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DECOMPOSITION THEOREMS FOR WEIGHT
FUNCTIONS IN TRANSFORMATION THEORY
FOR MEASURE SPACE.

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DECOMPOSITION THEOREMS FOR WEIGHT
FUNCTIONS IN TRANSFORMATION THEORY
FOR MEASURE SPACE
DISSERTATION

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INTRODUCTION

In 1961 Reichelderfer published a paper entitled "A Transformation Theory for Measure Space." The present work consists of contributions to this new abstract transformation theory. In turn Reichelderfer's paper has as its antecedent the book entitled Continuous Transformations in Analysis. This book was written jointly by Rado and Reichelderfer and was published in 1955. One of the main purposes of that book was to obtain powerful results concerning the transformation of integrals; the results obtained both include and extend the known classical transformation formulas.

The results in [2] (the number in [] refers to the bibliography) are developed in a setting which we shall now describe briefly; a thorough description of this setting is found in Chapter I of this paper. Let T be a function defined on a non-empty set $\underline{S}$ and suppose that $\underline{S}' = T\underline{S}$. Let $\{\underline{S}, \underline{M}, \mu\}$ and $\{\underline{S}', \underline{M}', \mu'\}$ be measure spaces. Suppose that $\underline{D}$ is a collection of subsets D of $\underline{S}$ and that W' is a non-negative function defined on $\underline{S}' \times \underline{D}$. Assume that T , $\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, $\underline{D}$, and W' satisfy certain standard conditions; under these conditions W' is termed a weight function for T . Within this basic setting Reichelderfer presents in [2] necessary and sufficient conditions in order that there exist a non-negative (extended real

valued) function f defined on $\underline{S}$ and μ -integrable such that

$$\int_{\underline{D}} f(s) \, d\mu = \int_{\underline{S}'} W'(s', D) \, d\mu', \quad D \in \underline{D}.$$

Whenever these conditions are satisfied it is shown that a transformation formula holds in the following sense:

Let H' be a real valued $\underline{M}'$ -measurable function defined on $\underline{S}'$. Then for each D in $\underline{D}$ the function $H'(s')W'(s', D), s' \in \underline{S}'$, is $\underline{M}'$ -measurable and the function $H'T(s)f(s), s \in \underline{S}$, is $\underline{M}$ -measurable. The former function is μ' -integrable over $\underline{S}'$ if and only if the latter function is μ -integrable over $\underline{D}$. When these functions are integrable the transformation formula

$$(1) \quad \int_{\underline{D}} H'T(s)f(s) \, d\mu = \int_{\underline{S}'} H'(s')W'(s', D) \, d\mu'$$

holds. The paper also examines an important special class of weight functions. It ends with an outline of the relationship between the results of [2] and [1] together with an application to Euclidean space. Indeed it is shown that the general transformation formula in [1] is a special case of (1).

The work in [2] constitutes not only an extensive generalization of large parts of [1] but also an unexpected simplification. Rado and Reichelderfer developed in [1] a transformation theory for continuous transformations defined on bounded domains in Euclidean n -space and having bounded ranges in Euclidean n -space. Additional conditions are imposed upon the weight functions W' . Moreover the topological properties of Euclidean n -space are used abundantly in [1]. Craft was able to remove these additional conditions on W' ; and then, in [2], Reichel-

derfer was able to eliminate all overt use of topology. Some of the standard hypotheses in [2] have obvious topological fore-runners. Nevertheless no topologies appear in the discussion. Thus the results in [2] remove the transformation formulas from the realm of Euclidean space and even from the much broader realm of topology. The consequences of this generalization have not yet been studied extensively. Furthermore there is more flexibility in Euclidean applications. For example, the results of [1] cannot be applied in the case of a continuous transformation from n -space to n' -space with $n \neq n'$; however, the results in [2] can be applied in this case.

The present work has [2] as its foundation. Throughout almost all of this dissertation we shall suppose that we are working with measure spaces $\{\underline{S}, \underline{M}, \mu\}$ and $\{\underline{S}', \underline{M}', \mu'\}$, a transformation T from $\underline{S}$ onto $\underline{S}'$, and families $\underline{D}$ and $\underline{B}'$ which satisfy the standard hypotheses H1 - H8 set forth in [2]. Our interest will be centered chiefly upon weight functions (for T). The main objective is to obtain decompositions of weight functions as sums of certain other weight functions each having special properties. These decompositions have as their model the classical Lebesgue Decomposition Theorem for measures (see [3]). In the course of establishing these results we shall prove some properties of weight functions which are themselves of independent value. There is also a result which falls into the realm of abstract transformation theory but which seems quite unrelated to the main objective. Thus the paper has a dual purpose. We

have not only the aim of obtaining certain decomposition theorems but also the broader goal of making general (sometimes scattered) contributions to the new transformation theory ([2]).

With these general remarks set forth we can now briefly describe the progression of the paper. In Chapter I certain preliminary concepts and lemmas are presented. Several broad conventions are established at the outset of this preliminary chapter. Then the standard hypotheses of [2] are listed along with some new standard hypotheses. The basic definitions, lemmas, and theorems of [2] are also listed for the reader's convenience. The chapter concludes with the presentation of some definitions and preliminary lemmas which appear here for the first time.

The first part of Chapter II is devoted to the proof of the General Decomposition Theorem for weight functions. This result has the virtue of being valid under very general conditions. It has a limitation also; one of the two "components" of the decomposition is not at all well classified. The next portion of Chapter II is devoted to the elimination of this drawback. The result is the Strong Decomposition Theorem. This last theorem requires however that a rather broad additional condition be imposed upon the weight function W' . The chapter concludes with the discussion of some germane examples.

Chapter III deals with the important special class of weight functions which satisfy H13 of [2]. This class was introduced first in [2] and it includes all of the weight functions

(or, admissible multiplicity functions) considered in [1]. In the first part of the chapter we prove several "measurability results" of far-reaching significance. Then we apply the results of Chapter II to obtain the Special Decomposition Theorem. We observe that the resulting decomposition assumes a particularly desirable form in this case. The chapter then concludes with a section that is unrelated to the main theme of the dissertation. The value of the results of this last section remains unknown.

Finally we present a concise Summary.

Some of the results of this paper are due entirely to Professor Reichelderfer, the writer's adviser. This fact is noted whenever such results appear.

CHAPTER I

PRELIMINARIES

In Chapter I the basic mathematical system with which this dissertation is concerned is set forth in detail. Section I.1 establishes some basic notational conventions. Section I.2 includes the basic hypotheses and definitions of transformation theory which are introduced in [2]. Theorems and lemmas from [2] which are used explicitly in this dissertation are listed in Section I.3. Finally, Sections I.4 through I.7 consist of certain preliminary conventions, definitions, and lemmas which are essential to this paper and which appear here for the first time.

I.1 General conventions

I.1.1. Remarks. Throughout this paper R^n denotes Euclidean n -space. R_+ always denotes the set $\{t \in R^1 : t \geq 0\} \cup \{\infty\}$. The symbols ∞ and $+\infty$ are used interchangeably. The symbol $-\infty$ is never used. We adopt the following conventions concerning ∞ :

- i) $\infty + a = \infty$ if a is in R^1 .
- ii) $\infty + \infty = \infty$.
- iii) $\infty - a = \infty$ if a is in R^1 .

$$\text{iv)} \quad \infty \cdot 0 = 0.$$

$$\text{v)} \quad \infty \cdot a = \infty \quad \text{if } a \text{ is in } R_+ \text{ and if } a > 0.$$

$$\text{vi)} \quad a < \infty \quad \text{if } a \text{ is in } R^1.$$

Other conventions are common but we shall not need them.

I.1.2. Remarks. Let M be a non-empty subset of $R^1 \cup R_+$.

The least upper bound of M or supremum of M is denoted by $\sup M$.

The greatest lower bound of M or infimum of M is denoted by $\inf M$.

I.1.3. Remarks. We shall frequently be concerned with functions which assume values in R^1 or in R_+ . A function whose values lie in R^1 is termed a real valued function. A function whose values lie in R_+ is termed a non-negative function.

I.1.4. Remarks. In this dissertation we shall often work with sets denoted by $\underline{S}$ and $\underline{S}'$. If $E \subseteq \underline{S}$ we define CE to be the set of objects in $\underline{S}$ but not in E . If $E' \subseteq \underline{S}'$ we define $C'E'$ to be the set of objects in $\underline{S}'$ but not in E' . If $E \subseteq \underline{S}$ then H_E denotes the characteristic function of E as a subset of $\underline{S}$; it is defined by

$$H_E(s) \begin{cases} = 1 & \text{if } s \in E \\ = 0 & \text{if } s \in CE \end{cases}, \quad s \in \underline{S}.$$

Similarly, if $E' \subseteq \underline{S}'$, then $H_{E'}'$ denotes the characteristic function of E' as a subset of $\underline{S}'$.

I.1.5. Remarks. Many concepts of measure theory are used in this paper. See reference [3] for definitions and theorems concerning the concepts of measure and integration which appear in this paper.

I.2 Standard hypotheses

In this section the basic hypotheses are introduced. They are listed as H1 through H15, H15i, H4*, H7*, and HC1 through HC4. Some basic definitions from [2] are presented and several notational conventions are established.

I.2.1. We list the standard hypotheses of [2] along with several definitions.

H1. $\{\underline{S}, \underline{M}, \mu\}$ is a σ -finite complete measure space.

H2. $\{\underline{S}', \underline{M}', \mu'\}$ is a σ -finite complete measure space.

H3. T is a single valued transformation from $\underline{S}$ onto $\underline{S}'$.

H4. $\underline{D}$ is a collection of subsets D of $\underline{S}$ with the following properties: The empty set $\emptyset$ and $\underline{S}$ are in $\underline{D}$, $\underline{D} \subseteq \underline{M}$, and $T\underline{D} \subseteq \underline{M}'$ where $T\underline{D} = \{TD : D \in \underline{D}\}$. If ${}_1D$ and ${}_2D$ are sets in $\underline{D}$ there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that ${}_1D \cap {}_2D = \bigcup D_i$. $\underline{S}$ can be expressed as the union of a countable number of sets $*D_j$ in $\underline{D}$ such that μ_*D_j is finite for every j , and such that if M is in $\underline{M}$ with $M \subseteq *D_j$ for some j then for every $\epsilon > 0$ there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that $M \subseteq \bigcup D_i$ and $\sum \mu D_i < \mu M + \epsilon$.

H4*. $\underline{D}$ is a collection of subsets D of $\underline{S}$ with the

following properties: The empty set $\emptyset$ and $\underline{S}$ are in $\underline{D}$, $\underline{D} \subseteq \underline{M}$, and $T\underline{D} \subseteq \underline{M}'$. If ${}_1D$ and ${}_2D$ are sets in $\underline{D}$ there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that ${}_1D \cap {}_2D = \bigcup D_i$. If M is in $\underline{M}$ and if $\zeta > 0$ then there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that $M \subseteq \bigcup D_i$ and $\sum \mu D_i \leq \mu M + \zeta$.

Remark. It is noted in [2] that $H4^*$ implies $H4$.

Definition. A set D_0 in $\underline{D}$ is said to be of type γ if it is one of a countable number of pairwise disjoint sets D_i , $i \geq 0$, for which there exist two subsets Y and Z of $\underline{S}$ such that

- i) $\underline{S} = Y \cup Z \cup \bigcup_{i \geq 0} D_i$;
- ii) $Y \in \underline{M}$, $\mu Y = 0$;
- iii) $TZ \in \underline{M}'$, $\mu' TZ = 0$.

H5. If D is in $\underline{D}$ there exists a sequence of sets D_j in $\underline{D}$ such that

- i) each D_j is of type γ ;
- ii) $D_j \subseteq D_{j+1}$, $j \geq 1$;
- iii) $D = \bigcup D_j$.

H6. $\underline{B}'$ is a σ -field of subsets B' of $\underline{S}'$ with the following properties: $\underline{B}' \subseteq \underline{M}'$ and $T^{-1}\underline{B}' \subseteq \underline{M}$. For each set M' in $\underline{M}'$ there exist sets B'_1 and B'_2 in $\underline{B}'$ such that $B'_1 \subseteq M' \subseteq B'_2$ and $\mu'B'_1 = \mu'B'_2$.

Remark. It is observed in [2] that we may even require $\mu'(B'_2 \cap C'B'_1) = 0$ in the last condition of H6.

Definition. Denote by $\mathbb{O}'$ all the subsets O' of $\underline{S}'$ for each of which there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that $T^{-1}O' = \bigcup D_i$.

H7. $\underline{S}'$ can be expressed as the union of a countable number of sets $*O'_j$ in $\mathbb{O}'$ such that $\mu' *O'_j$ is finite for every j and such that if M' is in $\underline{M}'$ with $M' \subseteq *O'_j$ for some j and if $\mu'M' = 0$ then for every $\varepsilon > 0$ there is a set O' in $\mathbb{O}'$ such that $M' \subseteq O'$ and $\mu'O' < \varepsilon$.

H7*. If M' is in $\underline{M}'$ and if $\zeta' > 0$ then there exists a set O' in $\mathbb{O}'$ such that $M' \subseteq O'$ and $\mu'O' \leq \mu'M' + \zeta'$.

Note that H7* implies H7.

Definition. An element O'_0 in $\mathbb{O}'$ is said to be of type γ' if it is one of a countable number of pairwise disjoint sets O'_i , $i \geq 0$, in $\mathbb{O}'$ for which there exist two subsets Y' and Z' of $\underline{S}'$ such that

- i) $\underline{S}' = Y' \cup Z' \cup \bigcup_{i \geq 0} O'_i$;
- ii) $T^{-1}Y'$ is in $\underline{M}$ with $\mu T^{-1}Y' = 0$;
- iii) Z' is in $\underline{M}'$ with $\mu'Z' = 0$.

Definition. A set M' in $\underline{M}'$ is said to be of type z' if there is a sequence of sets O'_j in $\mathbb{O}'$ such that O'_j is of type γ' , $j \geq 1$, $O'_j \subseteq O'_{j+1}$ for $j \geq 1$, and $M' = \bigcup O'_j$.

H8. If B' is a set in $\underline{B}'$ there are a sequence of sets M'_j in $\underline{M}'$ and two sets U' and V' in $\underline{M}'$ such that M'_j is of type z' for $j \geq 1$, $M'_j \supseteq M'_{j+1}$ for $j \geq 1$, $\mu'U' = 0 = \mu'V'$, and $\cap M'_j \cup U' = B' \cup V'$.

Definition. Let W' be a non-negative function defined on $\underline{S}' \times \underline{D}$. W' is termed a weight function for T if the following four conditions are satisfied:

- i) If D is in $\underline{D}$ and if s' is in $C'TD$ then $W'(s', D) = 0$.
- ii) If a set D contains a countable number of pairwise disjoint sets D_i then $\sum W'(s', D_i) \leq W'(s', D)$ for every s' in $\underline{S}'$.
- iii) If a set D is the union of a countable number of sets D_j such that $D_j \subseteq D_{j+1}$ for every j then $\lim_j W'(s', D_j) = W'(s', D)$ for every s' in $\underline{S}'$.
- iv) For each D in $\underline{D}$ the function $W'(s', D), s' \in \underline{S}'$, is $\underline{M}'$ -measurable.

H9. W' is a weight function for T . (Thus W' is a non-negative function defined on $\underline{S}' \times \underline{D}$ satisfying conditions i) - iv) of the definition just given.)

Remark. The remaining hypotheses are never considered unless H1 - H8 are already in effect.

H10. There exists a countable subset $*\underline{D}$ of $\underline{D}$ which contains the empty set $\emptyset$ and which has the following property.

Given a countable sequence of sets $D_j, j \geq 0$, in $\underline{D}$ such that

$D_0 \subseteq \bigcup_{j \geq 1} D_j$ then for every integer $j \geq 0$ there exists a countable sequence of sets $*D_{jk}$ in $*\underline{D}$ satisfying these conditions:

- i) $*D_{0k} \subseteq *D_{0, k+1}$ for every k and $\bigcup_k *D_{0k} = D_0$;
- ii) $*D_{0k} \subseteq \bigcup_{j \geq 1} *D_{jk}$ for every k , $*D_{jk} \subseteq D_j$ for $j \geq 1$ and $k \geq 1$;
- iii) for every positive integer k , there exists a positive integer j_k , depending on k , such that $*D_{jk} = \emptyset$ when $j > j_k$.

H11. If s_1 and s_2 are points in $\underline{S}$ for which there is a set D_0 in $\underline{D}$ such that $s_1 \in D_0$ and $s_2 \notin D_0$ then there are sets D_1 and D_2 in $\underline{D}$ such that $s_1 \in D_1$, $s_2 \in D_2$, and $D_1 \cap D_2 = \emptyset$.

Definition. Set

$$\mathfrak{S} = \{S \subseteq \underline{S} : S = \bigcap D, s \in D \in \underline{D}, \text{ for some } s \text{ in } \underline{S}\}.$$

H12. For each S in $\mathfrak{S}$ the set TS consists of a single point in $\underline{S}'$.

Definition. Set

$$\mathcal{E} = \{E \subseteq \underline{S} : \text{If } S \in \mathfrak{S} \text{ and } S \cap E \neq \emptyset \text{ then } S \subseteq E\}.$$

Definition. Assume H1 - H8, H11, and H12. Define

$\mathfrak{S}_{s'} = \{S \in \mathfrak{S} : TS = s'\}$, $s' \in \underline{S}'$. Let w be a non-negative function defined on $\mathfrak{S}$. Define a function

$$W_*(s', D, w), (s', D) \in \underline{S}' \times \underline{D},$$

by

$$W_*(s', D, w) = \Sigma wS, S \in \mathcal{S}_s, \text{ and } S \subseteq D, (s', D) \in \underline{S}' \times \underline{D};$$

here, the expression $\Sigma wS, S \in \mathcal{S}_s, \text{ and } S \subseteq D$, stands for the supremum of the set of numbers

$$\{0, \sum_{k=1}^p wS_k : S_k \in \mathcal{S}_s, S_k \subseteq D\}.$$

H13. W' is a weight function for T for which there exists a non-negative function w defined on $\mathcal{S}$ such that

$$W'(s', D) = W_*(s', D, w), (s', D) \in \underline{S}' \times \underline{D}.$$

H14. Assume H9. There exists a positive real number t such that $W'(s', D) \geq t$ whenever (s', D) is an element in $\underline{S}' \times \underline{D}$ for which $W'(s', D) > 0$.

H15. There exists a countable subset $\underline{D}^{**}$ of $\underline{D}$ having the following properties:

- i) For each S in $\mathcal{S}$ and D in $\underline{D}$ such that $S \subseteq D$ there is a set D^{**} in $\underline{D}^{**}$ such that $S \subseteq D^{**} \subseteq D$.
- ii) If for s' in $\underline{S}'$ and D in $\underline{D}$ there is a sequence of sets D_j^{**} in $\underline{D}^{**}$ such that $T^{-1}s' \cap D \subseteq \bigcup_j D_j^{**}$ then there is also a sequence of sets D_i^* in $\underline{D}^{**}$ such that $T^{-1}s' \cap \bigcup_i D_i^* = \emptyset$ and $D \subseteq \bigcup_j D_j^{**} \cup \bigcup_i D_i^*$.

Remark. The following statement is denoted by H15i for rather obvious reasons.

H15i. There exists a countable subset $\underline{D}^{**}$ of $\underline{D}$ containing $\underline{S}$ and having the property that for each S in $\mathcal{S}$ and D in $\underline{D}$ for

which $S \subseteq D$ there is a set D^{**} in $\underline{D}^{**}$ such that $S \subseteq D^{**} \subseteq D$.

I.2.2. The hypotheses H1 - H15 are introduced in [2].

Four additional hypotheses are introduced in this paper. They are denoted by HC1 - HC4.

HC1. $\underline{B}$ is a σ -field of subsets of $\underline{S}$ for which $\underline{D} \subseteq \underline{B} \subseteq \underline{M}$,

$T\underline{B} \subseteq \underline{M}'$, and $T^{-1}\underline{B}' \subseteq \underline{B}$.

HC2. For every D in $\underline{D}$ there is a countable sequence of sets D_i in $\underline{D}$ such that $D_i \subseteq D_{i+1}$ for every $i \geq 1$, $D = \bigcup D_i$, and such that if D^j , $j \geq 1$, is a countable number of sets in $\underline{D}$ such that

$D \subseteq \bigcup_{j \geq 1} D^j$, then for every positive integer i there exists a

positive integer $J(i)$ such that $D_i \subseteq \bigcup_{j=1}^{J(i)} D^j$.

HC3. $\underline{D}$ contains a countable subfamily $\underline{D}_{\#}$ with the property that every set D in $\underline{D}$ is the union of some sequence of sets $D_{\#j}$, $j \geq 1$ in $\underline{D}_{\#}$ for which $D_{\#j} \subseteq D_{\#j+1}$, $j \geq 1$.

HC4. $\underline{D}$ contains a countable subfamily $\underline{D}_*$ having the following properties:

- i) For every D in $\underline{D}$ there is a countable sequence of sets D_{*i} , $i \geq 1$, in $\underline{D}_*$ for which $D = \bigcup D_{*i}$ and $D_{*i} \subseteq D_{*i+1}$, $i \geq 1$.

- ii) If D in $\underline{D}$ contains a finite number of sets D_{*1} ,

..., D_{*n} in $\underline{D}_*$ then there exists a set D_* in $\underline{D}_*$

such that $\bigcup_{i=1}^n D_{*i} \subseteq D_* \subseteq D$.

I.3 Basic definitions from [2]

Throughout this section the standard hypotheses H1 - H8 are always assumed.

I.3.1. Definition. Let W' be a weight function for T . The function $W'(s', D)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for each D in $\underline{D}$. Set

$$WD = \int_{\underline{S}'} W'(s', D) d\mu', \quad D \in \underline{D}.$$

WD is termed the weight attached to D by W' .

I.3.2. Definition. Let W' be a weight function for T . The transformation T is said to be of bounded variation with respect to the attached weights W - briefly, T is BVW - if $WS = \int_{\underline{S}'} W'(s', \underline{S}) d\mu'$ is a finite real number.

I.3.3. Definition. Let W' be a weight function for T . Let $f(s)$, $s \in \underline{S}$, be a non-negative $\underline{M}$ -measurable function such that $\int_{\underline{D}} f(s) d\mu \leq WD$ for every D in $\underline{D}$. Such a function f is termed a lower bound function for the weights W - briefly, a l.b.f.W.

I.3.4. Definition. Let W' be a weight function for T . The transformation T is said to be absolutely continuous with respect to the attached weights W - briefly, T is ACW - if T

is BVW and if there exists a non-negative $\underline{M}$ -measurable function f defined on $\underline{S}$ which is a l.b.f.W and for which $\int_{\underline{S}} f(s) d\mu = W\underline{S}$.

Such a function f is termed a greatest lower bound function for the weights W - briefly, a g.l.b.f.W.

I.3.5. Definition. Let W' be a weight function for T . Then W - that is, the function which associates the weight WD to every D in $\underline{D}$ - is over additive (on $\underline{D}$) if $WD_0 \leq \sum_{i \geq 1} WD_i$ whenever $D_i, i \geq 0$, is a countable number of sets in $\underline{D}$ for which $D_0 \subseteq \bigcup_{i \geq 1} D_i$.

I.3.6. Definition. Let W' be a weight function for T . Then W is absolutely continuous with respect to the measure μ if for every M in $\underline{M}$ for which $\mu M = 0$ and for every $\epsilon > 0$ there exists a countable number of sets D_j in $\underline{D}$ such that $M \subseteq \bigcup D_j$ and $\sum WD_j < \epsilon$.

I.3.7. Remark. The sets $\mathcal{G}$, $\mathcal{E}$, and $\mathcal{G}_s$, were defined in Section I.2. Also, the expression $W'_*(s', D, w)$ was defined in Section I.2.

I.3.8. Definition. Assume H11 and H12. Let w be a non-negative function defined on $\mathcal{G}$. Suppose that W' is a weight function for T . W' is said to be generated by w if $W'(s', D) = W'_*(s', D, w), (s', D) \in \underline{S}' \times \underline{D}$.

I.4 Lemmas and theorems from [2]

We list those results from [2] which are used explicitly in this paper. The numbers preceding the results are those which appear with them in [2].

3.1. Lemma. Assume $H1 - H9$. Assume that T is BVW and that f is a l.b.f.W such that $\underline{WS} = \int_{\underline{S}} f(s) d\mu$. Then $\underline{WD} = \int_{\underline{D}} f(s) d\mu$ for every D in $\underline{D}$.

3.2. Lemma. Assume $H1 - H9$. Assume that f_1 and f_2 are non-negative μ -integrable functions defined on $\underline{S}$ for which $\int_{\underline{D}} f_1(s) d\mu \leq \int_{\underline{D}} f_2(s) d\mu$, $D \in \underline{D}$. Then $f_1(s) \leq f_2(s)$ a.e. μ on $\underline{S}$.

4.1. Lemma. Assume $H1 - H9$. Assume that T is ACW. Let D be a set in $\underline{D}$ for which there are a countable number of pairwise disjoint sets D_i in $\underline{D}$ and two subsets E and F of $\underline{S}$ satisfying the relations

- i) $\cup D_i \subseteq D$, $D \cap C(\cup D_i) = E \cup F$;
- ii) $E \in \underline{M}$, $\mu E = 0$;
- iii) $TF \in \underline{M}'$, $\mu' TF = 0$.

Then $\underline{WD} = \sum \underline{WD}_i$ and $W'(s', D) = \sum W'(s', D_i)$ a.e. μ' on $\underline{S}'$.

4.10. Theorem. Assume $H1 - H9$. Assume that T is ACW. Assume also that f is a g.l.b.f.W. Fix D in $\underline{D}$. Suppose that H' is a real valued $\underline{M}'$ -measurable function defined on $\underline{S}'$, for which either $H'(s')W'(s', D)$, $s' \in \underline{S}'$, is μ' -integrable or $H'T(s)f(s)$, $s \in D$, is μ -integrable: then

- i) $H'(s')W'(s',D)$, $s' \in \underline{S}'$, is μ' -integrable;
- ii) $H'T(s)f(s)$, $s \in D$, is μ -integrable;
- iii) the transformation formula

$$\int_D H'T(s)f(s) d\mu = \int_{\underline{S}'} H'(s')W'(s',D) d\mu'$$

is valid.

5.11. Corollary. Assume H1 - H9. Assume that T is BVW.

Then a necessary and sufficient condition that T be ACW is that W be over additive and absolutely continuous with respect to the measure μ .

6.1. Lemma. Assume H1 - H9. Assume that T is ACW. Then W' is an over additive function on $\underline{D}$ a.e. μ' on $\underline{S}'$ in the following sense: If D_0 is contained in the union of a countable number of sets D_j , $j \geq 1$, then there exists a set $X' = X'(D_j)$ in $\underline{M}'$ such that $\mu'X' = 0$ and such that $s' \in C'X'$ implies $W'(s',D_0) \leq \sum_{j \geq 1} W'(s',D_j)$. Also, if T is BVW and $W'(s',D)$ is over additive on $\underline{D}$ a.e. μ' on $\underline{S}'$ then W is over additive on $\underline{D}$.

7.2. Lemma. Assume H1 - H8. Assume also H11. The sets in $\mathcal{S}$ constitute a partition of $\underline{S}$ - that is, the sets in $\mathcal{S}$ are non-empty, pairwise disjoint, and their union is $\underline{S}$. If S in $\mathcal{S}$ and D in $\underline{D}$ are such that $S \cap D \neq \emptyset$ then $S \subseteq D$. For every S in $\mathcal{S}$ we have $S = \bigcap D$, $S \subseteq D \in \underline{D}$.

7.7. Lemma. Assume H1 - H8, H11, and H12. Let w be non-negative function defined on $\mathcal{S}$. Define $S^+ = \sum w(S)$, $S \in \mathcal{S}$ and

$wS > 0$. Then, given (s', D) in $\underline{S}' \times \underline{D}$, $W'_*(s', D, w) > 0$ if and only if s' is in $T(S^+ \cap D)$.

Remark. We list those results from 7.8 and 7.9 of [2] which we shall use.

Assume H1 - H8, H11, and H12. Let w be a non-negative function defined on $\mathcal{G}$. Define W' by

$$W'(s', D) = W'_*(s', D, w), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then:

- 1) Suppose a set D in $\underline{D}$ contains a countable number of pairwise disjoint sets D_i in $\underline{D}$. Then $\Sigma W'(s', D_i) \leq W'(s', D)$ for every s' in $\underline{S}'$.
- 2) Suppose a set D in $\underline{D}$ is contained in the union of a countable number of sets D_i in $\underline{D}$. Then $W'(s', D) \leq \Sigma W'(s', D_i)$ for every s' in $\underline{S}'$.
- 3) Suppose a set D in $\underline{D}$ is the union of a sequence of sets D_i in $\underline{D}$ for which $D_i \subseteq D_{i+1}$. Then $W'(s', D) = \lim_i W'(s', D_i)$ for every s' in $\underline{S}'$.
- 4) W' is a weight function for T if the function $W'(s', D)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$.

7.24. Theorem. Assume H1 - H8 and H11 - H13. Assume that T is BVW. Then T is ACW if and only if for every set M in $\underline{M}$ such that $M \subseteq \underline{U}S$, $S \in \mathcal{G}$, $wS > 0$, and $\mu M = 0$ it is true that TM is in $\underline{M}'$ and $\mu'TM = 0$.

8.2. Lemma. Assume H1 - H8. If D_k^* , $k = 1, \dots, p$, is a

finite sequence of sets in $\underline{D}$ then there exists a countable number of pairwise disjoint sets D_i in $\underline{D}$ such that $\bigcap_{k=1}^p D_k^* = \bigcup D_i$.

8.3. Lemma. Assume H1 - H8 and H11. If S is in $\mathcal{S}$ and if D_k , $k = 1, \dots, p$, is a finite sequence of sets in $\underline{D}$ such that $S \subseteq \bigcap_{k=1}^p D_k$ then there is a set D in $\underline{D}$ such that $S \subseteq D \subseteq \bigcap_{k=1}^p D_k$.

8.4. Lemma. Assume H1 - H8 and H11. If S_k , $k = 1, \dots, p$, is a finite sequence of distinct sets in $\mathcal{S}$ and if D is a set in $\underline{D}$ such that $\bigcup_{k=1}^p S_k \subseteq D$ then there is a sequence of pairwise disjoint sets D_k in $\underline{D}$ such that $S_k \subseteq D_k \subseteq D$ for $k = 1, \dots, p$.

8.5. Lemma. Assume H1 - H9 and H11 - H12. For each S in $\mathcal{S}$, TS is a point in $\underline{S'}$. Define a non-negative function $\hat{W}$ on $\mathcal{S}$ by $\hat{W}S = \inf\{W'(TS, D) : S \subseteq D \in \underline{D}\}$, $S \in \mathcal{S}$.

Then

$$W_*(s', D, \hat{W}) \leq W'(s', D), (s', D) \in \underline{S'} \times \underline{D}.$$

8.9. Theorem. Assume H1 - H12, H14, and H15. Assume that T is ACW. Define a non-negative function $\hat{W}$ on $\mathcal{S}$ by

$$\hat{W}S = \inf\{W'(TS, D) : S \subseteq D \in \underline{D}\}, S \in \mathcal{S}.$$

Then there is a set X' in $\underline{M'}$ such that $\mu'X' = 0$ and $W_*(s', D, \hat{W}) = W'(s', D)$ whenever s' is in $C'X'$ and D is in $\underline{D}$.

Remarks. The results of Section 9 of [2] are used in certain examples.

Remark. The following result is a modification of Lemma 2.12 of [2], the proof of which is omitted.

Lemma. Assume H1 - H8. Let H' be a non-negative $\underline{M}'$ -measurable function defined on $\underline{S}'$. There exists a non-negative $\underline{B}'$ -measurable function H'_0 defined on $\underline{S}'$ such that $H' = H'_0$ a.e. μ' on $\underline{S}'$.

I.5 Notational conventions

I.5.1. Remarks. We shall often consider many weight functions simultaneously. It is therefore insufficient to denote all weight functions merely by W' . However, all weight functions will be denoted by the basic symbol W' together with some subscript or even possibly a superscript. Thus, W' , W'_0 , W'_α , and $*W'$ denote weight functions (for T).

For each weight function there is the corresponding function which associates with each D in $\underline{D}$ its attached weight. This function will henceforth be denoted by the symbol for the corresponding weight function with the "prime" removed. Thus, W , W_0 , W_α , and $*W$ denote the weights induced by the weight functions W' , W'_0 , W'_α , and $*W'$.

Occasionally, we write "Assume H9" rather than "Assume W' is a weight function for T ." If we do write "Assume H9" it must be understood that W' (without other subscripts or superscripts) denotes the weight function thereby introduced.

I.5.2. Remarks. The entities $\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, T , $\underline{D}$, $\underline{B}'$, and $\underline{B}$ will always be fixed in any discussion in this paper. Thus such symbols as $\underline{S}$, $\underline{S}'$, μ , $\underline{D}$, etc., always refer to the same thing. This remark is also valid for such symbols as $\mathcal{C}$, $\mathcal{C}$, and $\mathcal{C}_{\underline{S}}$; for these sets are defined entirely in terms of $\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, T , $\underline{D}$, and $\underline{B}'$.

I.5.3. Convention. The expression "AC" is used as an abbreviation for both "absolutely continuous" and "absolute continuity".

I.5.4. Definitions. Assume H1 - H8. Let W'_1 and W'_2 be weight functions for T . We write $W'_1 \leq W'_2$ if and only if $W'_1(s', D) \leq W'_2(s', D)$ for every (s', D) in $\underline{S}' \times \underline{D}$. The corresponding weights are denoted by W_1 and W_2 . We write $W_1 \leq W_2$ if and only if $W_1 D \leq W_2 D$ for every D in $\underline{D}$. Let W'_0 be a third weight function for T . We write $W'_0 = W'_1 + W'_2$ if and only if $W'_0(s', D) = W'_1(s', D) + W'_2(s', D)$ for every (s', D) in $\underline{S}' \times \underline{D}$.

These definitions are in keeping with common usage.

I.5.5. Definition. Assume H1 - H8. Let $P(s', D)$ be a proposition defined for each (s', D) in $\underline{S}' \times \underline{D}$. We say that $P(s', D)$ holds a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$ if there is a set X' in $\underline{M}'$ such that $\mu' X' = 0$ and such that $P(s', D)$ holds whenever s' is in $C' X'$ and D is in $\underline{D}$. The conclusion of 8.9 of

[2] may be restated thus: Then $W'_*(s', D, \emptyset) = W'(s', D)$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$.

I.6 Basic definitions

Throughout this section H1 - H8 are assumed.

I.6.1. Definition. Let W' be a non-negative function defined on $\underline{S'} \times \underline{D}$. W' is said to be under additive on $\underline{D}$ if the following holds: Suppose a set D in $\underline{D}$ contains a countable number of pairwise disjoint sets D_i in $\underline{D}$; then

$$\sum W'(s', D_i) \leq W'(s', D), \quad s' \in \underline{S'}.$$

I.6.2. Remark. Often we say merely " W' is under additive" rather than " W' is under additive on $\underline{D}$ ".

I.6.3. Definition. Let W' be a non-negative function defined on $\underline{S'} \times \underline{D}$. W' is said to be inner continuous (on $\underline{D}$) if whenever a set D in $\underline{D}$ is the union of a sequence of sets D_i in $\underline{D}$ for which $D_i \subseteq D_{i+1}$, $i \geq 1$, we have $W'(s', D) = \lim_i W'(s', D_i)$, $s' \in \underline{S'}$.

I.6.4. Remarks. Condition ii) for weight functions may be restated as follows: W' is under additive (on $\underline{D}$). Condition iii) for weight functions may be restated as follows: W' is inner continuous (on $\underline{D}$).

I.6.5. Definition. Let W' be a non-negative function defined on $\underline{S}' \times \underline{D}$. W' is said to be almost under additive (on $\underline{D}$) if whenever a set D in $\underline{D}$ contains a countable number of pairwise disjoint sets D_i in $\underline{D}$ we have $\sum W'(s', D_i) \leq W'(s', D)$ a.e. μ' on $\underline{S}'$.

I.6.6. Definition. Let W' be a non-negative function defined on $\underline{S}' \times \underline{D}$. W' is said to be over additive (on $\underline{D}$) if whenever a set D is contained in the union of a countable number of sets D_i in $\underline{D}$ we have $W'(s', D) \leq \sum W'(s', D_i)$, $s' \in \underline{S}'$.

I.6.7. Definition. Let W' be a non-negative function defined on $\underline{S}' \times \underline{D}$. W' is said to be almost over additive (on $\underline{D}$) if whenever a set D in $\underline{D}$ is contained in the union of a countable number of sets D_i in $\underline{D}$ we have $W'(s', D) \leq \sum W'(s', D_i)$ a.e. μ' on $\underline{S}'$.

I.6.8. Definition. Let W' be a non-negative function defined on $\underline{S}' \times \underline{D}$. W' is said to be almost additive (on $\underline{D}$) if the following holds: Suppose D_i , $i \geq 0$, is a countable family of sets in $\underline{D}$ such that the sets D_i , $i \geq 1$, are pairwise disjoint / and such that there exist subsets Y and Z of $\underline{S}$ for which

$$i) \quad \bigcup_{i \geq 1} D_i \cup Y \cup Z = D_0;$$

$$ii) \quad Y \in \underline{M}, \mu Y = 0;$$

$$iii) \quad TZ \in \underline{M}', \mu' TZ = 0.$$

Then $W'(s', D_0) = \sum_{i \geq 1} W'(s', D_i)$ a.e. μ' on $\underline{S}'$.

I.6.9. Definition. Let W' and W'_A be weight functions for T . Assume that $W'_A \leq W'$. Assume also that T is ACW_A ; that is, assume that T is absolutely continuous with respect to the weights $W_A D$, $D \in \underline{D}$, induced by W'_A . Then W'_A is said to be an AC part of W' .

I.6.10. Definition. Let W' and W'_A be weight functions for T . Assume that W'_A is an AC part of W' . Then W'_A is said to be a maximal AC part of W' if $W'_A = W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$ whenever W'_a is any AC part of W' for which $W'_A \leq W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

I.6.11. Definition. Let W' and W'_A be weight functions for T . Assume that W'_A is an AC part of W' . Then W'_A is said to be a strongly maximal AC part of W' if $W'_a \leq W'_A$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$ whenever W'_a is an AC part of W' .

I.6.12. Definition. A weight function W' for T is said to be trivial if $W'(s', S) = 0$ a.e. μ' on $\underline{S}'$.

I.6.13. Definition. A weight function W' for T is said to be singular if every AC part of W' is trivial.

I.6.14. Definition. Assume also HCl. A weight function W' for T is said to be strongly singular if it is singular and if

there exists a set B_0 in $\underline{B}$ such that $\mu B_0 = 0$ and $W'(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0$.

I.6.15. Definition. Let W' be a weight function for T . Let W'_A and W'_S be weight functions for T . The ordered pair (W'_A, W'_S) is said to be a Lebesgue-type decomposition for W' if

- 1) $W' = W'_A + W'_S$;
- 2) T is ACW_A ;
- 3) W'_S is singular.

W'_A is said to be the AC component of the decomposition (W'_A, W'_S) .

W'_S is said to be the singular component of the decomposition (W'_A, W'_S) .

I.6.16. Definition. Assume HCl . Let W' be a weight function for T . Let W'_A and W'_S be weight functions for T . The ordered pair (W'_A, W'_S) is said to be a strong Lebesgue-type decomposition for W' if it is a Lebesgue-type decomposition for W' and if W'_S is strongly singular.

I.7 Preliminary lemmas

Some preliminary lemmas are presented in this section. The results are varied and, to some extent, unrelated.

I.7.1. Lemma. Assume $H1 - H8$. Let W' be a weight function for T and let H' be a non-negative $\underline{M}'$ -measurable function defined

on $\underline{S}'$. Define a non-negative function W'_0 on $\underline{S}' \times \underline{D}$ by

$$W'_0(s', D) = H'(s')W'(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then W'_0 is a weight function for T .

Next suppose W'_1 and W'_2 are weight functions for T . Define W'_0 by $W'_0 = W'_1 + W'_2$. Then W'_0 is a weight function for T .

Finally suppose that W'_1 and W'_2 are weight functions for T and suppose that H'_1 and H'_2 are non-negative real valued $\underline{M}'$ -measurable functions defined on $\underline{S}'$. Define W'_0 on $\underline{S}' \times \underline{D}$ by

$$W'_0(s', D) = H'_1(s')W'_1(s', D) + H'_2(s')W'_2(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then W'_0 is a weight function for T . (In particular, if M' is in $\underline{M}'$ and if we set $H'_1 = H'_{M'}$, and $H'_2 = H'_{C \setminus M'}$, then we see that the function W'_0 defined by

$$W'_0(s', D) = \begin{cases} W'_1(s', D), & \text{if } (s', D) \in M' \times \underline{D} \\ W'_2(s', D), & \text{if } (s', D) \in C \setminus M' \times \underline{D} \end{cases}$$

is a weight function for T .)

Remark. The last assertion of Lemma I.7.1 is often applied with W'_1 chosen to be identically 0. The set M' is quite often a set of μ' measure 0.

Lemma I.7.1 is due to Reichelderfer. The proof is omitted.

I.7.2. Lemma. Assume $H1 - H8$. Let W'_0 , W'_1 , and W'_2 be weight functions for T and suppose $W'_0 = W'_1 + W'_2$ a.e. μ' on $\underline{S}'$

uniformly with respect to $\underline{D}$. If T is BVW_1 and BVW_2 then T is BVW_0 . If T is ACW_1 and T is ACW_2 then T is ACW_0 . If T is ACW_0 and T is ACW_1 then T is ACW_2 .

The straightforward proof is omitted. Lemma 3.2 of [2] is required for the proof of the last assertion.

I.7.3. Remarks. Assume $H1 - H9$. Note that a strongly maximal AC part of W' is necessarily a maximal AC part. If W' is strongly singular then it is singular. W' is trivial if and only if $WS = 0$. Every strong Lebesgue-type decomposition for W' is a Lebesgue-type decomposition for W' .

I.7.4. Lemma. Assume $H1 - H8$. Suppose W'_1 and W'_2 are weight functions for T . Suppose that $W'_1(s', D) = W'_2(s', D)$ a.e. μ' on $\underline{S'}$ for every D in $\underline{D}$. Then:

- 1) $W_1 = W_2$.
- 2) T is BVW_1 if and only if T is BVW_2 .
- 3) T is ACW_1 if and only if T is ACW_2 .
- 4) W'_1 is almost over (under) additive if and only if W'_2 is almost over (under) additive.
- 5) W'_1 is almost additive if and only if W'_2 is almost additive.

Proof. The straightforward proof is omitted.

I.7.5. Lemma. Assume H1 - H8. Let W'_1 and W'_2 be weight functions for T. Assume that $W'_1 = W'_2$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. Then statements 1) - 5) of Lemma I.7.4 are valid. Moreover, W'_1 is singular if and only if W'_2 is singular. Assume now that HCl also holds. Then W'_1 is strongly singular if and only if W'_2 is strongly singular.

Proof. The proof is omitted. The Remark following Lemma I.7.1 is used.

I.7.6. Lemma. Assume H1 - H8. Let W' , W'_1 , and W'_2 be weight functions for T and suppose that $W'_1 \leq W'$ and $W'_2 \leq W'$. Assume also that $W'_1 = W'_2$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. Then W'_1 is a maximal AC part of W' if and only if W'_2 is a maximal AC part of W' . W'_1 is a strongly maximal AC part of W' if and only if W'_2 is a strongly maximal AC part of W' .

Proof. Obvious.

I.7.7. Lemma. Assume that H1 - H8 hold. Let W' , W'_A , and W'_S be weight functions for T and suppose $W' = W'_A + W'_S$. If W'_A is a maximal AC part of W' then W'_S is singular.

Proof. Assume that W'_A is a maximal AC part of W' . We wish to prove that W'_S is singular. Thus let W'_a be an AC part of W'_S . Hence $W'_a \leq W'_S$ and T is AC W'_a . Define a non-negative

function W'_0 on $\underline{S}' \times \underline{D}$ by $W'_0(s', D) = W'_A(s', D) + W'_a(s', D)$,
 $(s', D) \in \underline{S}' \times \underline{D}$. By Lemmas I.7.1 and I.7.2 W'_0 is a weight function
for T and T is ACW_0 . Moreover $W'_A \leq W'_0 = W'_A + W'_a \leq W'_A + W'_S = W'$.
But W'_A is a maximal AC part of W' . Hence $W'_A = W'_0$ a.e. μ' on
 $\underline{S}'$ uniformly with respect to $\underline{D}$. In particular $W'_A(s', \underline{S}) = W'_0(s', \underline{S})$
a.e. μ' on $\underline{S}'$. Consequently $W_{A\underline{S}} = W_{0\underline{S}}$. But also $W_{0\underline{S}} = W_{A\underline{S}} + W_{a\underline{S}}$.
As a result $W_{A\underline{S}} = W_{A\underline{S}} + W_{a\underline{S}}$. Since $W_{A\underline{S}}$ is finite $W_{a\underline{S}} = 0$.
Hence W'_a is trivial and so W'_S must be singular.

Remark. One may ask whether the conclusion of Lemma I.7.7
can be strengthened. In particular, can one show that W'_A must
be a maximal AC part of W' if W'_S is singular? This question is
answered in Chapter II.

I.7.8. Lemma. Assume H1 - H8. Suppose M_0 is an element
of $\underline{M}$. There exists a set B_0 which is in the smallest σ -field
containing $\underline{D}$ and for which we have $M_0 \subseteq B_0$ and $\mu(B_0 \cap CM_0) = 0$.

Proof. By H4 $\underline{S}$ can be written as the union of a countable
number of sets $*D_j$, $j \geq 1$, in $\underline{D}$ for which $\mu_* D_j < \infty$, $j \geq 1$,
and such that the following holds: Given a set M in $\underline{M}$ with
 $M \subseteq *D_j$ for some j and given $\epsilon > 0$, there is a countable number
of pairwise disjoint sets D_i in $\underline{D}$ such that $M \subseteq \bigcup D_i$ and
 $\sum \mu D_i < \mu M + \epsilon$.

Now let a set M_0 in $\underline{M}$ be given. Set $M_j = M_0 \cap *D_j$, $j \geq 1$.

Then M_j is in $\underline{M}$ and $M_j \subseteq \ast D_j$ for each j . For every pair of positive integers j and n there is a countable number of pairwise disjoint sets D_{nji} , $i \geq 1$, in $\underline{D}$ such that $M_j \subseteq \bigcup_{i \geq 1} D_{nji}$ and $\sum_i \mu D_{nji} < \mu M_j + \frac{1}{n}$. For each j , set $B_j = \bigcap_n \bigcup_i D_{nji}$. Then, for each j , B_j is in the smallest σ -field containing $\underline{D}$, $M_j \subseteq B_j$, and $\mu B_j = \mu M_j$. Indeed, since $\mu M_j < \infty$, it follows that $\mu(B_j \cap CM_j) = 0$, $j \geq 1$. Now set $B_0 = \bigcup_{j \geq 1} B_j$. Again, B_0 is in the smallest σ -field containing $\underline{D}$. Also $M_0 = \bigcup_{j \geq 1} M_j \subseteq \bigcup_{j \geq 1} B_j = B_0$. And, since $B_0 \cap CM_0 \subseteq \bigcup_{j \geq 1} (B_j \cap CM_j)$, it follows that $\mu(B_0 \cap CM_0) = 0$.

I.7.9. Lemma. Assume H1 - H8, H11, and H12. Let $c_1, \dots, c_n$ be positive real numbers and let $w_1, \dots, w_n$ be non-negative functions defined on $\mathcal{S}$. Then $W_*(s', D, \sum_{i=1}^n c_i w_i) = \sum_{i=1}^n c_i W_*(s', D, w_i)$, $(s', D) \in \underline{S} \times \underline{D}$.

Proof. The result follows at once from (b) and (c) in Section I.1.2 of [1].

I.7.10. Lemma. Assume H1 - H8, H11, and H12. Let $\{w_m\}_{m=1}^\infty$ be a sequence of non-negative functions defined on $\mathcal{S}$ for which $w_m \leq w_{m+1}$, $m \geq 1$. For each S in $\mathcal{S}$ the sequence $\{w_m S\}_{m=1}^\infty$ is convergent (perhaps to ∞). Define a non-negative function w_0 on

$\mathcal{S}$ by $w_0 S = \lim_m w_m S$, $S \in \mathcal{S}$. Then the sequence

$\{W_*(s', D, w_m)\}_{m=1}^{\infty}$ is non-decreasing for every (s', D) in $\underline{S}'XD$

and $\lim_m W_*(s', D, w_m) = W_*(s', D, w_0)$, $(s', D) \in \underline{S}'XD$.

Proof. A simple direct proof can be supplied or one can appeal to (a) and (b) in Section I.1.2 of [1].

I.7.11. Remarks. The standard hypotheses are H1 - H15, H4*, H7*, H15i, and HC1 - HC4. At no point in this dissertation are all of these hypotheses assumed to hold. Specific mention of those assumed in any situation is always made.

Some of the twenty-two standard hypotheses involve D alone. For example, H10, H15i, and HC4 are concerned only with D. Among these there are some relationships. We list them, without proof:

- 1) Assume H11. If HC3 holds so does H15i.
- 2) If H10 is satisfied so is HC3.
- 3) If HC4 is satisfied so is HC3.

No attempt has been made to discover all of the relationships among the standard hypotheses. 1) - 3) are merely the most obvious ones. They are sometimes used in the sequel without mention.

CHAPTER II

THE MAIN DECOMPOSITION THEOREMS

The main objective of Chapter II is the statement and proof of the "General" and "Strong" decomposition theorems. However, other interesting properties of weight functions will also be established.

In Sections II.1 through II.6 we shall assume that $\{\underline{S}, \underline{M}, \mu\}$ and $\{\underline{S}', \underline{M}', \mu'\}$ are measure spaces, that T is a single valued transformation from $\underline{S}$ onto $\underline{S}'$, that $\underline{D}$ is a family of subsets of $\underline{S}$, and that $\underline{B}'$ is a subfamily of $\underline{M}'$ for which the standard hypotheses H1 - H8 are satisfied. Throughout these sections then, $\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, T , $\underline{D}$, and $\underline{B}'$ remain fixed and hypotheses H1 - H8 are always in force. Whenever any of the other standard hypotheses is used, it is mentioned explicitly.

The notational conventions set forth in Chapter I will of course be observed faithfully.

II.1 The difference of two weight functions

We shall not consider the difference of two arbitrary weight functions. Indeed such a difference is likely not to be non-negative; it could not possibly be a weight function. We shall however consider the difference of two weight functions

satisfying certain conditions and show that this difference is essentially a weight function for T .

II.1.1. Definition. Suppose E is a subset of $\underline{S}$. The set E is said to be of type γ if there are a countable number of sets D_i , $i \geq 1$, in $\underline{D}$ and subsets Y and Z of $\underline{S}$ such that

- i) $E \cap D_i = \emptyset$, $i \geq 1$;
- ii) $D_i \cap D_j = \emptyset$, if $i > j \geq 1$;
- iii) $\underline{S} = E \cup \bigcup_i D_i \cup Y \cup Z$;
- iv) $Y \in \underline{M}$, $\mu Y = 0$;
- v) $TZ \in \underline{M}'$, $\mu' TZ = 0$.

II.1.2. Remark. In Chapter I we defined the statement "A set D_0 (in $\underline{D}$) is of type γ ". Examinations of that earlier definition and of the one just given reveal that a set D_0 in $\underline{D}$ is of type γ under the first definition if and only if it is of type γ under the definition just given. Hence this latter definition is an extension of the earlier one.

II.1.3. Definition. Let D_i , $i \geq 0$, be a countable family of sets in $\underline{D}$ and suppose that Y and Z are subsets of $\underline{S}$. Assume that

- i) $D_i \cap D_j = \emptyset$, if $i > j \geq 1$;
- ii) $Y \in \underline{M}$, $\mu Y = 0$;
- iii) $TZ \in \underline{M}'$, $\mu' TZ = 0$;
- iv) $D_0 = \bigcup_{i \geq 1} D_i \cup Y \cup Z$.

Then the triple $(\{D_i\}_{i \geq 1}, Y, Z)$ is said to be a D-partition of D_0 .

The set $\{D_i\}_{i \geq 1}$ is termed the D-component of the D-partition

$(\{D_i\}_{i \geq 1}, Y, Z)$.

Remark. Evidently a set D in $\underline{D}$ is of type γ if and only if it occurs in the D-component of some D-partition of $\underline{S}$.

II.1.4. Definition. Let $\{D_i\}_{i=1}^n$ be a finite sequence of sets in $\underline{D}$. The sequence $\{D_i\}_{i=1}^n$ is said to be of type γ if there is some D-partition of $\underline{S}$ the D-component of which includes the set $\{D_i\}_{i=1}^n$. The remaining sets in the D-component of such a D-partition are said to form a γ -complement of $\{D_i\}_{i=1}^n$.

Remark. If a sequence $\{D_i\}_{i=1}^n$ is of type γ then each D_i ($i = 1, \dots, n$) is of type γ .

II.1.5. Lemma. Suppose each of the subsets $E_1, \dots, E_n$ of $\underline{S}$ is of type γ . Then $\bigcup_{i=1}^n E_i$ is of type γ .

Proof. We shall prove the result by induction on n . There is nothing to prove if $n = 1$. It is convenient to prove the result for $n = 2$.

Thus suppose E_1 and E_2 are sets of type γ . For $i = 1, 2$, there are a sequence $\{D_{ik}\}_{k \geq 1}$ of sets in $\underline{D}$ and subsets Y_i and Z_i of $\underline{S}$ such that

- i) $E_i \cap D_{ik} = \emptyset, k \geq 1;$
- ii) $D_{ik_1} \cap D_{ik_2} = \emptyset, \text{ if } k_1 > k_2 \geq 1;$
- iii) $\underline{S} = E_i \cup \bigcup_k D_{ik} \cup Y_i \cup Z_i;$
- iv) $Y_i \in \underline{M}, \mu Y_i = 0;$
- v) $TZ_i \in \underline{M}', \mu' TZ_i = 0.$

For every pair of positive integers k_1 and k_2 there is by H^4 a countable sequence $D_{k_1 k_2 m}, m \geq 1$, of pairwise disjoint sets in $\underline{D}$ such that $D_{1k_1} \cap D_{2k_2} = \bigcup_{m \geq 1} D_{k_1 k_2 m}$. It follows from ii) that the sets $D_{k_1 k_2 m}, k_1 \geq 1, k_2 \geq 1, m \geq 1$, are pairwise disjoint. Next, note that, given any $D_{k_1 k_2 m}$, we have

$$(E_1 \cup E_2) \cap D_{k_1 k_2 m} \subseteq (E_1 \cap D_{1k_1}) \cup (E_2 \cap D_{2k_2}) = \emptyset.$$

Moreover the set $Y_1 \cup Y_2$ is in $\underline{M}$ and $\mu(Y_1 \cup Y_2) = 0$. The set $T(Z_1 \cup Z_2)$ is in $\underline{M}'$ with

$$\mu' T(Z_1 \cup Z_2) \leq \mu' TZ_1 + \mu' TZ_2 = 0.$$

Finally, it is easy to check that

$$(E_1 \cup E_2) \cup \bigcup_{k_1, k_2, m} D_{k_1 k_2 m} \cup (Y_1 \cup Y_2) \cup (Z_1 \cup Z_2) = \underline{S}.$$

The sets $D_{k_1 k_2 m}, k_1 \geq 1, k_2 \geq 1, m \geq 1$, and the subsets $Y_1 \cup Y_2$ and $Z_1 \cup Z_2$ of $\underline{S}$ satisfy the five conditions of Definition II.1.1. Hence the set $E_1 \cup E_2$ is of type γ .

We proceed to the induction step. Suppose the result is

true whenever $n = k$. Let $E_1, \dots, E_{k+1}$ be $k + 1$ sets of type γ .

By the induction hypothesis the set $E_1 \cup \dots \cup E_k$ is of type γ .

The union of two sets of type γ is of type γ and so

$$E_1 \cup \dots \cup E_{k+1} = (E_1 \cup \dots \cup E_k) \cup E_{k+1}$$

must be of type γ .

The proof is completed by an inductive argument.

Lemma II.1.5 is due to Reichelderfer.

II.1.6. Lemma. Let $\{D_i\}_{i=1}^n$ be a finite sequence of sets in $\underline{D}$ consisting of pairwise disjoint sets. Suppose that D_i , $i = 1, \dots, n$, is of type γ . Then the sequence $\{D_i\}_{i=1}^n$ is of type γ .

Proof. By Lemma II.1.5 the set $\bigcup_{i=1}^n D_i$ is of type γ . Hence there exist a countable number of sets D_i , $i \geq n + 1$, in $\underline{D}$ and subsets Y and Z of $\underline{S}$ such that

- i) $\bigcup_{k=1}^n D_k \cap D_i = \emptyset$, $i \geq n + 1$;
- ii) $D_{i_1} \cap D_{i_2} = \emptyset$, if $i_1 > i_2 \geq n + 1$;
- iii) $\underline{S} = \bigcup_{k=1}^n D_k \cup \bigcup_{i \geq n+1} D_i \cup Y \cup Z$;
- iv) $Y \in \underline{M}$, $\mu Y = 0$;
- v) $TZ \in \underline{M}'$, $\mu' TZ = 0$.

It follows from i) - v) and from Definition II.1.3 that the

triple $(\{D_i\}_{i \geq 1}, Y, Z)$ is a $\underline{D}$ -partition of $\underline{S}$. $\{D_i\}_{i=1}^n$ is surely

a subset of the $\underline{D}$ -component $\{D_i\}_{i \geq 1}$ of this $\underline{D}$ -partition.

According to Definition II.1.4, $\{D_i\}_{i=1}^n$ is of type γ .

II.1.7. Lemma. Let $\{D_i\}_{i=1}^n$ be a finite collection of pairwise disjoint sets in $\underline{D}$. There is, for $i = 1, \dots, n$, a sequence $\{D_{ij}\}_{j=1}^\infty$ of sets in $\underline{D}$ such that

- i) $D_{ij} \subseteq D_{i, j+1}$, if $1 \leq i \leq n$, $j \geq 1$;
- ii) the set $\{D_{ij}\}_{i=1}^n$ is of type γ , $j \geq 1$;
- iii) $D_i = \bigcup_{j \geq 1} D_{ij}$, $1 \leq i \leq n$.

Proof. For $i = 1, \dots, n$ there exists by H5 a sequence of sets D_{ij} , $j \geq 1$, in $\underline{D}$ such that

- a) $D_{ij} \subseteq D_{i, j+1}$, $j \geq 1$;
- b) each D_{ij} is of type γ , $j \geq 1$;
- c) $D_i = \bigcup_{j \geq 1} D_{ij}$.

Because the D_i 's are pairwise disjoint and because of b) and c), it follows that, for every $j \geq 1$, the sequence $\{D_{ij}\}_{i=1}^n$ is a subset of $\underline{D}$ consisting of pairwise disjoint sets of type γ . By Lemma II.1.6 the sequence $\{D_{ij}\}_{i=1}^n$ must be of type γ ; the desired conclusion follows from this statement and from a) and c).

II.1.8. Lemma. Let $\{D_i\}_{i=1}^I$ be a finite collection of pairwise disjoint sets in $\underline{D}$ each of which is of type γ . Suppose D_0 is in $\underline{D}$ and $\bigcup_{i=1}^I D_i \subseteq D_0$. Then there exists a $\underline{D}$ -partition of D_0 , the $\underline{D}$ -component of which includes $\{D_i\}_{i=1}^I$ as a subset.

Proof. By Lemma II.1.6 the sequence $\{D_i\}_{i=1}^I$ is of type γ . There is a $\underline{D}$ -partition of $\underline{S}$ - say, $(\{D_k^\#\}_{k \geq 1}, Y_\#, Z_\#)$ - for which $\{D_i\}_{i=1}^I \subseteq \{D_k^\#\}_{k \geq 1}$. We may as well assume

$$D_i = D_i^\#, i = 1, \dots, I.$$

Thus the sets $D_k^\#, k \geq 1$, are pairwise disjoint, $Y_\#$ is in $\underline{M}$ with $\mu Y_\# = 0$, and $TZ_\#$ is in $\underline{M}'$ with $\mu' TZ_\# = 0$. Finally

$\underline{S} = \bigcup_{k \geq 1} D_k^\# \cup Y_\# \cup Z_\#$. According to H^4 , there is, for $k \geq I + 1$, a countable number of pairwise disjoint sets D_{kj} , $j \geq 1$, in $\underline{D}$

such that $D_0 \cap D_k^\# = \bigcup_{j \geq 1} D_{kj}$. Set $Y = Y_\# \cap D_0$ and $Z = Z_\# \cap D_0$.

We now assert that the triple

$$(\{D_i\}_{i=1}^I \cup \{D_{kj}\}_{k \geq I+1, j \geq 1}, Y, Z)$$

is a $\underline{D}$ -partition of D_0 . According to Definition II.1.3, there are four conditions to check. It is easy to see that the sets

in the family $\{D_i\}_{i=1}^I \cup \{D_{kj}\}_{k \geq I+1, j \geq 1}$ are pairwise disjoint.

Observe that

$$\begin{aligned} & \bigcup_{i=1}^I D_i \cup \bigcup_{k,j} D_{kj} \cup Y \cup Z = \\ &= (D_0 \cap \bigcup_{i=1}^I D_i) \cup \bigcup_{k \geq I+1} (D_0 \cap D_k^\#) \cup (D_0 \cap Y_\#) \cup (D_0 \cap Z_\#) = \\ &= D_0 \cap \underline{S} = D_0. \end{aligned}$$

Moreover, Y is in $\underline{M}$ with $\mu Y = 0$ and TZ is in $\underline{M}'$ with $\mu' TZ = 0$.

Hence $(\{D_i\}_{i=1}^I \cup \{D_{kj}\}_{k \geq I+1, j \geq 1}, Y, Z)$ is indeed a $\underline{D}$ -partition for D_0 ; its $\underline{D}$ -component does include the sequence $\{D_i\}_{i=1}^I$.

II.1.9. Lemma. Suppose W' and W'_A are weight functions for T and suppose that W'_A is a finite valued AC part of W' . Define a non-negative function W'_Δ on $\underline{S}' \times \underline{D}$ by

$$W'_\Delta(s', D) = W'(s', D) - W'_A(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then W'_Δ satisfies the standard conditions i), iii), and iv) of H9 and is almost under additive.

Proof. First we check the standard conditions i), iii), and iv). Thus if D is in $\underline{D}$ and s' is in $C'TD$ then

$$W'_\Delta(s', D) = W'(s', D) - W'_A(s', D) = 0 - 0 = 0.$$

Next, suppose that D is the union of a sequence of sets D_i , $i \geq 1$, in $\underline{D}$ for which $D_i \subseteq D_{i+1}$, $i \geq 1$. Fix s' in $\underline{S}'$. We must show that $\lim_i W'_\Delta(s', D_i)$ exists and equals $W'_\Delta(s', D)$. Indeed

since $\lim_i W'_A(s', D_i) = W'_A(s', D)$ is finite it follows at once that

$$\begin{aligned} \lim_i W'_\Delta(s', D_i) &= \lim_i W'(s', D_i) - \lim_i W'_A(s', D_i) \\ &= W'(s', D) - W'_A(s', D) \\ &= W'_\Delta(s', D). \end{aligned}$$

Finally, for a fixed D in $\underline{D}$, the function $W'_\Delta(s', D)$, $s' \in \underline{S'}$, is the difference of two $\underline{M'}$ -measurable functions $W'(s', D)$, $s' \in \underline{S'}$, and $W'_A(s', D)$, $s' \in \underline{S'}$; hence it is $\underline{M'}$ -measurable.

Now we prove that the function W'_Δ is almost under additive.

Assume that $\{D_i\}_{i=0}^n$ is a finite subset of $\underline{D}$ for which the sets

D_i , $i = 1, \dots, n$, are pairwise disjoint and $\bigcup_{i=1}^n D_i \subseteq D_0$. Then

$\{D_i\}_{i=1}^n$ is a finite subset of $\underline{D}$ consisting of pairwise disjoint sets. By Lemma II.1.7, there is, for $1 \leq i \leq n$, a sequence of sets D_{ij} , $j \geq 1$, in $\underline{D}$ such that

- i) $D_{ij} \subseteq D_{i, j+1}$, $1 \leq i \leq n$ and $j \geq 1$;
- ii) the set $\{D_{ij}\}_{i=1}^n$ is of type γ , $j \geq 1$;
- iii) $D_i = \bigcup_{j \geq 1} D_{ij}$, $1 \leq i \leq n$.

Fix $j \geq 1$; by Lemma II.1.8 the set $\{D_{ij}\}_{i=1}^n$ is a subset of the $\underline{D}$ -component of some $\underline{D}$ -partition $(\{D_i^j\}_{i \geq 1}, Y_j, Z_j)$ of D_0 . We shall assume

$$D_{ij} = D_i^j, \quad 1 \leq i \leq n, \quad j \geq 1.$$

We have

$$(1) \quad \bigcup_{i \geq 1} D_i^j \subseteq D_0.$$

Using (1) and the under additivity of W' , we obtain

$$(2) \quad W'(s', D_0) \geq \sum_{i=1}^n W'(s', D_{ij}) + \sum_{i \geq n+1} W'(s', D_i^j),$$

$$s' \in \underline{S'}.$$

By Lemma 4.1 of [2] we obtain

$$(3) \quad W'_A(s', D_0) = \sum_{i=1}^n W'_A(s', D_{ij}) + \sum_{i \geq n+1} W'_A(s', D_i^j)$$

$$\text{a.e. } \mu' \text{ on } \underline{S'}.$$

From (2) and (3) it follows that

$$\begin{aligned} W'_A(s', D_0) &= W'(s', D_0) - W'_A(s', D_0) \\ &\geq \sum_{i=1}^n \{W'(s', D_{ij}) - W'_A(s', D_{ij})\} + \\ &\quad + \sum_{i \geq n+1} \{W'(s', D_i^j) - W'_A(s', D_i^j)\} \\ &\geq \sum_{i=1}^n \{W'(s', D_{ij}) - W'_A(s', D_{ij})\} \end{aligned}$$

$$\text{a.e. } \mu' \text{ on } \underline{S'}.$$

In short

$$(4) \quad W'_{\Delta}(s', D_0) \geq \sum_{i=1}^n W'_{\Delta}(s', D_{ij}) \quad \text{a.e. } \mu' \text{ on } \underline{S}', j \geq 1.$$

We have already proved that W'_{Δ} is inner continuous; hence, in view of i) and iii), it follows that $\lim_j W'_{\Delta}(s', D_{ij}) = W'_{\Delta}(s', D_i)$, $s' \in \underline{S}'$, $1 \leq i \leq n$. Consequently

$$(5) \quad \lim_j \sum_{i=1}^n W'_{\Delta}(s', D_{ij}) = \sum_{i=1}^n W'_{\Delta}(s', D_i), \quad s' \in \underline{S}'.$$

Because of (4) and (5) we have $\sum_{i=1}^n W'_{\Delta}(s', D_i) \leq W'_{\Delta}(s', D_0)$ a.e. μ' on $\underline{S}'$. Hence W'_{Δ} is almost under additive.

II.1.10. Remark. Because of Lemma II.1.9, the function W'_{Δ} might be described as a "pseudo" weight function for T . Only the required under additivity is lacking. From the point of view of integration this property is effectively present in the almost under additivity of W'_{Δ} . We shall now show that, subject to an additional hypothesis concerning the family $\underline{D}$, there does exist a weight function for T which agrees with W'_{Δ} a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

II.1.11. Lemma. Assume HC4. Let $\{D_i\}_{i=0}^I$ be a finite subset of $\underline{D}$ for which the sets D_i , $1 \leq i \leq I$, are pairwise disjoint and $\bigcup_{i=1}^I D_i \subseteq D_0$. For each $i = 0, 1, \dots, I$, there is a sequence of sets D_{ij}^* , $j \geq 1$, in $\underline{D}_*$ such that

- 1) $D_{ij}^* \subseteq D_{i, j+1}^*, \quad 0 \leq i \leq I \quad \text{and} \quad j \geq 1;$
- 2) $\bigcup_{j \geq 1} D_{ij}^* = D_i, \quad 0 \leq i \leq I;$
- 3) $\bigcup_{i=1}^I D_{ij}^* \subseteq D_{0j}^*, \quad j \geq 1.$

Proof. We shall construct the "double" sequence

$\{D_{ij}^*\}_{i=0}^I \quad j=1, 2, \dots$ by mathematical induction on j . According to

HC4, there is, for $0 \leq i \leq I$, a sequence of sets $D_{ij}^*, j \geq 1$, in $\underline{D}_*$ such that $D_{ij}^* \subseteq D_{i, j+1}^*, j \geq 1$, and $D_i = \bigcup_{j \geq 1} D_{ij}^*$. According to ii) of HC4, there exists a set D_{01}^* in $\underline{D}_*$ such that

$\bigcup_{i=0}^I D_{i1}^* \subseteq D_{01}^* \subseteq D_0$. Now set $D_{i1}^* = D_{i1}^*, 1 \leq i \leq I$. We must observe that we have

$$\text{ii)} \quad D_{i1}^* = D_{i1}^*, \quad 1 \leq i \leq I;$$

$$\text{iii)} \quad \bigcup_{i=1}^I D_{i1}^* \cup D_{01}^* \subseteq D_{01}^* \subseteq D_0.$$

Now assume that the family $\{D_{ij}^*\}_{i=0}^I$ has been extended to a

finite subset $\{D_{ij}^*\}_{i=0}^I \quad j=1, 2, \dots, k$ of $\underline{D}_*$ for which

$$\text{i}_k) \quad D_{ij}^* \subseteq D_{i, j+1}^*, \quad 0 \leq i \leq I, \quad 1 \leq j < k;$$

$$\text{ii}_k) \quad D_{ij}^* = D_{ij}^*, \quad 1 \leq i \leq I, \quad 1 \leq j \leq k;$$

$$\text{iii}_k) \quad \bigcup_{i=1}^I D_{ij}^* \cup D_{0j}^* \subseteq D_{0j}^* \subseteq D_0, \quad 1 \leq j \leq k.$$

We wish to extend the set $\{D_{ij}^*\}_{i=0}^I \quad j=1, 2, \dots, k$ to a larger subset

$\{D_{ij}^*\}_{i=0}^I \quad j=1, 2, \dots, k+1$ of $\underline{D}_*$ for which $\text{i}_k)$, $\text{ii}_k)$, and $\text{iii}_k)$ hold with k

replaced by $k + 1$. In order to carry out this extension, we use ii) of HC4 again. Thus, since $\bigcup_{i=0}^I {}^*D_{i \ k+1} \subseteq D_0$, there exists a set $D_{0 \ k+1}^*$ in $\underline{D}_*$ such that $D_{0k}^* \cup \bigcup_{i=0}^I {}^*D_{i \ k+1} \subseteq D_{0 \ k+1}^* \subseteq D_0$. Set $D_{i \ k+1}^* = {}^*D_{i \ k+1}$, $1 \leq i \leq I$. Conditions i_{k+1}), ii_{k+1}), and iii_{k+1}) are evidently satisfied by the set $\{D_{ij}^*\}_{i=0}^I \quad j=1$.

By induction we can then continue to extend the successive families $\{D_{ij}^*\}_{i=0}^I \quad j=1$ to obtain a subfamily $\{D_{ij}^*\}_{i=0}^I \quad j=1$ of $\underline{D}_*$ for which

- i) $D_{ij}^* \subseteq D_{i \ j+1}^*$, $0 \leq i \leq I$, $j \geq 1$;
- ii) $D_{ij}^* = {}^*D_{ij}$, $1 \leq i \leq I$, $j \geq 1$;
- iii) ${}^*D_{0j} \cup (\bigcup_{i=1}^I D_{ij}^*) \subseteq D_{0j}^* \subseteq D_0$, $j \geq 1$.

We assert that this family $\{D_{ij}^*\}_{i=0}^I \quad j=1$ satisfies conditions 1), 2), and 3). Indeed condition 1) of the conclusion is i) above. Condition 3) is an immediate consequence of iii).

Using ii), we observe that

$$\bigcup_{j \geq 1} D_{ij}^* = \bigcup_{j \geq 1} {}^*D_{ij} = D_i, \quad 1 \leq i \leq I;$$

using iii), we note that $D_0 = \bigcup_{j \geq 1} {}^*D_{0j} \subseteq \bigcup_{j \geq 1} D_{0j}^* \subseteq D_0$. Hence

the family $\{D_{ij}^*\}_{i=0}^I \quad j=1$ also satisfies 2).

II.1.12. Lemma. Assume HC4. Suppose W' is a non-negative function defined on $\underline{S}' \times \underline{D}$. Suppose that W' is inner continuous

and almost under additive. There exists a set M' in $\underline{M}'$ such that $\mu'M' = 0$ and such that the following holds: Whenever a set D in $\underline{D}$ contains a countable number of pairwise disjoint sets D_i in $\underline{D}$ then $\sum_{i \geq 1} W'(s', D_i) \leq W'(s', D)$, $s' \in C'M'$.

Proof. Denote by $\aleph$ the collection of all finite sequences $\{D_i^*\}_{i=0}^n$ of sets in $\underline{D}_*$ for which the sets D_i^* , $1 \leq i \leq n$, are pairwise disjoint and $\bigcup_{i=1}^n D_i^* \subseteq D_0^*$. Since $\underline{D}_*$ is countable the collection $\aleph$ is also countable. By assumption W' is almost under additive. Hence, for every set $\{D_i^*\}_{i=0}^n$ in $\aleph$, there exists a set $M'(\{D_i^*\}_{i=0}^n)$ in $\underline{M}'$ such that $\mu'M'(\{D_i^*\}_{i=0}^n) = 0$ and

$$\sum_{i=1}^n W'(s', D_i^*) \leq W'(s', D_0^*), \quad s' \in C'M'(\{D_i^*\}_{i=0}^n).$$

Define $M' = \bigcup M'(\{D_i^*\}_{i=0}^n)$, $\{D_i^*\}_{i=0}^n \in \aleph$. Because $\aleph$ is countable, it follows that M' is in $\underline{M}'$ and $\mu'M' = 0$. Furthermore

$$(1) \quad \sum_{i=1}^n W'(s', D_i^*) \leq W'(s', D_0^*), \quad s' \in C'M', \quad \{D_i^*\}_{i=0}^n \in \aleph.$$

Now let $\{D_i\}_{i=0}^I$ be any finite sequence of sets in $\underline{D}$ for which the sets D_i , $1 \leq i \leq I$, are pairwise disjoint and $\bigcup_{i=1}^I D_i \subseteq D_0$. By Lemma II.1.11 there exists a double sequence $\{D_{ij}^*\}_{i=0}^I \quad j=1 \quad \infty$ of sets in $\underline{D}_*$ with the properties listed as 1), 2), and 3) in the conclusion of Lemma II.1.11. The sets D_{ij}^* ,

$1 \leq i \leq I$, are necessarily pairwise disjoint for every j . Hence,

in view of 3), $\{D_{ij}^*\}_{i=0}^I$ is in $\mathcal{N}$ for $j \geq 1$. According to (1),

$$(2) \quad \sum_{i=1}^I W'(s', D_{ij}^*) \leq W'(s', D_{0j}^*), \quad s' \in C'M', \quad j \geq 1.$$

Using 1), 2), and the inner continuity of W' , we obtain

$$(3) \quad \lim_j W'(s', D_{ij}^*) = W'(s', D_i), \quad 0 \leq i \leq I, \quad s' \in \underline{S}'.$$

Hence the sequence $\{\sum_{i=1}^I W'(s', D_{ij}^*)\}_{j=1}^\infty$ is convergent for every

s' in $\underline{S}'$ and, using (2) and (3), we obtain

$$\begin{aligned} W'(s', D_0) &= \lim_j W'(s', D_{0j}^*) \\ &\geq \lim_j \sum_{i=1}^I W'(s', D_{ij}^*) \\ &\geq \sum_{i=1}^I W'(s', D_i) \end{aligned}$$

whenever s' is in $C'M'$; in brief

$$(4) \quad \sum_{i=1}^I W'(s', D_i) \leq W'(s', D_0), \quad s' \in C'M'.$$

Expression (4) holds whenever $\{D_i\}_{i=0}^I$ is any finite sequence of sets in $\underline{D}$ for which the sets D_i , $1 \leq i \leq I$, are pairwise disjoint

and $\bigcup_{i=1}^I D_i \subseteq D_0$.

Finally let D be a set in $\underline{D}$ which contains a countable number of pairwise disjoint sets D_i , $i \geq 1$. By the result

proved in the preceding paragraph, we have

$$\sum_{i=1}^I W'(s', D_i) \leq W'(s', D), \quad s' \in C'M',$$

for every positive integer I . Hence

$$\sum_{i \geq 1} W'(s', D_i) \leq W'(s', D), \quad s' \in C'M'.$$

II.1.13. Theorem. Assume HC4. Assume that W' and W'_A are weight functions for T . Assume that W'_A is a finite valued AC part of W' . Then there is a weight function $W'_\#$ for T for which $W' = W'_A + W'_\#$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$.
 $W' - W'_A = W'_\#$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$.

Proof. Define the function W'_Δ on the set $\underline{S'} \times \underline{D}$ by

$$W'_\Delta(s', D) = W'(s', D) - W'_A(s', D), \quad (s', D) \in \underline{S'} \times \underline{D}.$$

According to Lemma II.1.9, W'_Δ satisfies conditions i), iii), and iv) of H9 and is almost under additive. By Lemma II.1.12, there exists a set M' in $\underline{M'}$ such that $\mu'M' = 0$ and such that the following holds: Given a set D in $\underline{D}$ which contains a countable number of pairwise disjoint sets D_i in $\underline{D}$ then $\sum_i W'(s', D_i) \leq W'(s', D)$ for every s' in $C'M'$.

Now define a non-negative function $W'_\#$ on $\underline{S'} \times \underline{D}$ by

$$W'_\#(s', D) = \begin{cases} W'_\Delta(s', D) & \text{if } s' \in C'M', D \in \underline{D} \\ 0 & \text{if } s' \in M', D \in \underline{D} \end{cases}.$$

Observe that $W'_\# \leq W'_\Delta$. We assert that $W'_\#$ is a weight function for T . Condition i) of H9 follows from the facts that $W'_\# \leq W'_\Delta$ and that W'_Δ itself satisfies condition i) of H9. The inner

continuity of $W'_{\#}$ follows from the inner continuity of W'_{Δ} ; hence $W'_{\#}$ satisfies condition iii) of H9. Given D in $\underline{D}$, the function $W'_{\#}(s', D)$, $s' \in \underline{S}'$, is evidently $\underline{M}'$ -measurable; hence $W'_{\#}$ satisfies condition iv) of H9. In order to show that $W'_{\#}$ is a weight function for T , we now have only to show that it is under additive. Thus suppose a set D in $\underline{D}$ contains a countable number of pairwise disjoint sets D_i in $\underline{D}$. In case s' is in $\underline{M}'$, we have $\sum_i W'_{\#}(s', D_i) = 0 = W'_{\#}(s', D)$. But, in case s' is in $\underline{C}'\underline{M}'$, $\sum_i W'_{\#}(s', D_i) = \sum_i W'_{\Delta}(s', D_i) \leq W'_{\Delta}(s', D) = W'_{\#}(s', D)$. Hence $W'_{\#}$ is under additive. Hence $W'_{\#}$ is a weight function for T . Finally, observe that

$$\begin{aligned} W'(s', D) &= W'_{\Delta}(s', D) + W'_{\#}(s', D) \\ &= W'_{\Delta}(s', D) + W'_{\#}(s', D) \end{aligned}$$

if s' is in $\underline{C}'\underline{M}'$ and D is in $\underline{D}$. Since $\mu'\underline{M}' = 0$, it follows that $W' = W'_{\Delta} + W'_{\#}$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

II.1.14. Theorem. Assume HC4. Let W' and W'_a be weight functions for T and suppose that W'_a is an AC part of W' . Then there exist weight functions W'_{Δ} and $W'_{\#}$ for T such that

$$(a) \quad W' = W'_{\Delta} + W'_{\#};$$

$$(b) \quad W'_{\Delta} \leq W'_a;$$

(c) $W'_A = W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$;

(d) T is ACW_A .

Proof. By assumption T is ACW_a . Hence T is BVW_a . As a result, $W'_a(s', \underline{S}) < \infty$ a.e. μ' on $\underline{S}'$. Indeed, since W'_a is under additive, $W'_a(s', \underline{D}) < \infty$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. There exists a set M'_1 in $\underline{M}'$ such that $\mu'(M'_1) = 0$ and $W'_a(s', \underline{D}) < \infty$, $s' \in C'M'_1$, $\underline{D} \in \underline{D}$. Define a non-negative function W'_0 on $\underline{S}' \times \underline{D}$ by

$$(1) \quad W'_0(s', \underline{D}) = \begin{cases} W'_a(s', \underline{D}), & s' \in C'M'_1, \underline{D} \in \underline{D} \\ 0, & s' \in M'_1, \underline{D} \in \underline{D} \end{cases}.$$

By Lemma I.7.1 W'_0 is a finite valued weight function for T and $W'_0 \leq W'_a \leq W'$. Moreover, since $W'_0 = W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$, it follows from Lemma I.7.5 that T is ACW_0 .

The weight functions W' and W'_0 (for T) satisfy the hypotheses of Theorem II.1.13. Hence there exists a weight function $W'_\square$ for T for which $W' = W'_0 + W'_\square$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. There is a set M'_2 in $\underline{M}'$ such that $\mu'(M'_2) = 0$ and

$$(2) \quad W'(s', \underline{D}) = W'_0(s', \underline{D}) + W'_\square(s', \underline{D}), \quad s' \in C'M'_2, \underline{D} \in \underline{D}.$$

Define non-negative functions W'_A and $W'_{\#}$ on $\underline{S}' \times \underline{D}$ by

$$W'_A(s', D) = \begin{cases} W'_a(s', D), & \text{if } s' \in C'(M'_1 \cup M'_2), D \in \underline{D} \\ 0, & \text{if } s' \in M'_1 \cup M'_2, D \in \underline{D} \end{cases}$$

and

$$W'_{\#}(s', D) = \begin{cases} W'_{\square}(s', D), & \text{if } s' \in C'(M'_1 \cup M'_2), D \in \underline{D} \\ W'(s', D), & \text{if } s' \in M'_1 \cup M'_2, D \in \underline{D} \end{cases}.$$

By Lemma I.7.1 both W'_A and $W'_{\#}$ are weight functions for T .

We assert that the weight functions W'_A and $W'_{\#}$ satisfy conditions (a) - (d) of the conclusion. (a) follows from (1) and (2). Conditions (b) and (c) are easily verified. Condition (d) is an immediate consequence of Lemma I.7.5 and (c).

II.1.15. We prove a converse of Lemma I.7.7. One additional assumption is required.

Theorem. Assume HC4. Let W' , W'_A , and W'_S be weight functions for T and suppose that $W' = W'_A + W'_S$. Assume also that T is ACW_A . If W'_S is singular then W'_A is a maximal AC part of W' .

Proof. Suppose W'_a is an AC part of W' for which $W'_A \leq W'_a$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. There is a set M'_0 in $\underline{M'}$ such that $\mu'(M'_0) = 0$ and $0 \leq W'_A(s', D) \leq W'_a(s', D) < \infty$, $s' \in C'M'_0$, $D \in \underline{D}$. Define non-negative functions W'_1 and W'_2 on $\underline{S'} \times \underline{D}$ by

$$W'_1(s', D) = \begin{cases} W'_A(s', D), & s' \in C'M'_0, D \in \underline{D} \\ 0, & s' \in M'_0, D \in \underline{D} \end{cases}$$

and

$$W_2'(s', D) = \begin{cases} W_a'(s', D), & s' \in C'M_0', D \in \underline{D} \\ 0, & s' \in M_0', D \in \underline{D} \end{cases}.$$

By Lemma I.7.1 W_1' and W_2' are weight functions for T . They are finite valued. By Lemma I.7.5 T is ACW_2 and W_1' is an AC part of W_2' . By Theorem II.1.13 there is a weight function $W_{\#}'$ for T for which $W_1' + W_{\#}' = W_2'$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. By Lemma I.7.2, T is $ACW_{\#}$. There is a set M_1' in $\underline{M}'$ such that $\mu'M_1' = 0$ and $W_1'(s', D) + W_{\#}'(s', D) = W_2'(s', D)$, $s' \in C'M_1'$, $D \in \underline{D}$. Define a non-negative function W_0' on $\underline{S}' \times \underline{D}$ by

$$W_0'(s', D) = \begin{cases} W_{\#}'(s', D), & s' \in C'M_1', D \in \underline{D} \\ 0, & s' \in M_1', D \in \underline{D} \end{cases}.$$

By Lemma I.7.1 W_0' is a weight function for T . By Lemma I.7.5 T is ACW_0 . Moreover

$$W_0' \leq W_2' - W_1' \leq W_S'.$$

Hence W_0' is an AC part of W_S' and so W_0' must be trivial. Hence $W_{\#}'$ is trivial. As a result, $W_A' = W_a'$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

II.2 The general decomposition theorem

In this section we shall prove the "general" decomposition theorem for weight functions. First however, some lemmas are needed.

II.2.1. Lemma. Let $\{a_{ni}\}_{n=1}^{\infty} \{i=1}^{\infty}$ be a double sequence whose entries a_{ni} are real numbers or $+\infty$. Suppose $a_{ni} \leq a_{n+1 i}$, $i \geq 1$, $n \geq 1$, and $a_{ni} \leq a_{n i+1}$, $i \geq 1$, $n \geq 1$. Then the sequences $\{a_{ni}\}_{i=1}^{\infty}$ converge (possibly to $+\infty$) for $n \geq 1$ and the sequences $\{a_{ni}\}_{n=1}^{\infty}$ converge (possibly to $+\infty$) for $i \geq 1$. Moreover the sequences $\{\lim_i a_{ni}\}_{n=1}^{\infty}$ and $\{\lim_n a_{ni}\}_{i=1}^{\infty}$ converge (possibly to $+\infty$) and

$$\lim_n \lim_i a_{ni} = \lim_i \lim_n a_{ni}.$$

The simple proof is omitted.

II.2.2. Lemma. Assume $\{W'_n\}_{n=1}^{\infty}$ is a sequence of weight functions for T for which $W'_n \leq W'_{n+1}$, $n \geq 1$. (Thus the sequence $\{W'_n(s', D)\}_{n=1}^{\infty}$ is convergent for every (s', D) in $\underline{S'} \times \underline{D}$.) Define a non-negative function on $\underline{S'} \times \underline{D}$ by

$$W'_0(s', D) = \lim_n W'_n(s', D), \quad (s', D) \in \underline{S'} \times \underline{D}.$$

Then W'_0 is a weight function for T . If $\lim_n W'_n \underline{S} < \infty$, then T is BVW_0 . If $\lim_n W'_n \underline{S} < \infty$ and if T is ACW_n ($n \geq 1$), then T is ACW_0 .

Proof. The function W'_0 is indeed non-negative. To show that it is a weight function for T we shall check the four standard conditions of H9.

i) Suppose D is in $\underline{D}$ and s' is in $C'TD$. Then $W'_0(s', D) = \lim_n W'_n(s', D) = 0$.

ii) Let $\{D_i\}_{i=0}^m$ be a finite subset of $\underline{D}$ for which the sets D_i , $1 \leq i \leq m$, are pairwise disjoint and such that $\bigcup_{i=1}^m D_i \subseteq D_0$. Fix s' in $\underline{S'}$. Using the under additivity of each W'_n we obtain

$$\begin{aligned} \sum_{i=1}^m W'_0(s', D_i) &= \sum_{i=1}^m \lim_n W'_n(s', D_i) \\ &= \lim_n \sum_{i=1}^m W'_n(s', D_i) \\ &\leq \lim_n W'_n(s', D_0) \\ &\leq W'_0(s', D_0). \end{aligned}$$

Hence W'_0 is under additive.

iii) Suppose a set D in $\underline{D}$ is the union of a countable number of sets D_i in $\underline{D}$ for which $D_i \subseteq D_{i+1}$, $i \geq 1$. Fix s' in $\underline{S'}$. First of all, we have

$$(1) \quad W'_0(s', D) = \lim_n W'_n(s', D) = \lim_n \lim_i W'_n(s', D_i).$$

The "numbers" $a_{ni} = W'_n(s', D_i)$, $n \geq 1$, $i \geq 1$, satisfy the

requirements of Lemma II.2.1. Thus the sequences $\{W'_n(s', D_i)\}_{n=1}^\infty$ are convergent and the sequence $\{\lim_n W'_n(s', D_i)\}_{i=1}^\infty$ is convergent; and,

$$(2) \quad \lim_n \lim_i W'_n(s', D_i) = \lim_i \lim_n W'_n(s', D_i).$$

Combining (1) and (2) yields

$$W'_0(s', D) = \lim_i \lim_n W'_n(s', D_i) = \lim_i W'_0(s', D_i).$$

Hence W'_0 is inner continuous.

iv) Fix D in $\underline{D}$. Then $W'_0(s', D) = \lim_n W'_n(s', D)$, $s' \in \underline{S}'$. Hence the function $W'_0(s', D)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable because each of the functions $W'_n(s', D)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable.

In view of i) - iv), W'_0 is a weight function for T . Now assume also that $\lim_n W_n \underline{S} < \infty$. By the Lebesgue Monotone Theorem

$$\begin{aligned} W_0 \underline{S} &= \int_{\underline{S}'} W'_0(s', \underline{S}) \, d\mu' \\ &= \lim_n \int_{\underline{S}'} W'_n(s', \underline{S}) \, d\mu' \\ &= \lim_n W_n \underline{S}. \end{aligned}$$

Hence $W_0 \underline{S}$ is finite and so T is BW_0 .

Now assume that $\lim_n W_n \underline{S} < \infty$ and that T is ACW_n for every positive integer n . Observe that T is BW_0 . Since T is ACW_n , there is, for each n , a non-negative $\underline{M}$ -measurable function $f_n(s)$, $s \in \underline{S}$, such that $W_n D = \int_D f_n(s) \, d\mu$, $D \in \underline{D}$. The weight functions W'_n are monotonically non-decreasing; thus $W_n \leq W_{n+1}$ for every $n \geq 1$. Consequently,

$$\int_D f_n(s) \, d\mu \leq \int_D f_{n+1}(s) \, d\mu, \quad n \geq 1, D \in \underline{D}.$$

By Lemma 3.2 of [2]

$$(3) \quad f_n(s) \leq f_{n+1}(s) \quad \text{a.e. } \mu \text{ on } \underline{S},$$

for every $n \geq 1$. Define a non-negative function f_0 on $\underline{S}$ by

$$(4) \quad f_0(s) = \begin{cases} \lim_n f_n(s), & \text{if } \{f_n(s)\}_{n=1}^{\infty} \text{ is convergent} \\ 0 & , \text{ if } \{f_n(s)\}_{n=1}^{\infty} \text{ is divergent} \end{cases}.$$

Then f_0 is non-negative, $\underline{M}$ -measurable, and by the Lebesgue

Monotone Theorem we have

$$\begin{aligned} W_0 D &= \int_{\underline{S}} W'_0(s', D) \, d\mu' \\ &= \lim_n \int_{\underline{S}} W'_n(s', D) \, d\mu' \\ &= \lim_n \int_D f_n(s) \, d\mu \\ &= \int_D f_0(s) \, d\mu \end{aligned}$$

whenever D is in $\underline{D}$. Hence surely f_0 is a g.l.b.f. W_0 . Hence T is ACW_0 .

II.2.3. Lemma. Assume HC3. Let W'_1 and W'_2 be two weight functions for T for which T is ACW_1 and ACW_2 . Then the following three conditions are equivalent:

- i) $W'_1 \leq W'_2$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$;
- ii) $W'_1(s', D) \leq W'_2(s', D)$ a.e. μ' on $\underline{S}'$ for every D in $\underline{D}$;
- iii) $W_1 \leq W_2$.

Proof. It is clear that i) implies ii). That ii) implies iii) is equally clear.

It remains to show that iii) implies i). Thus assume $W_1 \leq W_2$. Let f_i be a g.l.b.f. W_i , $i = 1, 2$. It follows readily from Lemma 3.2 of [2] that $f_1(s) \leq f_2(s)$ a.e. μ on $\underline{S}$.

Suppose M' is in $\underline{M}'$ and D is in $\underline{D}$. As usual, let $H_{M'}^1$ denote the characteristic function of M' as a subset of $\underline{S}'$. Then, using the transformation formula (Theorem 4.10 of [2]), we obtain

$$\begin{aligned} \int_{M'} W_1^1(s', D) d\mu' &= \int_{\underline{S}'} H_{M'}^1(s') W_1^1(s', D) d\mu' \\ &= \int_D H_{M'}^1 T(s) f_1(s) d\mu \\ &\leq \int_D H_{M'}^1 T(s) f_2(s) d\mu \\ &\leq \int_{\underline{S}'} H_{M'}^1(s') W_2^1(s', D) d\mu' \\ &\leq \int_{M'} W_2^1(s', D) d\mu'. \end{aligned}$$

Using the usual measure theory argument, it follows that $W_1^1(s', D) \leq W_2^1(s', D)$ a.e. μ' on $\underline{S}'$. For every $D_{\#}$ in the countable family $\underline{D}_{\#}$ whose existence is guaranteed by HC3, there exists a set $M'(D_{\#})$ in $\underline{M}'$ such that $\mu' M'(D_{\#}) = 0$ and

$$W_1^1(s', D_{\#}) \leq W_2^1(s', D_{\#}), \quad s' \in C' M'(D_{\#}).$$

Now define $M'_0 = \bigcup M'(D_{\#})$, $D_{\#} \in \underline{D}_{\#}$. Since $\underline{D}_{\#}$ is countable M'_0 is in $\underline{M}'$ and, in fact, $\mu' M'_0 = 0$. Moreover

$$(1) \quad W_1^1(s', D_{\#}) \leq W_2^1(s', D_{\#}), \quad s' \in C' M'_0, D_{\#} \in \underline{D}_{\#}.$$

Using HC3 and the inner continuity of the weight functions W'_1 and W'_2 , it now follows from (1) that

$$(2) \quad W'_1(s', D) \leq W'_2(s', D), \quad s' \in C'M'_0, D \in \underline{D}.$$

i) now follows from (2) and the fact that $\mu'M'_0 = 0$.

Remark. If we assume HC3, assume that T is ACW_1 , assume that T is ACW_2 , and assume finally that $W_1 = W_2$, we can deduce that $W'_1 = W'_2$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. In other words, in this situation, the weights WD , $D \in \underline{D}$, determine the corresponding weight functions essentially uniquely.

II.2.4. Definition. Let U' be any family of weight functions for T . We define a relation on U' as follows: Suppose W'_1 and W'_2 are in U' . Then we write $W'_1 \prec W'_2$ if and only if $W_1 \leq W_2$.

II.2.5. Remarks. The relation $\prec$ is reflexive and transitive on U' . Hence $(U', \prec)$ is a partially ordered set. We say that W'_1 precedes W'_2 in the ordering $\prec$ if and only if $W'_1 \prec W'_2$. $\prec$ will be termed the standard partial ordering for the family U' of weight functions for T .

II.2.6. Lemma. Assume HC3. Assume that Ω' is a countable family of weight functions for T . Let $\prec$ be the standard partial ordering for Ω' . Assume that Ω' is linearly ordered by $\prec$. Assume further that T is ACW_a for every W'_a in Ω' .

Then there exists a weight function W'_0 for T for which

- 1) $W'_0(s', D) \leq \sup \{W'_a(s', D) : W'_a \in \Omega'\}$ for every (s', D) in $\underline{S'} \times \underline{D}$;
- 2) $W'_a \leq W'_0$ for every W'_a in Ω' ;
- 3) if $\sup \{W'_a : W'_a \in \Omega'\} < \infty$, then T is ACW'_0 .

Proof. Given two distinct weight functions W'_a and W'_c in Ω' with $W'_a < W'_c$, there exists by Lemma II.2.3 a set $M'(W'_a, W'_c)$ in $\underline{M'}$ such that $\mu'M'(W'_a, W'_c) = 0$ and such that

$$W'_a(s', D) \leq W'_c(s', D), \quad s' \in C'M'(W'_a, W'_c), \quad D \in \underline{D}.$$

Define

$$X' = \cup M'(W'_a, W'_c), \quad W'_a, W'_c \in \Omega' \text{ and } W'_a < W'_c.$$

Then, because Ω' is countable, X' is in $\underline{M'}$ and $\mu'X' = 0$. Moreover we have

$$(1) \quad W'_a(s', D) \leq W'_c(s', D), \quad s' \in C'X', \quad D \in \underline{D}, \quad W'_a < W'_c \in \Omega'.$$

Next, given W'_a in Ω' , we define a non-negative function $*W'_a$ on $\underline{S'} \times \underline{D}$ by

$$(2) \quad *W'_a(s', D) = \begin{cases} W'_a(s', D), & (s', D) \in C'X' \times \underline{D} \\ 0, & (s', D) \in X' \times \underline{D} \end{cases}.$$

Define $*\Omega'$ to be the family of all non-negative functions f' defined on $\underline{S'} \times \underline{D}$ for which $f' = *W'_a$ for some W'_a in Ω' . In view of (2) and of Lemma I.7.1, $*\Omega'$ consists of weight functions for T . We can introduce the standard partial ordering $\leq^*$ for the

family $^*\Omega'$ of weight functions for T . There is a natural mapping F from Ω' onto $^*\Omega'$ defined by

$$F(W'_a) = {}^*W'_a, \quad W'_a \in \Omega'.$$

Because of (2) and because of the fact that $\mu'X' = 0$ we can deduce that $W'_a = {}^*W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$ whenever W'_a is in Ω' and ${}^*W'_a = F(W'_a)$; by Lemma I.7.5, $W_a = {}^*W_a$ and thus T is AC^*W_a for every ${}^*W'_a$ in $^*\Omega'$. Furthermore it follows that $W'_a < W'_c$ if and only if $F(W'_a) <^* F(W'_c)$. Hence $<^*$ linearly orders $^*\Omega'$. Finally, it follows from (1) and (2) that ${}^*W'_a <^* {}^*W'_c$ if and only if ${}^*W'_a \leq {}^*W'_c$.

At last we are ready to introduce the desired function W'_0 .

We define a non-negative function W'_0 on $\underline{S}' \times \underline{D}$ by

$$W'_0(s', D) = \sup \{ {}^*W'_a(s', D) : {}^*W'_a \in {}^*\Omega' \},$$

for every (s', D) in $\underline{S}' \times \underline{D}$. First we shall prove that W'_0 is a weight function for T . We must check the four standard requirements of H9.

i) Suppose D is in $\underline{D}$ and s' is in $C'TD$. The family $^*\Omega'$ consists of weight functions for T so that we must have $W'_0(s', D) = \sup \{ {}^*W'_a(s', D) : {}^*W'_a \in {}^*\Omega' \} = 0$.

ii) Now we assert that W'_0 is under additive. Suppose D in $\underline{D}$ contains a finite number of pairwise disjoint sets D_i , $1 \leq i \leq I$. Fix s' in $\underline{S}'$. Two cases must be treated separately.

First we assume that there is a positive integer i_0 for

which $W'_0(s', D_{i_0}) = \infty$. In this situation we have $\infty = W'_0(s', D_{i_0}) =$
 $= \sup \{ *W'_a(s', D_{i_0}) : *W'_a \in *\Omega' \} \leq \sup \{ *W'_a(s', D) : *W'_a \in *\Omega' \} =$
 $= W'_0(s', D)$. As a result

$$(3) \quad \sum_{i=1}^I W'_0(s', D_i) = \infty = W'_0(s', D).$$

Next, we assume that $W'_0(s', D_i) < \infty$ for $i = 1, \dots, I$. Fix $\varepsilon > 0$. For each $i = 1, \dots, I$, there exists $*W'_i$ in $*\Omega'$ such that

$$*W'_i(s', D_i) > W'_0(s', D_i) - \frac{\varepsilon}{I}.$$

Let $*W'_a$ be one of the $*W'_i$'s which is preceded by the others in the linear ordering $<^*$. Recall that $*W'_a$ is under additive and that $*W'_i <^* *W'_a$ implies $*W'_i \leq *W'_a$ ($i = 1, \dots, I$); we obtain

$$\begin{aligned} \sum_{i=1}^I W'_0(s', D_i) &< \sum_{i=1}^I (*W'_i(s', D_i) + \frac{\varepsilon}{I}) \\ &< \sum_{i=1}^I (*W'_a(s', D_i) + \frac{\varepsilon}{I}) \\ &< *W'_a(s', D) + \varepsilon \\ &< W'_0(s', D) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary it follows that

$$(4) \quad \sum_{i=1}^I W'_0(s', D_i) \leq W'_0(s', D).$$

It follows from (3) and (4) that W'_0 is under additive.

iii) We assert that W'_0 is inner continuous. Thus

suppose D in $\underline{D}$ is the union of a sequence of sets D_i , $i \geq 1$, in $\underline{D}$ for which $D_i \subseteq D_{i+1}$, $i \geq 1$. Fix s' in $\underline{S}'$. W'_0 is under additive and so the sequence $\{W'_0(s', D_i)\}_{i=1}^{\infty}$ is convergent and $\lim_i W'_0(s', D_i) \leq W'_0(s', D)$. Suppose t is a real number for which $t < W'_0(s', D)$. There exists $*W'_a$ in $*\Omega'$ for which $t < *W'_a(s', D)$. $*W'_a$ is inner continuous and so there is a positive integer I such that $t < *W'_a(s', D_I)$. As a result $t < W'_0(s', D_I)$. The sequence $\{W'_0(s', D_i)\}_{i=1}^{\infty}$ is non-decreasing; we have therefore $t < W'_0(s', D_I) \leq \lim_i W'_0(s', D_i)$. It now follows at once that we must have $\lim_i W'_0(s', D_i) = W'_0(s', D)$.

iv) Fix D in $\underline{D}$. By definition

$$W'_0(s', D) = \sup \{ *W'_a(s', D) : *W'_a \in *\Omega' \}, s' \in \underline{S}'.$$

Because each of the functions $*W'_a(s', D)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable and because $*\Omega'$ is countable, we know that the function $W'_0(s', D)$, $s' \in \underline{S}'$, must be $\underline{M}'$ -measurable.

Now we wish to show that the weight function W'_0 satisfies conditions 1), 2), and 3) of the conclusion. That it satisfies 1) follows at once from (2). In order to verify 2) we fix W'_c in Ω' and D in $\underline{D}$. Then surely

$$W'_c D = *W'_c D = \int_{\underline{S}'} *W'_c(s', D) d\mu' \leq \int_{\underline{S}'} W'_0(s', D) d\mu' = W'_0 D.$$

Hence 2) holds.

Condition 3) remains to be checked. We assume that $\sup \{W_{\underline{a}} S : W'_{\underline{a}} \in \Omega'\} < \infty$. Set $t = \sup \{W_{\underline{a}} S : W'_{\underline{a}} \in \Omega'\}$. Surely $t = \sup \{^*W_{\underline{a}} S : ^*W'_{\underline{a}} \in ^*\Omega'\}$. Select $^*W'_1$ in $^*\Omega'$ so that $^*W'_1 S > t - 1$. Suppose that functions $^*W'_1, \dots, ^*W'_m$ in $^*\Omega'$ have been selected so that

$$^*W'_1 \prec^* ^*W'_2 \prec^* \dots \prec^* ^*W'_m$$

and

$$^*W'_{i-1} S > t - \frac{1}{i}, \quad i = 1, \dots, m.$$

Evidently there must exist $^*W'_{m+1}$ in $^*\Omega'$ such that

$$^*W'_m \prec^* ^*W'_{m+1}$$

and

$$^*W'_{m+1} S > t - \frac{1}{m+1}.$$

By induction there exists a sequence $\{^*W'_m\}_{m=1}^{\infty}$ of weight functions in $^*\Omega'$ such that

$$(5) \quad ^*W'_m \prec^* ^*W'_{m+1}, \quad m \geq 1,$$

and

$$(6) \quad ^*W'_m S > t - \frac{1}{m}, \quad m \geq 1.$$

Combining (5) and an earlier observation yields

$$(7) \quad ^*W'_m \preceq ^*W'_{m+1}, \quad m \geq 1.$$

The sequence $\{^*W'_m(s', D)\}_{m=1}^{\infty}$ is non-decreasing and hence is convergent for every (s', D) in $\underline{S}' \times \underline{D}$. Define a non-negative function $W'_{\#}$ on $\underline{S}' \times \underline{D}$ by

$$W_{\#}'(s', D) = \lim_m {}^*W_m'(s', D), \quad (s', D) \in S' \times \underline{D}.$$

By Lemma II.2.2 $W_{\#}'$ is a weight function for T ; moreover, since $\lim_m {}^*W_m \underline{S} = t < \infty$ and since T is AC^*W_m for every $m \geq 1$, it follows that T is $ACW_{\#}'$. Because of their respective definitions, it is evident that

$$(8) \quad W_{\#}' \leq W_0'.$$

In order to strengthen (8) we shall compare $W_{\#}'$ with the functions ${}^*W'$ in ${}^*\Omega'$. Thus let an arbitrary function ${}^*W'$ in ${}^*\Omega'$ be given. Two possibilities arise, because $<^*$ linearly orders ${}^*\Omega'$.

Case 1. Assume that there is an integer $m \geq 1$ such that ${}^*W' <^* {}^*W_m'$. Hence ${}^*W' \leq {}^*W_m'$ and so, using (7), ${}^*W' \leq {}^*W_m' \leq W_{\#}'$.

Hence

$$(9) \quad {}^*W' \leq W_{\#}'.$$

Case 2. Assume that ${}^*W_m' <^* {}^*W'$ for every $m \geq 1$. In this event, we have ${}^*W_m' \leq {}^*W'$, $m \geq 1$; clearly, $W_{\#}'(s', \underline{S}) \leq {}^*W'(s', \underline{S})$, $s' \in \underline{S}'$. But also, by (6), $W_{\#} \underline{S} \geq t \geq {}^*W \underline{S}$. It now follows at once from these remarks that

$$(10) \quad W_{\#}'(s', \underline{S}) = {}^*W'(s', \underline{S}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

According to (9) and (10), we have

$$(11) \quad {}^*W'(s', \underline{S}) \leq W_{\#}'(s', \underline{S}) \quad \text{a.e. } \mu' \text{ on } \underline{S}', \quad {}^*W' \in {}^*\Omega'.$$

Inasmuch as ${}^*\Omega'$ is countable, (11) implies

$$(12) \quad W_0'(s', \underline{S}) \leq W_{\#}'(s', \underline{S}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Using (8) and (12), we obtain

$$(13) \quad W'_0(s', \underline{S}) = W'_{\#}(s', \underline{S}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Now let $f_{\#}$ be a g.l.b.f. $W'_{\#}$. By (13)

$$(14) \quad W'_0 \underline{S} = \int_{\underline{S}} f_{\#}(s) \, d\mu.$$

But, again by (8),

$$(15) \quad \int_D f_{\#}(s) \, d\mu = W'_{\#} D \leq W'_0 D, \quad D \in \underline{D}.$$

From (14) and (15) it follows that T is ACW'_0 .

II.2.7. Theorem. Assume HC3. Let W'_0 and W'_a be weight functions for T . Suppose that W'_a is an AC part of W'_0 . Suppose that T is BVW'_0 . Then there exists a maximal AC part W'_A of W'_0 such that $W'_a \leq W'_A$.

Proof. Define U' to be the family of all weight functions W' for T for which

- a) $W'_a \leq W' \leq W'_0$;
- b) T is ACW .

Let $<$ denote the standard partial ordering of the family U' . Thus $(U', <)$ is a partially ordered set. Observe that the function W'_a is in U' . Hence $U' \neq \emptyset$.

Now we suppose that $\mathcal{G}'$ is a linearly ordered subfamily of U' . If we can show that $\mathcal{G}'$ has an upper bound in U' we may then apply Zorn's Lemma to $(U', <)$. Thus we seek to show that $\mathcal{G}'$ does have an upper bound in U' . The sets of the countable family

$\underline{D}_{\#}$ (guaranteed by HC3) can be listed: $D_{\#1}, D_{\#2}, \dots, D_{\#n}, \dots$

For every positive integer n and every W' in $\mathcal{G}'$ we have

$WD_{\#n} \leq W_{O\#n}^D \leq W_{O\underline{S}} < \infty$. Set $p_n = \sup \{WD_{\#n} : W' \in \mathcal{G}'\}$, $n \geq 1$.

As a result

$$p_n \leq W_{O\#n}^D < \infty, \quad n \geq 1.$$

Next, for every positive integer n , there is by induction a

sequence $\{W'_{nm}\}_{m=1}^{\infty}$ of functions in $\mathcal{G}'$ such that $W'_{nm} < W'_{n\ m+1}$,

$m \geq 1$, and such that $W_{nm}^D > p_n - \frac{1}{m}$. The subfamily Ω' of $\mathcal{G}'$

which consists of all of the W'_{nm} 's is countable. Ω' is linearly

ordered by $<$. Observe also that

$$\sup \{W_{\underline{S}} : W' \in \Omega'\} \leq W_{O\underline{S}} < \infty.$$

Hence Lemma II.2.6 applies. There exists a weight function $W'_{\#}$

for T for which

- i) $W'_{\#}(s', D) \leq \sup \{W'(s', D) : W' \in \Omega'\}$, for every
 (s', D) in $\underline{S}' \times \underline{D}$;
- ii) $W \leq W_{\#}$, $W' \in \Omega'$;
- iii) T is $ACW_{\#}$.

Since $\Omega' \subseteq \mathcal{U}'$ it follows from i) that $W'_{\#} \leq W'_0$. From ii) it

follows that $W_a \leq W_{\#}$. By Lemma II.2.3 $W'_a(s', D) \leq W'_{\#}(s', D)$ a.e.

μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. There exists a set X' in $\underline{M}'$ such that

$$(1) \quad \mu'X' = 0$$

and

$$(2) \quad W'_a(s', D) \leq W'_{\#}(s', D), \quad s' \in C'X', D \in \underline{D}.$$

At this point we define a non-negative function $W'_\square$ on $\underline{S}' \times \underline{D}$ by

$$W'_\square(s', D) = \begin{cases} W'_\#(s', D), & s' \in C'X', D \in \underline{D} \\ W'_a(s', D), & s' \in X', D \in \underline{D} \end{cases}.$$

By Lemma I.7.1 $W'_\square$ is a weight function for T . Since $W'_\# \leq W'_0$ and $W'_a \leq W'_0$ it is evident that $W'_\square \leq W'_0$. And, using (2), we see that $W'_a \leq W'_\square$. It follows from (1) and Lemma I.7.5 that $W_\square = W_\#$ and that T is $ACW_\square$. Consequently, $W'_\square$ is in U' . From ii), we have

$$(3) \quad W \leq W_\# = W_\square, \quad W' \in \Omega'.$$

It now follows that $W_{\square\#n}^D \geq W_{nm\#n}^D$ for every pair of positive integers m and n . Hence

$$(4) \quad W_{\square\#n}^D \geq p_n, \quad n \geq 1.$$

Now suppose that W' is in $\mathcal{Q}'$. Then, if we use (4) together with the definition of p_n , we obtain $W_{\square\#n}^D \leq W_{\square\#n}^D$, $n \geq 1$. If we use HC3 and the inner continuity of weights in general we obtain

$$(5) \quad W \leq W_\square.$$

We therefore proved that $W'_\square$ is in U' and, in view of (5), that $W' < W'_\square$ whenever W' is in $\mathcal{Q}'$. In other words the linearly ordered subfamily $\mathcal{Q}'$ of $(U', <)$ does admit an upper bound in U' . By Zorn's Lemma there is an element W'_A in U' which is maximal with respect to $<$; thus, if W' in U' is comparable with W'_A ,

W' must precede W'_A in the ordering $<$. Now, since W'_A is in U' , it follows that $W'_a \leq W'_A \leq W'_0$ and that T is ACW'_A . Hence W'_A is an AC part of W'_0 and if we can show that it is a maximal AC part of W'_0 the proof will be complete.

Thus suppose that $*W'$ is an AC part of W'_0 for which $W'_A \leq *W'$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. There must be a set M' in $\underline{M'}$ such that

$$(6) \quad \mu' M' = 0$$

and

$$(7) \quad W'_A(s', D) \leq *W'(s', D) \leq W'_0(s', D), \quad s' \in C'M', D \in \underline{D}.$$

Define a non-negative function $*W'$ on $\underline{S'} \times \underline{D}$ by

$$*W'(s', D) = \begin{cases} W'_A(s', D), & s' \in M', D \in \underline{D} \\ *W'(s', D), & s' \in C'M', D \in \underline{D} \end{cases}.$$

By Lemma I.7.1, $*W'$ is a weight function for T . From (7) and from the fact that $W'_A \leq W'_0$, we deduce that $W'_A \leq *W' \leq W'_0$. By

(6) and Lemma I.7.5, we know that

$$(8) \quad *W = *W$$

and that T is $AC*W$. Therefore $*W'$ is in U' . But $W'_A \leq *W'$ and

so $W'_A < *W'$. Using the maximality of W'_A , we obtain $*W' < W'_A$.

Therefore $*W = *W \leq W_A$ (by (8)). By Lemma II.2.3, $*W' \leq W'_A$

a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. In view of the

hypothesis concerning $*W'$, it follows now that $W'_A = *W'$ a.e. μ'

on $\underline{S}'$ uniformly with respect to $\underline{D}$. Thus W'_A is a maximal AC part of W'_0 .

II.2.8. The General Decomposition Theorem. Assume HC4. Let W'_0 be a weight function for T and suppose that T is BVW_0 . There exists a Lebesgue-type decomposition for W'_0 .

Proof. The function ${}_0W'$ on $\underline{S}' \times \underline{D}$ which is defined by ${}_0W'(s', D) = 0, (s', D) \in \underline{S}' \times \underline{D}$, is a weight function for T . Surely ${}_0W' \leq W'_0$; and, T is AC_0W . By Theorem II.2.7 there exists a maximal AC part W'_a of W'_0 . By Theorem II.1.14 there exist weight functions W'_A and W'_S for T such that

$$(a) \quad W'_0 = W'_A + W'_S;$$

$$(b) \quad W'_a = W'_A \quad \text{a.e. } \mu' \text{ on } \underline{S}' \text{ uniformly with respect to } \underline{D};$$

$$(c) \quad T \text{ is } ACW_A.$$

Evidently $W'_A \leq W'_0$. By Lemma I.7.6, W'_A is a maximal AC part of W'_0 .

We are now able to assert that (W'_A, W'_S) is a Lebesgue-type decomposition for W'_0 . Indeed this follows from (a), (c), the fact that W'_A is a maximal AC part of W'_0 , and Lemma I.7.7.

II.3 A sufficient condition for absolute continuity

II.3.1. Remarks. Theorem 7.24 of [2] gives a necessary and sufficient condition for T to be ACW ; the condition is expressed partly in terms of T . The result requires however that

H13 be satisfied. A similar result for which H13 is not required would be of great interest. The following theorem represents a preliminary step toward such a result.

We continue to assume H1 - H8 throughout.

II.3.2. Theorem. Assume H7*, H9, and HCl. Suppose that T is BVW and that W is over additive. Define $S'_+ = \{s' \in \underline{S}' : W'(s', \underline{S}) > 0\}$. Suppose also that whenever B is a set in $\underline{B}$ such that $B \subseteq T^{-1}S'_+$ and $\mu B = 0$ it is true that $\mu'TB = 0$. Then T is ACW.

Proof. According to 5.11 of [2], it suffices to prove that W is AC with respect to μ .

Note that, since T is BVW, there exists for every $t > 0$ a positive real number $\delta(t)$ such that if M' is any set in $\underline{M}'$ with $\mu'M' < \delta(t)$ then $\int_{M'} W'(s', \underline{S}) d\mu' < t$. Also, according to H7*, whenever M' is any element of $\underline{M}'$ and ζ' is any positive real number there exists a set O' in $\mathbb{O}'$ such that $M' \subseteq O'$ and $\mu'O' \leq \mu'M' + \zeta'$.

With these general remarks disposed of, we turn to the proof of the assertion that W is AC with respect to μ . Thus suppose M is any set in $\underline{M}$ with $\mu M = 0$. Let $\epsilon > 0$ be given. Using Lemma I.7.8 we obtain a set B in $\underline{B}$ such that $M \subseteq B$ and $\mu B = 0$. We know by HCl that TB is in $\underline{M}'$. The function $W'(s', \underline{S})$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable and so S'_+ is in $\underline{M}'$. By H6 there must be a set B'_+ in $\underline{B}'$ such that $B'_+ \subseteq S'_+$ and $\mu'(S'_+ \cap C'B'_+) = 0$.

According to HCl, $T^{-1}B'_+$ is in $\underline{B}$ and so $B \cap T^{-1}B'_+$ is in $\underline{B}$. And,
 $\mu(B \cap T^{-1}B'_+) = 0$.

Now we examine the set $T(B \cap T^{-1}B'_+)$. Evidently $T(B \cap T^{-1}B'_+) =$
 $= TB \cap B'_+$. Since $B \cap T^{-1}B'_+ \subseteq T^{-1}S'_+$, it follows from the hypo-
thesis that $\mu'(TB \cap B'_+) = 0$. Note that $TB \cap S'_+ \subseteq$
 $\subseteq (TB \cap B'_+) \cup (S'_+ \cap C'B'_+)$ so that

$$\mu'(TB \cap S'_+) \leq \mu'(TB \cap B'_+) + \mu'(S'_+ \cap C'B'_+).$$

Hence

$$(1) \quad \mu'(TB \cap S'_+) = 0.$$

By H7* there is a set O'_0 in $\mathbb{O}'$ such that $TB \cap S'_+ \subseteq O'_0$ and

$\mu'O'_0 < \delta(\frac{\epsilon}{2})$. There exists a countable number of pairwise dis-

joint sets D_i , $i \geq 1$, for which $T^{-1}O'_0 = \bigcup D_i$. Note that

$$\bigcup D_i = T^{-1}O'_0 \supseteq T^{-1}TB \cap T^{-1}S'_+ \supseteq B \cap T^{-1}S'_+ \text{ or, in brief,}$$

$$(2) \quad \bigcup D_i \supseteq B \cap T^{-1}S'_+.$$

Also $TD_i \subseteq O'_0$ for $i \geq 1$ and so

$$\begin{aligned} \sum W_{D_i} &= \sum \int_{TD_i} W'(s', D_i) d\mu' \\ &= \sum \int_{O'_0} W'(s', D_i) d\mu' \\ &= \int_{O'_0} \sum W'(s', D_i) d\mu' \leq \end{aligned}$$

$$\leq \int_{O'_0} W'(s', \underline{S}) d\mu'$$

or

$$(3) \quad \sum W D_i < \frac{\varepsilon}{2}.$$

Next we consider the set $B \cap CT^{-1}S'_+ = B \cap T^{-1}C'S'_+$. Note that $T(B \cap T^{-1}C'S'_+) = TB \cap C'S'_+$. The set $\underline{S}'$ can be written as the union of a countable number of sets B'_i , $i \geq 1$, in $\underline{B}'$ each of finite μ' measure. Define, for $i \geq 1$, $M'_i = B'_i \cap TB \cap C'S'_+$. Fix $i \geq 1$. There exists a set O'_i in $\mathcal{O}'$ such that $M'_i \subseteq O'_i$ and

$$\mu'(O'_i \cap C'M'_i) < \delta\left(\frac{\varepsilon}{2^{i+1}}\right).$$

There is a countable number of pairwise disjoint sets D_{ij} , $j \geq 1$, in $\underline{D}$ such that $\bigcup_j D_{ij} = T^{-1}O'_i$. Note that $\bigcup_{i,j} D_{ij} = \bigcup_i T^{-1}O'_i \supseteq \bigcup_i T^{-1}M'_i = T^{-1}TB \cap T^{-1}C'S'_+$, so that we have

$$(4) \quad \bigcup_{i,j} D_{ij} \supseteq B \cap CT^{-1}S'_+.$$

The positive integer i is again fixed. We have

$$O'_i \cap C'M'_i = (O'_i \cap C'B'_i) \cup (O'_i \cap C'TB) \cup (O'_i \cap S'_+)$$

so that $O'_i \cap C'M'_i \supseteq O'_i \cap S'_+$. Hence $\mu'(O'_i \cap S'_+) < \delta\left(\frac{\varepsilon}{2^{i+1}}\right)$. We

have $TD_{ij} \subseteq O'_i$ for $i \geq 1$ and $j \geq 1$. Hence

$$\sum_{i,j} W D_{ij} = \sum_i \sum_j \int_{TD_{ij}} W'(s', D_{ij}) d\mu' =$$

$$= \sum_i \sum_j \int_{O_i} W'(s', D_{ij}) d\mu'$$

$$\leq \sum_i \int_{O_i} W'(s', \underline{S}) d\mu'$$

$$\leq \sum_i \int_{O_i \cap S_+} W'(s', \underline{S}) d\mu'$$

$$< \sum_{i \geq 1} \frac{\varepsilon}{2^{i+1}}.$$

Thus we have

$$(5) \quad \sum_{i,j} W D_{ij} < \frac{\varepsilon}{2}.$$

In view of (2) and (4), we have $M \subseteq B = (B \cap T^{-1}S_+) \cup$

$$\cup (B \cap CT^{-1}S_+) \subseteq U_i D_i \cup U_{i,j} D_{ij} \quad \text{or}$$

$$(6) \quad M \subseteq U_i D_i \cup U_{i,j} D_{ij}.$$

And, by (3) and (5),

$$(7) \quad \sum_i W D_i + \sum_{i,j} W D_{ij} < \varepsilon.$$

It follows from (6) and (7) that W is AC with respect to μ .

Hence T is ACW.

II.4 On the difference of two almost over additive weight functions

We shall not examine the difference of two arbitrary almost over additive weight functions. Such a difference may not be

non-negative and so it could hardly be a weight function. We shall however examine the same special case as that considered in Section II.1. We shall suppose that W' and W'_A are weight functions for T and that W'_A is a finite valued AC part of W' . It will then be proved that, under certain conditions, $W' - W'_A$ is almost over additive if W' is almost over additive. We shall continue to assume H1 - H8.

II.4.1. Lemma. Assume $H4^*$ and H9. Suppose M is in $\underline{M}$ with $\mu M < \infty$ and suppose that D_0 is a set in $\underline{D}$ with $M \subseteq D_0$. Then there is a double sequence $\{D_{ki}\}_{k=1}^{\infty} \{i=1}^{\infty}$ of sets in $\underline{D}$ such that

$$(i) \quad M \subseteq \bigcup_{i \geq 1} D_{ki} \subseteq \bigcup_{i \geq 1} D_{k-1 i},$$

$$(ii) \quad \sum_{i \geq 1} W'(s', D_{ki}) \leq \sum_{i \geq 1} W'(s', D_{k-1 i}), \quad s' \in \underline{S},$$

for every $k > 1$ and such that

$$(iii) \quad \text{the sets } D_{ki} \ (i \geq 1) \text{ are pairwise disjoint,}$$

$$(iv) \quad \mu(\bigcup_{i \geq 1} D_{ki} \cap CM) < \frac{1}{k},$$

$$(v) \quad \bigcup_{i \geq 1} D_{ki} \subseteq D_0.$$

Proof. By $H4^*$ there is, for every positive integer k , a countable number of pairwise disjoint sets D_{ki}^* , $i \geq 1$, in $\underline{D}$ such that $M \subseteq \bigcup_{i \geq 1} D_{ki}^*$ and $\sum_{i \geq 1} \mu D_{ki}^* < \mu M + \frac{1}{k}$. For every positive integer i , there is a countable number of pairwise disjoint sets D_{li}^* , $m \geq 1$, such that $D_{li}^* \cap D_0 = \bigcup_{m \geq 1} D_{li}^*$. The sets D_{li}^* ,

$i \geq 1, m \geq 1$, are pairwise disjoint subsets of D_0 in $\underline{D}$ for which

$$M \subseteq D_0 \cap \bigcup_{i \geq 1} D_{1i}^{**} = \bigcup_{i,m} D_{im}^*$$

and

$$\sum_{i,m} \mu D_{im}^* \leq \sum_{i \geq 1} \mu D_{1i}^{**} < \mu M + \frac{1}{1}.$$

Hence there exist a countable number of sets D_{1i} , $i \geq 1$, in $\underline{D}$

for which (iii), (iv), and (v) hold when $1 \leq k \leq 1$ and

$$M \subseteq \bigcup_i D_{1i}.$$

Assume now that for all positive integers k which do not exceed p sets D_{ki} , $i \geq 1$, in $\underline{D}$ have been chosen so that (i) and

(ii) hold for $2 \leq k \leq p$ and (iii), (iv), and (v) hold for

$1 \leq k \leq p$. For every pair of positive integers i and m there is

a countable number of pairwise disjoint sets $D_{imu}^{(p)}$, $u \geq 1$, in $\underline{D}$

such that

$$D_{p+1 i}^{**} \cap D_{pm} = \bigcup_{u \geq 1} D_{imu}^{(p)}.$$

Note that the sets $D_{imu}^{(p)}$, $i \geq 1, m \geq 1, u \geq 1$, are pairwise dis-

joint sets in $\underline{D}$. Now also

$$\bigcup_{i,m,u} D_{imu}^{(p)} = \bigcup_{i \geq 1} D_{p+1 i}^{**} \cap \bigcup_{m \geq 1} D_{pm}.$$

Hence we have

$$M \subseteq \bigcup_{i,m,u} D_{imu}^{(p)} \subseteq \bigcup_{m \geq 1} D_{pm}$$

and

$$\mu(U_{i,m,u} D_{imu}^{(p)} \cap CM) \leq \mu(U_{i,p+1} D_{p+1}^{**} \cap CM) < \frac{1}{p+1}.$$

In addition

$$D_{imu}^{(p)} \subseteq D_{pm} \subseteq D_0, \quad i \geq 1, m \geq 1, u \geq 1.$$

Inasmuch as W' is under additive and since, for each $m \geq 1$, the

sets $D_{imu}^{(p)}$, $i \geq 1, u \geq 1$, are pairwise disjoint subsets of D_{pm}

it follows that

$$\sum_{i,m,u} W'(s', D_{imu}^{(p)}) \leq \sum_{m \geq 1} W'(s', D_{pm}), \quad s' \in \underline{S'}.$$

We list the sets $D_{imu}^{(p)}$, $i \geq 1, m \geq 1, u \geq 1$, as:

$$D_{p+1\ 1}, D_{p+1\ 2}, \dots, D_{p+1\ i}, \dots$$

It is evident from these remarks and from the induction hypothesis that the sets D_{ki} , $1 \leq k \leq p+1, i \geq 1$, satisfy (i) and (ii) for $2 \leq k \leq p+1$ and (iii), (iv), and (v) for $1 \leq k \leq p+1$.

The proof is completed by an inductive argument.

II.4.2. Lemma. Assume $H4^*$ and $H9$. Assume that T is ACW and let f be a g.l.b.f. W . Suppose M is a set in $\underline{M}$ with $\mu M < \infty$.

Let $\{D_{ki}\}_{k=1}^{\infty} \quad i=1}^{\infty}$ be a double sequence of sets in $\underline{D}$ for which

conditions (i) - (iv) of Lemma II.4.1 hold. Then the sequence

$\{\sum_{i \geq 1} W'(s', D_{ki})\}_{k=1}^{\infty}$ is convergent for every s' in $\underline{S'}$ and if we

define a non-negative function g' on $\underline{S'}$ by $g'(s') =$

$= \lim_k \sum_{i \geq 1} W'(s', D_{ki}), s' \in \underline{S}',$ then $g'(s'), s' \in \underline{S}',$ is $\underline{M}'$ -

measurable. Moreover

$$\int_{\underline{S}'} g'(s') d\mu' = \int_M f(s) d\mu.$$

Proof. The convergence of each sequence $\{\sum_{i \geq 1} W'(s', D_{ki})\}_{k=1}^{\infty}$

follows from (ii). The function $g'(s'), s' \in \underline{S}',$ is $\underline{M}'$ -measurable because it is the limit of a sequence of $\underline{M}'$ -measurable functions. The functions in the non-increasing sequence

$\{\sum_{i \geq 1} W'(s', D_{ki}), s' \in \underline{S}'\}_{k=1}^{\infty}$ are bounded by the μ' -integrable function $W'(s', \underline{S}), s' \in \underline{S}'.$ Hence it follows from the Lebesgue Theorem that

$$(1) \quad \int_{\underline{S}'} g'(s') d\mu' = \lim_k \int_{\underline{S}'} \sum_{i \geq 1} W'(s', D_{ki}) d\mu'.$$

Now f is a g.l.b.f. W so that we have

$$\begin{aligned} \int_{\underline{S}'} \sum_{i \geq 1} W'(s', D_{ki}) d\mu' &= \sum_{i \geq 1} \int_{\underline{S}'} W'(s', D_{ki}) d\mu' = \\ &= \sum_{i \geq 1} \int_{D_{ki}} f(s) d\mu = \int_{\bigcup_i D_{ki}} f(s) d\mu \end{aligned}$$

for every positive integer $k.$ Using (i) we obtain

$$(2) \quad \int_{\underline{S}'} \sum_{i \geq 1} W'(s', D_{ki}) d\mu' = \int_M f(s) d\mu + \int_{\bigcup_i D_{ki} \cap CM} f(s) d\mu.$$

Set $N_k = \bigcup_i D_{ki} \cap CM$ and $N = \bigcap_k \bigcup_i D_{ki} \cap CM.$ In view of (i) the

sequence $\{\int_{N_k} f(s) d\mu\}_{k=1}^{\infty}$ is non-increasing and hence convergent; indeed, since the integrals $\int_{N_k} f(s) d\mu$, $k \geq 1$, are finite, we have $\lim_k \int_{N_k} f(s) d\mu = \int_N f(s) d\mu$. But, by (iv) $\mu N = 0$. Hence

$$(3) \quad \lim_k \int_{\bigcup_{i=1}^{\infty} D_{ki} \cap CM} f(s) d\mu = 0.$$

Recall that (2) is valid for every $k \geq 1$. As a result

$$(4) \quad \begin{aligned} \lim_k \int_{\underline{S}'} \sum_{i \geq 1} W'(s', D_{ki}) d\mu' &= \\ &= \int_M f(s) d\mu + \lim_k \int_{\bigcup_{i=1}^{\infty} D_{ki} \cap CM} f(s) d\mu. \end{aligned}$$

It follows from (1), (4), and (3) that

$$\int_{\underline{S}'} g'(s') d\mu' = \int_M f(s) d\mu.$$

II.4.3. Lemma. Assume $H4^*$. Suppose that W' and W'_A are weight functions for T . Assume that W'_A is a finite valued AC part of W' . Assume also that W' is almost over additive. Define a non-negative function W'_Δ on $\underline{S}' \times \underline{D}$ by

$$W'_\Delta(s', D) = W'(s', D) - W'_A(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Let $\{D_k\}_{k=0}^n$ be a finite sequence of sets in $\underline{D}$ for which

$D_0 \subseteq \bigcup_{k=1}^n D_k$. Assume also that μD_0 is finite. Then

$$W'_\Delta(s', D_0) \leq \sum_{k=1}^n W'_\Delta(s', D_k) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Proof. By Lemma II.1.9 the function W'_Δ is almost under additive. If $n = 1$ the conclusion follows from the almost under additivity of W'_Δ . Assume then that $n \geq 2$.

Define $E_1 = D_0 \cap D_1$ and, for $k = 2, \dots, n$, define

$$E_k = D_0 \cap D_k \cap \bigcap_{i=1}^{k-1} CD_i. \text{ Each } E_k \text{ is in } \underline{M} \text{ and } \mu E_k \leq \mu D_0 < \infty.$$

The sets E_k , $k = 1, \dots, n$, are pairwise disjoint and

$$(1) \quad D_0 = \bigcup_{k=1}^n E_k.$$

Observe also that $E_k \subseteq D_k$, $1 \leq k \leq n$. By Lemma II.4.1 there exists, for $k = 1, \dots, n$, a double sequence of sets D_{kpi} ,

$p \geq 1$, $i \geq 1$, in $\underline{D}$ such that

$$(i) \quad E_k \subseteq \bigcup_{i \geq 1} D_{kpi} \subseteq \bigcup_{i \geq 1} D_{k \ p-1 \ i},$$

$$(ii) \quad \sum_{i \geq 1} W'_A(s', D_{kpi}) \leq \sum_{i \geq 1} W'_A(s', D_{k \ p-1 \ i}), \quad s' \in \underline{S'},$$

for every $p > 1$ and such that it is also true that

$$(iii) \quad \text{the sets } D_{kpi} \ (i \geq 1) \text{ are pairwise disjoint,}$$

$$(iv) \quad \mu(\bigcup_{i \geq 1} D_{kpi} \cap CE_k) < \frac{1}{p},$$

$$(v) \quad D_{kpi} \subseteq D_k, \quad i \geq 1,$$

for every $p \geq 1$. Let f_A be a g.l.b.f. W_A . By Lemma II.4.2 the

sequence $\{\sum_{i \geq 1} W'_A(s', D_{kpi})\}_{p=1}^\infty$ is convergent for every s' in $\underline{S'}$

and if

$$g_k'(s') = \lim_p \sum_{i \geq 1} W_A'(s', D_{kpi}), \quad s' \in \underline{S}',$$

then $g_k'(s')$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable; and,

$$(2) \quad \int_{\underline{S}'} g_k'(s') d\mu' = \int_{E_k} f_A(s) d\mu, \quad 1 \leq k \leq n.$$

At this point we fix the positive integer p . According to (iii) and (v), the sets D_{kpi} , $i \geq 1$, are pairwise disjoint subsets of D_k for each k . W_A' is almost under additive; hence

$$(3) \quad \sum_{i \geq 1} W_A'(s', D_{kpi}) \leq W_A'(s', D_k) \quad \text{a.e. } \mu' \text{ on } \underline{S}', \quad 1 \leq k \leq n.$$

In view of (1) and (i) we have

$$(4) \quad D_0 \subseteq \bigcup_{k=1}^n \bigcup_{i \geq 1} D_{kpi}.$$

The function W' is almost over additive. It follows that

$$W'(s', D_0) \leq \sum_{k=1}^n \sum_{i \geq 1} W'(s', D_{kpi}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Consequently

$$(5) \quad W_A'(s', D_0) \leq \sum_{k=1}^n \sum_{i \geq 1} W'(s', D_{kpi}) - W_A'(s', D_0) \\ \text{a.e. } \mu' \text{ on } \underline{S}'.$$

The function $\sum_{k=1}^n \sum_{i \geq 1} W_A'(s', D_{kpi})$, $s' \in \underline{S}'$, is real valued for

indeed

$$\sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi}) \leq n \cdot W'_A(s', \underline{S}) < \infty.$$

From (5) we obtain

$$\begin{aligned} W'_\Delta(s', D_0) &\leq \sum_{k=1}^n \sum_{i \geq 1} W'(s', D_{kpi}) - \sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi}) + \\ &+ \left\{ \sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi}) - W'_A(s', D_0) \right\} \text{ a.e. } \mu' \text{ on } \underline{S}' \end{aligned}$$

and so

$$\begin{aligned} (6) \quad W'_\Delta(s', D_0) &\leq \sum_{k=1}^n \sum_{i \geq 1} W'_\Delta(s', D_{kpi}) + \\ &+ \left\{ \sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi}) - W'_A(s', D_0) \right\} \text{ a.e. } \mu' \text{ on } \underline{S}'. \end{aligned}$$

Combining (3) and (6) yields

$$\begin{aligned} (7) \quad W'_\Delta(s', D_0) &\leq \sum_{k=1}^n W'_\Delta(s', D_k) + \\ &+ \left\{ \sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi}) - W'_A(s', D_0) \right\} \text{ a.e. } \mu' \text{ on } \underline{S}'. \end{aligned}$$

Expression (7) is valid for every positive integer p .

It follows now from (7) that

$$\begin{aligned} (8) \quad W'_\Delta(s', D_0) &\leq \sum_{k=1}^n W'_\Delta(s', D_k) + \sum_{k=1}^n g'_k(s') - W'_A(s', D_0) \\ &\text{a.e. } \mu' \text{ on } \underline{S}'. \end{aligned}$$

By 6.1 of [2] the function W'_A is almost over additive. Thus, in

view of (4), $W'_A(s', D_0) \leq \sum_{k=1}^n \sum_{i \geq 1} W'_A(s', D_{kpi})$ a.e. μ' on $\underline{S}'$ for every positive integer p . Hence, referring again to the definition of $g'_k(s')$, $s' \in \underline{S}'$, we obtain

$$(9) \quad W'_A(s', D_0) \leq \sum_{k=1}^n g'_k(s') \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Using (9), (2), and (1) successively, we note that

$$\begin{aligned} \int_{D_0} f_A(s) \, d\mu &= \int_{\underline{S}'} W'_A(s', D_0) \, d\mu' \leq \int_{\underline{S}'} \sum_{k=1}^n g'_k(s') \, d\mu' = \\ &= \sum_{k=1}^n \int_{\underline{S}'} g'_k(s') \, d\mu' = \sum_{k=1}^n \int_{E_k} f_A(s) \, d\mu = \\ &= \int_{D_0} f_A(s) \, d\mu < \infty. \end{aligned}$$

Hence

$$(10) \quad \int_{\underline{S}'} W'_A(s', D_0) \, d\mu' = \int_{\underline{S}'} \sum_{k=1}^n g'_k(s') \, d\mu' < \infty.$$

In view of (9) and (10) it follows at once that

$$(11) \quad W'_A(s', D_0) = \sum_{k=1}^n g'_k(s') \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Because of (8) and (11) we have finally

$$W'_\Delta(s', D_0) \leq \sum_{k=1}^n W'_\Delta(s', D_k) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

II.4.4. Lemma. Assume $H4^*$ and $HC2$. Suppose W' and W'_A

are weight functions for T . Suppose that W'_A is a finite valued

AC part of W' . Assume moreover that W' is almost over additive. Define a non-negative function W'_Δ on $\underline{S}' \times \underline{D}$ by $W'_\Delta(s', D) = W'(s', D) - W'_\Delta(s', D)$, $(s', D) \in \underline{S}' \times \underline{D}$. Then W'_Δ is almost over additive.

Proof. Let D^j , $j \geq 0$, be a countable family of sets in $\underline{D}$ for which

$$(1) \quad D^0 \subseteq \bigcup_{j \geq 1} D^j.$$

By HC2 there is a sequence of sets D_i , $i \geq 1$, in $\underline{D}$ for which the following properties hold:

- (i) $D_i \subseteq D_{i+1}$, $i \geq 1$;
- (ii) $D^0 = \bigcup_{i \geq 1} D_i$;
- (iii) if $\#D^j$, $j \geq 1$, is any countable family of sets in

$\underline{D}$ for which $D^0 \subseteq \bigcup_{j \geq 1} \#D^j$ then for every positive

integer i there exists a positive integer $J(i)$

such that $D_i \subseteq \bigcup_{j \geq 1}^{J(i)} \#D^j$.

By H4 there exists a countable family of sets $*D_j$, $j \geq 1$, in $\underline{D}$

such that $\underline{S} = \bigcup_{j \geq 1} *D_j$ and μ_*D_j is finite for $j \geq 1$. By (iii)

there is for every positive integer i a positive integer $J^*(i)$

such that $D_i \subseteq \bigcup_{j=1}^{J^*(i)} *D_j$. Thus $\mu D_i \leq \sum_{j=1}^{J^*(i)} \mu_*D_j$, $i \geq 1$. In

other words we have

$$(2) \quad \mu D_i < \infty, \quad i \geq 1.$$

Furthermore, according to (iii) and (1), there exists for every positive integer i a positive integer $j(i)$ such that

$$(3) \quad D_i \subseteq \bigcup_{j=1}^{j(i)} D^j, \quad i \geq 1.$$

Fix a positive integer I . From (2) and (3) we deduce

$$\mu D_I < \infty \text{ and } D_I \subseteq \bigcup_{j=1}^{j(I)} D^j. \text{ We apply Lemma II.4.3 to obtain}$$

$$W'_\Delta(s', D_I) \leq \sum_{j=1}^{j(I)} W'_\Delta(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'. \text{ Hence surely}$$

$$(4) \quad W'_\Delta(s', D_I) \leq \sum_{j \geq 1} W'_\Delta(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Expression (4) is valid for every positive integer I .

It was proved in Lemma II.1.9 that W'_Δ is inner continuous.

Using this fact together with (i), (ii), and (4), we obtain

$$W'_\Delta(s', D^0) \leq \sum_{j \geq 1} W'_\Delta(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'. \text{ Hence } W'_\Delta \text{ is almost over additive.}$$

II.5 The Strong Decomposition Theorem

First we prove a lemma about almost over additive weight functions which are singular. Then we proceed to the decomposition results.

II.5.1. Lemma. Assume H_9 , H_7^* , and H_{C1} . Assume that W' is almost over additive and that T is BVW. Suppose also that W' is singular. Then W' is strongly singular.

Proof. Since the measure space $(\underline{S}', \underline{M}', \mu')$ is σ -finite

there is a countable number of pairwise disjoint sets M'_n , $n \geq 1$, in $\underline{M}'$ for which $\underline{S}' = \bigcup_{n \geq 1} M'_n$ and $\mu' M'_n < \infty$, $n \geq 1$. Define

$$S'_+ = \{s' \in \underline{S}' : W'(s', \underline{S}) > 0\}.$$

Define Γ to be the family of all sets B in $\underline{B}$ for which $\mu B = 0$.

The union of a countable number of sets in Γ is again in Γ .

Also, for every positive integer n , define Γ'_n to be the family of all sets M' in $\underline{M}'$ for which $M' = M'_n \cap TB$ for some B in Γ . Set

$$p_n = \sup \{\mu' M' : M' \in \Gamma'_n\}, \quad n \geq 1. \quad \text{Note that } p_n \leq \mu' M'_n < \infty, \quad n \geq 1.$$

Now fix an arbitrary positive integer n . There exists a sequence of sets B_{nm} , $m \geq 1$, in Γ such that

$$\lim_m \mu'(M'_n \cap TB_{nm}) = p_n.$$

Introduce $B_n = \bigcup_{m \geq 1} B_{nm}$. Since Γ is closed under countable unions

it follows that B_n is in Γ . And, for each m , $M'_n \cap TB_{nm} \subseteq$

$\subseteq M'_n \cap TB_n$ so that $\mu'(M'_n \cap TB_{nm}) \leq \mu'(M'_n \cap TB_n)$. Hence

$$p_n = \lim_m \mu'(M'_n \cap TB_{nm}) \leq \mu'(M'_n \cap TB_n).$$

From the definition of p_n and from the fact that B_n is in Γ it follows that

$$(1) \quad \mu'(M'_n \cap TB_n) = p_n.$$

Next we set $B_0 = \bigcup_{n \geq 1} B_n$. B_0 is in Γ and, given $n \geq 1$,

surely $M'_n \cap TB_n \subseteq M'_n \cap TB_0$ and so $\mu'(M'_n \cap TB_n) \leq \mu'(M'_n \cap TB_0)$.

It follows as before from the definition of p_n that

$$(2) \quad \mu'(M'_n \cap TB_0) = p_n, \quad n \geq 1.$$

The set TB_0 is in $\underline{M}'$. By H6 there exists a set B'_0 in $\underline{B}'$ such that $TB_0 \subseteq B'_0$ and $\mu'(B'_0 \cap C'TB_0) = 0$. Define a non-negative function $W'_\#$ on $\underline{S}' \times \underline{D}$ by

$$W'_\#(s', D) = \begin{cases} W'(s', D), & (s', D) \in C'B'_0 \times \underline{D} \\ 0, & (s', D) \in B'_0 \times \underline{D} \end{cases}.$$

By Lemma I.7.1 $W'_\#$ is a weight function for T . W' is almost over additive and so $W'_\#$ is almost over additive. Hence $W'_\#$ is over additive. Moreover T is $BVW'_\#$. Define $S'_{\#+} = \{s' \in \underline{S}' : W'_\#(s', \underline{S}) > 0\}$. Note that $S'_{\#+} = S'_+ \cap C'B'_0$. Note further that we shall be able to apply Theorem II.3.2 if we can verify the following statement:

- (A) If B is a set in $\underline{B}$ for which $B \subseteq T^{-1}S'_{\#+}$ and $\mu B = 0$ then $\mu'TB = 0$.

In order to verify statement (A) we let B be a set in $\underline{B}$ for which $B \subseteq T^{-1}S'_{\#+}$ and $\mu B = 0$. We assert

- (3) $\mu'TB = 0$.

Fix a positive integer n . Surely B is in Γ and so $M'_n \cap TB$ is in Γ'_n . Note that

$$\begin{aligned} B &\subseteq T^{-1}S'_{\#+} = T^{-1}(S'_+ \cap C'B'_0) \\ &= T^{-1}S'_+ \cap T^{-1}C'B'_0 \\ &= T^{-1}S'_+ \cap CT^{-1}B'_0 \\ &\subseteq T^{-1}S'_+ \cap CT^{-1}TB_0. \end{aligned}$$

In short $B \subseteq CT^{-1}TB_0$. Hence $TB \cap TB_0 = \emptyset$. Consequently

$$(4) \quad (M'_n \cap TB_0) \cap (M'_n \cap TB) = \emptyset.$$

Now B_0 and B are in Γ so that $B_0 \cup B$ is in Γ . As a result,

$M'_n \cap T(B_0 \cup B)$ is in Γ'_n . We have, in view of (2) and (4),

$$\begin{aligned} p_n &= \mu'(M'_n \cap TB_0) \\ &\leq \mu'(M'_n \cap TB_0) + \mu'(M'_n \cap TB) \\ &\leq \mu'(M'_n \cap T(B_0 \cup B)) \\ &\leq p_n. \end{aligned}$$

Since p_n is finite we have $\mu'(M'_n \cap TB) = 0$. The integer n was arbitrary and so (3) must hold. Hence statement (A) is valid. According to Theorem II.3.2 T is $ACW_{\#}$. Since W' is singular and since $W'_{\#} \leq W'$ it follows that $W'_{\#}$ is trivial. Hence $W'_{\#}(s', \underline{S}) = 0$ a.e. μ' on $\underline{S}'$. It now follows from the fact that $\mu'(B'_0 \cap C'TB_0) = 0$ and from the definition of $W'_{\#}$ that $W'(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0$.

We have proved that B_0 is a set in $\underline{B}$ for which $\mu B_0 = 0$ and $W'(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0$. Hence the singular weight function W' is strongly singular.

II.5.2. Theorem. Assume $H9$, $H4^*$, $H7^*$, $HC1$, and $HC2$.

Assume that W' is almost over additive and that T is BVW. Then every Lebesgue-type decomposition for W' is a strong Lebesgue-

type decomposition for W' . Moreover the singular component of such a decomposition must be almost over additive.

Proof. Let (W'_A, W'_S) be a Lebesgue-type decomposition for W' .

Then $W' = W'_A + W'_S$; moreover T is ACW_A and W'_S is singular. Since T is BVW_A there exists a set M' in $\underline{M}'$ such that $\mu'M' = 0$ and

$$(1) \quad W'_A(s', D) < \infty, \quad s' \in C'M', \quad D \in \underline{D}.$$

We define two non-negative functions $\hat{W}'_A$ and $\hat{W}'_S$ on $\underline{S}' \times \underline{D}$ by

$$\hat{W}'_A(s', D) = \begin{cases} W'_A(s', D), & s' \in C'M', \quad D \in \underline{D} \\ 0, & s' \in M', \quad D \in \underline{D} \end{cases};$$

$$\hat{W}'_S(s', D) = \begin{cases} W'_S(s', D), & s' \in C'M', \quad D \in \underline{D} \\ W'(s', D), & s' \in M', \quad D \in \underline{D} \end{cases}.$$

By Lemma I.7.1 both $\hat{W}'_A$ and $\hat{W}'_S$ are weight functions for T .

Moreover

$$(2) \quad W' = \hat{W}'_A + \hat{W}'_S.$$

In view of the fact that $\mu'M' = 0$ it follows from Lemma I.7.5 that T is $AC\hat{W}'_A$ and that $\hat{W}'_S$ is singular. The weight function $\hat{W}'_A$ is finite valued. By Lemma II.4.4 $\hat{W}'_S$ must be almost over additive. Using Lemma I.7.5 again we see that W'_S must be almost over additive. By Lemma II.5.1 W'_S is strongly singular. Hence (W'_A, W'_S) is a strong Lebesgue-type decomposition for W' .

II.5.3. Corollary. (The Strong Decomposition Theorem)

Assume H_9 , H_4^* , H_7^* , and $HC1 - HC4$. Assume that T is BVW and that W' is almost over additive. Then there exists a strong Lebesgue-type decomposition for W' . Its singular component is almost over additive.

Proof. By Theorem II.2.8 there exists a Lebesgue-type decomposition (W'_A, W'_S) for W' . But, by Theorem II.5.2, (W'_A, W'_S) must be a strong Lebesgue-type decomposition for W' and W'_S must be almost over additive.

II.6 Almost additive weight functions

II.6.1. Remarks. Recently there has been an indication of some interest in almost additive weight functions. We now prove several basic results about such weight functions and establish a decomposition theorem for them.

II.6.2. Lemma. Assume that W' is a non-negative function defined on $\underline{S} \times \underline{D}$ which is inner continuous and almost additive. Then W' is almost under additive.

Proof. Let $\{D_i\}_{i=0}^I$ be a finite subset of $\underline{D}$ for which the sets D_i , $1 \leq i \leq I$, are pairwise disjoint and $\bigcup_{i=1}^I D_i \subseteq D_0$. By Lemma II.1.7 there exists, for $i = 1, \dots, I$, a sequence of sets D_{ij} , $j \geq 1$, in $\underline{D}$ with the following properties:

$$i) \quad D_{ij} \subseteq D_{i, j+1}, \quad 1 \leq i \leq I, \quad j \geq 1;$$

- ii) the set $\{D_{ij}\}_{i=1}^I$ is of type γ , $j \geq 1$;
 iii) $\bigcup_{j \geq 1} D_{ij} = D_i$, $1 \leq i \leq I$.

Now fix the positive integer j . The set $\{D_{ij}\}_{i=1}^I$ consists of pairwise disjoint subsets of D_0 , each of type γ . By Lemma II.1.8 there exists a $\underline{D}$ -partition $(\{D_{jk}^\#\}_{k \geq 1}, Y_j, Z_j)$ of D_0 for which $\{D_{ij}\}_{i=1}^I \subseteq \{D_{jk}^\#\}_{k \geq 1}$. We may as well assume that $D_{ij} = D_{ji}^\#$, $1 \leq i \leq I$. By assumption W' is almost additive. Consequently

$$\sum_{k \geq 1} W'(s', D_{jk}^\#) = W'(s', D_0) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Hence

$$\sum_{i=1}^I W'(s', D_{ij}) + \sum_{k \geq I+1} W'(s', D_{jk}^\#) = W'(s', D_0) \\ \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Since W' is non-negative we have

$$(1) \quad \sum_{i=1}^I W'(s', D_{ij}) \leq W'(s', D_0) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Statement (1) holds for every $j \geq 1$. It follows from (1), i), iii), and the inner continuity of W' that

$$\sum_{i=1}^I W'(s', D_i) \leq W'(s', D_0) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Thus W' is almost under additive.

II.6.3. Remark. It follows from Lemma II.6.2 that if W' satisfies conditions i), iii), and iv) for weight functions and if W' is almost additive then it is almost under additive. This fact does not guarantee that W' is a weight function for T . However there is reason to suspect that it may be possible to replace ii) of H9 by the requirement that W' be almost under additive without significantly altering the theory. If such a replacement were possible one could then say that a non-negative, almost additive function defined on $\underline{S}' \times \underline{D}$ is a weight function for T if it satisfies i), iii), and iv) of H9.

This matter remains unsettled. Thus, for the remainder of Section II.6, we shall require that W' be both a weight function for T and an almost additive function. It must be remembered that later work may produce a redundancy in this requirement. (See Remark II.1.10.)

II.6.4. Lemma. Assume that W' is an almost additive weight function for T . Let D_i , $i \geq 0$, be a countable family of sets in $\underline{D}$ such that $D_0 \subseteq \bigcup_{i \geq 1} D_i$. Suppose that D_i is of type γ for $i \geq 1$. Then

$$W'(s', D_0) \leq \sum_{i \geq 1} W'(s', D_i) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Proof. The conclusion follows from Lemma II.6.2 if there are only two sets, D_0 and D_1 , involved. We shall assume then that at least three sets - D_0 , D_1 , D_2 , and perhaps more - are

involved. We define $E_1^* = D_0 \cap D_1$, $E_2^* = D_0 \cap CD_1 \cap D_2$, and, in general,

$$E_k^* = D_0 \cap \bigcap_{i=1}^{k-1} CD_i \cap D_k, \quad k \geq 1.$$

The sets E_k^* , $k \geq 1$, are pairwise disjoint and their union is

D_0 . D_i is of type γ for $i \geq 1$. There exists a $\underline{D}$ -partition

$(\{D_{ij}\}_{j \geq 0}, Y_i, Z_i)$ of $\underline{S}$ for which $D_i \in \{D_{ij}\}_{j \geq 0}$; let us suppose

$D_i = D_{i0}$. It is also possible to require that $Y_i \cup Z_i = \bigcup_{j \geq 0} D_{ij}$.

There exist sets D_{1p} , $p \geq 1$, in $\underline{D}$ which are pairwise disjoint and for which $E_1^* = D_0 \cap D_1 = \bigcup_{p \geq 1} D_{1p}$. Now fix an integer

k with $k \geq 2$. Note that

$$E_k^* = D_0 \cap D_k \cap \bigcap_{i=1}^{k-1} (\bigcup_{j \geq 1} D_{ij} \cup Y_i \cup Z_i).$$

Let J_{k-1} be the family of all sequences of positive integers of length $k-1$. Observe that

$$(1) \quad D_0 \cap D_k \cap \bigcup_{\{j_1 \dots j_{k-1}\} \in J_{k-1}} (\bigcap_{i=1}^{k-1} D_{ij_i}) \subseteq E_k^*.$$

Also

$$(2) \quad E_k^* \subseteq [D_0 \cap D_k \cap \bigcup_{\{j_1 \dots j_{k-1}\} \in J_{k-1}} (\bigcap_{i=1}^{k-1} D_{ij_i})] \cup \bigcup_{i=1}^{k-1} (Y_i \cup Z_i).$$

For each $\{j_1 \dots j_{k-1}\}$ in J_{k-1} there is by Lemma 8.2 of [2] a

countable family of pairwise disjoint sets $D_p^{\{j_1 \dots j_{k-1}\}}$, $p \geq 1$,

in $\underline{D}$ for which $D_0 \cap D_k \cap \bigcap_{i=1}^{k-1} D_{ij_1} = \bigcup_{p \geq 1} D_p^{j_1 \dots j_{k-1}}$. In

view of (1) we have

$$(3) \quad \bigcup_{\{j_1 \dots j_{k-1}\}} \bigcup_{p \geq 1} D_p^{j_1 \dots j_{k-1}} \subseteq E_k^*.$$

(2) becomes

$$(4) \quad E_k^* \subseteq \bigcup_{j_1 \dots j_{k-1}} \bigcup_{p \geq 1} D_p^{j_1 \dots j_{k-1}} \cup \bigcup_{i=1}^{k-1} (Y_i \cup Z_i).$$

The sets in the family $\{D_p^{j_1 \dots j_{k-1}} : p \geq 1, \{j_1 \dots j_{k-1}\} \in J_{k-1}\}$ are evidently pairwise disjoint.

Since the sets E_k^* , $k \geq 1$, are pairwise disjoint it now follows from (3) and from the last sentence of the preceding paragraph that the sets in the family $\underline{D}_0 = \{D_{1p}\}_{p \geq 1} \cup$

$\bigcup_{\{j_1 \dots j_{k-1}\}} \{D_p^{j_1 \dots j_{k-1}} : p \geq 1, \{j_1 \dots j_{k-1}\} \in J_{k-1}\}_{k \geq 2}$ are pairwise disjoint. Moreover we obtain

$$(5) \quad \bigcup_{p \geq 1} D_{1p} \cup \bigcup_{k \geq 2} \left[\bigcup_{\{j_1 \dots j_{k-1}\}} D_p^{j_1 \dots j_{k-1}} \right] \subseteq \bigcup_{k \geq 1} E_k^* = D_0.$$

Using (4) we obtain $D_0 = \bigcup_{k \geq 1} E_k^* \subseteq \bigcup_{p \geq 1} D_{1p} \cup$

$\bigcup_{k \geq 2} \left[\bigcup_{\{j_1 \dots j_{k-1}\}} D_p^{j_1 \dots j_{k-1}} \cup \bigcup_{i=1}^{k-1} (Y_i \cup Z_i) \right]$ so that

$$(6) \quad D_0 \subseteq \bigcup_{p \geq 1} D_{1p} \cup \bigcup_{k \geq 2} \bigcup_{\{j_1 \dots j_{k-1}\}, p}^{D_p} \{j_1 \dots j_{k-1}\} \cup \bigcup_{k \geq 1} (Y_k \cup Z_k).$$

From (6) and (5) it follows that

$$(7) \quad D_0 = \bigcup_{p \geq 1} D_{1p} \cup \bigcup_{k \geq 2} \bigcup_{\{j_1 \dots j_{k-1}\}, p}^{D_p} \{j_1 \dots j_{k-1}\} \cup \bigcup_{k \geq 1} (D_0 \cap Y_k) \cup \bigcup_{k \geq 1} (D_0 \cap Z_k).$$

Note that $\bigcup_{k \geq 1} (D_0 \cap Y_k)$ is in $\underline{M}$ and

$$(8) \quad \mu \bigcup_{k \geq 1} (D_0 \cap Y_k) = 0.$$

Similarly $\bigcup_{k \geq 1} (D_0 \cap Z_k)$ is in $\underline{M}'$ and

$$(9) \quad \mu' \bigcup_{k \geq 1} (D_0 \cap Z_k) = 0.$$

It now follows from (7), (8), (9), and the first sentence of this paragraph that $(\underline{D}_0, \bigcup_{k \geq 1} (D_0 \cap Y_k), \bigcup_{k \geq 1} (D_0 \cap Z_k))$ is a $\underline{D}$ -partition of D_0 . Since W' is almost additive we have

$$(10) \quad W'(s', D_0) = \sum_{p \geq 1} W'(s', D_{1p}) + \sum_{k \geq 2} \sum_{\{j_1 \dots j_{k-1}\}, p} W'(s', D_p^{\{j_1 \dots j_{k-1}\}}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

The sets D_{1p} , $p \geq 1$, are pairwise disjoint subsets of D_1 so that, in view of the under additivity of W' , we have

$$(11) \quad \sum_{p \geq 1} W'(s', D_{1p}) \leq W'(s', D_1), \quad s' \in \underline{S}'.$$

Similarly we obtain

$$(12) \quad \sum_{\{j_1 \dots j_{k-1}\}, p} W'(s', D_p^{\{j_1 \dots j_{k-1}\}}) \leq W'(s', D_k),$$

$$s' \in \underline{S}', \quad k \geq 2.$$

Combining (10), (11), and (12) yields

$$W'(s', D_0) \leq W'(s', D_1) + \sum_{k \geq 2} W'(s', D_k) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

II.6.5. Lemma. Assume HC2. Assume that W' is an almost additive weight function for T . Then W' is almost over additive.

Proof. Let D^j , $j \geq 0$, be a countable family of sets in $\underline{D}$ for which $D^0 \subseteq \bigcup_{j \geq 1} D^j$. By HC2 there exists a sequence of sets

D_i , $i \geq 1$, in $\underline{D}$ with the following properties:

- i) $D_i \subseteq D_{i+1}$, $i \geq 1$;
- ii) $D^0 = \bigcup_{i \geq 1} D_i$;
- iii) if $\# D^j$, $j \geq 1$, is any countable family of sets in

$\underline{D}$ for which $D^0 \subseteq \bigcup_{j \geq 1} \# D^j$ then for every positive

integer i there exists a positive integer $J(i)$

such that $D_i \subseteq \bigcup_{j=1}^{J(i)} \# D^j$.

By H5 there is for $j \geq 1$ a sequence of sets D^{jp} , $p \geq 1$, in $\underline{D}$ with the following properties:

$$a) \quad D^{jp} \subseteq D^{j, p+1}, \quad p \geq 1;$$

$$b) \quad D^j = \bigcup_{p \geq 1} D^{jp};$$

$$c) \quad \text{each } D^{jp} \text{ is of type } \gamma.$$

Observe that $D^0 \subseteq \bigcup_{j \geq 1} D^j = \bigcup_{j, p} D^{jp}$.

At this point we fix the positive integer i . By iii) there exist positive integers $P(i)$ and $J(i)$ such that $D_i \subseteq \bigcup_{j=1}^{J(i)} \bigcup_{p=1}^{P(i)} D^{jp}$.

Using a) we obtain $D_i \subseteq \bigcup_{j=1}^{J(i)} D^{jP(i)}$. Each $D^{jP(i)}$ is of type γ and W' is almost additive; by Lemma II.6.4

$$(1) \quad W'(s', D_i) \leq \sum_{j=1}^{J(i)} W'(s', D^{jP(i)}) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Using b) and the under additivity of W' we deduce from (1)

$$(2) \quad W'(s', D_i) \leq \sum_{j=1}^{J(i)} W'(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

In view of (2), we see that we must surely have

$$(3) \quad W'(s', D_i) \leq \sum_{j \geq 1} W'(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Statement (3) holds for every positive integer i . In view of (3), i), ii), and the inner continuity of W' we have

$$W'(s', D^0) \leq \sum_{j \geq 1} W'(s', D^j) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

Hence W' is almost over additive.

II.6.6. The following decomposition theorem now becomes a corollary of Theorem II.5.2 and Corollary II.5.3.

Theorem. Assume H_9 , H_4^* , H_7^* , HC_1 , and HC_2 . Assume that W' is almost additive and that T is BVW.

1) Then every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' . The singular component of such a decomposition must also be almost additive.

2) Assume also HC_4 . Then there exists a strong Lebesgue-type decomposition for W' . Its singular component is almost additive.

Proof. First we prove 1). By Lemma II.6.5 W' is almost over additive. From Theorem II.5.2 it then follows that every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' . To see that the singular component of such a decomposition must be almost additive, let (W'_A, W'_S) be a Lebesgue-type decomposition for W' . W' is almost additive and, by Lemma 4.1 of [2], W'_A is almost additive. Recall that $W' = W'_A + W'_S$; it follows at once that W'_S is also almost additive.

2) follows at once from Corollary II.5.3, the fact that W' is almost over additive, and from 1).

II.6.7. Remarks. The decomposition theorem just proved for almost additive weight functions is essentially a mere special case of the results of Section II.5. It is of interest only because of the potential importance of almost additive weight functions. In Lemma II.6.5 we proved that an almost additive weight function is almost over additive (if HC2 is satisfied). One might ask whether an almost over additive weight function is necessarily almost additive. This question is answered in Section II.7.

II.7 Examples

II.7.1. Remarks. In this section we present several examples in order to illuminate a few points. The examples presented here are not the result of a systematic search. This section is not intended to be a comprehensive list of known examples. Indeed these examples were discovered rather incidentally. They do however serve as counter-examples for several interesting conjectures. It should be noted that Theorems III.2.3 and III.3.4 are used in this section. None of the results of Chapter III is dependent upon this section. It should also be noted that no a priori assumptions are made in this section.

II.7.2. Example. Let $\underline{S}$ be a non-empty bounded domain in $\mathbb{R}^2$ and let π denote the projection from $\underline{S}$ into the "X-axis"; π is defined by $\pi[(x,y)] = x$, $(x,y) \in \underline{S}$. Set $\underline{S}' = \pi \underline{S}$. Let L_1 denote one-dimensional Lebesgue measure and let L_2 denote

two-dimensional Lebesgue measure. Let $\underline{M}$ be the family of all L_2 -measurable sets which are subsets of $\underline{S}$ and let $\underline{M}'$ be the family of all L_1 -measurable sets which are subsets of $\underline{S}'$. Set $\mu = L_2|_{\underline{M}}$ and $\mu' = L_1|_{\underline{M}'}$. Let $\underline{D}$ be the family of all subsets of $\underline{S}$ which are domains. Suppose $\underline{B}'$ is the family of all Borel sets in R^1 which are subsets of $\underline{S}'$. According to 9. of [2], the measure spaces $(\underline{S}, \underline{M}, \mu)$ and $(\underline{S}', \underline{M}', \mu')$, the families $\underline{D}$ and $\underline{B}'$, and the continuous map π satisfy H1 - H8, H4*, H7*, H10 - H12, and H15. It follows from well-known theorems that HC2 - HC4 hold. Indeed if we define $\underline{D}_0$ to be the family of all domains D in $\underline{D}$ each of which is a 2-dimensional polyhedron having rational vertices then if we set $^*\underline{D} = \underline{D}^{**} = \underline{D}_{\#} = \underline{D}_* = \underline{D}_0$ we see that H10, H15, HC3, and HC4 are all satisfied. Moreover, define $\underline{B}$ to be the family of all Borel sets in R^2 which are subsets of $\underline{S}$. It follows again from well-known results that HC1 is satisfied. Thus all of the theory developed in [2] and in the present work can be applied here.

Now we define a weight function for π . Let K be a fixed positive number for which $\underline{S} \subseteq \{(x,y) \in R^2 : |x| \leq K, |y| \leq K\}$. Thus

$$(1) \quad \mu \underline{S} = L_2 \underline{S} \leq 4K^2$$

and

$$(2) \quad \mu' \underline{S}' = L_1 \underline{S}' \leq 2K.$$

For each s' in $\underline{S}'$ and open subset G of $\underline{S}$ the set G_s , defined by

$G_{s'} = \{y \in \mathbb{R}^1 : (s', y) \in G\}$ is an open set of real numbers and hence is surely L_1 -measurable. Moreover

$$(3) \quad L_1(G_{s'}) \leq 2K.$$

We define a non-negative function W' on $\underline{S}' \times \underline{D}$ by $W'(s', D) = L_1(D_{s'})$, $(s', D) \in \underline{S}' \times \underline{D}$. In view of (3) W' is bounded on $\underline{S}' \times \underline{D}$. In order to show that W' is a weight function for π we must check the four conditions of H9. First, we note that if D is in $\underline{D}$ and s' is in $C' \cap D$ then $D_{s'} = \emptyset$ and so $W'(s', D) = 0$. Using basic properties of measures (applied here for L_1) we see that W' is under additive and inner continuous; it is helpful to note that $G_{1s'} \cap G_{2s'} = \emptyset$ when $G_1 \cap G_2 = \emptyset$. Next, fix D in $\underline{D}$. It follows from the Fubini Theorem for the plane that the function $W'(s', D) = L_1(D_{s'})$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable. Hence W' is a weight function for π . As usual, we let WD , $D \in \underline{D}$, be the weights induced by W' . We apply the Fubini Theorem again to obtain $WD = \int_{\underline{S}'} W'(s', D) d\mu' = L_2 D = \mu D$, $D \in \underline{D}$, or, briefly,

$$(4) \quad WD = \mu D, \quad D \in \underline{D}.$$

Using (4) and (1) we obtain $\underline{WS} = \mu \underline{S} \leq 4K^2$. Hence π is BVW.

Moreover if we define a non-negative function f_1 on $\underline{S}$ by $f_1(s) = 1$, $s \in \underline{S}$, it follows from (4) that $WD = \int_{\underline{D}} f_1(s) d\mu$, $D \in \underline{D}$.

Hence π is ACW. We shall now use this example to discuss a possible extension of one theorem and the converse of another.

Theorem 8.9 of [2] asserts that if H1 - H12, H14, and H15

are satisfied and if T is ACW then W' "essentially" satisfies H13. In our present setting all of the hypotheses of Theorem 8.9 of [2] are satisfied except H14. We shall now show that the conclusion of Theorem 8.9 is not satisfied in the present situation; that is, we shall show that if a non-negative function $\hat{W}$ is defined on $\mathcal{S}$ by

$$\hat{W}S = \inf \{W'(\pi S, D) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{S},$$

then it is false that

$$W'(S', D) = \sum \hat{W}S, \quad S \in \mathcal{S}_{S'}, \quad S \subseteq D,$$

a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. To see this we shall show $\hat{W}$ is identically zero on $\mathcal{S}$.

Now according to 9. of [2] the family $\mathcal{S}$ consists of those subsets of $\underline{S}$ which contain exactly one point. Pick S_0 in $\mathcal{S}$.

The set S_0 consists of a single point s_0 . Define for all $r > 0$ the set $S(s_0, r)$ by

$$S(s_0, r) = \{(x, y) \in R^2 : d_2[(x, y), s_0] < r\}.$$

Here, d_2 is the conventional Euclidean metric for R^2 . For all integers n which are sufficiently large, say $n \geq N_0$, the set

$D^n = S(s_0, \frac{1}{n})$ is a subset of $\underline{S}$ and hence an element of $\underline{D}$. When-

ever $n \geq N_0$ we have $W'(\pi S_0, D^n) = L_1(D^n_{\pi S_0}) = \frac{2}{n}$; but also

$W'(\pi S_0, D^n) \geq \hat{W}S_0$. Hence $\hat{W}S_0 = 0$. Hence $\hat{W}$ is identically zero

on $\mathcal{S}$. Since it is false that $W' = 0$ a.e. μ' on $\underline{S}'$ uniformly

with respect to $\underline{D}$ it is clear that the conclusion of Theorem 8.9 does not hold here. The requirement that H14 hold cannot therefore be dropped entirely from Theorem 8.9 without making some sort of replacement.

The following statement is a converse to Theorem II.3.2:

Assume H1 - H9, H7*, and HC1. Assume that T is ACW. Then if B is a set in $\underline{B}$ such that $\mu B = 0$ and $B \subseteq T^{-1}\{s' \in \underline{S} : W'(s', \underline{S}) > 0\}$ it is true that $\mu' TB = 0$.

We can use the present example to show that this statement is false. It is convenient to make a definite choice for $\underline{S}$. We define $\underline{S} = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 2)^2 < 1\}$. We thereby obtain a special case of the example just presented. The hypotheses of the above "converse" are fulfilled since π is ACW. And yet, the conclusion is false. To see this, set $\lambda = \{(x, y) \in \underline{S} : y = 2\}$. Then λ is in $\underline{B}$, $\lambda \subseteq \pi^{-1}\{s' \in \underline{S} : W'(s', \underline{S}) > 0\}$, and $\mu\lambda = 0$. But $\mu'\pi\lambda = L_1[(-1, 1)] = 2$.

II.7.3. Example. In Section II.6 we investigated almost additive weight functions. We proved that (under certain conditions) every almost additive weight function is almost over additive. One might ask whether every almost over additive weight function is almost additive. One might also ask whether every weight function satisfying H13 is almost additive. The answers to these questions become clear from the following example.

In this example our choices for $\underline{S}$, $\underline{S}'$, $\underline{M}$, $\underline{M}'$, μ , μ' , π , $\underline{D}$, $\underline{B}$, and $\underline{B}'$ are the same as those made in Example II.7.2. However we again make a special choice for $\underline{S}$. In fact we again set

$\underline{S} = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-2)^2 < 1\}$. Then $\underline{S}' = (-1,1)$. Again

$\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, π , $\underline{D}$, $\underline{B}'$, and $\underline{B}$ satisfy the standard hypotheses H1 - H8, H4*, H7*, H10 - H12, H15, and HC1 - HC4.

Indeed the only departure from the last example is found in the choice of the weight function for π . Here we shall introduce a weight function which is generated by a non-negative function w defined on $\mathcal{G}$. Again define $\lambda = \{(x,y) \in \underline{S} : y = 2\}$. Again λ is in $\underline{B}$. As before, $\mathcal{G}$ consists of all those subsets of $\underline{S}$ which contain exactly one point; as a result, either $S \subseteq \lambda$ or $S \cap \lambda = \emptyset$ whenever S is in $\mathcal{G}$. We define a non-negative function w on $\mathcal{G}$ by

$$wS = \begin{cases} 1, & S \subseteq \lambda \\ 0, & S \cap \lambda = \emptyset \end{cases}.$$

Because λ is in $\underline{B}$ it follows at once from Theorem III.2.3 that the summatory function $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is a weight function for π . This weight function satisfies H13 (by construction) and hence is over additive (7.8 of [2]).

We now assert that this weight function is not almost additive. Define $D_1 = \{(x,y) \in \underline{S} : y > 2\}$ and $D_2 = \{(x,y) \in \underline{S} : y < 2\}$. Then D_1 and D_2 are in $\underline{D}$. Fix s' in $\underline{S}'$. Then

$$(1) \quad W_*(s', \underline{S}, w) = \sum wS, \quad S \subseteq \pi^{-1}s', \quad = w\{(s', 2)\} = 1$$

while also

$$(2) \quad W_*(s', D_1, w) = \sum wS, \quad S \in \mathcal{G}_{s'}, \quad S \subseteq D_1, \quad = 0, \quad i = 1, 2.$$

Since $\mu\lambda = 0$ and $\underline{S} = D_1 \cup D_2 \cup \lambda$ it is clear that $(\{D_i\}_{i=1}^2, \lambda, \emptyset)$ is a $\underline{D}$ -partition of $\underline{S}$. Thus if W_* were almost additive we would then have

$$(3) \quad W_*(s', \underline{S}, w) = W_*(s', D_1, w) + W_*(s', D_2, w) \quad \text{a.e. } \mu' \text{ on } \underline{S}'.$$

(1) and (2) hold for every s' in $\underline{S}'$. Hence (3) cannot hold unless $\mu'\underline{S}' = 0$. But $\mu'\underline{S}' = 2$. Hence W_* cannot be almost additive.

We have exhibited a weight function which is (almost) over additive and which even satisfies H13 and yet is not almost additive.

II.7.4. The following theorem gives three methods by which new weight functions can be constructed from given ones.

Theorem. Assume H1 - H8. Assume W' is a weight function for T . Let t be a positive real number.

1) Define a non-negative function ${}_tW'$ on $\underline{S}' \times \underline{D}$ by

$${}_tW'(s', D) = \begin{cases} W'(s', D), & \text{if } W'(s', D) > t \\ 0, & \text{if } W'(s', D) \leq t \end{cases}, \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then $0 \leq {}_tW' \leq W'$ and ${}_tW'$ is a weight function for T .

2) Define a non-negative function W'_t on $\underline{S}' \times \underline{D}$ by

$$W'_t(s', D) = \begin{cases} W'(s', D), & \text{if } \mu D > t \\ 0, & \text{if } \mu D \leq t \end{cases}, \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Then $0 \leq W'_t \leq W'$ and W'_t is a weight function for T .

3) Define a non-negative function W'_{tT} on $\underline{S}' \times \underline{D}$ by

$$W'_{tT}(s', D) = \begin{cases} W'(s', D), & \text{if } \mu' TD > t \\ 0, & \text{if } \mu' TD \leq t \end{cases}, (s', D) \in \underline{S}' \times \underline{D}.$$

Then $0 \leq W'_{tT} \leq W'$ and W'_{tT} is a weight function for T .

Proof. In each case the proof consists of a straightforward verification of i) - iv) of H9. The details are omitted. Note that the weight function ${}_t W'$ introduced in 1) always satisfies H14.

Parts 2) and 3) of Theorem II.7.4 are due to Reichelderfer.

II.7.5. Example. Using 2) of Theorem II.7.4 it is easy to construct an example which reveals that two conjectures are false.

We set $\underline{S} = \{s \in \mathbb{R}^1 : 0 < s < 1\}$ and $\underline{S}' = \{s' \in \mathbb{R}^1 : 0 < s' < 1\}$.

Let $\underline{M}$ consist of all L_1 -measurable sets which are subsets of $\underline{S}$

and let $\underline{M}'$ consist of all L_1 -measurable sets which are subsets

of $\underline{S}'$. Set $\mu = L_1|_{\underline{M}}$ and $\mu' = L_1|_{\underline{M}'}$. Define the mapping T from

$\underline{S}$ into $\underline{S}'$ by $Ts = s$, $s \in \underline{S}$. T is the identity mapping from $\underline{S}$

onto $\underline{S}' = \underline{S}$. Let $\underline{D}$ consist of all domains in $(0,1)$ and let $\underline{B}$

and $\underline{B}'$ be the families of all Borel sets in $\mathbb{R}^1$ which are subsets

of $\underline{S}$ and $\underline{S}'$ respectively. According to 9. of [2], $\{\underline{S}, \underline{M}, \mu\}$,

$\{\underline{S}', \underline{M}', \mu'\}$, T , $\underline{D}$, $\underline{B}'$, and $\underline{B}$ satisfy H1 - H8, H10 - H12, H15,

H4*, and H7*. It also follows from well-known theorems that

HCl - HC4 are satisfied.

The family $\mathcal{G}$ consists of those subsets of $\underline{S} = (0,1)$ which contain exactly one point. Define a non-negative function w

on $\mathcal{S}$ by $wS = 1$, $S \in \mathcal{S}$. By Theorem III.2.3 the function W' defined on $\underline{S'} \times \underline{D}$ by

$$W'(s', D) = W'_*(s', D, w), \quad (s', D) \in \underline{S'} \times \underline{D},$$

is a weight function for T . We now introduce the non-negative function $W'_{1/2}$ defined on $\underline{S'} \times \underline{D}$ by

$$W'_{1/2}(s', D) = \begin{cases} W'(s', D), & \text{if } \mu D > \frac{1}{2} \\ 0, & \text{if } \mu D \leq \frac{1}{2} \end{cases}, \quad (s', D) \in \underline{S'} \times \underline{D}.$$

By 2) of Theorem II.7.4 $W'_{1/2}$ is a weight function for T . We develop this example further and state several of its consequences in a series of remarks.

$$i) \quad WD = \mu D = \mu' TD, \quad D \in \underline{D}.$$

Proof. $W'(s', D) = 1$ if s' is in TD and is 0 if s' is in $C'TD$. Hence $WD = \mu' TD$. But also $\mu' TD = L_1 D = \mu D$.

$$ii) \quad W'_{1/2} \underline{S} = 1. \quad \text{Hence } T \text{ is } BVW_{1/2}.$$

Proof. Observe that $W'_{1/2} \underline{S} = \int_{\underline{S}} W'_{1/2}(s', \underline{S}) d\mu' = \int_{\underline{S}} W'(s', \underline{S}) d\mu' = W\underline{S} = 1$.

iii) Suppose W'_0 is a weight function for T for which $W'_0 \leq W'_{1/2}$ and W_0 is over additive. Then $W_0 = 0$.

Proof. The sets $D_1 = \{s \in \underline{S} : 0 < s < \frac{1}{2}\}$, $D_2 = \{s \in \underline{S} : \frac{1}{4} < s < \frac{3}{4}\}$, and $D_3 = \{s \in \underline{S} : \frac{1}{2} < s < 1\}$ are in $\underline{D}$. Evidently $W'_{1/2}(s', D_i) = 0$, $s' \in \underline{S'}$, for $i = 1, 2, 3$. Hence $W'_{1/2} D_i = 0$, for $i = 1, 2, 3$. Now $W'_0 \leq W'_{1/2}$ so that $W'_0 D_i = 0$ for $i = 1, 2, 3$. Since

$\underline{S} = D_1 \cup D_2 \cup D_3$ and W_0 is over additive it follows that $W_0 \underline{S} = 0$.

But then $W_0 = 0$.

iv) For every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\mu' TB < \epsilon$ whenever B is in $\underline{B}$ and $\mu B < \delta(\epsilon)$.

Proof. Note that $\mu' TB = \mu B$ for all B in $\underline{B}$. Set $\delta(\epsilon) = \epsilon$, $\epsilon > 0$.

v) It follows from ii) and iii) that $W_{1/2}$ is not over additive. By 5.11 of [2] T is not $ACW_{1/2}$. The following conjecture must therefore be false; for its hypotheses are satisfied in the present example and yet the conclusion is false here:

Conjecture. Assume $H1 - H12$, $H14$, $H15$, $H4^*$, $H7^*$, and $HC1 - HC4$. Suppose that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\mu' TB < \epsilon$ whenever B is in $\underline{B}$ and $\mu B < \delta(\epsilon)$. Assume also that T is BVW . Then T is ACW .

vi) W and $W_{1/2}$ are AC with respect to μ .

Proof. Since $W_{1/2} \leq W$ it suffices to show that W is AC with respect to μ . Thus suppose M_0 is a set in $\underline{M}$ with $\mu M_0 = 0$. Let $\epsilon > 0$ be given. There is a countable number of pairwise disjoint sets D_i , $i \geq 1$, in $\underline{D}$ for which $M_0 \subseteq \bigcup_{i \geq 1} D_i$ and $\mu(\bigcup_{i \geq 1} D_i) < \epsilon$. By i) $WD_i = \mu D_i$, $i \geq 1$. Hence $\sum_{i \geq 1} WD_i = \sum_{i \geq 1} \mu D_i < \epsilon$.

vii) $W'_{1/2}$ is singular.

Proof. Suppose W'_A is an AC part of $W'_{1/2}$. By 5.11 of [2] W_A is over additive. By iii), $W_A = 0$.

viii) Let W'_A and W'_S be weight functions for T for which $W'_A + W'_S \leq W'_{1/2}$, T is ACW_A , and W'_S is strongly singular. Then $W_A = W_S = 0$.

Proof. By vii) $W_A = 0$. Since W'_S is strongly singular there is a set B_0 in $\underline{B}$ such that $\mu B_0 = 0$ and $W'_S(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0$. Note that $\mu'TB_0 = \mu B_0 = 0$. Hence $W'_S(s', \underline{S}) = 0$ a.e. μ' on $\underline{S}'$. Hence $W_S = 0$.

ix) There is no strong Lebesgue-type decomposition for $W'_{1/2}$.

Proof. Use ii) and viii).

Thus there are weight functions for which there is no strong Lebesgue-type decomposition. We cannot hope to remove the requirement that W' be almost over additive from Corollary II.5.3 without making some sort of replacement.

II.7.6. Example. One may ask whether the classical Lebesgue decomposition theorem (see [3]) appears as a special case of our results. This classical theorem can be obtained from the Special Decomposition Theorem (III.3.4) but not in its most general form. Certain assumptions are required which are unnecessary in the customary approaches to the theorem. The rather awkward discussion follows.

Assume that $\{X, \underline{N}, \mu\}$ and $\{X, \underline{N}, \nu\}$ are measure spaces. Suppose that μ and ν are finite valued. In order to apply Theorem III.3.4 we must first choose $\underline{S}$, $\underline{S}'$, $\underline{M}$, $\underline{M}'$, μ , μ' , T , $\underline{D}$, $\underline{B}$, and $\underline{B}'$ in such a way that H1 - H8, H11 - H12, HC1, and HC4 are satisfied (see III.3.5). We set $\underline{S} = \underline{S}' = X$, $\underline{M} = \underline{M}' = \underline{B} = \underline{B}' = \underline{N}$, $\mu = \mu$, and $\mu' = \nu$. We define T to be the identity mapping from $\underline{S} = X$ onto $\underline{S}' = X$. Unfortunately we must assume that $\underline{D}$ can be chosen so that the hypotheses listed above are fulfilled; to carry out the discussion completely, we must also assume that H7* is satisfied. (The only natural choice for $\underline{D}$ in the abstract setting is $\underline{N}$. It must be observed however that those hypotheses which require that $\underline{D}$ have certain countable subfamilies (e.g., HC4) are not likely to be fulfilled if $\underline{D} = \underline{N}$.) Next, we define a non-negative function w on $\mathfrak{S}$ by $wS = 1$, $S \in \mathfrak{S}$. Define a non-negative function W' on $\underline{S}' \times \underline{D}$ by

$$W'(s', D) = W_*(s', D, w), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Evidently $W'(s', D) = H_D'(s')$, $(s', D) \in \underline{S}' \times \underline{D}$. It is easy to check that W' is a weight function for T . Of course H13 is satisfied.

As usual, let W denote the weights induced by W' . Then

$$WD = \int_{\underline{S}'} W'(s', D) d\mu' = \int_X H_D'(s') d\nu = \nu D, \quad D \in \underline{D}.$$

Hence T is BVW. By the Special Decomposition Theorem (III.3.4) there exists a strong Lebesgue-type decomposition (W_A', W_S') for W' . Then $W' = W_A' + W_S'$, T is ACW_A , and W_S' is strongly singular. For the corresponding weights we have $\nu|_{\underline{D}} = W = W_A + W_S$. There

exists a non-negative, μ -integrable function f_A defined on $\underline{S}$ such that

$$W_A D = \int_D f_A(s) d\mu, \quad D \in \underline{D}.$$

Moreover there is a set B_0 in $\underline{B}$ such that $\mu B_0 = 0$ and $W'_S(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0 = CB_0$.

Define a non-negative function ν_A on $\underline{N}$ by $\nu_A M = \int_M f_A(s) d\mu$, $M \in \underline{N}$. Then ν_A is a measure on $\underline{N}$, ν_A is AC with respect to μ , and $W_A = \nu_A|_{\underline{D}}$. It is clear that $\nu_A D \leq \nu D$, $D \in \underline{D}$. Now let any set M in $\underline{N}$ be given. Then M is in $\underline{M}'$ so that by H7* $\nu M = \inf \{\nu O' : O' \in \mathbb{O}', M \subseteq O'\}$. Since T is the identity mapping it is easily checked that $\mathbb{O}'$ consists of those subsets of X which can be written as the union of a countable number of pairwise disjoint sets D_i in $\underline{D}$. Hence

$$\begin{aligned} \nu M &= \inf \{ \sum \nu D_i : M \subseteq \cup D_i, D_i \cap D_k = \emptyset \text{ if } i \neq k \} \\ &\geq \inf \{ \sum \nu_A D_i : M \subseteq \cup D_i, D_i \cap D_k = \emptyset \text{ if } i \neq k \} \\ &\geq \nu_A M. \end{aligned}$$

Hence $\nu_A \leq \nu$. We may therefore define a non-negative function ν_S on $\underline{N}$ by $\nu_S M = \nu M - \nu_A M$, $M \in \underline{N}$. ν_S is a measure on $\underline{N}$ and $\nu_S|_{\underline{D}} = W_S$.

Finally, we assert that $\nu_S CB_0 = 0$. To see this let $\varepsilon > 0$ be given. Since $W'_S(s', \underline{S})$, $s' \in \underline{S}'$, is μ' -integrable there is for every $t > 0$ a number $\delta(t) > 0$ such that $\int_{M'} W'_S(s', \underline{S}) d\mu' < t$

if M' is in $\underline{M}'$ and $\mu'M' < \delta(t)$. By $H7^*$ there is a set O' in Φ' such that $CB_0 \subseteq O'$ and $\mu'O' < \mu'CB_0 + \delta(\epsilon)$. Hence

$\mu'(O' \cap B_0) < \delta(\epsilon)$. There is a countable number of pairwise

disjoint sets D_i in $\underline{D}$ such that $UD_i = O' \supseteq CB_0$. It follows that

$$\begin{aligned} v_S^{CB_0} &\leq \sum v_S^{D_i} = \sum W_S^{D_i} = \sum \int_{O'} W'_S(s', D_i) d\mu' \leq \int_{O'} W'_S(s', \underline{S}) d\mu' = \\ &= \int_{O' \cap B_0} W'_S(s', \underline{S}) d\mu' + \int_{CB_0} W'_S(s', \underline{S}) d\mu' < \epsilon + 0. \end{aligned}$$

Hence

$$v_S^{CB_0} = 0.$$

It should be noted that if $\underline{D} = \underline{N}$ the argument in the latter part of this discussion becomes much simpler. Moreover $H7^*$ is not required.

CHAPTER III

ON WEIGHT FUNCTIONS SATISFYING H13

Throughout this chapter we shall assume that $\{\underline{S}, \underline{M}, \mu\}$ and $\{\underline{S}', \underline{M}', \mu'\}$ are measure spaces, that T is a single valued transformation from $\underline{S}$ onto $\underline{S}'$, that $\underline{D}$ is a family of subsets of $\underline{S}$, and that $\underline{B}'$ is a subfamily of $\underline{M}'$ for which the standard hypotheses H1 - H8, H11, and H12 are satisfied. Thus, throughout this chapter, $\{\underline{S}, \underline{M}, \mu\}$, $\{\underline{S}', \underline{M}', \mu'\}$, T , $\underline{D}$, and $\underline{B}'$ remain fixed and hypotheses H1 - H8, H11, and H12 are always in force (unless it is specifically noted otherwise). The remaining standard hypotheses will be assumed only when they are explicitly mentioned.

Certain notational conventions were set forth in Chapter I. We shall repeat several of them now. Thus $\mathcal{G}$ always denotes the family $\{S \subseteq \underline{S} : S = \bigcap D, s \in D, \text{ for some } s \text{ in } \underline{S}\}$. $\mathcal{E}$ always denotes the family of all subsets E of $\underline{S}$ each of which includes every S in $\mathcal{G}$ that it intersects. In addition, given $A \subseteq \underline{S}$, $\mathcal{G}_A$ always denotes the set $\{S \in \mathcal{G} : S \cap A \neq \emptyset\}$. Finally, if w is a non-negative function defined on $\mathcal{G}$ then $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is always defined by

$$W_*(s', D, w) = \sum wS, \quad S \in \mathcal{G}_s, \quad S \subseteq D; \quad (s', D) \in \underline{S}' \times \underline{D}.$$

III.1 Preliminary lemmas

In this section we prove two simple but useful lemmas.

III.1.1. Lemma. Assume w is a non-negative function defined on $\mathcal{S}$. Define a function $\hat{w}$ on $\mathcal{S}$ by

$$\hat{w}S = \inf \{W_*(TS, D, w) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{S}.$$

1) Then $wS \leq \hat{w}S$ for every S in $\mathcal{S}$.

2) If S_0 is a set in $\mathcal{S}$ for which $\hat{w}S_0 < \infty$ then $\hat{w}S_0 = wS_0$.

Proof. First we prove 1). Fix S_0 in $\mathcal{S}$. Suppose D_0 is an element of $\underline{D}$ for which $S_0 \subseteq D_0$. Then $W_*(TS_0, D_0, w) = \sum wS$,

$$S \subseteq D_0 \cap T^{-1}TS_0, \quad \geq wS_0. \quad \text{Hence } \hat{w}S_0 \geq wS_0.$$

Now we prove 2). Since $\hat{w}S_0 < \infty$ there is a set D^* in $\underline{D}$ such that $S_0 \subseteq D^*$ and $W_*(TS_0, D^*, w) < \infty$. Let $\varepsilon > 0$ be given. There exist sets $S_1, \dots, S_m$ in $\mathcal{S}$ such that the sets $S_0, S_1, \dots, S_m$ are distinct, such that $S_i \subseteq D^*$ for $i = 1, \dots, m$, and such that $\sum_{i=0}^m wS_i > W_*(TS_0, D^*, w) - \varepsilon$. By Lemma 8.4 of [2] there is a sequence of pairwise disjoint sets D_i in $\underline{D}$ such that $S_i \subseteq D_i \subseteq D^*$ for $i = 0, 1, \dots, m$. Then

$$\begin{aligned} \hat{w}S_0 &\leq W_*(TS_0, D_0, w) \\ &\leq W_*(TS_0, D^*, w) - \sum_{i=1}^m W_*(TS_0, D_i, w) \\ &< \sum_{i=0}^m wS_i + \varepsilon - \sum_{i=1}^m wS_i \\ &< wS_0 + \varepsilon. \end{aligned}$$

Hence $\hat{w}S_0 = wS_0$.

Remark. Part 2) of Lemma III.1.1 is due to Reichelderfer.

III.1.2. Lemma. Assume w is a non-negative function defined on $\mathcal{G}$. Again define a function $\hat{w}$ on $\mathcal{G}$ by

$$\hat{w}S = \inf \{W_*(TS, D, w) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{G}.$$

Then the function $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is also generated by $\hat{w}$.

Proof. Using Lemma III.1.1 we have $W_*(s', D, w) \leq W_*(s', D, \hat{w})$, $(s', D) \in \underline{S}' \times \underline{D}$. According to 8.5 of [2], $W_*(s', D, \hat{w}) \leq W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$. (Actually Lemma 8.5 of [2] is not applicable here. In order to use it as it appears in [2] we must assume that $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is a weight function for T . However if one inspects the proof of 8.5 it becomes apparent that it requires only that W_* be under additive. This requirement is fulfilled.)

III.2 Measurability results

III.2.1. Remark. Let w be a non-negative function defined on $\mathcal{G}$. In view of 7.9 of [2], $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is a weight function for T provided only that the function $W_*(s', D, w)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. There are no theorems in [2] - except 8.9 - that give conditions under which the function $W_*(s', D, w)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. We shall now establish two such theorems, both of general interest. Theorem III.2.3 has as its forerunner the Theorem on page 217 of [1].

III.2.2. Definition. Let w be a non-negative function defined on $\mathcal{G}$. Define a non-negative function g_w on $\underline{S}$ by $g_w(s) = wS_s$, $s \in \underline{S}$, where S_s is that member of $\mathcal{G}$ containing s . (Note that in view of H11 $\mathcal{G}$ partitions $\underline{S}$.) The function g_w assumes a constant value on the subsets S (in $\mathcal{G}$) of $\underline{S}$. It is called the point function induced by w (or the induced point function).

III.2.3. Theorem. Assume HCl and H15i. Let w be a non-negative function defined on $\mathcal{G}$. Suppose that the induced point function $g_w(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable.

1) Then the function $W_*(s', D, w)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$.

2) $W_*(s', D, w)$, $(s', D) \in \underline{S}' \times \underline{D}$, is a weight function for T .

Proof. 2) follows from 1) and Remark 7.9 of [2]. It is sufficient to prove 1). The proof of 1) is carried out in three successively more general cases.

Case I. Assume g_w is the characteristic function for some set B in $\underline{B}$. Observe first that B must also be in $\mathcal{C}$. Define $N(s', T, D \cap B)$ to be the number of sets S in $\mathcal{G}_{D \cap B}$ which are mapped onto s' by T . The function $N(s', T, D \cap B)$, $(s', D) \in \underline{S}' \times \underline{D}$, is rather similar to the crude multiplicity function defined in [1]. It is easy to verify that in the present case we have

$$(1) \quad W_*(s', D, w) = N(s', T, D \cap B), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Our task now is to show that $N(s', T, D \cap B)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. To do this we fix D_0 in $\underline{D}$. Define

$\underline{D}^{**}(D_0)$ to be the family of all finite subsets $\{D_i^{**}\}_{i=1}^n$ of $\underline{D}^{**}$ consisting of pairwise disjoint subsets of D_0 ; here, $\underline{D}^{**}$ is the countable family whose existence is guaranteed by H15i. The family $\underline{D}^{**}(D_0)$ is countable. At this point we assert

$$(2) \quad N(s', T, D_0 \cap B) = \sup \left\{ 0, \sum_{i=1}^n H_{T(D_i^{**} \cap B)}^i(s') : \right. \\ \left. \{D_i^{**}\}_{i=1}^n \in \underline{D}^{**}(D_0) \right\},$$

for every s' in $\underline{S}'$.

We prove (2). First assume that $\{D_i^{**}\}_{i=1}^n$ is in $\underline{D}^{**}(D_0)$.

For each s' in $\underline{S}'$ we have

$$\begin{aligned} N(s', T, D_0 \cap B) &\geq \sum_{i=1}^n N(s', T, D_i^{**} \cap B) \\ &\geq \sum_{i=1}^n H_{T(D_i^{**} \cap B)}^i(s'). \end{aligned}$$

Hence

$$(3) \quad N(s', T, D_0 \cap B) \geq \sup \left\{ 0, \sum_{i=1}^n H_{T(D_i^{**} \cap B)}^i(s') : \right. \\ \left. \{D_i^{**}\}_{i=1}^n \in \underline{D}^{**}(D_0) \right\}$$

for every s' in $\underline{S}'$.

Now fix s' in $\underline{S}'$. If $N(s', T, D_0 \cap B) = 0$, then (2) follows from (3). Thus assume that $N(s', T, D_0 \cap B)$ is positive. Let m be any positive integer for which $N(s', T, D_0 \cap B) \geq m$. There

exist distinct sets $S_1, \dots, S_m$ in $\mathcal{S}$ such that $TS_i = s'$ and $S_i \subseteq D_0 \cap B$. According to Lemma 8.4 of [2] there exist pairwise disjoint sets $D_1, \dots, D_m$ in $\underline{D}$ such that $S_i \subseteq D_i \subseteq D_0$. By H15i there exist sets $D_1^{**}, \dots, D_m^{**}$ in $\underline{D}^{**}$ such that $S_i \subseteq D_i^{**} \subseteq D_i$ for $i = 1, \dots, m$. Evidently $\{D_i^{**}\}_{i=1}^m$ is in $\underline{D}^{**}(D_0)$. Moreover $H_{T(D_i^{**} \cap B)}^i(s') = 1$. Hence $\sum_{i=1}^m H_{T(D_i^{**} \cap B)}^i(s') = m$. It follows that $\sup \{0, \sum_{i=1}^n H_{T(D_i^{**} \cap B)}^i(s') : \{D_i^{**}\}_{i=1}^n \in \underline{D}^{**}(D_0)\} \geq N(s', T, D_0 \cap B)$. Hence (2) holds in this case also. Thus (2) holds for every s' in $\underline{S}'$. In view of (1), (2), and the fact that $\underline{D}^{**}(D_0)$ is countable, it is now apparent that it suffices to show that each function $H_{T(D_i^{**} \cap B)}^i(s')$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable. But by HCl each set $T(D_i^{**} \cap B)$ is in $\underline{M}'$ and so $H_{T(D_i^{**} \cap B)}^i$ is $\underline{M}'$ -measurable.

Case II. Assume that g_w is bounded. Thus we assume that there is a positive number t for which $g_w(s) \leq t$, $s \in \underline{S}$. We shall use the fact that 1) holds for Case I to prove it for the more general Case II.

Let m be a fixed positive integer. For each $j = 0, 1, \dots, 2^m$ we define

$$B_{mj} = \{s \in \underline{S} : \frac{jt}{2^m} \leq g_w(s) < \frac{(j+1)t}{2^m}\}.$$

Then B_{mj} is in $\underline{B}$ for each j . The characteristic function $H_{mj}(s)$,

$s \in \underline{S}$, of the subset B_{mj} of $\underline{S}$ is $\underline{B}$ -measurable. Each B_{mj} is also in $\mathcal{E}$ and so, given S in $\mathcal{G}$, $H_{mj}S$ consists of but one number $r_{mj}(S)$. Define a non-negative function w_{mj} on $\mathcal{G}$ by $w_{mj}S = r_{mj}(S)$, $S \in \mathcal{G}$. Surely $g_{w_{mj}} = H_{mj}$. Thus we may apply the result of Case I to conclude that the function $W_*(s', D, w_{mj})$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. We define a non-negative function w_m on $\mathcal{G}$ by

$$w_m S = \sum_{j=0}^{2^m} \frac{1}{2^m} w_{mj} S, \quad S \in \mathcal{G}.$$

By Lemma I.7.9 we have

$$W_*(s', D, w_m) = \sum_{j=0}^{2^m} \frac{1}{2^m} W_*(s', D, w_{mj}), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

It follows that the function $W_*(s', D, w_m)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$ and for every positive integer m . For each $m \geq 1$ the set $\underline{S}$ is the union of the pairwise disjoint sets B_{mj} , $j = 0, 1, \dots, 2^m$. And $B_{mj} = B_{m+1, 2j} \cup B_{m+1, 2j+1}$ for every m and j . It follows that $w_m \leq w_{m+1}$, $m \geq 1$, and

$0 \leq wS - w_m S < \frac{t}{2^m}$, $S \in \mathcal{G}$. Hence $\{w_m\}_{m=1}^{\infty}$ is a non-decreasing

sequence of functions and $wS = \lim_m w_m S$, $S \in \mathcal{G}$. By Lemma I.7.10

$$(4) \quad W_*(s', D, w) = \lim_m W_*(s', D, w_m), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Now fix D in $\underline{D}$. We have already observed that each function $W_*(s', D, w_m)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable. It follows from

(4) that $W_*(s', D, w)$, $s' \in \underline{S}'$, must also be $\underline{M}'$ -measurable.

We now know that 1) holds in Cases I and II. We shall use these facts to show that 1) holds in every case. Thus suppose g_w is an arbitrary non-negative, $\underline{B}$ -measurable function. Define a sequence $\{w_m\}_{m=1}^{\infty}$ of functions on $\mathcal{S}$ by $w_m S = \min(wS, m)$, $S \in \mathcal{S}$. For each $m \geq 1$, $W_*(s', D, w_m)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$; this follows from the result of Case II. Since $wS = \lim_m w_m S$, $S \in \mathcal{S}$, and since $w_m \leq w_{m+1}$, $m \geq 1$, it follows from Lemma I.7.10 that

$$\lim_m W_*(s', D, w_m) = W_*(s', D, w), \quad s' \in \underline{S}',$$

for every D in $\underline{D}$. Hence $W_*(s', D, w)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. The proof of 1) is complete.

III.2.4. Theorem. Assume H9, H15i, and HC1. Define a non-negative function $\hat{w}$ on $\mathcal{S}$ by

$$\hat{w}S = \inf \{W'(TS, D) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{S}.$$

1) There exists a non-negative function w defined on $\mathcal{S}$ such that the induced point function $g_w(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable and for which there exists a set B' in $\underline{B}'$ such that $\mu(B') = 0$ and $g_w(s) = g_{\hat{w}}(s)$, $s \in CT^{-1}B'$.

2) For every D in $\underline{D}$ the function $W_*(s', D, \hat{w})$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable.

3) $W_*(s', D, \hat{w})$, $(s', D) \in \underline{S}' \times \underline{D}$, is a weight function for T .

Proof. Note first that $W'(TS, D^{**}) \leq W'(TS, D)$ whenever

S is in $\mathcal{G}$ and $D^{**} \subseteq D$. Thus it follows from H15i that

$$(1) \quad \omega S = \inf \{W'(TS, D^{**}) : S \subseteq D^{**} \in \underline{D}^{**}\}, \quad S \in \mathcal{G}.$$

We first prove 1). By 2.12 of [2] (see I.4) there is, for every D in $\underline{D}$, a non-negative function h'_D on $\underline{S}'$ which is $\underline{B}'$ -measurable

and for which we have $h'_D(s') = W'(s', D)$ a.e. μ' on $\underline{S}'$. Define

$$X'_D = \{s' \in \underline{S}' : h'_D(s') \neq W'(s', D)\}. \quad \text{Then } X'_D \text{ is in } \underline{M}' \text{ and } \mu' X'_D = 0.$$

By H6 there is for every D a set B'_D in $\underline{B}'$ such that $X'_D \subseteq B'_D$

and $\mu' B'_D = 0$. Now we define $B' = \bigcup_{D^{**} \in \underline{D}^{**}} B'_{D^{**}}$, $D^{**} \in \underline{D}^{**}$. $\underline{D}^{**}$ is

countable and each $B'_{D^{**}}$ is in $\underline{B}'$. Hence B' is in $\underline{B}'$, and $\mu' B' = 0$.

Moreover

$$(2) \quad h'_{D^{**}}(s') = W'(s', D^{**}), \quad s' \in C' B', \quad D^{**} \in \underline{D}^{**}.$$

For each D^{**} in $\underline{D}^{**}$ we define a non-negative function $t_{D^{**}}$ on $\underline{S}$

by $t_{D^{**}}(s) = h'_{D^{**}}(Ts)$, $s \in \underline{S}$. Using the fact that each h'_D is

$\underline{B}'$ -measurable and the fact that $T^{-1} \underline{B}' \subseteq \underline{B}$ we can prove easily that

the functions $t_{D^{**}}(s)$, $s \in \underline{S}$, are $\underline{B}$ -measurable. Observe that

$$(3) \quad t_{D^{**}}(s) = W'(Ts, D^{**}), \quad s \in CT^{-1} B', \quad D^{**} \in \underline{D}^{**}.$$

Each function $t_{D^{**}}$ is constant on the sets S in $\mathcal{G}$. Hence for

each S in $\mathcal{G}$ the set $t_{D^{**}} S$ consists either of a single real number

or just of ∞ . Let $p(D^{**}, S)$ be the unique element of the set

$t_{D^{**}} S$. We define non-negative functions $w_{D^{**}}$ on $\mathcal{G}$ by

$$w_{D^{**}}S = \begin{cases} p(D^{**}, S) & \text{if } S \subseteq D^{**} \\ p(\underline{S}, S) & \text{if } S \cap D^{**} = \emptyset. \end{cases}$$

For the induced point functions we have

$$g_{w_{D^{**}}}(s) = \begin{cases} t_{D^{**}}(s), & \text{if } s \in D^{**} \\ t_{\underline{S}}(s), & \text{if } s \notin D^{**} \end{cases}.$$

Each function $g_{w_{D^{**}}}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable. Finally we can

introduce the desired function w . We define a non-negative function w on $\mathcal{S}$ by $wS = \inf \{w_{D^{**}}S : D^{**} \in \underline{D^{**}}\}$, $S \in \mathcal{S}$. Surely

$g_w(s) = \inf \{g_{w_{D^{**}}}(s) : D^{**} \in \underline{D^{**}}\}$, $s \in \underline{S}$. The function $g_w(s)$,

$s \in \underline{S}$, is evidently $\underline{B}$ -measurable. We now assert

$$(4) \quad wS = \hat{w}S, \quad S \subseteq CT^{-1}B'.$$

(Observe that it follows from H12 that either $S \subseteq T^{-1}B'$ or $S \subseteq CT^{-1}B'$ whenever S is in $\mathcal{S}$.)

To see (4) we assume $S_0 \subseteq CT^{-1}B'$. Let D^{**} be any element of $\underline{D^{**}}$ for which $S_0 \subseteq D^{**}$. Using (3) we obtain

$$W'(TS_0, D^{**}) = t_{D^{**}}S_0 = w_{D^{**}}S_0 \geq wS_0.$$

In view of (1)

$$(5) \quad \hat{w}S_0 \geq wS_0.$$

Now suppose that D^{**} is any set in $\underline{D^{**}}$. If $S_0 \subseteq D^{**}$ then

$w_{D^{**}}S_0 = t_{D^{**}}S_0 = W'(TS_0, D^{**}) \geq \hat{w}S_0$; but if $S_0 \cap D^{**} = \emptyset$ then

$w_{D^{**}}S_0 = t_{\underline{S}}S_0 = W'(TS_0, \underline{S}) \geq \hat{w}S_0$. In either event $w_{D^{**}}S_0 \geq \hat{w}S_0$.

Consequently

$$(6) \quad wS_0 \geq \hat{w}S_0.$$

From (5) and (6) we obtain $wS_0 = \hat{w}S_0$. Thus (4) is proved.

It follows from (4) that $g_w(s) = g_{\hat{w}}(s)$, $s \in CT^{-1}B'$. It is now clear that the function w fulfills the requirements of 1). Using this same function w together with (4) we shall now establish the truth of 2).

Thus fix D in $\underline{D}$. Suppose s' is in $C'B'$. Then $T^{-1}s' \cap T^{-1}B' = \emptyset$ so that if S is in $\mathfrak{S}_s$, it is true that $S \subseteq CT^{-1}B'$. It follows easily from (4) that $W_*(s', D, w) = W_*(s', D, \hat{w})$. Since $\mu'B' = 0$ we have evidently shown that $W_*(s', D, w) = W_*(s', D, \hat{w})$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. By Theorem III.2.3 the function $W_*(s', D, w)$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable for every D in $\underline{D}$. The measure space $\{\underline{S}', \underline{M}', \mu'\}$ is complete and so the function $W_*(s', D, \hat{w})$, $s' \in \underline{S}'$, must be $\underline{M}'$ -measurable for every D in $\underline{D}$. Thus 2) is proved. 3) follows from 2) and Remark 7.9 of [2].

III.2.5. Corollary. Assume H15i and HC1. Assume that W' is a weight function for T which satisfies H13. Then there exists a non-negative function w_0 defined on $\mathfrak{S}$ such that the induced point function $g_{w_0}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable, such that $W_*(s', D, w_0) \leq W'(s', D)$, $(s', D) \in \underline{S}' \times \underline{D}$, and such that $W_*(s', D, w_0) = W'(s', D)$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

Proof. We define a non-negative function $\hat{w}$ on $\mathcal{S}$ by

$$\hat{w}S = \inf \{W'(TS, D) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{S}.$$

By Lemma III.1.2 the function W' is generated by $\hat{w}$. By Theorem III.2.4 there exists a non-negative function $w_{\#}$ defined on $\mathcal{S}$ such that the induced point function $g_{w_{\#}}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable and for which there exists a set B' in $\underline{B}'$ such that $\mu'B' = 0$ and

$$(1) \quad g_{w_{\#}}(s) = g_{\hat{w}}(s), \quad s \in CT^{-1}B'.$$

(1) is equivalent to

$$(2) \quad w_{\#}S = \hat{w}S, \quad S \subseteq CT^{-1}B'.$$

Define a non-negative function w_0 on $\mathcal{S}$ by

$$w_0S = \begin{cases} w_{\#}S & \text{if } S \subseteq CT^{-1}B' \\ 0 & \text{if } S \cap CT^{-1}B' = \emptyset \end{cases}.$$

Surely $g_{w_0}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable. By (2) $W'_*(s', D, w_0) \leq W'_*(s', D)$, $(s', D) \in \underline{S}' \times \underline{D}$. Using the fact that $w_0S = \hat{w}S$ whenever $S \subseteq CT^{-1}B'$ it follows readily that

$$W'_*(s', D, w_0) = W'_*(s', D), \quad s' \in C'B', \quad D \in \underline{D}.$$

The proof is complete since $\mu'B' = 0$.

III.2.6. Remark. It follows from Corollary III.2.5 that every weight function satisfying H13 is effectively generated by some function w on $\mathcal{S}$ whose induced point function is $\underline{B}$ -measurable.

This result will be used several times in the sequel. It appears to be a significant tool in the study of those weight functions which satisfy H13.

III.3 The Special Decomposition Theorems

III.3.1. Remarks. According to 7.8 of [2], every weight function satisfying H13 is over additive and hence almost over additive. Evidently the results of Section II.5 can be applied directly to obtain decomposition theorems for weight functions of this special type. We shall adopt this approach. However it is also possible to obtain the desired result from the General Decomposition Theorem (Theorem II.2.8). The latter approach requires lengthier proofs but fewer of the standard hypotheses. We shall present an outline of the second approach, giving few details.

III.3.2. Lemma. Assume W'_0 is a weight function for T which satisfies H13. Suppose that W'_1 and W'_2 are weight functions for T for which $W'_0 = W'_1 + W'_2$. Define, for $i = 0, 1, 2$, a non-negative function $\hat{W}_i$ on $\mathcal{S}$ by

$$\hat{W}_i S = \inf \{W'_i(TS, D) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{S}.$$

Then

$$(1) \quad \hat{W}_0 S = \hat{W}_1 S + \hat{W}_2 S, \quad S \in \mathcal{S},$$

and

$$(2) \quad W'_0(s', D) = W'_1(s', D, \hat{W}_1) + W'_2(s', D, \hat{W}_2), \quad (s', D) \in \underline{S} \times \underline{D}.$$

Moreover if (s'_0, D_0) is an element of $\underline{S}' \times \underline{D}$ for which $W'_0(s'_0, D_0) < \infty$ then $W'_1(s'_0, D_0) = W'_*(s'_0, D_0, \hat{w}_1)$, $i = 1, 2$. Hence if T is BVW_0 then $W'_1(s', D) = W'_*(s', D, \hat{w}_1)$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

Proof. We prove (1) first. We assert that

$$(3) \quad \hat{w}_1 S + \hat{w}_2 S \leq \hat{w}_0 S, \quad S \in \mathcal{G}.$$

Surely (3) holds in case $\hat{w}_0 S = \infty$. Thus assume $\hat{w}_0 S < \infty$ and let p be a positive real number for which $\hat{w}_0 S < p$. There exists a set D_p in $\underline{D}$ such that $S \subseteq D_p$ and $W'_0(TS, D_p) < p$. It follows that $\hat{w}_1 S + \hat{w}_2 S \leq W'_1(TS, D_p) + W'_2(TS, D_p) = W'_0(TS, D_p) < p$. Hence (3) also holds in case $\hat{w}_0 S < \infty$. Next we prove

$$(4) \quad \hat{w}_0 S \leq \hat{w}_1 S + \hat{w}_2 S, \quad S \in \mathcal{G}.$$

Surely (4) holds in case $\hat{w}_1 S + \hat{w}_2 S = \infty$. Thus assume $\hat{w}_1 S + \hat{w}_2 S < \infty$ and suppose $\hat{w}_1 S + \hat{w}_2 S < p$. There exist sets D_1 and D_2 in $\underline{D}$ such that $S \subseteq D_1$, $S \subseteq D_2$, and $W'_1(TS, D_1) + W'_2(TS, D_2) < p$. By Lemma 8.3 of [2] there is a set D_p in $\underline{D}$ such that $S \subseteq D_p \subseteq D_1 \cap D_2$. Using the under additivity of the W'_i 's we obtain

$$\begin{aligned} \hat{w}_0 S &\leq W'_0(TS, D_p) \\ &\leq W'_1(TS, D_p) + W'_2(TS, D_p) \\ &\leq W'_1(TS, D_1) + W'_2(TS, D_2) < p. \end{aligned}$$

Hence (4) holds in case $\hat{w}_1 S + \hat{w}_2 S < \infty$. (1) now follows from (3) and (4). (2) follows from (1), Lemma I.7.9, and Lemma III.1.2.

Assume now that (s'_0, D_0) is an element of $\underline{S}' \times \underline{D}$ for which

$W'_0(s'_0, D_0) < \infty$. In view of (2) we have

$$(5) \quad W'_0(s'_0, D_0) = W'_*(s'_0, D_0, \hat{w}_1) + W'_*(s'_0, D_0, \hat{w}_2)$$

and, by hypothesis,

$$(6) \quad W'_0(s'_0, D_0) = W'_1(s'_0, D_0) + W'_2(s'_0, D_0).$$

Since $W'_0(s'_0, D_0) < \infty$ we can deduce from (5) and (6) that $W'_i(s'_0, D_0)$ and $W'_*(s'_0, D_0, \hat{w}_i)$ are finite ($i = 1, 2$) and finally that

$$(7) \quad 0 = (W'_1(s'_0, D_0) - W'_*(s'_0, D_0, \hat{w}_1)) + (W'_2(s'_0, D_0) - W'_*(s'_0, D_0, \hat{w}_2))$$

But, according to Lemma 8.5 of [2], we must have

$$(8) \quad W'_i(s'_0, D_0) - W'_*(s'_0, D_0, \hat{w}_i) \geq 0, \quad i = 1, 2.$$

It follows from (7) and (8) that $W'_i(s'_0, D_0) = W'_*(s'_0, D_0, \hat{w}_i)$,

$i = 1, 2$.

III.3.3. The following theorem may seem out of place in this section. However it is a result of independent interest.

Theorem. Assume HC⁴. Assume W' is a weight function for T which satisfies H13. Assume that T is BVW. Assume that W'_a is an AC part of W' . Then W'_a essentially satisfies H13 in the following sense: There exists a non-negative function w_0 defined on $\mathcal{S}$ such that $W'_*(s', D, w_0) \leq W'_a(s', D)$, $(s', D) \in \underline{S}' \times \underline{D}$, and such that $W'_*(s', D, w_0) = W'_a(s', D)$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$.

Proof. By Theorem II.1.14 there exist weight functions W'_A

and $W'_{\#}$ for T such that $W' = W'_A + W'_{\#}$, $W'_A \leq W'_a$, T is ACW_A , and such that $W'_A = W'_a$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. We define a non-negative function w_0 on $\mathcal{G}$ by $w_0 S = \inf \{W'_A(TS, D) : S \subseteq D \in \underline{D}\}$, $S \in \mathcal{G}$. Since T is BVW it follows from Lemma III.3.2 that $W'_A(s', D) = W'_*(s', D, w_0)$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. Moreover Lemma 8.5 of [2] guarantees that $W'_*(s', D, w_0) \leq W'_A(s', D)$, $(s', D) \in \underline{S}' \times \underline{D}$. It now follows at once that w_0 has the desired properties.

III.3.4. The special decomposition theorem. Assume H13, H4*, H7*, HC1, and HC2. Assume that T is BVW.

1) Then every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' . The components of such a decomposition must essentially satisfy H13 (see III.3.3) and hence are almost over additive.

2) Assume also HC4. Then there exists a strong Lebesgue-type decomposition (W'_a, W'_s) for W' . Its components satisfy H13.

Proof. By Lemma 7.8 of [2] W' is (almost) over additive. 1) follows at once from Theorem II.5.2 and Lemma III.3.2. We now prove 2). Applying the Strong Decomposition Theorem (Corollary II.5.3) we deduce that there exists a strong Lebesgue-type decomposition $(W'_A, W'_{\#})$ for W' . Thus, among other things, $W' = W'_A + W'_{\#}$. By Lemma III.3.2 there exist non-negative functions w_a and w_s defined on $\mathcal{G}$ for which

$$(1) \quad W'(s', D) = W'_*(s', D, w_a) + W'_*(s', D, w_s), \quad (s', D) \in \underline{S'} \times \underline{D};$$

moreover, $W'_A(s', D) = W'_*(s', D, w_a)$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$ and $W'_{\#}(s', D) = W'_*(s', D, w_s)$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. We define non-negative functions W'_a and W'_s on $\underline{S'} \times \underline{D}$ by

$$W'_a(s', D) = W'_*(s', D, w_a), \quad (s', D) \in \underline{S'} \times \underline{D},$$

and

$$W'_s(s', D) = W'_*(s', D, w_s), \quad (s', D) \in \underline{S'} \times \underline{D}.$$

Using H2, Remark 7.9 of [2], and the fact that W'_A and $W'_{\#}$ are weight functions for T , we deduce that W'_a and W'_s are weight functions for T . It follows from (1) and Lemma I.7.5 that (W'_a, W'_s) is a strong Lebesgue-type decomposition for W' . By construction the components W'_a and W'_s satisfy H13.

III.3.5. Remarks. It was stated in III.3.1 that the results of Theorem III.3.4 can be derived from the General Decomposition Theorem. We shall describe briefly the arguments used. Most details are omitted. Several preliminary results are needed.

i) Assume HCl. Assume that W' is a weight function for T generated by w . Suppose that the induced point function g_w is $\underline{B}$ -measurable. Define $S^+ = \cup S$, $S \in \mathcal{S}$ and $wS > 0$. Assume that T is BVW. Then T is ACW if and only if for every set B in $\underline{B} \cap \mathcal{C}$ for which $B \subseteq S^+$ and $\mu B = 0$ it is true that $\mu' TB = 0$.

The straightforward proof rests chiefly upon Theorem 7.24 of [2]. Lemma I.7.8 is also used at one point.

ii) Assume HCl and H15i. Suppose W' is a weight function for T satisfying H13. Assume that T is BVW and that W' is singular. Then W' is strongly singular.

We sketch a proof for ii). In view of Corollary III.2.5 and Lemma I.7.5 it is sufficient to prove ii) for weight functions W' generated by functions w whose induced point functions g_w are $\underline{B}$ -measurable. We write $\underline{S}'$ as the union of a countable number of pairwise disjoint sets M'_n , $n \geq 1$, in $\underline{M}'$ of finite μ' measure. Define $S^+ = \cup S$, $S \in \mathcal{G}$ and $wS > 0$. Define $\Gamma = \{B \in \underline{B} \cap \mathcal{E} : B \subseteq S^+ \text{ and } \mu B = 0\}$ and $\Gamma'_n = \{M' \in \underline{M}' : M' = M'_n \cap TB \text{ for some } B \text{ in } \Gamma\}$, $n \geq 1$. Set $p_n = \sup \{\mu' M' : M' \in \Gamma'_n\}$, $n \geq 1$. Using an argument rather similar to the one used early in the proof of Lemma II.5.1 we obtain a set B_0 in $\underline{B} \cap \mathcal{E}$ such that $\mu'(M'_n \cap TB_0) = p_n$, $n \geq 1$. The set TB_0 is in $\underline{M}'$. There exists a set B'_0 in $\underline{B}'$ such that $TB_0 \subseteq B'_0$ and $\mu'(B'_0 \cap C'TB_0) = 0$. The set $S^+ \cap CT^{-1}B'_0$ is in $\underline{B} \cap \mathcal{E}$. Define a non-negative function w_1 on $\mathcal{G}$ by

$$w_1 S = \begin{cases} wS, & \text{if } S \subseteq S^+ \cap CT^{-1}B'_0 \\ 0, & \text{if } S \cap (S^+ \cap CT^{-1}B'_0) = \emptyset \end{cases}.$$

Using Theorem III.2.3 and i) we prove that $W'_*(s', D, w_1)$, $(s', D) \in \underline{S}' \times \underline{D}$, is an AC part of W' . Hence $W'_*(s', \underline{S}, w_1) = 0$ a.e. μ'

on $\underline{S}'$. Using 7.7 of [2] it is now easy to see that $W'(s', \underline{S}) = 0$ a.e. μ' on $C'TB_0$.

iii) Assume H15i, HC1, and HC4. Assume that W' is a weight function for T satisfying H13. Assume that T is BVW.

1) Then every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' . The components of such a decomposition essentially satisfy H13 (see Theorem III.3.3) and hence are almost over additive.

2) There exists a strong Lebesgue-type decomposition for W' . Its components satisfy H13.

We discuss briefly the proof of iii). 1) follows chiefly from Lemma III.3.2, ii), and Lemma I.7.5. 2) follows readily from Theorem II.2.8, Lemma III.3.2, and 1). The argument is not difficult. It is rather similar to the one used in the proof of Theorem III.3.4.

iv) This development makes it possible to drop the assumptions HC2, H4*, and H7* from Theorem III.3.4.

III.4 On the AC component of the Special Decomposition Theorem

In this section we shall show that a maximal AC part of a weight function satisfying H13 is actually a strongly maximal AC part.

III.4.1. Lemma. Assume HC1 and H15i. Let w_1 and w_2 be non-negative functions defined on $\mathcal{G}$. Suppose that the induced point functions $g_{w_1}(s)$, $s \in \underline{S}$, and $g_{w_2}(s)$, $s \in \underline{S}$, are $\underline{B}$ -

measurable. Define a non-negative function w_0 on $\mathcal{S}$ by $w_0 S = \max \{w_1 S, w_2 S\}$, $S \in \mathcal{S}$. Define non-negative functions W'_i on $\underline{S}' \times \underline{D}$ by

$$W'_i(s', D) = W'_i(s', D, w_i), \quad (s', D) \in \underline{S}' \times \underline{D},$$

for $i = 0, 1, 2$. Then W'_i is a weight function for T , $i = 0, 1, 2$.

If T is BVW_1 and BVW_2 then T is BVW_0 . If T is ACW_1 and ACW_2 then T is ACW_0 .

Proof. For the induced point function $g_{w_0}(s)$, $s \in \underline{S}$, we have $g_{w_0} = \max \{g_{w_1}, g_{w_2}\}$. Hence $g_{w_0}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable. By Theorem III.2.3 each W'_i is a weight function for T . Now suppose that T is BVW_1 and BVW_2 . Note that $w_0 \leq w_1 + w_2$ so that, using Lemma I.7.9, $W'_0 \leq W'_1 + W'_2$. In particular $W'_0 \underline{S} \leq W'_1 \underline{S} + W'_2 \underline{S} < \infty$.

Assume now that T is ACW_1 and ACW_2 . Then surely T is BVW_0 . Define $S_i^+ = \{S \in \mathcal{S} : w_i S > 0\}$, for $i = 0, 1, 2$. Evidently

$$(1) \quad S_0^+ = S_1^+ \cup S_2^+.$$

To show that T is ACW_0 we shall use Theorem 7.24 of [2]. Thus let M be a set in $\underline{M}$ for which $M \subseteq S_0^+$ and $\mu M = 0$. Set $M_i = M \cap S_i^+$, $i = 1, 2$. Using Theorem 7.24 of [2] it follows that TM_1 and TM_2 are in $\underline{M}'$ and $\mu' TM_1 = \mu' TM_2 = 0$. In view of (1),

$TM = TM_1 \cup TM_2$. Hence TM is in $\underline{M}'$ and $\mu'TM = 0$. It follows from Theorem 7.24 of [2] that T is ACW_0 .

III.4.2. Theorem. Assume H15i, HC1, and HC4. Assume that W' is a weight function for T which satisfies H13. Suppose T is BVW. Every maximal AC part of W' is a strongly maximal AC part of W' .

Proof. Let W'_A be a maximal AC part of W' . Applying Theorem III.3.3 and then Corollary III.2.5 we find that there is a non-negative function w_a defined on $\mathcal{S}$ such that the induced point function $g_{w_a}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable and such that

$$(1) \quad W'_*(s', D, w_a) = W'_A(s', D) \quad \text{a.e. } \mu' \text{ on } \underline{S}' \text{ uniformly with respect to } \underline{D}; \quad W'_*(s', D, w_a) \leq W'_A(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Now let $W'_\#$ be an AC part of W' . Repeating the above reasoning, we obtain a non-negative function w_c defined on $\mathcal{S}$ such that the induced point function $g_{w_c}(s)$, $s \in \underline{S}$, is $\underline{B}$ -measurable and such that

$$(2) \quad W'_*(s', D, w_c) = W'_\#(s', D) \quad \text{a.e. } \mu' \text{ on } \underline{S}' \text{ uniformly with respect to } \underline{D}; \quad W'_*(s', D, w_c) \leq W'_\#(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

Define a non-negative function w_0 on $\mathcal{S}$ by $w_0 S = \max \{w_a S, w_c S\}$, $S \in \mathcal{S}$. Define a non-negative function W'_0 on $\underline{S}' \times \underline{D}$ by $W'_0(s', D) = W'_*(s', D, w_0)$, $(s', D) \in \underline{S}' \times \underline{D}$. By Lemma III.4.1 and by Lemma

I.7.5, W'_0 is a weight function for T and, moreover, T is ACW_0 .

Now we define a non-negative function $\hat{W}$ on $\mathcal{C}$ by

$$\hat{W}S = \inf \{W'(TS, D) : S \subseteq D \in \underline{D}\}, \quad S \in \mathcal{C}.$$

Using (1) and (2) we obtain

$$W'_*(s', D, w_a) \leq W'(s', D), \quad (s', D) \in \underline{S}' \times \underline{D},$$

and

$$W'_*(s', D, w_c) \leq W'(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

It follows from Lemma III.1.1 that $w_a \leq \hat{W}$ and $w_c \leq \hat{W}$. Hence

$w_0 \leq \hat{W}$ and so W'_0 is an AC part of W' (using Lemma III.1.2).

Moreover, surely

$$(3) \quad W'_*(s', D, w_a) \leq W'_0(s', D), \quad (s', D) \in \underline{S}' \times \underline{D},$$

and

$$(4) \quad W'_*(s', D, w_c) \leq W'_0(s', D), \quad (s', D) \in \underline{S}' \times \underline{D}.$$

From (1) and (3) it follows that $W'_A \leq W'_0$ a.e. μ' on $\underline{S}'$ uniformly

with respect to $\underline{D}$. Since W'_0 is an AC part of W' it follows that

$W'_A = W'_0$ a.e. μ' on $\underline{S}'$ uniformly with respect to $\underline{D}$. But then,

by (2) and (4), $W'_{\#} \leq W'_A$ a.e. μ' on $\underline{S}'$ uniformly with respect

to $\underline{D}$.

III.4.3. Corollary. Assume H15i, HCl, and HC4. Assume that W' is a weight function for T which satisfies H13. Assume that T is BVW. The AC component of any Lebesgue-type decomposition for W' is a strongly maximal AC part of W' .

Proof. By Theorem II.1.15 the AC component of any Lebesgue-

type decomposition for W' must be a maximal AC part of W' . By Theorem III.4.2 any maximal AC part of W' must be a strongly maximal AC part.

III.4.4. Corollary. Assume H15i, HCl, and HC4. Assume that W' is a weight function for T satisfying H13. Assume that T is BVW. There is essentially at most one Lebesgue-type decomposition for W' in the following sense: If $(W'_a, W'_\#)$ and (W'_A, W'_S) are Lebesgue-type decompositions for W' then $W'_a = W'_A$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$ and $W'_\# = W'_S$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$.

Proof. Both W'_a and W'_A are strongly maximal AC parts of W' . Hence $W'_a = W'_A$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. Since T is BVW it follows that $W'(s', D) < \infty$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. Thus since $W'_a + W'_\# = W' = W'_A + W'_S$ it follows that $W'_\# = W'_S$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$.

III.5 A property of weight functions of the special class

The material of this section is unrelated to the rest of the chapter. For the first two lemmas of this section it is not necessary to assume H11 and H12. Throughout this section $\{*_j^D\}_{j \geq 1}$ is a countable sequence of sets in $\underline{D}$ with the properties given in the last sentence of H4.

III.5.1. Lemma. Assume H9. Suppose that M is a set in $\underline{M}$ for which $M \subseteq \ast D_j$ for some j . Then there exists a double sequence $\{D_{ki}\}_{k=1}^{\infty} \{i=1}^{\infty}$ of sets in $\underline{D}$ having the properties

$$i) \quad M \subseteq \bigcup_{i \geq 1} D_{ki} \subseteq \bigcup_{i \geq 1} D_{k-1 i},$$

$$ii) \quad \sum_{i \geq 1} W'(s', D_{ki}) \leq \sum_{i \geq 1} W'(s', D_{k-1 i}), \quad s' \in \underline{S'},$$

for every $k > 1$ and such that

$$iii) \quad \text{the sets } D_{ki} \ (i \geq 1) \text{ are pairwise disjoint,}$$

$$iv) \quad \mu(\bigcup_{i \geq 1} D_{ki} \cap CM) < \frac{1}{k} \quad \text{for every } k \geq 1.$$

The proof is omitted. It is very similar to the proof of Lemma II.4.1. The main difference is that $H4$ is used in the situation here which corresponds to the situation in which $H4^*$ is used in the proof of Lemma II.4.1.

III.5.2. Lemma. Assume H9. Assume that T is ACW and let f be a g.l.b.f. W . Suppose that M is a set in $\underline{M}$ for which

$M \subseteq \ast D_j$ for some j . Let $\{D_{ki}\}_{k=1}^{\infty} \{i=1}^{\infty}$ be a double sequence of

sets in $\underline{D}$ for which conditions i) - iv) of Lemma III.5.1 hold.

Then the sequence

$$\{\sum_{i \geq 1} W'(s', D_{ki})\}_{k=1}^{\infty}$$

is convergent for s' in $\underline{S'}$. If a non-negative function g' is defined on $\underline{S'}$ by $g'(s') = \lim_k \sum_{i \geq 1} W'(s', D_{ki})$, $s' \in \underline{S'}$, then

$g'(s')$, $s' \in \underline{S'}$, is $\underline{M'}$ -measurable and

$$\int_{\underline{S}'} g'(s') d\mu' = \int_{\underline{M}} f(s) d\mu.$$

The proof is exactly like the proof of Lemma II.4.2.

III.5.3. Lemma. Assume H13. Assume that T is ACW and let f be a g.l.b.f. W . Let M be a set in $\underline{M}$ and suppose that $f(s) = 0$ a.e. μ on M . Then there exists a subset X' of $\underline{S}'$ such that $\mu'X' = 0$ and such that $W_S = 0$ whenever S is any set in $\mathcal{C}_M$ for which $S \subseteq CT^{-1}X'$.

Proof. Define $M_j = M \cap {}^*D_j$, $j \geq 1$. Then M_j is in $\underline{M}$ and $M_j \subseteq {}^*D_j$. By Lemma III.5.1 there is for each j a double sequence $\{D_{jki}\}_{k=1}^{\infty} \{i=1}^{\infty}$ of sets in $\underline{D}$ such that

$$i) \quad M_j \subseteq \bigcup_{i \geq 1} D_{jki} \subseteq \bigcup_{i \geq 1} D_{j \ k-1 \ i},$$

$$ii) \quad \sum_{i \geq 1} W'(s', D_{jki}) \leq \sum_{i \geq 1} W'(s', D_{j \ k-1 \ i}), \quad s' \in \underline{S}',$$

for every $k > 1$ and

$$iii) \quad \text{the sets } D_{jki} \ (i \geq 1) \text{ are pairwise disjoint,}$$

$$iv) \quad \mu(\bigcup_{i \geq 1} D_{jki} \cap CM_j) < \frac{1}{k},$$

for every $k \geq 1$. According to Lemma III.5.2 the sequence

$\{\sum_{i \geq 1} W'(s', D_{jki})\}_{k=1}^{\infty}$ is convergent for every s' in $\underline{S}'$ and $j \geq 1$.

We define a non-negative function g'_j on $\underline{S}'$ by $g'_j(s') =$

$$= \lim_k \sum_{i \geq 1} W'(s', D_{jki}), \quad s' \in \underline{S}', \quad j \geq 1. \quad \text{By Lemma III.5.2,}$$

$g'_j(s'), \ s' \in \underline{S}'$, is $\underline{M}'$ -measurable and

$$(1) \quad \int_{\underline{S}_j} g'_j(s') \, d\mu' = \int_{M_j} f(s) \, d\mu, \quad j \geq 1.$$

By hypothesis $\int_{M_j} f(s) \, d\mu = 0$, $j \geq 1$, and so, by (1),

$$(2) \quad g'_j(s') = 0 \quad \text{a.e. } \mu' \text{ on } \underline{S}', \quad j \geq 1.$$

Define $X'_j = \{s' \in \underline{S}' : g'_j(s') > 0\}$, $j \geq 1$. If we set $X' = \bigcup_{j \geq 1} X'_j$ then, in view of (2), X' is in $\underline{M}'$ and $\mu'X' = 0$.

Now let S_0 be any set in $\mathfrak{E}_M$ for which $S_0 \subseteq CT^{-1}X'$. Now S_0 is in $\mathfrak{E}_{M_j}$ for some j . Since TS_0 is in $C'X'$ we have

$$(3) \quad \lim_k \sum_{i \geq 1} W'(TS_0, D_{jki}) = 0.$$

Let k be any positive integer. Since $M_j \subseteq \bigcup_{i \geq 1} D_{jki}$ and

$S_0 \cap M_j \neq \emptyset$ there is an integer i_k such that $S_0 \subseteq D_{jki_k}$ (7.2 of

[2]). For each k we have $\sum_{i \geq 1} W'(TS_0, D_{jki}) \geq W'(TS_0, D_{jki_k}) \geq$

$\geq wS_0$. In view of (3), we must have $wS_0 = 0$.

III.5.4. Lemma. Assume H13. (Assume W' is generated by w .) Assume that T is ACW and let f be a g.l.b.f. W . Define $S^+ = \bigcup S$, $S \in \mathfrak{E}$ and $wS > 0$. Suppose that M is a set in $\underline{M}$ for which $M \cap S^+ = \emptyset$. Then $f(s) = 0$ a.e. μ on M .

Proof. Set $M_j = M \cap {}^*D_j$, $j \geq 1$. Since $\underline{S}' = \bigcup_{j \geq 1} {}^*D_j$ it suffices to prove that $f(s) = 0$ a.e. μ on M_j for every $j \geq 1$. Thus we fix j . Note that M_j is in $\underline{M}$ and $M_j \subseteq {}^*D_j$. By Lemma

III.5.1 there is a double sequence $\{D_{jki}\}_{k=1}^{\infty} \{i=1}^{\infty}$ of sets in $\underline{D}$ such that the properties listed as i) - iv) in the proof of Lemma III.5.3 are satisfied. According to Lemma III.5.2 the sequence $\{\sum_{i=1}^{\infty} W'(s', D_{jki})\}_{k=1}^{\infty}$ is convergent for every s' in $\underline{S}'$.

We define a non-negative function g'_j on $\underline{S}'$ by $g'_j(s') = \lim_k \sum_{i=1}^{\infty} W'(s', D_{jki})$, $s' \in \underline{S}'$. By Lemma III.5.2 $g'_j(s')$, $s' \in \underline{S}'$, is $\underline{M}'$ -measurable and

$$(1) \quad \int_{\underline{S}'} g'_j(s') d\mu' = \int_{M_j} f(s) d\mu.$$

Define $S'_j = \{s' \in \underline{S}': g'_j(s') > 0\}$ and $S'_{\infty} = \{s' \in \underline{S}': W'(s', \underline{S}) = \infty\}$.

We now wish to prove

$$(2) \quad S'_j \subseteq S'_{\infty} \cup T(S^+ \cap \bigcap_{k \geq 1} \bigcup_{i \geq 1} D_{jki}).$$

In order to prove (2) we let s'_0 be a point in S'_j but not in S'_{∞} .

Then, for $k \geq 1$, we have

$$\sum_{i=1}^{\infty} W'(s'_0, D_{jki}) = \sum wS, \quad S \in \mathcal{C}_{s'_0} \text{ and } S \subseteq \bigcup_{i \geq 1} D_{jki}.$$

Since s'_0 is in $C'S'_{\infty}$ we have

$$\sum wS, \quad S \in \mathcal{C}_{s'_0} \text{ and } S \subseteq \bigcup_{i \geq 1} D_{jki}, < \infty$$

for every $k \geq 1$. Consequently, the set $\mathcal{C}_0$ defined by $\mathcal{C}_0 =$

$\{S \in \mathcal{C}_{s'_0}: wS > 0 \text{ and } S \subseteq \bigcup_{i \geq 1} D_{jli}\}$ is countable. It cannot be

empty but it may be finite or infinite. Set $S_0 = \bigcup S, S \in \mathcal{C}_0$.

We now assert that

$$(3) \quad s_0 \cap s^+ \cap \bigcap_{k \geq 1} \bigcup_{i \geq 1} D_{jki} \neq \emptyset.$$

In order to prove (3) we must consider two cases.

Case I. Assume that $\mathfrak{S}_0$ is finite. We assume that (3) is false and derive a contradiction. List the elements of $\mathfrak{S}_0$ as $s_1, \dots, s_n$. Since (3) is false it follows that, for $u = 1, \dots, n$, there exists an integer k_u such that $s_u \not\subseteq \bigcup_{i \geq 1} D_{jk_u i}$ and hence $s_u \cap \bigcup_{i \geq 1} D_{jk_u i} = \emptyset$. Set $k_0 = \max \{k_1, \dots, k_n\}$. In view of i) $s_u \cap \bigcup_{i \geq 1} D_{jk_0 i} = \emptyset$, $u = 1, \dots, n$. Hence

$$(4) \quad s_u \cap D_{jk_0 i} = \emptyset, \quad i \geq 1, u = 1, \dots, n.$$

By i) the sets S in $\mathfrak{S}'_0$ for which $ws > 0$ and $S \subseteq \bigcup_{i \geq 1} D_{jk_0 i}$ must occur among $s_1, \dots, s_n$. In view of (4) it follows that $w'(s'_0, D_{jk_0 i}) = 0$, $i \geq 1$; hence

$$\sum_{i \geq 1} w'(s'_0, D_{jk_0 i}) = 0$$

and so $g'_j(s'_0) = 0$. But s'_0 is in S'_j . Hence we have a contradiction. Thus (3) holds in this case.

Case II. $\mathfrak{S}_0$ is denumerable. Again we assume (3) is false and derive a contradiction. List the elements of $\mathfrak{S}_0$ as $s_1, s_2, \dots$. Set $p = g'_j(s'_0) > 0$. Note that $\sum_{n \geq 1} ws_n = \sum_{i \geq 1} w'(s'_0, D_{jli}) < \infty$. There exists an integer N such that

$$(5) \quad \sum_{n \geq N} ws_n < p.$$

We are assuming (3) is false and so $U_{n \geq 1} S_n \cap S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki} = \emptyset$. Note that $U_{n \geq 1} S_n \subseteq S^+$. For each n we then have $S_n \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki} = \emptyset$. Hence for each n there exists a positive integer k_n such that $S_n \cap U_{i \geq 1} D_{jk_n i} = \emptyset$. Set $k_0 = \max \{k_1, \dots, k_N\}$. In view of i) the only sets S in $\mathcal{S}_{S'_0}$ for which $wS > 0$ and $S \subseteq U_{i \geq 1} D_{jk_0 i}$ occur among $S_{N+1}, S_{N+2}, \dots$. Using (5) we obtain

$$\sum_{i \geq 1} W'(s'_0, D_{jk_0 i}) \leq \sum_{n > N} wS_n < p.$$

Thus, by ii), $g'_j(s'_0) \leq \sum_{i \geq 1} W'(s'_0, D_{jk_0 i}) < p$. But this is a contradiction. Hence (3) is valid in this case also. With (3) established, (2) follows at once.

We now examine the sets S'_∞ and $T(S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki})$. Note first that S'_∞ is in $\underline{M}'$ and, since T is BVW, we have

$$(6) \quad \mu(S'_\infty) = 0.$$

Observe that $S^+ \subseteq CM_j$ and so $S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki} \subseteq \bigcap_{k \geq 1} U_{i \geq 1} D_{jki} \cap CM_j$. In view of iv) and i) we have $\mu(\bigcap_{k \geq 1} U_{i \geq 1} D_{jki} \cap CM_j) = 0$.

Hence $S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki}$ is in $\underline{M}$ and $\mu(S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki}) = 0$.

By Theorem 7.24 of [2] it follows that $T(S^+ \cap \bigcap_{k \geq 1} U_{i \geq 1} D_{jki})$ is in $\underline{M}'$ and

$$(7) \quad \mu'(T(S^+ \cap \bigcap_{k \geq 1} \bigcup_{i \geq 1} D_{jki})) = 0.$$

It now follows from (2), (6), and (7) that

$$(8) \quad \mu'(S'_j) = 0.$$

It follows from (8) and (1) that $f(s) = 0$ a.e. μ on M_j .

III.5.5. Theorem. Assume H9 and suppose in fact that W' is generated by a non-negative function w defined on $\mathcal{G}$. Set $S^+ = US$, $S \in \mathcal{G}$ and $wS > 0$. Assume that S^+ is in $\underline{M}$. (In view of Corollary III.2.5 this last assumption is not so restrictive.) Assume also that T is ACW and let f be a g.l.b.f. W . Then $f(s) = 0$ a.e. μ on CS^+ . There exists a set X' in $\underline{M}'$ such that $\mu'(X') = 0$ and such that $f(s) > 0$ whenever s is in $S^+ \cap CT^{-1}X'$.

Proof. The set S^+ is in $\underline{M}$ and hence CS^+ is in $\underline{M}$. By Lemma III.5.4, $f(s) = 0$ a.e. μ on CS^+ . Now define $M_0 = \{s \in \underline{S} : s \in S^+ \text{ and } f(s) = 0\}$. Since S^+ is in $\underline{M}$ and $f(s)$, $s \in \underline{S}$, is $\underline{M}$ -measurable it follows that M_0 is in $\underline{M}$. Surely $f(s) = 0$ on M_0 and so Lemma III.5.3 applies. There exists a set X' in $\underline{M}'$ such that $\mu'(X') = 0$ and such that $wS = 0$ whenever S is in $\mathcal{G}_{M_0}$ and $S \subseteq CT^{-1}X'$. It is easily verified that $M_0 \subseteq T^{-1}X'$. It follows that $f(s) > 0$ whenever s is in $S^+ \cap CT^{-1}X'$.

SUMMARY

We shall now repeat the most important results of this paper. These principal theorems will be reviewed in a rather informal manner. Precise statements of them have of course been given in the text. Comments regarding possible extensions of them and concerning related problems are interspersed with the statements of these principal results.

Theorems II.1.13 and II.1.14 comprise the main result of Section II.1. Actually, these theorems present the same result in two different forms. They are proved under the same assumptions; for each of them we assume H1-H9, HC4, and that W'_A is a finite valued AC part of W' . We prove in Theorem II.1.13 that there is a weight function $W'_\#$ for T such that $W' = W'_A + W'_\#$ a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. In Theorem II.1.14 the role of W'_A is played down somewhat in order to produce a modified conclusion; indeed we obtain two new weight functions W'_a and $W'_\#$ for T with the properties that $W' = W'_a + W'_\#$ and that W'_a is an AC part of W' which agrees with W'_A a.e. μ' on $\underline{S'}$ uniformly with respect to $\underline{D}$. Both of these results are presented because in a given situation the one may be more convenient than the other. Each is a "decomposition" result in its own right; together, they comprise an indispensable preliminary for the main decomposition theorems.

The remarks in II.1.10 and in II.63 suggest that it may be possible to remove HC4 from Theorems II.1.13 and II.1.14—and hence from the theory—by revising H9. Such a revision might prove fruitful. It is a problem that is likely to be attacked rather soon.

Next, we discuss the General Decomposition Theorem. The theorem can be stated as follows: If H1-H9 and HC4 are assumed and if T is assumed to be BVW, then W' can be expressed as the sum of two weight functions, one of which is an AC part of W' and the other of which is singular. This decomposition requires of W' only that T be BVW. Hence the requirements concerning W' are all but minimal. Unfortunately, the General Decomposition Theorem is weak because singular weight functions have never been described very well. The goal of the discussion which leads to the Strong Decomposition Theorem is precisely the discovery of properties possessed by singular weight functions. At the outset the specific aim was the discovery of some sort of relationship between singular weight functions (for T) and the transformation T itself. As a preliminary, the writer attempted to discover such a relationship for those singular weight functions which happen to satisfy H13. Theorem 7.24 of [2] proved to be an invaluable aid in this restricted situation. It was then natural to seek a counterpart of Theorem 7.24 which does not require H13. The examples in II.7 suggest in several ways that such a counterpart may be difficult to obtain. It remains a worthwhile (but unrealized) goal. For the purposes of this paper however, it is sufficient to work with a

rather modest counterpart - Theorem II.3.2 - of 7.24. Theorem II.3.2 asserts roughly that if we assume $H1-H9$, $H7^*$, and $HC1$, then, if W is over additive, T is BVW, and if T transforms certain sets of measure zero into sets of measure zero, it is true that T is ACW. An example in II.7 reveals that T can be ACW and yet not transform these certain sets of measure zero into sets of measure zero. In other words, the most natural converse of Theorem II.3.2 fails to hold. A better theorem of this type would be something like the following: Assume that T is BVW and that W is over additive; then T is ACW if and only if it transforms certain sets of measure zero into sets of measure zero. In view of Example II.7.5, it would seem to be very difficult to remove the overriding assumption that W be over additive from such a theorem.

Our strongest theorems concerning Lebesgue-type decompositions are Theorem II.5.2 and Corollary II.5.3. The following paragraph consists of a rather informal statement of these two closely related theorems.

Assume $H1-H9$, $H4^*$, $H7^*$, $HC1$, and $HC2$. Assume that T is BVW and that W' is almost over additive. Then every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' ; the singular component of such a decomposition must be almost over additive. Assume now that $HC4$ also holds. Then W' does admit a strong Lebesgue-type decomposition; in other words, W' can be written as the sum of two weight functions for T , one of which is an AC part of W' and the other of which is not only singular but also vanishes almost everywhere outside of the image of a subset of $\underline{S}$ having measure zero.

Thus we obtain a strong Lebesgue-type decomposition only for weight functions W' which are almost over additive and for which T is BVW. The weakening of the former requirement is a desirable goal. The example in II.7.5 shows that it cannot be dropped entirely. Indeed it would be worthwhile to establish conditions on W' which are necessary and sufficient for it to have a strong Lebesgue-type decomposition. Such a necessary and sufficient condition might prove valuable in other circumstances. Imposition of such a condition on weight functions in general might eliminate such "pathological" cases as the one described in II.7.5.

Chapter III contains theorems about the important special class consisting of those weight functions which satisfy H13. We discuss some measurability questions in the first part of Chapter III. The results are these. We first prove that if we assume H1-H8, H15i, HCl, and that w is a non-negative function defined on $\mathcal{G}$ whose induced point function g_w is $\underline{B}$ -measurable, then the summatory function generated by w is a weight function for T . This theorem can be applied to obtain a second valuable theorem. It is the following: If we assume H1-H9, H15i, and HCl and if we define a non-negative function $\hat{w}$ on $\mathcal{G}$ by

$$\hat{w}S = \inf \{W'(TS, D) : S \subseteq D \in \underline{D}\}, S \in \mathcal{G},$$

then there exists a non-negative function w defined on $\mathcal{G}$ whose induced point function is $\underline{B}$ -measurable and which agrees with $\hat{w}$ on the complement of the inverse of a set of μ' measure zero; moreover, the summatory function generated by $\hat{w}$ is a weight function

for T . There is a useful corollary to these two theorems. Its most prominent application is based upon the following vaguely stated feature: The class of weight functions satisfying $H13$ and having B -measurable induced point functions is "dense" in the class of all weight functions satisfying $H13$.

The next main result is the Special Decomposition Theorem. It can be obtained in two ways. The better result is this: If $H1-H8, H13, HC1$, and $HC4$ are assumed and if T is BVW , then every Lebesgue-type decomposition for W' is a strong Lebesgue-type decomposition for W' ; the components of such a decomposition "essentially satisfy" $H13$. Moreover, there does exist a strong Lebesgue-type decomposition for W' , the components of which satisfy $H13$.

We then establish several additional facts about the decomposition given by the Special Decomposition Theorem. Indeed it is shown that if $H1-H8, H13, HC1$, and $HC4$ are assumed, then the AC component of a Lebesgue-type decomposition for W' must be a strongly maximal AC part of W' ; it follows readily that there is in essence only one Lebesgue-type decomposition for W' in this case.

The results reviewed in the preceding paragraph strengthen substantially the conclusions of the Special Decomposition Theorem. We ask immediately whether we can establish similar results in conjunction with the General and Strong Decomposition Theorems. These problems are unsolved. Solution of them would surely give the General and Strong Decomposition Theorems a more definite character.

Finally, we repeat the principal conclusion of III.5.

Roughly speaking, it is shown that if H1-H8 and H13 are satisfied, if T is ACW, and if f is a g. l. b. f. W , then the functions g_w and f are either both zero or both positive at all points except those in the union of some set of μ measure zero and some set whose image under T has μ' measure zero. This theorem occupies a rather solitary position at present. No applications of it have been made.

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