(\tilde{t}_*, \tilde{x}_*)$ are merely truncated estimates of (t_*, x_*) .

This solution may require a detailed study of the t_* -estimator polynomial in conjunction with the confidence interval polynomial in order to determine which boundaries correspond to which solutions. If difficulties are still encountered, consult section 2.4.3.

2.4 Discussion of Practical Difficulties

The two theorems of sections 2.2 and 2.3 will suffice in many of the practical problems which arise. Yet, the difficulties encountered in the most recent example are symptomatic of difficulties which could occur in (but hopefully) a minority of the problems. We shall discuss these problems together with potential remedies.

It is not possible to foresee all difficulties which might arise, but at least a partial list can be given:

- 1) $\hat{\mu}_1$ and $\hat{\mu}_2$ have more than one intersection in I
- 2) $\hat{\mu}_1$ and $\hat{\mu}_2$ have real intersections, but none in I
- 3) $\hat{\mu}_1$ and $\hat{\mu}_2$ have no real intersections at all
- 4) the confidence interval polynomial fails to have $2k$ real roots, and/or the real roots which do occur are not readily identifiable with any intersection point t_* .

2.4.1 Examination of the Assumptions and the Sample

If difficulties do arise, some general inquiry should first be made.

Is our likelihood formulation correct ? In view of earlier discussions, the formulation is correct.

Are our assumptions correct ? Are our true regressions polynomials ? If not, this may be our source of difficulty and we should then study regression formulations not involving polynomials. Since the M.L. estimator requires extrapolation into an interval I , it could be that even though a polynomial approximation is sufficiently

accurate in the given interval of s 's and t 's , that it is inaccurate in I . To decrease such chances of failure, only low degree polynomials should be used. It could be that polynomial assumptions are adequate but that we have hypothesized the wrong degree polynomial. The degree can readily be changed. Note that odd degree polynomials will at least guarantee one real intersection.

If we are satisfied that our assumptions are valid enough, then the other source of difficulty must be the sample. If the sample regressions just barely fail to intersect, it could be that one entire set of data, either the x 's or y 's (or, in fact, both), is in error by a constant bias arising from the method of measurement. This situation should be studied and, if true, remedied. Otherwise the entire sample may be bad. Recalling that every sample value has the typical form

$$x_i = \mu_1(s_i) + \epsilon_i$$

this means that the collection of ϵ 's is unrepresentative of a normal population with zero mean. An attempt could be made to average out this difficulty. Using $\hat{\sigma}_1^2$ as a guide, construct a normal sample (λ_1) of size N_1 having zero means and (say) variance equal to $0.5\hat{\sigma}_1^2$. Then calculate an adjusted sample

$$x_i' = x_i + \lambda_i, \quad i = 1, \dots, N_1$$

The new sample admittedly has larger variance but it may also produce a $\hat{\mu}_1'$ which would intersect with $\hat{\mu}_2$ in or near enough to I . This technique could not be used repeatedly because the variance

estimates would grow too large; but one or two trials might produce desirable results. An attempt could also be made to edit the sample and remove observations producing significantly large residuals from $\hat{\mu}_1$ or $\hat{\mu}_2$. Then new estimates of μ_1 and μ_2 should be calculated from the reduced sample and the M.L. procedure performed again. In editing one must decide with some certainty that the observations being removed are indeed unrepresentative of the sample and not suggestive of some unforeseen change of events in the entire process.

The fact that difficulties arise is not altogether surprising when we consider that the estimator for t_* is based upon extrapolation. From a prediction standpoint we are thus working outside the most favorable regions, namely those intervals of definition for the members in the sample. Furthermore, in each of the two examples cited, the true solution was an end-point of I . We would expect estimates to fall on either side of t_* , so when our estimator falls barely on the left of $t = 0$ (in the examples) and hence outside I , it seems bad but actually is not surprising.

The cause of our difficulty is thus the bad sample. In simpler estimation problems a maximum likelihood procedure may well give a bad value of a good estimator. In our estimation problem the M.L. procedure may fail to give any value (in case of sufficiently bad samples) for an otherwise good estimator.

If after study of the formulation, assumption and the sample it is felt that nothing more can be done with these aspects of the situation, and that we must "live" with our sample results, good or bad, the problem is now how to derive the maximum amount of

information in view of the potential difficulties 1) to 4) .

2.4.2 More Than One Solution in I

Suppose that more than one real solution, $\hat{t}_*$, is found in I . Since L has already been maximized (and independently of I) there is no point in checking each $\hat{t}_*$ to see which gives the largest L . There is nothing to check. In the absence of additional information concerning which root to choose, some elementary random selection scheme should be used to select a $\hat{t}_*$.

2.4.3 Real Solutions, But None in I

This problem has been partially discussed in section 2.3 for the case when an estimate $\hat{t}_*$ falling outside I is sure to correspond to a true t_* in I and when the appropriate confidence intervals can be constructed. However, it could happen that the $\hat{t}_*$ obtained is clearly not associated with any t_* in I (as in figure 4 with degree 3 for the estimator polynomial) or that the confidence intervals could not be constructed (as in figure 5 with degree 6 for the confidence interval polynomials).

Thus, for our interests there are effectively no real roots at all. Alternative procedures for this section will therefore be treated in the next section.

2.4.4 No Real Solutions in I

Suppose that either there are no real solutions $\hat{t}_*$ at all, or that whatever real solutions are found clearly do not correspond to

Figure 4

Figure 5

a true t_* in I . Our approach will be to define an estimation procedure which will agree with maximum likelihood procedure when M.L. techniques produce a useful $\hat{t}_*$ and which will otherwise produce a sensible estimate with desirable properties. The intention is to procede in the spirit of M.L., though not quite in fact.

Consider the region R contained between the vertical lines $t = s_{N_1}$ and $t = t_1$ as illustrated in figure 6. Consider the plausibility of seeking as a solution a point $P(\hat{t}_*, \tilde{x}_*)$ which maximizes an associated likelihood type function

$$L_* = L \cdot \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{1}{2\sigma_1^2}[x_* - \sum \alpha_j t_*^j]^2} \cdot \frac{1}{\sqrt{2\pi} \sigma_2} e^{-\frac{1}{2\sigma_2^2}[x_* - \sum \beta_j t_*^j]^2}.$$

The definition of L_* is motivated by the thought of finding an optimal point P which could have been included in the estimate of both μ_1 and μ_2 . This idea will be referred to as the quasi-likelihood criterion, (Q.L.). The Q.L. function L_* is the ordinary M.L. function L multiplied by two distribution factors of the unknown point (t_*, x_*) ; Q.L. estimates will typically be denoted $\tilde{a}_1$. The sole purpose of Q.L. is to obtain a method of estimating (t_*, x_*) and not to get better estimates of the α 's, β 's and σ 's. The M.L. estimates for the latter parameters always exist and cannot be improved upon. For interpolation prediction the M.L. estimates should always be used, not the Q.L. estimates.

Treating L_* as a likelihood function, the estimates for $(\alpha_0, \dots, \beta_k, \sigma_1^2, \sigma_2^2, t_*, x_*)$ can be obtained by maximizing $\log L_*$.

Figure 6

But looking ahead, the estimates of σ_1^2 and σ_2^2 would involve the unknown $(\tilde{t}_*, \tilde{x}_*)$. This leads to serious computational difficulties because the $\hat{x}_*$ in turn depends on the σ 's. A reasonable alternative is to use the M.L. estimates (which don't involve $\tilde{t}_*$ or $\tilde{x}_*$) for σ_1 and σ_2 . Thus the σ 's in L_* are regarded as known constants (as the M.L. estimates). Then finding $\log L_*$ and differentiating gives the following system, where $r = 0, 1, \dots, k$ for the α 's and $r = 0, 1, \dots, q$ for the β 's.

$$\begin{aligned} \log L_* = & -N_1 \log 2\pi - N_1 \log \sigma_1 - \frac{1}{2\sigma_1^2} \sum_{i=1}^{N_1} [x_i - \sum_j \alpha_j s_i^j]^2 \\ & -N_2 \log 2\pi - N_2 \log \sigma_2 - \frac{1}{2\sigma_2^2} \sum_{i=1}^{N_2} [y_i - \sum_j \beta_j t_i^j]^2 \\ & - \log 2\pi - \log \sigma_1 - \frac{1}{2\sigma_1^2} [x_* - \sum_j \alpha_j t_*^j]^2 \\ & - \log 2\pi - \log \sigma_2 - \frac{1}{2\sigma_2^2} [x_* - \sum_j \beta_j t_*^j]^2 \end{aligned}$$

$$(1) \quad \frac{\partial \log L_*}{\partial \alpha_r} = -\frac{1}{2\sigma_1^2} \sum 2[x_i - \sum_j \alpha_j s_i^j](-s_i^r) - \frac{1}{2\sigma_1^2} 2[x_* - \sum_j \alpha_j t_*^j](-t_*^r) = 0$$

$$(2) \quad \frac{\partial \log L_*}{\partial \beta_r} = -\frac{1}{2\sigma_2^2} \sum 2[y_i - \sum_j \beta_j t_i^j](-t_i^r) - \frac{1}{2\sigma_2^2} 2[x_* - \sum_j \beta_j t_*^j](-t_*^r) = 0$$

$$(3) \quad \frac{\partial \log L_*}{\partial x_*} = -\frac{1}{2\sigma_1^2} (2)[x_* - \sum_j \alpha_j t_*^j] - \frac{1}{2\sigma_2^2} (2)[x_* - \sum_j \beta_j t_*^j] = 0$$

$$(4) \quad \frac{\partial \log L_*}{\partial t_*} = -\frac{1}{2\sigma_1^2} \cdot 2[x_* - \sum_j \alpha_j t_*^j](-\sum_j \alpha_j t_*^{j-1}) - \frac{1}{2\sigma_2^2} 2[x_* - \sum_j \beta_j t_*^j] \sum_j \beta_j t_*^{j-1} = 0$$

From (3) the estimate for $\tilde{x}_*$ is immediate.

$$\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right] \tilde{x}_* = \frac{1}{\sigma_1^2} \tilde{\mu}_1(\tilde{t}_*) + \frac{1}{\sigma_2^2} \tilde{\mu}_2(\tilde{t}_*)$$

$$\tilde{x}_* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tilde{\mu}_1(\tilde{t}_*) + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \tilde{\mu}_2(\tilde{t}_*) .$$

This solution for $\tilde{x}_*$ requires that $\tilde{t}_*$ and all the α 's and β 's be known. If $\sigma_1 = \sigma_2$ then $\tilde{x}_*$ is the numerical average of the two regressions $\hat{\mu}_1$ and $\hat{\mu}_2$ at any abscissa. If the sample regressions intersect then $\tilde{x}_*$ is the ordinate of this intersection. Note that (3) is of the form $A + B = 0$. One possible solution for this is $A = B = 0$. This means that (1) and (2) reduce to the original M.L. normal equations. This is not surprising since $(\tilde{t}_*, \tilde{x}_*)$ contribute exactly zero to the various sums of squares. Thus the Q.L. estimates are the M.L. estimates and $\tilde{t}_*$ is obtained as the intersection of the sample regressions if the intersection is in R . On the other hand if the sample regression intersects in R , then necessarily $A = B = 0$. This result, which serves as a consistency check on the value of Q.L. formulation, is summarized in the following theorem.

Theorem. The quasi-likelihood estimate for (t_*, x_*) is identical to the maximum likelihood estimate for (t_*, x_*) if and only if the sample regressions intersect in R , a vertical region of known confinement for t_* .

The other possible solution for (3) is that $A + B = 0$ with

neither being zero. This corresponds to non-intersecting sample regressions. In this case, and in the case where $\tilde{t}_*$ obviously does not correspond to R , we are virtually forced to maximize L_* by trial and error. This means that a choice of $(\tilde{t}_*, \tilde{x}_*)$ be made, α 's and β 's calculated, and the corresponding value of L_* be recorded. It is equivalent and easier to record merely

$$H = \sigma_2^2 [\tilde{x}_* - \hat{\mu}_1(\tilde{t}_*)]^2 + \sigma_1^2 [\tilde{x}_* - \hat{\mu}_2(\tilde{t}_*)]^2 .$$

These operations would have to be performed over an entire grid work of reasonable points in R . A first reasonable choice might well be the abscissa in R where the M.L. regressions $\hat{\mu}_1$ and $\hat{\mu}_2$ are closest, with $\tilde{x}_*$ calculated from (3). If near stabilization of the graphs of the H 's is observed, then the t_* in R corresponding to smooth or boundary minima of H should be tried. After all, L_* is a polynomial in $\tilde{t}_*$, and the actual $\tilde{t}_*$ must occur at a smooth extrema or a boundary abscissa of R . In fact the Q.L. estimation problem for t_* is that of finding a point which when used in the estimation of Q.L. regressions will produce an H function having that very point as an extrema. It seems geometrically obvious that at least one such point should exist in R , but finding it leads to the difficulties we have just discussed. Eventually the point producing the smallest H is to be taken as (an approximation to) the Q.L. estimate.

It might well be asked: 'why bother with the Q.L. estimates... why not merely use an abscissa in R , call it t_c , where $\hat{\mu}_1$ and $\hat{\mu}_2$ are closest ? ' The answer involves refinement, since a little

reflection reveals a shortcoming of the latter suggestion. Suppose that all observations $(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2-1})$ except y_{N_2} are error free and lie on μ_1 and $\hat{\mu}_2$ respectively. Then $\hat{\mu}_2$ can be changed merely by raising and lowering y_{N_2} and can thus be made to intersect μ_1 in R at an abscissa denoted by $\hat{t}_*$. But in manipulating y_{N_2} all sample regressions coefficients $(b_0, \dots, b_q)$ would change and $\hat{t}_*$ would rarely coincide with t_c . It should be noted that if the sample regressions intersect, then this method of closest regression recovers the M.L. estimates.

In summary, the justifications for using the Q.L. procedure are therefore (1) that Q.L. reduces to M.L. when P is in R , (2) that Q.L. is formulated by a logic similar to M.L., and (3) that Q.L. seems better than the computationally easier method of closest regressions in R .

No final distribution theory is yet available for the Q.L. estimator other than when Q.L. and M.L. agree.

2.4.5 The Method of Intersecting Confidence Regions

Another idea investigated was that of constructing confidence bands B_1 and B_2 of equal confidence around the M.L. regressions $\hat{\mu}_1$ and $\hat{\mu}_2$ and then taking the intersection C of B_1 , B_2 , and R (a known region of confinement for the true $P(t_*, x_*)$) as a confidence region for (t_*, x_*) . A point estimate could then be defined as the limit of C as the common confidence of B_1 and B_2 goes to zero.

This method was tried on the linear and quadratic examples of

section 2.3 . The results were not at all unreasonable and are shown in figures 7 and 8 .

This method was not investigated further since it seems that for larger degree k the confidence bands B_1 and B_2 (which are generalizations of hyperbolas) might double back too rapidly and hence not always define a bounded intersection C . The bands B_1 and B_2 tend to shy away from $\hat{\mu}_1$ and $\hat{\mu}_2$ at the end points, but that is sub-region where we are most interested.

In this method, if C were bounded and if the sample M.L. regressions were to intersect in R , then this estimate for (t_*, x_*) would agree with the M.L. estimate.

2.5 A Jump at the Intersection

In the previous section we have assumed that the branch regressions intersect so that t_* is defined as the abscissa such that $\mu_1(t_*) = \mu_2(t_*)$. In cases where μ_1 and μ_2 differ by a constant λ at t_* there may or may not be a solution.

If λ is unknown, the problem is ambiguous because t_* cannot be uniquely defined. Any t value in the interval $I: [s_{N_1}, t_{N_1+1}]$ will have a λ value corresponding to it and thus any (t, λ_t) would define a t_* .

If λ is known, there may or may not be a solution, depending on whether a unique t_* can be defined. It is geometrically obvious that if μ_1 and μ_2 are extrapolated throughout the subdomain R , and if in I the two branches are non-intersecting but parallel over any portion of it, then no unique t_* can be defined. If the two branches behave differently in I , then t_* is defined by $\mu_1(t_*) = \mu_2(t_*) + \lambda$.

Figure 7

Figure 8

2.6 N_1 and N_2 Not Known

Suppose that N_1 and N_2 are not known, although $N = N_1 + N_2$ is given. Then L is a function not only of the continuous variables $(\alpha_0, \dots, \alpha_r, \beta_0, \dots, \beta_s, \sigma_1^2, \sigma_2^2)$ but also of the single discrete variable N_1 . The number N_2 is known if N and N_1 are known. After obtaining the usual estimates for the continuous parameters our logic should be to further maximize L by proper choice of N_1 and N_2 . This can be done by examining the value of L for all choices of N_1 and N_2 . This means that for every choice of (N_1, N_2) a corresponding set of $(\alpha_0, \dots, \alpha_r, \beta_0, \dots, \beta_s, \sigma_1^2, \sigma_2^2)$ must be calculated and L computed. The pair (N_1, N_2) which maximizes L is picked as the solution and the corresponding parameter estimates are used to estimate t_* as in previous sections.

Example. In the example of section 2.3 there were 21 sample pairs; we will now be working under the assumption that we don't know that $(N_1 = 11, N_2 = 10)$ is the correct division between X's and Y's. It is still assumed that $\sigma_1^2 = \sigma_2^2$. The (N_1, N_2) choices investigated were $(17, 4), (16, 5), (15, 6), \dots, (6, 15), (5, 16), (4, 17)$. The smallest sample variance was .8212, corresponding to the choice $(N_1, N_2) = (11, 10)$, which corresponds to the t -interval $[0, 1]$. As seen before, the point estimate was $\hat{t}_* = -0.49$ and the 95% improved confidence interval was $[0.0, 0.07]$.

Ordinarily it is not necessary to try all possible pairs,

since by visual inspection of the graph of the observations one could delimit a reasonable range of pairs to try.

In making the choice of (N_1, N_2) based on maximum L there is always a chance of error in the selection. The most pertinent questions concern how good any procedure is in face of error. In problems of testing hypotheses these questions are answered by calculating the power of a given test. In problems of interval estimation these questions can be answered as follows. If under assumptions H we construct a confidence interval I having confidence $1-\alpha$, then under some alternative assumptions A we must calculate an alternative confidence coefficient $1-\beta$ for the I determined by H .

Now in our general problem with (N_1, N_2) known, the solution for estimating t_* is to form the intersection G of the 100% confidence interval $I: [s_{N_1}, t_1]$ with a $(1-\epsilon)100\%$ confidence interval J obtained from a sample statistic. But with (N_1, N_2) not exactly known, and hence estimated by maximizing L , the confidence for I , and hence for G , is reduced. Our next task is to discuss this reduction. This will be done by finding the probability that the likelihood computed under one choice of (N_1, N_2) exceeds the likelihood calculated under another choice of (N_1, N_2) . Consider first the two sample problem without regression assumptions.

2.6.1 No Regression Assumptions

Let $(x_1, \dots, x_{N_1})$ be an independent sample on a normal random variable X with unknown mean μ_1 and unknown variance σ^2 .

Let $(y_1, \dots, y_{N_2})$ be an independent sample on a normal random variable Y with unknown mean μ_2 and the same unknown variance σ^2 . Suppose these observations have occurred at ordered times $(t_1, \dots, t_{N_1}, t_{N_1+1}, \dots, t_{N_1+N_2})$. The t 's are not needed in the calculations but we wish to establish a regression point of view since we are searching for a time interval in which a change of regression takes place.

The likelihood function of the sample is

$$L(\mu_1, \mu_2, \sigma^2, N_1) = \prod_{i=1}^{N_1} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2\sigma^2} [x_i - \mu_1]^2} \cdot \prod_{i=1}^{N_2} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2\sigma^2} [y_i - \mu_2]^2}.$$

Suppose the choice $N_1 = m$ has produced maximum L . The estimates are

$$\hat{N}_1 = m$$

$$\hat{N}_2 = N - m = n$$

$$\hat{\mu}_1 = \bar{x}_m = \frac{1}{m} \sum_{i=1}^m x_i$$

$$\hat{\mu}_2 = \bar{y}_n = \frac{1}{n} \sum_{i=1}^n y_i$$

$$\hat{\sigma}^2 = \frac{1}{m+n} \left\{ \sum_{i=1}^m [x_i - \bar{x}_m]^2 + \sum_{i=1}^n [y_i - \bar{y}_n]^2 \right\}.$$

The value of the likelihood function is thus

$$L = \left(\frac{1}{\sqrt{2\pi} \hat{\sigma}} \right)^{m+n} e^{-\frac{1}{2}(m+n)^2} .$$

By the likelihood principle we would then assert that the most probable interval for the change of regression is $I: [t_m, t_{m+1}]$.

Let us determine the probability of this decision, namely,

$$P = \Pr(L_m > L_{m'} \mid \text{true } N_1 = m) ,$$

where L_m denotes the likelihood computed when $N_1 = m$ observations are treated as x 's. Thus $L_{m'}$ is the likelihood computed under some other choice of N_1 . Notice that if the true N_1 equals m , and if $N_1 \neq m$ is used in $L_{m'}$, then there is a small group of interior data points all on either side of x_m which have been misclassified as x 's or y 's. Denote these points by $z_1, \dots, z_s$. On the side away from the misclassification all classifications are correct. Thus whenever $N_1 = m' \neq m$ is used, the sample is being treated as having one of two forms:

$$(x_1, \dots, x_r, z_1, \dots, z_s, y_1, \dots, y_n) ,$$

where $r + s = m$ and the z 's are treated as y 's, or as

$$(x_1, \dots, x_m, z_1, \dots, z_s, y_1, \dots, y_r) ,$$

where $r + s = n$ and the z 's are treated as x 's. At this stage we must establish a useful decomposition identity.

Let there be given $(x_1, \dots, x_r, z_1, \dots, z_s, y_1, \dots, y_n)$. Define

$$m = r + s$$

$$\bar{x}_{r+k} = \frac{1}{r+k} \left[\sum_{i=1}^r x_i + \sum_{i=1}^k z_i \right], \quad k = 0, 1, \dots, s$$

$$\bar{z}_k = \frac{1}{k} \sum_{i=1}^k z_i, \quad k = 0, 1, \dots, s$$

$$\bar{y}_{k+n} = \frac{1}{k+n} \left[\sum_{i=s-k}^s z_i + \sum_{i=1}^n y_i \right], \quad k = 0, 1, \dots, s$$

$$S_1^2 = \sum_r [x_i - \bar{x}_{r+s}]^2 + \sum_s [z_i - \bar{x}_{r+s}]^2 + \sum_n [y_i - \bar{y}_n]^2$$

$$S_2^2 = \sum_r [x_i - \bar{x}_r]^2 + \sum_s [z_i - \bar{y}_{s+n}]^2 + \sum_n [y_i - \bar{y}_{s+n}]^2.$$

Now

$$\begin{aligned} \sum_r [x_i - \bar{x}_{r+s}]^2 &= \sum_r [x_i - \frac{r}{m} \bar{x}_r - \frac{s}{m} \bar{z}_s]^2 \\ &= \sum_r [x_i - \frac{r+s}{m} \bar{x}_r + \frac{s}{m} (\bar{x}_r - \bar{z}_s)]^2 \\ &= \sum_r [(x_i - \bar{x}_r) + \frac{s}{m} (\bar{x}_r - \bar{z}_s)]^2 \\ &= \sum_r (x_i - \bar{x}_r)^2 + \frac{rs^2}{m^2} (\bar{x}_r - \bar{z}_s)^2. \end{aligned}$$

Similarly

$$\sum_s [z_i - \bar{x}_{r+s}]^2 = \sum_s (z_i - \bar{z}_s)^2 + \frac{sr^2}{m^2} (\bar{z}_s - \bar{x}_r)^2$$

$$\sum^s [z_i - \bar{y}_{s+n}]^2 = \sum^s (z_i - \bar{z}_s)^2 + \frac{sn^2}{(s+n)^2} (\bar{z}_s - \bar{y}_n)^2$$

$$\sum^n [y_i - \bar{y}_{s+n}]^2 = \sum^n (y_i - \bar{y}_n)^2 + \frac{ns^2}{(n+s)^2} (\bar{y}_n - \bar{z}_s)^2 .$$

Making these substitutions and adding proves the following lemma.

Lemma.

$$S_1^2 = \sum^r (x_i - \bar{x}_r)^2 + \sum^s (z_i - \bar{z}_s)^2 + \sum^n (y_i - \bar{y}_n)^2 + \frac{rs}{r+s} (\bar{x}_r - \bar{z}_s)^2$$

$$S_2^2 = \sum^r (x_i - \bar{x}_r)^2 + \sum^s (z_i - \bar{z}_s)^2 + \sum^n (y_i - \bar{y}_n)^2 + \frac{ns}{n+s} (\bar{y}_n - \bar{z}_s)^2 .$$

It will soon be convenient to refer to the first three terms of S_1^2 and S_2^2 as C .

Now consider the first form of the sample, thus assuming that the entire right side (y's) is always correctly classified. Let L_{n-s} be the likelihood obtained when s z's are treated as y's . The estimates in this case are

$$\hat{\mu}_1 = \bar{x}_r = \frac{1}{r} \sum_{i=1}^r x_i$$

$$\hat{\mu}_2 = \bar{y}_{n+s} = \frac{1}{n+s} \left[\sum_{i=1}^s z_i + \sum_{i=1}^n y_i \right]$$

$$\hat{\sigma}^2 = \frac{1}{r+s+n} \left[\sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=1}^s (z_i - \bar{y}_{n+s})^2 + \sum_{i=1}^n (y_i - \bar{y}_{n+s})^2 \right]$$

The value of the likelihood is

$$L_{m-s} = \left[\frac{1}{\sqrt{2\pi} \hat{\sigma}} \right]^{r+s+n} e^{-\frac{1}{2}(r+s+n) \hat{\sigma}^2}.$$

The probability of the decision criterion is

$$\begin{aligned} P &= \Pr(L_m > L_{m-s} \mid \text{true } N_1 = m) \\ &= \Pr(\hat{\sigma}^2 > \hat{\sigma}^2 \mid \text{true } N_1 = m) \\ &= \Pr(S_2^2 > S_1^2 \mid \text{true } N_1 = m) \\ &= \Pr(-1 < \frac{S_1}{S_2} < 1 \mid N_1 = m) \\ &= \Pr(-1 < \frac{\frac{rs}{r+s} (\bar{x}_r - \bar{z}_s)}{\frac{ns}{n+s} (\bar{y}_n - \bar{z}_s)} < 1 \mid N_1 = m) . \end{aligned}$$

The last equality is obtained by using the decomposition lemma.

In P the numerator is normally distributed with

$$\text{mean} = 0$$

$$\text{variance} = \frac{rs}{r+s} \left(\frac{\sigma^2}{r} + \frac{\sigma^2}{s} \right) = \sigma^2 .$$

The denominator is normally distributed with

$$\text{mean} = \sqrt{\frac{ns}{n+s}} (\mu_2 - \mu_1)$$

$$\text{variance} = \frac{ns}{n+s} \left(\frac{\sigma^2}{n} + \frac{\sigma^2}{s} \right) = \sigma^2 .$$

The covariance between numerator and denominator is

$$\begin{aligned}
 \text{covariance} &= \sqrt{\frac{rs}{r+s}} \sqrt{\frac{ns}{n+s}} \text{cov}(\bar{x}_r - \bar{z}_s, \bar{y}_n - \bar{z}_s) \\
 &= \sqrt{\frac{rs \cdot ns}{r+s \cdot n+s}} \text{cov}(\bar{z}_s, \bar{z}_s) \\
 &= \sqrt{\frac{r \cdot n}{r+s \cdot n+s}} \sigma^2 .
 \end{aligned}$$

Dividing the numerator and denominator of P by σ gives the form

$$P = \Pr\left(-1 < \frac{w_1}{w_2} < 1 \mid N_1 = m\right) ,$$

where w_1 and w_2 are bivariate normal (bvn) components with means 0 and $\sqrt{\frac{ns}{n+s}}(\mu_2 - \mu_1)$, variances 1 and 1, and covariance $\sqrt{\frac{r \cdot n}{r+s \cdot n+s}}$. Thus

$$P = \iint_{-1 < \frac{w_1}{w_2} < 1} (\text{bvn density}) dw_1 dw_2 .$$

The region of integration is shaded below. All drawings will be made under the assumption $\mu_2 > \mu_1$. If $\mu_1 \leq \mu_2$ the mathematical equations are still correct, independent of the drawings.

By the transformation

$$u_1 = (w_1 - \mu_{w_1}) \cos \phi + (w_2 - \mu_{w_2}) \sin \phi$$

$$u_2 = -(w_1 - \mu_{w_1}) \sin \phi + (w_2 - \mu_{w_2}) \cos \phi$$

$$w_1 = \mu_{w_1} + u_1 \cos \phi - u_2 \sin \phi$$

$$w_2 = \mu_{w_2} + u_2 \sin \phi + u_1 \cos \phi$$

$$\tan 2\phi = \frac{2\sigma_{w_1 w_2}}{\sigma_{w_2}^2 - \sigma_{w_1}^2}$$

we can obtain uncorrelated and hence independent normal random variables u_1 and u_2 (Cramer, Mathematical Methods in Statistics, p. 279 .) Applying this result with $\phi = 45$ degrees gives means zero and variances

$$\sigma_{u_1}^2 = 1 + \sqrt{\frac{n \cdot r}{n+s} \frac{r}{r+s}}$$

$$\sigma_{u_2}^2 = 1 - \sqrt{\frac{n \cdot r}{n+s} \frac{r}{r+s}} \quad .$$

We transform further to independent normal(0,1) variates by

$$v_1 = \frac{u_1}{\sigma_{u_1}}$$

$$v_2 = \frac{u_2}{\sigma_{u_2}} \quad .$$

The point $(0,0)$ in w coordinates has been transformed into the point

$$(a,b) = \left(-\frac{\sqrt{\frac{ns}{n+s}} \frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 + \sqrt{\frac{n \cdot r}{n+s \cdot r+s}}}}, -\frac{\sqrt{\frac{ns}{n+s}} \frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 - \sqrt{\frac{n \cdot r}{n+s \cdot r+s}}}} \right) .$$

The new region of integration is shaded below:

Finally,

$$P = \int_{-\infty}^a n(0,1) dv_1 \int_{-\infty}^b n(0,1) dv_2 + \int_a^{\infty} n(0,1) dv_1 \int_b^{\infty} n(0,1) dv_2 .$$

Consider next the second form of the sample, in which the left side (m x's) are correctly classified. Let L_{m+s} be the likelihood obtained when s z's are treated as x's. The estimates in this case are

$$\hat{\mu}_1 = \bar{x}_{m+s} = \frac{1}{m+s} \left[\sum_{i=1}^m x_i + \sum_{i=1}^s z_i \right]$$

$$\hat{\mu}_2 = \bar{y}_r = \frac{1}{r} \sum_{i=1}^r y_i$$

$$\hat{\hat{\sigma}}^2 = \frac{1}{r+s+n} \left[\sum_{i=1}^m (x_i - \bar{x}_{m+s})^2 + \sum_{i=1}^s (z_i - \bar{x}_{m+s})^2 + \sum_{i=1}^r (y_i - \bar{y}_r)^2 \right]$$

The value of the likelihood is

$$L_{m+s} = \left(\frac{1}{\sqrt{2\pi} \hat{\hat{\sigma}}} \right)^{m+r+s} e^{-\frac{1}{2} \frac{(m+r+s)}{\hat{\hat{\sigma}}^2}}.$$

The probability of the decision criterion is

$$\begin{aligned} P &= \Pr(L_m > L_{m+s} \mid N_1 = m) \\ &= \Pr(\hat{\hat{\sigma}}^2 > \hat{\sigma}^2 \mid N_1 = m) . \end{aligned}$$

In this case the decomposition lemma is applied to obtain

$$\begin{aligned} (r+s+n) \hat{\hat{\sigma}}^2 &= \sum_{i=1}^m (x_i - \bar{x}_m)^2 + \sum_{i=1}^s (z_i - \bar{z}_s)^2 + \sum_{i=1}^r (y_i - \bar{y}_r)^2 \\ &\quad + \frac{ms}{m+s} (\bar{x}_m - \bar{z}_s)^2 . \end{aligned}$$

Then

$$\begin{aligned} P &= \Pr \left(\frac{ms}{m+s} (\bar{x}_m - \bar{z}_s)^2 > \frac{rs}{r+s} (\bar{y}_r - \bar{z}_s)^2 \mid N_1 = m \right) \\ &= \Pr \left(-1 < \frac{\sqrt{\frac{rs}{r+s}} (\bar{y}_r - \bar{z}_s)}{\sqrt{\frac{ms}{m+s}} (\bar{x}_m - \bar{z}_s)} < 1 \mid N_1 = m \right) . \end{aligned}$$

In this case the numerator is normally distributed with

$$\text{mean} = 0$$

$$\text{variance} = \sigma^2$$

The denominator is normally distributed with

$$\text{mean} = -\sqrt{\frac{ms}{m+s}} (\mu_2 - \mu_1)$$

$$\text{variance} = \sigma^2$$

The covariance is

$$\text{covariance} = \sqrt{\frac{r}{r+s} \cdot \frac{m}{m+s}} \sigma^2.$$

Letting w_1 and w_2 be bivariate normal components with the parameters given above, we see that

$$P = \Pr\left(-1 < \frac{w_1}{w_2} < 1 \mid N_1 = m\right)$$

$$= \iint_{-1 < \frac{w_1}{w_2} < 1} (\text{bvn density}) dw_1 dw_2$$

The region of integration is shaded below:

As before, we transform to uncorrelated variables and then to variates with unit variance. The point (0,0) in w-coordinates is transformed into v-coordinates of

$$(a,b) = \frac{\sqrt{\frac{ms}{m+s}} (\mu_2 - \mu_1) / \sqrt{2}}{\sqrt{1 + \sqrt{\frac{m}{m+s} \cdot \frac{n}{n+s}}}}, \frac{\sqrt{\frac{ms}{m+s}} (\mu_2 - \mu_1) / \sqrt{2}}{\sqrt{1 - \sqrt{\frac{m}{m+s} \cdot \frac{n}{n+s}}}} .$$

Then

$$P = \int_{-\infty}^a n(0,1) dv_1 \int_{-\infty}^b n(0,1) dv_2 + \int_a^{\infty} n(0,1) dv_1 \int_b^{\infty} n(0,1) dv_2 .$$

These two results can be summarized in the following theorems.

Hypotheses. Let $(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2})$ be independent observations on random variables X and Y having unknown means μ_1 and μ_2 and unknown but common variance σ^2 . Suppose N_1 is unknown although $N = N_1 + N_2$ is known. Let L_k denote the value of the likelihood function of the sample obtained by using the maximum likelihood estimates with $N_1 = k$. Let the particular choice $N_1 = m$ and the associated $\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2$ produce a maximum likelihood, L_m .

Theorem. $\Pr(L_m > L_{m-s} \mid \text{true } N_1 = m) = p_1 p_2 + (1 - p_1)(1 - p_2)$

where

$$p_1 = \int_{-\infty}^a n(0,1) dv_1 \quad \text{with} \quad a = \frac{\sqrt{\frac{ns}{n+s}} \frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 + \sqrt{\frac{r}{r+s} \cdot \frac{n}{n+s}}}}$$

$$p_2 = \int_{-\infty}^b n(0,1) dv_2 \quad \text{with } b = \frac{\sqrt{\frac{ns}{n+s} \frac{\mu_2 - \mu_1}{\sqrt{2}}}}{\sqrt{1 - \sqrt{\frac{n}{n+s} \cdot \frac{r}{r+s}}}} .$$

Theorem. $\Pr(L_m > L_{m+s} \mid \text{true } N_1 = m) = p_1 p_2 + (1 - p_1)(1 - p_2)$

where

$$p_1 = \int_{-\infty}^a n(0,1) dv_1 \quad \text{with } a = \frac{\sqrt{\frac{ms}{m+s} \frac{\mu_2 - \mu_1}{\sqrt{2}}}}{\sqrt{1 + \sqrt{\frac{m}{m+s} \cdot \frac{n}{n+s}}}}$$

$$p_2 = \int_{-\infty}^b n(0,1) dv_2 \quad \text{with } b = \frac{\sqrt{\frac{ms}{m+s} \frac{\mu_2 - \mu_1}{\sqrt{2}}}}{\sqrt{1 - \sqrt{\frac{m}{m+s} \cdot \frac{n}{n+s}}}} .$$

The choice $N_1 = m$ defines the interval $I: [t_{N_1}, t_{N_1+1}]$ most likely to contain the switch in regression from μ_1 to μ_2 . The confidence for I , for fixed s , is defined as either of the probabilities given in the previous two theorems. The power for I , for fixed s , is now defined as 1 minus either of the two previous probabilities when calculated under alternative values for N_1 . There are six cases to consider, and complete proofs will not be given because they are very similar to the proofs just presented. The variances and covariances remain the same. The rotation of 45 degrees remains the same. Only the means of the numerators and denominators change. Thus, explicit mention of these changes will constitute our proof.

Theorem . $\Pr(L_m > L_{m-s} \mid \text{various assumptions about } N_1) = p_1 p_2 + (1-p_1)(1-p_2)$

where $p_1 = \int_{-\infty}^a n(0,1) dv_1$ and $p_2 = \int_{-\infty}^b n(0,1) dv_2$.

Let $A = \frac{\frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 + \sqrt{\frac{r}{r+s} \cdot \frac{n}{n+s}}}}$ $B = \frac{\frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 - \sqrt{\frac{r}{r+s} \cdot \frac{n}{n+s}}}}$

$C = \frac{\frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 + \sqrt{\frac{r}{r+s} \cdot \frac{m}{m+s}}}}$ $D = \frac{\frac{\mu_2 - \mu_1}{\sqrt{2}}}{\sqrt{1 - \sqrt{\frac{r}{r+s} \cdot \frac{m}{m+s}}}}$.

Then

(i) For $N_1 = m - s - t$, $s > 0$, $t \geq 0$

$a = \frac{r-t}{r} \sqrt{\frac{rs}{r+s}} \cdot A$ $b = \frac{r-t}{r} \sqrt{\frac{rs}{r+s}} \cdot B$

(ii) For $N_1 = m - s + t$, $s > 0$, $0 \leq t \leq s$

$a = \left[\frac{s-t}{s} \sqrt{\frac{rs}{r+s}} - \frac{t}{s} \sqrt{\frac{ns}{n+s}} \right] \cdot A$ $b = \left[-\frac{s-t}{s} \sqrt{\frac{rs}{r+s}} - \frac{t}{s} \sqrt{\frac{ns}{n+s}} \right] \cdot B$

(iii) For $N_1 = m - s + t$, $t \geq s > 0$

$a = -\frac{n-t+s}{n} \sqrt{\frac{ns}{n+s}} \cdot A$ $b = -\frac{n-t+s}{n} \sqrt{\frac{ns}{n+s}} \cdot B$

Theorem. $\Pr(L_m > L_{m+s} \mid \text{various assumptions about } N_1) = p_1 p_2 + (1-p_1)(1-p_2)$

where p_1, p_2, A, B, C, D are as previously defined. Then

(iv) For $N_1 = m + s - t$, $s > 0$, $s \geq t \geq 0$

$$a = \left[-\frac{s-t}{s} \sqrt{\frac{rs}{r+s}} + \frac{t}{s} \sqrt{\frac{ms}{m+s}} \right] \cdot C \quad b = \left[\frac{s-t}{s} \sqrt{\frac{rs}{r+s}} + \frac{t}{s} \sqrt{\frac{ms}{m+s}} \right] \cdot D$$

(v) For $N_1 = m + s - t$, $t \geq s > 0$

$$a = \frac{m-(t-s)}{m} \sqrt{\frac{ms}{m+s}} \cdot C \quad b = \frac{m-(t-s)}{m} \sqrt{\frac{ms}{m+s}} \cdot D$$

(vi) For $N_1 = m + s + t$, $t > 0$, $s > 0$

$$a = -\frac{r-t}{r} \sqrt{\frac{rs}{r+s}} \cdot C \quad b = \frac{r-t}{r} \sqrt{\frac{rs}{r+s}} \cdot D$$

Proof. We will first be concerned with probability statements of the form

$$\begin{aligned} P &= \Pr(L_m > L_{m-s} \mid \text{various assumptions about } N_1) \\ &= \Pr \left(-1 < \frac{\sqrt{\frac{rs}{r+s}} (\tilde{x}_r - \bar{z}_s)}{\sqrt{\frac{ns}{n+s}} (\bar{y}_n - \bar{z}_s)} < 1 \mid \text{various assumptions about } N_1 \right) . \end{aligned}$$

As noted before we need only calculate the means of the above numerator and denominator; in fact, we will exhibit merely

$$\mu_N = E[\tilde{x}_r - \bar{z}_s] \quad \text{and} \quad \mu_D = E[\bar{y}_n - \bar{z}_s] .$$

(i)

$$\begin{aligned}
 \bar{x}_r &= \frac{1}{r} \left[\sum_{i=1}^{r-t} x_i + \sum_{i=r-t+1}^r x_i \right] \\
 &= \frac{r-t}{r} \bar{x}_{r-t} + \frac{t}{r} \bar{x}_t \\
 \mu_N &= \frac{r-t}{r} \mu_1 + \frac{t}{r} \mu_2 - \mu_2 = -\frac{r-t}{r} (\mu_2 - \mu_1)
 \end{aligned}$$

$$\mu_D = 0$$

(ii)

$$\begin{aligned}
 \bar{z}_s &= \frac{1}{s} \left[\sum_{i=1}^t z_i + \sum_{i=t+1}^s z_i \right] \\
 &= \frac{t}{s} \bar{z}_t + \frac{s-t}{s} \bar{z}_{s-t} \\
 \mu_N &= \mu_1 - \frac{t}{s} \mu_1 - \frac{s-t}{s} \mu_2 = -\frac{s-t}{s} (\mu_2 - \mu_1) \\
 \mu_D &= \mu_2 - \frac{t}{s} \mu_1 - \frac{s-t}{s} \mu_2 = \frac{t}{s} (\mu_2 - \mu_1)
 \end{aligned}$$

(iii)

$$\begin{aligned}
 \bar{y}_n &= \frac{1}{n} \left[\sum_{i=1}^{t-s} y_i + \sum_{i=t-s+1}^n y_i \right] \\
 &= \frac{t-s}{n} \bar{y}_{t-s} + \frac{n-t+s}{n} \bar{y}_{n-t+s}
 \end{aligned}$$

$$\mu_N = 0$$

$$\mu_D = \frac{t-s}{n} \mu_1 + \frac{n-t+s}{n} \mu_2 - \mu_1 = \frac{n-t+s}{n} (\mu_2 - \mu_1) .$$

We will next be concerned with probability statements of the form

$$\begin{aligned} P &= \Pr(L_m > L_{m+s} \mid \text{various assumptions about } N_1) \\ &= \Pr \left(-1 < \frac{\sqrt{\frac{rs}{r+s}} (\bar{y}_r - \bar{z}_s)}{\sqrt{\frac{ms}{m+s}} (\bar{x}_m - \bar{z}_s)} < 1 \mid \text{various assumptions about } N_1 \right) . \end{aligned}$$

We again demonstrate only μ_N and μ_D .

$$\begin{aligned} \text{(iv)} \quad \bar{z}_s &= \frac{1}{s} \left[\sum_{i=1}^{s-t} z_i + \sum_{i=s-t+1}^s z_i \right] \\ &= \frac{s-t}{s} \bar{z}_{s-t} + \frac{t}{s} \bar{z}_t \end{aligned}$$

$$\mu_N = \mu_2 - \frac{s-t}{s} \mu_1 - \frac{t}{s} \mu_2 = \frac{s-t}{s} (\mu_2 - \mu_1)$$

$$\mu_D = \mu_1 - \frac{s-t}{s} \mu_1 - \frac{t}{s} \mu_2 = -\frac{t}{s} (\mu_2 - \mu_1)$$

$$\begin{aligned} \text{(v)} \quad \bar{x}_m &= \frac{1}{m} \left[\sum_{i=1}^{m-(t-s)} x_i + \sum_{i=m-(t-s)+1}^m x_i \right] \\ &= \frac{m-(t-s)}{m} \bar{x}_{m+s-t} + \frac{t-s}{m} \bar{x}_{t-s} \end{aligned}$$

$$\mu_N = 0$$

$$\mu_D = \frac{m-(t-s)}{m} \mu_1 + \frac{t-s}{m} \mu_2 - \mu_2 = - \frac{m-(t-s)}{m} (\mu_2 - \mu_1)$$

(vi)

$$\begin{aligned} \bar{y}_r &= \frac{1}{r} \left[\sum_{i=1}^t y_i + \sum_{i=t+1}^r y_i \right] \\ &= \frac{t}{r} \bar{y}_t + \frac{r-t}{r} \bar{y}_{r-t} \end{aligned}$$

$$\mu_N = \frac{t}{r} \mu_1 + \frac{r-t}{r} \mu_2 - \mu_1 = \frac{r-t}{r} (\mu_2 - \mu_1)$$

$$\mu_D = 0$$

This completes the critical calculations for the proof.

All of the probabilities given in the previous theorems involve the unknown quantity $(\mu_2 - \mu_1)$, as they should. Various choices of this quantity can be made and plots of the various probabilities P as a function of $(\mu_2 - \mu_1)$ could be constructed to aid in further understanding.

2.6.2 With Regression Assumptions : Comments

If now regression assumptions are added the situation becomes very much more difficult. What is needed is a confidence for $I \wedge J$. This can be written

$$\text{Conf} [I \wedge J] = \text{Conf} [J | I] \cdot \text{Conf} [I] .$$

The factor $\text{Conf} [I]$ is associated with the probability P of the decision which defines I ,

$$P = \Pr(L_m > L_{m'} \mid N_1 = m) .$$

After appropriate cancellations and reductions, and by analogy to the non regression case just discussed, P becomes, for the first form of the sample,

$$\begin{aligned} P &= \Pr(S_2^2 > S_1^2 \mid N_1 = m) \\ &= \Pr(\frac{S_m^2}{S_{m-s}^2} < 1 \mid N_1 = m) \end{aligned}$$

where

$$\frac{S_m^2}{S_{m-s}^2} = \frac{r \sum [x_i - \hat{\mu}_1(s_i)]^2 + s \sum [z_i - \hat{\mu}_1(v_i)]^2 + n \sum [y_i - \hat{\mu}_2(t_i)]^2}{r \sum [x_i - \hat{\mu}_1(s_i)]^2 + s \sum [z_i - \hat{\mu}_2(v_i)]^2 + n \sum [y_i - \hat{\mu}_2(t_i)]^2} .$$

The variate S_m^2 / S_{m-s}^2 is the ratio of two dependent non-central chi-square variates, and its distribution is not readily determined.

In the non regression case this difficulty was overcome by additively decomposing S_m^2 and S_{m-s}^2 before forming the ratio and cancelling certain common terms. But this procedure would seem to be impossible in the regression case since both the observations x_i, z_i, y_i and the observable parameters s_i, v_i, t_i must be simultaneously decomposed. Typically, the difficulty arises with every a_i from $\hat{\mu}_1$. Such an a_i effectively contains the determinant of the matrix M of squares and cross products of s 's and v 's in its denominator. This is because of the nature of M^{-1} .

But the determinant of M contains s 's and v 's in a multiplicative way, and these two sets of variables cannot be separated in an additive manner. The problem is essentially that of an additive decomposition of, say, $(A + B)^{-1}$. It is doubtful whether any non additive decomposition, even if possible, would lead to manageable distribution results. Thus, chances for finding $\text{Conf} [I]$ seem small.

The factor $\text{Conf} [J | I]$ can be found by the methods of section 2.2.3 when I is correctly defined. But if I is incorrectly defined, and we want power-like statements for J , the non-decomposability problems again confront us. The best we have been able to do is to give some qualitative approximations to the reduction in confidence.

First recall that $\text{Conf} [J | I]$ requires calculations based on the student variate

$$t = \frac{\frac{u}{\sigma_u}}{\sqrt{\frac{\sum_{i=1}^r [x_i - \hat{\mu}_1(s_i)]^2 + \sum_{i=1}^s [z_i - \hat{\mu}_1(v_i)]^2 + \sum_{i=1}^n [y_i - \hat{\mu}_2(t_i)]^2}{m + n - 2k - 2}}}$$

where $m = r + s$

$$u = (a_0 + a_1 t_* + \dots + a_k t_*^k) - (b_0 + b_1 t_* + \dots + b_k t_*^k) \quad .$$

Let (H) refer to the situation that the z 's are correctly classified as x 's in the first form of the sample. Otherwise, (A), the z 's have been misclassified and should have been treated as y 's.

We have just noted that it is difficult to additively pry apart the contribution of the (v_i, z_i) to the determination of $\hat{\mu}_1$. Therefore we will consider an alternate student statistic which in no way involves the (v_i, z_i) .

Now under A consider the numerator, u . In particular consider any a_i . From 1.5 with the notation of 2.2.2

$$a_i = \sum_{j=0}^k r^{ij} c_j \quad i = 0, 1, \dots, k$$

where $(r^{ij}) = (r_{ij})^{-1}$

$$r_{ij} = \sum_{\epsilon=1}^r s_{\epsilon}^i s_{\epsilon}^j + \sum_{\epsilon=1}^s v_{\epsilon}^i v_{\epsilon}^j = r_{ij}^{(x)} + r_{ij}^{(z)}$$

$$c_j = \sum_{\epsilon=1}^r x_{\epsilon} s_{\epsilon}^j + \sum_{\epsilon=1}^s z_{\epsilon} v_{\epsilon}^j = c_j^{(x)} + c_j^{(z)}$$

$$j = 0, 1, \dots, k$$

$$\begin{aligned} \text{Then } E[a_i] &= \sum_j r^{ij} E(c_j) \\ &= \sum_j r^{ij} \left(\sum_{\alpha=1}^r s_{\alpha}^j E[x_{\alpha}] + \sum_{\alpha=1}^s v_{\alpha}^j E[z_{\alpha}] \right) \\ &= \sum_j r^{ij} \left(\sum_{\epsilon=1}^r s_{\epsilon}^j \sum_{p=0}^k \alpha_p s_{\epsilon}^p + \sum_{\epsilon=1}^s v_{\epsilon}^j \sum_{p=0}^k \beta_p v_{\epsilon}^p \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_j r^{ij} \left(\sum_{p=0}^k \alpha_p r_{jp}^{(x)} + \sum_{p=0}^k \beta_p r_{jp}^{(z)} \right) \\
&= \sum_j r^{ij} \sum_{p=0}^k \left[\alpha_p r_{jp}^{(x)} + \beta_p r_{jp}^{(z)} \right] .
\end{aligned}$$

$$\text{and} \quad E[u] = \sum_{i=0}^k [E(a_i) - \beta_i] t_*^i .$$

Since x 's and z 's have the same variances, and since variance expressions are functionally independent of means, then the variance of u is identical with the result obtained in 2.2.3 :

$$\sigma_u^2 = \sigma^2 \sum_{i=0}^k \sum_{j=0}^k (r^{ij} + s^{ij}) t_*^{i+j} .$$

Consider now the alternative variate

$$t = \frac{\frac{u}{\sigma_u}}{\sqrt{\frac{\sum_{i=1}^r [x_i - \hat{\mu}_r(s_i)]^2 + \sum_{i=1}^n [y_i - \hat{\mu}_2(t_i)]^2}{r + n - 2k - 2}}} ,$$

where $\hat{\mu}_r$ is calculated from only the x 's and s 's . The z 's and v 's are not being used. Under H , t is a student variate with $(r + n - 2k - 2)$ degrees of freedom. Under A , t is a non-central t variate with $(r + n - 2k - 2)$ degrees of freedom and

parameter

$$\delta = \frac{E[u]}{\sigma_u} .$$

Under H , fixed confidence intervals I_r for given $1-\epsilon$ can readily be constructed. Under A , tables of non-central t can be consulted for various δ to obtain an alternative confidence for the fixed interval I_r . It seems plausible that the reduction in confidence for I would be comparable to the reduction actually observed for I_r . For large m and n this reduction should be nearly negligible.

CHAPTER III
THE INTERSECTION OF SIMPLE LINEAR REGRESSIONS
CONTINUOUS MODEL

3.1 Definition of the Problem

Let there be given two sample functions $y_1(t)$ and $y_2(t)$ from two intersecting processes $Y_t^{(1)}$ and $Y_t^{(2)}$. Assume

$$Y_t^{(1)} = m_1(t) + \varepsilon_t^{(1)} = a_0 + a_1 t + \varepsilon_t^{(1)}, \quad t_1 \leq t \leq t_2$$

$$Y_t^{(2)} = m_2(t) + \varepsilon_t^{(2)} = b_0 + b_1 t + \varepsilon_t^{(2)}, \quad t_3 \leq t \leq t_4,$$

where $\varepsilon_t^{(1)}$ and $\varepsilon_t^{(2)}$ are independent processes with covariance functions $\sigma_{jk}^{(1)}$ and $\sigma_{jk}^{(2)}$, where a_0, a_1, b_0, b_1 are constants to be estimated, and where t_1, t_2, t_3, t_4 are known times with $t_2 < t_3$. Our problem is to estimate t_* , the abscissa of intersection of m_1 and m_2 , and to provide some distribution theory for the estimator.

For the continuous model we select maximum likelihood estimation whenever possible, and otherwise least squares estimators. The choice is dictated by how much is known or unknown about the given processes. This in turn depends partly on the covariance structures $\sigma_{jk}^{(1)}$ and $\sigma_{jk}^{(2)}$. We consider explicitly the case when the $\varepsilon_t^{(i)}$ have the structure of a Fundamental Random Process (FRP) (see Mann, 1953).

It will be assumed that for practical calculations some theory or form of physical equipment is available to perform integrations when introduced.

3.2 Fundamental Random Process

Assume that the $Y_t^{(i)}$ are processes for which the $\epsilon_t^{(i)}$ are FRP's. Let $\epsilon_{t,\tau}^{(i)} = \epsilon_{t+\tau}^{(i)} - \epsilon_t^{(i)}$. Then $\sigma_{\epsilon_{t,\tau}^{(i)}} = c_i \tau$, $i = 1, 2$. According to Theorem 2.3 of Mann (1953) the constants c_i can be estimated with arbitrarily high precision by

$$\hat{c}_i = \frac{1}{N_i} \sum_{n=1}^{N_i} [y_i(n \cdot \Delta t_n) - y_i(\overline{n-1} \cdot \Delta t_n)]^2, \quad i = 1, 2,$$

where the intervals $[t_1, t_2]$ and $[t_3, t_4]$ have been divided into N_i equal parts of length Δt_n . The expected value of $\hat{c}_i$ is asymptotically c_i , and $\hat{c}_i$ converges in the mean to c_i . The variable $(\hat{c}_i - c_i)$ is stochastically of the order $N_i^{-1/2}$. The arbitrarily high precision is obtained by taking each N_i as large as possible. Thus from the statistical estimation point of view we may regard the c_i as known.

Consider now maximum likelihood estimation of the a_i . The estimation for the b_i will be identical except for change in the time interval of definition for the process. We thus suppress the superscript (i) distinction. For a subdivision $S_n: [t_1 = s_0, s_1, s_2, \dots, s_n = t_2]$ we may assume, because we are dealing with an FRP, that the $Y_{s_j} - Y_{s_{j-1}}$ are independently and normally distributed

with mean $m(s_j) - m(s_{j-1})$ and variance $c \cdot (s_j - s_{j-1})$. The likelihood function is

$$L(S_n; a_0, a_1) = \prod_{j=0}^n \frac{1}{\sqrt{2\pi c(s_j - s_{j-1})}} \cdot e^{-\frac{[(Y_{s_j} - Y_{s_{j-1}}) - (a_0 - a_0) - a_1(s_j - s_{j-1})]^2}{2c(s_j - s_{j-1})}}.$$

Notice that a_0 vanishes by subtraction, and that the likelihood is thus a function of the single parameter a_1 . By maximizing the logarithm of L we obtain the single equation

$$\left[\sum_{j=1}^n (Y_{s_j} - Y_{s_{j-1}}) - a_1(s_j - s_{j-1}) \right] \left[\frac{s_j - s_{j-1}}{s_j - s_{j-1}} \right] = 0.$$

Since Y_t is a process of second order we may pass to the stochastic limit in the mean to obtain

$$\int_{t_1}^{t_2} dY_t = a_1 \int_{t_1}^{t_2} dt.$$

Hence

$$\hat{a}_1 = \frac{Y_{t_2} - Y_{t_1}}{t_2 - t_1}.$$

This estimate is normally distributed with zero mean and variance $c_i/(t_2 - t_1)$.

In practice one is given a single sample record $y(t)$, and the estimate is then

$$\hat{a}_1 = \frac{y(t_2) - y(t_1)}{t_2 - t_1} .$$

This replacement can be logically justified. Introduce $I, \sum_n, f_n$, and the basic sample space Ω with elements ω by the extended identity

$$I = \int_{t_1}^{t_2} dY_t = \text{l.i.m.} \sum_{j=1}^n [Y_{t_j}(\omega) - Y_{t_{j-1}}(\omega)] = \lim_{n \rightarrow \infty} \int_{\Omega} \sum_n(\omega) dP .$$

This means

$$\lim_{n \rightarrow \infty} \int_{\Omega} [\sum_n(\omega) - I]^2 dP = 0 ,$$

or that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n^2(\omega) dP = 0 .$$

From Hobson (Theory of Functions of a Real Variable, 2, Dover (1957), paragraphs 168-172) the existence of such limits implies that $f_n(\omega)$ converges to zero at almost every point in Ω . That is to say, the ordinary Riemann integral of almost every sample function exists. Thus $\int_{t_1}^{t_2}$ exists for almost all $y(t)$ and equals $y(t_2) - y(t_1)$. The estimation of b_1 would proceed in like fashion. Mann has shown that the estimates are unbiased.

It is disappointing that the maximum likelihood procedure as formulated here, using the FRP assumptions that changes in Y_t

are independent and normal, does not lead to direct estimation of a_0 . If we attempt to formulate M.L. at the Y_t level, matters become quite complicated for an FRP because we can no longer assume that consecutive observations are independent. Furthermore, assumptions would need now be made about the covariance of Y_t , whereas before such assumptions had only to be made about changes in Y_t . Other writers have been able to avoid this problem by assuming that at time t_1 the process is virtually constant and thus a_0 can be estimated directly by

$$\tilde{a}_0 = y(t_1) .$$

Adopting this point of view we would set

$$\tilde{a}_0 = y_1(t_1) \quad \text{and} \quad \tilde{b}_0 = y_2(t_1) .$$

Thus in our model, a particle starts out with virtually no variation and pursues a linear course in which its random deviations from the course get larger. Then at some unknown time, t_* , it is influenced to alter its direction and continue in a linear course, and does so until its random variation becomes negligible.

Finally, the estimate for t_* is obtained by equating m_1 and m_2 :

$$\begin{aligned} \tilde{a}_0 + \tilde{a}_1 t_* &= \tilde{b}_0 + \tilde{b}_1 t_* \\ \tilde{t}_* &= \frac{\tilde{a}_0 - \tilde{b}_0}{\tilde{b}_1 - \tilde{a}_1} . \end{aligned}$$

This random variable is the ratio of the constant $h = \tilde{a}_0 - \tilde{b}_0$ and a

normal variate $z = \tilde{b}_1 - \tilde{a}_1$ having mean $\mu_z = b_1 - a_1$ and known variance $\sigma_z^2 = \sigma_{b_1}^2 + \sigma_{a_1}^2 = c_2/(t_4 - t_3) + c_1/(t_2 - t_1)$. Three cases must be considered in stating the distribution of t_* .

Suppose first that $h > 0$. Let k be a fixed point on the distribution of z . Then

$$\Pr \left[\frac{h}{z} < k \right] = \begin{cases} \frac{1}{2} + \int_{h/k}^{\infty} dN & \text{for } k > 0 \\ \frac{1}{2} & \text{for } k = 0 \\ \frac{1}{2} - \int_{-\infty}^{h/k} dN & \text{for } k < 0 \end{cases},$$

where N denotes the distribution of z .

Secondly, if $h < 0$ we rewrite t_* as follows :

$$\tilde{t}_* = \frac{b_0 - a_0}{\tilde{a}_1 - \tilde{b}_1}.$$

Again the numerator is constant, but this time the denominator is the normal variate $z_* = \tilde{a}_1 - \tilde{b}_1$. The distribution of $\tilde{t}_*$ is obtained from the previous result by replacing dN by dN_* , where N_* is the distribution of z_* .

Thirdly, if $h = 0$, the solution for t_* is

$$\tilde{t}_* = \frac{\tilde{b}_1}{\tilde{a}_1}.$$

This time $\tilde{t}_*$ is a ratio of independent normal variates.

Let

$$A = \frac{E[\tilde{b}_1]}{E[\tilde{a}_1]} = \frac{b_1}{a_1}$$

$$u = \tilde{b}_1 - A \cdot \tilde{a}_1 \quad .$$

Then u is normally distributed with zero mean and variance

$$\sigma_u^2 = \sigma_{b_1}^2 + A^2 \sigma_{a_1}^2 = \frac{c_2}{t_4 - t_3} + \frac{b_1^2}{a_1^2} \cdot \frac{c_1}{t_2 - t_1} \quad .$$

Confidence limits for A can be found from the distribution of u in a standard manner.

3.3 Other Investigations

In 1951 Robbins and Monro presented a distribution-free stochastic method for solving the equation $f(x) = \alpha$ for x . The function $f(x)$ is an unknown regression, and observations of random variables about $f(x)$ are available at any stage x . The constant α is known. The solution consists of constructing a sequence of approximations which converge in some probabilistic sense to the true value $x = \theta$. In 1952 Kiefer and Wolfowitz modified the RM procedure to estimate maxima of $f(x)$. In 1956 Derman summarized all work on RM in a single paper.

If the objective in our problem of estimating intersections of regressions is to locate a max or a min, then the results of Kiefer and Wolfowitz may solve the problem in the continuous case. There

Figure 9

Figure 10

could be practical difficulties if slow convergence is encountered. However, our interest in intersecting regressions may be to locate a change point (θ, α) of the form shown in figure 9 . The functions M_1 and M_2 are branch regressions of $f(x)$. The RM procedure could be used to approximate (θ, α) as follows. Relative to M_1 estimate several values θ_1 corresponding to various given values of α_1 . Do the same for M_2 . Plot these results, as in figure 10 . Then estimate the point of intersection of best fit curves through the two clusters of points. If the clusters are wide spread, repeated estimates at any α_1 can be obtained by using a different starting point in the calculation. Multiple estimates at any α_1 can then be averaged and the spread of the clusters condensed. A thorough study of this approach could be the topic of another paper.

In 1959 E.J. Williams published his Regression Analysis . He does not consider problems of intersecting regressions, but some of the intermediate results of this thesis are similar in nature to Williams's work on zeros of polynomials.

In 1954 Box and Hunter determined confidence regions for solutions to simultaneous linear equations. One application of their results is the location of extrema.

In conclusion we note that Williams, Box and Hunter, and the author of this thesis have all encountered practical difficulties when a sample is just bad enough to destroy the expected shapes of needed confidence regions.

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AUTOBIOGRAPHY

I, Donald Edward Robison, was born in Alturas, California, July 1, 1931. I attended grade school and two years of high school in San Gabriel, California. I completed my high school education in Oregon where I graduated from Springfield High School in 1948. In 1952 I received the degree Bachelor of Science in Mathematics from the University of Oregon. From 1952 to 1954 I was in the United States Army. I was stationed at White Sands Proving Grounds in New Mexico, working under the military occupation speciality of Mathematics Research Assistant. During this time I began my graduate work as a special student at New Mexico A. and M. College. In 1954 I came to Ohio State University and in 1956 I was granted the degree Master of Science in Mathematics. My thesis was written in the field of mathematical statistics. The years 1956 to 1959 were spent at Ohio State University where I was preparing for the Ph.D. degree in Mathematics in the field of Mathematical Statistics. During this time I held positions of graduate assistant, research assistant, research associate, and assistant instructor. In 1959, after passing the general examination for the Ph.D. degree, I joined the technical staff of Space Technology Laboratories of Los Angeles as a Mathematical Statistician. Granted a leave of absence from that organization for 1961-1962, I returned to Ohio State University to complete my doctoral dissertation.