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TORSIONAL RESTRAINT OF LATERAL BUCKLING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

The Ohio State University
1964

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NOMENCLATURE

Symbol	
x, y, z	Fixed rectangular coordinates
ξ, η, ζ	Rectangular coordinates through centroid of cross section
u, v	Displacements in x, y direction
β	Angle of rotation of cross section
u_c, v_c	Displacements at center of a span
$\bar{u}, \bar{v}$	Displacements in ξ, η direction
u_t	Displacement of flange
u', v', β'	First derivatives with respect to z
E	Modulus of elasticity in tension and compression
G	Modulus of elasticity in shear
I_1, I_2	Minimum and maximum moments of inertia of cross section respectively
C_t	St. Venant's torsional constant
$B_1 = EI_1$	Minimum flexural rigidity
$B_2 = EI_2$	Maximum flexural rigidity
$C = GC_t$	Torsional rigidity
$D \approx \frac{B_1}{2}$	Flexural rigidity of one flange
L	Span length

Symbol

h	Depth of cross section
$y = \frac{I_2 - I_1}{I_2}$	Cross section property
$a^2 = \frac{E_1 h^2}{4C} = \frac{Oh^2}{2c}$	Property of cross section and material
E_r, I_r, L_r	Modulus of elasticity, moment of inertia and length of a restraining member
K_r	Stiffness of restraining member (ft.-lbs./rad. or ft.-lbs./rad./ft.)
K_b	Torsional stiffness of beam (ft.-lbs./rad. or ft.-lbs./rad./ft.)
$K_c = \frac{K_r}{K_b}$	Continuous torsional restraint stiffness factor (non-dimensional)
$K_p = \frac{K_r}{K_b}$	Point torsional restraint stiffness factor (non-dimensional)
T	Magnitude of a restraining couple
q	Positive interger
r	Length of a beam segment
M	End moment load (ft.-pounds)
P	Concentrated load (pounds)
w	Uniform load (pounds per ft.)
m	Factor in $M_{cr} = \frac{m}{L} \sqrt{\frac{B_c}{y}}$, $P_{cr} = \frac{m}{L^2} \sqrt{\frac{B_c}{y}}$ and $(\omega L)_{cr} = \frac{m}{L^2} \sqrt{\frac{B_c}{y}}$
$s = \frac{m}{(m)_{K_c=0}}$	Critical stress factor

CHAPTER I

INTRODUCTION

A beam bent about its major axis and having a flexural rigidity about this axis which is much greater than its flexural rigidity about its minor axis may buckle by twisting and simultaneously deflecting in the lateral direction. This action may occur long before the stresses due to bending exceed the proportional limit and is especially true if the beam is not restrained or braced along its length or at its ends. The load that is required to maintain such a laterally deflected equilibrium position of the beam is called the critical load and the type of instability is called lateral buckling.

Bleich(1) has provided a historical sketch of the important early contributions to lateral buckling theory. Since the publication of Bleich's book in 1952, there have been many additional important contributions which are summarized and evaluated by Lee(2).

In 1899, Prandtl(3) and Michell(4) each published, independently of each other, a theoretical analysis of

the lateral stability problem of narrow rectangular beams. They arrived at substantially the same second order linear differential equation with variable coefficients to define the phenomenon. Since then many investigators have added to the research in this field. Timoshenko(5) derived the basic fourth order differential equation that controls the problem of lateral buckling of deep symmetric I-beams. Timoshenko(5) also developed an energy method and used it to solve lateral buckling problems. Bleich(1) considered the case of an I-beam with unequal flanges and also the case of a beam with its tension flanges fixed against lateral displacement. Several cases were solved by Prescott(6), including a beam loaded by both equal end moments and thrust. He also considered beams with various support conditions. The general theory of the torsional and flexural buckling of thin bars of open cross section was presented by Goodier(7), Timoshenko(8) and Vlasov(9). The theory developed by the latter three investigators is not restricted to symmetrical cross sections.

Flint(10) has considered the influence of restraints on the lateral stability of beams, especially restraints at the supports and point torsional restraints at the center of rectangular beams. Flint(10) also obtained solutions for rectangular cross section beams when a horizontal elastic restraint exists at mid-span. The loadings considered by Flint(10) are rather general. They include equal end

moments, concentrated and uniformly distributed loads, and combinations of loads that are symmetrically placed on the span. He obtained his results by using energy methods as well as by exact solutions of the differential equations.

Zuk(11) considered the problem of a horizontal continuous elastic restraint, a point horizontal elastic restraint at mid-span and a fixed horizontal restraint at mid-span. All of his solutions are for I-beams loaded by equal end moments. He obtained solutions by energy methods and exact solutions of the differential equations. An important feature of Zuk's work is that he determined magnitudes of the bracing forces in terms of the initial deformations of the beam and the stiffness of the brace.

Salvadori(12, 13) solved the problem of a beam loaded by unequal moments at the ends. He considered both rectangular beams and I-beams. His work is very complete in that it covers very general loading situations. He used an energy method for his solutions. The most important case he considered was that of an I-beam with unequal end moments.

Austin, Vegian and Tung(14) achieved an interesting solution by using finite differences for the case of I-beams loaded with both concentrated and uniformly distributed loads. A discussion of the influence of restraints at the supports was given. The cases of a cantilever beam and an overhanging beam with a concentrated

load was given by Kerensky, Flint and Brown(15) along with a determination of the influence of restraints. The case of a cantilever I-beam with a uniform load was considered by Poley(16), who used the finite differences method.

Winter(17, 18) contributed solutions for beams with unsymmetrical cross sections for both concentrated and uniform loads. He used an energy method. Winter also considered the case of a beam with its tension flange restrained against lateral displacement.

De Vries(19) developed simple rules for the design of I-beams. His L_d/bt formula is well-known and extensively used. Although such a simple method for a complicated problem can lead only to approximate results, it has a rational basis. This was brought out by Winter(20) in a discussion of De Vries' paper. He started with Timoshenko's solution for I-beams loaded by equal end moments and showed that one term was of a higher order for beams with usual flange width to thickness ratios. The result was that De Vries' conclusions were substantiated. De Vries' recommendations were adopted by the A.I.S.C.(21) and are used extensively in this country.

Many experiments have been made to verify the theoretical results. Using small aluminum alloy test specimens, Flint(10) obtained test results to verify his analysis. Hill and Clark(22, 23) have conducted many tests on aluminum alloy I-beams to verify existing

analytical solutions (especially Salvadori's). Other tests were performed by Clark and Jombock(24) to verify Salvadori's(13) solutions for the case of unequal end moments. The greatest difficulty in the above-mentioned tests seemed to be in providing the same end conditions in the laboratory as were assumed in analysis. A second difficulty in conducting the tests was the presence of initial deformations in the test specimens. In spite of these difficulties, the investigators have reported results that are in good agreement with the theory.

The solutions referred to are restricted to elastic material. That is, it is assumed that the greatest stress at the instant of buckling does not exceed the proportional limit. If the proportional limit is exceeded prior to the instant of buckling, the modulus of elasticity changes and the elastic solutions do not apply. With the increase in the use of plastic design in recent years and the assumption of severe inelastic deformations before failure, there resulted a need for more knowledge of inelastic lateral buckling. Timoshenko(5) discussed the problem and suggested solutions based upon simplifying assumptions. More precise solutions have been provided by Neal(25), White(26) and Galambos(27). Neal's work was the first inelastic solution, but it applied only to a beam of rectangular cross section. He used the reduced modulus concept. White obtained a solution for I-beams with variable end moments,

and the value of his work is that it provides a basis for locating lateral braces in the vicinity of a plastic hinge. Galambos obtained an inelastic solution for WF-beam columns. He used the tangent modulus concept and considered the effect of residual stresses.

There is one type of restraint of lateral buckling for which it is felt not enough research has been done. Figure 1 shows the cross section of two parallel I-beams which are rigidly connected at their webs by a filler joist. The dotted lines show the position of the beams and joist before the critical load has been reached. The solid lines show the same beams and joist just after they have started to buckle laterally. The secondary member is seen to apply a restoring couple which is proportional to the rotation of the main beams and opposite in direction to the rotation. The greater the flexural stiffness of the secondary member the greater this restoring torque will be.

It is the purpose of this dissertation to explore analytically the problem described in the preceding paragraph. The problem can be idealized as the lateral buckling of I-beams restrained by elastic torsional couples. In the case of many such restraining couples along the length of a beam such as might be provided by closely spaced bar joists, the restraint will be considered to be a continuous elastic torsional restraint. In the case of

Fig. 1.--Parallel Beams Restrained by a Filler Joist

a single secondary member, there exists a point elastic torsional restraint.

With the solution of this problem, the designer will be able to design in a rational way when the restraint provided by a secondary member is either a horizontal elastic restraint or a torsional elastic restraint, since the former problem has been solved by Flint(10) and Zuk(11). Note that in Figure 1, no horizontal restraint is provided by the secondary member. The problem of a point torsional restraint has been solved by Flint(10) for the special case of a rectangular beam. Vlasov(9) has considered lateral buckling of beams embedded in an elastic medium. His general system of differential equations can be reduced to cover the case of a continuous elastic torsional restraint. By using energy methods, he has completed solutions for several important cases. They include the case of a beam loaded by equal end moments with a continuous torsional restraint and this is considered again in this dissertation.

The following approximations and assumptions will be made in this investigation:

(a) The beam cross section is a symmetric I-beam. (The solutions will permit specialization to the case of a narrow rectangular beam.)

(b) All loads act at the centroid (also the shear

center) and remain vertical regardless of the possible change in position of the centroid.

(c) The beams are perfectly straight with no initial deformations.

(d) Nonlinear terms are dropped from equations that also contain linear terms. These terms are considered to be of a higher order since the displacements are assumed to be small.

(e) The beams are linearly elastic. An analysis for a critical load will not apply if the proportional limit is exceeded at some point on the beam.

In the solutions that are to follow, the beam will be considered to be free to rotate with respect to the principal axes at the supports, but rotation about the longitudinal axis of the beam at the supports is considered to be prevented. Although assumption (d) is usually made in lateral buckling problems, Masur(28) has found that inclusion of the nonlinear terms results in a stiffening of the beam or an increase in the critical load. By making this assumption, results should be conservative or on the safe side. In regard to (b), Timoshenko(5) has shown that if the load is applied above the centroid, the critical load is reduced and has supplied an analysis for this case.

Although a great number of lateral buckling problems

have been solved by energy methods, the writer will attempt his solutions by direct attack on the differential equations involved. It is felt that this approach gives a better appreciation of the approximations and assumptions that are made.

Table 1 is supplied to make it easy to find the solutions that are considered in this dissertation.

TABLE 1
SUMMARY OF SOLUTIONS

Case	Loading	Torsional Restraint Type	Buckling Formula	Stiffness Table
1	Equal End Moments	Continuous	$M_{cr} = \frac{m}{L} \sqrt{\frac{B_1 C}{\gamma}}$	Table 4 Page 45
2	Concentrated at Mid-Span	Continuous	$P_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}$	Table 5 Page 64
3	Uniform	Continuous	$(\omega L)_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}$	Table 6 Page 77
4	Equal End Moments	Point	$M_{cr} = \frac{m}{L} \sqrt{\frac{B_1 C}{\gamma}}$	Table 9 Page 101
5	Concentrated at Mid-Span	Point	$P_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}$	Table 12 Page 117
6	Uniform	Point	$(\omega L)_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}$	Table 15 Page 128

CHAPTER II

EQUILIBRIUM EQUATIONS

Figure 2 shows three views of an I-beam which has buckled to the side and twisted as a result of vertical loads (not shown) which were originally in the yz plane. After buckling the loads remain parallel to the y axis but may be displaced parallel to the x axis if their points of application on the beam move parallel to the x axis. The x, y, z axes are fixed axes through the left end of the beam. The ξ, η, ζ axes are a movable coordinate system having an origin that coincides with the centroid of any cross section a distance, z , from the origin of the fixed axes. The ξ, η axes coincide with the principal axes of the cross section. Only the elastic curve of the beam is shown in the top and front views. The deformation of the beam is defined by the two displacement components, u and v , in the x and y directions (negative as shown) and the angle of rotation, β . The right-hand screw sign convention is used for the sign of a rotation. Thus a rotation from x to y

Fig 2.--Coordinates and Displacements

would turn a right-hand screw in the positive z direction. Thus β in this direction is positive as shown.

In the analysis that follows in later chapters, it will be necessary to know the cosines of the angles between the x, y, z axes and the ξ, η, ζ axes (direction cosines) in terms of the displacement components, u , v , and β . Since this analysis will be limited to cases where the displacements are very small, the usual approximations are made: $\cos \beta = 1$, $\sin \beta = \beta$, etc. By dropping quantities of a higher order, Table 2, the direction cosine table, is obtained.

TABLE 2

DIRECTION COSINES FOR ξ, η, ζ AXES

	x	y	z
ξ	1	β	$-\frac{du}{dz}$
η	$-\beta$	1	$-\frac{dv}{dz}$
ζ	$\frac{du}{dz}$	$\frac{dv}{dz}$	1

In Figure 3(a) the displacements, u and v , in the x and y directions are shown. Also shown are the displacements, $\bar{u}$ and $\bar{v}$, which are the displacements in the ξ and η directions. From the geometry of the figure,

Fig. 3.--Displacements and Curvatures

with β negative as shown, the following equations are obtained:

$$\begin{aligned} u &= \bar{u} \cos \beta - \bar{v} \sin \beta \\ v &= \bar{u} \sin \beta + \bar{v} \cos \beta. \end{aligned} \quad (2-1)$$

When Eqs. (2-1) are differentiated twice and usual approximations are made that $\sin \beta = \beta$ and $\cos \beta = 1$, they become

$$\begin{aligned} u'' &= \bar{u}'' - \beta \bar{v}'' - 2\beta' \bar{v}' - \beta'' \bar{v} \\ v'' &= \beta \bar{u}'' + \bar{v}'' + 2\beta' \bar{u}' + \beta'' \bar{u}. \end{aligned} \quad (2-2)$$

The last two terms of both Eqs. (2-2) are thrown out although it is not obvious that they are small in comparison with the other terms. The justification of such a procedure lies in comparison with tests. Eqs. (2-2) then become

$$\begin{aligned} u'' &= \bar{u}'' - \beta \bar{v}'' \\ v'' &= \beta \bar{u}'' + \bar{v}''. \end{aligned} \quad (2-3)$$

If the displacements are small as has been assumed, the curvatures in the xz , yz , ξz and ηz directions are u'' , v'' , $\bar{u}''$, and $\bar{v}''$ respectively. The curvatures can be considered as vectors drawn parallel to the planes of curvature as shown in Figure 3(b) (see Bleich(1), Page 156). The geometry of Figure 3(b) also gives Eqs. (2-3).

When Eqs. (2-3) are solved for $\bar{u}''$ and $\bar{v}''$ the result is

$$\begin{aligned}\bar{u}'' &= (u'' + \beta v'') \frac{1}{(1 + \beta^2)} \\ \bar{v}'' &= (v'' - \beta u'') \frac{1}{(1 + \beta^2)} .\end{aligned}\tag{2-4}$$

From the elementary theories of bending and torsion,

$$\begin{aligned}B_1 \bar{u}'' &= M_\eta \\ B_2 \bar{v}'' &= -M_\xi \\ C \beta' &= M_\gamma .\end{aligned}\tag{2-5}$$

In Eqs. (2-5), B_1 and B_2 are the minimum and maximum flexural rigidities, C is the torsional rigidity and β' is the angle of twist per unit length. M_η , M_ξ , and M_γ are the components of the moment vector at the section. When Eqs. (2-4) are substituted into Eqs. (2-5) and rearranged the result is

$$\begin{aligned}B_1 u'' &= M_\eta + \frac{I_1'}{I_2} \beta M_\xi \\ B_2 v'' &= -M_\xi + \frac{I_2'}{I_1} \beta M_\eta \\ C \beta' &= M_\gamma .\end{aligned}\tag{2-6}$$

These equations are the same as Flint's(8) Eqs. (10) and are suitable for the lateral buckling of narrow rectangular beams, but are not accurate enough for application to I-beams. The third of Eqs. (2-6) assumes that all cross sections are free to warp. For solid cross sections, the prevention of warping produces a negligible effect on the angle of twist (see Timoshenko and Goodier(29), Page 302). With I-beams, channels and other thin-walled

members, the prevention of warping is accompanied by bending of the flanges and this may have considerable effect on the angle of twist.

Timoshenko(30) has provided the analysis for torsion of an I-beam. When warping is prevented, the external torque is balanced in part by the torsional stresses and in part by shearing stresses that result from the bending of the flanges. The portion balanced by the torsional stresses is

$$C\beta' \quad (2-7)$$

which is already included in the third of Eqs. (2-6). To include warping, consider Figure 4, which shows a cross section of a twisted I-beam and the shearing forces, Q , in the flanges. The displacement of the center of the top flange is

$$u_T = -\beta \frac{h}{2} \quad (2-8)$$

assuming the flange thickness is small in proportion to the beam depth, h . If D is the flexural rigidity of one flange (which is usually taken as equal to one-half of B_1), then from the elementary theory of bending

$$Q = D u_T''' \quad (2-9)$$

Fig. 4.--Torsion of an I-Beam

Differentiation of Eq. (2-8) with respect to z and substitution in Eq. (2-9) gives

$$Q = - \frac{Dh}{2} \beta'''$$

There is an equal force, Q , of opposite sign in the bottom flange and these two forces together form a couple, the moment of which is

$$- \frac{Dh^2}{2} \beta'''. \quad (2-10)$$

The two portions of the torque in the I-beam are given by (2-7) and (2-10), or

$$C\beta' - \frac{Dh^2}{2} \beta'''$$

Therefore, in order to make the third of Eqs. (2-6) applicable to I-beams, it is only necessary to add (2-10) to the left side. The result is

$$\begin{aligned} B_1 u'' &= M_\eta + \frac{I_1}{I_2} \beta M_\xi \\ B_2 v'' &= -M_\xi + \frac{I_2}{I_1} \beta M_\eta \\ C\beta' - \frac{Dh^2}{2} \beta''' &= M_T. \end{aligned} \quad (2-11)$$

Equations (2-11) are the final equations of equilibrium and are applicable to I-beams. For a particular problem, it will be necessary to determine the components of the moment to the right of the section about the x, y, z

axes, to resolve this moment about the ξ, η, ζ axes by using the direction cosines in Table 2 and to substitute this into Eqs. (2-11). In this way the differential equations are obtained.

CHAPTER III

TWO TYPES OF TORSIONAL RESTRAINT

Figure 1 showed the general kind of restraint considered. The restraining couples, T , act on the sides of the main beams. Figure 5 shows the elastic curve of a restraining member with couples, T , acting at the ends. Below this are the shear and moment diagrams. The moment of inertia, modulus of elasticity, and length of this member are designated: I_r , E_r and L_r , respectively.

By use of the area moment method for analysis of an elastic curve, it is found that the vertical distance from point 2 on the elastic curve to a tangent drawn to the curve at point 1 equals

$$\frac{TL_r^2}{12E_rI_r}.$$

When the above distance is divided by $\frac{L_r}{2}$, the slope at the end of the restraining beam is obtained which also equals β , the angle of twist of the main beam.

$$\beta = \frac{TL_r}{6E_rI_r}. \quad (3-1)$$

Fig. 5.--Elastic Curve, Shear and Moment Diagrams for a Restraining Member

K_r is now introduced as the stiffness of a restraint. It is defined as the moment, T , that will make the angle at the end equal to unity. By letting $\beta=1$ in Eq. (3-1) and solving for T , the result is

$$K_r = \frac{G E_r I_r}{L_r}. \quad (3-2a)$$

Two types of torsional restraint are considered. In the first type, there is a single restraining beam and this will be called a point torsional restraint. The torsional stiffness of the main beam is defined as

$$K_b = \frac{G C_T}{L}. \quad (3-3a)$$

A non-dimensional factor, K_p , is introduced and defined as the ratio of the stiffness of the restraint to the stiffness of the beam.

$$K_p = \frac{K_r}{K_b} = \frac{G E_r I_r L}{L_r G C_T}. \quad (3-4)$$

In the second type of torsional restraint, there are many closely spaced restraining beams, and this will be called a continuous torsional restraint. The stiffness of the restraint per unit length of the main beam will be

$$K_r = \frac{G E_r}{L_r} \left(\frac{I_r}{L} \right), \quad (3-2b)$$

with I_r the moment of inertia of all the closely spaced restraining beams. The torsional stiffness of the main beam per unit length of the main beam is

$$K_b = \frac{GC_T}{L^2}. \quad (3-3b)$$

Defining K_C , a non-dimensional factor for the continuous restraint, as the ratio of the stiffness of the restraint to the stiffness of the beam, we have

$$K_C = \frac{K_r}{K_b} = \frac{GE_r I_r L}{L_r G C_T}, \quad (3-4) \text{ Repeat}$$

which is the same result that was obtained for K_p .

Therefore, one can calculate K_p or K_C by Eq. (3-4) if one is careful to consider I_r as the sum of the moments of inertia of all the restraining beams, whether one or many. However, it will be necessary to understand the analysis, to think of K_r and K_b as per unit length of the main beam for a continuous restraint, and this is the reason for having two non-dimensional factors, K_p and K_C .

In order to get an idea of the order of magnitude of the factors K_C and K_p , two numerical examples are considered. Figure 6 shows the plan of a bay of a building frame. The closely spaced, lightweight open joists support a floor which must be designed for a total live and dead load of 200 lbs. per ft.². The bay is 24' x 24', and the

Fig. 6.--Light Weight Joists Restraining Two Main Girders

dotted lines show the possible laterally buckled positions of the main beams. Because the joists are closely spaced, the restraint is considered to be equivalent to a continuous torsional restraint. If the joists are designed as simple beams, the maximum moment taken by the joists in a one-foot strip perpendicular to the main beams is

$$\frac{200(24)^2}{8} = 14,400 \text{ ft.-lbs. per ft.},$$

and if the joists are 12 in. deep, I_r equals

$$\frac{14,400(12)6}{20,000} = 51.84 \text{ in.}^4 \text{ per ft.}$$

The last calculation was made by using that specification from the A.I.S.C. Manual(18), which limits the allowable stress in bending to 20,000 p.s.i. for A7 steel. Using the same specification for the design of the main girders, a 16 WF 64 is selected. (Incidentally, this beam would not be satisfactory without torsional restraint because $I_d/bt = 756$. Since this is greater than the permissible 600, the allowable stress is reduced.) Using Eq. (3-4) with $E = 29 \times 10^6$ p.s.i., $G = 11 \times 10^6$ p.s.i. and $C_T = 2.85 \text{ in.}^4$, as determined from the Bethlehem Steel Co. Handbook(28) of torsional properties,

$$K_C = \frac{G E_r I_r L}{L_r G C_T} = \frac{6 \cdot 29 \times 10^6 (51.84 \times 24) 24 \times 12}{24 \times 12 \cdot 11 \times 10^6 \cdot 2.85}.$$

$$K_C = 6905.$$

Note that the substitution for I_r , 51.84×24 , included all restraining joists.

As an example of a point torsional restraint, consider two 14 WF 64 main girders, 24 ft. in length, and restrained in the center by a filler joist as shown in Figure 1. The filler joist is a 12 WF 27, is non-load-carrying and is 12 ft. in length. The moment of inertia of this joist, I_r , is 204.1 in.^4 . Eq. (3-4) gives the value of the point torsional stiffness constant as follows:

$$K_p = \frac{G E_r I_r L}{L_r G C_T} = \frac{6 \cdot 29 \times 10^6 (204.1) 24 \times 12}{12 \times 12 \cdot 11 \times 10^6 \cdot 2.85}$$

$$K_p = 2265.$$

These numerical calculations were made to get an idea of the approximate size of K_c and K_p in practical cases. Actually, it cannot always be assumed that the connection between the main beam and restraining beam is 100 per cent rigid. However, it will be shown that even if only one per cent of the available restraint was provided in these cases, lateral buckling would be suppressed and design can be made on a stress basis only.

CHAPTER IV

EQUAL END MOMENTS, CONTINUOUS TORSIONAL RESTRAINT

In Figure 7 the elastic curve of an I-beam is shown. The load consists of two equal end couples, M , which are in the yz plane. The beam is shown after it has buckled laterally. In this position it is in equilibrium under the influence of a continuous elastic torsional restraint and the applied load. Equal reactive couples at the ends, called T , which are in the xy plane result from the moments applied by the restraint.

The restraint exerts a couple on an element, dz , of the beam which is

$$K_r \beta dz = \frac{K_c C}{L^2} \beta dz.$$

This can be integrated over half the beam to give T .

$$T = \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz.$$

The shear diagram of Figure 5 shows that there is a shearing force in the restraint which equals twice the couple at the ends divided by L_r . This shear force is

Fig. 7.--Elastic Curve, Equal End Moments, Continuous Torsional Restraint

up on one of the main beams and down on the other, but it will act up on the beam shown in Figure 7. Therefore on any element, dz , of the beam, the restraint exerts a force in the y direction that equals

$$\frac{z}{L_r} K_r \beta dz = \frac{z}{L_r} \frac{K_c C}{L^2} \beta dz.$$

This can be integrated over half the beam to give the reactive forces at the ends of the main beam due to this shearing force in the restraint.

$$\frac{z}{L_r} \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz = \frac{z T}{L_r}.$$

The above force is shown in Figure 7 acting in the negative y direction at a point on the x axis, a distance, u_r , from the origin. The displacement, u_r , equals the distance in the x direction to the center of gravity of the shear forces on half the beam.

The three components of the moment vector of all forces and couples to the left of the section due to the restraint alone are

$$\begin{aligned} M_x &= \frac{z}{L_r} z \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz - \frac{z}{L_r} \int_0^z \frac{K_c C}{L^2} \beta \frac{t}{2} dt \\ M_y &= 0 \\ M_z &= - \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz + \int_0^z \frac{K_c C}{L^2} \beta dt \\ &\quad - \frac{z(u_r - u)}{L_r} \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz - \frac{z(u - u_g)}{L_r} \int_0^z \frac{K_c C}{L^2} \beta dt. \end{aligned} \tag{4-1}$$

The displacement, u_g , is the distance to the center of gravity of the shear forces to the left of the section in the x direction. The sign convention that was used in Eqs. (4-1) and will be used from now on is as follows:

"When a moment vector acting on a section to the left is directed, as determined by the right-hand screw convention, in the negative direction of an axis, it is positive." The last two terms in the third of Eqs. (4-1) contain products of small displacements, such as u_β , $u_r\beta$, $u_g\beta$. Therefore, these terms are dropped. Eqs. (4-1), which include the moments of the restraint forces and couples only, become

$$\begin{aligned} M_x &= \frac{2}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta dt \\ M_y &= 0 \\ M_z &= -T + \int_0^z \frac{K_c C}{L^2} \beta dt. \end{aligned} \quad (4-2)$$

When the moments of the applied load are included, Eqs. (4-2) become

$$\begin{aligned} M_x &= -M + \frac{2}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta dt \\ M_y &= 0 \\ M_z &= -T + \int_0^z \frac{K_c C}{L^2} \beta dt. \end{aligned} \quad (4-3)$$

To resolve these components to the ξ, η, ζ , axes, the direction cosines from Table 2 are used giving

$$\begin{aligned}
 M_{\xi} &= -M + \frac{z}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt \\
 &\quad + u' T - u' \frac{K_c C}{L^2} \int_0^z \beta dt \\
 M_{\eta} &= \beta M + v' T - v' \frac{K_c C}{L^2} \int_0^z \beta dt \\
 &\quad - \beta \frac{z}{L_r} z T - \frac{\beta}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt \\
 M_{\zeta} &= -u' M - T + \frac{K_c C}{L^2} \int_0^z \beta dt + u' \frac{z}{L_r} z T \\
 &\quad - \frac{u'}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt.
 \end{aligned} \tag{4-4}$$

All nonlinear terms, such as terms containing $u'\beta$, $v'\beta$, and β^2 , are dropped from Eqs. (4-4). Displacements are considered to be very small, so these terms are of a higher order than the linear terms in the same equations. Since T is a linear function of β , Eqs. (4-4) become

$$\begin{aligned}
 M_{\xi} &= -M + \frac{z}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt \\
 M_{\eta} &= \beta M \\
 M_{\zeta} &= -u' M - T + \frac{K_c C}{L^2} \int_0^z \beta dt.
 \end{aligned} \tag{4-5}$$

Substitution in the equilibrium equations, Eqs. (2-11), gives

$$\begin{aligned}
 B_1 u'' &= \beta M + \frac{I_1}{I_2} \beta \left[-M + \frac{z}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta + dt \right] \\
 B_2 v'' &= M - \frac{z}{L_r} z T + \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta + dt \\
 &\quad + \frac{I_2}{I_1} \beta^2 M \\
 C \beta' - \frac{D h^2}{2} \beta''' &= -u' M - T + \frac{K_c C}{L^2} \int_0^z \beta + dt.
 \end{aligned} \tag{4-6}$$

Again nonlinear terms are dropped to give Eqs. (4-7).

$$\begin{aligned}
 B_1 u'' &= \beta M - \frac{I_1}{I_2} \beta M = \beta \gamma M \\
 B_2 v'' &= M - \frac{z}{L_r} z T + \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta + dt \\
 C \beta' - \frac{D h^2}{2} \beta''' &= -u' M - T + \frac{K_c C}{L^2} \int_0^z \beta + dt.
 \end{aligned} \tag{4-7}$$

γ is defined as $\frac{I_2 - I_1}{I_2}$. The second of Eqs. (4-7) does contain terms that result from the shear forces in the restraint, but the first and third of Eqs. (4-7) do not. Since the latter two equations are the ones that will be used to determine the buckling load, the shear forces in the restraint do not influence the solution in this linear analysis. It also follows that it is immaterial whether one considers a beam with the restraint shear force acting up or down.

To solve Eqs. (4-7) the third is differentiated with respect to z , remembering that T is not a function of z .

$$C \beta'' - \frac{D h^2}{2} \beta^{IV} = -u'' M + \frac{K_c C}{L^2} \beta.$$

Into this is substituted the value of μ'' from the first of Eqs. (4-7) giving

$$C\beta'' - \frac{Dh^2}{2}\beta^{IV} = -\frac{\beta\gamma M^2}{B_1} + \frac{K_c C}{L^2}\beta.$$

The latter equation is rearranged to give

$$\beta^{IV} - \frac{2C}{Dh^2}\beta'' - \left(\frac{2\gamma M^2}{B_1 Dh^2} - \frac{2K_c C}{L^2 Dh^2}\right)\beta = 0. \quad (4-8)$$

Eq. (4-8) is a fourth order linear differential equation in β with constant coefficients and can be solved by the usual methods. To begin, Eq. (4-8) is written as follows:

$$\beta^{IV} - \frac{1}{a^2}\beta'' - \frac{1}{d^4}\beta = 0, \quad (4-9)$$

with

$$a^2 = \frac{Dh^2}{2C}, \quad (4-10)$$

and

$$d^4 = \frac{B_1 Dh^2 L^2}{2\gamma M^2 L^2 - 2K_c C B_1}. \quad (4-11)$$

The boundary conditions are as follows:

- (a) $z = 0, \beta = 0$
- (b) $z = 0, \beta'' = 0$
- (c) $z = L, \beta = 0$
- (d) $z = L, \beta'' = 0.$

Boundary conditions (a) and (c) follow from the ends being completely restrained from rotating about the z axis. The displacement of the flanges, u_T , and the other displacements, u and β , are related as follows:

$$u_T = u \pm \frac{h}{2} \beta, \quad (4-12)$$

h being the depth of the beam. When Eq. (4-12) is differentiated twice, the result is

$$u_T'' = u'' \pm \frac{h}{2} \beta'' \quad (4-13)$$

Since the beam is free to rotate about the y axis at the ends, $u'' = 0$. Also the flange ends are free to rotate about the minor axis of the beam, consequently $u_T'' = 0$. From this, it may be concluded that $\beta'' = 0$ at $z = 0$ and $z = L$. These are boundary conditions (b) and (d).

Assuming that $\beta = e^{rz}$ is a solution of Eq. (4-9), substitution in this equation gives

$$e^{rz} \left(r^4 - \frac{1}{a^2} r^2 - \frac{1}{d^4} \right) = 0.$$

Solving for the roots

$$r^2 = \frac{1}{2a^2} \pm \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}}. \quad (4-14)$$

The term under the radical in Eq. (4-14) can be positive, negative, or zero. First, assuming that the term is zero, the roots of Eq. (4-14) are all real since a^2 is real and positive. There are two distinct real roots and two

repeated real roots giving the general solution to Eq. (4-9) as

$$\beta = Ae^{rz} + Bze^{rz} + Ce^{-rz} + Dze^{-rz},$$

with

$$r = +\sqrt{\frac{1}{2a_2}}.$$

The general solution can be put in the following form:

$$\beta = C_1 \sinh rz + C_2 z \sinh rz + C_3 \cosh rz + C_4 z \cosh rz. \quad (4-15)$$

Boundary condition (a) gives $C_3 = 0$. When Eq. (4-15) is differentiated twice

$$\begin{aligned} \beta'' = & C_1 r^2 \sinh rz + 2C_2 r \cosh rz \\ & + C_2 z r^2 \sinh rz + 2C_4 r \sinh rz \\ & + C_4 z r^2 \cosh rz. \end{aligned} \quad (4-16)$$

Boundary condition (b) gives $C_2 = 0$. When boundary conditions (c) and (d) are substituted into Eqs. (4-15) and (4-16), the result is the following two homogeneous linear equations:

$$0 = C_1 \sinh rL + C_4 L \cosh rL$$

$$0 = C_1 r^2 \sinh rL + C_4 (2r \sinh rL + r^2 L \cosh rL).$$

The determinant of the coefficient matrix is

$$2r \sinh^2 rL,$$

and the only solution that makes it equal to zero is $r = 0$. This is not possible since a^2 is real and greater than zero. Therefore the term under the radical of Eq. (4-14) cannot be zero as assumed.

If the aforesaid term is assumed to be negative, the four roots of Eq. (4-14) can be expressed as:

$$r_{1,2} = \pm \sqrt{s+t\iota} \quad , \quad r_{3,4} = \pm \sqrt{s-t\iota} \quad ,$$

with

$$s = \frac{1}{2a^2} \quad \text{and} \quad t = \sqrt{\left(\frac{1}{4a^2} + \frac{1}{d^2}\right)}.$$

Both s and t are positive and real and the general solution to Eq. (4-9) is

$$\begin{aligned} \beta = & A e^{\sqrt{s+t\iota}z} + B e^{-\sqrt{s+t\iota}z} \\ & + C e^{\sqrt{s-t\iota}z} + D e^{-\sqrt{s-t\iota}z}. \end{aligned} \quad (4-17)$$

When Eq. (4-17) is differentiated twice

$$\begin{aligned} \beta'' = & A(s+t\iota)e^{\sqrt{s+t\iota}z} + B(s+t\iota)e^{-\sqrt{s+t\iota}z} \\ & + C(s-t\iota)e^{\sqrt{s-t\iota}z} + D(s-t\iota)e^{-\sqrt{s-t\iota}z}. \end{aligned} \quad (4-18)$$

Substitution of boundary conditions (a) and (b) into Eq. (4-17) and (4-18) gives

$$\begin{aligned} 0 &= A + B + C + D \\ 0 &= A(s+t\iota) + B(s+t\iota) + C(s-t\iota) + D(s-t\iota). \end{aligned} \quad (4-19)$$

When the first of Eq. (4-19) is multiplied by $-(s+t\iota)$ and added to the second

$$C = -D \quad \text{and} \quad A = -B.$$

Boundary conditions (c) and (d) and the last result give

$$\begin{aligned}
 & B(e^{-\sqrt{s+\tau\dot{\mu}}L} - e^{\sqrt{s+\tau\dot{\mu}}L}) + D(e^{-\sqrt{s-\tau\dot{\mu}}L} - e^{\sqrt{s-\tau\dot{\mu}}L}) = 0 \\
 & B[(s+\tau\dot{\mu})e^{-\sqrt{s+\tau\dot{\mu}}L} - (s+\tau\dot{\mu})e^{\sqrt{s+\tau\dot{\mu}}L}] \\
 & + D[(s-\tau\dot{\mu})e^{-\sqrt{s-\tau\dot{\mu}}L} - (s-\tau\dot{\mu})e^{\sqrt{s-\tau\dot{\mu}}L}] = 0.
 \end{aligned} \tag{4-20}$$

When the first of Eq. (4-20) is multiplied by $-(s+\tau\dot{\mu})$ and the two equations are added

$$-2D\tau\dot{\mu}[e^{-\sqrt{s-\tau\dot{\mu}}L} - e^{\sqrt{s-\tau\dot{\mu}}L}] = 0,$$

or

$$-2D\tau\dot{\mu} = 0,$$

since the term represented by the brackets is not equal to zero. Since $2\tau\dot{\mu} \neq 0$, $D = 0$. Now from the first of Eqs. (4-20)

$$B[e^{-\sqrt{s+\tau\dot{\mu}}L} - e^{\sqrt{s+\tau\dot{\mu}}L}] = 0$$

or $B = 0$ since the term represented by the brackets is not equal to zero. Thus $A = B = C = D = 0$, and there is no solution when the term under the radical of Eq. (4-14) is negative. The only possibility left is that it is positive.

The roots of Eq. (4-14) can be expressed now as follows:

$$n_{1,2} = \pm \sqrt{\frac{1}{2a^2} + \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}}}$$

$$t_{1,2} = \pm \sqrt{\frac{1}{2a^2} - \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}}}.$$

The roots, $n_{1,2}$, are known now to be real. However, the roots $t_{1,2}$, can be real or imaginary. Assuming they are real, the solution to Eq. (4-9) can be put in the following form:

$$\beta = A \sinh n z + B \cosh n z + C \sinh t z + D \cosh t z.$$

Differentiating the above twice gives the following equation:

$$\beta'' = A n^2 \sinh n z + B n^2 \cosh n z + C t^2 \sinh t z + D t^2 \cosh t z.$$

Substitution of boundary conditions (a) and (b) gives

$$0 = B + D$$

$$0 = n^2 B + t^2 D,$$

or

$$t^2 - n^2 = 0.$$

From this

$$\sqrt{\frac{1}{4a^4} + \frac{1}{d^4}} = 0,$$

and we have already proved that this term cannot be zero.

Therefore $B = D = 0$. Substitution of boundary conditions (c) and (d) gives

$$0 = A \sinh nL + C \sinh tL$$

$$0 = An^2 \sinh nL + Ct^2 \sinh tL,$$

or

$$\sinh nL(t^2 \sinh tL) - n^2 \sinh nL(\sinh tL) = 0,$$

or again

$$t^2 - n^2 = 0.$$

Therefore $A = C = 0$. Only the trivial solution is possible if the roots, $t_{1,2}$, are real.

The general solution of Eq. (4-9) can be expressed as follows:

$$\beta = C_1 \sin mz + C_2 \cos mz + C_3 e^{nz} + C_4 e^{-nz}, \quad (4-21)$$

with

$$m = \sqrt{-\frac{1}{2a^2} + \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}}}, \quad (4-22)$$

and

$$n = \sqrt{\frac{1}{2a^2} + \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}}}.$$

Both m and n are assumed to be positive and real since this is the only possibility that is left.

Boundary conditions (a) and (b) substituted into Eq. (4-21) give two equations as follows:

$$\begin{aligned} C_2 + C_3 + C_4 &= 0 \\ -m^2 C_2 + n^2 C_3 + n^2 C_4 &= 0, \end{aligned}$$

from which $C_2 = 0$ and $C_3 = -C_4$. Eq. (4-21) becomes

$$\beta = C_1 \sin mz + 2C_3 \sinh nz. \quad (4-23)$$

Now substituting boundary conditions (c) and (d) in Eq. (4-23) gives

$$\begin{aligned} C_1 \sin mL + 2C_3 \sinh nL &= 0 \\ -m^2 C_1 \sin mL + n^2 2C_3 \sinh nL &= 0. \end{aligned}$$

The above two homogeneous linear equations in C_1 and C_3 must have a coefficient matrix whose determinant equals zero for a non-trivial solution. In other words

$$\sin mL (2m^2 + 2n^2) \sinh nL = 0.$$

In order to satisfy this equation, since $\sinh nL$ may not equal zero for all n , $\sin mL = 0$, and

$$mL = q\pi$$

with q an integer. Squaring both sides of Eq. (4-22) and substitution of m^2 from the above gives

$$-\frac{1}{2a^2} + \sqrt{\frac{1}{4a^4} + \frac{1}{d^4}} = \frac{q^2 \pi^2}{L^2}.$$

Using the notation from Eqs. (4-10) and (4-11), letting

$M = M_{cr}$, where M_{cr} is the critical value of the moment which causes buckling

$$M_{cr}^2 = \frac{B_1 C}{y} \frac{\pi^2}{L^2} \left[\frac{a^2}{L^2} q^4 m^2 + q^2 + \frac{K_c}{\pi^2} \right]. \quad (4-24)$$

This result agrees with a solution provided by Vlasov(9). His Eq. (3-6), Page 363, reduces to Eq. (4-24), if his horizontal restraint constant is equated to zero and the length of the beam is taken as L .

Eq. (4-24) shows that M_{cr} will take on its smallest value when $q = 1$, regardless of the value of K_c . This is an unusual result because in other stability problems (for example, see Timoshenko(5), Page 100) buckling in more than one half-wave will result if the elastic spring constant is large enough. However, in this case only the one half-wave solution is of practical interest. Substitution of $q = 1$ in Eq. (4-24) gives the following result:

$$M_{cr} = \frac{\pi}{L} \sqrt{\frac{B_1 C}{y}} \sqrt{1 + \frac{\pi^2 a^2}{L^2} + \frac{K_c}{\pi^2}}. \quad (4-25)$$

When $K_c = 0$, or a condition of no restraint, Eq. (4-25) reduces to Timoshenko's(5) Eq. 159, Page 261, except for y . ($y \approx 1$ if I_1 is very small in comparison to I_2 .) Also, if L^2/a^2 is very large, as it is for a long narrow beam, Eq. (4-25) reduces further to Timoshenko's(5)

solution for a deep, narrow beam of rectangular cross section (Eq. 141, Page 242).

Using a form similar to Timoshenko's(5), the results are expressed in the following way:

$$M_{cr} = \frac{m}{L} \sqrt{\frac{B_1 C}{y}} \quad (4-26)$$

$$m = \pi \sqrt{1 + \frac{\pi^2 a^2}{L^2} + \frac{K_c}{\pi^2}} \quad (4-27)$$

A study was made of several standard steel I-beams to determine typical values of the parameter, $(L/a)^2$. This parameter occurs in Eq. (4-27) and in later chapters. In Table 3, "a" was taken from the Bethlehem Steel Co. Handbook(31). For each section considered, $(L/a)^2$ was determined for two limiting span lengths, L. The smaller span is that at which Ld/b_t equals 600. The larger span was taken as twenty-four times the beam depth, since this is often a limiting criterion for deflection. The engineer will be concerned with lateral buckling only when the span length is within this range. From this table, one can see that all the values of $(L/a)^2$ were between 2 and 55 for this typical but limited sample of beams.

TABLE 3

TYPICAL VALUES OF THE DIMENSIONLESS
PARAMETER, L^2/a^2

Section	a (in.)	L (ft.)	L^2/a^2
36 WF 300	119.1	72	52.6
		38	14.7
33 WF 141	122.7	67	42.9
		16.5	2.6
24 WF 130	118.4	48	23.7
		26	6.9
18 WF 64	72.60	36	35.4
		16.5	7.4
16 WF 40	69.62	32	30.4
		11	3.6
14 WF 30	70.93	28	22.4
		9.5	2.6
12 WF 27	60.96	24	22.3
		10.5	4.3

Values of the factor, m , in Eq. (4-26) were calculated by Eq. (4-27) for various L^2/a^2 and K_c , and the results are tabulated in Table 4. The table is used by first obtaining "a" from the Bethlehem Steel Co. Handbook(31), computing

TABLE 4

VALUES OF m IN EQ. (4-26) FOR DIFFERENT L^2/a^2 AND K_c

$K_c \backslash L^2/a^2$	0.1	1	2	3	4	6
0	31.37	10.36	7.65	6.51	5.85	5.11
20	31.69	11.28	8.86	7.90	7.36	6.79
40	32.00	12.14	9.23	9.07	8.62	8.13
60	32.31	12.93	10.89	10.12	9.71	9.28
80	32.62	13.69	11.77	11.06	10.69	10.30
100	32.92	14.40	12.59	11.93	11.59	11.23
120	33.23	15.08	13.36	12.74	12.42	12.09
140	33.53	15.73	14.09	13.50	13.20	12.89
160	33.82	16.35	14.78	14.22	13.94	13.64
180	34.12	16.95	15.45	14.91	14.64	14.36
200	34.41	17.53	16.08	15.57	15.30	15.04
220	34.70	18.09	16.69	16.20	15.94	15.69
240	34.99	18.64	17.28	16.80	16.56	16.31
260	35.27	19.16	17.85	17.39	17.15	16.91
280	35.55	19.68	18.40	17.95	17.72	17.50
300	35.83	20.18	18.94	18.50	18.28	18.06
320	36.11	20.67	19.46	19.04	18.82	18.60
340	36.39	21.15	19.96	19.55	19.34	19.13
360	36.66	21.62	20.46	20.06	19.85	19.65
380	36.93	22.07	20.94	20.55	20.35	20.15
400	37.20	22.52	21.41	21.03	20.84	20.64
420	37.47	22.96	21.88	21.50	21.31	21.12
440	37.73	23.39	22.33	21.96	21.78	21.59
460	38.00	23.82	22.77	22.41	22.23	
480	38.26	24.23	23.21	22.85	22.68	
500	38.52	24.64	23.63	23.29		
520	38.78	25.04	24.05	23.71		
540	39.04					
560	39.29					

TABLE 4 - Continued

$K_c \backslash L^2/a^2$	8	10	12	16	20	24
0	4.70	4.43	4.24	3.99	3.84	3.73
20	6.48	6.29	6.16	6.00	5.89	5.82
40	7.88	7.72	7.61	7.48	7.40	7.34
60	9.06	8.92	8.83	8.72	8.65	8.60
80	10.10	9.98	9.90	9.80	9.73	9.69
100	11.05	10.94	10.86	10.77	10.71	10.67
120	11.92	11.82	11.75	11.66	11.61	11.57
140	12.73	12.63	12.57	12.49	12.44	12.41
160	13.49	13.40	13.34	13.26	13.22	13.19
180	14.21	14.13	14.07	14.00	13.95	13.93
200	14.90	14.82	14.76	14.70	14.65	14.63
220	15.56	15.48	15.43	15.36	15.32	15.29
240	16.19	16.11	16.06	16.00	15.96	15.94
260	16.79	16.72	16.67	16.61	16.58	16.55
280	17.38	17.31	17.26	17.20	17.17	17.14
300	17.95	17.88	17.83	17.78	17.74	17.72
320	18.49	18.43	18.38	18.33	18.30	18.27
340	19.03	18.96	18.92	18.87	18.83	18.81
360	19.55	19.48	19.44	19.39	19.36	19.34
380	20.05	19.99	19.95	19.90	19.87	
400	20.54	20.48	20.44	20.40	20.37	

TABLE 4 - Continued

$K_c \backslash L^2/a^2$	28	32	36	40	100
0	3.65	3.59	3.55	3.51	3.29
20	5.77	5.74	5.71	5.68	5.55
40	7.30	7.27	7.25	7.23	7.13
60	8.56	8.54	8.52	8.50	8.42
80	9.66	9.64	9.62	9.61	9.53
100	10.65	10.63	10.61	10.60	10.53
120	11.55	11.53	11.51	11.50	11.44
140	12.38	12.37	12.35	12.34	12.28
160	13.17	13.15	13.14	13.12	13.07
180	13.90	13.89	13.88	13.87	13.81
200	14.61	14.59	14.58	14.57	14.52
220	15.28	15.26	15.25	15.24	15.19
240	15.92	15.90	15.89	15.88	15.84
260	16.53	16.52	16.51	16.50	16.46
280	17.13	17.11	17.10	17.10	17.05
300	17.70	17.69	17.68	17.67	17.63
320	18.25	18.24	18.24	18.23	18.19
340	18.80	18.79	18.78	18.77	18.73
360	19.32	19.31	19.31	19.30	19.26

L^2/a^2 , and then finding "m" by interpolation for the known value of K_c .

A factor, called s, the Critical Stress Factor, is introduced as follows:

$$S = \frac{m}{(m)_{K_c=0}} = \frac{M_{cr}}{(M_{cr})_{K_c=0}} \quad (4-28)$$

A similar factor was used by Flint(10) to show the effect of a restraint. The factor can be thought of as a multiplier which, when applied to the critical buckling stress without restraint, yields the buckling stress with restraint. In Figure 8, values of s are plotted against K_c for several values of L^2/a^2 . Keeping in mind the usual range of values of the non-dimensional stiffness factor, K_c , the figure shows that the critical buckling stress will be multiplied several times by a relatively small K_c . For a small L^2/a^2 , the effect of K_c is less; however in these cases, design would probably be made on the basis of a permissible stress, which is not reduced by stability considerations (see Table 3 for values of L^2/a^2 when $Ld/bt = 600$). Also note from Table 4 that in the case of L^2/a^2 as small as 2, K_c would have to be only 180 to double the buckling load ($s = 2$). It is recommended that Table 4 be used for quantitative results because of the greater accuracy possible from tabular interpolation.

Fig. 8.--The Effect of Continuous Torsional Restraint on The Critical Stress Factor for Equal End Moments

CHAPTER V

CONCENTRATED LOAD AT MID-SPAN, CONTINUOUS TORSIONAL RESTRAINT

Figure 9 shows the elastic curve of an I-beam with a continuous elastic torsional restraint after it has been buckled by a load, P , applied at the center of the beam. The restraint results in equal reactive couples, T , supplied to the ends in the xy plane. The forces on the beam due to the shear force in the restraint are not shown for clarity, but they would appear as was shown in Figure 7, where a different applied load was considered. At any section, a distance z from the left end, the moments due to all the restraint forces and couples are

$$M_x = \frac{z}{L_r} T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt$$

$$M_y = 0$$

$$M_z = -T + \int_0^z \frac{K_c C}{L^2} \beta dt,$$

(4-2)
Repeat

with $T = \int_0^{\frac{L}{2}} \frac{K_c C}{L^2} \beta dz.$

Fig. 9.--Elastic Curve, Concentrated Load, Continuous Torsional Restraint

The moments due to the applied load alone are

$$M_x = -\frac{P}{2} z$$

$$M_y = 0$$

$$M_z = -\frac{P}{2} (\mu_c - \mu),$$

since both μ and μ_c are shown as negative in Figure 9. The total moments therefore are

$$M_x = -\frac{P}{2} z + \frac{2}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta dt$$

$$M_y = 0$$

$$M_z = -\frac{P}{2} (\mu_c - \mu) - T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

Table 2 is used to get the moments about the ξ , η , ζ axes. After the small nonlinear terms are dropped, they become

$$M_\xi = -\frac{P}{2} z + \frac{2}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta dt$$

$$M_\eta = \frac{P}{2} z \beta$$

(5-1)

$$M_\zeta = -\frac{P}{2} z \mu' - \frac{P}{2} (\mu_c - \mu) - T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

Eqs. (5-1) are substituted into the equilibrium equations, Eqs. (2-11), and nonlinear terms are dropped to give

$$\begin{aligned}
 B_1 u'' &= \frac{P}{2} z \beta - \frac{I_2}{I_1} \beta \frac{P}{2} z \\
 B_2 v'' &= \frac{P}{2} z - \frac{2}{L_r} z T + \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt \\
 C \beta' - \frac{D h^2}{2} \beta''' &= -\frac{P}{2} z u' - \frac{P}{2} (\mu_c - \mu) - T \\
 &\quad + \int_0^z \frac{K_c C}{L^2} \beta dt.
 \end{aligned} \tag{5-2}$$

The first of Eqs. (5-2) becomes

$$B_1 u'' = \frac{P}{2} z \beta \gamma, \tag{5-3}$$

with $\gamma = \frac{I_2 - I_1}{I_2}$. Differentiating the third of the same equations gives

$$C \beta'' - \frac{D h^2}{2} \beta^{IV} = -\frac{P}{2} z u'' + \frac{K_c C}{L^2} \beta. \tag{5-4}$$

Eliminating u'' from Eqs. (5-3) and (5-4) gives

$$C \beta'' - \frac{D h^2}{2} \beta^{IV} = -\left(\frac{P}{2} z\right)^2 \frac{\beta \gamma}{B_1} + \frac{K_c C}{L^2} \beta,$$

which when rearranged is

$$\beta^{IV} - \frac{2C}{D h^2} \beta'' - \left[\frac{P^2 \gamma z^2}{B_1^2 h^2} - \frac{2 K_c C}{L^2 D h^2} \right] \beta = 0. \tag{5-5}$$

Eq. (5-5) is a linear differential equation in β with variable coefficients. Since a numerical solution is planned, it is written in non-dimensional form by introducing $z' = \frac{z}{L}$, substituting and then dropping the primes. The result is

$$\beta^{IV} - \frac{2L^2 C}{Dh^2} \beta'' - \left[\frac{P^2 \gamma L^6}{B_1^2 h^2} z^2 - \frac{2K_c CL^2}{Dh^2} \right] \beta = 0. \quad (5-6)$$

Using the previously introduced notation for a^2 and taking the solution for P_{cr} in the following form,

$$P_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}, \quad (5-7)$$

Eqs. (5-6) and (5-7) can be reduced to

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - \left[\frac{m^2}{4} \frac{L^2}{a^2} z^2 - K_c \frac{L^2}{a^2} \right] \beta = 0. \quad (5-8)$$

Eq. (5-3) was also put in non-dimensional form by letting $z' = \frac{z}{L}$ and $u' = \frac{u}{L}$, substituting, and then dropping the primes.

$$\frac{L}{h\sqrt{\gamma}} u'' = \frac{m}{4} \frac{L}{a} z \beta. \quad (5-9)$$

Eqs. (5-8) and (5-9) were solved with the following boundary conditions:

- (a) $z = 0, \beta = 0$
- (b) $z = 0, \beta'' = 0$
- (c) $z = 0, \mu = 0$
- (d) $z = 0.5, \beta' = 0$
- (e) $z = 0.5, \mu' = 0$

and special condition (f)

- (f) $z = 0.5, \beta''' = 0.$

Since the conditions at the ends of the beam are the same as they were for the problem in Chapter IV, boundary conditions (a) and (b) are also the same. Since the beam is not free to translate in the x direction at the ends, boundary condition (c) is true. Boundary conditions (d) and (e) follow from the assumed symmetry at the center of the beam. Special condition (f) comes from the third of Eqs. (5-3), since $\mu' = \beta' = (\mu_c - \mu) = 0$ and $T = \int_0^z \frac{K_c C}{L^2} \beta dt$ when $z = \frac{L}{2}$.

The method used to solve the boundary value problem above was originally due to Young(32), generalized by Hoblit(33) and adapted for use with a high speed computer by Ojalvo and Rajagopalan(34). The half-beam was broken into a number of equal segments. At $z = 0$, or station 0 along the beam, $\beta_0 = 0$ and $\beta_0'' = 0$ by boundary conditions (a) and (b). β_0' was taken equal to 1.0. (The value of β_0' chosen will not influence the calculations because in the

stability problem the magnitudes of the displacements are indeterminate.) β_0''' was unknown at the start and was taken ~~to~~ equal to k. (k will be determined later from special condition (f) at $z = 0.5$.) β^{IV} , β^{V} , and β^{VI} were determined at all stations by the differential equation (Eq. (5-8)) and the two equations that result when Eq. (5-8) is differentiated twice. To go from one station along the beam to another, the following form of the Taylor Series was used, with r the length of the beam segment or distance between stations:

$$\begin{aligned}\beta_i &= \beta_{i-1} + \beta_{i-1}' r + \beta_{i-1}'' \frac{r^2}{2!} + \beta_{i-1}''' \frac{r^3}{3!} + \beta_{i-1}^{IV} \frac{r^4}{4!} \\ &\quad + \beta_{i-1}^V \frac{r^5}{5!} + \beta_{i-1}^{VI} \frac{r^6}{6!} \\ \beta_i' &= \beta_{i-1}' + \beta_{i-1}'' r + \beta_{i-1}''' \frac{r^2}{2!} + \beta_{i-1}^{IV} \frac{r^3}{3!} + \beta_{i-1}^V \frac{r^4}{4!} \\ &\quad + \beta_{i-1}^{VI} \frac{r^5}{5!} \\ \beta_i'' &= \beta_{i-1}'' + \beta_{i-1}''' r + \beta_{i-1}^{IV} \frac{r^2}{2!} + \beta_{i-1}^V \frac{r^3}{3!} + \beta_{i-1}^{VI} \frac{r^4}{4!} \\ \beta_i''' &= \beta_{i-1}''' + \beta_{i-1}^{IV} r + \beta_{i-1}^V \frac{r^2}{2!} + \beta_{i-1}^{VI} \frac{r^3}{3!}.\end{aligned}$$

In the equations above, superscripts refer to the derivative and subscripts refer to the station along the beam. Numerical values were chosen for the parameters, L^2/a^2 and K_c , in Eq. (5-8). m was chosen as a guess or trial value. Storage space in the computer was reserved for two multi-dimensional arrays. Into one array was put all

derivatives of β at every station starting with $\beta'_0 = 1.0$ and $\beta''_0 = 0$. The same thing was done with the other array, starting with $\beta'_0 = 0$ and $\beta''_0 = 1.0$. The constant, k , was determined as a multiplier for the second array so that special condition (f) was satisfied at the center of the beam. The value of any derivative at a station was the first array minus the second array times k . In this fashion, the value of β' at the center was determined for the trial value of m (or P_{cr}). If β' was positive, P_{cr} was too small and therefore m was increased by 0.1 of m , and all calculations were repeated. This operation was continued until β' at $z = 0.5$ became negative, at which time m was decreased by 0.01 of m . This was continued until β' at $z = 0.5$ became positive again. m was then increased by 0.001 of m until the sign of β' changed again. This could have been continued to any accuracy desired, but it was decided that m to an accuracy of 0.1 of one per cent was sufficient. At the end of this step, k was again determined along with β and all six derivatives of β at every station.

Eq. (5-9) was used to determine the values of u and the derivatives of u at all stations. Since m and all the values of β had been determined at this point by the program and L^2/a^2 had a given numerical quantity, the left side of Eq. (5-9) was determined. (Results must be

divided by L and multiplied by $h\sqrt{s}$, but henceforth will be referred to as u , u' , etc., for simplicity.)

Starting at the left end of the beam, the first station, $u_0 = 0$ was taken as boundary condition (c) and $u'_0 = -1.0$ was arbitrarily selected to start with. (Correct value to be determined by boundary condition (e) at the center of the beam.) At all other stations, u and u' were determined by using seven and six terms of the Taylor Series as follows:

$$\begin{aligned}
 u_i &= u_{i-1} + u'_{i-1} r + u''_{i-1} \frac{r^2}{2!} + u'''_{i-1} \frac{r^3}{3!} + u^{IV}_{i-1} \frac{r^4}{4!} \\
 &\quad + u^{V}_{i-1} \frac{r^5}{5!} + u^{VI}_{i-1} \frac{r^6}{6!} \\
 u'_i &= u'_{i-1} + u''_{i-1} r + u'''_{i-1} \frac{r^2}{2!} + u^{IV}_{i-1} \frac{r^3}{3!} + u^{V}_{i-1} \frac{r^4}{4!} \\
 &\quad + u^{VI}_{i-1} \frac{r^5}{5!}
 \end{aligned}$$

Higher derivatives of u were determined from Eq. (5-9) and the equations that result when Eq. (5-9) is repeatedly differentiated. When the center of the beam was reached, u' at $z = 0.5$ was calculated and compared with the boundary condition (e), $u' = 0$ at $z = 0.5$. The difference was subtracted from the starting value of $u' = -1.0$ at the first station and all values of u and derivatives of u were recalculated.

At this point, all the calculations had been

completed and the computer read out as data the value of m and β , β' , u , and u' at all stations. From this data it was possible to check to see whether β remained positive and u negative for the entire half of the beam. In this way the possibility of obtaining m for a half-wave solution greater than one was eliminated.

Errors in the above numerical calculations come from two sources. First there is the roundoff error when m is narrowed down to 0.1 of one per cent, and second there is the error from the truncation of the Taylor Series. To check on the convergence, the program was run for the four cases of $L^2/a^2 = 16$ and $K_c = 0, 20, 40$ and 60 , using 5, 10, 25 and 50 segments for the half-beam. It was expected that the greater the number of segments, the smaller the segment length, r , and the smaller the terms in the Taylor Series containing r^7 , r^8 , etc., would become. The results showed no difference at all in m to eight significant figures when 10, 25 and 50 segments were used, and a small difference when 5 segments were used in all four cases. There was also good agreement in the values of β , β' , u , and u' at like stations.

The method described above was used for the solution of this and other problems and henceforth will be referred to as the "Taylor Series Method." Another method was used as a check, and it is described in what follows

and will be referred to as the "Recursion Formula Method."

Let us assume that the solution to Eq. (5-8) can be taken in the form of a power series, $\beta = \sum_{n=0}^{\infty} b_n z^n$. By boundary conditions (a) and (b), b_0 and b_2 will both have to equal zero. By differentiation and substitution in Eq. (5-8), the following is obtained:

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) b_n z^{n-4} - \frac{L^2}{a^2} \sum_{n=2}^{\infty} n(n-1) b_n z^{n-2} - \frac{m^2 L^2}{4 a^2} \sum_{n=0}^{\infty} b_n z^{n+2} + K_c \frac{L^2}{a^2} \sum_{n=0}^{\infty} b_n z^n = 0.$$

Shifting the index, n , so that all powers of z are the same gives

$$\sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1) b_{n+4} z^n - \frac{L^2}{a^2} \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} z^n - \frac{m^2 L^2}{4 a^2} \sum_{n=2}^{\infty} b_{n-2} z^n + K_c \frac{L^2}{a^2} \sum_{n=0}^{\infty} b_n z^n = 0,$$

and by letting $n = 0, 2, 4, 6, \dots$ it is noted that all even coefficients of the series are zero. This is because b_0 and b_2 are zero by boundary conditions (a) and (b). By letting $n = 1$, the following equation results:

$$b_5 = \frac{6 \frac{L^2}{a^2} b_3 - K_c \frac{L^2}{a^2} b_1}{120}, \quad (5-10)$$

and in general for $n \geq 3$ there is the following recursion formula:

$$b_{n+4} = \frac{\frac{L^2}{a^2} b_{n+2}}{(n+4)(n+3)} + \frac{\frac{m^2 L^2}{4 a^2} b_{n-2} - K_c \frac{L^2}{a^2} b_n}{(n+4)(n+3)(n+2)(n+1)}. \quad (5-11)$$

Using Eqs. (5-10) and (5-11), any coefficient of the series can be obtained in terms of b_1 and b_3 . With boundary conditions (d) and special condition (f), the following series were obtained from $\beta = \sum_{n=0}^{\infty} b_n z^n$:

$$0 = b_1 + 3(2)^{-2}b_3 + 5(2)^{-4}b_5 + 7(2)^{-6}b_7 + 9(2)^{-8}b_9 + \dots$$

$$\begin{aligned} 0 = (3)(2)b_3 + (5)(4)(3)(2)^{-2}b_5 + (7)(6)(5)(2)^{-4}b_7 \\ + (9)(8)(7)(2)^{-6}b_9 + \dots \end{aligned} \quad (5-12)$$

Since all coefficients after b_3 can be expressed in terms of b_1 and b_3 , the two above series, when the terms are added up take on the following form:

$$0 = Ab_1 + Bb_3$$

$$0 = Cb_1 + Db_3$$

The two linear homogeneous equations above in b_1 and b_3 must have a coefficient matrix whose determinant equals zero, or

$$AD - CB = 0.$$

The IBM-7090 computer was used to advantage to solve this problem also. Numerical values were picked for L^2/a^2 and K_c , and a trial value of m was chosen. For a predetermined number of terms of the two series (Eqs. (5-12)), the value of the determinant, $AD - CB$, was calculated as outlined above. If this was positive, m was increased by 0.1 of m , and the calculations were repeated and the whole

operation was repeated until the determinant became negative. Then m was reduced by 0.01 of m and so on. The value of m was narrowed down to 0.1 of one per cent in exactly the same way as was done in the Taylor Series Method. It was necessary in this method, however, to test the effect of increasing and decreasing m on the value of the determinant, while in the Taylor Series Method it was known that a positive slope at the center meant m was too small.

Errors in this method come from the truncation of the series and the roundoff of m . The program was tested for $L^2/a^2 = 16$, and four cases where K_c was equal to 0, 20, 40 and 60. Each case was run with the last included term of the series being term number 11, then 13, then 15, and so on up to term number 23 of the series. It was found that by using term 15, or more of the series, identical results for m to eight significant figures were obtained in all four cases.

A comparison of results for the two methods showed them to agree to the eighth significant figure. That is, by using 10 or more segments for the beam or 15 or more terms of the series, one gets the same results when $L^2/a^2 = 16$ and $K_c = 0, 20, 40$ and 60. (A more complete check was made on the final runs.) This was to be expected since m was iterated in the same way in both cases.

Table 5 gives values of m in Eq. (5-7) for different K_c and L^2/a^2 . These values of m were determined by the Taylor Series Method, using 25 beam segments and then over half were checked by the Recursion Formula Method using up to the 21st term of the series. In no case was a difference of over 0.1 of one per cent discovered.

Figure 10 shows s plotted against K_c for several L^2/a^2 . s is defined as before as follows:

$$s = \frac{m}{(m)_{K_c=0}} = \frac{P_{cr}}{(P_{cr})_{K_c=0}} \quad \begin{array}{l} (4-28) \\ \text{Repeat} \end{array}$$

Qualitatively, the results are similar to those obtained for equal end moments. A relatively small K_c results in a large increase in the buckling load. This increase is much less for small L^2/a^2 . For a very large L^2/a^2 , or a long slender beam, the curves run together, and L^2/a^2 ceases to be important. However, in most cases of standard steel I-beams, L^2/a^2 will fall between 2 and 50.

The programs were tested for a very large K_c in the particular case where L^2/a^2 was equal to 16. K_c was increased in steps of 40 up to 8000, and m was determined in each case. Figure 11 shows m plotted against K_c for this case. Although the increase in m becomes less as K_c becomes larger, in all cases m did increase up to

TABLE 5

VALUES OF m IN EQ. (5-7) FOR DIFFERENT L^2/a^2 AND K_c

$K_c \backslash L^2/a^2$	0.4	1.0	2.0	4.0	6.0
0	86.90	56.54	41.79	31.92	27.87
20	90.27	61.57	48.37	40.15	36.99
40	93.50	66.21	54.15	46.92	44.22
60	96.64	70.56	59.35	52.81	50.40
80	99.64	74.63	64.12	58.07	55.86
100	102.57	78.50	68.54	62.88	60.79
120	105.44	82.17	72.71	67.35	65.35
140	108.22	85.70	76.64	71.52	69.57
160	110.90	89.07	80.33	75.42	73.54
180	113.54	92.30	83.92	79.14	77.28
200	116.13	95.45	87.31	82.67	80.85
220	118.65	98.46	90.56	86.05	84.24
240	121.11	101.42	93.70	89.26	87.50
260	123.56	104.25	96.75	92.40	90.59
280	125.93	107.05	99.70	95.41	93.58
320	130.54	112.33	105.30	101.09	99.27
360	134.92	117.46	110.65		
400	139.25	122.27	115.70		
440	143.42	126.95			
480	147.43	131.41			
520	151.39				
560	155.23				

TABLE 5 - Continued

$K_c \backslash L^2/a^2$	8.0	12	16	24	32
0	25.61	23.13	21.77	20.33	19.56
20	35.30	33.51	32.56	31.57	31.04
40	42.80	41.28	40.48	39.60	39.11
60	49.12	47.74	47.00	46.14	45.66
80	54.66	53.38	52.64	51.80	51.27
100	59.67	58.43	57.67	56.81	56.26
120	64.27	63.02	62.27	61.34	60.77
140	68.52	67.26	66.50	65.54	64.89
160	72.48	71.21	70.44	69.42	68.74
180	76.24	74.95			
200	79.80	78.48	77.63	76.49	75.72
220	83.19				
240	86.41	85.02			
280	92.45				

$K_c \backslash L^2/a^2$	48	64	96	240	400
0	18.76	18.34	17.91	17.35	17.20
20	30.46	30.14	29.80	29.31	29.15
40	38.55	38.23	37.85	37.25	37.04
60	45.06	44.71	44.26	43.55	43.29
80	50.64	50.22	49.72	48.88	48.58
100	55.57	55.10	54.53	53.58	53.22
120	60.00	59.50	58.87	57.82	57.44
140	64.07	63.53			
160	67.86	67.31	66.61	65.44	64.73
200	74.75	74.15	73.41	72.02	70.95

Fig. 1Q--The Effect of Continuous Torsional Restraint on The Critical Stress Factor for a Concentrated Load at Mid-Span

Fig. 11.--The Effect of a Large Continuous Torsional Restraint on the Value of m

twenty times the buckling value of m with no restraint. The conclusion was that the beam will buckle in one half-wave regardless of the magnitude of the restraint as in the case of equal end moments. However, the numerical solution does not provide positive proof.

It was found that the Recursion Formula Method required less computer time than the Taylor Series Method. However, in neither case, was the time excessive. Also the Recursion Formula Method gave an answer for m only, while the Taylor Series Method gave m and β , β' , u and u' for every station along the beam, thus providing a complete picture of the beam displacements.

CHAPTER VI

UNIFORM LOAD, CONTINUOUS TORSIONAL RESTRAINT

The problem of lateral buckling with a continuous elastic torsional restraint is continued by considering the case where a beam is loaded by a uniform load of intensity, w per unit length. Figure 12 shows the elastic curve of such a beam after it has buckled. The restraining couples and shear forces are omitted from this figure for clarity, since they are exactly as shown in Figure 7, Chapter IV.

At an arbitrary section, a distance z along the beam, the displacement components are u , v , and β (u is shown). The two equivalent beam reactions, each equal in magnitude to $\frac{wL}{2}$, will be located as shown, a distance of Δ in the x direction from the point of support. The value of Δ is equal to the value of u at the center of gravity of the load on one half of the beam. Thus

$$\Delta = \frac{\int_0^L w u \, dz}{wL}. \quad (6-1)$$

Fig. 12.--Elastic Curve, Uniform Load, Continuous Torsional Restraint

It also will be necessary to determine δ , the value of u at the center of gravity of the load to the left of the section.

$$\delta = \frac{\int_0^z w u dt}{w z}. \quad (6-2)$$

Both Eqs. (6-1) and (6-2) are written assuming distances along the elastic curve are equal to distances along the z axis.

For the load only, taking moments to the left of the section

$$M_z = -\frac{wL}{2}(\Delta - u) - wz(u - \delta),$$

since u is shown as negative in Figure 12. Substitution of Δ and δ from Eqs. (6-1) and (6-2) gives

$$M_x = -\frac{wL}{2}z + \frac{wz^2}{2}$$

$$M_y = 0$$

$$M_z = -\frac{1}{2}\int_0^L w u dz + \frac{wL}{2}u - wz u + \int_0^z w u dt.$$

For the continuous elastic restraint alone, as was done in Chapter IV

$$M_x = \frac{2}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt$$

$$M_y = 0$$

$$M_z = -T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

(4-2)
Repeat

Therefore the total moments are

$$M_x = -\frac{\omega L}{2} z + \frac{\omega z^2}{2} + \frac{z}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt$$

$$M_y = 0$$

$$M_z = -\frac{1}{2} \int_0^L \omega u dz + \frac{\omega L}{2} u - \omega z u + \int_0^z \omega u dt - T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

Using Table 2 and dropping higher order terms

$$M_\xi = -\frac{\omega L}{2} z + \frac{\omega z^2}{2} + \frac{z}{L_r} z T - \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt$$

$$M_\eta = -\beta \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right)$$

$$M_\xi = u' \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right) - \frac{1}{2} \int_0^L \omega u dz + \frac{\omega L}{2} u - \omega z u + \int_0^z \omega u dt - T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

These moments are substituted into the equilibrium equations, Eq. (2-11), and after higher order terms are dropped, the following three equations are obtained:

$$B_1 u'' = \beta \frac{\omega L}{2} z - \beta \frac{\omega z^2}{2} + \frac{I_1}{I_2} \beta \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right)$$

$$B_2 v'' = \frac{\omega L}{2} z - \frac{\omega z^2}{2} - \frac{z}{L_r} z T + \frac{1}{L_r} \int_0^z \frac{K_c C}{L^2} \beta t dt \quad (6-3)$$

$$C \beta' - \frac{D h^2}{2} \beta''' = u' \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right) - \frac{1}{2} \int_0^L \omega u dz + \frac{\omega L}{2} u - \omega z u + \int_0^z \omega u dt - T + \int_0^z \frac{K_c C}{L^2} \beta dt.$$

The first of Eqs. (6-3) is rewritten and the third is differentiated, giving

$$B_1 u'' = \beta \gamma \left(\frac{\omega L}{2} z - \frac{\omega z^2}{2} \right),$$

and

$$C \beta'' - \frac{D h^2}{2} \beta^{IV} = u'' \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right) + \frac{K_c C}{L^2} \beta.$$

These two equations in u and β are combined to eliminate u'' giving

$$C \beta'' - \frac{D h^2}{2} \beta^{IV} = -\frac{\beta \gamma}{B_1} \left(\frac{\omega L}{2} z - \frac{\omega z^2}{2} \right)^2 + \frac{K_c C}{L^2} \beta,$$

which when rearranged is

$$\begin{aligned} \beta^{IV} - \frac{2C}{D h^2} \beta'' - \frac{4\gamma}{B_1^2 h^2} \left[\frac{\omega^2 L^2}{4} z^2 - \frac{\omega^2 L}{2} z^3 + \frac{\omega^2}{4} z^4 \right] \beta \\ + \frac{2K_c C}{L^2 D h^2} \beta = 0. \end{aligned}$$

The last equation is a linear differential equation with variable coefficients. Writing it and the boundary conditions (a), (b), (c) and special condition (d) in non-dimensional form

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - \frac{\pi^2 L^2}{4 a^2} (z^2 - 2z^3 + z^4) \beta + \frac{L^2}{a^2} K_c \beta = 0$$

(6-4)

$$(a) \quad z = 0, \quad \beta = 0$$

$$(b) \quad z = 0, \quad \beta'' = 0$$

$$(c) \quad z = 0.5, \quad \beta' = 0$$

$$(d) \quad z = 0.5, \quad \beta''' = 0$$

The boundary conditions are the same as were considered in Chapter V. Special condition (d) comes from the last of Eqs. (6-3). a^2 has the same meaning as before and m is from Eq. (6-5) below.

$$(\omega L)_{cr} = \frac{m}{L^2} \sqrt{\frac{B_c C}{\gamma}}. \quad (6-5)$$

The boundary value problem represented by Eqs. (6-4) was solved by the Taylor Series Method as described for the previous problem. The half-beam was broken into equal segments, and β and the derivatives of β through the third were calculated at every station using 7, 6, 5, and 4 terms of the Taylor Series, respectively. Eq. (6-4) and the two equations that come from differentiating Eq. (6-4) twice were used for the other derivatives. Otherwise, the boundary conditions and the iteration of m were handled in the same way as was done for the previous problem.

A check on the program was run with $L^2/a^2 = 32$, and $K_c = 0, 20, 40$ and 60 with 5, 10 and 50 beam segments. The results obtained using 10 segments were identical with eight significant figures to those obtained using 50 segments.

The Recursion Formula Method was used as a check. The solution of Eq. (6-4) was taken in the form of a series

$$\beta = \sum_{n=0}^{\infty} b_n z^n. \quad (6-6)$$

Boundary conditions (a) and (b) require that b_0 and b_2 equal zero. Substitution of Eq. (6-6) into Eq. (6-4), shifting the index, n , until all powers of z are alike gives

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1) b_{n+4} z^n - \frac{L^2}{a^2} \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} z^n \\ & - \frac{m^2 L^2}{4 a^2} \sum_{n=2}^{\infty} b_{n-2} z^n + \frac{L^2}{a^2} K_c \sum_{n=0}^{\infty} b_n z^n \\ & + \frac{m^2}{2} \frac{L^2}{a^2} \sum_{n=3}^{\infty} b_{n-3} z^n - \frac{m^2}{4} \frac{L^2}{a^2} \sum_{n=4}^{\infty} b_{n-4} z^n = 0. \end{aligned} \quad (6-7)$$

With $n = 0$ and $n = 2$, b_4 and b_6 equal zero since b_0 and b_2 are zero. With $n = 1$, Eq. (6-7) gives

$$b_5 = \frac{L^2}{a^2} \left(\frac{6b_3 - K_c b_1}{120} \right), \quad (6-8)$$

and with $n = 3$

$$b_7 = \frac{L^2}{a^2} \left[\frac{\frac{L^2}{a^2} b_3 - K_c b_3 + \frac{m^2}{4} b_1}{840} - \frac{\frac{L^2}{a^2} K_c b_1}{5040} \right], \quad (6-9)$$

and with $n \geq 5$

$$\begin{aligned} b_{n+4} = \frac{L^2}{a^2} & \left[\frac{b_{n+2}}{(n+4)(n+3)} \right. \\ & \left. - \frac{K_c b_n + \frac{m^2}{4} b_{n-2} - \frac{m^2}{2} b_{n-3} + \frac{m^2}{4} b_{n-4}}{(n+4)(n+3)(n+2)(n+1)} \right]. \end{aligned} \quad (6-10)$$

Eqs. (6-8), (6-9) and (6-10) make it possible to get any term of the series in terms of b_1 and b_3 . Note that only four even terms of the series and none of the odd terms are zero this time.

Using boundary condition (c), special condition (d), and series (6-6), two new series are obtained. If a finite number of terms are added up in each series, two linear homogeneous equations in b_1 and b_3 are obtained. The high speed computer was used to add the terms. The determinant of the coefficient matrix must equal zero, and m was iterated as before.

A test run with $L^2/a^2 = 32$, and $K_c = 0, 20, 40$ and 60 in turn was made using different numbers of terms of the series. It was found in this problem it was necessary to add at least 27 terms before the results became identical to eight significant figures to the results obtained when more terms are used. The results converged to the same values obtained by the Taylor Series Method for these four cases.

Results are tabulated in Table 6, which gives the value of m in Eq. (6-5) for various K_c and L^2/a^2 . The data recorded in Table 6 was obtained by the Taylor Series Method using 25 beam segments, and over half of the results were checked by the Recursion Formula Method using 45 terms of the series. No differences greater than 0.1 of one per cent were noted, and in most cases the results were the same.

TABLE 6

VALUES OF m IN EQ. (6-5) FOR DIFFERENT L^2/a^2 AND K_c

$K_c \backslash L^2/a^2$	0.4	1.0	2.0	4.0	6.0
0	144.23	93.85	69.35	52.99	46.29
20	149.83	102.20	80.29	66.67	61.48
40	155.18	109.96	89.92	77.97	73.58
60	160.40	117.12	98.61	87.85	83.91
80	165.45	123.95	106.58	96.68	93.08
100	170.33	130.37	113.98	104.76	101.41
120	175.09	136.53	120.92	112.26	109.11
140	179.70	142.40	127.51	119.21	116.28
160	184.16	148.00	133.73	125.83	122.99
180	188.55	153.44	139.70	132.10	129.37
200	192.84	158.68	145.41	138.06	135.41
220	197.03	163.68	150.91	143.78	141.23
240	201.21	168.59	156.14	149.28	146.71
260	205.18	173.39	161.31	154.61	152.11
280	209.11	178.05	166.23	159.73	157.22
300	213.02				
320	216.88	186.98	175.74	169.42	167.02
360	224.29	195.52	184.67	178.57	176.22
400	231.48	203.62	193.20	187.28	
440	238.42	211.52			
480	245.20	219.07			
520	251.66				
560	258.17				

TABLE 6 - Continued

$K_c \backslash L^2/a^2$	8.0	12	16	24	32
0	42.52	38.40	36.18	33.78	32.53
20	58.68	55.75	54.23	52.62	51.78
40	71.24	68.82	67.55	66.22	65.52
60	81.86	79.71	78.60	77.37	76.73
80	91.22	89.27	88.21	87.06	86.41
100	99.69	97.85	96.83	95.72	95.10
120	107.47	105.70	104.71	103.61	102.94
140	114.70	112.99	112.04	110.92	110.20
160	121.44	119.81	118.86	117.67	117.02
180	127.87	126.21	125.27		
200	133.98	132.31	131.13	130.18	129.39
220	139.74	138.13			
240	145.31	143.71	142.68	141.40	140.62
280	155.79	154.20	153.13		
320	165.66	163.92			

$K_c \backslash L^2/a^2$	48	64	128	280	400
0	31.21	30.53	29.47	28.87	28.71
20	50.92	50.46	49.75	49.30	49.18
40	64.74	64.36	63.63	63.16	63.04
60	76.01	75.60	74.82	74.31	74.16
80	85.69	85.26	84.43	83.85	83.68
100	94.30	93.83	92.96	92.32	92.14
120	102.17	101.67	100.72	99.98	99.79
140	109.38	108.84	107.83		
160	116.14	115.56	114.49	113.65	113.37
200	128.45	127.81	126.62	125.70	125.26
240	139.53	138.90	137.54	136.54	135.99

Figure 13 shows s plotted against K_c for a few values of L^2/a^2 . As before, s is defined as follows:

$$S = \frac{m}{(m)_{K_c=0}} = \frac{\omega L}{(\omega L)_{K_c=0}} \quad (4-28)$$

Repeat

Again it is seen from Figure 13 that a relatively small K_c greatly increases the buckling load, but this effect is less when L^2/a^2 is small.

A comparison of the results of this problem and the previous problem shows that lateral buckling occurs at greater total load when the load is distributed uniformly than when it is concentrated at the center of the beam. This is true for any L^2/a^2 and K_c . The same conclusion was reached by Timoshenko(5).

When $K_c = 0$, the case of a beam with no restraint is obtained. These results agree with Timoshenko's(5) results of the same problem for both the concentrated load and the uniform load. Timoshenko used an energy method for his solutions.

Fig. 13--The Effect of Continuous Torsional Restraint on The Critical Stress Factor for Uniform Load

CHAPTER VII

EQUAL END MOMENTS, POINT TORSIONAL RESTRAINT

Figure 14 shows the elastic curve of a beam after it has buckled laterally. The load consists of equal end moments, M , applied in the yz plane. A point elastic torsional restraint exists at the center where the rotation and displacement are β_c and μ_c . Since the restraint at the center is elastic, it exerts a torque on the beam of magnitude

$$K_r \beta_c = \frac{K_p C}{L} \beta_c.$$

Equal reactive couples with magnitudes of

$$\frac{K_p C}{2L} \beta_c$$

will act at the ends as shown. Because of the shear in the restraint, a force with magnitude

$$\frac{2 K_p C}{L L_r} \beta_c$$

also acts at the center of the beam in the positive y direction. The reactions at the ends of the beam due to this force are equivalent to a force with magnitude

$$\frac{K_p C}{L L_r} \beta_c$$

Fig. 14.--Elastic Curve, Equal End Moments, Point Torsional Restraint

acting in the negative y direction at a point, a distance, u_c , along the x axis. The moments of all the restraint forces and couples acting alone at a distance, z , from the left end are

$$\begin{aligned} M_x &= \frac{K_P C}{LL_r} \beta_c z \\ M_y &= 0 \\ M_z &= -\frac{K_P C}{2L} \beta_c + \frac{K_P C}{LL_r} \beta_c (u_c - u). \end{aligned} \tag{7-1}$$

The last term of the third of Eqs. (7-1) is of higher order and is dropped, giving the following equations for the moments due to the restraint only:

$$\begin{aligned} M_x &= \frac{K_P C}{LL_r} \beta_c z \\ M_y &= 0 \\ M_z &= -\frac{K_P C}{2L} \beta_c. \end{aligned} \tag{7-2}$$

When the moments due to the applied load are included, Eqs. (7-2) become

$$\begin{aligned} M_x &= -M + \frac{K_P C}{LL_r} \beta_c z \\ M_y &= 0 \\ M_z &= -\frac{K_P C}{2L} \beta_c. \end{aligned}$$

Using Table 2 and dropping nonlinear terms

$$\begin{aligned} M_{\xi} &= -M + \frac{K_P C}{L L_r} \beta_c z \\ M_{\eta} &= \beta M \\ M_{\zeta} &= -\mu' M - \frac{K_P C}{2L} \beta_c. \end{aligned}$$

Substituting in the equilibrium equations, Eqs. (2-11), and again dropping nonlinear, higher order terms

$$\begin{aligned} B_1 u'' &= \beta M - \frac{I_1}{I_2} \beta M = \beta \gamma M \\ B_2 v'' &= M - \frac{K_P C}{L L_r} \beta_c z \\ C \beta' - \frac{D h^2}{2} \beta''' &= -\mu' M - \frac{K_P C}{2L} \beta_c. \end{aligned} \quad (7-3)$$

Differentiation of the third of Eqs. (7-3) and elimination of u'' by using the first gives the following differential equation:

$$\beta^{IV} - \frac{2C}{D h^2} \beta'' - \frac{2M^2 \gamma}{B_1 D h^2} \beta = 0. \quad (7-4)$$

Eq. (7-4) is written in non-dimensional form by letting $z = \frac{z'}{L}$ and then dropping the primes.

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - m^2 \frac{L^2}{a^2} \beta = 0. \quad (7-5)$$

In Eq. (7-5), a^2 is defined as before and m is defined by the following equation:

$$M_{cr} = \frac{m}{L} \sqrt{\frac{B_1 C}{\gamma}}. \quad (4-26)$$

Repeat

The first of Eqs. (7-3) can be written in non-dimensional form as follows:

$$\frac{L}{h\sqrt{g}} u'' = \frac{m}{2} \frac{L}{a} \beta. \quad (7-6)$$

The following boundary conditions:

- (a) $z = 0, \beta = 0$
- (b) $z = 0, \beta'' = 0$
- (c) $z = 0, u = 0$
- (d) $z = 0.5, \beta' = 0$
- (e) $z = 0.5, u' = 0$

and special condition

$$(f) z = 0.5, \beta''' = \frac{K_p \beta_c}{2} \frac{L^2}{a^2}$$

were first used for u and β , and will be referred to henceforth as Case I. Boundary conditions (a), (b), and (c) follow from the assumed conditions at the ends of the beam and have been discussed before. Boundary conditions (d) and (e) are true because of the assumed symmetry of the beam at the center. Special condition (f) comes from the third of Eqs. (7-3) with $\beta' = u' = 0$ at $z = 0.5$, after writing it in non-dimensional form.

A computer program for a numerical solution of the boundary value problem by the Taylor Series Method was written. The half-beam was divided into 50 equal segments since this was found to be an accurate procedure for other

solutions (see Chapters V and VI). Seven terms of the Taylor Series were used for calculating β at every station, six terms for β' , five terms for β'' , and four terms for β''' . Eq. (7-5) was used for calculating β^{IV} at every station, and β^V and β^{VI} were calculated at every station by using the two equations that result from differentiating Eq. (7-5) twice. Calculations were started at station 0, the left end of the beam, with β_0 and $\beta_0'' = 0$ by boundary conditions (a) and (b). β_0' was taken equal to 1.0 and β_0''' was taken equal to k . When the center of the beam was reached, k was determined by boundary condition (d), and all the values of β and its derivatives were recalculated. The recalculations were made for a complete solution; however, K_p can be determined as soon as β''' at $z = 0.5$ is known.

To get the values of u in the Taylor Series Solution, seven terms of the Taylor Series were used for u at every station and six terms for u' . Eq. (7-6) was used for u'' , and u''' , u^{IV} , u^V and u^{VI} were obtained by repeatedly differentiating Eq. (7-6). Note that all values of β were already known. Also all the results for the values of u must be divided by L and multiplied by h/δ since $\frac{L}{h\delta}$ in Eq. (7-6) was ignored. Calculations were started at station 0 with $u_0 = 0$ by boundary condition (c), and u_0' was taken equal to $-k$, with $-k$ an undetermined constant.

When the center of the beam was reached, k was determined, using boundary condition (e), and all the values of u were recalculated. At this point, the computer read out β , β' , u , and u' at every station and K_p with

$$K_p = \frac{2\beta_c'''}{\beta_c} \frac{a^2}{L^2} \quad (7-7)$$

which comes from special condition (f).

The program was tested as follows: Fifty segments for the half-beam were used, L^2/a^2 was selected as 8, and m was varied but was started as 4.70. This value was used because a previous solution showed this to be correct for a one half-wave solution with no restraint (see Timoshenko(5), Page 262, also see Table 4). With this input data, $K_p = 0$ was the result, and β and u are plotted for the half-beam in Figure 15. The value of m was increased by 0.1 of m and all the calculations were repeated. When m reached 16.21, K_p became equal to 183.1, and u and β are shown in Figure 16. Finally, when m became equal to 22.87, K_p was infinite and u and β plot as shown in Figure 17. This value of $m = 22.87$ was determined by iterating m to 0.1 of one per cent with $\beta_c = 0$. The variation of K_p with m for $L^2/a^2 = 8$ is shown in Figure 18. The curves in Figure 18 show that a relatively small K_p means a large increase in m and in the buckling load. In fact, with

Fig. 15.--Values of u and β Plotted for the Half-Beam, Equal End Moments, $L^2/a^2 = 8$, $m = 4.70$, $K_p = 0$ - Case I Boundary Conditions

Fig. 16.--Values of u and β Plotted for the Half-Beam, Equal End Moments, $L^2/a^2 = 8$, $m = 16.21$, $K_p = 183.1$ - Case I Boundary Conditions

Fig. 17.--Values of u and β Plotted for the Half-Beam, Equal End Moments, $L^2/a^2 = 8$, $m = 22.87$, $K_p = \infty$ - Case I Boundary Conditions

Fig. 18.--The Effect of a Point Torsional Restraint on the Value of m in Eq. (4-26), $L^2/a^2 = 8$, Case I Boundary Conditions

$K_p = 500$, approximately 85 per cent of the rise in m has resulted. However, for $m = 22.87$, K_p must be infinite.

When m is larger than 22.87, K_p becomes negative, which simply means no equilibrium position is possible for the beam. However, if m is large enough, another equilibrium position is possible. With $m = 32.8$, K_p and β_c''' became zero again, and u and β plot as shown in Figure 19. Note this divides the beam into three equal portions, symmetrical about the center line, and is a three half-wave solution. With m greater than 32.8, K_p becomes positive again indicating equilibrium positions are possible. However, the above has no practical significance but is interesting.

The computer program was used to determine m for an infinite K_p for several different L^2/a^2 , and the results are in Table 7. The values of m when $K_p = 0$ also are included in Table 7. Boundary conditions for Case I were used, and all values of m were iterated to 0.1 of one per cent. The values of m when $K_p = 0$ should check the values of m in Table 4 when $K_c = 0$, which they do. They also check with the results published by Timoshenko(5).

The next question considered was whether a lower buckling load is possible with a different set of boundary conditions than those used in Case I. In Case I, $\beta_L' = \beta_R' = 0$ at the center was used and resulted in

TABLE 7

VALUE OF m IN EQ. (4-26) FOR $K_p = 0$, $K_p = \infty$ AND
DIFFERENT L^2/a^2 - CASE I BOUNDARY CONDITIONS

L^2/a^2	$K_p = 0$	$K_p = \infty$
0.1	31.37	195.51
1	10.36	62.15
2	7.65	44.20
3	6.51	36.32
4	5.85	31.62
6	5.11	26.11
8	4.70	22.87
10	4.43	20.68
12	4.24	19.08
16	4.00	16.87
20	3.84	15.39
24	3.73	14.32
28	3.65	13.49
32	3.59	12.85
36	3.55	12.32
40	3.51	11.88
100	3.29	9.14

symmetry with respect to the center, but more generally the requirement is that $\beta'_L = \beta'_R$. To prove this for the special case of an I-beam, consider the following equation relating the displacements of the center of the

Fig. 19.--Values of u and β Plotted for the Half-Beam, Equal End Moments, $L^2/a^2 = 8$, $m = 32.8$, $K_p = 0$ - Case I Boundary Conditions

flanges, u_T , to the other beam displacements, u and β :

$$u_T = u \pm \frac{h}{2} \beta. \quad (4-12)$$

Repeat

Differentiation with respect to z gives

$$u'_T = u' \pm \frac{h}{2} \beta'.$$

At any section of the beam, $u'_{T_L} = u'_{T_R}$ and $u'_L = u'_R$. Therefore $\beta'_L = \beta'_R$ follows from the above equation. In the more general case of a thin-walled open section subjected to nonuniform torsion, Timoshenko(8) shows that the displacements in the longitudinal direction, w , are related to β' in the following way:

$$w = \Theta \beta'.$$

Θ is a function of the cross section and defines the warping. For continuity of longitudinal displacements, $w_L = w_R$, and β'_L must equal β'_R at any cross section. Therefore, anti-symmetry about the center was assumed, and the boundary conditions that follow are called Case II.

- (a) $z=0, \beta=0$
- (b) $z=0, \beta''=0$
- (c) $z=0, u=0$
- (d) $z=0.5, \beta=0$
- (e) $z=0.5, \beta''=0$
- (f) $z=0.5, u=0$

With $\beta = 0$ at the point of location of the restraint, the restraining couple and restraint shear force will be zero and the moment equations are

$$M_x = -M$$

$$M_y = 0$$

$$M_z = 0.$$

Using Table 2

$$M_\xi = -M$$

$$M_\eta = \beta M$$

$$M_\zeta = -u' M.$$

Substituting in the equations of equilibrium, Eqs. (2-11)

$$B_1 u'' = \beta M + \frac{I_1}{I_2} \beta (-M)$$

$$B_2 v'' = M + \frac{I_2}{I_1} \beta (\beta M)$$

$$C \beta' - \frac{D h^2}{2} \beta''' = -u' M.$$

Dropping higher order terms

$$B_1 u'' = \beta \gamma M$$

$$B_2 v'' = M$$

$$C \beta' - \frac{D h^2}{2} \beta''' = -u' M.$$

Differentiating the third, eliminating u'' with the first, writing the result in non-dimensional form along with the first gives

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - m^2 \frac{L^2}{a^2} \beta = 0$$

(7-5)
Repeat

$$\frac{L}{h\sqrt{\gamma}} u'' = \frac{m}{2} \frac{L}{a} \beta.$$

(7-6)
Repeat

The result is exactly the same differential equations in u and β as before, the only difference being in the boundary conditions.

A computer program using the Taylor Series Method was written as before except k was determined by boundary condition (e) and m was iterated to 0.1 of one per cent by using boundary condition (d). The computer read out values of β , β' , β'' , u , u' and u'' at all stations as well as the value of m . The program was tested with $L^2/a^2 = 8$ and 50 segments for the half-beam were used. The result was almost perfect symmetry in all values of u and β about the quarter-beam point for half the beam. (Anti-symmetry about the mid-point for the whole beam.) A plot that shows u and β is given in Figure 20. The value of m was equal to 15.31. This is a two half-wave solution with equal end moments and is ordinarily of no practical interest. However in this case $m = 15.31$ is less than $m = 22.87$, which is what was obtained using the boundary conditions in Case I with $K_p = \infty$. With $m = 15.31$ and Case I boundary conditions, $K_p = 150.4$ was the result. This means with $L^2/a^2 = 8$ and K_p less than 150.4, the beam will buckle as in Case I with m less than 15.31; but if K_p is greater than or equal to 150.4, the beam will buckle as in Case II with m equal to 15.31.

The second program with Case II conditions was run

Fig. 20.--Values of u and β Plotted for the Half-Beam, Equal End Moments, $L^2/a^2 = 8$, $m = 15.31$ - Case II Boundary Conditions

for several L^2/a^2 and the results are in Table 8. In all cases m was less than m with Case I boundary conditions and $K_p = \infty$.

TABLE 8

VALUES OF m IN EQ. (4-26) FOR DIFFERENT L^2/a^2
CASE II BOUNDARY CONDITIONS

L^2/a^2	m
0.1	125.10
1	40.01
2	28.64
3	23.65
4	20.73
6	17.31
8	15.31
10	13.99
12	13.02
16	11.71
20	10.85
24	10.23
28	9.76
32	9.40
36	9.11
40	8.87
100	7.43

For several L^2/a^2 and m varying up to m for the two half-wave solution, K_p was determined using the first program with Case I boundary conditions. The results are shown in Table 9. If the actual value of K_p in a given problem is greater than the largest given, as will usually be the case, then the largest m given in Table 9 should be used, since lateral buckling with Case II conditions will govern. It is seen from the table that a relatively small value of K_p will give the maximum m for any L^2/a^2 with the exception of $L^2/a^2 = 0.1$, and this L^2/a^2 is unusually low. Lateral buckling is prevented by a torsional restraint to a greater extent when L^2/a^2 is large.

TABLE 9
VALUES OF m IN EQ. (4-26) AND K_p FOR DIFFERENT L^2/a^2
CASE I BOUNDARY CONDITIONS

$L^2/a^2 = 0.1$		$L^2/a^2 = 1$		$L^2/a^2 = 2$		$L^2/a^2 = 3$		$L^2/a^2 = 4$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
31.37	0	10.36	0	7.65	0	6.51	0	5.85	0
34.51	104	13.39	11	8.42	6.2	7.16	4.5	6.44	3.6
37.95	230	12.53	25	9.26	13.7	7.87	9.9	7.08	8.0
41.75	384	13.79	42	10.19	22.9	8.66	16.6	7.79	13.4
45.93	572	15.17	63	11.20	34.2	9.53	24.8	8.56	20.0
50.52	804	16.68	88	12.33	48.1	10.48	34.8	9.42	28.2
55.57	1,088	18.35	119	13.56	65.2	11.53	47.2	10.36	38.3
61.13	1,438	20.18	157	14.92	86.3	12.68	62.7	11.40	50.9
67.24	1,874	22.20	205	16.41	112.8	13.94	81.9	12.54	66.6
73.79	2,417	24.42	265	18.05	146.0	15.34	106.3	13.79	86.5
81.36	3,102	26.87	341	19.85	188.2	16.88	137.3	15.17	112.1
89.50	3,974	29.55	438	21.84	242.6	18.57	177.7	16.69	145.5
98.45	5,103	32.51	565	24.02	314.2	20.42	231.3	18.36	190.4
108.29	6,598	35.76	735	26.42	411.5	22.46	305.0	20.20	253.1
119.12	8,643	39.33	971	28.64	524.6	23.66	357.9	20.73	274.8
125.10	10,008	40.01	1,024						

TABLE 9 - Continued

$L^2/a^2 = 6$		$L^2/a^2 = 8$		$L^2/a^2 = 10$		$L^2/a^2 = 12$		$L^2/a^2 = 16$		$L^2/a^2 = 20$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
5.11	0	4.70	0	4.43	0	4.24	0	4.00	0	3.84	0
5.62	2.8	5.18	2.4	4.87	2.1	4.67	1.9	4.39	1.7	4.22	1.6
6.18	6.1	5.70	5.3	5.36	4.6	5.13	4.2	4.83	3.8	4.65	3.5
6.80	10.3	6.27	8.8	5.89	7.7	5.65	7.1	5.32	6.3	5.11	5.9
7.48	15.3	6.90	13.1	6.48	11.6	6.21	10.7	5.85	9.5	5.62	8.8
8.23	21.6	7.59	18.5	7.13	16.4	6.83	15.1	6.43	13.5	6.18	12.5
9.05	29.4	8.34	25.2	7.85	22.3	7.51	20.6	7.08	18.5	6.80	17.3
9.96	39.1	9.18	33.6	8.63	29.9	8.27	27.6	7.79	24.9	7.48	23.3
10.95	51.4	10.10	44.2	9.49	39.4	9.09	36.6	8.56	33.1	8.23	31.3
12.05	67.0	11.11	57.9	10.44	51.8	10.00	48.3	9.42	44.1	9.05	41.9
13.25	87.3	12.22	75.8	11.49	68.3	11.00	63.9	10.36	59.1	9.96	57.0
14.58	114.1	13.44	100.0	12.64	90.7	12.10	85.7	11.40	80.8	10.85	76.7
16.04	150.9	14.78	134.0	13.90	123.1	13.02	109.2	11.71	88.8		
17.31	191.8	15.31	150.4	13.99	125.7						

TABLE 9 - Continued

$L^2/a^2 = 24$		$L^2/a^2 = 28$		$L^2/a^2 = 32$		$L^2/a^2 = 36$		$L^2/a^2 = 40$		$L^2/a^2 = 100$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
3.73	0	3.65	0	3.59	0	3.55	0	3.51	0	3.29	0
4.11	1.5	4.02	1.4	3.95	1.4	3.90	1.3	3.86	1.3	3.62	1.2
4.52	3.3	4.42	3.2	4.35	3.1	4.29	3.0	4.24	2.9	3.98	2.6
4.97	5.6	4.86	5.3	4.78	5.2	4.72	5.1	4.67	5.0	4.38	4.5
5.47	8.4	5.35	8.1	5.26	7.8	5.19	7.7	5.14	7.5	4.82	7.0
6.01	11.9	5.88	11.5	5.79	11.2	5.71	11.0	5.65	10.8	5.30	10.2
6.61	16.5	6.47	15.9	6.37	15.6	6.28	15.3	6.21	15.1	5.83	14.8
7.27	22.4	7.12	21.7	7.00	21.3	6.91	21.0	6.84	20.8	6.42	21.6
8.00	30.1	7.83	29.5	7.70	29.1	7.60	28.9	7.52	28.8	7.06	32.7
8.80	40.8	8.62	40.3	8.47	40.1	8.36	40.2	8.27	40.5	7.43	42.4
9.68	56.2	9.48	56.3	9.32	57.0	9.11	55.7	8.87	53.2		
10.23	68.6	9.76	63.0	9.40	58.8						

CHAPTER VIII

CONCENTRATED LOAD AT MID-SPAN, POINT TORSIONAL RESTRAINT

Figure 21 shows the elastic curve of a beam in the buckled position. This time the load consists of a concentrated load, P , at the center. The displacements at the center are u_c and β_c . The reactions at the ends due to the load P are equivalent to forces $P/2$, with a line of action that is a distance, u_c , from the ends in the x direction. The restraint exerts a torque on the beam at the center equal to $\frac{K_P C}{L} \beta_c$. Reactive couples due to this torque act at the ends as shown, with a magnitude equal to $\frac{K_P C}{2L} \beta_c$. The force due to the shear in the restraint and its reactions are not shown for clarity. However, they appear as shown in Figure 14. The moments, a distance z from the left end, due to the restraint forces and couples only are

$$M_x = \frac{K_P C}{2L} \beta_c z$$

$$M_y = 0$$

$$M_z = - \frac{K_P C}{2L} \beta_c$$

(7-2)
Repeat

Fig. 21.--Elastic Curve, Concentrated Load at Mid-Span,
Point Torsional Restraint

When the moments due to the applied load are included,
Eqs. (7-2) become

$$M_x = -\frac{P}{2}z + \frac{K_P C}{L L_r} \beta_c z$$

$$M_y = 0$$

$$M_z = -\frac{P}{2}(\mu_c - \mu) - \frac{K_P C}{2L} \beta_c.$$

Using Table 2 and dropping higher order terms, the following equations are obtained:

$$M_\xi = -\frac{P}{2}z + \frac{K_P C}{L L_r} \beta_c z$$

$$M_\eta = \beta \frac{P}{2} z$$

$$M_\chi = -\mu' \frac{P}{2} z - \frac{P}{2}(\mu_c - \mu) - \frac{K_P C}{2L} \beta_c.$$

The latter three equations are substituted into the equilibrium equations, Eqs. (2-11), and higher order terms are again dropped.

$$B_1 \mu'' = \frac{P}{2} z \beta - \frac{I_1}{I_2} \beta \frac{P}{2} z$$

$$B_2 v'' = \frac{P}{2} z - \frac{K_P C}{L L_r} \beta_c z \quad (8-1)$$

$$C \beta' - \frac{D h^2}{2} \beta''' = -\mu' \frac{P}{2} z - \frac{P}{2}(\mu_c - \mu) - \frac{K_P C}{2L} \beta_c.$$

Since $\gamma = \frac{I_2 - I_1}{I_2}$, the first of Eqs. (8-1) becomes

$$B, \mu'' = \frac{P}{2} \gamma \beta. \quad (8-2)$$

Differentiating the third of Eqs. (8-1)

$$C\beta'' - \frac{Dh^2}{2} \beta^{IV} = -\frac{P}{2} \gamma \mu''. \quad (8-3)$$

Eliminating μ'' between Eqs. (8-2) and (8-3) and rearranging

$$\beta^{IV} - \frac{2C}{Dh^2} \beta'' - \frac{P^2 \gamma}{B^2 h^2} \gamma^2 \beta = 0. \quad (8-4)$$

In non-dimensional form, Eqs. (8-2) and (8-4) are

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - \frac{m^2}{4} \frac{L^2}{a^2} \gamma^2 \beta = 0, \quad (8-5)$$

and

$$\frac{L}{h\sqrt{\gamma}} \mu'' = \frac{m}{4} \frac{L}{a} \gamma \beta, \quad (5-9)$$

Repeat

with m as in the following equation

$$P_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}.$$

(5-7)
Repeat

Case I boundary conditions

$$(a) \quad z=0, \beta=0$$

$$(b) \quad z=0, \beta''=0$$

$$(c) \quad z=0, \mu=0$$

$$(d) \quad z=0.5, \beta'=0$$

$$(e) \quad z=0.5, \mu'=0$$

and special condition

$$(f) \quad z=0.5, \beta''' = \frac{K_p \beta_c}{2} \frac{L^2}{a^2}$$

were used. Boundary conditions (d) and (e) are by symmetry, and special condition (f) comes from the third of Eqs.

(8-1) with $\beta' = 0$, $u' = 0$, $u = u_c$ at $z = 0.5$ after writing it in non-dimensional form.

A computer program was written using the Taylor Series Method to solve this boundary value problem. The only difference between this program and the one used in Chapter VII was that Eq. (8-5) was used to get the higher derivatives of β . The program was tested using 50 segments for the half-beam and $L^2/a^2 = 16$. The value of m was varied by increments of 0.1 of m , but was started at $m = 21.78$. This value of m was used because it was the result obtained in Chapter V, when $L^2/a^2 = 16$ and $K_c = 0$ (see Table 5). This time with

$$K_p = \frac{2 \beta_c'''}{\beta_c} \frac{a^2}{L^2}$$

(7-7)
Repeat

from special condition (f), the result was $K_p = 0$. Thus a partial check on the program was obtained. For this case, the values of u and β plot as shown in Figure 22. As m was increased, β_c became smaller and K_p larger. Finally K_p approached infinity as β_c approached zero. This value of m was obtained by iteration to 0.1 of one per cent. The values of β and u for these conditions are shown in Figure 23 with $m = 144.32$. A curve that shows how m varied with K_p for $L^2/a^2 = 16$ is drawn in Figure 24. The results shown by Figures 22, 23 and 24 are qualitatively similar to those obtained when the load consisted of equal end moments (see Chapter VII).

Using the above program, m was determined at $K_p = \infty$ for several L^2/a^2 . The results are shown in Table 10, along with the value of m when $K_p = 0$.

To obtain m for a two half-wave solution, Case II boundary conditions, as given in Chapter VII, were used as follows:

- (a) $z = 0, \beta = 0$
- (b) $z = 0, \beta'' = 0$
- (c) $z = 0, \mu = 0$
- (d) $z = 0.5, \beta = 0$
- (e) $z = 0.5, \beta'' = 0$
- (f) $z = 0.5, \mu = 0$.

TABLE 10

VALUES OF m IN EQ. (5-7) FOR $K_p = 0$, $K_p = \infty$ AND
DIFFERENT L^2/a^2 - CASE I BOUNDARY CONDITIONS

L^2/a^2	$K_p = 0$	$K_p = \infty$
0.4	86.91	830.57
1	56.55	527.03
2	41.79	375.25
4	31.93	268.63
6	27.88	222.20
8	25.61	194.70
12	23.14	162.84
16	21.78	144.32
24	20.34	123.27
32	19.57	111.33
48	18.77	98.55
64	18.35	92.37

With $\beta = 0$ at $z = 0.5$, the restraining couple is zero. However the differential equations for u and β turn out to be identical to Eqs. (8-5) and (5-10). The Taylor Series solution to this problem with the computer was similar to the one with Case I boundary conditions except in the following ways: The constant k used in the calculations for β was determined by boundary condition (e) and m was iterated to satisfy boundary

Fig. 22.--Values of u and β Plotted for the Half-Beam, Concentrated Load at Mid-Span, $L^2/a^2 = 16$, $m = 21.78$, $K_p = 0$ - Case I Boundary Conditions

Fig. 23.-- Values of u and β Plotted for the Half-Beam,
 Concentrated Load at Mid-Span, $L^2/a^2 = 16$, $m = 144.32$,
 $K_p = \infty$ - Case I Boundary Conditions

Fig. 24.--The Effect of a Point Torsional Restraint on the Value of m in Eq. (5-7), $L^2/a^2 = 16$, Case I Boundary Conditions

condition (d). Boundary condition (f) was satisfied by iterating the initial value of u' at $z = 0$. The program was run with $L^2/a^2 = 16$, using 50 segments for the half-beam. The value of m was 87.48 and u and β are plotted in Figure 25. The curves in Figure 25 show that this time they are not symmetrical about the quarter point, since $\beta' = 0$ and $u' = 0$ occur at values of z greater than 0.25. However there is anti-symmetry about the mid-point and $\beta'_L = \beta'_R$, $u'_L = u'_R$ at $z = 0.5$. This is the two half-wave solution for lateral buckling with a concentrated load and ordinarily is of little practical interest. However it is of interest here because $m = 87.48$ is less than $m = 144.32$ for $K_p = \infty$ with Case I boundary conditions. This means with $L^2/a^2 = 16$, the beam cannot be loaded with m exceeding 87.48, no matter what the torsional restraint stiffness at the center.

The problem above was solved for several different values of L^2/a^2 and the results are shown in Table 11. Note that all values of m in this table are less than those in Table 10 for Case I boundary conditions and $K_p = \infty$.

Table 12 was then prepared for the designer. For several L^2/a^2 , K_p was determined using Case I conditions for values of m up to and including those in Table 11. If K_p is larger than the largest given in Table 12, no

Fig. 25.--Values of u and β Plotted for the Half-Beam,
Concentrated Load at Mid-Span, $L^2/a^2 = 16$, $m = 87.48$, Case II
Boundary Conditions

additional increase in the buckling load will be obtained,
and Table 11 should be used for m .

TABLE 11

VALUES OF m IN EQ. (5-7) FOR DIFFERENT L^2/a^2
CASE II BOUNDARY CONDITIONS

L^2/a^2	m
0.4	466.11
1	297.09
2	212.76
4	154.17
6	128.82
8	114.10
12	97.16
16	87.48
24	76.71
32	70.76
48	64.54
64	61.48
96	59.36

TABLE 12
VALUES OF m IN EQ. (5-7) AND K_p FOR DIFFERENT L^2/a^2
CASE I BOUNDARY CONDITIONS

$L^2/a^2 = 0.4$		$L^2/a^2 = 1$		$L^2/a^2 = 2$		$L^2/a^2 = 4$		$L^2/a^2 = 6$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
86.91	0	56.55	0	41.79	0	31.93	0	27.88	0
95.60	27	62.21	11	45.97	6	35.12	3.6	30.67	2.7
105.16	59	68.43	25	50.57	14	38.64	7.9	33.73	6.0
115.68	98	75.27	41	55.62	23	42.50	13.1	37.11	10.0
127.24	145	82.79	61	61.18	34	46.75	19.5	40.82	14.9
139.97	202	91.07	86	67.30	47	51.42	27.3	44.90	20.8
153.97	227	100.18	115	74.03	63	56.57	36.7	49.39	28.0
169.36	357	110.20	151	81.44	83	62.22	48.3	54.33	36.8
186.30	461	121.22	195	89.58	107	68.44	62.4	59.76	46.7
204.93	588	133.34	249	98.54	136	75.29	79.7	65.74	61.0
225.42	744	146.68	315	108.39	173	82.82	101.1	72.31	77.3
247.96	935	161.34	397	119.23	217	91.10	127.4	79.54	97.7
272.76	1,171	177.48	497	131.15	272	100.21	160.2	87.50	123.0
300.04	1,464	195.23	622	144.27	341	110.23	201.2	96.25	154.9
330.04	1,829	214.75	778	158.69	428	121.25	253.0	105.87	195.5
363.04	2,290	236.22	975	174.57	537	133.38	319.5	116.46	248.1
399.35	2,878	259.85	1,227	192.02	679	146.72	406.3	128.11	317.8
439.29	3,641	285.83	1,559	211.23	865	154.17	462.4	128.82	322.5
466.11	4,237	297.09	1,720	212.76	882				

TABLE 12 - Continued

$L^2/a^2 = 8$		$L^2/a^2 = 12$		$L^2/a^2 = 16$		$L^2/a^2 = 24$	
m	K_p	m	K_p	m	K_p	m	K_p
25.61	0	23.14	0	21.78	0	20.34	0
28.17	2.3	25.45	1.9	23.96	1.6	22.37	1.4
30.99	5.1	28.00	4.1	26.35	3.6	24.61	3.2
34.09	8.4	30.80	6.9	28.99	6.1	27.07	5.3
37.50	12.5	33.88	10.2	31.87	9.0	29.78	7.9
41.25	17.5	37.27	14.3	35.08	12.7	32.76	11.0
45.37	23.6	40.99	19.3	38.58	17.1	36.03	14.9
49.91	31.1	45.09	25.4	42.44	22.6	39.64	19.7
54.90	40.3	49.60	33.0	46.69	29.3	43.60	25.6
60.39	51.5	54.56	42.3	51.35	37.6	47.96	33.0
66.43	65.5	60.02	53.8	56.49	48.0	52.75	42.3
73.07	82.8	66.02	68.3	62.14	61.0	58.03	54.0
80.38	104.5	72.62	86.5	68.35	77.5	63.84	69.1
88.41	131.9	79.88	109.8	75.19	98.9	70.22	89.0
97.25	167.1	87.87	140.0	82.71	127.1	76.71	113.7
106.98	213.1	96.66	180.5	87.48	148.2		
114.10	252.8	97.16	183.0				

TABLE 12 - Continued

$L^2/a^2 = 32$		$L^2/a^2 = 48$		$L^2/a^2 = 64$		$L^2/a^2 = 96$	
m	K_p	m	K_p	m	K_p	m	K_p
19.57	0	18.77	0	18.35	0	17.92	0
21.53	1.3	20.65	1.2	20.18	1.1	19.71	1.0
23.68	2.9	22.71	2.6	22.20	2.5	21.68	2.3
26.05	4.9	24.98	4.4	24.42	4.2	23.85	3.9
28.65	7.2	27.48	6.6	26.87	6.3	26.24	5.9
31.52	10.2	30.23	9.3	29.55	8.8	28.86	8.3
34.67	13.8	33.25	12.6	32.51	12.0	31.75	11.2
38.14	18.2	36.58	16.8	35.76	15.9	34.92	14.9
41.95	23.8	40.24	21.9	39.33	20.8	38.41	19.4
46.15	30.7	44.26	28.3	43.27	27.0	42.25	25.1
50.76	39.4	48.68	36.6	47.60	34.9	46.48	32.3
55.84	50.6	53.55	47.2	52.35	45.1	51.13	41.2
61.42	65.1	58.91	61.2	57.59	58.7	56.24	52.5
67.56	84.5	64.54	79.5	61.48	70.7	59.36	59.9
70.76	96.5						

CHAPTER IX

UNIFORM LOAD, POINT TORSIONAL RESTRAINT

The case of a point elastic torsional restraint at the center is extended for a uniform load over the entire span as shown in Figure 26. The torque exerted by the restraint on the beam is $\frac{K_p C}{L} \beta_c$, and there results therefrom reactive couples of $\frac{K_p C}{2L} \beta_c$ as before. The forces on the beam due to the shear in the restraint are not shown for clarity, but they would appear exactly as shown in Figure 14. The moments at a section, a distance, z , from the left end due to the restraint forces and couples only are

$$M_x = \frac{K_p C}{L L_r} \beta_c z$$

$$M_y = 0$$

$$M_z = -\frac{K_p C}{2L} \beta_c.$$

(7-2)
Repeat

The applied load has an intensity, w per unit length. The reactions due to this applied load are equivalent to a force, $\frac{wL}{2}$, located a distance, Δ , in the negative x direction from the point of support as shown in Figure 26.

Fig. 26.--Elastic Curve, Uniform Load, Point Torsional Restraint

The distance Δ is equal to the value of u to the center of gravity of the load on one half of the beam.

$$\Delta = \frac{\int_0^L w u dz}{wL}. \quad \begin{array}{l} (6-1) \\ \text{Repeat} \end{array}$$

The distance, δ , is the value of u when u is the distance to the center of gravity of the load to the left of the section a distance z from the support.

$$\delta = \frac{\int_0^z w u dt}{wz}. \quad \begin{array}{l} (6-2) \\ \text{Repeat} \end{array}$$

The moments at the section due to the applied load only are

$$M_x = -\frac{wL}{2} z + \frac{wz^2}{2}$$

$$M_y = 0$$

$$M_z = -\frac{wL}{2} (\Delta - u) - wz(u - \delta)$$

since u is shown as negative. The total moments at the section are

$$M_x = -\frac{wL}{2} z + \frac{wz^2}{2} + \frac{K_p C}{LL_r} \beta_c z$$

$$M_y = 0$$

$$M_z = -\frac{1}{2} \int_0^L w u dz + \frac{wL}{2} u + \int_0^z w u dt - wz u - \frac{K_p C}{2L} \beta_c$$

after substitutions for Δ and δ are made from Eqs. (6-1) and (6-2). Using Table 2 and dropping higher order, non-linear terms

$$M_{\xi} = -\frac{\omega L}{2} z + \frac{\omega z^2}{2} + \frac{K_{PC}}{LL_r} \beta_c z$$

$$M_{\eta} = \beta \frac{\omega L}{2} z - \beta \frac{\omega z^2}{2}$$

$$M_{\xi} = -\mu' \frac{\omega L}{2} z + \mu' \frac{\omega z^2}{2} - \frac{1}{2} \int_0^L \omega u dz + \frac{\omega L}{2} \mu + \int_0^z \omega u dt - \omega z \mu - \frac{K_{PC}}{2L} \beta_c.$$

Substitution into the equations of equilibrium, Eqs.

(2-11), and then dropping higher order terms

$$B_1 u'' = \beta \frac{\omega L}{2} z - \beta \frac{\omega z^2}{2} + \frac{I_1}{I_2} \beta \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right)$$

$$B_2 v'' = \frac{\omega L}{2} z - \frac{\omega z^2}{2} - \frac{K_{PC}}{LL_r} \beta_c z \quad (9-1)$$

$$C \beta' - \frac{Dh^2}{2} \beta''' = \mu' \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right) - \frac{1}{2} \int_0^L \omega u dz + \frac{\omega L}{2} \mu + \int_0^z \omega u dt - \omega z \mu - \frac{K_{PC}}{2L} \beta_c.$$

Rewriting the first of Eqs. (9-1)

$$B_1 u'' = \beta \gamma \left(\frac{\omega L}{2} z - \frac{\omega z^2}{2} \right). \quad (9-2)$$

Differentiating the third of Eq. (9-1)

$$C \beta'' - \frac{Dh^2}{2} \beta^{IV} = \mu'' \left(-\frac{\omega L}{2} z + \frac{\omega z^2}{2} \right). \quad (9-3)$$

Elimination of u'' with Eqs. (9-2) and (9-3) and rearranging gives

$$\beta^{IV} - \frac{2C}{Dh^2} \beta'' - \frac{\gamma}{D^2 h^2} \left(\frac{\omega^2 L^2}{4} z^2 - \frac{\omega^2 L}{2} z^3 + \frac{\omega^2}{4} z^4 \right) \beta = 0. \quad (9-4)$$

Eq. (9-4) is a linear differential equation in β with variable coefficients. Writing it in non-dimensional form

$$\beta^{IV} - \frac{L^2}{a^2} \beta'' - \frac{m^2 L^2}{4 a^2} (z^2 - 2z^3 + z^4) \beta = 0, \quad (9-5)$$

where m is in

$$(\omega L)_{cr} = \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}}. \quad (6-5) \text{ Repeat}$$

Eq. (9-5) was solved by the Taylor Series Method with Case I boundary conditions (a), (b) and (c) and special condition (d) as follows:

$$(a) \quad z=0, \beta=0$$

$$(b) \quad z=0, \beta''=0$$

$$(c) \quad z=0.5, \beta'=0$$

$$(d) \quad z=0.5, \beta''' = \frac{K_p \beta_c}{2} \frac{L^2}{a^2}.$$

Special condition (d) comes from the third of Eqs. (9-1) and the boundary condition, $u' = 0$ at $z = \frac{L}{2}$. Values of u were not determined for this loading as they are not essential for obtaining the results. This could have

been done with Eq. (9-2). The results of this solution with $K_p = 0$ and $K_p = \infty$ are shown in Table 13.

TABLE 13

VALUES OF m IN EQ. (6-5) FOR $K_p = 0$, $K_p = \infty$ AND DIFFERENT L^2/a^2 - CASE I BOUNDARY CONDITIONS

L^2/a^2	$K_p = 0$	$K_p = \infty$
0.4	144.24	1,143.84
1	93.86	725.83
2	69.36	516.27
4	52.99	369.17
6	46.30	304.66
8	42.53	266.67
12	38.41	222.05
16	36.19	196.22
24	33.79	166.05
32	32.54	148.77
48	31.22	129.10
64	30.54	117.95
128	29.48	98.45
280	28.88	85.24
400	28.72	81.20

Case II boundary conditions were as follows:

- (a) $z = 0, \beta = 0$
- (b) $z = 0, \beta'' = 0$
- (c) $z = 0.5, \beta = 0$
- (d) $z = 0.5, \beta'' = 0.$

With $\beta = 0$ at $z = 0.5$, the restraining torque on the beam is again zero. The differential equation for β is the same as in Case I, Eq. (9-5). This equation and Case II boundary conditions were solved with the Taylor Series Method and the results for the two half-wave solution are given in Table 14. As before, 50 segments for the half-beam were used and m was iterated to 0.1 of one per cent. All values of m in Table 14 are less than those in Table 13 with $K_p = \infty$, indicating that buckling in two half-waves will occur if K_p is large enough. Table 15 was prepared using Case I boundary conditions, varying m up to the value of m determined for the two half-wave solution. The results are in the same pattern as was determined for the other two types of loadings.

TABLE 14

VALUES OF m IN EQ. (6-5) FOR DIFFERENT L^2/a^2
CASE II BOUNDARY CONDITIONS

L^2/a^2	m
0.4	673.63
1	429.19
2	307.15
4	222.37
6	185.59
8	164.22
12	139.52
16	125.39
24	109.42
32	100.51
48	90.63
64	85.29
128	76.47
280	71.08
400	69.63

TABLE 15
VALUES OF m IN EQ. (6-5) AND K_p FOR DIFFERENT L^2/a^2
CASE I BOUNDARY CONDITIONS

$L^2/a^2 = 0.4$		$L^2/a^2 = 1$		$L^2/a^2 = 2$		$L^2/a^2 = 4$		$L^2/a^2 = 6$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
144.24	0	93.86	0	69.36	0	52.99	0	46.30	0
158.66	27	103.25	11	76.30	6	58.29	3.6	50.93	2.8
174.53	59	113.57	25	83.93	14	64.12	7.9	56.02	6.1
191.98	98	124.93	42	92.32	23	70.53	13.2	61.63	10.1
211.18	146	137.42	62	101.55	34	77.58	19.7	67.79	15.1
232.30	204	151.16	86	111.70	47	85.34	27.6	74.57	21.1
255.53	275	166.28	117	122.88	64	93.88	37.3	82.02	28.5
281.08	362	182.91	153	135.16	84	103.26	49.2	90.23	37.7
309.19	469	201.20	199	148.68	109	113.59	63.8	99.25	49.0
340.11	600	221.32	255	163.55	140	124.95	81.9	109.17	63.0
374.12	762	243.45	324	179.90	178	137.44	104.6	120.09	80.6
411.53	964	267.79	410	197.89	225	151.19	132.9	132.10	102.8
452.69	1,217	294.57	518	217.68	285	166.31	169.0	145.31	131.2
497.96	1,536	324.03	655	239.45	361	182.94	215.4	159.84	168.2
547.75	1,945	356.43	831	263.39	460	201.23	276.5	175.82	217.7
602.53	2,479	392.08	1,062	289.73	591	221.35	359.3	185.59	253.6
662.78	3,193	429.19	1,355	307.15	694	222.37	364.0		
673.63	3,339								

TABLE 15 - Continued

$L^2/a^2 = 8$		$L^2/a^2 = 12$		$L^2/a^2 = 16$		$L^2/a^2 = 24$		$L^2/a^2 = 32$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
42.53	0	38.41	0	36.19	0	33.79	0	32.54	0
46.78	2.3	42.25	1.9	39.81	1.7	37.17	1.5	35.79	1.4
51.46	5.1	46.48	4.2	43.79	3.7	40.89	3.2	39.37	3.0
56.61	8.5	51.12	7.0	48.17	6.2	44.97	5.4	43.31	5.0
62.27	12.7	56.24	10.4	52.99	9.2	49.47	8.1	47.64	7.5
68.49	17.8	61.86	14.6	58.28	13.0	54.42	11.4	52.41	10.6
75.34	24.1	68.05	19.8	64.11	17.6	59.86	15.5	57.65	14.5
82.88	31.9	74.85	26.2	70.52	23.4	65.85	20.6	63.41	19.4
91.17	41.5	82.34	34.2	77.58	30.6	72.43	27.2	69.75	25.6
100.28	53.5	90.57	44.2	85.33	39.8	79.68	35.5	76.73	33.7
110.31	68.6	99.63	56.9	93.87	51.4	87.64	46.4	84.40	44.4
121.34	87.7	109.59	73.3	103.25	66.7	96.41	60.9	92.84	59.2
133.48	112.5	120.55	94.8	113.58	87.0	106.05	81.2	100.51	76.5
146.83	145.1	132.60	123.9	124.94	115.4	109.42	89.7		
161.51	189.4	139.52	143.9	125.39	116.7				
164.22	198.8								

TABLE 15 - Continued

$L^2/a^2 = 48$		$L^2/a^2 = 64$		$L^2/a^2 = 128$		$L^2/a^2 = 280$		$L^2/a^2 = 400$	
m	K_p	m	K_p	m	K_p	m	K_p	m	K_p
31.22	0	30.54	0	29.48	0	28.88	0	28.72	0
34.34	1.2	33.59	1.2	32.43	1.1	31.77	1.1	31.59	1.0
37.78	2.7	36.95	2.6	35.67	2.5	34.94	2.4	34.75	2.4
41.55	4.6	40.65	4.4	39.24	4.2	38.44	4.0	38.23	4.0
45.71	7.0	44.71	6.7	43.16	6.3	42.28	6.2	42.05	6.2
50.28	9.9	49.18	9.5	47.48	9.1	46.51	9.0	46.25	9.1
55.31	13.5	54.10	13.1	52.23	12.7	51.16	12.9	50.88	13.1
60.84	18.2	59.51	17.8	57.45	17.5	56.28	18.4	55.97	19.0
66.92	24.3	65.47	23.9	63.19	24.3	61.91	26.8	61.56	28.6
73.62	32.4	72.01	32.3	69.51	34.4	68.10	41.6	67.72	47.1
80.98	43.6	79.21	44.3	76.46	51.2	71.08	52.9	69.63	56.5
89.07	59.9	85.29	57.9	76.47	51.3				
90.63	63.7								

CHAPTER X

CONCLUSION

Although no attempt will be made to summarize in detail the findings, since this was done for each problem, some of them should be repeated. Three types of loading and two types of torsional restraint were considered and striking similarities can be seen in the equations and in the shape of the curves. In all cases, torsional restraint is seen to be very effective in increasing the critical buckling load, especially at large values of L^2/a^2 where it is most needed. With a continuous torsional restraint, the critical load on a beam will depend only on the restraint stiffness, but with a point torsional restraint the critical load will be limited by the two half-wave solution of lateral buckling.

In Chapter III, a numerical example was considered in which a 16 WF 64 beam, 24 ft. in length, carried a uniform load of 2400 lbs. per ft. It was chosen on the basis of an allowable stress of 20,000 p.s.i. The value of Ld/b_t for this section and span length is 756. Since this is greater than 600, the allowable stress must be

reduced as required by the A.I.S.C. Specifications(21).

Therefore this beam would be unsatisfactory if no restraint is present. The actual load on the span is $wL = 2400 (24) = 57,600$ lbs., or 57.6 Kips. The value of $L^2/a^2 = (24 \times 12/60.35)^2 = 22.8$, with $a = 60.35$ in. as taken from the Bethlehem Steel Co. Handbook(31). To determine the critical load with no restraint, Eq. (6-5) is used and m is obtained from Table 6 by interpolation as 34.1.

Therefore

$$\begin{aligned} (wL)_{cr} &= \frac{m}{L^2} \sqrt{\frac{B_1 C}{\gamma}} \\ &= \frac{34.1}{(24 \times 12)^2} \sqrt{\frac{29 \times 10^6 \cdot 68.4 \cdot 11 \times 10^6 \cdot 2.85}{0.917}} \\ &= 107.2 \text{ Kips,} \end{aligned}$$

since $\gamma = \frac{I_2 - I_1}{I_2} = \frac{833.8 - 68.4}{833.8} = 0.917$, $G = 11 \times 10^6$ and

$E = 29 \times 10^6$. The factor of safety against lateral buckling with no restraint is

$$\frac{107.2}{57.6} = 1.86.$$

A statement was made on Page 27, that if only one per cent of the available torsional restraint was effective, as the result of partially rigid connections, lateral buckling would be suppressed and design could be made on a stress basis. The accuracy of this statement will be investigated. Since K_c for the example considered in

Chapter III was equal to 6,905, one per cent of K_c is 69.05. Interpolation in Table 5 with $L^2/a^2 = 22.8$ gives $m = 81.9$. Substitution in Eq. (6-5) gives

$$(WL)_{cr} = 107.2 \left(\frac{81.9}{34.1} \right) = 257.2 \text{ Kips}$$

or a factor of safety of

$$\frac{257.2}{57.6} = 4.47$$

which is certainly more than enough. If instead of a continuous torsional restraint, a point torsional restraint is present, as in the second example considered in Chapter III, $K_p = 2265$. One per cent of this is 22.65. From Table 15 it is seen that this is not enough to produce two half-waves of lateral buckling, but $m = 68.2$ by interpolation. Substitution in Eq. (6-5) gives

$$(WL)_{cr} = 107.2 \left(\frac{68.2}{34.1} \right) = 200 \text{ Kips},$$

or a factor of safety of

$$\frac{200}{57.6} = 3.47.$$

This example not only shows that in this case a torsional restraint is extremely effective, but also it shows how easy it is to use the tables to calculate the critical loads. It should be noted also that multiplication of the stiffness factors by 0.01 is probably conservative, depending on the rigidity of the connections.

Further research on the problem of torsional restraint of lateral buckling could be continued along one or more of the following lines:

(a) Laboratory test results could be obtained to support the analytical solutions obtained here.

(b) An investigation could be made to evaluate the restraint supplied by members that are likely to provide the restraint, such as standard steel bar joists.

(c) Other kinds of loading might be considered, including loading of the top flange.

(d) Extension could be made of the results to beyond the elastic region to include cases where the proportional limit is exceeded before the critical load is reached and also the case of a beam with a variable I (non-prismatic beams). Both would be a case of variable stiffness compared with the constant stiffness considered here.

(e) The question of more than one point torsional restraint could be considered and quantitative answers could be obtained for the question: "How close together must the torsional restraints be, to be considered a continuous torsional restraint?"

(f) The magnitude of the restraining couples might be determined so that the restraining members may be designed. (As Zuk(11) has determined the restraining force for other types of restraints.) This will require consideration of the initial deformations.

The success achieved with the Taylor Series Method in the problems considered here indicates that it would be successful also in solving some of the latter problems. This method could be used to check other existing solutions to lateral buckling problems as well. It could be used to check on the validity of the assumption that the nonlinear terms are negligibly small, to determine half-wave solutions of lateral buckling greater than one, and to solve problems involving lateral buckling of beams on elastic supports and beams with unsymmetrical cross sections.

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