STUDENT DISCOVERY OF ALGEBRAIC PRINCIPLES AS A
MEANS OF DEVELOPING ABILITY TO GENERALIZE

Dissertation

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CHAPTER I

INTRODUCTION

The Problem

One problem confronting secondary schools is the determination of a suitable mathematics program for students in the ninth grade. This problem becomes more acute when all ninth grade students are enrolled in the same mathematics class. In such a case it is important that both the content and the methods of teaching have a wealth of educational value for all students. The task of this dissertation is to present such a single mathematics course which has as its central theme the development of students' ability to generalize.

The proposed program is an outgrowth of experimental investigation into the teaching of algebra at the University School, The Ohio State University. Instructional procedures were such as to encourage students to extend their previously acquired mathematical principles, and to discover new algebraic principles from their present and past experiences. Throughout the course, emphasis is placed on student discovery. The teacher's role is to organize material and to direct class thinking so that the students can discover generalizations. He also has the

responsibility for guiding students to a realization that essentially the same method is used in forming generalizations in other areas of learning as in formulating mathematical generalizations. This, of course, involves the question of transfer to which careful consideration should be given in teaching procedures. The mathematical content of the course was selected so as to:

1. Illustrate various generalizing procedures
2. Present an overview of mathematics and to show the interrelatedness of its many branches
3. Include topics the students make use of now and most likely will use in the future
4. Review and extend arithmetic understandings.

Although these criteria were used as a basis of selection no organized attempt is made to determine which content most closely adheres to them. The primary concern in this study is the teaching procedures used in presenting the content to the students.

In the development of this course, the following tasks were assumed to be the most significant:

1. Analyzing the different processes of generalizing and determining the characteristics of a superior generalizer
2. Designing lesson sheets and other procedures which can be used in aiding students to develop their ability to generalize in both mathematical and non-mathematical situations

3. Evaluating in terms of the characteristics of a superior generalizer any improvement made by students enrolled in the experimental class
4. Evaluating the mathematical achievement of the experimental class.

A measure of the improvement made by the experimental students in their ability to generalize was obtained from the results of a generalization test given at the beginning and end of the school year. Since it is possible that any change the experimental students might undergo could be the result of their normal maturation, the generalization test was also given at approximately the same time to a group of algebra students from several algebra classes in the Columbus Public Schools. In the study this group of students is referred to as the status group. Conclusions concerning the effectiveness of the teaching procedures in the development of generalizing abilities are then made in terms of the test results from both groups of students. Other sources of data for evaluating the experimental course* were observer reports, student notebooks, the writer's notes concerning each class meeting, and responses on student reaction sheets. A measure of the mastery of mathematics is also judged in terms of tests given the experimental and public school classes at the beginning and end of the year.

*These other sources do not include data from the status group.

Background

During the first quarter of the century algebra was almost universally used as the mathematics course for all ninth grade students. Around 1930 Betz seriously questioned the value of algebra as it was being taught, and in his plea for revision of both content and teaching methods offered these four basic criticisms:

1. Absence of a central theme
2. Curse of "Empty Symbolism"
3. Imitative, mechanical learning
4. Dissociation of technique from applications¹

Algebraic content has been greatly revised in the past twenty-five years. Nevertheless, in considering any changes to be made in the teaching of algebra, Betz's criticism should still be seriously considered. An examination of algebra textbooks for the past half century suggests another major obstacle, namely, dissociation from major general education objectives, which prevents algebra from being considered a part of general education.

A further search into the literature on the teaching of mathematics reveals that even before the time of Betz, leading educators² were giving positive advice on the im-

¹Betz, William, "Whither Algebra," The Mathematics Teacher, XXXII (February, 1930), pp 104-05.

²Perry, John, "The Teaching of Mathematics," Educational Review, 28 (February, 1902), pp 158-181.
 Moore, E. H., "On the Foundations of Mathematics," First Yearbook of the National Council of Teachers of Mathematics, 1926, pp 32-57.

provement of teaching mathematics. If this advice had been heeded Betz's criticism probably would never have been made. These suggestions were:

1. Students should be led to discover as many mathematical principles for themselves as possible.
2. The high school mathematics curriculum should be reorganized so that the most useful concepts be taught first. The reorganized curriculum should disregard the conventional boundaries of algebra, geometry, solid geometry, trigonometry, analytic geometry, and calculus.
3. Applications should occupy a much more dominant role at all levels.

Twenty years ago Suzanne Langer, a professor of philosophy at Radcliff College, clearly indicated a direction algebraic instruction could follow if it was to have general education value. The flavor of her remarks is reflected in the following quotations:

For algebra, rightly understood, gives us much more than a technique for computing certain scientific problems; it increases our power of conception to an astounding degree, and liberates us from the concrete, visual and tactual habits of thought that bind the naive mind so closely to Mother Earth.³

No other school subject is designed to expand our mental powers in this direction; the whole task of training the discursive reason, which lifts man above the cleverest of other creatures, devolves upon the teachers of algebra.⁴

³Langer, Suzanne K., "Algebra and the Development of Reason," The Mathematics Teacher, XXIV (May, 1931), p 286.

⁴Ibid., p 288.

If we were not in too great a hurry to get to mechanical devices in algebra and "do examples," any child of high school mentality could learn the principles of correct generalization.⁵

Whether algebra can be taught on these principles to the average adolescent mind, I do not know; the scheme may be Utopian. Only careful pedagogical experiment can decide such a question. But if algebra cannot be imparted in this or some similar way to everybody, then it should not be a required subject in a general arts course.⁶

The emphasis of instruction in algebra, according to Miss Langer, should be on developing students' ability to generalize. It seems that such an emphasis should not only serve as an excellent central theme for the teaching of algebra, but also aid in a further elimination of the other three criticisms mentioned by Betz and in the implementation of the three suggestions offered by leading educators for the improvement of mathematics. A hypothesis of this dissertation is that the ability to generalize can be developed if students are provided with mathematical and non-mathematical situations in which they are able to generalize and at the same time determine the validity of their generalizations. The number of opportunities for discovering mathematical principles to be learned are greatly increased as students generalize with mathematical content. To become increasingly more proficient in discovering mathematical generalizations students are almost compelled to search for meanings behind the symbols they are using. If students

⁵Ibid., p 296.

⁶Ibid., p 295.

discover, at least to some degree, the mathematical principles they learn, and search for meanings of the symbols they use, it is not likely that their learning will be imitative and mechanical. Applications aid in developing generalizing ability; they provide many settings for discovering mathematical generalizations and have motivating power in that students are greatly encouraged when they realize their discoveries are of practical importance.

The development of the ability to generalize, from the point of view of the mathematics educator, seems to be an excellent central theme for the teaching of algebra. Also to be considered are the general purposes of the school which provide the setting and educational climate of the experiment. The general purposes of the University School are based upon a conception of the nature and meaning of democracy and are summarized as follows:

A. Continuous Curriculum Experiences Directly Related to Democratic Values

1. Developing social sensitivity
2. Developing co-operativeness
3. Developing the ability and zeal to utilize the method of intelligence in solving all problems of human concern
4. Developing creativeness
5. Developing skills in democratic living
6. Interpreting democracy
7. Developing self-direction

B. Continuous Curriculum Experiences Implied by Democratic Values

1. Developing communication skills and appreciations
2. Developing skills in measurement and the use of quantitative symbols
3. Developing skills in utilizing goods and services
4. Promoting social adjustments
5. Promoting health and safety
6. Developing vocational adjustments and standards
7. Developing adequate recreational outlets
8. Developing standards of personal appearances and grooming⁷

Developing the generalizing ability of students seems to be closely associated with "developing creativeness," "developing self-direction," and "developing the ability and zeal to utilize the method of intelligence in all human concerns." If group discovery of generalizations is used as an important teaching procedure, also concomitant with the pursuit of the central theme are certain aspects of "developing co-operativeness" and "developing skills in democratic living." Even a cursory examination of the above purposes rather conclusively indicates that development of generalizing ability is consistent with the purposes of the University School.

⁷The Faculty of the University School, The Philosophy and Purposes of the University School. Columbus, Ohio: The Ohio State University, 1948, pp 9-11.

Definition of Terms

Student discovery, algebraic principles, and generalize are key words and phrases in this study. Student discovery is a teaching procedure in which students are given the type of guidance and mathematical experiences which will enable them to become conscious of the emergence of "new" mathematical concepts and principles from their past and present learning. It should not be assumed that student discovery means students are expected to rediscover the mathematics it took centuries for the human race to discover. The teacher has an important role in the learning process in that he seeks to remove obstacles that prevent students from generalizing, and encourages them by supplying timely hints in order that they may explore the value of mathematics to the extent their capabilities will allow.

Algebraic principles are the mathematical generalizations which can be classified under the four criteria for selecting the content studied by the experimental class. A listing of the algebraic principles studied can be found in Chapter V which discusses the content in considerable detail. The final word, generalizing, is difficult to define because it has so many shades of meaning. For this reason the next chapter is devoted to the analysis of many of the ways in which the term is used.

Basic Assumptions

Every experimental study is based on certain assumptions. Those for this study are listed below.

1. An objective of major educational value to all students is the development of their ability to generalize.

2. The growth of the experimental class in the ability to generalize, had they been taught in the conventional manner, would have been little different than that of the status group.

3. Any significant improvement made by the experimental group when compared with the status group is partially due to the teaching procedures in the mathematics classroom.

Limitations of the Study

1. The status group was composed of students in the Columbus Public Schools. No elaborate procedures were used in selecting students for the status group so that they were equal in ability to students in the experimental group.

2. The experimental class was conducted in a school where all teachers seek to improve their students' ability to generalize. Therefore, gains made by the experimental class cannot be said to be due entirely to teaching procedures used in any single class.

3. The experimental class consisted of only a small number of students.

4. The estimate of the ability of the status group to generalize is derived from scores from one evaluation device, a paper and pencil generalization test.

5. The generalization test measured only four of the fifteen abilities and behaviors that are made explicit in Chapter II.

6. The estimate of the mathematical achievements of the status and experimental groups is obtained from results on a forty-minute test given at the beginning of the year and another forty-minute test given at the end of the year.

7. Teachers in the public school, since they knew the purposes of this study, may also have made a special effort to improve their students' generalizing abilities. No attempt was made to secure information about their teaching procedures.

Related Studies

Several studies have been made in the field of algebra which helped in the preliminary speculation of the course to be described in this dissertation. The most recent study to be mentioned here is the one carried out by Jackson.⁸ In his two classes, composed of students whose

⁸Jackson, W. N., "The Role of Algebra in the Development of Relational Thinking," Unpublished Doctoral dissertation, The Ohio State University, 1952.

I.Q.'s were generally below normal, he employed methods of student discovery. General theme of the content was upon the discovery of relationships in non-mathematical materials as well as mathematical. The aspect of relationship which he evaluated was that of interpreting data. His classes showed a significant gain at the end of the school year in this particular ability. No control group was used.

Willet attempted quite a radical but interesting departure from the usual procedures of teaching ninth grade mathematics. The central theme was problem solving. Solution of problems, such as, "What is the Federal Wage-Hour law?" "How does the Federal Income Tax work?" and "When does water boil?" were sought. Mathematics was brought in as it related to the problems. These problems were not selected by the students, but determined in advance by the teacher. Control and experimental groups were used in the experiment. Willet's conclusions were:

1. Pupils' ability to analyze materials of a non-mathematical nature can be improved if instruction is directed toward that end.

2. Experimental group, although of less ability before the course was given, proved to be better able to handle verbal problems, but this was done at the sacrifice of mechanics.⁹

⁹Willet, William M., "New Objectives for Ninth Grade Mathematics: An Exposition and Appraisal," Unpublished Doctoral dissertation, Temple University, 1944.

Shuster's study,¹⁰ another one involving experimentation, reports the success of a course in algebra organized around functional thinking, the theme advocated by the National Committee on Mathematical Requirements,¹¹ and, to some extent, around the method of student discovery. The experimental group made as much progress in algebra as the control group, but considerably more improvement in functional thinking as measured by the then new Progressive Education Association tests of the Eight-Year Study.

Another group of studies has been made which did not involve planned experimentation but which proved helpful. The best known of these is the analytical study made by Hamley.¹² He first analyzes and sketches the history of the function concept and then outlines a course consistent with his analysis. The study includes tests on mathematical relations and a statistical analysis of the results given before students began the study of algebra. No suggestions were made concerning the teaching procedures to be used in promoting functional thinking.

¹⁰Shuster, John F., "A Study of Functional Thinking in Elementary Algebra at Shaker Heights," Unpublished Master's thesis, The Ohio State University, 1938.

¹¹The Reorganization of Mathematics in Secondary Education, North Association of America, 1923.

¹²Hamley, Herbert R., "Relations and Functional Thinking in Mathematics," Ninth Yearbook of the National Council of Teachers of Mathematics, 1934.

Several master's theses at The Ohio State University have been written which, in general, agree with the approach of the teaching of algebra as advocated in this dissertation. James Despinasse¹³ advocates the method of induction with strong emphasis on functional relation. His study includes many guide sheets illustrative of the approach he advocates. Verna Newman¹⁴ made a similar study. Her emphasis was on induction with non-mathematical materials. Bernice Haywood's study¹⁵ placed primary emphasis on the development of meaningful generalizations through the use of concrete experiences for understanding abstract meanings. The thesis also includes sources of materials and suggestions for a mathematics laboratory.

The above studies stress student discovery by inductive procedures. Student discovery is defined in a somewhat different way in this dissertation. Inductive procedures in the above studies meant generalizing many facts concerning particulars into one inclusive statement. Such a definition from one point of view is not broad

¹³Despinasse, James, "Building Concepts Through the Process of Induction," Unpublished Master's thesis, The Ohio State University, 1946.

¹⁴Newman, Verna, "The Teaching of Inductive Reasoning Through the Channel of Algebra," Unpublished Master's thesis, The Ohio State University, 1945.

¹⁵Haywood, Bernice, "Laboratory Procedures in the Study of Algebra," Unpublished Master's thesis, The Ohio State University, 1947.

enough to include all the ways by which mathematical discoveries can be made. For example, it would exclude discoveries students can make by deduction. From another point of view the definition is too inclusive. It does not define the process well enough to give direction to teachers who are interested in the details of procedure. Because of the vagueness of the expression, induction is not used to describe procedures in this study. It is the purpose of the next chapter to determine with at least some degree of clarity the meaning of generalizing and to show how it is employed in both mathematical and non-mathematical contexts.

CHAPTER II

AN ANALYSIS OF THE PROCESSES OF GENERALIZING

Before it is possible to make a concerted effort to improve the generalizing ability of students, something must be known about the process itself. Generalizing is defined as the process used in arriving at generalizations. Such a definition focuses the attention on the result rather than on the process of generalizing. A generalization, as defined by Max Black is "an assertion about some or all members of a class of objects."¹ Two types of generalizations are suggested by this definition. The first, an assertion about all members of a class of objects, Black calls uniform generalizations.² Examples of this type are: Acid turns blue litmus red, Metals expand when heated, and All crows are black. Generalizations of this type are the backbone of science. The second, an assertion about some members of a class, he calls statistical generalizations.³ These are more numerous and play a large role in the social

¹Black, Max, Critical Thinking, An Introduction to Logic and Scientific Method, New York; Prentice Hall, Inc., 1946, p 256.

²Ibid., p 297.

³Ibid., p 298.

sciences and every-day activities. Examples such as "Fishermen visit certain streams because they usually find fish there," "Doctors prescribe penicillin because it brings relief in a majority of the cases of infection," should bring many more to the mind of the reader. Uniform generalizations are exact and therefore accurate in their predictions concerning the class and individuals of the class, and lend themselves nicely to mathematical treatment. Statistical generalizations are more troublesome to use and there is less likelihood of obtaining successful results when making predictions, particularly for individuals of the class. But they do possess the advantage of being easier to establish and of not being destroyed by only one contradictory instance.

To analyze the processes of generalizing means to analyze the ways by which generalizations are reached. Methods of arriving at generalizations, regardless of whether the generalizations are true or false, are to be considered in this chapter. A student can improve his ability to generalize by developing techniques of discovery and by developing a desire for proof before accepting generalizations. Of primary concern here, however, is an analysis of the discovery aspect in generalizing. Proof is referred to only as needed in the discussion and is generally limited to answering such questions as: "Does this generalization make sense?" "Does it work when I apply it to the world around me?"

What are the ways by which generalizations are suggested? These ways can be divided into two broad categories, one way being empirical, the other rational. Therefore, in speaking of generalizations, those reached by empirical means will be referred to as empirical generalizations;⁴ those reached by rational means, as rational generalizations.

Empirical Generalizations

Empirical generalizations are the most basic and also the most precarious type. They are basic because they are suggested from an examination of observable facts.⁵ Science, as we know it today, began when scientists considered as a major approach to research the discovery of generalizations from observable facts. Empirical generalizations are basic also in another way. The study of the facts of any science would be a near impossibility if they were not bound together by generalizations. It is much easier to remember generalizations than a variety of isolated facts, and generalizations are more universally applicable to different situations.

Empirical generalizations are precarious because they are often accepted as true before they are carefully verified. In the scientific laboratory, careful verification

⁴Stebbins, L.S., A Modern Introduction to Logic, New York; The Humanities Press, 1933, p 242.

⁵Ibid., p 230-31.

is the rule rather than the exception and even here the history of science shows that many of these generalizations had to be discarded as the science advanced. Examples of a few of these discarded generalizations are: Nature abhors a vacuum, All swans are white, Cold weather will cause perennial plants to enter the dormant state, and Heat is a fluid. As a rule, outside the scientific laboratory and other research centers only feeble attempts are made to verify generalizations used. Therefore, the chances that these generalizations even approach the truth are extremely small. Examples of such generalizations are: Men drivers are better than woman drivers, Negroes are naturally less intelligent than white people, and People with red hair are hot-tempered.

Rational Generalizations

Before proceeding with the discussion of rational generalizations, it will be helpful to illustrate the meaning of the expression, structure⁶ of a generalization. This structure has both material⁶ and form.⁶ In a certain sense, this structure can be compared to a building. The architectural drawings or floor plans are the pictures of the form of the structure; the substance such as brick, wood, and cement, is the material. A point to remember in this analogy as the discussion proceeds is that form and materials are inseparable, one does not exist without the other. Yet it is

⁶Terms taken from Max Black, op.cit., p 42.

possible to analyze either separately, largely, if not entirely, ignoring the other. It is possible for the form in two situations to be the same, but the material different. In like manner, it is possible for the form to be different but the material the same. Carrying the above analogy a step forward to include these two possibilities, one could say: It is possible for two houses to have the same architectural drawings (form) yet the siding material of one be brick and the other stucco. On the other hand, it is possible for the material of two houses to be brick yet the architectural drawings (form) be different. In the realm of generalizations an example of constancy of form is:

All barns are red

All crows are black

Both statements are of the form

All x are y

where x is a class of objects and y is the color of an individual in the class. The material in the first case, barns and red, is obviously different from the material in the second, crows and black.

An example of constancy of material in two generalizations is

Dogs chase rabbits

Dogs eat rabbits

The form in the first case:

x chase y

the second case:

x eat y

where x is the class of all dogs and y the class of all rabbits.

Rational generalizations are suggested entirely by generalizations already formed or by generalizations and observable data. The material is more dominant in observable facts than in generalizations. Rational generalizations are further removed from observable facts than are empirical generalizations. The material is in the background and the form is dominant. For example, data (material) collected by Tycho Brahe led Kepler to the empirical generalization that the movement of Mars followed an elliptical path with the sun situated at one of the foci. It was only natural for Kepler to make this discovery sufficiently general to include all the planets moving around the sun. From this more inclusive generalization, Newton deduced the rational generalization that the intensity of the attractive forces between the sun and a planet vary inversely as the square of the planet's distance from the sun. This deduction was possible because of the logical form into which the previous observable facts of Tycho Brahe and others fit.⁷

Form, whether mathematical, logical, musical, or otherwise, rather than the material, plays a more dominant

⁷For a more detailed account of these discoveries see Ball, Sir Robert, Great Astronomers, Philadelphia, Lippencott and Co., 1895, pp 133-134.

role in rational generalizations than in empirical generalizations. A detailed discussion concerning the role of form will follow later when the sub-types of both rational and empirical generalizations are mentioned. The faith that the universe is orderly,⁸ and the fact that rational generalizations are often made and at some later date observable facts are found which verify a given generalization, account for the importance of rational methods in science and mathematics.

Role of Empirical and Rational Generalizing Methods
in the Discovery of Mathematics

Many believe the difference between mathematics and natural science is that mathematics concerns itself with rational and disregards empirical generalizations. This may be partially true, but the real difference is not that empirical generalizations are never useful, but that mathematical generalizations need not be substantiated with observable facts to be of importance to the mathematician. His first concern is the logical consistency of his generalizations with those he has previously accepted. Logical consistency is accomplished by focusing attention on the mathematical or, more broadly speaking, the logical form of his generalizations. Questions foremost in the mathematician's mind are: "Is this generalization deducible from generalizations I already know?" "What are further implications of this generalization?" "From what more inclusive

⁸ Whitehead, A.N., Science and the Modern World, New York, The Macmillan Co., 1925, p 20.

generalizations is this theorem deducible?" The difference between the mathematician and the scientist is that while the mathematician's problem is solved when this consistency is established, the scientist must take a further step. He must determine by various means whether the generalization actually includes all related observable facts and whether it can be used to predict facts that are capable of observation.

The mathematician is, nevertheless, concerned about the relation of his generalizations to observable facts of natural phenomena. The history of mathematics clearly shows that many of the important problems solved by him originated with observable facts. Kasner and Newman⁹ claim the branch of mathematics called Topology was started when Euler solved the Konigsberg bridge problem and then extended the theory so he could solve all kinds of problems of the same type. Likewise, Fermat and Pascal presented the first papers on the theory of probability after certain gambling problems were proposed to Pascal by some fellow countryman.¹⁰ In these situations the first step these mathematicians seek is to express the observable facts in mathematical terms. By so doing, the relationships between the variables (form) become

⁹Kasner, Edward, and Newman, James, Mathematics and the Imagination, New York: Simon and Schuster, Inc., pp 260-61.

¹⁰Smith, David Eugene, History of Mathematics, Boston: Ginn and Co., 1925, p 529.

prominent and the individual character of the variables fade into the background. These relationships are then studied for further implied relationships, and these latter relationships for still further relationships and so on until a new branch of mathematics is fairly well formulated. If the resulting new generalizations, when put to the test, prove they also describe observable facts, then the new mathematics has applications. But even if the new generalizations do not describe observable facts, the time of the mathematician still may not have been wasted. In the future these generalizations may find applications. Because of this latter possibility, institutions in our society which promote research find they can afford to let mathematicians spend at least part of their time developing their science with little regard for observable facts. There are many examples of branches of mathematics that were developed long before any applications were known. The theory of conic sections was developed by the Greeks nearly two thousand years before Kepler made use of it in describing the motion of Mars around the sun. Also, the theory of complex numbers was well developed by mathematicians before they were used by Maxwell to suggest the existence of the wireless telegraph.¹¹ And again, Riemannian Geometry was gathering dust on a shelf before Einstein found it useful in the

¹¹Shaw, James B., Lectures on the Philosophy of Mathematics, Chicago: The Open Court Publishing House, 1918, p 1.

formulation of the theory of relativity. However, most mathematicians are paid primarily to develop mathematics with applications immediately apparent or to adapt mathematics already developed to the situation at hand. The mathematician therefore is concerned that mathematical generalizations form a structure which closely corresponds to the structure of developing science, whether the science be business, chemistry, engineering, or psychology. For the closer the correspondence, the greater is the applicability of mathematics in making predictions and solving problems.

The relation between empirical and rational procedures as it relates to the development of mathematics has been the subject of controversy for 2500 years. However, present-day mathematicians regard mathematics as being the result of the interrelation of both procedures and that research is the most fruitful when neither procedure is used in isolation. As Katsoff summarizes it:

Mathematics and its application furnish an argument neither for pure empiricism nor for pure rationalism. It rather shows the close interrelation of the two in a unique fashion. The development of mathematical theory and the investigation of the roots of mathematics have shown the complementarity of both approaches to each other. Critical and speculative philosophy of mathematics and pure and applied mathematics are both aspects and not separate spheres.

The concept implied here is a dialectical interpretation of the relation of mathematics to reality. They are not to be considered to be separate realms but rather two aspects, to know either of which is to lead to the other. Mathematics may be abstracted and dealt with deductively as a static body of propositions, but to understand

it as a phenomena it is necessary to treat it dynamically--as leading to other fields. The same thing holds true of the study of reality. Mathematics is not merely a body of symbols, but is a construct of humans which is evolved in a society and which has relation to empirical events.¹²

The following quotation by Courant and Robbins also gives importance to non-rational procedures:

It sometimes happens that such a concentration on the formal structure rather than on the intuitive meaning of the concepts makes it easier to find generalizations and applications that might have been overlooked in a more intuitive approach. But a significant discovery or an illuminating insight is rarely obtained by an exclusively axiomatic procedure. Constructive thinking, guided by the intuition*, is the true source of mathematical dynamics. Although the axiomatic form is an ideal, it is a dangerous fallacy to believe that axiomatics constitutes the essence of mathematics. The constructive intuition of the mathematician brings to mathematics a non-deductive and irrational element which makes it comparable to music and art.¹³

Shaw's quotation below shows that he recognizes the importance of observable facts in experience.

Mathematics is a creation of the mind and it is not due to the generalization of experiences or to their analysis, nor is it due to an innate form or mold which the mind compels experience to assume, but it is the outcome of an evolution, the determining factors of which are the creative ability of the mind and the environment in which it finds the problems which it has to solve in some manner to some degree.¹⁴

¹²Katsoff, Louis O., A Philosophy of Mathematics, Iowa State College Press, 1948, p 253.

*It is not altogether clear what the author means by intuition, but it does include empirical generalizations.

¹³Courant, Richard, and Robbins, Herbert, What is Mathematics?, New York: Oxford Univ. Press, 1941, p 216.

¹⁴Shaw, op. cit., p 30.

Whether the mathematician likes it or not he does not transcend to any great extent empirical generalizations concerning the world of reality. Empirical generalizations concerning reality give him clues for the discovery of mathematics he does not know and inform him when a sphere of mathematics corresponds to generalizations concerning reality.

A closeness to observable facts of the natural phenomena suggests new areas of research for mathematicians and new possibilities for application of mathematical concepts. Use of empirical procedures, within the symbolism of mathematics, can also suggest to the mathematicians generalizations to be proved mathematically. For example, in number theory a study of the mathematical observable facts, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, $1 + 3 + 5 + 7 + 9 = 25$, ----- suggests the statement, "Any number that is a perfect square can be expressed as the sum of successive odd numbers." Also, repeated experience with closed curves would suggest even to the layman the topological theorem (Jordon curve theorem) "A simple closed curve C in a plane divides the plane into exactly two domains, an inside and an outside."¹⁵

It is hoped that the above discussion shows that even the mathematician, who is so deeply concerned with logical accuracy, also makes use of often logically inaccurate empirical procedures.

¹⁵Courant and Robbins, op. cit., p 245.

Simple Enumeration As An
Empirical Generalizing Method

The process of discovering empirical generalizations can be classified for the purpose of discussion into four distinct methods. These methods are given the names of simple enumeration, analogy, extension of form (extrapolation and interpolation), and statistical.

Generalization by simple enumeration is an assertion made from a number of examined instances which are not assumed to constitute all the instances of the given class.¹⁶ This method could be symbolized in the following manner:

I_1, I_2, I_3, I_4 ----- suggests G_I
where I_1, I_2, I_3, I_4 ---- are all members
of class I tried and G_I is a generalization
concerning all the members in class I.

This symbolization implies that no contradictory instance was found. Generalizations that have been made and verified in this way are: All crows are black, Iron rusts in moist air, and The boiling point of alcohol at sea level is 78.3° C. Generalizations that have been made and have been proven false by contradictory instances are: All swans are white, All feeble-mindedness is hereditary, and All redheads have bad tempers.

The truth of a generalization by simple enumeration depends upon the number of cases and upon an analysis of the

¹⁶Stebbins, op. cit., p 245.

types of cases in the class concerned. In general, as the number of cases increases, the more certain the generalization. However, the number of cases in itself is of little value in arriving at certainty if it does not include some cases of each type in the class; in other words, cases in the sample must be representative of all the cases in the class. The number of confirmatory instances strengthens any empirical generalization, whether uniform or statistical. However, it takes only one contradictory instance to destroy a uniform generalization.

An example taken from an algebra class will illustrate how discoveries in algebra can be made by simple enumeration. The problem for the class was to develop the rule for addition of signed numbers. Students at the ninth grade level, of course, are more or less familiar with addition as it occurs in arithmetic. One of the major difficulties in teaching algebra is to guide the students in revising their concept of addition and explicitly formulating a generalization (or rule) that can be applied to any addition problem involving positive and negative numbers. Since students have little difficulty solving problems involving the addition of positive and negative numbers when "story" problems provide the setting, the topic should commence with the solving of such problems. Below are a few examples of these story problems which, in order to be consistent with the symbolic representation of simple enumeration, we will

call I_1, I_2, I_3 -----

In the first quarter of a football game, Janowicz carried the ball five times. What was his total yardage if the yardage for each try was

I_1 - 5 yds.gain, 3 yds.gain, 10 yds.gain, 7 yds.gain, 4 yds.gain?

I_2 - 2 yd.loss, 4 yds.loss, 3 yds.loss, 1 yd.loss, 5 yds.loss?

I_3 - 10 yds.gain, 5 yds.loss, 7 yds.gain, 3 yds.loss, 12 yds.gain?

I_4 - 6 yds.loss, 7 yds.loss, 9 yds.gain, 3 yds.loss, 2 yds.gain?

In forecasting weather, to know the hourly change in temperature is often more helpful than the actual temperature. What is the total change in temperature if the hourly change has been reported to be

I_4 - up 2, up 5, up 4, up 1

I_5 - down 1, down 3, down 2, down 6

I_6 - down 2, down 1, up 2, up 3

I_7 - up 3, up 2, down 1, down 4

What is the total change in value for the past week of one share of A.T. & T. stock if the daily changes are

I_9 - down 1, down $\frac{1}{2}$, down $\frac{1}{8}$, down $\frac{3}{8}$, down $\frac{1}{8}$, down $\frac{1}{4}$

I_{10} - up $\frac{1}{8}$, up $\frac{3}{8}$, up 1, up $\frac{1}{2}$, up $\frac{1}{8}$, up $\frac{5}{8}$

I_{11} - down 1, down $1\frac{1}{2}$, up 2, down $\frac{1}{4}$, up $\frac{3}{8}$, up $\frac{1}{2}$

I_{12} - up 1, down $\frac{1}{8}$, down $\frac{1}{2}$, up $\frac{1}{4}$, down 2, down 1

After students have worked out the answers, have written I_1, I_2 , ----- in a more concise form (e.g., I_1 would be written +5, +3, +10, +7, +4), after they realize that all of these problems are concerned with "the total" and therefore are addition problems, and after they have worked abstract addition problems (such as +5, -2, +7, -2) by placing these abstractions in story problems*, they are

*This step is extremely important for it is in this way students can be made to feel that the generalization is really a short-cut.

ready to think about a rule* which will solve any addition problem involving signed numbers.

In formulating this rule there will be opportunity for the teacher to point out the importance of considering a sample of all possible addition problems involving signed numbers. More than likely there will be opportunity to demonstrate how one contradictory case can destroy a generalization. It often happens that the first rules formulated must be discarded for those that are more general.

Other topics in algebra, such as the development of the rule for subtracting with signed numbers, the relation between the y -intercept and the slope of a line to m and b in $y = mx + b$, and the unit on relationships in general, lend themselves quite easily to treatment in this manner. The recurrence of the method is good because it gives an opportunity to discuss in an incidental fashion such ideas as sampling, analysis, classification, and hasty generalizations. (More will be said about some of these ideas later.)

Analogy As An Empirical Generalizing Method

Generalization by analogy¹⁷ is an assertion concerning an instance because it is like some other instance in

*In general, two steps should be taken in the formulation of the rule. Students should discover a way any problem which involves their revised concept of addition can be solved by use of the number scale. After they have worked and understand problems by using this method, they are ready to discover a rule that will dispense with the visual aid--the number scale.

¹⁷A good discussion concerning this type is found both in Stebbings, op. cit., pp 249-256, and Black, op.cit., pp 292-4

which the generalization is true. An example quoted by Black illustrates quite vividly what the definition says. "The herring we ate last week was improved in flavor by being cooked in vinegar; the same can be expected from treating this mackerel in the same way."¹⁸ The probability that the argument is true depends directly upon the number of properties both instances have in common. The likelihood of the truth of the example quoted generally depends upon the number of properties "herring" and "mackerel" have in common. However, the value of analogy as suggestive of a generalization does not increase as the degree of likeness (number of properties in common) between the two instances increases. "Mackerel" and "herring" certainly have more in common than electricity and running water or propagation of light and the formation of fluid waves. Nevertheless, the last two analogies have been much more fruitful in leading to useful generalizations than the one concerning the two varieties of salt water fish. To further the analysis of analogy, the following representation may be helpful:

¹⁸ Ibid., p 291.

R has properties $P_1, P_2, P_3, \dots, P_a, P_{a+1}, \dots, P_{a+n}, r_1, r_2, \dots, r_k$
 S is known to have properties $P_1, P_2, P_3, \dots, P_a, s_1, s_2, \dots, s_k$
 S is then expected to have properties
 $P_{a+1}, P_{a+2}, \dots, P_{a+n}$
 where R and S are classes and $P_1, P_2, P_3, \dots, P_a$ are
 properties common to both classes (called positive
 analogy) and $r_1, r_2, \dots$ in class R and $s_1, s_2, \dots$
 in class S are properties not in common (called
 negative analogy).

In the example quoted above, the herring could be represented by class R, the mackerel by class S; the positive analogy P_1 and P_2 , salt water fish and food fish; the negative analogy r_1 and r_2 , average length 13 inches and moving about in great schools; and s_1 and s_2 , average length of 18 inches and steel blue back with blackish bars and silvery underside. The property P_{a+1} for R (herring) is that something which causes R to be improved in flavor by being cooked in vinegar. The new generalization is that mackerel, because they are alike in so many other respects, also possess the property P_{a+1} .

The cited generalization above is true if being a food fish and a salt water fish are relevant factors in the positive analogy, and if length, color, and moving in great schools and all other properties in either negative analogy are irrelevant properties. In other words "improved flavor by cooking in vinegar" should depend upon properties "food

fish" and "salt water fish" and not upon the length, color, or whether fish move about in great schools.

The truth of the generalization by analogy is certain if the total positive analogy contains all the properties relevant to the generalization, and the total negative analogy contains only the irrelevant properties. To meet these criteria a complete careful analysis of the two cases and a knowledge of which of the factors are relevant or irrelevant to the generalization are required. The last two statements suggest that this method of analogy arrives at certainty. The conditions for certainty are stated, but even in verification it is rare that all the conditions can be met. The conditions were stated to indicate a direction to move in approaching certainty.

In an algebra class, as the students' knowledge of factoring is extended, there are opportunities to make use of analogies. Students have but little difficulty in factoring such expressions as $x^2 + 5x + 6$, $x^2 + 7x + 12$, and $x^2 + 9x + 8$.^{*} They quickly catch on to the principle that the coefficient of the x term is the sum of the factors of the constant term. By analogy they apply the same idea to $y^2 + 5y + 6$, $a^2 + 7a + 12$, and $b^2 + 6b + 8$. The principle holds because the likeness of these examples contains all

^{*}Students often fail to see how this expression is like the other two. In this example, they see that the coefficient of the middle term is larger than the constant term. They apparently feel this is a relevant difference when, of course, in this case it is not.

the relevant similarities. Representing the unknown with y , a , and b are irrelevant differences. By analogy the same principle holds for examples of four other types such as (1) $x^2 - 6x + 9$, $a^2 - 9a + 8$, and $b^2 - 9b + 20$, (2) $x^2 - x - 6$, $b^2 - 2b - 8$, $c^2 - 3c - 10$, (3) $b^2 + b - 20$, $d^2 + 5d - 14$, $y^2 + 4y - 5$, and (4) $a^2 - 9$, $b^2 - 4$, $x^2 - 25$. In these four types the differences all prove to be irrelevant when compared to the type first mentioned (i.e., $x^2 + 5x + 6$). When students are first asked to factor expressions where the coefficient of the squared term is other than one, such as $2x^2 + 7x + 12$ most students will apply the above rule since it has been so successful in all the types previously tried. Of course, the rule breaks down. The difference (coefficient of x^2 being other than one) proves to be a relevant one. This incident makes an ideal teaching situation to show the danger of analogy and to show again the main contribution of analogy, namely the suggesting of generalizations.*

Interrelationship of Simple Enumeration and Analogy

On first examination it appears that the methods of simple enumeration and analogy are quite distinct, but careful examination of generalizations by both means quoted

*In the classroom after the breakdown of analogy has been discussed, examples of analogies outside the subject matter limits of mathematics should be discussed and students should be encouraged to hunt for such examples to bring to class.

above shows that the other is present and that the difference is really one of emphasis rather than kind. For example, when scientists made the assertion, "All crows are black," they were really dealing with more resemblances between I_1 , I_2 , I_3 ,----- than the one, blackness.* The truth of the assertion is dependent upon the number of instances observed, but it also depends upon the number of properties all crows have in common. This recognition of resemblances between all instances of crows makes the assertion partially dependent upon analogy. Stebbings' concluding remarks concerning simple enumeration state this dependence very clearly:

Simple enumeration is not, then, to be regarded as a process simply of counting; it is a counting of instances recognized as having certain properties in common. The inference is dependent upon recognition of resemblances. An inference from resemblance is an inference from analogy.¹⁹

Likewise, when making the assertion that mackerel, like herring, will be improved in flavor if it is cooked in vinegar it is suggested that more than one mackerel and more than one herring are involved. The use of the class name suggests a large number of instances. Class names are used to tie together a large number of objects that possess a large number of common significant properties. The large number of instances suggested by this generalization immediately

*The mention of a class name, as crow, immediately suggests a group of objects all possessing a large number of properties in common.

¹⁹Stebbing, op. cit., p 249.

implies at least a partial dependence on simple enumeration. When obtaining the suggestion for a generalization the discoverer may be consciously employing either simple enumeration or analogy. However, when testing the generalization for its truth the generalization must be applied to a large number of similar instances.

Classifying and Generalizing

Class names were involved both with simple enumeration and analogy. Just what is the connection between generalization and classification? Classification means a grouping together of objects according to properties possessed in common. By this grouping an assertion can be made concerning all members of the various invented classes. From this point of view then, a classification implies a generalization, but the generalization might be said to be a forced one or a generalization by definition. The statement, "All whales are mammals" is a generalization, but it is one of the same type as "All quadrilaterals have four sides." These statements are true because it was found convenient to group the objects in that manner. Such generalizations are not directly suggestive of new knowledge--they only bind together or organize knowledge already known in a way that it can be made to serve some purpose. Careful classification, however, of the subject matter of science does make easier the discovery of new knowledge. One function of this careful classification is to make both simple enumeration and analogy

more fruitful. Usually the most useful scheme of classification is one that groups together the widest variety of objects with the largest number of important properties in common. In addition, this classification should also have a logical structure. An example of a highly structured classification would be the way in which man is classified:

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Chordata
  Vertebrata
    Mammalia
      Primates
        Hominidae
          Homo
            Sapiens
              John Doe20

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Classification is present in the subject matter of algebra. Expressions are classified according to the number of terms, i.e., monomial, binomial, trinomial, and polynomial, according to degree, i.e., 1st, 2nd, 3rd, etc., and sometimes according to the way in which they can be factored. Classification is also involved when considering all possible types that are to be covered by a generalization that is in the process of being formulated by the students. Classification has a greater opportunity to be discussed if a unit on statistics (collecting data) is included as part of the algebra course. More will be said about the content of an algebra course emphasizing generalizing in a later chapter. Although it is not possible to illustrate with algebra all the important details to remember when classifying, it is

²⁰Bowers, Henry, Thinking for Yourself. Toronto: J. M. Dent & Sons, 1946 p 76.

possible and worthwhile to spend some time considering the purpose behind any classifications that are used and the role they play in the formulation of any algebraic concepts being developed.

Statements of Relationships

Before discussing the third type of empirical generalization, it is appropriate to say a word about statements of a relationship. Generalizations and statements of relations* are both propositions.²¹ Two objects or more are present when a relation exists. These objects, however, do not have to be elements (objects that can be isolated) as might be suggested by the term object itself. In the relation, Man's voluntary movements are accompanied by consciousness, "Man's voluntary movements," and "consciousness" are not elements. The statement of a relationship may be a generalization as the above example illustrates, but it also may be just a statement of an observable fact as in the statement, My son Douglas is taller than his friend Gary.

*The word, relation, is an undefined term in the logical sense of the word. It is hoped that the use of relation in the text is made clear by talking about relation and giving examples of the way it is used.

²¹According to Stebbings, op. cit., p 33, "a proposition is anything, believed, disbelieved, doubted, or supposed." A sentence which expresses something that is either true or false. A statement that lends itself to the methods of verification.

Relations are of two basic kinds: relations made explicit (or understood) by a single proposition, and relations that cannot be made explicit unless more than one proposition is present. The latter is a relation between propositions. Relationships that are generalizations which have been reached through the simple enumeration procedure are of the former type. Such generalizations differ from the statement of the observable fact²² only in that the class names of the objects are substituted for the particular objects themselves. An example should make clear what is implied by the last sentence.

The sum of the interior angles of triangle A is equal to 180°

The sum of the interior angles of triangle B is equal to 180°

The sum of the interior angles of any triangle is equal to 180°

The final statement is a generalization and a statement of relationship. The other statements of relation concerning particular triangles are simple propositions.

General relationships suggested by analogy are concerned with the relation between two instances and, therefore, could be thought of as a relation between two propositions. A correspondence is established between like properties (positive analogy) in each case. From the strength of this

²²A statement of an observable fact is often called a simple proposition. See Stebbings, op. cit., p 37.

likeness, an assertion that is known to be true concerning the first instance is made about the second instance. The instances in this case could be a class of objects. By way of illustration in the following analogy:

If you saw a clock you would infer the existence of a clockmaker. The human body is a machine even more intricate than the best watch ever made. It must have been designed by some Master Craftsman.²³

the instances are the clock and human body. The generalization suggested is: the human body must have been designed by a Craftsman.

Extension of Form As An Empirical Generalizing Method

The first two empirical methods disregarded the form in which the material of the generalization was found. The instances in simple enumeration and the common properties of the two instances in analogy are not taken in any particular order. The primary concern was the material. The third method of arriving at generalizations, generalization by continuity of form,²⁴ examines the relation between instances or simple propositions that have been ordered according to some system, notes the peculiarities that result and then makes predictions. Such generalizing seems to go to the heart of man's desire to find a system to things he observes. Sometimes these systems are rather blind, as in the following example:

²³Black, op. cit., p 301.

²⁴Terminology suggested in Wertheimer, Max, Productive Thinking. New York: Harpers and Brothers Publishers 1946, p164.

In the spring of 1951, many rabid Yankee baseball fans were hopeful that rookie outfielder, Mickey Mantle, would be a great professional baseball player. It so happened that the following observations had been made:

Babe Ruth, a great baseball player, wore the Yankee suit, No. 3.

Lou Gehrig, a great baseball player, wore the Yankee suit, No. 4.

Joe DiMaggio, a great baseball player, wore the Yankee suit, No. 5

Mickey Mantle wore Yankee suit, No. 6.

The prediction that fits into the above form is Mickey Mantle will be a great baseball player. The generalization, New York Yankee baseball players are great if their suit numbers are consecutive with past great Yankee baseball players, is not explicitly stated in either the prediction or the three statements of fact, but it must be in the mind of the predictor before the prediction can be made. It is well to point out at this time that the rather subconscious generalization above is unlike those formulated by simple enumeration in that it cannot be stated by merely substituting class names in the statements of observable fact. The single proposition is not a special case or an instance which by itself can verify the implicit generalization.

Of course, the baseball illustration just cited suggests a generalization that cannot be taken seriously. Number order on baseball suits is an irrelevant variable of great Yankee baseball players. However, this example does illustrate a method of discovery which often is effective.

Such a method was used by Halley when he predicted the appearance of the comet which now is named after him.

Halley plotted the paths of 24 bright comets appearing between 1337 and 1698 A.D. Halley selected three of these because their paths closely resembled each other and noted their dates of occurrence: 1531, 1607 (seen by Kepler), and 1682 (seen by Halley). The interval of 75 or 76 years between these occurrences suggested to Halley that they might be the same comet. Acting on this hypothesis, Halley found in ancient records that a bright comet had occurred at each of the following dates in addition to the ones listed above: 1305, 1380, 1456, 1531, 1607, and 1682.

Noting the continuity of form in the above dates and concluding these comet appearances were due to the same comet following a well-defined orbit, Halley concluded that this comet was due to return 75 or 76 years after 1682, which would be 1757 or 1758.----The fulfillment of this prophecy after his death assured Halley a place among the immortals of science. Later appearances in 1835 and 1910 further confirmed Halley's prediction.²⁵

The above record of discovery makes use of all three types of empirical generalizations, but the most prominent one is that of continuity of form. The outline below illustrates the form followed.

A bright comet with a peculiar path occurred in 1531.

A bright comet with the same peculiar path occurred in 1607.

A bright comet with the same peculiar path occurred in 1682.

This data suggest: A bright comet will occur in 1757 or 1758. The generalization that is implicit within the manner the data are organized is that a bright comet occurs every 75 or 76 years.

²⁵Myers, Sheldon, "Applications," The Mathematics Teacher, XLV (May, 1952), pp 365-6.

The two illustrations given thus far concerning generalization by continuity of form involve extrapolations of data already observed. The next illustration, an account of Galileo's generalization concerning horizontal motion, is one of continuity of form by interpolation or filling in the gap(s). Galileo made observations from which he concluded

- a) As spheres roll down an inclined plane the velocity becomes increasingly greater (positive acceleration)
- b) As spheres roll up an inclined plane the velocity becomes increasingly less (negative acceleration)

To complete the picture some statement must be made concerning horizontal motion. By interpolation Galileo concluded that the motion of a sphere moving in a purely horizontal direction must have an acceleration of zero, or stated in a different way:

If a body moves horizontally in a given direction, it will continue to move at a constant velocity--for all eternity, if no external force changes its state of motion.²⁶

The implicit generalization which concerns the form in which the observable data of Galileo's first two generalizations can be placed and which must have been in the back of Galileo's mind was that as the sphere rolled on the inclined plane with a steep angle of declination to a steep angle of inclination, the acceleration decreased from a positive number with a large absolute value to a negative number with a large absolute value. Galileo's third explicitly stated generalization

²⁶ Wertheimer, op. cit., p 164.

(on horizontal motion) is, as are the first two, merely a special case of implicit generalization concerning form.

In passing, it might be mentioned that this generalization concerning horizontal motion was considered nonsense, even heresy, at the time it was made (early seventeenth century), but as any science student of today knows, the development of modern physics began when such a generalization was finally accepted.

As a summary of what has been said thus far concerning generalizations by continuity of form, the following two symbolic representations are presented, the first for extrapolation, the second for interpolation:

1.

2.

where in both cases each individual I represents simple propositions involving data that have been ordered according to some scheme, and I_n represents a predicted proposition of the same inclusiveness as all the known I's. The prediction was made on the basis of an observed relation r between successive instances.

The method of generalizing by continuity of form can be used to good advantage by the teacher in guiding students to discover mathematical generalizations for themselves. The rule for multiplication of signed numbers lends itself nicely for student discovery in such a way. The technique²⁷ really amounts to extending the multiplication tables.

$$\begin{array}{rcl} (4)(4) & = & 16 \\ (3)(4) & = & 12 \\ (2)(4) & = & 8 \\ (1)(4) & = & 4 \\ (0)(4) & = & 0 \end{array}$$

If the form of the above observable mathematical data is studied, implicit generalizations are sensed by the students so as to extend the table to include the products of the following factors:

$$\begin{array}{l} (-1)(4) \\ (-2)(4) \\ (-3)(4) \end{array}$$

The jump or extrapolation in passing from multiplication of two positive signed numbers to multiplication of a positive and a negative factor is usually taken in stride. But making a similar jump from multiplication of a positive and a negative

²⁷ This method of introducing multiplication of signed numbers is discussed in the following articles:

Bruns, Walter J., "The Introduction of the Negative Number," The Mathematics Teacher, XXX, (Oct. 1946), pp 262-269.

Christofferson, H. C., "Teaching Directed Numbers," School Science and Mathematics, XLVII, (May 1947), p 451.

Jones, Phillip S., "Mathematical Recreations," The Mathematics Teacher, XLIII, (Dec. 1950), pp 417-422.

number to multiplication of two negative factors usually disturbs the students, even though in the example below they sense that the answer to $(-4)(-1)$ must be $+4$ if the continuity of form is maintained:*

$$\begin{array}{ll} (-4)(+3) = -12 & (-4)(0) = 0 \\ (-4)(+2) = -8 & (-4)(-1) = +4 \\ (-4)(+1) = -4 & (-4)(-2) = +8 \end{array}$$

Another example will be given which will not only show the use of continuity of form, but also a way in which mathematical symbols are born. In textbooks on semantics and in books on pedagogy one often finds statements which imply that symbols should not be introduced until students have had some experience in the world of reality with the referents of those symbols. It is wise in many, if not in most, cases to follow such a rule. But ruling out entirely the introduction of the symbol first and searching for referents or applications later ignores certain aspects of

* If carefully thought through, this topic will provide the teacher with several opportunities to promote the generalizing ability of the students. Students accept the extension of the system in the first case, but not in the second. Why? Some students will rebel because of the suggestion that the product of two negative numbers will be equal to a positive number--an example of prejudice against new ideas (as in Galileo's time). Quite often, when the explicit formulation of the rule actually is undertaken, hasty generalization results. For example, one student in a class taught by the writer offered the following rule and many in the class were willing to accept it: "If a positive and a negative number are multiplied together, the sign of the product is the sign of the number with the smaller absolute value." The rule was made before answers to problems like $(-5)(4)$ were determined.

the inventiveness of the human intellect. A mathematician in making his generalizations often finds the domain in which he has been working has been limited in some way. For the sake of generality he seeks to remove these limitations by creating new entities,²⁸ and thereby make his generalization more general.

The negative number, for example, is a symbolic invention which came into being for the purpose of making subtraction possible when the subtrahend was larger than the minuend. The mathematical meanings given this invention are not arbitrary but are chosen to extend the form already defined in the original limited domain. The examples

$$\begin{array}{rcl} 5 - 2 & = & 3 \\ 5 - 3 & = & 2 \\ 5 - 4 & = & 1 \\ 5 - 5 & = & 0 \end{array}$$

already set the pattern. Continuity of form suggests that the first term of a fifth expression will be 5, and the second, 6. The term making up the right members of the four equations suggests that each following term is one less than that preceding it. Therefore, the right member of the fifth equation is one less than zero, the right member of the fourth equation. The term, -1, was invented to impart this mathematical meaning. Other mathematical symbols invented because of similar reasons are the ideal point in modern geometry and the imaginary number, i.

²⁸Shaw, op. cit., p 181.

Strictly speaking, if one rigidly followed the definition of rational generalizations, the method of discovering the rule for multiplication of signed numbers and the method leading to the invention of the negative number would have to be considered as rational generalizations. The form is quite prominent in both cases and the statements of mathematical observable data, $(4 \times 4) = 16$ and $5 - 2 = 3$, are really generalizations. However, the non-mathematical application of continuity of form, Halley and the comet, Galileo and horizontal motion, are so directly concerned with observable facts that it seemed best to consider continuity of form, at least in part, an empirical method. Also, facts such as the individual combinations in the multiplication tables are so universally applicable that one usually thinks of them as observable facts rather than generalizations. The difference between rational and empirical generalizations is not distinct, but is really one of degree. The extreme subtypes, simple enumeration and the first rational method, deduction, are easily classified, but the method, continuity of form, is in the fringe area. Later in the chapter, several other examples of continuity of form will be cited which, without much doubt, are examples of rational procedures.

Functional Relationships

Teaching algebra with emphasis on functional relation, a special kind of relation, has been recommended for the past

three or four decades. Functional relation is difficult to define. The usually accepted definition is that a functional relation is a correspondence between two ordered classes.²⁹ Accordingly, there is a functional relationship expressed in the table below between Siamese imports (in millions of dollars) from the United States each year from 1929 to 1933 and the number of marriages in New York City in those same years:³⁰

<u>Siamese Imports</u>	<u>Marriages in New York City</u>
3.1	65,780
2.4	63,337
1.3	61,574
.7	58,957
.8	60,116

There are two classes of data that are ordered and there is a one-to-one correspondence between the variables of the class according to the year in which the data were collected. That is all that is required by the definition of functional relation. However, when such a definition allows two classes of variables to be regarded as a functional relation when common sense dictates that there is no physical relationship between the corresponding variables, then that definition is lacking. An important value of a functional relation about which the definition does not say a word is its predictive capacity. A formula that would fit the above

²⁹Hamley, Herbert R., op. cit., p 20

³⁰Cooley, Hollis R., et.al., Introduction to Mathematics. Boston: Houghton Mifflin Co., p 262.

data could be determined but the probability of predicting either the Siamese imports or marriages in New York City for succeeding years would be exceedingly small. Why has the definition been made so general? The answer according to Cooley:

If the definition of a functional relation had been such as to require the variables to be physically related, then each time we said that there was a functional relation between two certain variables we would be obliged to prove that the variables were physically related. It would be extremely difficult, if not impossible, to give such a proof, even in the simplest of situations. The definition of a functional relation avoids this difficulty.³¹

Statements of a relation are discovered by any of the three empirical procedures. However, if the relation is discovered by the method, continuity of form, the relation must be classified as a functional relation.* Both this method and functional relation involve order and correspondence between two classes. Such being the case implies that a ninth grade mathematics course with emphasis on student discovery should include the discovery of functional relations as a vital part of the program. Such experiments in a mathematics class as the determination of the cooling curve of a liquid, the relation between the Fahrenheit and Centigrade scales,³² and the relation between

³¹Ibid., p 263.

*The converse, if a relation is classified as a functional relation, the method by which the relationship was discovered was continuity of form, is not true.

³²See article, Myers, S. S., Mack, J. E., and Reynolds, Jack, "Making Concrete Several Kinds of Variation in Algebra," School Science and Mathematics XLVIII, (Nov. 1948), pp 639-646.

stress and strain on a spring are examples of content which can be presented by student discovery techniques so as to highlight the nature of the functional relation. In addition, such experiments arouse considerable interest because they involve physical activity and offer a change from the usual routine.

Statistical Generalizing Procedures

Earlier mention was made of uniform and statistical generalizations. The distinction here was based on whether or not the resulting generalization implied certainty rather than on whether the method was characteristic of a procedure. Considered in such a way uniform and statistical generalizations could be classified according to any of the sub-types under either empirical or rational procedures. Nevertheless, there is a statistical procedure of arriving at empirical generalizations. This procedure makes use of such statistical concepts as mean, median, mode, and correlation coefficient in the formulation of generalizations. Suppose a mathematics test were given to a class of 16 students, 11 boys and 5 girls. The scores were as follows:

<u>Boys</u>	<u>Girls</u>
50	86
98	89
91	80
55	87
97	85
95	
60	
93	
55	
90	
70	

If the method of simple enumeration were used with the above data, the generalizations possible would be:

1. Some boys made better grades on the mathematics test than any girl.
2. Some boys made poorer grades on the mathematics test than any girl.
3. Some girls made better scores than some of the boys, and conversely.

These three conclusions are confusing, for none say anything about the groups of boys and girls in the class at large. Each of the three conclusions includes only parts of the data and none can be used as representative of all the data. By statistical procedures one can summarize all the above data with the following generalization which will have at least some degree of conclusiveness:

In general, the girls in the class did better on the test than the boys in the class.

In describing the statistical method, requisites such as collecting representative, comparable, and relevant data, organizing and presenting data, interpreting and predicting from the data, are usually mentioned. These requisites not only make generalizing easier by the statistical method but also make easier the use of the other empirical methods. Therefore, it seems appropriate to say something more about each requisite. Generally, it is not possible to collect data from all members of the class about which generalizations are made. In such cases, data are collected from a representative sample of all the members of the class.

In the above data if the mathematics class were composed of a representative sample of all boys and all girls taking that particular course then the generalization could be extended to:

Girls who take a particular mathematics course do better on a certain test than do boys.

To be certain a sample is representative there should be data from all types of cases within the class and the per cent of each type in the sample should be the same as the per cent of the type in the whole population. A sample is also helpful when discovering generalizations by simple enumeration, but to know the per cent of each type in the population is unnecessary. In discovering mathematical generalizations this per cent is meaningless. The number of cases in the total population and in each type is infinite.

Comparable data are data that have been collected under similar circumstances, using the same units of measure and representing the same degree of precision in counting or measuring. In the test data above, examples of similar circumstances are: each person should have had the same amount of time available to him in working the test; all tests should have been graded by using the same standards and graded by the same person. If comparisons are to be made, all test scores should be determined by the same general scoring scheme. If part of the tests were graded on the per cent right basis and others given marks, A, B, C, D, or E, generalizations concerning the test scores would be

nearly impossible until some conversion factor was introduced. Finally, if the possible score range was from zero to 100 points and if some papers were scored with the nearest number divisible by ten and others to the nearest whole number, different degrees of precision would be in use. As far as generalizations concerning the above test scores are concerned, such lack of consistency in precision would not be much of a handicap. However, in other situations such lack would be a great handicap. For example, records in which heights of basketball players in a league were recorded to the nearest inch for one team and to the nearest foot on other teams would be worthless in determining which school had the team with the greater height.

To collect data indiscriminately would be foolish. Much of it would be useless and the bulk itself would hide what data really were significant. Therefore, it is wise to analyze the problem to determine which kind of data is relevant before collection is made. No hard and fast rules can be set down for this determination. In the final analysis the choice rests upon experience, preferably the experience of those who have given the most study to the problem situation. The selection of relevant factors is directly related to the basic question left unanswered in the discussion of functional relation (see page 51). The question to be answered in the affirmative in selecting relevant factors is, "Is this factor physically related in any way to the problem

at hand?" Examples of relevant data to be collected in answer to the question, "Who are better drivers, men or women?" are the number of accidents in which men and women are involved as drivers and the number of hours spent driving by men and women.

Organizing and presenting data help in the formulation of generalizations. The test data above were organized and presented in table form; other data could be presented graphically. Organization according to different schemes are suggestive of relationships that might be present. Also, organization enables one to determine with less difficulty statistical measures such as mean, median, and mode. An excellent example of organization of data which led to a generalization was the organization of the multiplication facts (see page 46). The organization rather than the data itself suggested that the product of two negative numbers was a positive number.

To discuss in detail the process of interpretation and prediction from data would be to repeat the discussion of all the discovery procedures mentioned in this chapter. The interpretation of data is nothing more than making use of generalizing procedures in summarizing the data at hand. Prediction from the data is applying the generalization to situations for which no specific data are available.

Many aspects of statistical procedures can be taught when they relate to other topics of algebra such as graphs,

formulas, and the discovery of the rules for operating with signed numbers (see Chapter IV and the Appendix), but if statistics is to receive the attention it deserves, it also should be included as a special topic for study.³³ Many algebra texts do include units on statistics, but often the units are quite unrelated to the rest of the work in the course. With development of generalizing ability through student discovery procedures as the central theme, statistics becomes more of an integral part of the study of algebra.

Deduction as a Rational Generalizing Method

Rational generalizations, the second major type, have already been discussed to some extent--they are suggested by speculating with generalizations already verified or assumed to be true. They have been further classified into the following sub-types, again according to the way the generalizations are suggested: generalizing by deduction, generalizing by variation, generalizing by formal analogy, and generalizing by inverse deduction.

Probably the best known procedure is generalizing by deduction, and is used when "new" generalizations are deduced from generalizations already accepted. Some logicians may

³³ Materials used in studying statistics by the experimental class were selected from the following texts:

Schorling, Raleigh, Clark, John R., Lankford, Francis G., Mathematics for the Consumer. New York: World Book Company, 1947, p 438.

Mitchell, U. G., Walker, Helen, Algebra, A Way of Thinking. New York: Harcourt, Brace and Company, 1936, p400.

Snader, Daniel, Algebra Meaning and Mastery, Book I. Philadelphia: John C. Winston Co., 1949, p 502.

argue that this is not a method of discovery since the "new" generalization was already implied in the parent generalization. For practical purposes the generalization really is not born until it has been made explicit and is open for inspection and ready for use. However, to assume that this method--the method by which the mathematician demonstrates the validity of his generalizations--is the only method of discovery he uses would show an ignorance of the way mathematics has developed.³⁴ It was by deduction that Galileo discovered the proposition: The distances freely falling bodies traverse are proportional to the square of the time of their fall.³⁵ He deduced this generalizations from the assumption that the change in velocity of a freely falling body during an interval of time is proportional to that interval. This assumption could not be directly tested; therefore, Galileo deduced from it a generalization which could be. Since this technique enables scientists to indirectly verify generalizations otherwise entirely unverifiable, it is a discovery method of some importance. Often, however, no new suggested generalizations result, only those that have been known for some time. For example, Newton deduced from his laws of motion generalizations concerning tides. Certain laws followed by tides were determined empirically long before the time of Newton. The implied

³⁴Shaw, op. cit., p 169.

³⁵Cohen, Morris R and Nagel, Ernest, An Introduction to Logic and Scientific Method. New York: Harcourt, Brace and Company, 1934, pp 204-205.

generalizations of concern in this dissertation, however, are those which produce generalizations not already known. Relatively speaking, only minor generalizations are discovered by deduction. One handicap is not knowing which way to direct ones thinking in order to find new generalizations. Building a deductive proof is much easier; the goal is already precisely defined and, therefore, the direction is more or less defined. Generally, the new generalizations, as in Galileo's discovery concerning the distance traversed by a falling body, result from deductions that involve only a syllogism or two. Generalizations which would require several steps are usually discovered by other methods.

Deduced generalizations result from the logical form established by the parent assumptions or general propositions. The same sort of deduction would be possible regardless of the material of the parent propositions as long as the logical form remained constant. The "new" generalization is, of course, still within this form set up by the parent generalization. The boundaries of the domain of knowledge have not been changed in any way. What has been done is a filling in of the gaps within the boundaries already established.* The material of the parent generalization

*Mathematicians often think of the process of generalizing as extending the system of concern to a more inclusive domain. Deductive generalizations are the result of interpolations within the logical system. If such a view is taken, the results of deductions could not be considered generalizations.

however, does play an important role. It makes the proposition less abstract and, therefore, aids greatly in the discovery of the newly deduced generalization.

The general form of deduced generalizations can be symbolized as follows:

$$A, B, C, - - - \xrightarrow{\text{implies}} G$$

where A, B, C, - - - - are assumed generalizations, verified generalizations, or simple propositions, and G is the generalization discovered.

The illustrations of deduced generalizations cited above were taken from the field of science. The illustrations below should bring to the mind of the reader many similar illustrations:

1. Pets will die if they eat rat poison.
2. Dogs are pets.
3. Therefore, dogs will die if they eat rat poison.

This illustration could be symbolized as

$$A, B, \longrightarrow G$$

where A and B are premises and G the generalization.

In the first example, statements 1 and 2 are the premises A and B, and statement 3 is the generalization G. The second example is more complex, but it is probably more illustrative of examples actually a part of everyday thinking.

One day when rumors of the surrender of Germany were circulated in Washington, a newspaper correspondent saw Fleet Admiral Leahy (Chief of Staff at the time) going to the park with his grandson who was carrying a baseball bat. He at once wired to his paper that no official news of surrender could be expected that day.³⁶

Analysis of the second example will be something as follows:

A is the generalization, If official news of surrender were expected, Admiral Leahy would have hurried to the office;

B is the simple proposition, Admiral Leahy was not hurrying to the office; and G is the generalization, No news of surrender will be forthcoming today.³⁷

Students can discover by deduction the rule for division of signed numbers. The procedure follows the above symbolic form where A is the rule for multiplication of signed numbers, the product of two numbers, x and y is a positive number, p, if both numbers are positive or both negative; the product, p, negative if either x or y is negative and the other positive; where B is the generalization, If x multiplied by y equal p, then p divided by y equal x, and p divided by x equal y, and G, the rules for division of signed numbers. Of course, students do not discover the rule for division from such explicitly formulated statements. Nevertheless, the premises must be sensed by the student before he can use the method. This example will

³⁶Black, op. cit., p 10.

³⁷This deduction involves an obverted contrapositive which is an aspect of deduction. (Stebbings, op. cit., p 66.)

again illustrate that it is not always necessary to verbally formulate generalizations before they can be used.³⁸

The setting for student discovery of the rule for division must be concrete to the students. The first step is for students to solve problems similar to those listed below, as part of their practice work dealing with multiplication:

1. $+15 = (-3)(?)$	5. $(16)(?) = -96$
2. $(-3)(?) = -18$	6. $225 = (-5)(?)$
3. $(?)(6) = -24$	7. $162 = (?) (9)$
4. $+156 = (12)(?)$	8. $(?) (-64) = -512$

The next step is for the students to write the above exercises in a different way, namely $\frac{+15}{-3} = ?$ or $(-18) \div (-3) = ?$. The first step is dependent upon the students' knowledge of A; the second is dependent upon their knowledge of B.

Usually after working exercises involving both steps shown above, students are ready to formulate verbally the rule for division of signed numbers. The working of several exercises in each step is for the purpose of assisting the students in formulating a verbal statement of a division rule that will solve all the examples. In general, students have less difficulty discovering this rule than the other rules of operation with signed numbers, mainly because of its similarity to the rule for multiplication.

³⁸ For a discussion of the important role of un verbalized generalizations see the article: Hendrix, Gertrude, "A New Clue to Transfer of Training," Elementary School Journal 48, (Dec. 1947), p 197; and Ebert, Rueben S., "Generalization Abilities in Mathematics," Journal of Educational Research XXXIX, (May 1946), pp 671-681.

Variation as a Rational Generalizing Method

A second sub-type of rational generalization is suggested in the following quotation:

But there is another mode also which leads to generalization, and that is the removal of premises in arguments, or at least parts of premises.³⁹

The method suggested is made even more general if the quotation were stated "another mode that leads to generalization is varying the given conditions by subtraction, addition, and substitution of premises." In his dissertation,⁴⁰ Clarence Heinke suggested the name, generalizing by variation, for this type. Before this method is discussed further, an example will be given from geometric subject matter in which there are many possible applications of this method. The established (proven) generalization,

The diagonals of a square are perpendicular bisectors of each other

can be analyzed as follows:

Data: Square ABCD
Diagonals AC and BD

Conclusion: $AC \perp BD$
AC bisects BD

³⁹Shaw, op. cit., p 172.

⁴⁰Heinke, Clarence, Discovery in Geometry through the Process of Variation; Generation of New Theorems and Exercises in Geometry by Performing Certain Operations upon Either the Data or the Conclusion, or Both, of a Known Theorem or Exercise. Unpublished Doctor's dissertation. Ohio State University, 1953.

The data can be broken down further by looking at the following definition of a square:

A square is a parallelogram with a right angle and a pair of adjacent sides equal.

The data then would become

Parallelogram ABCD
 $\angle ADC$ is a right angle
 $AD = DC$
 Diagonals AC and BD

Suppose the premise, $\angle ADC$ is a right angle, is removed, or in other words, the square in the original proposition is changed to rhombus. Such a change results in a "new" generalization which needs to be tested. Verification or nonverification can be established by deduction and simple enumeration. Even though the new generalization may be false, it is still valuable in that it directs thinking along avenues which are likely to produce a true generalization. It happens in this instance that the change from square to rhombus can be made in the given conditions of the original proposition without affecting the truth of the conclusion. Suppose the premise $AD = DC$ is subtracted instead, or stated differently, square in the original proposition is changed to rectangle, what happens to the conclusion? In this variation the conclusion of the original proposition no longer applies. Nevertheless, the conclusion of the original proposition directs the generalizer to a conclusion that does apply: AC bisects but is no longer perpendicular to BD.

Heinke's method of variation is even more inclusive than the definition cited above. It also includes varying the conclusion and noting what changes could or must be made in the given conditions. Going back to the geometric example, "the diagonals of a square are perpendicular bisectors of each other," if the conclusion were made less restrictive by excluding "perpendicular" the original proposition could be made more general. The premises of the proposition, as students could discover, would not only include all squares but all other parallelograms as well. In varying the conclusion, restrictions could be added to rather than lifted from the conclusion. For example, if the original proposition had been, "the diagonals of a rectangle bisect each other, and its conclusion would have been changed to perpendicular bisector, then the given conditions of rectangle must be changed to rhombus.

In allowing the method of variation to include changing the conclusion as well as the premises, the method is broad enough to include those generalizations suggested by the converse statements* of generalizations already accepted. The writer has not found this method too useful in ninth grade mathematics classes, but there are situations where its use suggests ideas to be explored. For example, it may be of some advantage to discuss the converse of the

*A converse of a statement is another statement in which there has been at least a partial interchange between the given conditions and the conclusion(s).

following generalizations:

1. If two numbers are positive their product is positive (a converse--if the product of two numbers is positive, then the two numbers are positive).
2. If a linear equation is in the slope y-intercept form, the coefficient of the x term is the slope of the line (a converse--if the coefficient of the x term of a linear equation is the slope of a line, then the equation is in the slope y-intercept form).
3. If a graph of an equation crosses the x-axis twice, it is a second degree equation (a converse--if an equation is of the second degree, the graph of the equation crosses the x-axis twice).
4. If a triangle is a right triangle the sum of the squares of the two legs equal the square of the hypotenuse (a converse--if the sum of the squares of the two joining sides of a triangle equal the square of the third side, then the triangle contains a right angle).

Anyone who has studied geometry knows some converse statements are true, some false. The same is true of the statements above.

In teaching algebra a teacher could direct the students to the discovery of $(a + b)^2 = a^2 + 2ab + b^2$ by using a method of variation. The procedure might guide students in making the following steps:

1. Represent a product as $4 \times 4 = 16$ geometrically where 4 is the length of a side and 16 the area of the square.

2. Represent the product $2a \times 2a$ geometrically.

3. Represent the product $(a + b)(a + b)$ geometrically. The above squares had areas of 16 and $4a$ respectively. How would you represent the area of this square symbolically other than by $(a + b)(a + b)$?

4. Through the geometrical representation in step 3 you discovered another way to express $(a + b)(a + b)$. How could you arrive at this expression through algebraic procedures?

The method of variation can also be employed to good advantage in a unit on graphing as is illustrated by the exercises below. For example, it will be noted that all the exercises under number 1 differ one from the other in that the coefficients of x have varying values. In all other respects the exercises are identical.

1. In what ways are the equations below alike? How different?

$$\begin{aligned} y &= 3x + 5 \\ y &= 4x + 5 \\ y &= 5x + 5 \\ y &= 2x + 5 \\ y &= -3x + 5 \\ y &= x + 5 \end{aligned}$$

$$y = \frac{1}{3}x + 5$$

$$y = \frac{1}{2}x + 5$$

2. Plot the equations in exercise 1 on the same chart. What similarities and differences do you observe?
3. In what ways are the equations below alike? How different?

$$y = 3x + 5$$

$$y = 3x + 6$$

$$y = 3x + 8$$

$$y = 3x + 3$$

$$y = 3x - 5$$

$$y = 3x - 2$$

$$y = 3x$$

4. Plot the equations in exercise 3 on the same chart. What similarities and differences do you observe?
5. Discover by graphing $y = 3x^2 + 5$ how its pattern is alike or different from $y = 3x + 5$.
6. Discover by graphing $y = 3x^3 + 5$ how its pattern is alike or different from $y = 3x + 5$ and $y = 3x^2 + 5$.

With a little imagination many other questions of the above type can be asked. Better yet, students can be encouraged to ask similar questions of themselves.

Examples of the use of the method of variation in everyday life are numerous. In a cookbook the following standard recipe was given for plain biscuits:

2 cups flour
 1/2 teaspoon salt
 3 teaspoons baking powder
 3 to 4 tablespoons shortening
 3/4 cup milk

By varying the above recipe it is possible to make Butter-section Biscuits, Drop Biscuits, Orange Biscuits, Pinwheel biscuits, or Whole Wheat Biscuits. Of course, if one varies a recipe too much, disaster may result, as it would in the recipe above if 3/4 cup sour milk were substituted for 3/4 cup of sweet milk. However, if when using sour milk the

right quantity of soda were substituted for the baking powder, the variation would probably still produce good biscuits. Innovations on automobiles, radios, television sets, refrigerators are all examples of the use of the method of variation in everyday life.

Formal Analogy as a Rational Generalizing Method

Another method of arriving at rational generalizations is through formal analogy. Black cites the following quotation from Alice in Wonderland to make clear what he means by logical analogy (an aspect of formal analogy):

"Then you should say what you mean," the March Hare went on. "I do," Alice hastily replied; "at least--at least I mean what I say--that's the same thing, you know."

Immediately she gets hauled over the coals by her infuriatingly logical companions:

"Not the same thing a bit," said the Hatter.

"Why you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"

"You might just as well say," added the March Hare, "That 'I like what I get' is the same thing as 'I get what I like'!"⁴¹

The material or subject matter in all three instances was quite different and the discussion was not concerned with any likeness in the actual process of eating and seeing, sleeping and bathing, and saying and meaning. What is of concern is the form in which the subject matter occurs in the different propositions. Alice's statement (which is a generalization), I say what I mean is the same thing as I mean what I say, is shown to be an inconclusive argument by

⁴¹Black, op. cit., p 39.

other statements which have the same form but whose subject matter makes them obviously incorrect.

The above example indicated by analogy that a generalization was incorrect. It is also possible, as the example below indicates, to arrive at a valid generalization by substituting different material into the same form of a known generalization. Stebbings gives an example which illustrates the actual use of such procedure:

Maxwell recognized an analogy between certain problems in the theory of gravitational attraction and certain problems in electrostatics. Thus the gravitational formula is the law of the inverse square; the formula for electrical attraction is also the law of the inverse square, although the nature of the attraction is different. There is between these two sets of phenomena what might be called structural* identity. By developing this analogical, or, as it might be called, structural hypothesis, Maxwell was led to formulate his electromagnetic theory of light.⁴²

In a less formal sense the scientist obtained suggestions of new generalizations concerning the way in which an atom is constructed (its form) from generalizations concerning solar systems. The subject matter of the atom (e.g. nucleus, electrons) is substituted for the subject matter of the solar system (sun, planets).

The method of formal analogy differs from the subtype generalization by analogy, discussed under empirical generalizations, in that the latter is concerned with similar

⁴²Stebbing, op. cit., p 309.

*The word structural here excludes the reference to material or subject matter. For definition of structure as used in this dissertation, see page 19.

properties of the material in two different situations while the former is concerned with the relation between the forms (whether logical or otherwise) of the two situations. In the illustration concerning the solar system and atoms, by way of contrast, an empirical generalization by analogy may be: Our solar system is not completely filled with substance, but is mostly composed of a void. The atom is expected to possess the same characteristic.

The discussion above may seem to indicate a procedure that offers little help because of its complexity in aiding ninth grade students to discover algebraic generalizations. There are, however, some generalizations which the students can discover with ease after they become aware of a situation analogous in form to the problem at hand--at least with as much ease as they could follow the arguments used by Alice's companions to point out her logical fallacy. Introducing students to the analytical method of solving equations is aided greatly by comparing the undoing of the operations performed upon the unknown to the undoing of the operations performed to get at the unknown contents of a wrapped package.* In both cases, order of operation and inverse operations are involved.** The use of such analogy enables students to solve immediately problems as complicated as $\frac{5(x + \frac{1}{2})}{4} - 3 = 2$ without first solving problems of

*To the writer's knowledge, this analogy was first used by Dr. Fawcett in his classes at the University School, The Ohio State University.

**A more detailed discussion will be found in the chapter on selected procedures.

traditional types, such as: $ax = b$, $x - a = b$, $\frac{x}{a} = b$, and $x + a = b$. Also, since students cannot determine the answer very easily by a guess-and-check method, they immediately sense the usefulness of the new generalized technique.

The example of representing the product of two like factors, cited under the method of variation, by a geometric figure is really a contribution of that method and the method of analogy. The likeness between the symbols $(a + b)(a + b)$ and a square is that of form. The product of $(a + b)(a + b)$ is first suggested by working with the square. With that suggestion before the students they find it easier to discover a valid generalized algebraic procedure which will get the same result.

Applications of mathematics, in general, are the result of formal analogy. The places where mathematics is applicable to the world of reality are where the form symbolized by mathematics is analogous to the form of the material of the real world. The more alike the two forms, the more accurately the situation can be symbolized mathematically and, therefore, the more accurately problems within the situations can be solved.

Inverse Deduction as a Rational Generalizing Method

Generalizations reached by the final rational method, inverse deduction,* are those which seek an answer to the

*Inverse is used here in the same sense as in the statement, "division is the inverse operation of multiplication

question: "From what generalizations could these facts and/or generalizations be deduced?" Often, the given information can be deduced from many different and sometimes conflicting generalizations. The generalizations finally accepted are the result of one's experiences, of philosophical convictions, of imagination, and of what Whitehead calls the "climate of opinion" in which one finds oneself. For example:

The scientific traveler on a high plateau of the Andes and his native guide view in different ways the impossibility of getting their potatoes to cook. To the latter, the impossibility is due to the simple fact that "the cursed pot," doubtless owing to the devil in it, "did not wish to cook the potatoes," to the former it is an interesting example of the dependence of the boiling point upon the pressure.⁴³

The stimulus for using this method is man's desire for an explanation of things. Usually when the method is used the explanation is composed of generalizations already discovered. Such was the case of the scientific traveler in the quoted example above. Occasionally, an explanation of the given information cannot be deduced from generalizations already accepted. Then generalizations are invented from which the given information can be deduced and inverse deduction becomes an important method of discovery. Since it is so easy to invent generalizations that later prove false or fruitless, the scientist first gives serious consideration to those invented generalizations which are not inconsistent

⁴³ Nunn, T. P., Education: Its Data and First Principles. New York: Langmans, Green and Co., 1930, p 206.

with the system of generalizations already verified, which help explain facts and generalizations other than the given information, and which are of such a nature as to be capable of either direct or indirect verification.

In symbolic form the generalization by inverse deduction can be symbolized as

$$G, B, C, \text{ ---- } \longrightarrow A$$

where A is a generalization already accepted or an observable fact, G as the new generalization, and B, C, ---- are other facts and/or accepted generalizations needed in order to deduce A. Both G and A may be more than one statement of generalization.

Many of the great steps toward understanding the universe were made when generalizations from seemingly unrelated spheres of knowledge were unified by being bound together with more inclusive generalizations. Newton's laws of motion bound together Kepler's laws of planetary motion and laws concerning terrestrial motion. A symbolic representation of the method employed by Newton is a double application of the symbolism above:

$$\begin{array}{lcl} & B_1, C_1 & \longrightarrow A_1 \\ G \swarrow & & \\ & B_2, C_2 & \longrightarrow A_2 \end{array}$$

where G symbolizes Newton's laws of motion, A₁ - Kepler's laws of motion, A₂ - laws of terrestrial motion; and B₁, C₁ and B₂, C₂ are data necessary to deduce A₁ and A₂ respectively from G. Another well known example of this procedure is

Einstein's Unified Field Theory which unites Newton's laws and electro-magnetic theory. And at the present time it seems as if Einstein also has been successful in unifying quantum phenomena and his Unified Field Theory into a still more inclusive theory.⁴⁴

One rather simple but clear-cut example of inverse deduction in operation is the account of the class' search for a verbal rule for the multiplication of two signed numbers. One student suggested the following phrasings:

The product of two positive numbers is a positive number.

The product of two negative numbers is a positive number.

When the class was asked if they could make one rule out of the two, the resultant rule was as follows:

The product of two numbers with the same sign is positive.

The probable line of thought used in arriving at the generalization was: From what more inclusive rule can these rules be deduced?

Inverse deduction is often closely allied with the method of variation. Suppose an application of a generalization is made and the result is verified. Within the setting of the application there is a major and a minor premise and a conclusion. The minor premise is substituted for another minor premise in some way similar, and it is noted the original generalization does not apply. So far, the

⁴⁴Life. January 9, 1950, p 22.

method involved is the method of variation. Now, an attempt is made to limit the generalization to a domain in which it does apply or to extend it so it is applicable in both instances. The method of concern now is inverse deduction. For instance, suppose a ninth grade mathematics class agreed upon the following generalization:

Every first degree equation with one unknown has one solution.

If the equation, $5x + 7 = 17$, is the minor premise, then the conclusion, there is one value for x , happens to be true. Suppose the minor premise were $5x + 15 = 5(x + 3)$ instead. It is discovered that the value for x can be any value. It is, therefore, necessary to limit the domain of application of the generalization by revising it to:

Every conditional first degree equation with one unknown has one solution.

In this instance a first degree equation was invented which contradicted the original generalization. It is quite common for mathematicians to purposely seek and attempt to invent exceptional cases that are contradictions of generalizations previously made. In this way⁴⁵ new frontiers for research are opened. Opportunities of this kind occur often in a mathematics class if the general teaching procedure is one in which students are encouraged to verbalize generalizations they discover.*

*For one example, see footnote on page 47. Other examples are given in Chapter IV, pp 127-28 and p 228.

⁴⁵For examples, see Shaw, op. cit., pp 182-3.

The following example, which again makes use of the methods of variation and inverse deduction, can be used to good advantage in either the eighth or ninth grades when formulas for various geometric figures are studied. After the students have formulated the rule for finding the area of a rectangle, $A = LW$, they can be asked if such a formula would apply if the rectangle, A, were changed to a

parallelogram, B (application of the method of variation). To answer this question students would have to discover the area of B by some other means, probably by counting the number of unit squares B contains. It so happens that the same assertion can be made concerning the general parallelogram as well, but the width must be obtained by measuring the distance between the appropriate set of opposite sides rather than by measuring the shortest side as in the rectangle. This reinterpretation of width is an invariant regardless of the size of angle "a" in Figure A. If instead the students had formulated the following rule for the area of a rectangle,

The area of a rectangle is equal to the product of the length of one side by the length of either adjacent side

then the rule would not apply to the parallelogram. In such a case it is necessary, as Shaw says, to restate the

"theorem so that in the new form it will apply to a wider domain."⁴⁶ Whether the concept of width is revised or whether the generalization is restated, the method being employed can be called inverse deduction.

Regardless of which type of formulation students make a reinterpretation is required by them before the concept can be extended to include the parallelogram. Such reorientation requires students to analyze the original situation from a different point of view,⁴⁷ to analyze the "new" situation for the purpose of finding an invariant relevant to the discussion (in this case, an invariant that can be used to determine area), and then to word a generalization so as to make it applicable in both situations. Whether the new generalization is made more restrictive, as in the statement concerning the solution of first degree equations in one unknown, or less restrictive as in the formula for the area of a parallelogram, a reorientation requiring further analysis is necessary. Such procedure is quite common with the method of inverse deduction.

⁴⁶ Shaw, op. cit.

⁴⁷ Looking at situations from a different point of view is a general education objective which might be furthered in the mathematics classroom if it is the desire of the teacher to emphasize it whenever appropriate situations occur. Also, Max Wertheimer suggests in Productive Thinking, p 124, that the ability to reinterpret or to reorient oneself is necessary if concepts learned are to be of maximum value in application.

It should be apparent to the reader by now that generalizations are not discovered by a single application of one method. The extension of the trigonometric functions to angles greater than 90° makes use of three different methods. The method of variation is involved when the condition, "an acute angle," in the following definition, the sine of an acute angle of a right triangle is the ratio of the side opposite the acute angle to the hypotenuse, is changed to "an obtuse angle." Such a change requires one to define the sine ratio in a different way. The reformulation of the definition into one more inclusive is an example of the use of inverse deduction. A guiding influence in the search for a definition of the sine ratio which applies to both acute and obtuse angles is found by making use of the method, continuity of form. The final definition of sine must establish a form into which the sines of various acute and obtuse angles fit.

Invention of Symbols and Generalizing

The discussion of empirical and rational generalizations will be concluded by completing the discussion begun earlier concerning the search for exceptions to generalizations. It was said a successful search meant the original generalization must be limited or extended (see page 76). Sometimes, however, neither course is taken, but new symbols are invented so a conclusion is more universally applicable. Suppose that students, before being introduced to the

imaginary number, made the generalization: Every conditional quadratic equation in one unknown has a solution. The selection of $x^2 + 7x + 14 = 0$ as an application of the generalization proves to be an exception. The number 1 was invented by mathematicians so as to make a solution possible. One might say the generalization is forced upon the exceptional case. Use of the symbol 1 opens up a whole new realm of application for the generalization.

Only the better mathematics students in the ninth grade have an opportunity to study quadratic equations involving imaginary numbers. However, opportunity for showing the role of invented symbols presents itself after students have discovered the rule for division of powers of numbers with the same base ($a^m \div a^n = a^{m-n}$). Students first discover the rule when the exponent in the divisor is smaller than the exponent in the dividend (both exponents, of course, being positive). After being explicitly formulated the rule is forced to apply to cases where n is equal to or larger than m. The result is the invention of zero and negative numbers as exponents.*

Technically, from a rigid interpretation of the definition given generalization at the beginning of the chapter, the invention of symbols as discussed above did not result

*The invention of zero and negative numbers as exponents can also be suggested by continuity of form. The fact that the same suggestions are arrived at through different procedures adds to the assurance that the invention will be useful.

in a new generalization. (The wording was not changed in any way.) Nevertheless, the domains symbolized by the original generalizations in the examples cited above were extended and the effect was the same as if the generalization had been worded differently to include more cases. Since the effect is the important consideration, one can consider such procedure as a method of generalizing. If classified under any of the sub-types, it probably should be under the method "continuity of form"--the rational aspect of continuity of form (see page 49).

Summary

There are many procedures used in arriving at generalizations. In the preceding discussion these procedures were divided into two broad categories: empirical and rational. The empirical generalizations were analyzed into four sub-types, namely:

1. Simple enumeration - A large number of members of a class are observed to possess a certain property; this fact suggests that all members of the class possess that property.

2. Analogy - Two similar classes of objects are observed to have at least one property in common. One class is observed to have still another property and this fact suggests that the second class also possesses that property.

3. Continuity of form - A series of similar instances or simple propositions are organized according to some system by which they are related to each other; other concomitant relations between them are also observed to result; finally, predictions are made that certain other instances or propositions will fit into the established system.

4. Statistical - A large number of facts are collected relevant to some topic of interest; statistical measures such as mean, median, correlation, and coefficient are determined; finally, statistical generalizations are made which best summarize the collected data.

Rational generalizing procedures were analyzed into the four following sub-categories:

1. Deduction - From generalizations and facts already accepted "new" generalizations are deduced.

2. Variation - The given conditions and conclusions of accepted generalizations are varied in turn or simultaneously for the purpose of obtaining suggestions for new generalizations that might also be accepted as true.

3. Formal analogy - A large number of accepted generalizations and simple propositions concerning some topic are known to be so related to each other as to establish a certain form (logical or otherwise); a limited number of accepted generalizations and simple propositions concerning a vastly different topic are observed to have a certain

correspondence as to form; new generalizations are suggested which make this correspondence more complete.

4. Inverse deduction - In answering the question, "From what generalizations could certain accepted generalizations and simple propositions be deduced," new generalizations are often suggested.

As a person becomes successful in employing the methods of generalizing, certain types of behaviors and abilities should be noticeable as he engages in the processes independently or in group situations. An analysis of these methods reveals that a superior generalizer has the following behaviors and abilities:

1. He is able to detect likenesses and differences between situations and then to group together those situations which possess common properties.
2. He is able to correctly generalize with certainty where certainty is justified, but refrains from generalizing when it is unwarranted.
3. He is able to determine trends from available data and from them make reasonable extrapolations and interpolations.
4. He is cautious in accepting generalizations, but is willing to consider any generalization as a hypothesis.
5. He can distinguish generalizations from observable facts.
6. He uses generalizing methods independently to further his understanding of the world around him.
7. He makes use of generalizations which he senses through non-verbal means.

8. He is able to verbalize generalizations and methods used in discovering them.
9. He continually searches for examples which are exceptions to formulated generalizations and then reformulates them to include the exceptions.
10. He investigates implications of generalizations under consideration.
11. He searches for explanations for generalizations discovered empirically.
12. He empirically tests or searches for an increasing number of applications to generalizations that have been suggested by rational procedures.
13. He searches for and recognizes relevant factors responsible for happenings or for behavior of objects in certain situations.
14. He studies accepted generalizations from different points of view.
15. He uses statistical measures in making generalizations.

The above explicit formulation of behaviors and abilities is of utmost importance to the teacher who has as one of his objectives the development of a student's ability to generalize. To know the type of behavior or ability gives direction to teaching and, in addition, provides a basis for evaluation. Of concern in evaluation, also, are the conditions of the experiment. The conditions of this experiment will be given in the following chapter.

CHAPTER III

DATA AND CONDITIONS OF THE EXPERIMENT

In carrying out the experiment, data were collected from two different groups of students, the experimental class and a status group. The experimental class was composed of all ninth-grade students at the University School, The Ohio State University, Columbus, Ohio. A large per cent of the students' parents were engaged in professional occupations and their economic status was considerably above that of the average family. However, there were several students from families less fortunate educationally, socially, and economically. Students' homes were located in many sections of the city and county. Approximately 90 per cent of the students had attended University School in the seventh and eighth grades, and nearly two-thirds of the students had attended University School in the lower grades.

The schedule of classes for the experimental students is given in the chart¹ below:

¹The Faculty of the University School, A Description of Curriculum Experiences, The Upper School. Columbus, Ohio: The Ohio State University, 1952, p 64.

Chart

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00	Core	-----	Dancing	Music	Guided Reading
10:00	Core Art	----- Music	-----	-----	-----
11:00	Core	-----	-----	-----	-----
12:00	Lunch				
1:00	Math	-----	-----	-----	-----
2:00	Physical Education				
3:00	Arts Elective				

The distribution of chosen electives during the arts elective period were:

Typing	26
Science	5
Related Arts (Fine and Industrial)	2
Home Arts	1
Work Experience	1

During the year the mathematics class met 155 times for a period of 55 minutes. Of this period approximately 25 minutes were reserved for a work period, since no time is allowed in the schedule for separate study periods.

The program of the University School junior high, which includes the ninth grade, is unique in that a three-hour core period is scheduled in which the students study broad topics cutting across subject matter lines. These

topics are selected by the students and teacher after considering the resource materials available, the past units studied by the class, the educational background of the teacher, and the adaptability of the unit to the individual interests of students in the class. During the particular year in which this study was made the first core unit of study was The Crime Problem in this Country, and the second was Life in the United States Today.

Mathematics is required of every student in the seventh, eighth, and ninth grades and is scheduled for a 55 minute period in each of these years. The other courses in the mathematics program are Algebra II, Nature of Proof, Senior Mathematics (Trigonometry, College Algebra, Solid Geometry) and General Mathematics.* The description of the mathematics program² is briefly described as follows:

There are six fundamental concepts which may be thought of as vertical threads running through our entire program from kindergarten through grade twelve--the horizontal threads being the grade levels. These concepts along with the mathematics classes, where strong emphasis is placed on them, are as follows:

Symbolism - - - - -	Algebra, Nature of Proof
Number - - - - -	Grade 7, Algebra, General Mathematics
Operation - - - - -	Grade 7-8, Algebra, General Mathematics
Measurement - - - - -	Grades 7-8
Function or relationship -	Algebra, Senior Mathematics
Proof - - - - -	Nature of Proof

*General Mathematics can be elected in the tenth, eleventh, or twelfth grades by any student. The large majority of the students are those who have had difficulty in mathematics in previous years.

²The Faculty of the University School, op.cit., p 28.

The status group included the students of six classes with four different teachers in three Columbus Junior High Schools. No measures other than selecting schools from communities of different socio-economic levels were taken to get an accurate sample of ninth grade algebra students in the city of Columbus. At the beginning of the year students in the six classes numbered 177; at the end of the year due to failures and mid-year shifting of students, the number dropped to 127. Students in these classes had elected to study algebra. During the school year each of the six classes met for 185 periods, each period being 40 minutes long. The class period was devoted to group work almost exclusively and individual studying was done during the students' daily two or three scheduled study hall periods. Students in these classes studied algebra in the conventional way. The algebraic content was that included in the adopted text³ in which generalizations are given and are to be applied by the students to the solution of textbook problems.

A comparison of the data on the experimental and public school groups reveals two rather important differences. First, the students in the experimental class were required to study algebra, while in the public

³Mallory, Virgil S., A First Course in Algebra. Chicago: Benj. H. Sanborn and Co., 1943.

schools students elected algebra. Second, the public school students had 30 more class periods during the year than did the experimental class. The experimental class periods were 15 minutes longer than those in the public schools so that the total class time of the experimental students was slightly more than that of the public school students. However, the total school time public school students could justifiably reserve for the study of mathematics (one-fourth of their study time plus the class period) was considerably greater than the school time available to the experimental students.

Intelligence test (Hemmon-Nelson) scores were available on 118 of the 127 public school students. Most of these students took this test when they entered the seventh grade. Test scores, mostly from Binet Intelligence Tests, were available on 33 of the 35 students in the experimental class. However, these tests were not given to all the students during the same year. Most of them were taken while the students were still in the elementary grades. Table I gives the range, median, and first and third quartiles for both groups. Needless to say, one must be extremely cautious in generalizing from these intelligence test data. At best they can give only a rough estimate of the mental ability of each group.

TABLE I

Summary of Intelligence Test Results

	Experimental Class	Public School Classes
Range	78 - 154	74 - 154
First quartile	97	100
Median	112	112
Third quartile	126	121

These results indicate that the median intelligence quotient of each group was the same, but the first and third quartiles indicate that the experimental class consisted of a larger percentage of students with I.Q.'s below 100 and above 121, a not uncommon distribution for the University School. The table below gives the age distribution of both groups.

TABLE II

Ages of Students

Age	Experimental Class		Public School Classes	
	Number	% of Total No.	Number	% of Total No.
12			1	1
13	11	31	50	39
14	19	54	64	50
15	5	14	9	7
16			3	2
	34	99	127	99

These results indicate that the ages of both groups were similar, although in the experimental group the ages seem to cluster more closely around the age of 14 than in the public school classes. This clustering is probably

due to the grade promotion policy which, although not a rigid one, tends to regard chronological age rather than mental age as a better criterion for grouping.

Two tests, the Iowa Algebra Aptitude Test* and a generalization test, were given to both groups at the beginning of the year for the purpose of securing additional information. Before even tentative conclusions could be made concerning methods of teaching algebra to the experimental and public school groups, it was felt that some information, crude as it might be, had to be collected so as to give some indication of the mathematical potential of the two groups at the beginning of the year. The Iowa Algebra Aptitude Test is considered⁴ as one of the best of its kind that can be given during a forty-minute class period. The results of this test are given in Table III.

*Published by Bureau of Educational Research and Service, Extension Division, University of Iowa, Iowa City.

⁴Buros, Oscar k., The Fourth Mental Measurements Yearbook. Highland Park, New Jersey: The Gryphon Press, 1953, pp 500-02.

TABLE III

Summary of Iowa Algebra Aptitude Test
Percentile Scores

Per- centile	Experimental Class			Public School Classes		
	Fre- quency	Per Cent in Class	Cumu- la- tive %	Fre- quency	Per Cent in Class	Cumu- la- tive %
90-99	2	6.1	100.1	8	6.3	100.0
80-89	2	6.1	94.0	11	8.7	93.7
70-79	3	9.1	87.9	15	11.7	85.0
60-69	2	6.1	78.8	12	9.5	73.3
50-59	0	0	72.7	20	15.7	63.8
40-49	4	12.1	72.7	19	15.0	48.1
30-39	5	15.2	60.6	16	12.6	33.1
20-29	8	24.2	45.4	13	10.3	20.5
10-19	3	9.1	21.2	8	6.3	10.2
0-9	4	12.1	12.1	5	3.9	3.9

Raw Scores

	Experimental Class	Public School Classes
Range	18-89	22-79
First quartile	35	43
Median	43	52
Third quartile	67	62
Mean	46.27	51.76
Standard deviation	15.69	13.11
Standard error of mean	2.73	1.16

Mean difference 5.49

Standard error 2.96

Difference significant to the 5% level

Median difference 9.00

Standard error 3.71

Difference is significant

These results indicate that the public school classes made significantly better scores than did the experimental class. A comparison of the frequency distribution according to percentile scores indicates that, in general, the public school students made scores approximating the test norms. The most noticeable exception was that only 20.5 per cent of the students scored below the 30th percentile. Other less noticeable deviations occurred, but all the deviations indicate a greater than normal clustering around the 50th percentile mark. The distribution of the experimental class scores show little clustering around the 50th percentile mark. In fact, 45.4% of the students made marks below the 30th percentile and 72.7% made scores below the 50th percentile. One probable reason for this peculiarity is the tendency of the school to enroll a larger percentage of students with abilities above or below the I.Q. range of 100-110 than a chance selection would allow. Another probable factor of importance is the nature of the University School curriculum. The students in the experimental class who attended the University Elementary School did not have a regularly scheduled time for arithmetic, but time for mathematical topics was taken when they happened to relate best to the unit of study. A daily scheduled time for the study of arithmetic, for the most part unrelated to the large unit students were studying, did not begin until the seventh

grade. This late start in studying arithmetic as a system of interrelated ideas from the point of view of arithmetic and in purposely introducing many situations for practicing certain arithmetic procedures could account in large measure for the significantly lower scores by the experimental class.

An important objective of this study is to determine how much more improvement, if any, the experimental class made in generalizing than did the public school group. Before any conclusions could be drawn it was important to collect data concerning the generalizing abilities of both groups at the beginning of the year. Since no test was available for measuring generalizing ability, a test was designed especially for this study. The steps used in constructing the test were:

1. Determining those aspects of the process of generalizing to be tested.
2. Preparation of test items which measure these aspects.
3. Selecting those items measuring each ability which are the most discriminating.
4. Compilation of the test so as to include only the most discriminating items.

Fifteen behavior characteristics of a superior generalizer are listed in the preceding chapter. Of these, only four were chosen as the bases for selecting items to

be included in the generalization test. It would be practically impossible to construct a forty-minute test that would satisfactorily measure all fifteen abilities and behaviors. These four, which are listed below, were selected because they were more tangible than the other abilities and behaviors.

1. He is able to detect likenesses and differences between situations and then to group together those situations which possess common properties.
2. He recognizes generalizations which are or are not supported by the given data.
3. He is able to determine trends from available data and from them make reasonable extrapolations and interpolations.
4. He is cautious in accepting generalizations, but this degree of caution does not extend to the consideration of generalizations which serve as hypotheses.

Suggestions for the test items and some of the items themselves were taken from other tests, among which were:

The Ohio Thinking Check-Up. For Junior High Schools.
Issued by the Bureau of Educational Research, The Ohio State University.

Social Beliefs 4.21. Progressive Education Association, Evaluation in the Eight-Year Study, University of Chicago, Chicago, Illinois.

Watson-Glaser Tests of Critical Thinking. Published by World Book Company, Yonkers on Hudson, New York.

American Council on Education Psychological Examination for High School Students. American Council on Education, 1744 Jackson Place, Washington, D.C.

The items selected for the test were largely non-mathematical in nature so as to secure some indication of transfer of learning. The test itself and a discussion of its scoring can be found in Appendix A.

Originally, two generalization tests were constructed. Form A was given to 220 ninth grade public school and University School students in the fall of the year preceding the experiment, and Form B was given to the same group of students in the spring of that same school year. The most discriminating items from these two forms made up the final generalization test, Form C. Selection was so made that the final form contained those items which were answered correctly most often by the students scoring in the upper quarter and answered incorrectly most often by the students scoring in the lower quarter. The number of items on the final test was limited to ninety-two with the expectation that a large number of students would finish in thirty or forty minutes.

Table IV summarizes the results of the generalization test given at the beginning of the year to the experimental and public school students.

TABLE IV

Summary of Generalization Test Results
of Students in the Experimental and Public School Classes

Score (Per Cent right)	Experimental Class			Public School Classes		
	Number	Per Cent	Cumu- lative Per Cent	Number	Per Cent	Cumu- lative Per Cent
80-84	2	6.1	99.9	1	.8	100.0
75-79	1	3.0	93.8	1	.8	99.2
70-74	2	6.1	90.8	2	1.6	98.4
65-69	5	15.1	84.7	9	7.1	96.8
60-64	4	12.1	69.6	12	9.4	89.7
55-59	6	18.1	57.5	12	9.4	80.3
50-54	3	9.1	39.4	15	11.8	70.9
45-49	4	12.1	30.3	22	17.3	59.1
40-44	2	6.1	18.2	20	15.8	41.8
35-39	3	9.1	12.1	16	12.6	26.0
30-34	1	3.0	3.0	8	6.3	13.4
25-29				8	6.3	7.1
20-24						
15-19				1	.8	
10-14						
	<u>33</u>			<u>127</u>		

Range	33-84	14-81
First quartile	48	39
Median	57	47
Third quartile	68	56
Mean	57.00	47.55

The mean difference is significant

These results, unlike those of the Iowa Algebra Aptitude Test, were significantly in favor of the experimental class. The distribution of scores shows that only 30.3 per cent of the experimental class made below the 50 per cent mark on the test, while 59.1 per cent of the public school classes made below the 50 per cent mark. This significant difference in favor of the experimental class at the beginning of the year is not surprising. Nearly all the teachers in the University School conscientiously work toward the objective of improving the student's ability to generalize. Also, since the educational level of the parents of the students in the experimental class was generally greater than that of the parents of the public school students, it is reasonable to assume that the home environment would tend to direct students toward the improvement of their generalizing abilities.

By way of summary, then, the experimental and public school classes are roughly similar as to age and intelligence quotients. However, at the beginning of the year it appears that the experimental class was more competent than the public school classes in generalizing, but that the public school classes had more algebraic aptitude. What then might be expected at the end of the year if the public school classes emphasized as a primary goal the mastery of algebraic concepts, while in the experimental class the dual goals of improvement in generalizing and

and mastery of algebraic concepts are actively pursued? If the experimental procedures are successful the results at the end of the year should indicate significantly larger gains made by the experimental class in ability to generalize and a mastery of algebra by both groups as predicted by the test results on the Iowa Algebra Aptitude Test. It would seem, however, that since the students in the public schools could spend considerably more school time on algebra than the experimental group, an attainment of the predicted algebraic mastery by the experimental class would indicate a greater achievement on their part than that of the public school classes, should they also attain their predicted mastery.

CHAPTER IV

SELECTED ILLUSTRATIONS OF CLASSROOM PROCEDURES

Two major goals were kept in mind as the instructor initiated the course in ninth grade mathematics--helping students to master mathematics and to develop their generalizing abilities. The students had in mind only the first objective, and therefore it was felt that it would be unwise to deal explicitly with the second objective in the first weeks of the course. Instead, situations in which students were learning algebra would be controlled so that the second objective would gradually assume importance for the student, and then when the time was appropriate it could be dealt with explicitly.

To make the student discovery procedure more effective, students were required to keep a notebook of new terms they had defined, mathematical generalizations they had discovered individually or as a class, and examples of the types of problems they had learned to solve. They were also strongly urged to include generalizations they had found in current periodicals and in their everyday living, accounts of how they as individuals or as a class had made mathematical discoveries, accounts of how others outside the classroom had made discoveries, and any other optional work they had done. They were urged to think of this notebook as a source

to which they could refer when they for some purpose needed to review a generalization or procedure. Since the notebook was to be an individual project, only a minimum number of directions were given to the students on the nature of their notebooks. The more imagination displayed in a student's notebook, the more valuable it would be in evaluating his ability to generalize and his understanding of mathematics.

Textbooks are not designed for the student discovery approach to teaching and no textbook sequence was followed. The greatest handicap of a text is that all rules and generalizations are stated explicitly and leave little opportunity for student discovery. Nevertheless, each student was issued a textbook and it proved to be a great timesaver for the instructor in that it served as a valuable source of problem materials. These problem materials were the most useful when students were learning how to solve equations and story problems and were studying indirect measurement.

When students meet for their first class in algebra, they bring with them a mixture of feelings toward the subject. Not only are they curious about algebra but because of adverse impressions received from others, they are frequently afraid of failure. In addition, many students, particularly those not interested in science, are skeptical about the value of algebra to them now and in their everyday living later. To counter such feelings of fear and skepti-

cism the content, at least in the beginning, must be easy, close to students' past experiences and rich with applications.

Considering this mixture of feelings, the study of signed numbers seemed to be the most appropriate initial topic. No other unit in algebra offers as good an opportunity for student generalizing. In a period of three or four weeks on this topic three different methods of generalizing can be introduced and contrasted. Also, the subject matter is new but with enough relationship to students' previous arithmetic experiences so as to be real enough and easy for even the slow students to make discoveries. In addition, it is easy for students to find applications of signed numbers in life around them.

Not all the lessons studied by students in the experimental class are discussed in this chapter. The main basis of selection was the extent to which the lesson proved illustrative of student discovery in action. For the purpose of presentation in this chapter the selected lessons are organized under the following topics:

1. Signed Numbers
2. Graphs and Signed Numbers
3. Graphs and Formulas
4. Algebraic Grammar
5. Translating Situations into Formulas
6. Solving Equations
7. Proportion
8. The Right Triangle
9. Non-mathematical generalizations

It must be emphasized that the order of presentation is not the exact order the lessons were taken up in class. For example, all lessons on Signed Numbers were not studied before introducing lessons under Graphs and Signed Numbers; all lessons on Graphs and Formulas were not studied before introducing lessons under Algebraic Grammar. The exact order in which lessons actually were studied depended to a large measure on the direction taken by the day-to-day group thinking of the class.

Signed Numbers

Discussion during the first day's lesson centered around numbers. Topics such as history of number, kinds of number, and number systems were mentioned. It was not long until someone mentioned the negative number. Most students had heard of this kind of number and had a correct idea of how it came into existence--namely by subtracting a larger from a smaller number. As a summary, the writer had the following table placed on the board:

$$\begin{array}{rcl} 5 - 1 & = & 4 \\ 5 - 2 & = & 3 \\ 5 - 3 & = & 2 \\ 5 - 4 & = & 1 \\ 5 - 5 & = & 0 \\ 5 - 6 & = & -1 \end{array}$$

From this table students could observe number patterns that would be impossible to detect if only one of the number facts were present. Also discussed was the tendency of mathematicians to make their work more general by eliminating certain restrictions placed upon it. The introduction of the

negative number eliminates the arithmetic restriction that the subtrahend must be smaller than the minuend. Such a discussion begins to give students a better idea of the nature of research in mathematics. The discussion was directed to the use of positive and negative numbers in solving problems by asking the students to think of problem situations in which signed numbers were either the answer, part of the given information, or both. The problems they gave were built around the thermometer or indebtedness. For the remainder of the class period students worked on the lesson sheet, Signed Numbers 1.

SIGNED NUMBERS 1

1. What is the reading on the thermometer?
2. Suppose the thermometer read -23 . What would be the meaning of the reading.
3. Locate and label the following readings on the thermometer scale at the left of this sheet.
 - a. 46° b. -3° c. 78° d. 63° e. -27°
 - f. $15\frac{1}{2}^{\circ}$ g. 32° h. $-8\frac{1}{2}^{\circ}$ i. $13\frac{1}{2}^{\circ}$ j. 21°
4. What is the point halfway between these points?
 - a. 74° and 110°
 - b. 0° and 10°
 - c. -8° and 8°
 - d. 4° and -2°
 - e. -3° and -9°
 - f. 6° and -8°
5. Which number is farthest from 0?
 - a. 6° or -10°
 - b. 26° or 18°
6. Which number is farthest from 12° ?
 - a. 18° or 8°
 - b. 26° or -5°
7. Which number is farthest from -2° ?
 - a. 7° or -10°
 - b. 2° or -8°
8. If the thermometer starts at 40° and moves to 47° how far has it moved? What was the direction of the change, up or down?
9. If the thermometer starts at 40° and moves to 33° , how far has it moved? What was the direction of change?

Signed Numbers 1 - continued

10. If the thermometer stands at 10° and then drops 16° , what will the new reading be?
11. If the thermometer stands at -12° and rises 5° , what will the new reading be?
12. If the thermometer stands at 23° and falls 27° , what will the new reading be?
13. If the thermometer moves from 14° to -10° , how far and in what direction (up or down) did it move?
14. If the temperature at the present time is -10° and we are told that the temperature has dropped steadily at the rate of $\frac{1}{2}^{\circ}$ every hour, was the temperature higher or lower six hours ago and by how much?
15. At 8:00 a.m. the temperature was 40° . If the temperature rises at the rate of 2° every hour, what is the temperature at 2:00 p.m.? If after 2:00 p.m. the temperature drops $\frac{3}{4}^{\circ}$ every hour, what will the temperature be at 6:00 a.m.?

Students experienced little difficulty in solving the problems on Signed Numbers 1 even though the answers appear to require the application of the rules for the four fundamental operations with signed numbers. They determined the answers without being aware of adding, subtracting, multiplying, or dividing, but worked the problems by visualizing the problem setting. Since it is important for students to know kinds of situations in which the various operations do apply, time was spent at the beginning of the next mathematics class period naming the operations suggested by each problem. Although it is not important for students to make explicit the rules for operating with signed numbers in solving such problems it is necessary if they are to have a reasonable mastery of elementary algebra. At this time students were introduced to the idea that they would be discovering for themselves as much as possible the principles they would learn in algebra, that such a procedure when generalized is important in non-mathematical situations, and that the first principles they would discover would be the rules for operating with signed numbers. During the discussion several illustrations were given which mentioned both faulty and good generalizations and how they would be discovered. The illustrations were examples of the procedure which was to be used in discovering the rules for adding and subtracting signed numbers. Since the idea of student discovery was novel the students were willing, if not some-

what eager, to try their hand at discovering the rule for addition of signed numbers.

Students had little, if any, difficulty using the simple enumeration procedures (see page 28) in discovering a method for solving abstract addition problems.* They did, however, have considerable trouble verbalizing their procedure so that the words actually said what was meant. Introducing the expression absolute value at the appropriate time helped. Verbalizing depends on analyzing, and analyzing made students wonder in some cases whether they were adding or subtracting. Because they discovered that the sum did not always have a value greater than any of the addends, it was necessary to examine the meaning of addition. Difficulty in verbalizing, together with the necessity for a revised concept of addition, disturbed students and led many of them to doubt the rationality of algebra and the value of student discovery as a teaching procedure. At such times the class makes little progress toward the discovery of a generalization, and since it is important for students to feel that they are making progress the writer tended to help the students more than he wanted to in the wording of the rule. Student confidence in ability to discover and recognition of

*The procedures for discovering the rules for addition of signed numbers were referred to in Chapter II and, therefore, just a few important points are mentioned here. If the reader is interested in the procedural details, they can be found, for the most part, in the discussion of the subtraction procedures which are discussed here in some detail.

the value of that ability are goals for the course, not prerequisites.

Teaching the subtraction rule involves four stages, each of which is more abstract than the one preceding it. In the first stage students work problems in real settings (such as shown in lesson sheet Signed Numbers 5) and then describe them with mathematical symbols.

SIGNED NUMBERS 5

1. The present temperature reading is in Column A and the previous hour's is in Column B. Find the difference in temperature and indicate whether the direction of the difference is up or down.

	Column A (is)	Column B (was)	Difference	Direction of Difference (change)
a	+55	+45		
b	-10	-15		
c	+16	+23		
d	-13	- 7		
e	+ 6	0		
f	+12	- 5		
g	0	+11		
h	- 3	+10		
i	-14	0		
j	+ 6	- 9		
k	0	-12		
l	-10	+ 2		

2. Write parts a through h in problem 1 as abstract problems. Write each in both horizontal and vertical form.

a. (e.g.) $+55 - (+45) = +10$

$$\begin{array}{r} - +55 \\ +45 \\ \hline +10 \end{array}$$

b.

c.

d.

e.

f.

g. ..

h.

3. Some hay farmers make it a practice to fasten a tab on the wire of a bale of hay. On this tab there would be a positive sign if the weight were above 100 lbs. and a negative sign if the weight were below. Thus +7 on a tab meant 7 pounds over the standard weight, and -7 meant 7 pounds under.

Column A represents the weight of a bale of hay your friend received. The corresponding number in Column B represents the weight of a bale of hay you received.

Signed Numbers 5 - continued

	Column A (his)	Column B (yours)	Difference	More or Less Than You?
a	+ 8	+ 5		
b	+ 6	+ 9		
c	-10	0		
d	+ 5	- 2		
e	-10	- 5		
f	- 1	-10		
g	0	+10		
h	- 7	- 5		
i	+ 2	+ 3		
j	- 8	- 6		
k	- 7	- 3		
l	- 2	- 9		

4. Write parts e through l in problem 3 as abstract problems. Write each in both horizontal and vertical form.

f.

g.

b.

1.

1.

k.

1.

5. Subtract:

$$\begin{array}{r} a. \quad +8 \\ \quad +4 \\ \hline \end{array}$$

b. + 9
+ 11
20

$$\begin{array}{r} c. + 5 \\ - 10 \\ \hline \end{array}$$

d. $+10$
 $- 6$
 4

$$\begin{array}{r} e. \quad -7 \\ +9 \\ \hline \end{array}$$

f. -6
-3

g. $\begin{array}{r} 0 \\ -8 \\ \hline \end{array}$

h. $\frac{-7}{0}$

6. Add:

$$\begin{array}{r} \text{a. } -125 \\ -214 \\ +251 \\ -12 \\ +440 \\ \hline \end{array}$$

b. $325 + (-100) =$

c. $+17 + 0 + (-2) =$

d. $-3\frac{1}{3} + (-1\frac{1}{3}) =$

e. $-19\frac{1}{2} + (+21) =$

$$\begin{array}{r} \text{f. } -15\frac{3}{4} \\ +17\frac{1}{2} \\ -21 \\ +19\frac{1}{2} \end{array}$$

Students are able to solve the problems because they can visualize the settings, not because they are able to apply a mathematical rule. Providing students with the opportunity to state problems in abstract terms familiarizes them with symbolic notation as it relates to the concrete setting and is helpful if students are to master operations with signed numbers. Problem 5 lists several abstract exercises to be solved. If individuals are to solve these exercises it is desirable for them to picture each in a concrete setting and then to solve them as they take on meaning in these settings. It is as important for the understanding of operations with symbols to invent settings for abstract problems as it is to describe problem settings in abstract terms. Problem 6 lists addition exercises which serve as a review and give the students an opportunity to contrast subtraction with addition. By seeing this contrast at the beginning, students will less likely confuse the two processes when used later in situations completely abstract. Signed Numbers 5 was given to the students without any group discussion. The problems were simple enough that all students made some progress in working them. Later, when the answers to the lesson sheet were checked, three types of errors were found to be the most common. First, students, probably because they did not read problem 1 closely enough, moved from the number in Column A to its correspondent in Column B to determine the direction of difference. Second, students

in answering problem 2 expressed the numbers in an incorrect order. Both these errors emphasize that care must be taken in selecting the correct number for the subtrahend and minuend in subtraction. Third, in solving the abstract subtraction exercises in problem 5 students obtained answers smaller than the minuend. For example, when they solved the subtraction problem, $\begin{array}{r} +10 \\ -6 \\ \hline \end{array}$ they obtained as an answer $+4$ rather than $+16$. To them the answer $+16$ meant they were adding, not subtracting. Such confusion calls for a new look at the meaning of subtraction.

A number scale is an excellent device through which a student's concept of subtraction can be generalized. The use of the number scale to solve subtraction problems was the second stage in learning the rule. A device such as the number scale helps the student to visualize not only the importance of order in subtraction but also when the difference between two numbers can be larger than the minuend. The general method for subtraction by use of the number scale was discovered by the class through group discussion.¹ Students were asked to solve as their first problem one in

¹The number scale method for subtracting signed numbers was the one suggested by J. P. Everett in his dissertation, The Fundamental Skills of Algebra, Teachers College Contribution to Education, No. 324. The method as applied to $+6 - (-4)$ is as follows: Locate $+6$ and -4 on the number scale. The absolute value of the difference is the number of units between the points $+6$ and -4 , in this case 10 units. The sign of the difference is determined by the direction of moving from -4 , the subtrahend, to $+6$, the minuend. In this case the sign of the difference is $+$ since the direction of movement is to the right. The answer to the subtraction problem, $+6 - (-4)$, is then $+10$.

which the answer was readily determined by arithmetic, e.g., $\begin{array}{r} +13 \\ +5 \\ \hline \end{array}$. After a method to obtain the answer on the number scale was found, students were asked to use the same method in determining answers to other types of abstract problems for which arithmetic did not provide the answer (either or both the minuend and subtrahend negative, or the value of the subtrahend larger than the minuend). These answers were checked by giving the original abstract problems concrete meaning and then by determining whether the number scale answer made sense. After the number scale procedure was understood by the class, students were given the lesson sheet, Signed Numbers 6, to work and turn in at the end of the class period.

SIGNED NUMBERS 6

$$\begin{array}{r} 1. \text{ Subtract} \\ +20 \\ -3 \\ \hline \end{array}$$

$$\begin{array}{r} 2. \text{ Subtract} \\ -17 \\ +12 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 0 - (+19) = \\ 4. +14 - (+7) = \\ 5. +10 - (-17) = \end{array}$$

$$\begin{array}{r} 6. \text{ Subtract} \\ 0 \\ -6 \\ \hline \end{array}$$

$$\begin{array}{r} 7. \text{ Subtract} \\ -22 \\ -16 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \text{ Subtract} \\ -13 \\ -13 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \text{ Subtract} \\ +17 \\ 0 \\ \hline \end{array}$$

$$\begin{array}{r} 10. -7 - (+15) = \\ 11. +21 - (-21) = \\ 12. +8 - (+13) = \\ 13. -5 - (0) = \end{array}$$

$$\begin{array}{r} 14. \text{ Subtract} \\ -4 \\ -10 \\ \hline \end{array}$$

$$\begin{array}{r} 15. \text{ Subtract} \\ -7 \\ +7 \\ \hline \end{array}$$

$$\begin{array}{r} 16. \text{ Subtract} \\ +4 \\ +4 \\ \hline \end{array}$$

$$\begin{array}{r} 17. \text{ Subtract} \\ +31 \\ -25 \\ \hline \end{array}$$

$$\begin{array}{r} 18. \text{ Subtract} \\ -27 \\ +19 \\ \hline \end{array}$$

$$\begin{array}{r} 19. +33 - (-13) = \\ 20. +29 - (+17) = \\ 21. 0 - (+32) = \\ 22. -23 - (-23) = \end{array}$$

$$\begin{array}{r} 23. -18 \\ -33 \\ \hline \end{array}$$

$$\begin{array}{r} \text{Subtract} \\ 24. +17 \\ +25 \\ \hline \end{array} \quad \begin{array}{r} 25. -16 \\ +23 \\ \hline \end{array}$$

$$\begin{array}{r} 26. \text{ Add} \\ +243 \\ -312 \\ +61 \\ -23 \\ \hline \end{array}$$

$$\begin{array}{r} 27. \text{ Add} \\ -1\frac{1}{8} \\ +\frac{1}{8} \\ -2\frac{1}{4} \\ +3\frac{1}{2} \\ \hline \end{array}$$

$$\begin{array}{r} 28. -7 + (+15) = \\ 29. +21 + (-21) = \\ 30. +8 + (+13) = \\ 31. -5 + (0) = \end{array}$$

$$\begin{array}{r} 32.* \text{ Subtract} \\ +98.12 \\ -35.15 \\ \hline \end{array}$$

$$\begin{array}{r} 33.* \text{ Subtract} \\ +37.236 \\ +55.987 \\ \hline \end{array}$$

$$34.* +6 + (-7) - (-9) + (+10) =$$

35.* Write a rule which will tell you how to work problems 1-25 without referring to the number scale.

*Optional

Because the central purpose of this lesson was to give students practice in solving abstract problems by the number scale, exercise 35 which calls for a generalization of the process was optional.

To discover the rule for subtraction, the third stage in this development calls for more extended study than the rule for addition. The reason, of course, is that in subtraction there are more than twice as many significant types of problems to consider as in addition. Lesson sheet Signed Numbers 7 was therefore highly structured so students could see the relation between subtraction and addition and be guided to the discovery of the usual rule with a minimum of difficulty.

SIGNED NUMBERS 7

Add

$$\begin{array}{r} 1. \ +11 \\ \quad -3 \\ \hline \end{array}$$

$$\begin{array}{r} 2. \ +4 \\ \quad +11 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \ (+9) + (+4) = \\ 4. \ (-9) + (+6) = \end{array}$$

Add

$$\begin{array}{r} 5. \ -19 \\ \quad -11 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \ -32 \\ \quad +16 \\ \hline \end{array}$$

$$7. \ (-5) + (-3) =$$

Subtract

$$8. \ (+20) + (-2) =$$

$$\begin{array}{r} 9. \ +11 \\ \quad +3 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \ +4 \\ \quad -11 \\ \hline \end{array}$$

$$11. \ (+9) - (-4) =$$

$$12. \ (-9) - (-6) =$$

Subtract

$$\begin{array}{r} 13. \ -19 \\ \quad +11 \\ \hline \end{array}$$

$$\begin{array}{r} 14. \ -32 \\ \quad -16 \\ \hline \end{array}$$

$$15. \ (-5) - (+3) =$$

$$16. \ (+20) - (+2) =$$

17. Write a rule which will tell you how to work the subtraction problems above.

18. Make up a story problem for exercise 13.

Subtract

$$\begin{array}{r} 19. \ -\frac{1}{4} \\ \quad \frac{1}{2} \\ \hline \end{array}$$

$$\begin{array}{r} 20. \ -\frac{2}{3} \\ \quad -\frac{3}{4} \\ \hline \end{array}$$

$$\begin{array}{r} 21. \ -6\frac{1}{2} \\ \quad +3\frac{2}{3} \\ \hline \end{array}$$

$$22. \ -.8 - (-6) =$$

$$23. \ +1.3 - (2.4) =$$

$$24. \ +.26 - (-5.04) =$$

Find the averages of:

$$25. \ 12^{\circ}, -3^{\circ}, +8^{\circ}, +6^{\circ}, -14^{\circ}, 0^{\circ}$$

$$26. \ +6, +0, -2, -9, -1, +12, -7, +10$$

This lesson sheet is so designed as to emphasize the nature of the subtraction rule:

1. Problems 1 through 8 are problems of addition while problems 9 through 16 are problems of subtraction.
2. The absolute values of the expressions in problems 1 and 9, 2 and 10, and so on up to 8 and 16 are the same, and, in addition, the answers in each pair are identical.

Students were to determine the answers to problems 9 through 16 by using the number scale and afterwards were to study the sixteen problems for clues that might help them formulate a rule. In using a lesson sheet composed of separate problems such as this one, it was difficult to get students to examine the problems as a group rather than as separate entities. As long as the student's frame of mind is such that he proceeds from problem to problem by working one and then forgetting about it, he will make little progress in discovering general principles.

An examination of the rules invented by several of the students as optional work on the lesson sheet Signed Numbers 6 revealed the following formulation which verbalized a new approach to the rule:

1. If you have a problem with a negative and positive number in it you add arithmetically the absolute values of the numbers and follow rule 3.

2. If you have a problem with both the same sign subtract, arithmetically, the smaller absolute value from the larger absolute value and follow rule 3.
3. If the subtrahend (the number that is being subtracted from the other) is larger,* the answer will be negative. If the subtrahend is smaller, the answer will be positive.

Before the work on lesson sheet Signed Numbers 7 was discussed, the class was invited to formulate as many different types of subtraction problems as possible. As they were mentioned they were placed on the board. With but little help students were able to discover the seventeen types below:

Types of Subtraction Problems

-6	-6	+6	+6	-8	-8	+8	+8	
<u>-8</u>	<u>+8</u>	<u>-8</u>	<u>+8</u>	<u>-6</u>	<u>+6</u>	<u>-6</u>	<u>+6</u>	
+8	-8	-8	+8	-6	0	+6	0	0
<u>+8</u>	<u>-8</u>	<u>+8</u>	<u>-8</u>	<u>0</u>	<u>-6</u>	<u>0</u>	<u>+6</u>	<u>0</u>

Determination of the types not only served as a basis for testing the rules students formulated during the lesson of the preceding day, but also provided an opportunity for teaching the use of a sample in the formulation of both mathematical and non-mathematical generalizations.

Most of the students who had formulated a rule mentioned the usual one. However, in spite of the highly structured lesson sheet aimed at discovering the usual subtraction rule, another student gave one similar to the three-

*The students' use of the word "larger" means larger in value, not larger in absolute value. For example, +6 is larger than -10.

step rule above. When asked which they preferred, the class chose the three-step rule. At first it was difficult to understand why they should prefer the longer rule, but upon reflection the reason seems fairly obvious. First of all, it avoids the statement which to them says, "If you want to subtract, add," and second, the longer rule is more directly an extension of the arithmetic concept of subtraction. By applying the three-step rule to the subtraction problem $\begin{array}{r} 5 \\ -2 \\ \hline \end{array}$, which in algebra is $\begin{array}{r} +5 \\ +2 \\ \hline \end{array}$, the problem is still worked the same way as it was in arithmetic. Also, upon examining rule 3, the procedure is consistent with the way in which students were introduced to negative numbers at the beginning of the unit, namely, by subtracting a larger number from a smaller number.*

After the rule for subtraction was accepted, the class proceeded to the following lesson sheet, Signed Numbers 8, which involves a series of abstract problems to be worked by applying the rules for addition and subtraction of signed numbers.

*Formulation of the three-step rule by this particular student apparently helped to change her from one who merely wanted to be told the rule into a student who enjoyed doing "research" on problems not required of all. The previous day she had openly rebelled at having to discover for herself the rule for addition. Later in the year she did "research" on graphing equations with two unknowns. Her discoveries led her to the study of conic sections and to the graphing of equations in three unknowns. Finally, with the aid of her family, she constructed a three-dimensional graph of a cone.

SIGNED NUMBERS 8

Perform the indicated operations by applying the appropriate rules.

$$1. \begin{array}{r} + -4 \\ +7 \end{array} \quad 2. \begin{array}{r} - +8 \\ -6 \end{array} \quad 3. \begin{array}{r} - -7 \\ -5 \end{array} \quad 4. \begin{array}{r} + +6 \\ +2 \end{array} \quad 5. \quad 8 + (-6) =$$

$$6. +18 - (+11) =$$

$$7. \begin{array}{r} - -19 \\ +6 \end{array} \quad 8. \begin{array}{r} + +16 \\ -9 \end{array} \quad 9. \begin{array}{r} + -17 \\ -12 \end{array} \quad 10. \begin{array}{r} - +22 \\ +8 \end{array} \quad 11. +25 - (-7) =$$

$$12. +19 - (+25) =$$

$$13. \begin{array}{r} - 0 \\ -12 \end{array} \quad 14. -18 - (-23) = \quad 16. \begin{array}{r} - -13 \\ -16 \end{array} \quad 17. \begin{array}{r} + +86 \\ -114 \end{array}$$

$$15. -32 + (-45) =$$

$$18. -28 - (+64) = \quad 20. \begin{array}{r} - -16 \\ +16 \end{array} \quad 21. \begin{array}{r} - +52 \\ +63 \end{array} \quad 22. \begin{array}{r} - -29 \\ -29 \end{array}$$

$$19. -14 - (0) =$$

$$23. \begin{array}{r} + +73 \\ +73 \end{array} \quad 25. +16 + (+19) =$$

$$26. -36 - (-40) =$$

$$27. +16 + (+17) + (-13) + (-6) =$$

$$28. +18 - (+16) - (+3) + (+16) =$$

$$29. -45 + (-15) - (-6) + (-4) =$$

$$30. -19 - (-5) - (-7) - (-22) =$$

$$31. \text{ Make up story problems for exercises 14 and 17.}$$

It is important to point out that students were again asked to create story problems (question 31) that could be symbolized by certain given abstract problems. This lesson sheet covers the fourth stage in developing the subtraction rule, namely, the direct application of the rule to abstract subtraction problems.

Teaching by the method of student discovery is frequently suggestive of other ways subject matter might be developed. The three-step subtraction rule divorces subtraction from addition and therefore suggests that the subtraction of signed numbers might be taught before addition. Such a reversal of procedure might very well encourage students to do more independent thinking. Both textbooks and parents would then be unable to tell them the rule. Textbooks almost universally advocate the use of the "change the sign" rule for subtraction and most parents learned that rule when they studied algebra. Also, if the negative number is introduced as an invention to make subtraction always possible, as it was in the experimental class, it would seem pedagogically sound to go into the study of subtraction as the next step in the study of signed number.

Rules for multiplication are not easily discovered by use of simple enumeration as described for subtraction. Constructing multiplication problems from concrete settings is somewhat artificial and more difficult for students to

comprehend. Therefore, it is fortunate that another method, continuity of form (see page 41), lends itself so nicely to the discovery of the multiplication rules. As an introduction to this method the account of the discovery of Halley's comet (see page 43) was related to the class. The story not only clearly illustrated the difference in procedures but also aroused considerable interest. After the story the following sequence of multiplication facts was placed on the board:

$$\begin{array}{ll} 5 \times 3 = 15 & 3 \times 8 = 24 \\ 5 \times 2 = 10 & 2 \times 8 = 16 \\ 5 \times 1 = 5 & 1 \times 8 = 8 \\ 5 \times 0 = 0 & 0 \times 8 = 0 \end{array}$$

Students easily generalized both sequences to include

$$\begin{array}{ll} 5 \times (-1) = -5 & (-1) \times 8 = -8 \\ 5 \times (-2) = -10 & (-2) \times 8 = -16 \\ 5 \times (-3) = -15 & (-3) \times 8 = -24 \end{array}$$

They also were ready with concrete examples when asked "Do the answers to these combinations make sense?" Examples they cited were problems based on the thermometer and a football game. One class member suggested that $5 \times (-1) = -5$ made sense because the problem meant to add -1 five times. When the class was asked what would be a good rule to follow to get the answer, the following three were suggested:

To multiply two numbers with different signs multiply the absolute values and the sign of the product is

1. the sign of the smaller number
2. the sign of the number with the smaller absolute value
3. always negative

As students mentioned the first two rules, other students eliminated them by citing examples that did not apply. The fact that two faulty generalizations were suggested first again emphasized the importance of listing many possible types of problems before generalizing. The students seemed to be quite pleased with themselves that they had discovered a rule so quickly.

The method of discovering the rule for the product of two negative numbers was similar to that of discovering the product of a positive and a negative number. The following sequences were placed on the board:

$$\begin{array}{ll}
 4 \times (-2) = -8 & (-3) \times 4 = -12 \\
 3 \times (-2) = -6 & (-3) \times 3 = -9 \\
 2 \times (-2) = -4 & (-3) \times 2 = -6 \\
 1 \times (-2) = -2 & (-3) \times 1 = -3 \\
 0 \times (-2) = 0 & (-3) \times 0 = 0
 \end{array}$$

Students sensed the number patterns in both cases, but they were annoyed when the extension of the patterns resulted in the following:

$$\begin{array}{ll}
 (-1) \times (-2) = +2 & (-3) \times (-1) = +3 \\
 (-2) \times (-2) = +4 & (-3) \times (-2) = +6
 \end{array}$$

The class admitted that the extension was consistent with the original pattern, but for them this fact was not proof enough that the product of two negative numbers should be a positive number. They wanted examples in which such number facts would make sense. Skepticism on the part of the class to this extension and a call for applications was the proper attitude a discoverer of ideas should assume upon sensing a new idea. But this contact with a new idea

radically different from past experience shook their confidence, weak as it was, in the student discovery approach. The writer sensed this and rather than asking for examples in which the product of two negative numbers is positive he gave several similar to the one below:

A water tank was punctured by a rifle shot three hours ago and is losing on the average of 5 gallons an hour. How much more or less water was there in the tank 3 hours ago than now?

Because of the artificiality of the problem and the class temper, such examples were not of much help. As a last resort an example was given* which was based on their arithmetic experiences. Everyone knew that when $(+6)$ is multiplied by $(+5)$ the answer is $(+30)$. Another way in which this answer could be obtained is to work the problem as follows:

$$\begin{array}{r}
 10 + (-4) \\
 10 + (-5) \\
 \hline
 (-50) + (?) \\
 100 + (-40) \\
 \hline
 100 + (-90) + (?)
 \end{array}$$

The value of $(-4) \cdot (-5)$ must be equal to $+20$ if the value of $100 + (-90) + (?)$ is to be $+30$.

This example was convincing; at least after citing the example, the class was willing to make an attempt at verbalizing the rule for multiplication. The first rule given was:

*Suggested to the writer by Dr. Nathan Lazar, Department of Education, Ohio State University.

To find the product of negative numbers, multiply the absolute values and give the answer a positive sign.

After this rule was examined and corrected to:

To find the product of two negative numbers, multiply the absolute values. The sign of the answer will be positive.

the following rule was added:

To find the product of two positive numbers, multiply the absolute values. The sign of the answer will be positive.

These two along with the rule for the product of two numbers with unlike signs made three multiplication rules for the class to learn. When the class was asked to enter the three rules in their notebooks, someone suggested that two of the rules could be made into one by the following statement:

To find the product of two numbers with the same sign multiply the absolute values. The sign of the answer will be positive.

A lesson sheet, Signed Numbers 13, was distributed which was designed to give the class practice in applying the rules for multiplication of signed numbers and set the stage for the discovery of a rule that would predict the sign for the product of three or more factors.

Signed Numbers 13 (in part)

	<u>Answers*</u>
1. $(-2)(-4)(+6) =$	+48
2. $(-3)(-3)(-3)(-3) =$	+81
3. $(+7)(+7)(+7)(-6)(-6) =$	+12348
4. $(-2)(-2)(-2) =$	-8
5. $(-5)(-5) =$	+25
6. $(-6)(-6)(-6)(-6)(-6) =$	-7776
7. $(+2)(+2)(+2)(-3)(-3)(-3) =$	-216
8. $(-1)(-6)(+2)(-3)(+7)(0) =$	0
9. $(-7)(+3)(-2)(+4)(-5)(-6)(-3) =$	-15120
10. $(7)(-1)(-1)(+2)(+6)(3) =$	+252

After the students had checked and corrected their answers, they were asked the following question: "Do any of you have a theory as to how the sign of the product could be quickly determined for each of these ten cases?" From that time until the rule was discovered no lead questions were asked by the writer. Five tries at formulating the rule were made by five different students before the rule was discovered. After each try the rule was tested with examples from the list of the ten problems already worked. Testing was done automatically by the students without any prompting from the writer. The five tries as worded by the students and recorded by an observer were:

1. The sign of the product will have the same sign as the last number (factor).
2. If there are an even number of numbers (factors), the answer will be positive.
3. If there are more negatives than positives the answer is negative.

*The answers were not on the lesson sheet, but are given here for the convenience of the reader.

4. Subtract the number of signed numbers from each other (number of positive factors from number of negative factors), if the difference is odd the answer is negative.
5. Look at the numbers (factors), if there is an even number of negatives the answer will be positive, if there is an odd number of negatives, the answer will be negative.

This lesson seemed to be enjoyed. Most of the class participated and the experience was remembered. That which was remembered was not so much the rule discovered, which is really rather insignificant, but the reflective process of arriving at the rule. A lesson of this type was appropriate at this time. Previous to this lesson, class members had made faulty mathematical generalizations, and examples of faulty non-mathematical generalizations had been mentioned in class. This lesson, however, afforded an excellent opportunity to discuss faulty generalizing in some detail. Not only did group discovery of this rule emphasize that it is easy to generalize incorrectly in mathematics, but it also clearly shows that faulty generalizations can be helpful in leading to the eventual discovery of a good generalization.

The discovery procedure predominant in the lesson account of Signed Numbers 13 was simple enumeration. Ten problems involving a series of factors and their respective products served as the observable facts from which the rule was discovered. The first step was to search each observable fact for a common property in the given series of factors

which would be usable in predicting the sign of the product. After this property finally was discovered the next step was to verbalize the property and transform it into a rule of operation. Other methods* of discovery were also present, as evidenced by the students' five tries at determining the sign of the product from the signs of the factors. Each try, with the exception of the first, suggested the following one. Try number two suggests that an even number of factors is related to a positive product; try number three suggests that the number of positive and number of negative factors should be dealt with rather than the total number of factors; try number four makes an attempt at combining the suggestions in the two preceding tries; try number five finally makes the correct generalization by ignoring the number of positive factors, an irrelevant fact and applying the "odd and even" idea only to the number of negative factors. This lesson provides an excellent illustration of group thinking.

Signed Numbers 14 was given to the students to work before the class at large discussed and verbalized the rule for the division of signed numbers.

*The methods were probably a combination of variation and inverse deduction. See pages 63 and 72.

SIGNED NUMBERS 14

1. $(+7)(\quad) = +35$
2. $(-8)(\quad) = -32$
3. $(-5)(\quad) = +30$
4. $(+4)(\quad) = -56$
5. $(-3)(\quad) = +93$
6. $(+6)(\quad) = -5.4$
7. $(-15)(\quad) = 240$
8. $(.2)(\quad) = 10$
9. $(24)(\quad) = -256$
10. $(-.2)(\quad) = 3.2$
11. $(-96)(\quad) = +12$
12. $(56)(\quad) = -49$
13. How did you find the missing numbers in exercises 1 through 12?
14. What do you think the rule for division of signed numbers should be?
15. Write problems 1 through 6 as division problems.
16. $\frac{39}{-3} =$
17. $\frac{-64}{-16} =$
18. $\frac{-84}{7} =$
19. $\frac{0}{6} =$
20. $\frac{8}{0} =$
21. Write story problems for exercises 16, 17, 18.

When the lesson sheet was discussed students almost immediately verbalized the division rule. Many of the students had already written out the rule as part of their assignment and most of the others sensed that the wording would be similar to that of the multiplication rule. The method students used in discovering the rule differed from the other two used thus far. From their knowledge of the relationship between multiplication and division, they deduced the division rule from the multiplication rule. After students had discovered the rule the following symbolization of their probable chain of reasoning was placed on the board to give them some idea about the nature of deduction:

1.	If $(-a)(-b) = +c$	Major premise
	$(-3)(?) = 39$	Minor premise
	The sign of the unknown
number is negative	Conclusion
2.	If $ab = c$, then $\frac{c}{a} = b$	Major premise
	$-3(?) = 39$	Minor premise
	$\frac{39}{-3} = ?$	Conclusion

If such a chain of reasoning is followed the sign of the quotient is determined by syllogism 1, and the absolute value by syllogism 2. To further clarify and show the importance of the syllogism in reasoning, the following non-mathematical example, which was taken from an article in a news magazine, was examined:

Eating foods containing oxalic acid is
harmful to the human body	(Major premise)
Spinach, as food, contains
oxalic acid.	(Minor premise)
Eating spinach is harmful to the
human body.	(Conclusion)

As in the discovery of the rule for multiplication, the discovery of the division rule was not derived from observable facts of everyday affairs. However, in order to prevent algebra from becoming nothing more than the manipulation of symbols lacking in meaning, it is important that students search for and find applications of the division rule in everyday affairs. One way of encouraging this search for applications is by asking students to invent problems solvable by the application of the division rule. In illustrating the division problem $(-60) \div (-3)$ one student provided the following physical setting:

Three downs before making a touchdown, a football team was sixty yards from the opponent's goal line. What was the average yardage per down?

The student thought of "three downs before" as -3 , the divisor, and "sixty yards to go" as -60 , the dividend. The answer, of course, is $+20$ since average yardage needed for a touchdown must be gained, not lost. To illustrate $(-60) \div (+3)$ another student gave:

A football team lost 60 yards (-60) in 3 downs $(+3)$. What was the average yardage per try?

The most difficult type to illustrate is the division of a positive by a negative number. After several attempts one of the better students, remembering the illustration given for $(-3) \times (-5)$ finally illustrated $\frac{+15}{-3}$ as follows:

Three hours ago (-3) a tank held 15 gallons more water $(+15)$ than it does now. What was the average change per hour?

The answer is a loss of five gallons (-5) per hour.

The problem which caused the most difficulty on Signed Numbers 14 was $8 \div 0$. Answers for the problem were about evenly distributed between eight and zero. Regardless of which answer a student had, each was certain he was right and no one sensed that the operation was not possible. In order to convince students their answers were incorrect they were asked to express this problem, $8 \div 0$, in the form used for the first twelve problems of the lesson sheet. After they changed the problem to "What number multiplied by zero will give eight?", they recognized there was no answer. To give division by zero some meaning, the following series of number facts was placed on the board.

$$\begin{aligned} 8 \div 8 &= 1 \\ 8 \div 4 &= 2 \\ 8 \div 2 &= 4 \\ 8 \div 1 &= 8 \\ 8 \div \frac{1}{2} &= 16 \\ 8 \div \frac{1}{4} &= 32 \\ 8 \div \frac{1}{8} &= 64 \end{aligned}$$

In generalizing by continuity of form students saw that as the divisor approaches the limit zero the quotient becomes infinitely large. For most of the students this was the first time they had heard the terms infinitely large and limit used in a purely mathematical setting.

It requires more than the discovery of rules to master operations with signed numbers. Practice is also

needed and it was, to some extent, provided for between discoveries. Within the framework of the practice, however, students also were given graphs involving signed numbers to interpret and were introduced to such ideas as order of operations and the use of parentheses. The opportunity for order of operations presented itself when the writer unintentionally gave an incorrect answer to a problem on one of the early lesson sheets. This incorrect answer resulted from working the series of additions and subtractions in a different order than from left to right. To capitalize on this error, the problem $24 + 3 \times 2$ was placed on the board and all students agreed the answer was 16 (the correct answer). The writer then gave an answer of +4 and asked if they knew how he had arrived at that figure. When students indicated the order of operations used to obtain his answer he told them that mathematicians usually performed the order of operations in such cases as this from left to right and that if they wanted to keep the numbers in the same order but wished to indicate multiplication was to be performed first, they would use parentheses or brackets around the 3×2 , for example, $24 + (3 \times 2)$. As a start in showing students when order of operations really makes a difference, students were given the following exercises to solve as part of the lesson sheet, Signed Numbers 9.*

*The lesson sheet makes use of the methods of variation and simple enumeration, see pages 28 and 63.

Signed Numbers 9 (in part)

1. $+8 + (+5) - (+6) =$
 2. $+8 + [(+5) - (+6)] =$
 3. $+6 + (-8) - (-5) =$
 4. $+6 + [(-8) - (-5)] =$
 5. $+12 - (-8) + (+9) =$
 6. $+12 - [(-8) + (+9)] =$
 7. $-3 - (-13) - (-21) =$
 8. $-3 - [(-13) - (-21)] =$
 9. $+7 + (-15) + (-22) =$
 10. $+7 + [(-15) + (-22)] =$
 11. $+16 - (-26) + (+18) =$
 12. $+16 - [(-26) + (+18)] =$
13. As you probably have noticed the problems above are given in pairs. The only difference between pairs is that their order of operations is different. Does the order of operations make a difference? Is this always true? If not, what are the peculiarities of the problems where order does and does not make a difference.

Several days later the example $10 - 6 \cdot 2 + 8$ was placed on the board for the class to solve. Class members agreed the answer was 16. When asked if they could get any other answers from the problem and justify them, members of the class gave the answers 4 and -50. It was here that the students saw the need of agreement and the rule was given, specifically:

If a series of operations is to be performed, unless otherwise indicated by parentheses or brackets, the multiplications and divisions are to be worked in order from left to right before the additions and subtractions.

If given a choice the students would undoubtedly have suggested that the order used should be from left to right as it always had been in arithmetic when they were given problems to solve mentally. The fact that the rule was

different annoyed many of the students and one expressed her annoyance by asking the question, "Why didn't you tell us ahead of time how the problems are to be worked?"

Several students, in response to her remark indicated that it was important to discover that several answers were possible in order for them to see the necessity for an agreement.

Introducing signed numbers extends the meaning of the signs - and + . In arithmetic the signs were used primarily as signs of operation. Now the signs indicate two other meanings--the sign of a number and direction of movement on a number scale. In order to perform algebraic operations it is necessary for students to use these meanings when appropriate and for them to know how all these meanings are related. Before studying signed numbers no ninth grade student would have had trouble solving a problem like $9 - 7 + 6 - 2 + 18$. After the introduction of new meanings for the signs - and +, the problem caused difficulty because the + and - signs are ambiguous. Lesson sheet, Signed Numbers 10,* was designed as a first step in helping students to acquire understanding in the correct use of signs.

*The lesson sheet makes use of the methods of variation and simple enumeration procedures. See pages 28 and 63.

SIGNED NUMBERS 10

1. $-9 + (+8) =$
2. $-9 - (-8) =$
3. $+6 - (-2) + (+8) =$
4. $+6 + (+2) + (+8) =$
5. $+6 - (-2) + (+8) =$
6. $+6 - (-2) + (+8) =$
7. $(-6) - (+9) + (+11) - (+27) =$
8. $(-6) + (-9) - (-11) + (-27) =$
9. $(+4) - (-5) + (-3) + (-13) =$
10. $(+4) + (+5) - (+3) - (+13) =$
11. $-10 - (+4) - (-9) - (+5) =$
12. $+10 + (-4) - (-9) - (+5) =$
13. $+10 + (-4) + (-9) - (+5) =$
14. $-25 + (+36) - (-15) - (+9) =$
15. $-25 - (-36) - (+15) + (-9) =$
16. $-25 + (-36) - (+15) - (+9) =$
17. As you probably have noticed several problems in succession, although they are different, are made up of numbers having the same absolute values. Neatly circle each similar group of problems. In each group indicate whether the answers are all the same, all different, mostly the same or mostly different. How could you have predicted this comparison by examining the problems in each group?

Solve the following problems, then rewrite the problem in two other ways using the same absolute values and the same order so that the answers will still be the same. Be certain to solve the problems you invent.

18. $+15 + (-7) + (-11) =$
19. $-11 - (+6) - (-16) =$
20. $+18 - (+11) - (+18) =$
21. $-15 - (+27) + (-9) =$
22. In how many ways can problem 18 be rewritten?

Students solved problems one through sixteen quite readily. However, the remaining problems caused some difficulty for about half the class. The successful solution of the last five problems required students to carefully analyze the problems they had already worked. Analysis requires a certain amount of self-discipline which a good fraction of the class had not as yet developed. Knowing this to be the case, a longer study period was given so the writer could give added individual attention. Time was spent the next day in discussing the generalization they had discovered while working the lesson sheet. Students were not required to verbalize the generalization but to illustrate it in their notebooks by examples. With the extra time for individual help and the class discussions which followed, a large majority of the class understood the idea for which the lesson sheet was designed.

Lesson sheet, Signed numbers 11 was designed to contrast further the relationship between the sign of operation and the sign of a number.

SIGNED NUMBERS 11

In solving problems involving a series of operations with addition and subtraction of signed numbers, it is easier to think of the problem either as an addition problem entirely or as a problem requiring the use of addition and subtraction of positive numbers. The exercises below will help you use both of these methods.

- I. Solve the problems then rewrite them as in the preceding lesson but so the operation to be performed throughout is addition. Be certain to solve the new problem to see if the answer really is the same.

1. $+8 - (+6) =$	5. $+6 - (+9) + (-4) + (+6) =$
2. $15 - (-7) =$	6. $-4 + (-7) - (-15) - (+6) =$
3. $-18 - (-27) + (+5) =$	7. $14 + (+3) + (+5) - (+16) =$
4. $-17 + (-11) - (+17) =$	8. $+16 + (15) + (+13) + (+1) =$

- II. Again rewrite the above problems, but this time make the sign of each number (with the possible exception of the first number in the series) positive. Be certain to solve the new problem to see if the answer is really the same.

1.	5.
2.	6.
3.	7.
4.	8.

- III. In arithmetic a series of operations, as $7 - 6 + 5 - 2$, the signs indicate the operation to be performed. If both sign of operation and sign of the number had to be indicated how would the arithmetic expressions below be written?

1. $3 + 6 + 2$	3. $10 - 6$
2. $9 - 5 + 13 - 2$	4. $17 + 2 - 7 + 10 - 2$

- IV. Except for keeping the same order of absolute values, entirely rewrite the series you wrote in Part III, but so that the answers are still the same.

1.	3.
2.	4.

- V. Rewrite the problems in Part I so they are in the familiar arithmetic form where but one sign is expressed between absolute values. Work the new problem to see if its answer is the same as the answer in Part I.

1.	5.
2.	6.
3.	7.
4.	8.

What sign is eliminated in each case?

In addition to the lesson sheet work some discussion was necessary before the class at large clearly sensed the following generalizations:

1. Any expressed subtraction of signed numbers may be rewritten without changing its value so that the sign of operation is +. Example:
 $+6 - (+4) = +6 + (-4)$.
2. Any expressed addition or subtraction of signed numbers may also be rewritten without changing its value so that the sign of the number added or subtracted is +. Example:
 $5 + (-6) = 5 - (+6)$; $5 - (-6) = 5 + (+6)$
3. If only one sign is expressed in an indicated addition or subtraction of signed numbers, the sign missing is +. The indicated sign can be assumed to indicate either the sign of operation or the sign of the number; depending on which is the more convenient.*
 Example: $6 - 3 = 6 + (-3) = 6 - (+3)$

Symbolic Thinking

Student discovery of algebraic generalizations is not possible unless the symbols of algebra have meaning for the students, and, in general, it seems fair to say that the more meanings are understood, the more successful students are in discovery. Therefore, the teacher who uses a student discovery approach should design lesson sheets so that correct mathematical meanings will be formulating within the minds of the students. The unit on signed numbers made use of most of the types of experiences which the writer considered necessary if students were to use symbols intelligently.

These types of experiences are summarized below; others will

*Here again, students were not required to include the generalizations in their notebooks in so many words, but were required to illustrate the ideas by examples.

be listed later in the chapter when the topics of study are more appropriate for the explicit statement of the principles:

1. Students should have experiences which provide practice in selecting suitable mathematical symbols that describe as accurately as possible the quantitative aspects of a wide variety of problem situations. (Signed Numbers 2 and 5, see page 110 and appendix.)

2. Students should have experiences which provide practice in finding and imagining situations in which quantitative aspects can be described by given mathematical symbols. For example, students were asked to verbalize problems based on such expressions as $+5 - (+2)$, $+6 + (-3)$, $(-5)(+2)$ and $(-6) + (-3)$.

3. Students should have experiences which provide practice in reading correct mathematical meanings into mathematical expressions. For example, they should be able to think of $(-1)(5)$ as $(-1) + (-1) + (-1) + (-1) + (-1)$ or should recognize that $5 - 2$ can be expressed as $5 - (+2)$ or $5 + (-2)$.

4. Students should have experiences which provide practice in manipulating symbols according to rules and procedures they have discovered. Practice must be provided if the manipulation of symbols is to become automatic. The degree of automatism required of each student and the amount of practice needed to attain such response depends upon the

interests, abilities and purposes of the individual student. In general, however, automatic response, attained in an intelligent manner, frees the mind for advanced discoveries.² Because important future discoveries in algebra are dependent upon a mastery of signed numbers, all students should be able to perform automatically, at least to some degree, the operations with signed numbers.

5. Students should have experiences which illustrate how principles and concepts can be extended by the invention of symbols. In the unit on signed numbers, subtraction was extended by the invention of the negative number to include cases where the minuend is smaller than the subtrahend.

6. Students should have experiences which illustrate how the meaning of certain mathematical symbols can be extended. In multiplication of signed numbers the meaning of positive number was extended to include the product of two negative numbers.

7. Students should have experiences in working with mathematical symbols with more than one meaning and in determining from the mathematical context which meaning is applicable. For example, $+$ can mean the sign of a number, the sign of operation, direction of movement, or a little more than. Juxtaposition means multiplication in $(-2)(-3)$,

²Whitehead, A. N., Introduction to Mathematics.
Oxford: Oxford University Press, 1911, p 42.

addition in $2\frac{1}{3}$, and place value in 23. (Mathematical synonyms)

8. Students should have experience in working with different symbols that have the same mathematical meaning. For example, multiplication can be indicated by a dot ($\cdot$), the times sign ($\times$), or juxtaposition; order of operations by parentheses or brackets. (Mathematical synonyms)

9. Students should have experience in working with mathematical symbols that have vague meanings and in symbolizing vaguely when it is advantageous to do so. For example, the expression, $7 - 3$, is vague about the sign of operation and sign of the number. The expressed sign could be either one of operation as in $7 - (+3)$ or the sign of the number as in $7 + (-3)$. Such vagueness, however, is justifiable only when a more definite meaning is not necessary to obtain the answer to a problem. Vagueness about irrelevant details often makes generalizations more inclusive. The expression $7 - 3$ includes both $7 - (+3)$ and $7 + (-3)$.

Graphs and Signed Numbers

A means by which operating with signed numbers can be made meaningful to students is to confront them with many concrete situations in which signed numbers are used--situations that have been largely overlooked by them before. Also, it is important for students to see how the negative number fits into the mathematical ideas to be encountered

next--graphical representation. The principles of mathematics are interrelated; therefore, the subject matter should be so organized and taught that this interrelatedness is readily apparent. The two lesson sheets which follow fulfill both criteria. Questions were based on a chart published by Dun and Bradstreet, Inc., Wartime and Postwar Shifts in U. S. Population and Businesses for Nine Geographical Regions. After the functions of a corporation such as Dun and Bradstreet were discussed, each student was given a copy of the chart to examine. This particular chart was especially effective for the purpose of discussing the properties of a good graph since it was colorful, easily read, clearly labeled, titled, and included a table of values and verbal description of the data presented. Students "discovered" these properties by examining the graph itself. After discussing the chart students were given lesson sheet Graphs and Signed Numbers 2.*

*Students worked this lesson sheet as their next assignment following the discovery of the rule for subtraction.

GRAPHS AND SIGNED NUMBERS 2

1. Which regions lost population during the years 1940 to 1944? Which gained?
2. The per cent change in population for all the United States from 1940 to 1944 was +.7%. List each region (abbreviate if you wish) and indicate whether its change was greater or less than the change for the United States at large. In each case also indicate how much greater and how much less.
3. Which regions lost businesses during the war years? Which gained?
4. The per cent change in number of businesses in the United States from 1940-1944 was -14%. List each region and indicate whether its change was greater or less than the change for the United States at large. In each case also indicate how much greater and how much less.
5. Which three regions had the highest per cent gain in population between 1944 and 1948? Which three the least? Would you be willing to say all states gained in population between 1944 and 1948? Why or why not?
6. List each region and indicate whether its per cent change in population from 1944 to 1948 was greater or less than the change in the United States at large.
7. Which three regions had the highest per cent gain in number of businesses from 1944 to 1948? How many per cent greater were these changes than the per cent change in number of businesses in the United States at large?
8. List the first three regions that had the most people move in between the years 1940 and 1944. In each case how many more people moved in.
*Can you account for this shift? Explain.
9. List the first three regions that had the largest number of businesses move in between 1944 and 1948. In each case list the number moving in.
*Can you also account for this shift? Explain.

*Optional

Graphs and Signed Numbers 2 - continued

- *10. If you were a person hunting for a job and free to move any place in the United States, which region would you most likely find a job? Explain.
- *11. If you planned on opening a small business such as a drug store or a grocery store, which region would you most likely be successful? Explain.

*Optional

Graphs and Signed Numbers 2 - continued

(For copy of chart, Wartime and Postwar
Shifts in U. S. Population and Businesses
for Nine Geographical Regions, see
envelope inside back cover.)

The next day answers were given and the starred questions discussed at some length. These starred questions were optional work since nothing as yet had been mentioned in class about drawing conclusions from graphical materials of this type.

Signed numbers are useful in situations dealing with opposites (e.g., below and above sea level; moving to the right and left). The next lesson* to be mentioned deals with this quality of oppositeness and also offers a familiar medium for the introduction of the coordinate system. In the discussion of opposite magnitudes, the writer made certain that the examples, north and south latitude and east and west longitude, were mentioned. When asked which directions should be positive, the class, apparently because it seemed consistent with past experience, readily suggested north and east. For the work period, students were given a small blank mercator projection map of the world and lesson sheet Graphs and Signed Numbers 4 in which they were to locate certain given countries by the coordinates of a point within the country and to name the countries in which given points (expressed by coordinates) would be found. A few of the examples were worked by class members on the large class map in front of the room, but the remaining ones they worked by themselves. Students enjoyed the lesson sheet, probably because it was a different kind of lesson and because they found it fairly easy.

*This lesson immediately followed the unit on signed numbers.

GRAPHS AND SIGNED NUMBERS 4

Countries and their Locations
in Terms of Latitude and Longitude

Country		Degrees Longitude	Degrees Latitude
1.		- 100	+ 40
2.	Brazil		
3.	China		
4.		- 60	- 20
5.	Australia		
6.		- 160	+ 60
7.		+ 40	+ 60
8.		+ 20	- 20
9.	Turkey		
10.	Mexico		
11.		+ 140	+ 36
12.	Tibet		
13.		+ 80	+ 20
14.		- 65	- 30
15.	Germany		
16.	Sweden		
17.		- 80	+ 50
18.		- 10	0
19.	Iran		
20.		- 65	- 18
21.	Norway		
22.		105	20
23.		104	63
24.		+ 25	- 32
25.	Egypt		
26.	Chile		
27.	Spain		
28.		+ 3	46
29.	Manchuria		
30.	Peru		

Graphs and Formulas

The study of formulas ties in nicely with both signed numbers and graphical representation. Formulas chosen for class study provided enough material on which a whole lesson or more could be built. In introducing this topic to beginning students, most textbooks use geometric settings. Geometric formulas, although they can be made quite concrete to the students, were not given much emphasis in this course, mainly because they usually involve more than two variables, which makes them inappropriate for graphing, and also because they are already familiar enough to the students to be considered "old stuff." There are many formulas with two variables derived from other settings which are new to the students, and these were used as a basis for designing lesson sheets. The purpose of the lesson sheets under this topic, Graphs and Formulas, is to give students

1. practice in evaluating formulas
2. opportunities for the discovery of the nature of relationships expressed in formulas, tables of values, and graphs
3. opportunities for determining the range of applicability of generalizations expressed by formulas, tables of values, and graphs
4. practice in plotting data and determining the line which is the best generalization of the

- data (line of best fit)
5. opportunities for making mathematical graphs from collected data
 6. opportunities for determining formulas from tables of values (the formula is a generalization of the tables of values)
 7. opportunities for interpreting graphs and charts
 8. opportunities for determining factors which are responsible for the occurrence of incidents, both mathematical and non-mathematical in nature (a search for relevant factors)
 9. experiences which show the importance of statistical procedures in the affairs of the ordinary citizen.

The intermediate lesson interrelating graphs, signed numbers, and formulas was based on the Fahrenheit-Centigrade relationship. To acquire a feeling for the relationship the class observed an experiment³ carried out by several students in the class. Several beakers were filled with water and then each heated or cooled so the water in each beaker was of a different temperature. The water temperatures were then each taken simultaneously by two thermometers, one calibrated in degrees Fahrenheit, the other in degrees Centigrade. As the data were collected

³Myers, Mark, and Reynolds, op. cit.

the class recorded these values in a table and then plotted the values on a graph. The range of temperatures varied from 95°C to -3°C. Students were surprised to see that the plotted points fell on a straight line. Before distributing Graphs and Formulas 1, the conversion formula, $C = \frac{5}{9} (F - 32)$, was introduced to the class and the procedure for evaluating formulas was discussed.

GRAPHIC AND FORMULAS 1

1. The temperature for converting Fahrenheit to Centigrade is given by the formula:

$$C = \frac{5}{9} (F - 32)$$

Fill in the values in the table below.

F	-4	+5	+23	0	-40	68	104	212
C								

2. According to Prime Minister Churchill the heat at the time of the explosion of the A-bomb in Australia was one million degrees Fahrenheit. What was this temperature in degrees Centigrade?
3. The temperature for converting Centigrade to Fahrenheit is given by the formula

$$F = \frac{9}{5} C + 32.$$

Fill in the table below.

C	0°	-10°	20°	-270°	100°	50°	-11°
F							

4. Examine the graph. At what temperature does the Centigrade reading equal the Fahrenheit reading?
5. Plot as many values for C and F (taken from 1 and 3 above) as your graph will allow. How does the line formed by these points compare with the line made by the points which describe the experimental data? How do you account for the comparison?
6. The extreme world temperatures and for the United States are given below:

	<u>World</u>	<u>United States</u>
High	136°F Azizia, Libya Sept. 13, 1922	134°F Death Valley July 10, 1913
Low	-90°F Verkhoyansk, Siberia Feb. 5 and 7, 1892	-66°F Yellowstone Park February 9, 1933
What are these temperature readings in Centigrade?		

Graphs and Formulas 1 - continued

7. In the small village of Mont Louis in the French Pyrenees a big burning glass has been constructed. This reflecting glass gathers together enough sun rays so that when they are focused on a metallic object its temperature is raised to 2500° Centigrade. What is the temperature in degrees Fahrenheit?
8. From the graphical picture of the relationship between Fahrenheit and Centigrade determine the values for
a. C if $F = -6^{\circ}$	c. F if $C = -6^{\circ}$
b. F if $C = 85^{\circ}$	d. C if $F = 20^{\circ}$
9. Review:
a. $-3 - 7 + 12 - 9 =$	f. $-13 - 6(5 - 10) =$
b. $+5 + 6(8 - 3) + 9 =$	g. $6(8 + 6) - 5(-8 + 15) =$
c. $-5 + (-9) - (-10) - (+6) =$	h. $15 - (-5) - (+17) =$
d. $15 + 5 \cdot 4 + 7 =$	i. $35 - 6 \cdot 7 + 39 + 3 =$
e. $+15 - 56 + 16 - 5 + 66 =$	j. $-8 - 9 + 2(7 - 10) =$
10. Write two interpretations of the following problems:
a. $-7 = 18 - 9 - 17$
b. $+3 + 9 + 6 + 5$
11. Write the problems without the parenthesis and with but one + or - sign between two successive numbers. Also give the answer to the problems.
a. $-7 + (-8) - (+6) + (+8) =$
b. $-15 - (+18) - (-10) + (-16) =$

The lesson sheets which follow are self-explanatory and were given to the students without any preliminary discussion.

GRAPHS AND FORMULAS 2

An approximation to the normal weight of an individual is given by the formula⁴

$$w = 5.5(h - 60) + 110$$

where "w" represents the number of pounds the individual weighs, and "h" represents his height in inches.

Give the first question a good try. If you cannot answer it, try again after working the second question.

1. a. According to this formula how tall would you have to be to weigh 110 lbs.? _____
- b. If a person weighs over 110 lbs., he would have to be at least how many inches tall? _____
- c. If a person weighs 220 lbs., how tall would he be? _____
2. Find the approximate normal weight of persons having the heights given in the table.

h	65	60	78	55	72	50	63	68	40	30
w										

Do you think the formula above is a good one? Why or why not?

If you had made the table above, how would you have organized it?

3. Make a graph showing the values for "h" and "w" in question 2. Be certain to label your graph properly, give it a title and include a table of values.
4. Read from your graph the value for "w" when h = 70 _____. When h = 58 _____. Determine the values for "w" by substituting the same values for "h" in the formula. Do the values agree? Explain.

⁴National Council of Teachers of Mathematics, A Source Book of Mathematical Applications, Seventeenth Year-book. New York: Bureau of Publications, Teachers College, Columbia University, 1942, p 120.

Graphs and Formulas 2 - continued

5. Plot your weight and height on the graph. Does the point representing your measurements fit on the curve? Plot the weight and height of three other persons on the graph. Try to get measurements quite different from yours.
6. How would you suggest we find out how good an approximation the formula is?
7. The actual statistics as taken from a calorie-counter-diet book and a baby book for the normal weight of a person according to his height are given in the table below:

h	20	25	30	35	40	45	50	55	60	65	70	75
w	7	15	22	28	38	46	59	88	115	130	150	180

Plot these points on the same graph you plotted the formula $w = 5.5(h - 60) + 110$ and then draw a smooth curve through all the new points. For what heights is the weight-height formula a fairly accurate predictor of the weight? . These values for the height for which the predicted values for the weight are reasonable are called the range for the formula. All other values, since they make little or no sense, are ignored.

8. On the board there is a list of weights and heights of many of your classmates. Plot these points on your graph. Would you say the line is fairly close to all the points and that there are about as many points on one side the curved line as the other? If so, the statistician would say the curved line was the "line of best fit." If the curved line does not fit the points as well as you think it should, draw a smooth line that is a better fit. The better the line fits, generally the more accurate one can make predictions. Often in medical and other science magazines, and occasionally in newspapers, one can find plotted data with their line of best fit. If you find any such graphs, please bring them to class before including them in your notebook.

GRAPHS AND FORMULAS 3

1. The maximum blood pressure within man's arteries is called systolic pressure. Below are the average systolic pressures for several different ages. These averages were taken from data collected at the Columbus Blood Center and from the nurse's office of the University School.

a	12	13	14	15	16	18	20	22	24	26
p	106	108	111	118	107	117	125	125	121	122

a	28	30	32	34	36	38	40	42	44	48	50
p	129	123	126	125	124	122	126	124	117	124	129

- Plot these values on graph paper.
- Draw a single straight line which fits this data. This line of best fit should have as many points on one side as on the other.
- Would you say the formula for this line (after it has been determined) would be a good formula for doctors to use in determining whether a person's blood pressure were normal or not? Why or why not?
- The formula⁵, $p = 110 + \frac{a}{2}$, is supposed to tell how a person's systolic blood pressure can be calculated if his age is given. Make a picture of this formula on the same graph as a above after first filling in the table of values below:

a	14	24	42	20	30	48	36	54
p								

- How could you find out which line was the best to use for prediction, the line you drew in b or the picture of the formula $p = 110 + \frac{a}{2}$?
2. The minimum blood pressure within man's arteries is called diastolic pressure. Below are the average pressures for several different ages.

⁵ Ibid, p 121

Graphs and Formulas 3 - continued

a	12	13	14	15	16	19	23	25	27	29
p	65	71	73	67	68	74	73	74	72	74

a	31	33	37	39	41	43	45	47	50	52	35
p	74	76	79	75	77	82	83	78	79	87	81

- Plot these values on the same graph as problem 1a.
- Draw the line of best fit as in problem 1b.
- After extending the line of best fit as far as you can, read the graph for your diastolic blood pressure. Would you say this would come fairly close to your true diastolic pressure? Why or why not?
- A formula for the normal diastolic pressure is $p = 71 + \frac{a}{5}$. Make a picture of this formula by filling out the table of values below.

*a							
p							

*Choose any values you wish for the independent variable, a , and then calculate the corresponding value for p , the dependent variable.

- Is this new line in nearly the same position as the first one? Which is more correct? Why?
3. In what way are these "curves" like the picture of the formula for height and weight? In what ways different?

Similar procedures were used in Graphs and Formulas 2 and Graphs and Formulas 3. Both deal with collecting data, graphing data, determining the line of best fit, and with the formula for the line of best fit. The difference was the order in which these activities were taken up in each lesson. In Graphs and Formulas 2 the formula was given, then data collected from various sources and graphed to see if the formula was a good one. In Graphs and Formulas 3 the order was changed to one more nearly approximating the procedures used by statisticians to determine a formula which best describes empirical data. Briefly, data concerning the blood pressures of individuals of different ages were plotted and the line of best fit estimated before the formula was introduced. The formula was not given to the students until they understood that part of the statistician's job is the actual discovery of the formula. It was made clear to the students that they were given the formula because it was too difficult a task for them at this time to discover it for themselves. The formulas involved in the two lessons are of a similar type; both express linear relationships and are examples of variation in the same sense⁶ (as one variable increases the other also increases). The lesson sheets which follow are based on formulas of different types. Again, each of the lessons is designed so students can begin work on them immediately without the aid of introductory remarks.

⁶This terminology was adopted from J. A. Swenson's text, Integral Mathematics with Special Application to Algebra. Ann Arbor, Mich.: Edwards Brothers, 1935, p 215.

GRAPHS AND FORMULAS 4

The following formula⁷ has been found which determines the number of hours of sleep a person should receive according to his age:

$$h = 8 + \frac{18 - a}{2}$$

1. Give these questions a good try. If you cannot answer them, try again after working the second question.
 - a. Is the above formula one which can be applied to all ages? If so, give a reason. If not, give the range of ages for which you think it would apply fairly closely.
 - b. According to this formula, at what age should you be before you need only 8 hours sleep?
 - c. According to this formula, how many hours of sleep should you get? Do you actually get that much sleep?
 - d. How much sleep should a new-born child get (according to formula)?
2. Make a graph of the above formula after first filling in the table below.

a									
b									

Extrapolate the curve as far as your graph will allow. Clearly mark that part of the graph for which the values are reasonable. Give a reason for your limits.

3. How many variables are there in the formula above? Name both the independent and dependent variables. How many variables do you think there actually are in determining the average amount of sleep a person should receive? If you believe there are more than age, name some of the variables. List the constants in the formula.
4. Compare the graph of the sleep formula with the graphs of the weight-height and age-blood pressure formulas. In what ways are they alike; what ways are they different?

⁷ National Council of Teachers of Mathematics,
op. cit., p 121.

GRAPHS AND FORMULAS 6

A formula used in engineering for determining the wind pressure on a billboard or on the side of a building is given by the formula⁸

$$p = .003 vv$$

where p is the pressure per square foot and v the velocity or speed of the wind in miles per hour.

1. Fill in the values in the table below.

v	10	20	30	40	50	60
p						

Plot these values on graph paper and draw a line that includes all the plotted points. Be certain to label the axes of your graph, give the graph a title, and include a table of values on the graph.

2. Read from your graph the pressure per square foot on a billboard if the wind is blowing at the rate of 15 m.p.h. (Substitute for v in the above formula and determine the value of p .) Did the value you read from your graph agree with the value of p as determined by the formula? If it does then the line you have drawn to include all the points in the table has been correctly drawn to include at least one other value for v and p .
3. If the wind is blowing at the rate of 35 m.p.h., what is the total pressure against a billboard 10 ft. wide and 18 ft. long?
4. In what way is the graph of $p = .003 vv$ like the graph of $w = 5.5(h - 60) + 110$? In what ways is this graph different?
5. All values for h and w that fit the formula $w = 5.5(h - 60) + 110$ fit on a single straight line. The values for v and p do not. How do you suppose this could be predicted by examining the formula?
- *6. In problem 1 how would you interpret a negative velocity for the wind? If you can find an interpretation, continue the line you have drawn to include the points where $v = -10, -20, -30, -40$, and -50 .

⁸Ibid., p 111.

*Optional

GRAPHS AND FORMULAS 8

The formula⁹ used in radio work to change kilocycles to wave length is

$$w = \frac{300,000}{k}$$

where w is the wave length in meters and k the number of kilocycles.

1. From examining the formula above would you say its picture was a straight line or curved line?

2. Fill in the table below.

k	200									1500
w										

3. After examining the table do you still agree with the answer you gave to question 1?
4. Plot the values in the table and then draw the line of best fit.
5. How is the "curve" like the "curve" of the formula $p = .003v^2$? How different?
6. How is the "curve" like the "curve" of the formula $h = 8 + \frac{18-a}{2}$? How different?
7. How is the "curve" like the "curve" of the formula $w = 5.5(h - 60) + 110$? How different?
8. By reading the graph determine the wave lengths used by the following radio stations.
 - a. WBNS 1460 kilocycles _____
 - b. WCOL 1230 kilocycles _____
 - c. WHKC 610 kilocycles _____
 - d. WOSU 820 kilocycles _____
 - e. WRFD 880 kilocycles _____
 - f. WLW 700 kilocycles _____
 - g. WVKO 1580 kilocycles _____

⁹Ibid., p 122.

Different types of formulas are dealt with in each of the above three lesson sheets. The formula, $h = 8 + \frac{18 - a}{2}$, is a linear relationship like $p = 110 + \frac{a}{2}$ and $w = 5.5(h - 60) + 110$, but is different in that it is an example of variation in the opposite sense¹⁰ (as a increases h decreases). When Graphs and Formulas 4 was discussed and comparisons made between the tables of values and the respective graphs of their formula and those on the lessons preceding it, the class discovered the differences and how they could have been predicted if only the table of values, the graph, or formula had been given.

The formula, $p = .003 vv$,* is a curvilinear relationship but is like $p = 110 + \frac{a}{2}$ in that for positive values of the independent variable the formula is an example of variation in the same sense. In the discussion following Graphs and Formulas 6, comparisons were made as before with the relationships previously studied. In addition, optional question 6, concerning the interpretation of negative values for the velocity, was discussed. Several class members answered the question by suggesting that a negative wind velocity could have the same speed but from the opposite direction as a positive velocity. Most of the students

*The notation vv was used because the concept of exponent had not as yet been introduced. Also, the writer feels that the use of the notation as vv should be used for some time. In this way a readiness is built for the study of the exponential notation.

¹⁰Swenson, J. A., Op. cit., p 215.

were willing to accept this interpretation, but a few objected. They felt that if the negative velocity meant wind from the opposite direction the formula, because pressure would be on the opposite side of the billboard, should give a negative value for pressure. Since substitution of negative wind velocity into the formula gave a positive value for pressure these few students were not willing to extend the domain of application into the realm of negative velocities. The thinking displayed by these few students was praiseworthy. Since the purpose of suggesting the possibility of negative velocities was to introduce the class to the parabola it was important that all class members accept the possibility of a negative interpretation for the wind velocity. To overcome the objections to a negative interpretation, the expressions vector and scalar quantities were introduced. Since in physics, velocity is defined as a quantity in which direction must be considered (vector quantity) and pressure a quantity in which direction is usually not considered (a scalar quantity) then the negative values for wind velocity do make sense. A discussion of interpretations for mathematical symbols is important. As more interpretations are found for mathematics the more meaningful mathematics is to the students. The more meaningful mathematics becomes, the greater the possibilities are for application.

Graphs and Formulas 8 deals with a fourth type of formula. The formula, $w = \frac{300,000}{k}$, expresses a curvilinear relationship and is an example of variation in the opposite sense. In the discussion following the completion of this lesson sheet, all the different types of relationships studied thus far were discussed and compared. To facilitate the comparison, sheets with the graphs and tables of the four different types of formulas were duplicated and given to each student. In making the comparisons students seemed to think it was reasonable that a formula such as $p = .003vv$ should be curvilinear, feeling that the double use of v in one term was responsible for the line curving. That the graph of the formula, $w = \frac{300,000}{k}$, also happened to be curvilinear surprised some of the more thoughtful students. Apparently they felt that an equation not containing a term making multiple use of a particular variable would be linear. Because of the students' insufficient algebraic background, formulas were not at this time analyzed to the point of determining the characteristic that would enable them to tell whether a formula expressed a linear or curvilinear relationship. It was pointed out, however, that because a formula containing a term wherein a particular variable occurs more than once is curvilinear, does not mean that the converse, a formula that is curvilinear contains a term wherein one variable occurs more than once, is true. To help students sense the meaning of

converse several examples of a non-mathematical source were cited (see page 230).

Formulas studied thus far contained only two variables. The next lesson sheet was based on a formula containing three variables.* This lesson happened to be one for the day following the football banquet which several of the boys in the class had attended. As an introduction to the lesson a discussion was started by asking the question, "Upon what factors does getting a football letter next year depend?" Such factors as weight, team spirit, and speed were mentioned. After other timely situations involving dependence were mentioned, the following question was, "Upon which factors do the seriousness of an automobile accident depend?" Answers given were: speed, weight, driving ability, where car is hit, number of people in car, weather and road conditions. When asked which two factors were generally the most important, students indicated speed and weight. The discussion was brought to an end and for the rest of the hour the class worked on Graphs and Formulas 9.

*Suggestion for the use of this formula and most of the lesson sheet based on it was made by Katharine Jones, a student in the Department of Education, Ohio State University.

GRAPHS AND FORMULAS 9

The seriousness of an accident can be expressed by the following formula:

$$X = \frac{WSS}{2}$$

where X is a number representing the seriousness of the accident, W is the weight of the car in pounds and S is the speed of the car in miles per hour.

1. Give these questions a good try. If you cannot answer them, try again after working problems 2 and 3.
 - a. How many times greater would the seriousness of an accident be if the weight were doubled? _____
 - b. How many times greater would the seriousness of an accident be if the speed were doubled? _____
 - c. How many times greater would the seriousness of an accident be if both speed and weight were doubled?
 - d. Explain the reason for your answers on the back of this sheet to the above three questions.
2. If your family car weighs 3200 lbs. make a graph if the speeds are:

S	10	20	30	40	50	60	70	80	90
X									

From the graph read the value X when the speed of your car is 15 m.p.h. _____. 45 m.p.h. _____.

3. Now, suppose the speed of any car you drive on the open road is always the same (40 m.p.h.). Make a graph if the weight of the car you are driving is

W	3000	3200	3400	3600	4000	4200
X						
4. How do the two graphs compare? How could you have predicted their differences by examining the formula?
- *5. In the formula above, the unit used in measuring weight, W, is pounds; the unit for speed, S, is miles per hour. What is the unit for X, the seriousness of an accident?

*Optional

After completing the above lesson sheet a discussion ensued in which the following ideas were considered:

1. Why the graph was curvilinear when the weight was held constant
2. Why the graph was linear when the speed was held constant
3. What effect doubling or tripling the speed has upon the seriousness of the accident
4. From what basic formula was the seriousness of the accident formula derived (formula for kinetic energy, $F = \frac{mv^2}{2}$)
5. What type of units are necessary if the variables in the formula, $X = \frac{WSS}{2}$, are to make sense

At the beginning of one class period, the formula $A = LW$ was placed on the board. The question was asked: "How would you make a picture of this formula?" One person suggested that it be plotted, as had been the other formulas, by letting one variable assume different values. This suggestion was followed until the student encountered difficulties. Then another person, apparently sensing how this formula was different from the others studied, suggested that values had to be assumed for two variables instead of one. A table of values was made. Someone else wondered how the three variables could be plotted on a graph with only two axes. The nature of other suggestions was such that there had to be a third axis and that this axis could not be in the plane of the board but perpendicular to the board at the origin. Finally, someone remembered that he had plotted a graph of three variables, $X = \frac{WSS}{2}$. The class

referred to Graphs and Formulas 9 and the graphing procedure was discussed. It was discovered that two graphs were made, one in which W was kept constant and the other in which S was held constant. Further discussion led to the conclusion that in an equation of three variables, three different graphs were possible. At this point students were ready for the lesson sheet Graphs and Formulas 11.

GRAPHS AND FORMULAS 11

As all of you know $A = LW$ is the formula for the area of a rectangle. If the same system for graphing a formula of two variables is to be used to graph this formula, it is necessary to assign a constant value to one of the variables thereby changing the formula to one with two variables.

1. All on the same chart, graph the picture of the formula $A = LW$ when
 - a. $L = 4$
 - b. $L = 8$
 - c. $L = 11$

Be certain to include all tables of values on the same sheet as the chart.

2. Study the relationship between the graphs in problem 1 and then without plotting the formula tell as much as you can about the picture of $A = LW$ when $L = 6$.
3. On a second sheet of graph paper, graph the formula $A = LW$ when
 - a. $W = 5$
 - b. $W = 9$
 - c. $W = 12$

Be certain to include all tables of values on the same sheet as the chart.

4. Study the relationships between the graphs in problem 3 and then without plotting the formula tell as much as you can about the picture of the formula $A = LW$ when $W = 4$.

5. On a third chart, graph the formula $A = LW$ when
 - a. $A = 72$
 - b. $A = 16$
 - c. $A = 48$

Be certain to include all tables of values on the same sheet as the chart.

6. Study the relationship between the graphs in problem 5 and then without plotting the formula tell as much as you can about the picture of the formula $A = LW$ when $A = 96$.
7. What are the characteristic curves for the formula $A = LW$?
8. List below all other formulas you can think of that have the same characteristic curves as $A = LW$.

The lesson sheet above proved to be fruitful in that it provided a setting from which several important ideas could be developed or extended. First, the lesson made extensive use of a common discovery procedure, variation, and the subject matter of the lesson was simple enough for students to analyze and from such an analysis make explicit this discovery method. Before students could graph $A = LW$, the formula had to be changed into a two-variable formula. Such a step illustrates a common problem solving approach, sometimes called transformation, in which an original problem, the solution of which is difficult or unknown, is transformed into one that can readily be solved. Briefly, the use of variation here changes the given conditions of the original problem into one whose nature or peculiarities are known. Graphs and Formulas 11 also uses the method of variation in another sense wherein the given conditions of the original problem in which the known nature or peculiarities are deliberately varied for the purpose of discovering its effect upon the final result. In 1(a) the formula $A = LW$ is graphed when $L = 4$. In the exercises immediately following 1(a), the given conditions are changed to graphing $A = LW$ when $L = 8$ and $L = 11$, and then the students are directed toward determining how each graph differs from the others. To help make explicit this latter use of variation, examples were mentioned in non-mathematical situations such as the effects

of using different soils on the growth of tomatoes or the effect of using different kinds of inoculations in the control of polio.

Second, the lesson extends the students' background knowledge so that they are in a better position to discover which characteristics of a formula are responsible for it being classified as expressing a linear or curvilinear relationship. As the reader will remember, Graphs and Formulas 6 left this characteristic undiscovered because at that time students did not realize that the formula $w = \frac{300,000}{k}$ and $wk = 300,000$ are equivalent. Plotting the curve for $A = LW$ when $A = 72$ (question 5, Graphs and Formulas 8) provided students with additional information that was necessary before they could discover that characteristic which tells whether a formula expresses a relation that is linear or curvilinear. In the discussion of question 5, the students' attention was directed toward examining the graphical patterns of $72 = LW$ and $w = \frac{300,000}{k}$. Students had no observable difficulty in seeing the similarity of the graphs. To help them see the similarities between the formulas, students were asked to express $72 = LW$ in different ways, and without too much hesitation they suggested the following:

$$\frac{72}{W} = L$$

$$L = \frac{72}{W}$$

$$\frac{72}{L} = W$$

$$W = \frac{72}{L}$$

The next step in discovering these "linear or curvilinear" characteristics was not made in connection with this lesson sheet but with the Algebraic Grammar lesson sheet dealing with the classification of algebraic expressions.*

Third, this lesson sheet is designed to help extend students' concept of mathematical variation. Question 8 which asks for examples of the same type as $A = LW$ was used as the point of departure for leading to the formulation of an important generalization concerning mathematical variation, namely, all formulas with three variables such that the product of two of them is equal to the third have similar graphical behavior patterns. Many of the students left question 8 unanswered, and to help them the following formulas were placed on the board:

1. $V = \pi r^2 h$	6. $C = nc$
2. $a^2 + b^2 = c^2$	7. $E = IR$
3. $V = at$	8. $c = Id^2$
4. $D = rt$	9. $W = pt$
5. $S = at^2$	

Four questions were then asked of the class:

Are these formulas alike in any way?

*As was indicated earlier all lesson sheets classified under Graphs and Formulas were not studied before lessons on Algebraic Grammar were introduced. Lessons from different topics were studied whenever a need for them occurred. In this case, the lesson sheet Algebraic Grammar 5 (see page 190) was one of the next lessons following Graphs and Formulas 11.

If you were to classify them, which would you have in the same group?

Which groups are made up of formulas like $A = LW$?

What property do these formulas have in common?

The discussion concerning Graphs and Formulas 11 terminated with the classification of all formulas grouped with $A = LW$ into the general type, $xy = z$.

A similar lesson followed in which the general type of formula, $x + y = z$, was studied. This type was introduced to the students by a simple visual device designed by Sheldon Myers.¹¹

A simple pulley was mounted on a board and a length of cord weighted at both ends was hung around the pulley. In the illustration above the "x" end of the cord is at 5, and the "y" end at 3. As the "x" end is changed to 4, 6, and 3, the "y" end changes respectively to 4, 2, and 5. These values were recorded in tabular form.

¹¹Myers, Mark, and Reynolds, op. cit.

x	y
5	3
4	4
6	2
3	5

Students quickly grasp that the values in the table all satisfy the equation $x + y = 8$. After this formula had been determined other lengths of cord were hung around the pulley to give new equations, e.g., $x + y = 4$, $x + y = -3$. To conclude the lesson all these formulas were generalized into the type, $x + y = z$.

Graphs and Formulas 10 served as a good starting point from which individual students could do "mathematical research" of their own. Students planning to study mathematics further were asked to invent different general types of formulas involving three variables and to plot their patterns. Two students accepted the challenge, invented equations as $x^2 + y^2 = z^2$, and became involved in the graphing of conic sections. Another student made a detailed study of the form $Z = \frac{5}{x^2 - y}$.

While the class studied patterns made by equations, the importance of patterns in everyday experience was also discussed. It was pointed out that man instinctively searches for patterns. Sometimes the patterns he finds are quite reliable such as finger prints, graphs made by a lie detector machine, a cardiograph or a seismograph; other times they are not, such as systems some horse race enthusiasts use in selecting "winners," and "reading" the

palm of an individual's hand to see what his past and future indicate.

Several important considerations were kept in mind when selecting the foregoing lesson sheets. Formulas were taken from a variety of situations most likely to be close to the lives of students. Also, formulas were selected which probably would fulfill several of the previously listed purposes of the unit on Graphs and Formulas. For example, formulas of different mathematical types were selected so that students, in addition to acquiring skill in graphing and evaluating formulas, would also acquire some background for reading directly from the formula whether it express a linear or curvilinear relationship.

Of greatest concern to the ordinary citizen are statistical graphs encountered in his daily reading of newspapers and magazines. He seldom constructs graphs and rarer still determines formulas for the graphs, but almost daily he interprets them in his reading. The inclusion of statistical charts and graphs, then, not only gives students experience with a type of subject matter of considerable practical value but also gives them an excellent media through which their generalizing abilities can be developed. The following sheet is designed to give students practice in interpreting statistical graphs.

INTERPRETING GRAPHS III

Buick Special with Dynaflow

Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

- ___ 1. The faster the Buick Special (with Dynaflow) travels, the more miles per gallon of gasoline.
- ___ 2. It is more economical to drive a Buick than a Cadillac.
- ___ 3. The number of miles per gallon for a Buick Special (with Dynaflow) averaging 80 m.p.h. is 11.9 miles per gallon.
- ___ 4. Buick is the best buy on the market.
- ___ 5. The number of miles per gallon of gasoline for 30 m.p.h. is twice as much as it is for 60 m.p.h.
- ___ 6. The faster a car travels, the less the number of miles per gallon of gasoline.
- ___ 7. The number of miles per gallon at 35 m.p.h. is approximately 21 miles.
- ___ 8. At twenty miles per hour the Buick Special (with Dynaflow) uses less gasoline per mile than at 25 m.p.h.
- ___ 9. The Buick Special (with Dynaflow) uses more and more gasoline per mile as the speeds increase from 25 m.p.h. to 75 m.p.h.
- ___ 10. Buicks without the Dynaflow get more mileage per gallon than Buicks with Dynaflow.

This lesson required no preliminary discussion before students were able to answer the questions, but considerable time was spent in considering student responses. A lively discussion resulted when their responses differed from those given on the lesson key. The class also studied the given data and then discussed the nature of conclusions that could be drawn from it with certainty, reasonable certainty (justifiable extrapolations and interpolations), or no certainty at all (irrelevant conclusions).

Algebraic Grammar

English grammar includes, among other things, the study of accepted forms for conveying ideas in English and the meanings given words and symbols because of their position in a written expression. In this same sense there is also a grammar for algebraic symbols, and in order to work intelligently with these symbols it is necessary to spend some time on algebraic grammar. The concepts symbolized by such expressions as exponent, factor, coefficient, term, and similar terms must be clear before the student can use the language of algebra intelligently. To attain such proficiency students not only must work with formulas that make use of such concepts, but must spend some time in breaking down the formula into its component parts and in working with each part outside the formula context. Two principles were kept in mind in organizing and presenting lesson sheets on algebraic grammar. First, aspects of

grammar studied by the students should evolve from other types of lessons such as the Graph and Formula lesson sheets just discussed and from solving equation lessons which will be discussed later. Second, seldom should a full day's lesson be devoted to algebraic grammar. Surprisingly, most students seemed to enjoy the study of this topic when it was presented in this manner.

Many errors in symbolic manipulation result when the referents of the words, coefficient and exponent, are not firmly established within the student's mind. One measure taken, which seemed to help students grasp the meaning of and distinguish between the concepts, was to introduce these terms after they had had considerable practice interpreting, evaluating, and graphing a variety of formulas. Preliminary to the introduction of algebraic grammar the following formulas, which had already been studied, were listed on the board:

$$F = \frac{9}{5} C + 32$$

$$W = 5.5(h - 60) + 110$$

$$P = 110 + \frac{1}{2} a$$

$$h = 8 + \frac{18 - a}{2}$$

$$p = .003vv$$

$$d = .045rr + 1.1r^*$$

*This formula was the basis of Graphs and Formulas 7 (See Appendix)

Many students remembered using exponents in connection with geometric formulas in the eighth grade and insisted upon writing the last two formulas as

$$p = .003 v^2$$

$$d = .045 r^2 + 1.1r$$

This circumstance provided an opening for the introduction of the word, exponent. The word coefficient was introduced when some student in the class made the inevitable mistake of thinking, for example, that the exponent 2 in v^2 meant 2 times v . Students were then asked to point out exponents and coefficients in the above formulas or in others that came to their minds. An important step in understanding the meanings of these terms is to recognize the relationship as indicated in the following:

$$a + a + a + a = 4a^*$$

$$a \cdot a \cdot a \cdot a = a^4$$

Algebraic Grammar 1, part of which is given below, was designed to give students practice in working with exponents and coefficients. As will be noticed two important teaching principles were followed in preparing this lesson sheet. First, each exercise required students to contrast the meaning of coefficient and exponent and then to choose the meaning applicable to the exercise at hand. Second, exer-

*Up until this time, students had thought of $4a$ as meaning 4 times a . This meaning is general and includes $a + a + a + a$, but such a fact does not mean students are aware of it.

cises within the same lesson required students to illustrate clearly the referent when the symbol is given and to select the proper symbol when the referent is given.

Algebraic Grammar 1 (in part)

1. Rewrite each of the following, using maximum use of exponents and coefficients.

- a. $b b b =$
- b. $b + b + b =$
- c. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 =$
- f. $3 x x =$
- h. $acc + acc + acc =$
- i. $m n m n m n =$
- j. $(a - b) + (a - b) + (a - b) =$
- k. $(a - b)(a - b)(a - b) =$

2. Rewrite each of the following in the long form (without using either coefficient or exponents).

a. $5m$	g. m^3n
b. x^4	h. $6m + 5m$
c. $(mn)^3$	j. $(m - n)^4$
f. mn^3	k. $5(m - n)$

To work the problems above, students had to generalize their original concept of exponent and coefficient. Their generalizing often resulted in faulty conclusions. For example, after working $(a - b) + (a - b) + (a - b)$ and getting the answer $3a - 3b$, a number of students felt $(a - b)(a - b)(a - b)$ should be $a^3 - b^3$. Such an instance provided an excellent setting for students to rediscover the ease in which one can overgeneralize and the importance of proof. The proof, as suggested by several students, was nothing more than substituting arithmetic values for the variables. However, in

emphasizing the importance of proof and the ease of over-generalizing, the teacher should guard against discouraging students from "playing their hunches." The teacher who makes use of student discovery procedures must endeavor to maintain a class atmosphere which on the one hand will encourage students to make certain hypotheses and then at the proper time to communicate these hunches to their class members, and on the other hand, to treat these hunches as such until some sort of proof has been established.

After the discussion of Algebraic Grammar 1, class attention was directed toward the definitions of coefficient and exponent. The first definitions students offered included the following phrasings:

A coefficient indicates the number of times a number is added.

An exponent indicates the number of times a number is multiplied.

Almost immediately several of the more alert students objected to this wording. To them the phrasings as applied to $3a$ and a^3 meant respectively 3 additions or $a + a + a + a$, and 3 multiplications or $a \cdot a \cdot a \cdot a$. To overcome this criticism a student suggested that the above phrasing should be changed to:

A coefficient indicates the number of times a number is used in a problem in which addition is the only operation involved.

An exponent indicates the number of times a number is used in a problem in which multiplication is the

only operation involved.*

According to the above definition of coefficient both 3 and a in the expression $3a$ could be considered coefficients. Since several students recognized, without any prompting, this possibility, the adjectives numerical and literal were introduced so that they could distinguish between the types of coefficients. As a summary of the discussion concerning coefficient students were asked to include in their notebooks the revised phrasing pertaining to coefficient, and also notes resembling those below:

$4a$

$4a$ can mean $a + a + a + a$;

4 is the numerical coefficient of a

a is the literal coefficient of 4

$5a^2b$

$5a^2b$ can mean $a^2b + a^2b + a^2b + a^2b + a^2b$

5 is the numerical coefficient of a^2b

a^2b is the literal coefficient of 5

a^2 is a literal coefficient of $5b$

b is a literal coefficient of $5a^2$

However, the meaning of coefficient alone without qualifying adjectives is that of numerical coefficient.

A more general meaning for coefficient

A coefficient is the multiplier of another number.

For example, the coefficient in $4a$ means 4 is multiplied by a; in $5ab$, 5 is multiplied by ab.

As a summary of the discussion of exponents students included the revised phrasing pertaining to exponent and the following examples:

*This definition applies only when the exponent is a positive integral number.

a^3 means $a \cdot a \cdot a$

ab^3 means $a \cdot b \cdot b \cdot b$

a^3b means $a \cdot a \cdot a \cdot b$

a^3b^3 means $a \cdot a \cdot a \cdot b \cdot b \cdot b$

$(ab)^3$ means $a \cdot b \cdot a \cdot b \cdot a \cdot b$

a^3b^3 can stand for the same number as $(ab)^3$.

These notes indicate the extent to which the concepts, exponent and coefficient, were developed during the first months of the course. Although no single verbal statement could be made that would define either of these concepts, the time spent on attempting to verbalize them broadened and clarified the students' understanding of the terms and also provided an evaluation device for determining the students' understanding of the concepts.

Other words confusing to students are term and factor. The introduction of the word term evolved from the discussion concerning the grammar of the formulas listed earlier (see page 180); the word factor rather naturally evolves from the study of the concepts of exponent, coefficient, and term--a coefficient is really a factor of a single term; an exponent indicates the number of times a number is used as a factor. The difficulty students have with term and factor seems to originate from the fact that they are not able to determine whether an algebraic expression is a series of factors, a series of terms, or a combination of both, and from their tendency to apply to a series of terms the rules of operation pertaining to a

series of factors, and vice versa. Algebraic Grammar 2, part of which is given below, shows how the writer coped with these difficulties. The theory behind the construction of the lesson sheet was that students will be able to make proper distinction between term and factor much better by working with these concepts from the very beginning rather than having separate lessons on the two concepts.

Algebraic Grammar 2 (in part)

1. If possible write these expressions in the shortest form possible.
 - a. $3t + 3t + 3t =$
 - b. $(3x)(3x)(3x) =$
 - c. $b \cdot b \cdot b \cdot b \cdot b =$
 - f. $5b + 3b + 2c =$
 - g. $2x + 3y + 4z =$
 - l. $(7d)(3c)(4a) =$
2. Write the following expressions without actually using a coefficient or an exponent.
 - d. $3a^3$
 - e. $c^3 - d^4$
 - f. $5mx^4$
 - h. $c^3 + d^3$
 - i. $(c + d)^3$

When answers to the lesson were checked, the procedure followed for most of the problems was to list on the board the answers students gave. In some cases there was disagreement. In asking students to support their answers two different procedures were used; the first, to check their answers by substituting actual values for the variables, the second, to show the line of reasoning used in arriving at the answer. For example in the latter case, the answer to $2a + 3a + 4a$ was obtained by the following procedure:

$$2a = a + a$$

$$3a = a + a + a$$

$$4a = a + a + a + a$$

$$2a + 3a + 4a = a + a + a + a + a + a + a + a = 9a$$

This procedure is the deductive method of discovery in action and provides a good opportunity to say a word about the deductive method also as a method of proof and to contrast it with the empirical approach to proof. The deductive approach proves the answer $9a$ to be correct regardless of the actual value of the number a . The empirical approach actually proves the answer $9a$ to be correct for only one value of a at a time. In applying the deductive approach, however, one must be certain that each step can be supported by good reasons. If it happens that the reasons are not applied correctly or if the reasons themselves are faulty, the deductive proof is really no proof at all. Therefore, an answer is more likely to be correct if verified by both approaches to proof. Each reinforces the other at their respective weak points

Two other problems on Algebraic Grammar 2 also proved to be worthy of discussion. Because they considered the likeness of $(7d)(3c)(4a)$ to $2x + 3y + 4z$ to be of greater significance than the difference, many students generalized by analogy from $2x + 3y + 4z$ that $(7d)(3c)(4a)$ could not be written in simpler form. In order to guide students toward the realization that $(7d)(3c)(4a)$ could be expressed as $84dca$ the following questions were asked:

What kind of operations are involved in the problem?

In a series of multiplications in what order or orders can the series be worked?

After 84doe was determined it was checked empirically.

The second problem that provided a good discussion was: Write $5mx^4$ without actually using coefficients and exponents. Five different answers were given, among which were $m + m + m + m + mxxxx$ and $(m + m + m + m + m)xxxx$. Since the answer the writer had in mind, namely $xxxxx + xxxxx + xxxxx + xxxxx + xxxxx$, was not suggested and also since he did not realize for the moment that a correct answer had been given, he persisted in asking for more answers. Feeling that a correct answer already had been presented, three or four students insisted that the answers on the board should be checked. The responses were checked and the correct one, namely $(m + m + m + m + m)xxxx$, was left on the board and beside it was placed the answer the writer had in mind. Students had not as yet discovered the distributive law, and the fact that two such different but correct answers could exist surprised them. Here again the actual happenings in the class are illustrative of the unpredictableness of the student discovery approach. Such an approach makes teaching interesting but requires the teacher to be constantly alert and to respect student suggestions.

The final algebraic grammar sheet to be discussed focuses student attention on the roles likeness and difference play in classification. Through this lesson sheet the writer introduced students to the words and phrases, monomial, binomial, trinomial, polynomial, the degree of an expression, and similar terms.

ALGEBRAIC GRAMMAR 5

1. Classify the following expressions

a. $2x + 3y$

h. $a^2 + b^2 + 3c^2 - 4d^2$

b. $2b^2 + 6a^2 + 7$

i. $abc + 5abc - 3abc + 2abc$

c. y

j. $V = \frac{4}{3} \pi r^3$

d. $3x - y = 6$

k. $4a^3$

e. $3x - y^2$

l. $5x^2y$

f. $a + b + c$

m. $4 + 5 = 9$

g. xy

n. $2 + y = 5$

2. Classify the following terms in as many ways as you can.

a. abc

j. $2b^3c$

b. $2x$

k. $6abc$

c. $2a$

l. $7x$

d. $5c$

m. $5a$

e. $5m^3$

n. $6bc^3$

f. 5

o. $14\mu m^3$

g. $5b^3c$

p. c

h. $7b$

q. $3b$

i. $5y$

r. 3

The first question in Algebraic Grammar 5 was used as the basis for a group thinking period planned for that hour. Students were asked to select expressions alike in some way, and as these were mentioned they were placed on the board. When students had made explicit the common property, the name was given to that type of expression, for example, first or second degree expression, monomial, binomial, or trinomial. The second question was given as preparation for the next day's development of the concept, similar terms. Common characteristics recognized by the students in making their classifications were the number of literal factors, the value of the numerical coefficient, and the same literal factor or factors. Several possible bases of classification provided an opportunity for discussing the problem of classification in general. Classifications which are significant are those which serve some important purpose, such as to classify previous knowledge or objects so they can be found quickly when needed, or to organize knowledge so it aids in the discovery of new knowledge. To focus the students' attention on the purposes of classification two non-mathematical examples were used. The first was read from an article concerning the desirability of classifying all equipment of armed forces into one catalog. Such an effort would be a time-consuming and costly task, but in the long run would be cheaper to the

taxpayer in that it would avoid order duplication and aid in interdepartmental requisitioning. The second example was based on the recent edict given out by the Pope that whales could be classified as fish and therefore eaten by Roman Catholics on Fridays. During the discussion students gave other examples of classifications useful in everyday living, suggesting items such as library books and telephone numbers, both of which are classified in different ways for the convenience of the public. After their attention was directed to the usefulness of classifying items according to some purpose, students were asked to look again at the classifications of the algebraic expressions on the board. At that time students were informed that because it was useful in working with algebraic expressions, classification of terms according to identical literal factors was given a special name, such terms being called similar.

After students had been introduced to the classification of algebraic expressions according to degree, they possessed enough background material to conclude with some success their search for the property of a formula which determines whether the formula expresses a relation that is linear or curvilinear (see page 174). All the formulas for which students had made graphs were again placed on the board and beside each formula was noted whether its graph was linear or curvilinear. After only a few minutes of

class discussion, several of the students recognized the determining property, namely:

The graph of a two-variable first degree formula is linear, while the graph of a two-variable formula of two or more degrees is a curved line.

This generalization, like the definition of exponent, is not adequate* when students, for example, study second or third year algebra and analytical geometry. Nevertheless, this fact should not prevent them from making temporary use of the property expressed in the generalization. Of course, students should be aware of the fact that mathematical generalizations like those they make every day may need to be revised as they acquire more knowledge and experience.

Translating Situations into Formulas

Another topic, introduced as Graphs and Formulas sheets were being studied, was the construction of formulas from various kinds of information. When a student is able to translate a variety of quantitative situations into algebraic language, he has mastered an important and one of the most difficult parts of algebra. It is important in that mathematics will be useful to the degree that the quantitative aspects of everyday affairs can be translated in mathematical terms. It is difficult because the construction of formulas depends to a large measure on an ability that takes a long time to develop, namely the ability to

*Examples of contradictions are $x^2 - y^2 = 0$ and $x^2 + 3y^2 = 0$.

think with symbols.* Many types of exercises, such as constructing formulas from verbal statements, from tables of values, and from geometric drawings, given at various times over a long period of time help to build proficiency in describing situations mathematically. Two types of lessons which proved especially interesting to students were those which directed them in their search for formulas describing properties of arithmetic series and the Tower of Hanoi puzzle.

At the beginning of one class period the arithmetic series,¹² 1, 2, 3, 4, 5, 6, 7, 8, 9, was placed on the board and students were asked to find the sum. When various individuals were asked to tell the group in what order they added, it was found that most of them added in order from left to right. There were two who added the numbers as follows:

$$\begin{array}{rcl} 1 + 9 & = & 10 \\ 2 + 8 & = & 10 \\ 3 + 7 & = & 10 \\ 4 + 6 & = & 10 \\ 5 & = & \frac{5}{45} \end{array}$$

Many class members expressed surprise and commented on the cleverness displayed by the students who had used this method. The class at large was convinced that this method

*For principles to be used in promoting symbolic thinking, see pages 140-143 of this chapter.

¹²Lesson suggested by Wertheimer, Max, Productive Thinking. New York: Harper and Brothers Publishers, p 89.

of finding a sum was much better than adding in order from left to right. To give them all a chance to use the method the following series were placed on the board for them to add:

3, 6, 9, 12, 15, 18, 21, 24, 27

2, 6, 10, 14, 18, 22, 26

8, 13, 18, 23, 28, 33, 38, 43

Students had little difficulty using the new method for finding these sums. It is important to note that the method they used in finding the sum of an odd number of terms differed from the method of finding the sum of an even number in that for an odd number the value of the middle term must be known. Students were then asked to find the sum of an arithmetic series when only the following information was given:

<u>First term</u>	<u>Last Term</u>	<u>Number of Terms</u>
4	48	12
4	48	23
111	102	8

Since the first exercise had an even number of terms the majority of the class readily obtained the sum, 312. They added the first and last terms and multiplied the sum by one-half the number of terms. But the series in the second exercise contained an odd number of terms and they felt they needed to know the value of the middle term before they could find the sum. The students were asked if they could determine the value of the middle term from the given infor-

mation. To help them answer this question the following complete series with an odd number of terms were placed on the board with the first, last, and middle terms underlined and the number of terms indicated.

<u>1</u> , 2, 3, 4, <u>5</u> , 6, 7, 8, <u>9</u>	9 terms
<u>3</u> , 6, 9, 12, <u>15</u> , 18, 21, 24, <u>27</u>	9 terms
<u>2</u> , 6, 10, <u>14</u> , 18, 22, <u>26</u>	7 terms
<u>51</u> , 64, <u>77</u> , 90, <u>103</u>	5 terms

With this help students discovered by the simple enumeration procedure how they could determine the middle term from the first and last terms. Of course, the procedure was to add the first and last terms together and divide the sum by two. By knowing this relationship they were also able to find the sum of an odd number of terms. Next, students were asked to generalize this procedure for finding the sum of an arithmetic series for any value A, for the first term; any value L, for the last term; and n, the number of terms. Almost immediately they suggested $A + L$ as part of the formula, but since they did not know whether n was odd or even, they had difficulty with the remainder of the formula. The class struggled a bit with the problem and then it was suggested that they determine two formulas, one when n is even and another when n is odd. Such an exercise was now merely a matter of substituting the literal number for the actual values in the exercises they had just solved, and it did not take them long to ob-

tain the following formulas:

n = any even number

n = any odd number

$$S = (A + L) \frac{n}{2}$$

$$S = (A + L) \frac{(n-1)}{2} + \frac{A+L}{2}$$

When both formulas were placed on the board one student discovered, when the class was asked to use it for finding the sum of several given arithmetic series, that the formula $S = (A + L) \frac{n}{2}$ would also determine the sum when n was an odd number. This fact suggested to several other students that perhaps the formula derived for an odd number of terms would also work for an even number. When the class found that it did work for several examples they were willing to accept the formulas as equivalent* and readily agreed that the simplest one, namely $S = (A + L) \frac{n}{2}$, should be used for finding the sum of an arithmetic series with either an odd or even number of terms. This discussion prepared the students for the lesson sheet, Arithmetic Series 1.

*Students were not yet able to simplify the formula for an odd number of terms and, therefore, were not able to prove by deductive reasoning that the two formulas were equivalent.

ARITHMETIC SERIES 1

1. Find the sum of the following series:

a. $18, 25, 32, 39, 46, 53, 60, 67, 74, 81 =$

b. $26, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, 71 =$
(16 terms)

c. $186, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, 355 =$
(14 terms)

d. $18, 10, 2, -6, -14, -22, -30, -38 =$

e. $27, 18, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, -54 =$

2. Complete the series to the term indicated:

a. $2, 5, 8, 11, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}. \quad (7 \text{ terms})$

b. $11, 17, 23, 29, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}, \underline{\hspace{1cm}}. \quad (9 \text{ terms})$

c. first term 7, each succeeding term is increased by 8. Write 10 terms of the series.

3. What is the indicated term of the following series:

a. $3, 6, 9, 12 \text{ --- tenth term? } \underline{\hspace{2cm}}$

b. $5, 14, 23, 32 \text{ --- fifteenth term? } \underline{\hspace{2cm}}$

c. $68, 71, 74, 77 \text{ --- twenty-fifth term? } \underline{\hspace{2cm}}$

d. $7, 11, 15, 19 \text{ --- hundredth term? } \underline{\hspace{2cm}}$

What is a quick way for finding the indicated terms in the four problems above?

e. $a, a + d, a + 2d, a + 3d \text{ --- nth term? } \underline{\hspace{2cm}}$

4.* What is the sum of fifty terms of an arithmetic series if

a. the first term is 5 and the difference between each succeeding term is 3.

b. the first term is 108 and the difference between each succeeding term is 17.

*Optional

The discussion of the lesson sheet during the next class period revealed that a large number of students had discovered a formula for finding the last term of a series when the first term, the difference between each succeeding term, and the number of terms were known. A few were even able to use the formula they derived in problem 4 for finding the sum of a series when the last term was not given. To stimulate further work on arithmetic series, the following story Bell tells of Gauss was read to the class:

Nothing extraordinary happened during the first two years (of school). Then, in his tenth year (10 years old), Gauss was admitted to the class in arithmetic. As it was the beginning class none of the boys had ever heard of arithmetic progression. It was easy for the heroic Büttner to give out a long problem in addition whose answer he could find by a formula in a few seconds. The problem was of the following sort, $81297 + 81495 + 81693 + \text{-----} + 100899$, where the step from one number to the next is the same all along (here 198) and a given number of terms (here 100) are to be added.

It was the custom of the school for the boy who first got the answer to lay his slate on the table, the next laid his slate on top the first, and so on. Büttner had barely finished stating the problem when Gauss flung his slate on the table. "There it is," he said--"Ligget se" in his peasant dialect. Then for the ensuing hour, while the other boys toiled, he sat with his hands folded, favored now and then by a sarcastic glance from Büttner, who imagined the youngest pupil in the class was just another block-head. At the end of the period Büttner looked over the slates. On Gauss' slate there appeared but a single number. To the end of his days Gauss loved to tell how the one number he had written was the correct answer and how all the others were wrong.¹³

¹³Bell, E. T., Men of Mathematics. New York: Simon and Schuster, 1937, p 221.

When it became apparent that most students were able to use the formulas $d = A + (n - 1)d$, and $S = (A + L)\frac{n}{2}$ in solving abstract problems they were given a lesson sheet, Arithmetic Series 2, which contained two practical problems, solutions of which depended upon the application of these formulas.

ARITHMETIC SERIES 2

1. A store selling television sets advertizes that a set may be taken home for a down payment of \$35. The next month's payment would be \$33, the next \$31 and so on for a total of 16 payments. What was the amount of the final payment and how much is paid for the set? If the set could be bought for \$275 what was the carrying charge? What per cent was the carrying charge of the list price? Would you say the carrying charge was reasonable? Why or why not?
2. In Mr. Peck's home the steps to the basement were open on one side and were, therefore, hazardous for his two year old daughter. To eliminate the danger he planned to nail slats on the open side as shown in the drawing below. In order to purchase slats he needed to know their total length. How many slats did he need to cut? What was the total length of the slats? How much longer must the second slat be than the first?¹⁴

¹⁴Suggestion found in Myers, Sheldon, "Applications,"
The Mathematics Teacher XLIII , December 1950.

To answer the question in Problem 2, "How much larger must the second slat be than the first," students had to solve the equation $99 = 3 + 16d$. A problem such as this was provided purposely to direct the students' thinking toward the next large unit they were to study, the solution of equations. Many students did solve the equation, however, by interpreting it as 99 is equal to 3 plus 16 times some number and then employing a refined guess-and-check system for determining the value of d.

One day near the end of the fall quarter when nearly all the class members were involved in the production of a Christmas program and had difficulty in settling down, the Tower of Hanoi puzzle was introduced to them as a subject for study. Since the puzzle had been available to them in the mathematics office, nearly every student had at one time or another tried working it. The puzzle (see figure below) was made up of three posts mounted perpendicularly on a rectangular base. On one post there was a series of flat blocks, square in shape with a hole in the middle, each square different in size. All the blocks forming a pyramid on one post were to be moved, one by one, never placing a larger block upon a smaller one, to another post.

When students had tried working the puzzle, few concerned themselves with transferring the blocks in the least number of moves. With this different twist to the problem, class interest in the puzzle was again revived. First of all, only one block was placed on the first post, the number of moves necessary being one; then two blocks, then three and on up to six blocks were placed on the first post and the number of moves was recorded in a table like the one below:

b	m
1	1
2	3
3	7
4	15
5	31
6	63

In working the puzzle a member of the class held the puzzle so it could be seen by all students and made the moves while the class dictated and recorded the moves. Before a new number of blocks was transferred, several members in the class were asked to predict the number of required moves. Guesses became more accurate as more data were placed in the table, the class discovering that as the number of blocks was increased by one, the number of moves could be predicted

by doubling that required for the last group of blocks and then subtracting one. When asked if they had used this method of discovery before, many of the students indicated that it was the same method they employed in discovering the rule for multiplication of signed numbers. Finding the formula, $M = 2^b - 1$, to fit the data in the table is too much to expect of ninth graders. Nevertheless, three tries were made and the last resulted in the formula, $M = b^2 - (b - 1)$, which worked for the first three values in the table.

The lessons on arithmetic series and the Tower of Hanoi puzzle illustrate an interesting point to be remembered when considering motivation. It is not always necessary for students to be aware of an application of the mathematics they are to learn before they become interested. Often, a lesson presented as a puzzle arouses much more interest than a lesson filled with highly practical problems. The lesson above was presented with a large degree of success as a puzzle with no attempt being made, at least in the initial stages, to convince the students the topic had practical worth. Puzzles usually arouse interest if the setting of the puzzle is understood by the students, but to maintain interest the puzzle must be neither too easy, which would bring no pleasure at solution, nor too difficult, which would discourage students from making an effort at solution. Although applications of practical significance are not required for stimulating student interest, such

applications should not be ignored. Stimulating interest in the topic is not the only pedagogical principle to be considered; to be able to use mathematical principles when needed is also of great importance. The best assurance that this principle will be fulfilled is to introduce to students and have them search for applications of the mathematical principles studied.

Solving Equations

From the early stages in the study of Graphs and Formulas the three following uses of graphs were emphasized:

1. To present data clearly and quickly to the consumer
2. To help in discovering trends and relationships
3. To help in making predictions

Near the end of the unit a fourth use,

4. To aid in solving problems

was presented as an introduction to the unit on solving equations. Graphs and Problems 2 is an example of one such lesson sheet.

GRAPHS AND PROBLEMS 2

Make a graph which will solve the following problem:

The Fairfield-Suisun Airfield in California received a message from Chicago, 2025 miles away, to refuel a B-36 that had left Chicago at noon (Pacific time) for Hawaii and which was traveling at an average ground speed of 325 m.p.h. The refueling plane at an average ground speed of 235 m.p.h. left the Fairfield-Suisun Airfield at 5:00 p.m. How far away from Fairfield-Suisun and at what time did the two planes meet?

B-36	
Hour of day	Distance from F-S Airfield
12	
1	
2	
3	
4	
5	
6	
7	

Refueling Plane	
Hour of day	Distance from F-S Airfield
5	
6	
7	
8	

Find formulas which will express the relationship between the time traveled and the distance traveled.

Nearly all students experienced difficulty in expressing by formula the relationship between distance and the hour of day. To help them express this relationship the following tables were placed on the board. Students filled in the specific values in each table.

h_1 = number of hours B-36 was in the air

h_2 = number of hours refueling plane was in the air

D = distance from Fairfield Suisun Airfield

h_1	D	h_2	D	Time	h_1	h_2
0	2025	0	0	5 p.m.	5	0
1	1700	1	235	6 p.m.	6	1
2	1375	2	470	7 p.m.	7	2
3	1050	3	705	8 p.m.	8	3
4	725	4	940			
5	400					
6	75					
7	-250					
8	-575					

$$D = 2025 - 325 h_1$$

$$D = 235 h_2$$

$$h_1 = h_2 - 5$$

Students gave formulas for the second and third tables without much hesitation. Before giving the correct formula for the first table they were asked to verbalize the procedure they used in obtaining the specific values for D for each value of h_1 . The procedure verbalized was not of much help because they determined the values for D by subtracting 325 from the previous value for D. Finally, a student gave the formula after the class had considered how several distances could be determined when only the number of hours in the air was given and no other values for D were known. After the three formulas were derived the analytical solution to the problem was placed on the board. The purpose

of the solution was not to teach students the procedure but to acquaint them with the fact that there was another method of solution. A large majority of the graphical solutions obtained by students were close approximations to the analytical solution. In discussing the degree of accuracy of the solution obtained by graphical means as compared to the one obtained by analytical means, it was mentioned that the graphical was probably as valuable as the analytical solution in the practical situation. Variables in the actual situation, such as velocity and distance, accuracy of reading instruments, and navigational deviations from a straight line flight plan, would cause as much, if not greater, variation in the mathematical solution as would any error in reading the graph.

Introduction of the analytical method of solving equations with such simple types as $x + 3 = 6$, $3x + 9$, $x - 7 = 10$, and $\frac{y}{5} = 2$, does not convince students of the power of the method. Since all these equations can be solved by inspection merely by asking the question implicit in the equation (in the first equation the question would be, "What number added to 3 is equal to 6?") they feel no need for any method, not even a guess-and-check method. If students are discouraged from asking themselves what the algebraic statement means, they are likely to operate mechanically with empty symbols. As the equations become more difficult, students are almost required to find some

system of solution if they are to be successful in discovering the value of the unknown. To solve problems in the lesson sheet Solving Equations 3, students employed a guess-and-check method. For example, in finding the value of $\underline{k}$ in the equation $\frac{7k - 5}{3} = 10$, a guess, say 3, is made and is substituted for $\underline{k}$ in the right member of the equation. The value of that expression then becomes $5\frac{1}{3}$. Since the value should have been 10, $\underline{k}$ cannot be equal to 3, but must have a larger value. Other guesses are made until the correct value for $\underline{k}$ (namely $k = 5$) when substituted in the left member of the equation gives 10, the same value as the right member. Spending time on the guess-and-check method is not only a readiness activity for the analytical method, but also fosters a desirable attitude if students are to become discoverers of generalizations that accurately describe the world about them. The lesson sheet, Solving Equations 3, was given to students to be solved by the guess-and-check method.

SOLVING EQUATIONS 3

1. Solve the following problems by the guess-and-check method.

a. $9d + 6 = 60$

g. $\frac{y + 6}{8} = 5$

b. $\frac{s}{6} - 10 = 3$

h. $\frac{3k}{4} = 18$

c. $\frac{s}{4} + 3 = 11$

i. $\frac{1}{2}n - 8 = 12$

d. $11x - 21 = 23$

j. $\frac{7k - 5}{3} = 10$

e. $\frac{c - 7}{2} = 6$

*k. $.06p + 5 = 10$

f. $\frac{n}{10} + \frac{1}{2} = 2\frac{3}{4}$

*l. $.4y - .3 = 5.7$

*m. $4n - 6 = n + 3$

2. Translate the following statements into algebraic statements (equations).

a. The length of a tennis court is 42 ft. more than the width.

b. The total number of baseball games a team won and lost one season was 154.

c. The team won 30 more games than it lost.

3. Make a "story" problem which will fit the following equations.

a. $\frac{x}{25} = 3$

*b. $\frac{x}{9} - 3 = 15$

*c. $3a + 6 = 24$

Following the lesson on Solving Equations 3, the equation $\frac{250 - 7}{3.5} + 4.6 = 252.6$ was placed on the board for students to solve by the guess-and-check method. Several unsuccessful attempts were made and the class was asked if they would like to discover a more systematic and direct method of solution, to which suggestion a widespread feeling of relief was immediately evident. They all sensed that the use of the guess-and-check method with such difficult equations was time consuming and unsatisfactory. As an aid for their discovery of the analytical method of solution, students were told that the method was analagous to the unwrapping of a package.* (At this time a package wrapped with paper and ribbon was shown.) First the operations performed in wrapping the package were listed:

1. Unknown contents placed in the box
2. Cover placed on the box
3. Box wrapped with paper
4. Ribbon placed around the wrapped box.

To undo these operations the students readily suggested the following:

1. Untie ribbon
2. Take off the paper
3. Take off the cover
4. Take contents out of box.

*Dr. H. P. Fawcett suggested in his education course at Ohio State University the use of this analogy in introducing the solution of equations by the analytical method.

At this time students noted that the undoing of the operations was in reverse order to that in which they were performed. The next step for the students was to analyze the operations performed upon the unknown number in the equation $\frac{25C - 7}{3.5} + 2.6 = 252.6$. They easily recognized the operations needed and the order in which they were to be performed. In fact, some members could have determined the unknown without further help from the group discussion. The mathematical operations and their inverse were placed on the board beside those for the package:

1. Unknown number, C, multiplied by 25
2. 7 subtracted from 25C
3. (25C - 7) divided by 3.5
4. 2.6 added to $\frac{25C - 7}{3.5}$

The undoing of the operations involves:

1. Subtracting 2.6 from $\frac{25C - 7}{3.5} + 2.6$
2. Multiplying $\frac{25C - 7}{3.5}$ by 3.5
3. Adding 7 to 25C - 7
4. Dividing 25C by 25

Before students could solve an equation, however, they needed to know something about the mathematical relationship of equality. Previously, students had discussed how an expression such as $3x - 4 = 6$ differed from $3x - 4$ and learned that the former expression, because of the equal sign, was called an equation. When at this time students were asked what equal meant, someone during the discussion mentioned

that in the United States everyone was equal. This remark was denied by many other members, especially those belonging to minority groups. Everyone in the United States does not have the same economic status, the same advantages and disadvantages, the same height and weight. To direct the discussion in a different vein the writer asked, "What then does the statement from the Declaration of Independence mean: 'We hold these truths to be self-evident, that all men are created equal'?" Finally, one member of the class suggested that "equal" in that sense meant that as an ideal all members in our society were equal in the eyes of the law, and the class agreed. The class was then informed that the equal sign in the equation had a similar meaning, each member of the equation being equal in the eyes of algebraic law. What was done to one member to change its value must also be done to the other member. As operations performed upon the unknown were undone in the equation the package was also unwrapped. The value of C in the equation was 35 and the package contained 35 pieces of candy. After checking the value for the unknown the class was asked to compare the guess-and-check and the analytical methods. Here again there was an opportunity to contrast an empirical with a deductive approach to discovery.

A few minutes were taken after solving the equation to discuss the procedure (formal analogy) used in discovering

the analytical solution. It was pointed out that even though the box and the equation were vastly different in substance, operating on the box was helpful in directing attention to the operations to be performed on the equation. Similar in the two situations were the ways in which the substances lent themselves to the performance of certain operations. Also mentioned were other examples where formal analogy had been helpful in making discoveries. For example, meteorologists find generalizations concerning the movement of water in the ocean useful in discovering generalizations concerning the movement of air in the atmosphere; atomic physicists find astronomical generalizations helpful in discovering generalizations concerning the atom. Before students worked on the lesson sheet Solving Equations 4 individually, two problems were solved on the board through the joint efforts of the students. These solutions were examined for common characteristics and it was noted that the procedure was to eliminate all except the unknown number from one member of the equation. Before solving the equations on this lesson sheet students were advised to list the operations performed upon the unknown. Such a listing helped the slower students considerably. In passing from table to table as students worked through the lesson sheet, the writer was gratified to find that even in these early stages the majority of the class was quite successful in using the analytical method.

SOLVING EQUATIONS 4

- 1.* On another sheet of paper solve and check the following equations (use algebraic form).

a. $\frac{2}{3}k - 1 = 18$

f. $\frac{d + 6}{2} = 2$

b. $\frac{1}{3}n - 8 = 12$

g. $3k + \frac{3}{2} = 9$

c. $12x - 21 = 23$

h. $.07x + 5 = 9$

d. $\frac{3c - 5}{2} = 6$

i. $5\frac{1}{6} = x - 2\frac{1}{3}$

e. $54 = 7d + 7$

j. $5 + \frac{x}{2} = 6$

k. $7 - 3x = 14\frac{1}{2}$

2. Under each of the following "story" problems, three equations are given. In each case select the equation which correctly represents the conditions of the problem.

- a. By selling eggs for 67¢ a dozen, a boy made a profit of 9¢. Find the cost per dozen.

$9c = 67$

$c - 9 = 67$

$c + 9 = 67$

- b. Agnes found that during one week she paid \$1.70 for her five school lunches. What was the average amount she spent per day?

$\frac{a}{5} = 170$

$5a = 170$

$a + 5 = 170$

- c. After working for 20 minutes, a typist found that she had completed one-fourth of the material which she was copying. How many minutes would she require to type all the material?

$m - \frac{1}{4} = 20$

$4m = 20$

$\frac{m}{4} = 20$

3. Write one or more equations that fit the conditions of the following "story" problems.

- a. A high school basketball team played 25 games. It won 7 more games than it lost. How many games did it win?

Solving Equations 4 - continued

- b. I bought some 2¢ stamps and four times as many 3¢ stamps. If I paid 70¢ for all, how many of each kind did I buy?

*Equations were selected which were difficult to solve by the guess-and-check method and which were examples of equations involving various combinations of fundamental operations.

When the unit on signed numbers was studied, several lessons were designed for the purpose of giving students an understanding of the relationship between the sign of operation and the sign of the number. Such an understanding enables students to be flexible in their use and interpretation of the + or - signs. Solving the equation, $7 - 3x = 14\frac{1}{2}$ requires such flexibility. During the process of solving equations, the negative sign, as in the above equation, is usually interpreted to be a sign of operation. However, after eliminating the number seven from the left member it is best to think of the sign as the sign of the number. The students had discovered that $7 - 3x$ could mean either $7 - (+3x)$ and $7 + (-3x)$ without changing the actual value of the expression, and the shift in meaning of the sign caused little difficulty. It is important to search for and then devote some class time to situations in which flexibility in thinking helps in problem solving. The situation here illustrates two kinds of flexibility; first, looking at a situation from a different point of view, a characteristic possessed by a superior generalizer (see page 84), and, second, changing a problem which is difficult to solve into an equivalent one which is easy to solve, an ability possessed by one who is skillful in solving mathematics problems.

It will be noted that on the lesson sheets Solving Equations 3 and 4, there were exercises other than solving

equations. These were included to give students additional practice in describing situations in mathematical terms and in inventing actual situations which can be described by certain given mathematical expressions. The principle of practice followed throughout the school year is to give practice at regular intervals and at such times the skill, concept, or principle being practiced is related to the new topic of study. Translating concrete problem situations into equations enables one to use equation solving techniques in determining numbers which, when interpreted in the language of concrete situations, are the solutions.

Proportion

During the last twenty minutes of a class period students were given the following quiz:

EQUIVALENT FRACTIONS (Quiz)

1. Fill in the missing number so the first fraction in each case is equal to the second.

a. $\frac{2}{3} = \frac{4}{\quad}$

b. $\frac{5}{\quad} = \frac{15}{8}$

c. $\frac{16}{48} = \frac{\quad}{12}$

d. $\frac{\quad}{35} = \frac{7}{5}$

e. $\frac{6}{15} = \frac{\quad}{5}$

f. $\frac{18}{32} = \frac{9}{\quad}$

g. $\frac{51}{17} = \frac{\quad}{51}$

h. $\frac{3}{\quad} = \frac{18}{24}$

i. $\frac{45}{25} = \frac{\quad}{100}$

j. $\frac{4}{5} = \frac{\quad}{100}$

k. $\frac{\quad}{100} = \frac{16}{40}$

l. $\frac{36}{\quad} = \frac{6}{7}$

2. Find and describe a single method of obtaining the missing number in every one of the twelve examples.
3. If both fractions in each example are equal there will be another relationship existing between the numbers in each example. What is it?

Everyone in class solved correctly the problems in Part 1 and the majority of the class made some attempt to verbalize the processes by which they obtained their answers (Part 2). None of them discovered a second way to find their answers (Part 3). Parts 1 and 2 clearly illustrate again that it is not necessary for students to verbalize a generalization before they can use it successfully.¹⁵ None of the students, not even those who worked all the problems in Part 1 correctly, were able to word the generalization in Part 2 so that it could be applied to all the problems in Part 1.

The next day the following proportions were placed on the board:

$$\begin{array}{cccc} \frac{1}{2} = \frac{2}{4} & \frac{1}{3} = \frac{2}{6} & \frac{2}{3} = \frac{8}{12} & \frac{9}{12} = \frac{3}{4} \\ \frac{3}{5} = \frac{12}{20} & \frac{2}{7} = \frac{16}{56} & \frac{8}{10} = \frac{16}{20} & \frac{6}{14} = \frac{9}{21} \end{array}$$

The students were asked, as they examined each proportion, to make attempts at discovering a principle that would relate the four numbers in the proportions. Their replies were all directly related to the basic principle concerning equivalent fractions--If you multiply or divide both numerator and denominator of a fraction by the same number, the value of the fraction remains the same. The students were pressed for other relationships and finally one of the students with average mathematical ability discovered the

¹⁵Hendrix, op. cit.

cross-multiplication relationship. Many of the other class members were chagrined because they had overlooked such a relationship. After testing the generalization by applying it to a number of cases, the proportion $\frac{a}{b} = \frac{c}{d}$ was placed on the board. Students readily suggested that cross multiplication as applied to this general case was $ad = bc$, and recognized that such an approach to a generalization was empirical. The students were then told that there was also a deductive approach to the statement $ad = bc$, but they were in no mood to discover this for themselves. Apparently, they had had enough discovery for the day. Rather than pass up the opportunity to contrast the empirical with the deductive approach, the second approach was presented to the class and the better mathematics students at least recognized the value of the approach in this case. As part of their individual work students were asked to try wording this new method for finding the missing number of a proportion and to hand it in at the end of the class period. When several of the better statements were read in class the students decided that the following statement was the one they understood best:

Take the number that does not have a number under or over it, and multiply it by the number diagonally from it. Then divide the product by the number that is left.

While the class was struggling with the discovery of the cross-multiplication relation, one student mentioned the following characteristic she had observed to be true. Her discovery illustrates the quality of thinking certain students can do when their minds are set free to explore the topic under discussion.

If a new fraction is formed so that its numerator and denominator are respectively equal to the sum of the numerators and denominators of the equivalent fractions in the proportion, then this new fraction is equal in value to either of the original fractions.*

Proportion problems are easy for students to solve after the quantitative aspects of the problem situation have been correctly translated into the form $\frac{a}{b} = \frac{c}{d}$. What is difficult for many students is the translation of the problem so that it expresses mathematically the correct correspondence between the given and unknown data. This proportion technique can be applied to two general types of problem situations, one in which the corresponding variables vary in the same sense $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, the other in which they vary in the opposite sense, $\frac{a_1}{a_2} = \frac{b_2}{b_1}$. The first type has for the ordinary citizen far more applications than the second. In most problems involving per cent, a proportion can be set up which will solve or help to solve the problem. Many students, especially those who have found per cent problems

*This wording represents the idea presented by the student. Since the student's wording was not recorded, the above is that of the writer.

difficult, prefer the proportion method. This preference stems from the fact that after the proportion has been set up for the particular per cent problem, the three main types of per cent problems* can be solved the same way. Most ninth grade students still have difficulty with per cent and several lessons were devoted to the solving of per cent problems in different settings. Introducing a new method of solution might give students a better understanding of per cent and at the same time does provide practice in setting up proportions. However, to ignore the inverse proportion would be to disregard the method of contrast as a teaching principle. A concept or principle is made more explicit to the learner if he also becomes familiar with a contrasting concept or principle. The law of the lever is an application of the second type of proportion problem. This application is most appropriate; not only is it close to the lives of the students but it also can be discovered directly from its concrete setting.

Before the class met, the following diagram and table were placed on the board:

*For the reader's information these three types are illustrated by the following abstract problems:
 1. 5% of 60 is ?. 2. ?% of 60 is 12. 3. 5% of ? is 12.

	w_1	w_2	d_1	d_2
1.	144 gr		10 cm	20 cm
2.	22 gr	110 gr	40 cm	
3.	270 gr	360 gr		30 cm
4.		172 gr	30 cm	20 cm
5.				
6.				

With equipment borrowed from the science room, the missing values in the table were determined. Values for the last two rows in the table were readings taken from the lever when two different students, in turn, were allowed to change the values w_1 , w_2 , d_1 , and d_2 as they so wished. The only restriction placed upon their selection was that the lever must balance. With these data before them, the students' attention was directed at the discovery of the relationship between the variables w_1 , w_2 , d_1 , and d_2 that would apply in all six cases (simple enumeration method of discovery). The relationship suggested and accepted by the class as applying was $\frac{w_1}{w_2} = \frac{d_2}{d_1}$.

At the time the class discovered the formula for the lever they realized that lever problems were, mathematically speaking, proportion problems, but they did not sense that they were of a different type than per cent problems. To help students sense the two types of proportion problems,

they were asked to solve the following:

1. The cost of 5 gallons of gasoline was \$1.36. By using the proportion technique find out how much 8 gallons of that same brand of gasoline would cost.
2. The cost of n_1 gallons of gasoline was C_1 and the cost of n_2 gallons was C_2 . Write a formula expressing the relationship between n_1 , n_2 , C_1 , and C_2 .

A comparison of the formulas, $\frac{n_1}{n_2} = \frac{C_1}{C_2}$ and $\frac{w_1}{w_2} = \frac{d_2}{d_1}$ revealed the basic difference between the two general types of proportion problems and also emphasized the necessity for exercising care in setting up proportions. While discussing the two types of proportion a comparison was made with the variation concepts studied in the unit Graphs and Formulas, namely, varying in the same and opposite sense.

Earlier in the chapter, nine types of experiences were listed which would help students to think more intelligently with symbols. This unit on ratio and proportion suggests three more types:

10. Students should have experiences which lead them to recognize the concise nature of mathematical symbols. The mathematical statement, if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ is certainly more concise than if it were given in words.

11. Students should have the type of practice which will lead them to recognize the importance of using, wherever possible, symbols indicative of their referents. The formula

$\frac{w_1}{w_2} = \frac{d_2}{d_1}$ gives more direction in setting up a proportion, as it applies to lever problems, than does $\frac{a}{b} = \frac{c}{d}$.

12. Students should have experiences which lead them to recognize that properly selected symbols often suggest new generalizations. The formulas $\frac{n_1}{n_2} = \frac{c_1}{c_2}$ and $\frac{w_1}{w_2} = \frac{d_2}{d_1}$ reveal generalizations concerning the correspondence of the variables in each formula. Discovering this correspondence by examining formulas such as $\frac{a}{b} = \frac{c}{d}$ and $\frac{w}{x} = \frac{y}{z}$ would be considerably more difficult.

The Right Triangle

The only illustration given here in the study of the right triangle is the account of the procedure used in the discovery of the Pythagorean relation. As an aid in making the discovery, each student was given the following assortment of right triangles, all drawn according to scale:

RIGHT TRIANGLES

- I. Find, if you can, a single equation that states the relationship between the sides of all the right triangles.

- II. Determine whether the triangles are right triangles.

Students recognized that all the triangles in part I were right triangles. It was suggested that for these particular triangles there might be a definite relationship between the lengths of the sides. In order to investigate this possibility a student suggested that the known data on each triangle be organized into the following tabular form:

Triangle	a	b	c
A	3	4	5
B	5	12	13
C	8	15	17
D	20	21	29
E	7	24	25
F	28	45	53
G	6	8	10

Various members of the class suggested the following relationships:

$$1. \frac{b+c}{a} = a$$

$$3. a^2 = b + c$$

$$2. (a+b) - (a-1) = c$$

$$4. a^2 + b^2 = c^2$$

The first three relationships are true for three of the right triangles A, B, and E. The third relationship, $a^2 = b + c$, was suggested by one of the four students for whom this was the first year in the University School and who had had previous contact with the Pythagorean relation. His suggested relationship was the result of faulty recollection of his previous study of this topic. Nevertheless, the suggestion of squaring the length of a side led to the relationship $a^2 + b^2 = c^2$. (During the discovery period it did not occur to the students that the statements

$\frac{b+c}{a} = a$ and $a^2 = b+c$ were merely different ways of stating the same relationship.) The statement, $a^2 + b^2 = c^2$, was checked for the data in all the seven examples in the above table. Even then, one student commented that the relationship might not be true in general, but, in response, another student said, "yes, but we are more certain it is true after checking it with seven cases than we would be if we had tried it with just three." To be still more certain, students suggested, the relationship must be checked by applying it to many other right triangles and a quick way of doing this would be for each student to draw a right triangle, measure the sides and substitute the values in $a^2 + b^2 = c^2$. Checking the relationship in this way led to a discussion of the approximate nature of measurement and the difficulties involved in making accurate drawings. After all the students were convinced the relationship applied to their triangle, they were told that Pythagoras had proved deductively that this relationship was true for any right triangle, and that they would become familiar with such a proof when they studied demonstrative geometry.

The statement of the Pythagorean relationship provided an opportunity to discuss the converse of an accepted generalization. Previously, the class had rejected the converse of the generalization, "If an equation makes multiple use of a particular variable in one of its terms, then its graph is a curved line." (See page 166). Because

the converse is false a general discussion of converse had been delayed until such a time as now when it happened to be true. As an introduction to the concept of converse, several statements as those below were placed on the board and their converse given:

1. Most dogs chase cats.
2. Most boys in their late teens are interested in girls.
3. If a polygon has three sides, it is a triangle.
4. If it is night, we cannot see the sun.

Students readily agreed that two of the converse statements were true and the other two false. After several attempts, the class was able to state the converse of the Pythagorean relation. A discussion followed concerning the truth of the statement and the practical value of the generalization. The statement was accepted as true after each member in class found that angle C was a right angle when the lengths of the sides of a triangle each had constructed were such that $a^2 + b^2 = c^2$.

Non-mathematical Generalizations

Throughout this chapter, illustrations of methods of discovery in areas other than mathematical were cited when appropriate. For example, when the method of discovery, continuity of form, was used in discovering the rule for multiplication of signed numbers the story was told of a similar method used by Halley in predicting the recurrence

of a comet every 75 years. When classification of algebraic expressions was the topic of study mention was made of the classification of all armed forces equipment into one catalog. In addition to the mention of non-mathematical discoveries at the time students made mathematical discoveries, interesting current discoveries in non-mathematical fields of knowledge were considered. Examples of such discoveries were the use of Gamma Globulin in preventing certain forms of polio and the use of the drug, ACTH, in the treatment of burns. Non-mathematical generalizations were also mentioned in class when they happened to be the subject of statistical graphs being studied. In fact, one basis of selection of the graphs for study was their general interest to other fields of knowledge. One example of such a graph was included in this chapter (see page 146), and others, such as Salary Treadmill and Automobile Registration and Cost, can be found in the Appendix. Still another way of studying generalizations in everyday life was to have students actually work lesson sheets based on such generalizations as in Generalizations 1.

GENERALIZATIONS 1

The article below was taken from the newspaper, The Columbus Citizen, November 28, 1952. The experiment described was a step in the direction of answering the question, "Does smoking increase a person's chances of getting lung cancer." Read the article, then answer the questions.

Half a lifetime of smoking a cigaret once every hour for a 12-hour day increases the chance of getting lung cancer by about one-third--for mice, that is, with a hereditary tendency to lung cancer.

This finding is from one of the very few experiments in which animals have been used to try to settle the controversy over whether tobacco smoke causes lung cancer. The experiment was made by Dr. J. M. Essenberg of the Chicago Medical School.

The mice, of course, did not actually smoke the cigarets. They were housed in a special cage with a specially designed automatic smoking machine. This machine had a rotary cigaret carriage holding 12 cigarets. An automatic electrical cigaret lighter connected to a program clock lighted a cigaret every hour for 12 hours. A vacuum pump created just enough suction to burn the cigaret and circulate fresh air.

Of 23 mice in this cage with the hourly cigaret smoke for one year, 21 developed lung cancer. Of 32 mice of the same strain, housed and fed the same but not "smoked," 19 developed cancer. The difference, 31.9 per cent, is statistically significant. The probability of its occurring by chance is less than one in 100.

Besides getting more cancer, the smoked mice grew more slowly and failed "by a large margin" to reach the weight of the unsmoked controls. The smoked mice had no young, while the controls, reproduced freely, Dr. Essenberg states in his report to the Journal, Science.

1. What does the evidence of the experiment actually show?
2. Would you say 31.9% more mice get lung cancer when living in a smoke filled room than in a room with circulating fresh air? Explain your answer.

Generalizations 1 - continued

3. Does this experiment provide evidence that persons who smoke have a greater chance of getting lung cancer than those who do not smoke? If your answer is no explain why not. If your answer is yes, what method of generalizing led you to such conclusion? Have we used this method in arriving at mathematical generalizations? Which generalizations?
4. How do you suppose scientists got the hunch that smoking causes lung cancer? What mathematical generalizations were arrived at through similar method?
5. What kind of experiments would have to be made before you would be willing to say that smoking increases a person's chances of getting lung cancer?
6. Why did scientists experiment with mice when their real concern is with human beings?

A note of caution must be mentioned in connection with the use of non-mathematical material. Some use of such material stimulates interest, helps to insure transfer of training, and provides for more flexibility in dealing with individual differences, but the introduction of too much of this material causes some concern among the students, particularly those who have a strong liking for mathematics. In general, the students do not seem to be satisfied with their progress in a mathematics class unless they are convinced they are learning as much mathematics as they are capable of handling.

The illustrations given in this chapter include accounts of sample situations in which student discovery procedures were used. These accounts also mention some of the concepts and principles studied in the course, but the reader should not assume they are a representative sample of the topics studied. A more complete picture of the content can be found in the following chapter.

Teaching Principles for the Experimental Course

In summarizing the illustrations of this chapter, it is well to make explicit the guiding principles used in designing and teaching this course in ninth grade mathematics.

1. As the teacher becomes more familiar with the student's level of understanding, the more intelligently he can guide that student toward discovering verifiable mathe-

mathematical and non-mathematical generalizations. The teacher must project himself into the thinking of the student in order to be familiar with his level of understanding. This projection becomes more accurate as the number of responses increases, but only if these responses are a product of the student's own thinking. Such responses can be observed from the student's work on carefully designed lesson sheets and tests, from group discussion periods, from individual conferences in which students feel free to express themselves, from the organization of student notebooks, and from other optional work and student-initiated projects.

2. The more referents students know for a symbol and the more intelligently these referents are used, the more successful a student will be in discovering generalizations (see pages 140-3 and 225-6).

3. Effective and continuous organization by the student of his previous generalizations helps him to remember them when needed and aids in the discovery of new generalizations. Such organizing implies that students see and know the nature of relationships between concepts and principles previously acquired. Organizing becomes more refined as the student becomes aware of the why, how, when, and where of the generalizations formed. However, the learner does not acquire such complete understanding all at once. Understandings grow as the generalization recurs in the experience of the learner.

4. The role of the teacher is to organize experiences only as much as needed so that students can make discoveries frequently enough to give them confidence in their ability to generalize. As they gain confidence the teacher should organize these experiences even less. Nevertheless, the amount of organization by the teacher must be enough for students to acquire those agreements which make mathematics the universal language it is. Therefore, the problem of the teacher is to organize experiences so that students may conform with these agreements without losing any of their originality and creativity.

5. Class atmosphere for the discovery of generalizations should be one in which caution is exercised in accepting generalizations rather than one in which caution is displayed in making generalizations. Such an emphasis gives proof an important role. Empirical proofs should be given for every student-discovered generalization and wherever possible students should be introduced to the deductive proof. One type of proof should not be considered better than the other; both have a vital role to play before certain generalizations can be accepted.

6. Definitions and agreements should not be introduced until the students sense a need for them. Topics to be studied should, wherever possible, be the outgrowth of some previous topic the class has studied and should provide experiences which can be used as the basis for the develop-

ment of future topics.

7. Student discovery in itself is not a strong motivating factor for most students. Other motivating influences should be employed, such as:

a. Selection of study materials which are challenging, but not too difficult for successful experience and which are timely, practical, and interesting to the students.

b. Keeping the student familiar with the quality of his progress, acknowledging his efforts, providing progressive goals to work toward, and giving the student a feeling his participation is essential in the selection of materials, selection of class and individual activities, and in determining the policy toward class discipline.

8. Newly discovered generalizations and processes by which they were discovered should be integrally related to the present, past, and probable future experiences of the students, and to significant everyday happenings in the world if both process and product are to have carry-over value for present and future living. Applications of the process of discovery and the generalization itself should be considered both when the generalization is discovered and when such applications are prominent in current affairs.

9. To allow for individual differences, lessons should be suggestive of other possibilities to be explored. Work required of all students should not be of such great quantity that individual students have no time to explore some topic of their own choosing. Selection of required units of work to be studied by all students should avoid units highly sequential in nature. Units depending greatly on the mastery of a preceding one should be considered optional work or only required of those students who plan to elect Algebra II.

CHAPTER V

CONTENT OF THE EXPERIMENTAL COURSE

Several considerations were kept in mind when the content for the experimental course was selected. Most important was that the content should be illustrative of the various generalizing procedures discussed in Chapter II and give students many and a wide variety of opportunities to improve their ability to generalize. The development of algebraic content, as already indicated in Chapter II, is especially suitable for illustrating generalizing procedures.

Another important consideration was that this mathematics course should give the students an overview of mathematics and make them aware of the interrelatedness of its many branches. This consideration is especially important to the students in helping them decide the nature of their future academic study. From the point of view of mathematics educators,¹ to show the interrelatedness of mathematics is sound education. Therefore, basic ideas from other branches such as arithmetic, geometry, trigonometry, analytical geometry, and statistics were also included as part of the content to be studied.

¹Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics. The Place of Mathematics in Secondary Education. New York: Bureau of Publications, Teachers College, Columbia University, 1940, pp 57-58.

Still another consideration is that the content should consist of principles, concepts, and procedures which students in the experimental class can use and most likely will use in the future. Such a consideration implies that the day-to-day content of the experimental course could not be determined before students met for their first class. Lesson plans for each day were flexible so as to allow for any direction the thinking of the class might take and were based on class events of the preceding day. In addition, a special effort was made to include relevant content of current interest to all students. Selection of content which the experimental students most likely would use in the future was based on studies² which attempted to determine the mathematics of greatest value to the ordinary citizen, and upon college preparatory requirements since, on the average, 85% to 95% of University School graduates enroll in college.

²Stalzer, Elsie, Contributions of Mathematics to a Proposal for Reorganizing General Education in Secondary Schools on the Basis of a Core Program. Unpublished Doctors Dissertation, Ohio State University. Columbus: 1952.

Commission on Post War Plans of the National Council of Teachers of Mathematics, Guidance Report of the Commission on Post War Plans, The Mathematics Teacher, Nov. 1947, pp 315-339.

Commission on Secondary School Curriculum, Progressive Education Association, Mathematics in General Education. New York: D. Appleton-Century Company, p 423.

Joint Commission of the Mathematics Association of America and the National Council of Teachers of Mathematics, op. cit.

A final consideration in the selection of content was that topics should be studied which review and extend students' understandings of arithmetic. Arithmetic understandings will probably be of greater value to all the students than any other branch of mathematics and, therefore, should not be neglected. Reviews can often be a waste of time, especially when they are nothing more than a quick re-teaching of the concepts, principles, and procedures that were to be learned previously. Reviews are more likely to be effective if the concepts are viewed differently. If thought of as generalized arithmetic, it seems that algebra can serve as an excellent medium through which the basic ideas of arithmetic can be reinforced. It not only gives students an immediate reason for reviewing arithmetic but also provides a setting in which students can think of arithmetic from a more general point of view.

The mathematical content studied by all members of the experimental class is listed below under five major divisions: Number and Operation, Graphs and Formulas, Equations and Problem Solving, Proportion and Indirect Measurement, and Statistics. Concepts, operational procedures, and principles that were studied are not listed in the order in which they were taught. Neither should it be assumed that all topics under one division were taught before proceeding to the topics under another division. The

greatest factor in determining the order of introduction was the trend established by the class in its thinking, or relevant events in current happenings that attracted the students' attention. Not all the principles listed below were verbalized. Nevertheless, it was a goal of instruction that all students should attain a reasonable mastery of all the principles listed.

DIVISION

I. Number and Operation

Concepts	Operational Procedures	Principles
Number base	Adding and subtracting on a number scale	Commutative laws of addition and multiplication
Positive number		
Negative number	Using the rules for addition, subtraction, multiplication, and division of signed numbers	Associative laws of addition and multiplication
Absolute value of a number		Distributive law
Value of a number	Adding and subtracting terms	Rules for operating with signed numbers
Number scale	Multiplying monomials	
The symbols, $+$ and $-$, as signs of: a number operation direction	Multiplying a polynomial by a monomial	The addition problem, $a + (-b)$ has the same answer as the subtraction problem, $a - (b)$ and vice versa
The arithmetic operations addition subtraction multiplication division square root	Multiplying a binomial by a binomial	The addition problem, $a + (b)$ has the same answer as the subtraction problem, $a - (-b)$ and vice versa
The algebraic operations addition subtraction multiplication division square root	Dividing an algebraic expression by a monomial	
	Simplifying algebraic expressions	
	Factoring simple algebraic expressions	The algebraic expression $a - b$ can be thought of as either $a - (b)$ or $a + (-b)$ and vice versa
	$ax + ay$	
	$a^2 - b^2$	
	$x^2 + (a + b) + ab$	The algebraic expression $a + b$ can be thought of as either $a - (-b)$ or $a + (b)$ and vice versa
Dividing by zero	Adding and subtracting algebraic fractions	
Direction of difference or change		Rule for squaring a binomial
Signs for multiplication juxtaposition the dot ($\cdot$)	Multiplying and dividing algebraic fractions	Properties of a trinomial perfect square
Limit		Fundamental law of fractions
Infinitely large		
Coefficient, literal and numerical		Law of exponents
Term, like and unlike		$a^m \cdot a^n = a^{m+n}$
		$a^m \div a^n = a^{m-n}$
		$(a^m)^n = a^{mn}$
Arithmetic expression		
Algebraic expression		
Monomial, binomial, trinomial, polynomial		
Factor		
Prime factor		
Multiple		
Degree of a term		
Degree of an algebraic expression		

I. Number and Operation - continued

Concepts	Operational Procedures	Principles
Algebraic expression in its simplest form		
Base		
Power		

II. Graphs and Formulas

Concepts	Operational Procedures	Principles
Title of a graph	Substituting values in a formula	Properties of a graph that present data well
Table of values	Evaluating formulas	If two variables vary in the same sense, the graphical representation of the relationship is a line (curved or straight) which slopes upward when moving from left to right.
Coordinate axes, x and y axes	Determining the coordinates of a point on a plane	If two variables vary in the opposite sense, the graphical representation of the relationship is a line (curved or straight) which slopes downward when moving from left to right.
Origin	Locating points on a plane when its coordinates are given	
Quadrant		
Coordinates of a point	Translating verbal statements into algebraic expressions and vice versa	
Scale of an axis	Translating geometrical drawings into algebraic expressions	
Formula		
Equation		
Variable		
Dependent variable	Inventing formulas that describe certain given data	The graphical picture of a first degree equation in one or two variables is a straight line.
Independent variable	Organizing data into a table of values and then predicting other appropriate sets of values	All equations for a straight line can be placed in the form $y = mx + b$, where m is the slope of the line and b is the y-intercept.
Constant	Drawing a line that best describes the points on a graph	As the absolute value of m in $y = mx + b$ increases its graphical picture makes a larger angle with the axis.
Vary in the same sense	Drawing a graphical picture of the relationship expressed in a formula	If m in $y = mx + b$ is positive, x and y vary in the same sense and if negative, x and y vary in the opposite sense.
Vary in the opposite sense	Using the slope, y-intercept method for determining the graphical representation of a formula that expresses a linear relationship	
Line of best fit		
Parabola		
Hyperbola		
Slope of a line		
Positive slope		
Negative slope		
Zero slope		
Dependence		
Standard form of a formula that expresses a linear relationship $y = mx + b$	Determining a formula that expresses the relationship pictured by a straight line on a graph	Two variable second degree formulas in which not more than one variable is raised to the second power is represented on a graph by a curved line.

III. Equations

Concepts	Operational Procedures	Principles
Left member	Solving equation by guess-and-check method	Law of equalities
Right member		Laws of inverse operations
Guess-and-check method	Solving a first degree equation in one unknown by analytical procedures	a. To undo the addition of a certain number, subtract that number from the sum.
Undoing operations		b. To undo the subtraction of a certain number, add that number to the difference.
Reverse order	Solving literal equations for certain unknowns that are of the first degree	c. To undo the multiplication by a certain number, divide the product by that number (zero multiplier excluded)
Equality		d. To undo the division by a certain number, multiply the answer by the same number (zero divisor excluded)
Identical equation	Solving simultaneous equations by graphical method	
Conditional equation	substitution method	
Equation with one variable	addition method	
	subtraction method	
Equation with two or more variables	Translating story problems into algebraic language	
	investment problems	
	age problems	
Simultaneous equations	mixture problems	
	navigation problems	
Inconsistent set of equations	number problems	
	distance problems	
	geometry problems	
Dependent set of equations	Inventing story problems that are described by certain equations	The value of x in $f(x) = 0$ can be found by graphing $f(x) = y$. The solution(s) for $f(x) = 0$ are the value(s) where $f(x) = y$ crosses the x -axis.
Consistent set of equations		
First degree equation	Solving quadratic equations by graphical methods	
Second degree equation	Solving quadratic equations by factoring	
Quadratic equation		
Literal equation		
Changing the subject of a formula		

IV. Proportion and Indirect Measurement

Concepts	Operational Procedures	Principles
Ratio	Interpreting the three basic types of per cent problems as proportions with one missing item	If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$
Proportion		The law of the lever, $\frac{w_1}{w_2} = \frac{d_2}{d_1}$
Cross product		
Subscript notation	Solving the basic types of per cent problems and many other appropriate arithmetic problems by applying the cross product relationship	The sum of the interior angles of a triangle equal 180°
Inverse proportion		
Direct proportion		
Direct measurement	Solving abstract algebraic proportion problems of the types: $\frac{ax}{b} = \frac{cx}{d}; \quad \frac{a}{x+b} = \frac{c}{x+d};$ $\frac{a}{bx} = \frac{c}{dx}$	If two angles of one triangle equal two angles of another, the triangles are similar
Degree of accuracy in measurement		
Indirect measurement		
Similar triangles	Solving lever problems	If two triangles are similar the ratio of two sides of one triangle is equal to the ratio of the corresponding sides of the other triangle.
Corresponding sides of similar triangles		
Trigonometric ratio		
Right triangle	Using the tangent ratio in solving problems	Pythagorean relation
Legs of a right triangle		
Angle of concern		
Side opposite the angle of concern		Converse of the Pythagorean relation
Side adjacent to the angle of concern		
Hypotenuse		
Tangent		
Angle of elevation		
Angle of depression		
Pythagorean relationship		
Pythagorean numbers		
Square root		

V. Statistics

Concepts	Operational Procedures	Principles
Collecting data	Tabulating data and making a frequency distribution	A statistical generalization is not necessarily destroyed because of certain exceptions.
Sample		
Representative sample	Making histograms and other statistical graphs	
Relevant data		A statistical generalization applies to the class at large but not necessarily to all members in the class.
Comparable data	Interpreting and making predictions from graphs, charts, and other data	
Organizing data		
Tabulating data	Determining the mean, median and mode of collected data	The mean as a measure of central tendency has the disadvantage of being greatly affected by items that are extreme deviates from the middle term.
Frequency		
Frequency distribution	Determining the type of data needed for the answering of a particular question or the solving of a certain problem	
Class interval		The mode as a measure of central tendency is the only measure of central tendency that will always reveal the most popular item(s) or class(es).
Class mark		
Histogram	Selecting the measures of central tendency most appropriate for the answering of certain questions	
Measures of central tendency		The median is the only measure of central tendency that gives equal weight to all items under consideration.
Average		
Mean		
Median		
Mode		
Trend		
Prediction		
Line of best fit		
Approximate or statistical relationship		
Range of a formula		
Relevant factors		
Extrapolation		
Interpolation		

In addition to the content listed above students were encouraged to do more than that which was required. The nature of such work could take almost any form as long as it was related in some way to class work. Several of the better mathematics students chose to further their understandings of mathematics by pursuing some topic in which they had become interested or by working available optional lesson sheets. Below are listed all the concepts, operational procedures, and principles selected by various students for study:

Optional Content

Concepts	Operational Procedures	Principle
Number system with base 5	Performing operations with numbers to the base of 5 and 12	Fundamental law of radicals
Number system with base 12	Determining the graphical patterns of certain second and third degree equations	If the first two terms of a trinomial perfect square are given, the last term may be determined by dividing the second term by 2, squaring the quotient, and finally dividing this result by the first term.
Conic sections	Expanding the binomial theorem	The roots of a quadratic equation $ax^2 + bx + c = 0$ can be determined by the following formula:
Ellipse	Solving quadratic equations by completing the square	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Standard form for the general second degree expression	Solving quadratics by use of the formula	In $ax^2 + bx + c = 0$ there are no real roots if $4ac > b^2$; one root if $b^2 = 4ac$; and two roots if $b^2 < 4ac$
Binomial theorem	Adding and subtracting fractions with binomial denominators	Any number (with the possible exception of zero) raised to the zero power equal one.
Imaginary numbers	Operating with algebraic expressions containing radicals	In the expansion of $(a+b)^n$ there are $n+1$ terms the coefficient of the first and last term is one the coefficient of the second and next to the last is n the coefficients of the other terms can be obtained from Pascal's triangle
Pascal's triangle	Dividing a polynomial by a polynomial	the power of b in any term is one less than the number of the term
Radical sign	Extracting the square root of a number	the power of a in any term is the number of the term subtracted from $n+1$
Asymptote	Factoring trinomials of the type $ax^2 + bx + c$ where a is some value other than 1	
Transformation of axes	Solving fractional equations with polynomial divisors	
Fractional exponents		
Negative exponents		
Zero exponent		

Several students included in their notebooks content which was less mathematical in nature. Most of the topics indicated below were an outgrowth of ideas discussed in class.

- Anecdotes from the history of numbers
- Anecdotes from the lives and works of great mathematicians
- Anecdotes from the lives and works of physicists
- Recent discoveries in the field of science and social sciences
- Generalizations from current periodicals
- Different types of graphs
- Applications of mathematics to everyday life

The procedures a student employs in the discovery of a mathematics generalization are of relatively little value to him if he is not able to use similar procedures outside the mathematics classroom. To aid this generalization of procedures, illustrations of non-mathematical discoveries were frequently discussed in class. In the previous and following chapters examples are given which show how these illustrations were used. In general, students engaged first in the discovery of mathematical generalizations, and then illustrations from other fields were cited to help students become more aware of the discovery process. Not infrequently did students engage in the generalizing process and then become aware of the fact that their generalization was faulty. At such times non-mathematical illustrations were also cited. It is as important for superior generalizers to be aware of the pitfalls of the various generalizing procedures as well as its successes. As the year progressed generalizations from other fields were briefly discussed

when they were of current interest. At such times the discussion centered around the procedures used in arriving at these generalizations. Below are listed many of the generalizations brought to the students' attention:

Physical Sciences

Discoveries made by Galileo

Law of horizontal motion

Law of falling bodies

Law of the pendulum

Discovery of Halley's comet

Archimedes principle

Locating a water well by using a dowsing rod (idea taken from an advertisement)

Invention of an underwater radar device for locating sunken outboard motors, lost fishing gear and other salvage, and extending its use to other purposes (Time magazine)

Pacific Ocean as the hole left when the moon split away from the earth

Effect of monsoons on the movement of the north pole (newspaper article)

Biological Sciences

Poliomyelitis caused by lack of salt in body (medical journal)

Poliomyelitis caused by eating food from tin cans (letter to newspaper editor)

Poliomyelitis caused by humidity (newspaper article)

Gamma Globulin as a preventative for certain types of poliomyelitis (Time magazine)

Extending the use of ACTH to the treatment of third degree burns (Time magazine)

Experiments showing tobacco smoke causes lung cancer in mice (newspaper article)

Explanation for skin trouble by Cleveland doctor. "Many Americans," he said, "take too many baths." (Time magazine)

Effect of phases of the moon on the number of insane people admitted to asylums (Time magazine)

Effect of phases of the moon on the number of births (newspaper article)

Social Sciences

Life expectancy of people living in northern and southern states

Cost of raising a child from infancy to age of eighteen (newspaper article)

Social Sciences - continued

- Generalizations concerning occupations and salaries of college graduates (Time magazine)
- Generalization concerning the relationship between a girl's popularity and whether she smokes and drinks (letter printed in Dr. George Crane column, newspaper)
- Generalizations made in a newspaper feature, "Predict Your Own Future."
- Gallup poll generalizations concerning MacArthur's removal from command
- presidential nominations of the Republican and Democratic parties
- Prediction concerning the prosperity of England during Queen Elizabeth II reign (England was prosperous during the reigns of Queen Victoria and Elizabeth I) (newspaper article)
- Prediction concerning General Eisenhower's fate as a politician (West Point Generals usually fare badly in politics) (newspaper article)

Sports

- Ambiguous rules in a softball league concerning player's eligibility to play in the league tournament (newspaper article)
- Hopes of Yank officials concerning Mickey Mantle's possibilities as a baseball player during his first spring training with the Yankee ball club (Babe Ruth wore No. 3 shirt - Gehrig, No. 4 - DiMaggio, No. 5 - Mantle, No. 6) (Collier's magazine)

Religion

- Edict issued by Pope stating that whale could be classified as a fish and therefore could be eaten on Fridays by Roman Catholics (Time magazine)

Miscellaneous

- Predicting the fortune (numerology) of an individual by knowing his home address
- The story: "Last week I got into trouble through imbibing too much whiskey and soda. The other day it was brandy and soda, and two months ago I spent a sorry day after an evening with rum and soda. I see, accordingly, that it is the soda that must be responsible."

These examples illustrate the variety of non-mathematical subjects brought into the class discussion, and they also show that the newspaper and current periodicals are valuable

sources for non-mathematical generalizations.

Although the mathematical content was selected to illustrate generalizing procedures, to give students an overview of the field of mathematics, to include topics the ordinary citizen should understand, and to further the students' understanding of arithmetic, no claim is made that the content studied was most appropriate for all ninth grade students. Because of individual differences between groups it is highly improbable that the detailed content to be studied by every ninth grade class could be determined in advance. However, if the above criteria of selection were accepted, the general topics could probably be determined. Even then, there is need for considerable experimentation before such topics can be enumerated with any degree of authority. As to the content used in this study, it became apparent before the end of the year that probably more emphasis should have been given to the study of ratio and proportion, indirect measurement, and statistics. In turn, less emphasis probably should have been placed on factoring, special products, and algebraic fractions. There is also evidence to indicate that the verbalization of the laws of exponents should have been deferred until Algebra II. These topics, which should have received less emphasis, are quite suitable for illustrating generalizing procedures, but they seem to do little in the way of meeting the other criteria of selection.

CHAPTER VI

EVALUATION OF THE TEACHING PROCEDURES USED IN THE EXPERIMENTAL CLASS

The main objective of the experiment was to use student discovery procedures in helping students to develop their ability to generalize and gain a mastery of basic algebraic principles. In evaluating these procedures it is helpful, if one assumes that students learn by doing, to give accounts of class events and work in which students made use of various aspects of generalizing. Such records can be obtained from student notebooks, class observers' notes, the instructor's preplanning notes and actual accounts of each class session. The chapter illustrating procedures gave many of these accounts, but a few will be mentioned here again as they relate to the various generalizing abilities and behaviors.

It is necessary to have some information on the students' ability to generalize at the beginning and at the end of the course in order to detect any change. One source of helpful information is the results of the generalization test given at the beginning of the program and after the completion of the course. Other sources are the written

observations of a graduate student* who visited the experimental class over thirty times throughout the school year, the preplanning notes and accounts of class events, and student reaction to each class session. In order to determine with some degree of certainty whether any improvement (if noted) was the result of student discovery teaching procedures, it is necessary to compare the improvement made by the experimental class with that made by a status group. And so a status group (Columbus Public School students) studying algebra in a more or less conventional manner was also given the same generalization test at the same time. If the improvement made by the experimental group on the final generalization test is found to be significantly larger statistically than that of the status group, then it seems reasonable to assume that the methods used to teach algebraic principles in the experimental class were a factor contributing to that greater improvement.

Also incorporated in the main objective is to determine how well student discovery procedures contributed to the students' mastery of algebraic principles. Both the experimental and status groups were just beginning their systematic study of algebra and, therefore, information con-

*This individual, Guy W. Buddemeyer, is an experienced high school teacher and at the time of this study was a graduate student, interested in evaluation, at The Ohio State University.

cerning the growth of each group toward the mastery of algebraic principles could be obtained by giving an algebra achievement test at the end of the year. Again, before judgment can be made as to whether one teaching method proved more successful than the other, information on the algebraic aptitude of both groups is helpful. For this reason, the Iowa Algebra Aptitude Test was given to both groups at the beginning of the year.

Other information such as student age and I.Q. were also available for both groups. These data (see page 92) provide evidence on their maturity and general native ability. Reaction of students to the teaching methods is also to be considered. Indirect sources of information for these reactions were the observations of the graduate student mentioned above and other observers, the instructor's preplanning notes and actual accounts of class sessions. A more direct source came from a questionnaire filled out by students in the experimental class and turned in at the end of the last day of school.

Another factor concerning the suitability of student discovery teaching procedures is how well they reflect such pedagogical considerations as:

1. Do they encourage students to seek knowledge independently and to think for themselves?

2. Do they provide for enough and a variety of practice so that the main principles of the course will be remembered?

3. Do they provide for the type of experiences that will enable students to think intelligently with symbols?

4. Do they give adequate attention to desirable motivating devices?

5. Are they flexible enough to allow for the wide range of individual differences that result from varying abilities and purposes?

6. Do they give adequate attention to "transfer of learning?"

The way in which these considerations are met depends on the teacher's philosophy of education and views on the psychology of learning. In the chapters on selected illustrations and content some evidence is given on how these considerations were provided for in the teaching of algebraic principles to the experimental class. Still additional evidence will be found in excerpts from observer's notes and student notebooks to be quoted in this chapter.

Data collected and used in evaluating this study will be presented in two parts. The first part, given in this chapter, considers largely subjective data (observer's notes, student reactions, student notebooks) collected from the experimental group only. The second part, given in Chapter VII, collected from both the status and experimental groups, considers a more objective type data (generalization test, standardized algebra tests).

Student Discovery as a Procedure for Improving Generalizing Ability

Fifteen behaviors and abilities of a superior generalizer were listed in a previous chapter. Evidence will now be presented to indicate how student discovery procedures encourage growth toward the realization of these abilities and behaviors.

He is able to detect likenesses and differences between situations and then to group together those situations which possess common properties. This ability, as can be seen by the discussion in Chapter II, is particularly important when generalizations are discovered empirically. Since a large percentage of the lesson sheets directed students toward the empirical discovery of algebraic principles, students had many opportunities to develop this ability. Many of these lessons have been discussed in some detail in previous chapters and some are listed below for the convenience of the reader.

Formulating the rule for adding positive and negative numbers (see pages 29)

Formulating the rule for subtracting positive and negative numbers (see pages 109-22)

Discovering a quick way to determine the sign of the product of three or more factors (see page 127)

Discovering when order of operations is important in problems involving addition and subtraction of signed numbers (see pages 134-36)

Discovering when changing the sign of operations and sign of the number affect the answers of a series of additions and subtractions (see page 140)

Classifying algebraic expressions (see page 189)

Discovering a formula from a table of values (Tower of Hanoi puzzle) (see page 203)

Discovering the cross-multiplication relationship of a proportion (see page 220)

Discovering the law of the lever (see page 224)

Discovering the Pythagorean relation (see page 225)

Discovering the peculiarities of equations which when graphed determine lines with the same y-intercept or slope (see page 67)

He is able to generalize correctly with certainty where certainty is justified, but refrains from generalizing when it is unwarranted. This ability is of particular importance while the generalizer is in the process of forming and reforming generalizations so that they best describe the given data. In a society where the ordinary citizen makes a large number of decisions for himself it is of primary importance that he be able to interpret correctly the data confronting him, and to do so depends to a great extent upon this ability. Two lesson sheets which have been discussed in Chapter IV, Graphs and Signed Numbers 2 (see page 145) and Interpreting Graphs 3 (see page 178) as well as several others that can be found in the Appendix, have as their primary purpose the development of this ability. Additional practice in this ability was provided every time

students engaged in the discovery of mathematical generalizations. Some of the opportunities especially effective were:

Formulation of the multiplication rule
(see page 126)

Formulation of the rule for multiplying a series
of factors (see pages 127-129)

Attempts at discovering the formula for the
Tower of Hanoi puzzle (page 203)

Discovery of the Pythagorean relation (see
page 228)

During the course of the year students were encouraged but not required to include in their notebooks generalizations and a few pertinent remarks concerning them. Examples of such remarks as found in the notebooks of three different students are:

If you went to the store and bought a sack of apples and they were all bad, then the apples you buy at that store will always be bad. This is a bad generalization because there are not enough facts to draw your conclusions.

A bad generalization: Bob likes to swim, therefore all boys like to swim.

This I think is a very good example of a generalization. How good a generalization this is I don't know, but I wouldn't think that everything said in the article would be true for the thousands of people who have their birthday on May 20th. (A comment concerning the horoscope for May 20.)

The above evidence indicates that there were opportunities for the development of this ability and that some students sensed the type of situations in which generalizing is unwarranted.

He is able to determine trends from available data and from them make reasonable extrapolations and interpolations. This ability is basic to the generalization method, continuity of form; it is also important to the ordinary citizen when he makes predictions from data, an aspect of interpreting data. In addition to providing lesson sheets, such as Interpreting Graphs 3 (see page 178) and Generalizations 1 (see page 232) many opportunities for emphasizing this ability were provided, among which were the following mathematics lessons:

Extending subtraction to include a subtrahend larger than the minuend (see page 103)

Extending the multiplication tables to include negative factors (see page 123)

Giving division by zero a meaning (see page 133)

Predicting additional values for a table of values (Tower of Hanoi puzzle) (see page 203)

The following quotations, the first from a student's notebook and the second from an observer's notes, further indicate that students actually engaged in predicting:

This is a generalization pertaining to cars. The person who drew these cars generalized what the car of 195x would look like by seeing what the cars of other years looked like. (A comment made concerning a series of pictures of car models from 1902-1952 and a picture of the car for 195x)

Observation, January 31, 1953

Students were asked to plot on the same chart the equations:

$$1. \quad 6a + 3p = 51$$

$$2. \quad 3a + 6p = 57$$

Various students were asked to help construct the table of values:

a	p
2	13
4	9
6	5

Teacher asked the students to determine other values for a and p without substituting values in the equation. Students did so with little hesitation.

Teacher: "What are we doing here?" A number of students responded, "generalizing."

Teacher: "What kind?" A few students knew-- "extrapolating."

In selecting values for the second equations, several students wanted to start generalizing after the first set of values was obtained. Majority of class responded, "no." After one a and one p quantity had been secured, teacher intimated that generalizing might take place. Most students objected, sensing that they did not have sufficient data to establish a trend.

He is cautious in accepting generalizations but is willing to consider any generalization as a hypothesis.

A proper attitude toward caution is probably more vitally associated with all the different generalizing procedures mentioned in the second chapter than any other ability or behavior. Such a cautious attitude could be shown on a scale as near the center with extreme caution being shown at one end and extreme lack of caution at the other. Opportunities for the development of proper regard for caution are provided for in the interpreting data lesson sheets, in lessons where generalizations are verbalized, and in other lessons such as those listed below.

Extending the multiplication to include the product of two negative numbers (see page 123)

Extending subtraction to include problems such as $10 - (-6)$ (see page 113)

Giving division by zero a meaning (see page 133)

Determining the domain of application of a formula (see pages 157, 158, 161)

Simplifying and interpreting the meanings of algebraic expressions (see pages 182, 187)

Finding formulas from tables of values (see pages 203, 228)

The three quotations below, the first from a student's notebook and the last two from an observer's notes, further indicate that students were considering caution in connection with their generalizing in mathematics and in their general reading.

Dr. Papez has found the same type of virus in cases of polio, heart disease, and mental illness. He made the generalization that these virus might be the cause for these diseases. More research must be done however before a positive statement can be made. His generalization might be false because all his specimens are dead tissue and the virus might be some virus that gets in after the tissue is dead. However, I think it is a good generalization and it may lead to a great discovery.

Observation, October 22, 1952:

The teacher returned to the process and its application to data other than mathematical quantities and operations. Astrology and Astronomy were discussed. During the discussion, danger of overgeneralizing and the consequent need for cautiousness were emphasized. One student had brought a newspaper clipping concerning the arrest of a secretary for embezzlement and forgery. He pointed out that there was danger of overgeneralizing in assuming that the secretary was guilty.

Observation, January 23, 1953

1. Example: $2(x + 6) = 3(4 + x) + 2^*$

2. Example: $5(2 + 3)$

Teacher: "How could we get 25 for an answer by some other method than first adding 2 and 3 and then multiplying by 5?"

Students at first did not see another method. They just sat in silence considering the situation. As hands came up the teacher encouraged continuation of thinking.

Don P. was finally called on to explain the method. He got blocked halfway through. Teacher urged him to continue and he finally clearly suggested a different method.

When students were asked whether they would be willing to make a general rule from the three or four examples placed on the board a few students agreed. One girl, however, commented that she would not say because on other occasions the teacher had thought of cases which proved to be exceptions to the rule.

The distributive law was then applied to the equation. The equation was solved and then checked.

The teacher indicated that mathematicians are concerned with determining an alternative method when confronted with difficulties that cannot be handled by an already known method.

He can distinguish generalizations from observable facts. It seems reasonable to assume that if a student did not attain this ability he had not made much progress in developing his generalizing ability. Opportunities for identifying generalizations from facts were provided in the lesson sheets on

Interpreting graphs (see pages 145, 178)

Generalizations (see page 232)

*Class members could not solve the equation because of the previous agreements accepted concerning order of operations which told them to add x and 6 and then multiply the sum by two. Students' attention was then directed to the second example.

Finding formulas from tables of value
(see pages 204, 228)

Student notebooks provided some evidence that students actually were able to distinguish an observable fact from a generalization. The inclusion of non-mathematical generalizations which had been noticed outside the classroom constituted the major part of the non-required materials for the notebooks. Out of thirty student notebooks, nineteen contained such material. Some examples along with the students' comments have already been quoted in this chapter. Other students merely included generalizations they had found in current periodicals, examples of which, taken from the notebooks of seven students, are given below:

Last week the National Bureau of Standards was proclaiming that high frequency waves do reflect from the ionosphere and can get around the earth's curve.

There are more mouth breathers to an acre in England than in any other country.

Many folks bathe too often.

The average American city of some 300,000 has about 400,000 pounds of garbage daily.

Summer babies have a better chance of being born free from defects and of surviving their first year of life.

Church-going reaches its high point every nine years.

Maritime nations have a great advantage over those with no outlets.

Every nation is the lengthened shadow of its people.

In general, according to the instructor's diary of class events, most members of the class became generalization conscious by the end of the year.

He uses generalizing methods independently to further his understanding of the world around him. One of the reasons for using student discovery procedures was to help the students become independent thinkers. Of course, if they are unable to use generalizing methods creatively then the methods used in the experimental class have accomplished very little. There were opportunities for students to use generalizing methods independently in discovering "new" mathematical principles. Among these were

Performing the fundamental operations with numbers with a base other than ten (see page 103)

Discovering the properties of an expanded binomial (see Appendix C)

Investigating the patterns (graphs) made by various two and three variable formulas (see page 176)

Discovering the formula for solving the general quadratic equation (see Appendix C)

In addition to these topics, students were also encouraged to include in their notebooks an expansion of any topic which they felt was related to generalizing. Examples of such expansions are as follows:

Analysis of the properties of various types of graphs found in current periodicals and the peculiar properties of each type

A critical analysis of various types of advertisements, articles, and editorials in newspapers for different types of generalizing

Students also included in their notebooks any independent mathematical "research" they conducted. When equations of the type $xy = z$ and $x + y = z$ (see page 174) were studied by the entire class, students realized that other three variable equation types existed. Several were mentioned, among which were $x^2 + y^2 = z^2$ and $z = \frac{5}{x^2 - y}$. One student chose to examine the patterns of the first equation and another to examine the second equation for inclusion in their notebooks. To carry on their research both students needed to acquire some understanding of imaginary numbers. The student who studied $x^2 + y^2 = z^2$ made several two-dimensional graphs of variations of the equation as each variable was, in turn, assumed to be constant. Her research led into an introductory study of conic sections and culminated with the construction of a three-dimensional graph of a cone. She also determined the formula describing the cone. The student who chose to examine the formula $z = \frac{5}{x^2 - y}$ made a series of sixteen two-dimensional graphs which were in some way variations of the original equation. Her first group of graphs described patterns made by the original equation as each variable in turn assumed constant values. Her next step was to change the basic form of the original equation to:

$$1. \quad z = \frac{5}{x - y^2}$$

$$2. \quad z = \frac{5}{x^2 - y^2}$$

$$3. \quad z = \frac{5}{x^2 + y^2}$$

She then determined their respective patterns as she had done with the original equation. Her study introduced her to the translations of axes, asymptotes, and other analytical geometric concepts. This girl, now two years later, is still continuing her study of graphical patterns of different types of three-variable equations. At the present time she is studying the type $x^y = z$. The variation of great concern to her is $-2^y = z$ which, as the reader will recognize, is difficult to represent graphically.

Optional lesson sheets were prepared which would guide the mathematically gifted in pursuing certain topics. One such lesson sheet was based on the expansion of the binomial. Three students included this "research" as part of their notebooks and one of them wrote in addition a page or two on the relationship of the Pascal triangle to the expansion of the binomial and to the laws of probability. Other lesson sheets directed students toward discovering the solution for the general quadratic. One of these students went deeply enough into the study to be able to determine without solving the equation whether the roots were real, equal, or imaginary.

Thus far, most of the data presented under this ability is mathematical in nature. Student notebooks did furnish data on situations in which students made independent use of the various generalizing abilities in non-mathematical situations. Those which have already been mentioned under each of these abilities are listed below:

Student comments concerning horoscope readings, sack of apples, and swimming boys (see page 260)

Student comments concerning the car model for 195x (see page 261)

Student comments concerning Dr. Papez and polio (see page 263)

Comments concerning a secretary who was indicted for embezzlement and forgery (see page 263)

Clippings of generalizations from newspapers and periodicals (see page 265)

Other non-mathematical samples will be given along with some of the remaining generalizing abilities.

He makes use of generalizations which he senses through nonverbal means. It is reasonably certain that learning is easier for the individual if he becomes aware of generalizations through this means rather than through verbal means. Therefore, many of the algebraic principles and concepts were introduced by nonverbal means before considering the verbalization of the principle. Quite often principles of concepts were introduced and used

throughout the year without being verbalized at all.¹ Some of the situations in which students sensed generalizations through nonverbal means are listed below:

Solving problems involving positive and negative numbers without the use of rules (see pages 105-110)

Sensing trends and then making predictions (Lesson sheets on interpreting data) (see page 178)

Solving abstract division problems involving positive and negative numbers (see page 62)

Introducing the principles for solving equations through the analogy of unwrapping a package (see pages 211-14)

Using the square in directing students to the discovery that $(a + b)^2 = a^2 + 2ab + b^2$. (see page 67)

He is able to verbalize generalizations and methods used in discovering them. Even though generalizations can be applied without being verbalized it is still important to the superior generalizer that he verbalize his generalizations. Verbalization enables him to make a record of his discoveries and, therefore, relieves his mind for more creative efforts than memorization. Also, it is easier to examine a verbalized generalization for suggestions in making new generalizations. Most of the basic algebraic

¹ Most educators believe that too early emphasis on verbalization encourages students to manipulate symbols blindly and mechanically. Some educators even are inclined to believe that verbalization of generalizations should not be introduced at all to most students. (see article, Hendrix, op. cit.)

generalizations that were discovered were verbalized through the joint efforts of the class. The chapter illustrating procedures includes accounts of many such lessons, among which are the rules for operating with signed numbers. The accounts below of the class verbalizing a generalization were taken from an observer's notes and the instructor's record of class happenings. They also illustrate how group and individual thinking were employed in developing principles.

After students had solved correctly problems involving addition of signed numbers in real settings (e.g., temperature, football, and stock market report statistics) and had solved abstract addition problems by using the number scale, they were asked, as part of their assignment for the next day, to formulate a rule that would solve any abstract addition problem. Some of the rules as taken from the students' papers were:

1. Add the - and the + and subtract. Answer will be the same as the largest signed number.
2. I subtract the smaller number from the larger. Then, depending on which was the larger, I label the answer + or - .
3. Add together all the pluses, then add together all minuses, then subtract the smaller from the larger. The sign of the difference will be the sign of whichever had the larger amount.
4. Subtract the smaller number from the larger number. If the larger number is + the answer will be + . If the larger number is - the answer will be - .

Observation, October 10, 1952

Teacher interpreted rule no. 1 as meaning the number of positive and number of negative numbers in the problem.

Teacher suggested that "absolute value" might be a useful term.

For rule no. 1, Dick P. suggested that the rule should be: (a) Add the absolute values of all negative numbers, (b) Add the absolute values of all positive numbers.

Sally D. followed with (c) Subtract the smaller sum from the larger sum.

Next step to the rule in doubt--no student ready to advance a statement.

Teacher pointed out that the class was engaging in generalizing.

Jack W. finally worded the final step: (d) If the larger sum is positive the amount will be positive. If the larger sum is negative, the answer will be negative.

Teacher asked class to critically review statements as an assignment to see whether the rule holds for all cases.

Observation, October 11, 1952

Teacher asked for examples of uses of positive and negative numbers to be placed on bulletin board. Reminded students that examples of the uses of positive and negative numbers should be listed in their notebooks.

Example of misunderstood baseball rules presented by the teacher (newspaper clippings).

Returned to the wording of the rules started Oct. 10. Dick P. pointed out that the rule was not particularly applicable to adding +6 and -6.

Several addition problems were worked by the teacher and class. Karin S. pointed out that the aspect of the rule that says the "difference takes the sign of the larger number" was incorrectly worded. Absolute values have no sign.

Jane B. pointed out that the kind of adding required as stated in the rule was not clear. The adding to be done must be arithmetic adding.

Paula W. pointed out that part (c) of the rule was not clear, that "smaller sum" and "larger sum" should refer to "smaller absolute value" and "larger absolute value." Teacher assisted in clarifying thought.

The final wording as a result of the above individual and group thinking was:

- (a) Add, arithmetically, the absolute values of all positive numbers.
- (b) Add, arithmetically, the absolute values of all negative numbers.
- (c) Subtract, arithmetically, the sum with the smaller absolute value from the sum with the larger absolute value.
- (d) The sign of the final answer will be the same as the sign of the arithmetic sum with the larger absolute value.

Observation, October 18, 1952

Halley's comet was discussed. The students were told that the method used in discovering the comet would be used to discover the rule for multiplication of signed numbers.

First the teacher explained that multiplication is indicated in different ways in algebra.

$(4)(3) = 12$	Teacher cautioned that the
$(4)(2) = 8$	mere fact that this table
$(4)(1) = 4$	seemed logical did not
$(4)(-1) = ? (-4)$	necessarily make it true.
$(4)(-2) = ? (-8)$	Last two problems were
	checked by applying addition.

Don P. attempted to give a story problem applicable to $4 \times (-2)$. Example not clear.

Dick P. gave example of temperature dropping two degrees each hour for four hours.

Teacher: "Would anyone like to formulate a rule?"

Steve K: "Multiply absolute values and give the product the negative sign."

Debby M: "What happens when you have $(-3) \times (-4)$?"

Dick P: "Steve made an error in thinking that the larger absolute values sign was applied to the answer."

Sally D: "If one of the two numbers being multiplied is negative the answer is negative."

Observation, February 6, 1953

Class made rules for removing parenthesis in the following problems from the work assigned the previous day.

1. $9 - (6 + 5) = 9 - 6 - 5$	4. $17 + (-3 - 8) = 17 - 3 - 8$
2. $15 + (8 - 5) = 15 + 8 - 5$	5. $-23 - (-3 + 8) = 23 + 3 - 8$
3. $19 - (-5 - 7) = 19 + 5 + 7$	6. $5 - (x - 6) = 5 - x + 6$

"If a negative sign comes before the parenthesis all signs falling within the parenthesis are changed."

The above generalization was developed on the basis of contributions of about a dozen students.

Later when students were told that these generalizations were to be placed in their notebooks they wanted to go through the process of stating them exactly. The teacher agreed to do this.

Students sensed difficulty in making a statement that was comprehensive enough for the generalization. Kay T. pointed out that the statement did not include what happened to the minus sign appearing in front of the parenthesis. Kay also thought the statement in the textbook was too general.*

Students were encouraged to verbalize for themselves generalizations which were left un verbalized in class. Below are listed examples of such verbalized statements found in the notebooks of four students.**

Student A

Distributive Law: It makes no difference whether you multiply numbers by a certain number and then add the products or whether you add the numbers and then multiply the sum by the certain number.

Student B

To find the square of any number ending in five, raise the first digit by one and multiply the original digit by the new one. That is the first part of the answer. Then square the five. That is the last part of the answer.***

Examples: $(55)^2 = 3025$
 $(75)^2 = 5625$
 $(125)^2 = 15625$
 $(4\frac{1}{2})^2 = 20\frac{1}{4}$

*After the above experience with this aspect of algebraic grammar, the writer concluded that the removing of parentheses should be approached in a manner that is less empirical. See Appendix B.

**Students included other worded statements of mathematical generalizations when they were not required, but most of these statements were copies from texts. Such an experience, although not as valuable as the actual formulation of statements, was not discouraged to any great extent. After all, being able to find information when it is needed is also an important educational objective.

***As the reader will observe, the wording does not say exactly what was meant.

Student C

If, in adding, you have a straight run of numbers, they are calculated from left to right as in any other adding problem.

If, in subtracting, you have a straight run of numbers they are calculated from left to right as in any other subtracting problem.

If, in multiplication, you have a straight run of numbers they are calculated from left to right as in any other multiplication problem.

If, in division, you have a straight run of numbers, they are calculated as in any other division problem, from left to right.

If, in a mixed problem, which involves several calculations, you calculate the multiplication and division parts first (in order from left to right), then the addition and subtraction (in order from left to right), unless there are brackets (and parentheses) involved. If there are, see the page pertaining to them.

The class as a group worded the rule for addition of signed numbers as follows:

(a) Add, arithmetically, the absolute values of all positive numbers.

(b) Add, arithmetically, the absolute values of all negative numbers.

(c) Subtract, arithmetically, the sum with the smaller absolute value from the sum with the larger absolute value.

(d) The sign of the final answer will be the same as the sign of the arithmetic sum with the larger absolute value.

One student was not satisfied with the wording and changed rules (c) and (d) as follows:

Student D

(c) Subtract, arithmetically, the smaller sum from the larger.

(d) If the sum of the absolute values of the positive numbers is larger, the final answer will be positive. If it is smaller, the answer will be negative.

Several students included in their notebooks accounts of the class efforts at generalizing and summarized their understanding of effective generalizing. Quoted below are six such accounts taken from the notebooks of five students.

Student E

Commutative law means that two numbers when multiplied will give the same answer whether written $5 \cdot 2$ or $2 \cdot 5$. The class discovered it by empirical method.

Student F

When in a problem you would see an equation like this:

$$5(2 + 3) = 25$$

you would naturally add 2 and 3 and then multiply it by 5 to get your 25. But what if you had one like this?

$$5(x + 3) = 25$$

You couldn't very well add x and 3, but you can do this:

$$5 \text{ times } x \text{ and } 5 \text{ times } 3 = 25$$

You have done away with the parenthesis and you can now work it (the equation) the way you are used to. Whenever you want to multiply or divide a number by the sum of two or more letters you have to use the distributive law.

Student G

After using the distributive law in such cases as $4(3 + 2)$, some members of the class attempted to use it in such cases as $4(3 \cdot 2)$.

Student H

The class over-generalized in finding the products below. They did not give enough significance to the additional plus sign in the third problem:

1. $(a - 2)(a + 2) = a^2 - 4$
2. $(10 - b)(10 + b) = 10 - b^2$
3. $(z + 6)(z + 6) = z^2 + 36$

Student I

Statistical and Mathematical Generalizations

One difference between a statistical and a mathematical generalization is that in a statistical generalization there can be a few cases that the generalization does not hold true. While in a mathematics generalization if there is one case that does not hold true for your generalization your whole generalization is wrong. For instance, to say that 4×5 is the same as 5×4 is correct. But if there is one exception to the rule (commutative law) it is wrong. But if you would say it is cold in winter you still would be correct if there was one warm day.

In collecting data and making a statistical generalization your information must be relevant (student meant representative). That is to say-- if you wanted to find out which is preferred, ice cream or cake, you have to get a sample opinion from different parts of the country. This is because in Florida where it is hot a cold treat may be preferred. You would have to take a sample of the different seasons for the same reason above. You would have to take data for the different age levels. Children might prefer ice cream and adults, cake. Perhaps after all of this you could discover which is preferred.

In making a statistical or regular generalization you must watch you don't over-generalize. For instance if there is a group of people who tend toward the communist way of living, this does not mean that every individual in the group is communistic.

As I have said before, in making a mathematical generalization there can be no exception to the rule. One way to make a generalization is to make a whole set of examples. Perhaps you will see something in these that are exactly alike. For example:

1. $4 \times 5 = 20$; $5 \times 4 = 20$
2. $3 \times 6 = 18$; $6 \times 3 = 18$
3. $10 \times 5 = 50$; $5 \times 10 = 50$

You see here that no matter which way you write it (order the factors are used) it always has the same answer. Now try a few more examples to make sure your generalization is correct. In this case, it is.

Another way is to first come to a conclusion and then test it. This (way) was done with things as the atom bomb. This is the kind of thing Einstein does.

Beware of making faulty mathematical generalizations though. This is quite easy to do especially when you do not have all the facts.

One of the good generalizations our class made this year was one about cross multiplication. We generalized this by putting (examining) a lot of examples on the board. Examples:

$$\frac{3}{5} = \frac{6}{10} \qquad \frac{4}{5} = \frac{8}{10}$$

The rule is that when the two sides are equal, by multiplying diagonally you will get the same answer.

Analytical Method for Solving Equations

This method is the most widely used. (In reference to the analytical method of solving first degree equations.) It is the type that is worked like the opening of a box. When you wrap a present first you close the box. Then you put the paper on it. Then you put a ribbon on it. But the person who gets your present first takes the ribbon off. Then takes off the paper. Then opens the box. So it is with solving equations the analytical way. In wrapping the present it is like constructing an equation. For example, this equation, $5x - 6 = 10$, was constructed by first multiplying by five and then subtracting six. We reverse the problem so that we first add six and then divide by five. But we must be sure that whatever we do to one side we do to the other. For if we had something like this, $2 = 2$, and we took one from the left member, it would read like this, $1 = 2$. That, of course, is not right. But if we take one from both members it would read like this: $1 = 1$. That is correct.

He continually searches for examples which are exceptions to formulated generalizations and then reformulates them to include the exceptions. Basically, this ability partially describes the method of inverse deduction discussed in a preceding chapter. A search for exceptions is more fruitful when conducted systematically, and one such way is to apply the original generalization to a representative

sample of situations relevant to the given conditions of the original generalization. Another would be to extend the given conditions to include more types of situations, as described under the method of variation, and then to note if the conclusions of the original generalizations still apply to the added types. In the chapter illustrating procedures and also in this chapter, accounts are given of situations in which students took advantage of opportunities for selecting representative samples, searching for exceptions, and reformulating generalizations. Some of these situations are listed below:

Formulating the rule for adding signed numbers
(see pages 108, 271)

Formulating the rule for subtracting signed numbers
(see pages 103, 113)

Formulating the rule for multiplying signed numbers
(see pages 123, 126)

Formulating the rule for determining the sign of the product of three or more factors
(see pages 127-9)

Discovering the property of formula which determines whether or not it is curvilinear
(see pages 166, 173, 192)

Discovering the formula for finding the sum of a series
(see page 196)

Attempting to discover the formula for the Tower of Hanoi puzzle
(see page 203)

Discovering the Pythagorean relation
(see page 228)

He investigates implications of generalizations under consideration. This ability is especially important

for the discovery of generalizations by rational procedures and, of course, is essential if one is to prove generalizations logically. Listed below are lessons in which students took advantage of opportunities for studying implications of certain accepted principles.

Discovering the rule for dividing signed numbers (see page 131)

Using the basic meaning of exponent, coefficient, and term in discovering rules for simplifying algebraic expressions (see page 187)

Using the basic principles concerning equalities and fundamental operations in the solution of equations (see pages 215, 217)

Explaining the cross-multiplication principle concerning proportion in terms of principles of equation solving. (see page 221)

The following account was taken from an observer's notes.

Observation, November 20, 1952

On the homework for today, the students were asked to write $8x^2$ the long way. This caused a little trouble. One of the boys wanted to know if you raised your literal factor to the power indicated and then multiplied by the coefficient, or if you did it vice versa. The way you do it, he said, would make a difference in your answer. This shows a sign of thinking taking place.

He searches for explanations for generalizations discovered empirically. The procedure involved in this search is essentially inverse deduction. Generalizations are sought from which at least the general trend of the empirically discovered generalizations can be deduced. Lessons, some of which are listed below, were provided in which

students were to explain empirical discoveries.

Explaining the shifts in population and businesses (see page 144)

Explaining the relationship between the gasoline mileage of a car and its speed (see page 178)

Explaining the relationship between number of cricket chirps and the temperature (see Appendix A)

Explaining the tendency for blood pressure to increase with age (see pages 158, 159)

Explaining the mathematical reason for the cross-multiplication property of proportions (see page 221)

Find the reason why $(7d)(3c)(4a)$ can be simplified but $2x + 3y + 4a$ cannot (see page 187)

The paragraph below, which was taken from a student's notebook, illustrates an application of this behavior.

This is a generalization I heard on the radio some time ago. Taking human nature into consideration, I would say the generalization was pretty logical.

According to a man on the radio, when people complain about it raining just after they wash their car, there is something to be said for the complaint. People usually pick a dry spell for the time to wash their car. It takes the people several days to decide that it is a dry spell. Then they wash their car, and then the dry spell ends and it rains.

He empirically tests or searches for an increasing number of applications to generalizations that have been suggested by rational procedures. New generalizations may be logically consistent with old generalizations yet be of little value in understanding the world in which we live. The test for the worth of a new generalization is whether

it can now or eventually be applied. Below are listed some lessons which provided students with opportunities to search for applications.

Searching for applications of signed numbers
(see page 104)

Inventing problems in which operations with
signed numbers make sense (see pages 104, 125, 132)

Inventing problems that are described by
equations (see page 210)

Testing simplifications of algebraic ex-
pressions by substituting values for the
variables in the expressions (see page 182)

Checking the solution of equations (see page 215)

He searches for and recognizes relevant factors responsible for happenings or for behavior of objects in certain situations. This ability is of particular importance to the generalizing method of analogy and to the understanding of functional relationships. A study of the necessary precautions to be followed while collecting data which will answer questions or will support certain conclusions affords many opportunities for the discussion of relevant factors. Discussion concerning the Gallup Poll and lessons based on research articles on such questions as, "Does smoking increase the probability that an individual will develop lung cancer?" (see page 232) and, "Who are better drivers, men or women?" afforded excellent opportunities for considering both the collection of data and relevant factors. Below are listed other lessons which provided students with opportunities to search for relevant

factors.

Seriousness of an accident formula (see page 168)

Discovering the characteristic of a two-variable formula which makes it linear or curvilinear (see pages 166, 173, 192)

Discovering the characteristics of a series of factors which determine the sign of the product (see page 129)

He studies data and accepted generalizations from different points of view. This behavior is difficult for students to develop but especially so, if the generalizations have been accepted for a long period of time. However, the difficulty is worth combating, for in developing the ability students greatly increase their power to generalize. When students can examine generalizations and data from a different viewpoint they often become aware of possibilities that are exceptions to or extensions of the accepted generalization. Below are listed some class situations in which students did look at accepted generalizations from a different point of view.

Subtracting a larger from a smaller number (see page 103)

Writing numbers with a base other than 10 (see page 103)

Obtaining a sum smaller than an addend (see page 108)

Obtaining a difference greater than the minuend (see page 113)

Obtaining a positive product when the factors are negative (see page 124)

Notes from an observer also give an account of lessons in which looking at problems from a different point of view facilitates generalizing.

Observation, May 7, 1952

The teacher discussed and demonstrated parallel problems similar to the ones students worked individually the day before.*

$$\text{e.g. } \frac{7 + \frac{2}{7}}{3 + \frac{5}{7}} \quad \text{and} \quad \frac{c + \frac{2}{c}}{3 + \frac{5}{c}}$$

It was noted how it was extremely confusing, if not impossible, to generalize from the arithmetic examples to similar algebraic problems by making use of the arithmetic mixed numbers idea. He then explained to the class that it was often necessary to make new approaches when certain ones failed to yield results.

He uses statistical devices in making generalizations. Statistical procedures for generalizing can be applied to data where other procedures are nonproductive (see page 52). Some of the opportunities for using statistical devices are found in the list that follows.

Interpreting the chart, Wartime Shifts in Populations and Businesses (see page 144)

Determining the range of application of a formula (see page 157)

Determining the line of best fit (see page 157)

*The previous day, students were given the following problems to solve. No instructions were given other than a reminder that heavy horizontal lines meant division.

$$\frac{6}{1 - \frac{2}{3}}$$

$$\frac{7 - \frac{2}{3}}{7 + \frac{3}{4}}$$

$$\frac{\frac{1}{6} - 2}{3 + \frac{5}{6}}$$

$$\frac{c + \frac{2}{c}}{3 + \frac{1}{c}}$$

$$\frac{1 - \frac{x}{y}}{y - \frac{y^2}{x}}$$

Organizing and presenting data (see page 156)

Interpreting data (see page 203)

An observer mentioned in his notes still other statistical topics that were studied.

Observation, Spring Quarter, 1952

The purpose of this observation was to note how mathematics was related to life. I found interesting and profitable class discussion of the uses of mean, median, and mode as guides to interpreting data. Further discussion followed concerning valid generalizations that might be drawn from the percentages listed in a recent Gallup Poll. All the problems were real, the discussion quite logical and informative, and learning outcomes excellent.

The data under each of the generalizing abilities and behaviors illustrate not only that student discovery of algebraic principles affords many opportunities for the development of generalizing abilities and behaviors, but also that the students at large took advantage of some of these opportunities.

Student Reaction to Discovery Procedures

In considering the use of a teaching method it is of great importance to take cognizance of student reaction to the selected procedure. One source of information for such reaction was the observation notes of an experienced teacher who, at the time of the experiment, was a graduate student majoring in evaluation. Throughout the year he observed the class thirty times. The following quotations are excerpts from his notes:

Observation, October 11, 1952

Subtraction was introduced by using the number scale. Class tended to be inattentive and confused, displayed an attitude of incredulity and unwillingness to accept an answer to the problem: subtract -3 from +20. Students felt the answer contradicted their concept of subtraction.

Observation, October 16, 1952

New idea presented at beginning of class. Those students who had completed the assignment were asked to work on this new idea--the conversion of numbers in bases other than 10 to numbers to the base 10. Class reacted to this idea with a mixture of wonderment, surprise, frustration, and disbelief. (Teacher commented that he was beginning to believe that new ideas hurt.)

Many students need improvement in work habits, particularly when confronted with frustrating new situations and ideas.

Observation, October 19, 1952

The prediction method of illustrating the multiplication of signed numbers was extended to include the multiplication of two negative numbers. This is a logical* type of prediction. Consequently, it appears to be correct when presented in tabular form. The students, however, explained, "It doesn't make sense; if it doesn't make sense, it isn't right." Faced with the problem of seeming to be logically* correct but apparently in error, the teacher used several examples to show that the logic was applicable to specific cases.

Although still present to a marked degree, the antagonism and frustration noted during previous periods when new concepts were developed was not as pronounced as heretofore. Students were a little more willing to examine the logic and the evidence presented in the discussion of this concept and to use the concept in working problems.

Observation, October 23, 1952

(Discovering the rule for determining the sign of a series of numbers.) Class was much more receptive to new ideas and more willing to think

*Logical here is used loosely.

critically about examples, illustrations, and hypotheses.

Observation, May 7, 1953

Students again seemed very interested and attentive. Seemed to be gaining an understanding of reduction and simplification of algebraic fractions. Students seem to have a critical sense of knowing when a basic or fundamental idea is being studied.

Next portion of period involved checking and discussing formulas written by students for various types of geometric figures. Two students asked about alternate formulas which they had devised but which proved to be incorrect. Both boys seemed much more willing than earlier in the year to look at merits of their formula and attempt to identify the error or omission.

These observations seem to indicate that new mathematical ideas do disturb students; the discovery of an idea which violated the intuitive notions of the students often disturbed them so much that use of the group thinking technique was ineffective for the remainder of the hour. The notes, however, indicated that the students did become more tolerant and open-minded toward ideas in mathematics.

Students in this experimental class, even though they have received most of their formal education in a school where subject matter lines are largely disregarded, were somewhat concerned at the beginning of the year when they learned that improving their ability to generalize was a major objective of the course. Apparently most of them felt that the teacher's role is to "tell," rather than to help them become independent thinkers. The final quotes from this observer's notes mention their lack of acceptance

which he noted at the beginning of the year and how this attitude changed as the course progressed.

Observation, May 6, 1953

Students much less critical or scornful of the term and idea of generalization. Early part of the year students ridiculed and made fun of the idea of generalizing, particularly when it was referred to as a major purpose of the course.

In addition to the observer's notes, students were asked to respond to a simple questionnaire concerning the course. In the discussion which follows, each question is listed as it appeared on the questionnaire, and immediately following it is a summary of the students' reactions.

Question 1: What did you like about this course and the way it was taught?

Twenty-five students responded to the question. Listed below is a tabulation of the responses occurring more than once:

The use of student discovery methods of teaching	9
Opportunity to receive extra help from teacher	4
Extra credit assignments	2
Quizzes	2
"Unwrapping the box" approach to the solution of an equation	2

Question 2: What did you dislike about this course and the way it was taught?

Twenty-five students responded. Dislikes were:

Homework	13
Class too noisy	3
Class moved too slowly	2
Class moved too fast	2
Keeping notebook	2
Unit on statistics	2
Teacher not strict enough	2

Question 3: What is your feeling about emphasizing generalizing and algebra in the same course as we did this year? Do you see any way in which generalizing ties in with algebra? Explain by giving illustrations.

A summary of the twenty responses to the first aspect of the question shows:

Students stated explicitly that they liked the emphasis on generalizing	5
Students implied that at least some emphasis should be placed on generalizing (4 students, however, thought the emphasis was more than was justified)	10
Students saw no relation between learning algebra and generalizing	2
Students were noncommittal	3

Twelve students indicated in their remarks that they saw the relationship of generalizing to algebra. Eight of these twelve cited situations such as the formulation of rules, interpretation of graphs, and the solution of problems as examples where generalizing is an important factor in the learning of algebra.

It is important to mention here that student discovery teaching procedures as described in this dissertation were quite different from those used in teaching students the mathematics they had previously learned. Up until this time the principle goal had been to give them a mastery of practical mathematical content, with little or no emphasis placed on mathematics as a way of thinking. It should not be surprising, then, if at first these new procedures are viewed by students with wariness and skepticism.

In general, the following conclusions concerning student reaction to student discovery and the emphasis on generalizing are:

1. At the beginning of the year student discovery procedures and the emphasis on generalizing should be introduced slowly. Apparently, students have to live with and gain confidence in such teaching procedures before they see their worth.
2. If students sense they are making progress in the discovery of a generalization, their reaction to student discovery procedures is highly favorable.
3. Successful use of student discovery gives students considerable satisfaction.
4. By the end of the year a large percentage of the students liked the emphasis on student discovery, and the class at large felt the emphasis on generalizing was desirable.
5. Student discovery procedures can arouse students emotionally. This is especially noticeable whenever some generalization result contradicts a preconceived notion of what they had expected to find. An example of such a generalization is the rule for multiplying two negative numbers.
6. The success of student discovery procedures on any particular day is more dependent than other procedures upon the general emotional tenor of the class.

Student Discovery and
Certain Pedagogical Considerations

Substantial evidence has already been given to indicate how well certain pedagogical considerations were met. The data below, most of which are from the notes of seven different class observers, serve as further evidence concerning the effectiveness of general teaching procedures employed in the class.

Observation, November 7, 1952

As they substituted values for letters, they were asked to explain this process. Then they were asked for one word that would define the process. I liked the way the class came up with the word substitution. It was not asked of them in a direct manner but rather indirect. The teacher first made sure that all knew what went on in this process before asking for the word substitution.

Observation, October 22, 1952

In the latter part of the period, problems involving a sequence of operations were written on the board and students were asked to work them. These were discussed when the work was completed and it became obvious that agreement must be reached concerning what the symbols mean when placed in sequential order (i.e., rules concerning order of operations). One student wanted to know why the teacher did not tell them how the problems were to be worked. The teacher explained that he was interested in establishing in the student's mind the importance of agreements in working with number symbols. Such a procedure by the teacher indicated to these students the need for prior agreement in mathematics regarding the meaning of symbols, and therefore illustrated that everything in mathematics is not absolutely determined by the nature of the quantities.

Observation, October 13, 1952

- - - -Of even more interest was the methodology that was used. The class as a whole was trying to organize a rule concerning signed numbers rather than having the rule given to them. . . .This also entailed a discussion of why certain rules were necessary and this is something that would carry over into other aspects. Varied examples were then available to show the use of rules.

The students had plenty of chance to talk and move around (which caused some problems, too). They had no inhibitions about expressing themselves and there were at least three evidences of some original thinking.

As a new topic was introduced it was related to the topic just previously studied as well as to other work in arithmetic.

Observation, November 19, 1952

The class had been given a problem concerning a telephone switchboard. They were given the numbers of phones that led into this switchboard and then they were to find the possible number of connections or combinations that could be made. From this developed the formula:

$$C = \frac{n(n-1)}{2}$$

I like the way these formulas were worked out by the class. The idea of being able to generalize from a given problem is completely different from the way I learned such formulae.

Observation, Fall Quarter, 1952

The majority of the class thought the answer to $5mx^4 = m + m + m + m + mxxxx$. The teacher took time out and let $m = 2$ and $x = 3$, and had the class substitute the values in the problem to see if this answer was correct.

Observation, October 10, 1952

A very friendly atmosphere was noticed and it seemed to me that the students were learning--not just receiving meaningless material.----I could see that the purpose of the instructor was exchange of ideas and judgments regarding the algebraic addition problems.

Observation, October 17, 1952

Some basic addition and subtraction operations were briefly reviewed by using the number scale. This practice period revealed some errors and misunderstandings on the part of several students. Thereafter, the emphasis in teaching was more on rules than on learning to generalize or gain insight into the operations involved.

During the course of the period, the teacher made a number of comments such as, "You've got to learn the rules." Although several short articles (at the beginning of the period) were read illustrating tentative generalizations regarding the cause of polio as deduced from certain facts, the major part of the period was given to rules and then application. This seemed to be a departure from the general purpose which the teacher proposed in the teaching of signed numbers, that is, promoting the ability to generalize through the use of algebraic concepts.*

Observation, February 5, 1953

Teacher: "Let us solve the following problem in tomorrow's lesson. (First time such fractional equations were introduced.)

$$\frac{5x}{6} - 3 = \frac{2x}{3}$$

Ted O. suggested that the first step would be to multiply each side by the lowest common denominator.

Sally G. wondered whether this would leave the equation expressing the same relationship. The teacher referred to the principle developed earlier in the year. (Golden rule of algebra.)

From this problem the question arose again of the difference between $\frac{5(18)}{6}$ and $\frac{5 + 18}{6}$. The

*The practice with the rules here occurred after the class had spent time on the discovery and verbal formulation of rules through the process of both individual and group thinking. The writer feels such practice is necessary for mastery of the concepts. Mastery, in turn, is necessary if the student is to be freed from the concrete in order to discover additional concepts where signed numbers are involved.

teacher pointed out the danger of overgeneralizing. Overgeneralizing was then discussed with reference to several newspaper articles and election statements. Bill W. wondered whether there would be many instances of overgeneralizing in the election campaign.

Observation, November 26, 1952

Classification, what does this mean? To put in order? The class gave examples of where classification is used--in the library, in the army, directories, dictionaries, offices, and in biology. Classification is done in algebra, too. The class looked at a group of expressions and picked out those things which were similar to them. They discovered some of the expressions had the same number of terms. They soon caught on that those with one term were called monomial, those with two, binomial; those with three, trinomial; and those with two or more, polynomial.

Observation, November 5, 1952

The class discussed various theories of how polio develops. The teacher brought in an article from Time magazine on the topic. Also, articles were brought in on statistics of body development and its influence on careers. The class then compared these generalizations to those in mathematics. This led to a discussion of what they associate with algebra.

Observation, May 12, 1953

An example of a statistical generalization was presented by the teacher which stated that a new-born child would cost \$7700 by the time it was 18 years of age. One student commented that girls cost more than boys. Another student remarked, "That is a hasty generalization."

Several students presented generalizations of various kinds that they had found. Students were interested, attentive, and made critical comments.

One boy presented the generalization that at elections, turnouts had been found to be correlated with the amount of rainfall in a certain state.

Earlier in the chapter six pedagogical principles were listed which were of particular concern in this experiment. These considerations will now be adjudged

separately in light of the evidence already presented.

Do the teaching procedures encourage students to seek knowledge independently and to think for themselves?

Involved in this consideration are both group and individual thinking. An important aspect of group thinking is the degree to which all members in the class participate. The regular observer's notes, of which only part are quoted in this chapter, indicated that twenty-three of the thirty-six students made contributions at one time or another. Although the majority of the twenty-three were the better mathematics students, there were nine who were average or below. About two-thirds of the students not contributing were below average achievers. From the writer's account of each class period, the observer's record of participation was close to a representative sample of the individuals participating throughout the year. Sixty-four per cent participation indicates that to rely on group thinking entirely as a teaching technique does not insure that each individual's powers of thinking will be used in the discovery of mathematical principles. If liberal use is made of individual guide sheets the percentage of those taking part in the discovery of concepts is increased. Often, such sheets were used as an assignment due just before the process of group thinking was employed. Listed below are examples of instruction which illustrate the interplay of group and individual thinking.

Discovery of the rule for subtraction of signed numbers (see page 115)

Discovery of the rule for division of signed numbers (see page 130)

Discovery of the importance of order in adding and subtracting numbers (see page 135)

Discovery of the relation between + and - used as a sign of a number and sign of operation (see page 137)

Discovery of the cross-product relation of a proportion (see page 219)

The evidence provided in this chapter and in the chapter illustrating procedures leaves little doubt that the teaching procedures of the experimental class were based on the objective, encouraging students to seek knowledge independently and to think for themselves.

Do the teaching procedures provide for enough and a variety of practice so that the main principles of the course will be remembered? Practice was provided for in two main ways. The first, in which the basic principles of the course were taught as related ideas, is illustrated in the following lesson sheets.

Graphical representation and signed numbers
(see page 144)

Graphs and formulas (see pages 153, 156, 158, 161)

Graphs and the solution of equations (see page 206)

Formulas and algebraic grammar (see page 180)

Correspondence between the peculiarities of a particular relationship as represented by a formula, a graph, or a table of values (see page 166)

The second type of practice, in which students were given a single lesson work sheet with many types of problems, are illustrated in the following lesson sheets.

Signed Numbers 5 (see page 110)

Signed Numbers 8 (see page 121)

Algebraic Grammar 1 (see page 181)

Algebraic Grammar 2 (see page 186)

Solving Equations 4 (see page 215)

A careful examination of these lessons indicates that student discovery procedures offer many opportunities for practice.

Do teaching procedures provide for the type of experiences that will enable students to think intelligently with symbols? A special effort was made to help students think with symbols. For a more detailed discussion of these procedures, see Chapter IV, pages 140-43.

Do the teaching procedures give adequate attention to desirable motivating devices? No matter how many motivating devices the teacher may employ he can always use more. Below are listed some of the devices utilized in the experimental class.

The performance of experiments with science equipment (see pages 151, 224)

Problem materials based on information close to the lives of the students (see pages 145, 158, 232)

Application of mathematical principles to other areas of learning and life in general (see pages 183, 201)

Lesson sheets based on puzzles (see pages 198,204)

A unique presentation of a lesson (see pages 211-14)

History of mathematics (see page 199)

Student discovery procedures do not restrict the use of motivational devices. In fact, discovery of a principle affords students a certain feeling of satisfaction, which acts as a motivating device, not present when he has not had the opportunity to discover.

Are the teaching procedures flexible enough to allow for the wide range of individual differences that result from varying abilities and purposes? Individualizing instruction in a normal-sized class is always a challenging problem when it is also important to use group thinking in the development of the subject matter being studied.

Teaching procedures which encourage students to strike out on their own in extending certain phases of the unit of study offer a good way in which instruction can be individualized. Earlier in this chapter (see page 266) and in the chapter on content (see page 249) various opportunities were cited in which students did do some "research" on their own.

In addition to this "research," students were encouraged to develop their notebooks in any way they saw fit. It will be remembered that part of this notebook was to include the mathematical generalizations discovered, a list of all terms used, and examples of problems they had learned

to solve. In addition, students were encouraged to include non-mathematical generalizations, rewording of generalizations discovered by the class, discussion of class procedures and a comparison of certain of these procedures with the discovery of non-mathematical generalizations, the use and discussion of discovery procedures in finding "new" mathematical concepts, and anything else they felt was appropriate. This nonrequired part of the students' notebooks provided valuable evidence on the effectiveness of class instruction, and much of it has been quoted in this chapter. Of the class of thirty-five students, thirty handed in notebooks; of these thirty, three notebooks contained only that which was required. Of the remaining twenty-seven, which included at least some extra materials, nine indicated a considerable amount of initiative and creativeness on the part of the students. Such a class response to work not required is an indication that the notebook was a valuable device in providing for individual differences.

Do the teaching procedures give adequate attention to "transfer of learning?" Many of the observers' notes quoted (see pages 291-4) and the discussion in Chapter IV (see pages 230-4) give some indication of the procedures used in allowing for transfer. In addition, the subject matter, as can be observed from the lesson sheets included in Chapter IV, was chosen so as to emphasize applications

in a wide variety of situations. For the most part, the view toward transfer throughout the year of the experiment was one as described by Stroud:

Courses of study are justifiable only if the facts learned permit the students to make generalizations about something; or if the knowledge, skill, attitude, and method may be sufficiently abstracted as to be generally useful in other courses of study or in life outside of school. To the extent that training is generalized it will transfer.²

In general, these data suggest that the student discovery procedures as used in the experimental course were in a large measure merely an application of the above-mentioned pedagogical considerations to the teaching of algebra.

What evidence is there that transfer actually did take place? One piece of evidence is an account of a class discussion following the showing of a March of Time film, Americans All, to the experimental class students during a core period. The topic of the film was on attaining racial and religious tolerance during World War II.

At one point in the film a statement, among others made, on a speech on how to cut down racial and religious tensions and antagonisms was, "Don't generalize." Because of experience with concept of generalization in the mathematics class, the students laughed at the statement. When asked if the man meant the statement, the students said they thought he meant, "Don't over-generalize." When pushed for a meaning for over-generalize an answer given was, "Means jumping to conclusions." I

²Stroud, J. B., "The Role of Practice in Learning," The Psychology of Learning, 41st Yearbook, National Society for the Study of Education. Chicago: University of Chicago Press, 1942, p 366.

pushed him further; "Means generalizing on insufficient data." I asked someone to make this idea more concrete. His example was concerning the habit of many people judging a whole group of people by the behavior of one or two acquaintances in this group.* Several other students elaborated on this point.*

In addition to this account, many other examples of transfer to non-mathematical areas of learning occurred in the students' notebooks and in observer accounts. All of these examples have already been listed under the various abilities and can be found on the following pages.

Comments concerning horoscope reading,
sack of apples, and swimming boys (see page 260)

Comments concerning the car model for 195x
(see page 261)

Comments concerning Dr. Papez and polio (see
page 263)

Comments concerning the indictment of a secretary for embezzlement and forgery (see page 263)

Generalizations from newspapers and periodicals
(see page 265)

Comment concerning an opinion sample (see page
277)

Comment concerning a warm day in the winter
(see page 277)

Comment concerning rain and a newly washed
car (see page 281)

Comments concerning the election campaign and
over-generalizing (see page 294)

* These notes were written by Mrs. Katharine Jones,
the core teacher.

Comments concerning election returns and amount of rainfall on election day (see page 294)

The many accounts of teaching procedures and comments by students indicate that a conscious effort was made to teach for transfer and that students made use of their generalizing abilities in non-mathematical situations.

CHAPTER VII

A COMPARISON BETWEEN THE GROWTHS OF THE EXPERIMENTAL AND THE STATUS GROUPS

Before it is possible to formulate conclusions with any degree of certainty concerning the effectiveness of teaching procedures, it is necessary to show that students taught by such procedures actually made significantly greater growth toward instructional goals than would be expected if such procedures were not used. The main goal of the teaching procedures used in this study was to help students develop their ability to generalize. It is assumed that the growth toward this ability by students in the status group, who were taught algebra in the conventional manner, is approximately the same as would be the growth of the experimental class had they been taught algebra in the conventional manner. The generalization test (see Appendix A) which was given to both groups of students provides some evidence concerning the growth made by both groups on several of the characteristics of the superior generalizer. These characteristics and data concerning the growth made by both groups are listed below.

Test Results on Generalization Items
Measuring Certain Generalizing Abilities

He is able to detect likenesses and differences between situations and then to group together those situations which possess common properties.

The generalization test provided evidence on the development both groups made in their ability to detect likenesses and differences. A summary of results on test items which measure this ability are given in Table V.

TABLE V

Summary of Results on the
Detection of Likenesses and Differences

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	36	36	36	36
Range	13-33	15-35	3-32	5-33
First quartile	18	20	15	17
Median	21	26	18	21
Third quartile	27	29	22	25
Mean	22.00	25.00	18.34	21.24
Standard deviation	6.00	5.28	5.76	6.27
Standard error of mean	1.01	.92	.51	.56
Mean difference	3.00		2.90	
Standard error	1.37		.76	
	Difference is significant to the 2% level.		The difference is significant	

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P^* = 22.00 - 18.34 = \underline{3.66}$	$M_E - M_P = 25.00 - 21.24 = \underline{3.76}$
Standard error = 1.14	Standard error = 1.08
Both differences are significant**	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-2 to +8	-3 to +12
Mean gain	2.79	2.98
Standard deviation	2.57	3.12
Standard error of mean	.45	.28
Mean difference	-.19	
Standard error	.53	
The difference is <u>not</u> significant		

* M_E = Mean of the Experiment Group

M_P = Mean of the Public School Group

**When significant is not qualified on this or any of the following tables, the significance is beyond the 1% level.

Briefly, the table shows that at the beginning and end of the year students in the experimental class scored significantly higher than students in the public school classes. It is important to note that the mean score and percentile marks made by the experimental group in the fall were slightly better than the corresponding marks made by the status group in the spring. Also to be noticed on the fall test is that for all practical purposes the first quartile mark and the median score of the experimental class could be considered respectively equal to the median score and third quartile mark of the status group. A further statistical treatment of the gains show that both groups made significant gains, but the difference between them is not significant. These results would indicate either that the students in the experimental class made no greater gains in developing their ability to detect likenesses and differences than students in the status group, or that the generalization test was not sensitive enough to measure the strength of the gain made by the experimental students. Considering the amount of practice the experimental students received for the purpose of developing this ability, the latter possibility seems to be a more reasonable assumption than the former. Another factor supporting this assumption is the significant gain made by the experimental students even though on the initial test they scored significantly higher than the status group. It would seem that a significant gain on a test where there is

little room for improvement is a much greater accomplishment than a significant gain where there is much room for improvement.

He is able to generalize correctly with certainty where certainty is justified, but refrains from generalizing when it is unwarranted.

Two different types of items were included in the generalization test to measure growth toward this ability. Test results on the first type of item, those which measured the recognition of situations which could be answered with certainty, are in Table VI.

TABLE VI

Summary of Results on Items
Which Could Be Answered with Certainty

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	44	44	44	44
Range	14-40	18-41	3-40	6-41
First quartile	22	24	16	19
Median	28	30	21	25
Third quartile	32	34	26	29
Mean	26.38	29.09	21.11	24.46
Standard deviation	6.78	6.24	7.23	7.51
Standard error of mean	1.19	1.10	.64	.67
Mean difference	2.71		3.35	
Standard error	1.62		.93	
	Difference is significant to the 5% level		The difference is significant	

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 26.38 - 21.11 = 5.27$	$M_E - M_P = 29.09 - 24.46 = 4.63$
Standard error = 1.35	Standard error = 1.29
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-4 to +9	-4 to +13
Mean gain	2.97	3.28
Standard deviation	2.25	3.46
Standard error of mean	.40	.31
Mean difference	-.31	
Standard error	.51	
The difference is not significant		

Again, these results show that both at the beginning and end of the year the experimental students scored significantly higher than students in the public school. Both groups made gains of significance, and the difference between these group gains was slightly, but far from significantly, in favor of the status group.

The test results on the second type of items, measuring the recognition of situations in which neither extrapolation nor interpolation is warranted, are in Table VII.

TABLE VII

Summary of Results on the Recognition of Situations
in which Extrapolation or Interpolation is Unwarranted

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	31	31	127	127
Possible score	15	15	15	15
Range	0-15	3-14	0-15	0-14
First quartile	2	7	2	3
Median	6	9	4	6
Third quartile	12	11	6	9
Mean	6.84	9.26	4.42	5.81
Standard deviation	4.57	3.14	3.51	3.97
Standard error of mean	.81	.56	.31	.35
Mean difference	2.42		1.39	
Standard error	.98		.47	
Both differences are significant				

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 6.84 - 4.42 = \underline{2.42}$	$M_E - M_P = 9.26 - 5.81 = \underline{3.45}$
Standard error = .87	Standard error = .66
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-3 to +8	-5 to +13
Mean gain	2.42	1.28
Standard deviation	2.91	2.86
Standard error of mean	.51	.25
Mean difference	1.14	
Standard error	.57	
The difference is significant to the 2 level		

The results on these items again show that at the beginning and end of the year the students in the experimental class scored significantly higher than public school students. Both groups made significant gains, but the difference in the gains was significant (2% level) and in favor of the experimental group. A further study of Table VII shows a large increase in the first quartile mark and a small decrease in the third quartile mark by the experimental group. Such a peculiar change stimulated an inspection of the gains made by each individual, results of which are as follows:

1. Experimental Group: 22 out of 31 students (71%)* increased their score on the spring test.
Public School Group: 69 out of 127 (54%)* increased their score on the spring test.
2. Experimental Group: 17 out of the 17 (100%) who made scores equal to the median or below in the fall increased their score in the spring.
Public School Group: only 31 out of 71 (44%) who made scores equal to the median or below in the fall increased their score in the spring.
3. Experimental Group: 2 out of 9 (22%) who made scores in the upper quarter in the fall increased their scores in the spring.

*The difference between 54% and 71% is significant to the 5% level (standard error of difference, 9.3%).

Public School Group: 13 out of 31 (42%) who made scores in the upper quarter in the fall increased their scores in the spring.

The data as noted under point 3 show that only a small per cent of the experimental students scoring in the upper quartile on the fall test improved their scores in the spring. Why is this? A probable partial explanation is that the students' original scores were very high and, therefore, could not be greatly increased. And, too, many of these students were overly cautious in the fall, but much less so in the spring. Such a behavior change could account for decreased scores for some students.

He is able to determine trends from available data and from them make reasonable extrapolations and interpolations.

Two different types of items were included in the generalization test to measure growth toward this ability. Test results on the items which measured the testee's ability to go beyond data justifiably are given in Table VIII.

TABLE VIII

Summary of Results on Justifiable
Extrapolation and Interpolation

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	32	32	127	127
Possible score	13	13	13	13
Range	0-10	0-13	0-9	0-10
First quartile	1	2	1	1
Median	2	5	2	2
Third quartile	3	8	2	3
Mean	2.44	5.00	1.91	2.31
Standard deviation	2.18	3.12	1.55	1.87
Standard error of mean	.39	.55	.14	.17
Mean difference	2.56		.40	
Standard error	.68		.22	
	The difference is significant		The difference is <u>not</u> significant	

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 2.44 - 1.91 = .53$	$M_E - M_P = 5.00 - 2.31 = 2.69$
Standard error = .41	Standard error = .58
The difference is <u>not</u> significant	The difference is significant

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-4 to +9	-5 to +5
Mean gain	2.56	.41
Standard deviation	3.63	1.68
Standard error of mean	.64	.15
Mean difference	2.15	
Standard error	.67	
The difference is significant		

These results are strongly in favor of the experimental group. At the beginning of the year the difference between the scores of the experimental group and the status group was not significant, but at the end of the year the difference was significantly in favor of the experimental group. The gain made by the status group was not significant, but that of the experimental group was significant. Also, the difference in the gains between the experimental group and the status group was significantly in favor of the experimental group. A more detailed study of the initial and final scores for each student shows that 21 out of 32 (66%) of the experimental students made gains, while only 54 out of 127 (43%) of the public school students made gains.*

Test results on the second type of item, those which measured going beyond the data when not justified, are in Table IX.

*The difference between the percentages is significant (standard error of difference, 9.7%).

TABLE IX

Summary of Results on Unjustifiable
Extrapolation and Interpolation

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	32	32	127	127
Perfect score	0	0	0	0
Range	21-2	24-1	27-1	25-0
First quartile	14	10	15	12
Median	10	8	10	8
Third quartile	6	4	6	6
Mean	10.44	7.34	11.07	9.23
Standard deviation	5.67	4.47	5.70	5.18
Standard error of mean	1.00	.79	.51	.46
Mean difference	-3.10		-1.84	
Standard error	1.27		.69	
Both differences are significant				

Comparison of the Means for Each Group

<u>Fall</u>	<u>Spring</u>
$M_E - M_P = 10.44 - 11.07 = \underline{-.63}$	$M_E - M_P = 7.34 - 9.23 = \underline{1.89}$
Standard error = 1.12	Standard error = .91
The difference is <u>not</u> significant	The difference is significant to the 5% level

Comparison of the Negative Gains for
Each Group

	Experimental	Public School
Gain range	+3 to -15	11 to -17
Mean gain	-3.63	-1.74
Standard deviation	3.68	4.59
Standard error of mean	.65	.47
Mean difference		-1.89
Standard error		.80
The difference is significant		

These results show that there was a noticeable decrease in the number of times the experimental group extrapolated or interpolated unjustifiably. Briefly, both groups made a significant decrease in number of unjustifiable predictions on the spring test, but the decreases of the experimental students were significantly greater. In comparing the fall and spring tests for each student it was found that on the spring test 27 out of 32 (84%) of experimental students made decreases, and only 76 out of 127 (60%) of public school students decreased in number their unjustifiable extrapolations.*

He is cautious in accepting generalizations, but is willing to consider any generalization as a hypothesis.

The generalization test provided considerable evidence concerning the changes students made in caution. This evidence is given in four different tables, the first of which summarizes the results on "climate of opinion" items.

*The differences between the percentages is significant (standard error of difference, 7.8%).

TABLE X

Summary of Results on Climate of Opinion Items

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	19	19	19	19
Range	4-16	3-16	0-16	0-11
First quartile	7	8	5	6
Median	10	11	6	8
Third quartile	12	13	9	9
Mean	9.58	10.00	6.66	7.41
Standard deviation	3.20	3.26	2.82	2.25
Standard error of mean	.56	.57	.25	.25
Mean difference	.42		.75	
Standard error	.69		.35	
	The difference is <u>not</u> significant		The difference is significant to the 2% level	

Comparison of the Means for Each Group

<u>Fall</u>	<u>Spring</u>
$M_E - M_P = 9.53 - 6.66 = \underline{2.92}$	$M_E - M_P = 10.00 - 7.41 = \underline{2.59}$
Standard error = .61	Standard error = .62
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-5 to +6	-7 to +9
Mean gain	.45	.37
Standard deviation	2.55	2.35
Standard error of mean	.44	.25
Mean difference	-.42	
Standard error	.51	
The difference is <u>not</u> significant		

Briefly, the table shows that both at the beginning and end of the year the experimental class scored significantly higher than the status group on the "climate of opinion" items. On the spring test the status group made a greater gain than the experimental class, but it is important to note that the scores made by the experimental class in the fall were significantly higher than those made by the status group in the spring. However, the fact that the experimental class did make gains at all seems in itself to be of some significance. More will be said about these results later.

Results showing the students' willingness to go beyond data are given in Table XI.

TABLE XI

Summary of Results on Willingness to Go Beyond Data

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	32	32	120	120
Possible score	28	28	28	28
Range	4-26	3-17	2-28	1-28
First quartile	8	10	8	8
Median	13	12	12	11
Third quartile	16	15	17	15
Mean	13.06	12.13	12.80	11.85
Standard deviation	6.12	4.41	6.31	5.64
Standard error of mean	1.07	.77	.56	.50
Mean difference	-.93		-.95	
Standard error	1.32		.71	
	The difference is <u>not</u> significant		The difference is substantial but not significant	

Comparison of the Means for Each Group

<u>Fall</u>	<u>Spring</u>
$M_E - M_P = 13.06 - 12.80 = .26$	$M_E - M_P = 12.13 - 11.85 = .28$
Standard error = 1.21	Standard error = .92
The difference is not significant	The difference is not significant

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-9 to +8	-22 to +11
Mean gain	-.81	-1.71
Standard deviation	5.22	4.33
Standard error of mean	.92	.38
Mean difference	+.90	
Standard error	.99	
The difference is not significant		

These test results suggest that student discovery procedures in the experimental class had little effect on the behavior, willingness to go beyond the data. However, the behavior of the first quartile indicated that there were changes taking place in the experimental group that were quite different from those in the status group. The results of further investigation show:

1. Experimental Group: 17 out of 32 students (53%) were less willing to go beyond the data on the final test than on the initial test.
Public School Group: 66 out of 127 (52%) were less willing to go beyond data on the final test than on the initial test.
2. Experimental Group: 9 out of 9 (100%) scoring in the upper quarter on the fall test scored lower on the spring test.
Public School Group: 27 out of 34 (79%) scoring in the upper quarter on the fall test scored lower on the spring test.
3. Experimental Group: 7 out of 8 (88%) scoring in the lower quarter on the fall test scored higher on the spring test.
Public School Group: 19 out of 34 (56%) scoring in the lower quarter on the fall test scored higher on the spring test.

The data seem to indicate that in the experimental more than in the public school group, those students most willing to go beyond data in the fall were less willing in the spring, and those most reluctant to go beyond data in the fall were more willing to go beyond data in the spring. It seems such a change would be desirable from the standpoint of improving generalizing ability. Some but not all situations warrant going beyond the data, and a superior generalizer's attitude toward this particular characteristic should be one of discretion. Results as recorded above are such as one might expect of students who had learned to exercise more discretion in going beyond the data.

Results on the climate of opinion items when scored from the point of view of caution (in Part I of generalization test) are given in Table XII.

TABLE XII

Summary of Results on the Caution Items in Part I
of the Generalization Test

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	57	57	57	57
Range	15-41	13-48	6-36	7-40
First quartile	21	24	16	19
Median	25	31	21	23
Third quartile	34	35	25	27
Mean	26.56	29.91	20.28	23.33
Standard deviation	8.21	8.20	6.72	7.05
Standard error of mean	1.44	1.43	.59	.63
Mean difference	3.35		3.05	
Standard error	2.03		.85	
	Difference is significant to the 5% level		The difference is significant	

Comparison of the Means for Each Group

<u>Fall</u>	<u>Spring</u>
$M_E - M_P = 26.56 - 20.28 = \underline{6.28}$	$M_E - M_P = 29.91 - 23.33 = \underline{6.58}$
Standard error = 1.55	Standard error = 1.53
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-9 to +19	-8 to +24
Mean gain	2.49	3.02
Standard deviation	6.97	5.71
Standard error of mean	1.21	.51
Mean difference	-.53	
Standard error	1.31	
The difference is <u>not</u> significant		

These results show that both at the beginning and end of the year the experimental class scored significantly higher than the status group on caution items in the first part of the test. Both groups made significant gains on the spring test, but the difference in gains, although favoring the status group, was not significant. The standard deviations and quartile marks suggest that there might still be changes occurring in the experimental class significantly different than those in the public school classes. Further investigation shows:

1. Experimental Group: 16 out of 33 students (48%)* scored less on the spring test than on the fall test.
Public School Group: 34 out of 127 (27%)* scored less on the spring test than on the fall test.
2. Experimental Group: 8 out of 11 (73%) scoring in the upper quarter on the fall test scored lower on the spring test.
Public School Group: 15 out of 33 (45%) scoring in the upper quarter on the fall test scored lower on the spring test.
3. Experimental Group: 6 out of 9 (67%) scoring in the lower quarter on the fall test scored higher on the spring test.

*Difference between 27% and 48% is significant to the 2 per cent level (standard error of difference, 9.6%).

Public School Group: 25 out of 32 (78%)

scoring in the lower quarter on the fall test
scored higher on the spring test.

This latter analysis indicates that a significantly larger per cent of the experimental class became less cautious in answering "climate of opinion" items than did the status group, and this loss of caution was especially pronounced with those experimental students scoring high on caution on the fall test. It is important to note, however, that even with the large per cent of students becoming less cautious the class mean increased substantially on the spring test. Also, the class mean increased slightly when these items were scored from the point of view of being right or wrong (see Table IX). These two sets of scores on the same items suggest that the experimental class had become more discriminating in their use of caution.

Results on the "interpreting graph and chart" items when scored for caution (in Part II of the generalization test) are given in Table XIII.

TABLE XIII

Summary of Results on the Caution Items
in Part II of Generalizing Test

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	32	32	120	120
Possible score	41	41	41	41
Range	3-34	3-33	0-35	0-36
First quartile	11	16	7	9
Median	19	22	13	15
Third quartile	25	24	17	21
Mean	18.50	20.47	12.92	15.30
Standard deviation	9.09	6.18*	7.23	7.95
Standard error of mean	1.60	1.09	.66	.73
Mean difference	1.97		2.38	
Standard error	1.93		.99	
	Difference is not significant		Difference is significant	

Comparison of the Means of Each Group

<u>Fall</u>	<u>Spring</u>
$M_E - M_P = 18.50 - 12.92 = \underline{5.58}$	$M_E - M_P = 20.47 - 15.30 = \underline{5.17}$
Standard error = 1.73	Standard error = 1.32
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-7 to +16	-9 to +24
Mean gain	1.81	2.33
Standard deviation	5.95	6.22
Standard error of mean	1.05	.57
Mean difference	-.52	
Standard error	1.19	
The difference is <u>not</u> significant		

*Difference between the standard deviation of the Experimental Class is significant to the 5% level.

These results show that at both the beginning and end of the year the experimental class scored significantly higher than the status group. However, the gain in caution made by the status group was significant, while the gain in the experimental group was not. Nevertheless, the difference in gains between the two groups is not significant. Again, the significant decrease in the standard deviation and the peculiar behavior of the quartile marks of the experimental class suggest that a substantial change had taken place within the experimental group. The results of further investigation show:

1. Experimental Group: 12 out of 32 students (38%) became less cautious, but 94%* increased their score on these items when they were scored as part of the complete generalization test.**
Public School Group: 42 out of 120 (35%) became less cautious, but only 66%** increased their score on these items when they were scored as a part of the complete generalization test.
2. Experimental Group: 7 out of 8 (88%) scoring in the upper quarter on the fall test scored lower on the spring test.

*The difference between 94% and 66% is significant (standard error of difference, 6.2%).

**The method of scoring items for caution differed from the method using in determining the total generalization score.

Public School Group: 18 out of 35 (51%) scoring in the upper quarter on the fall test scored lower on the spring test.

3. Experimental Group: 8 out of 9 (89%) scoring in the lower quarter on the fall test scored higher on the spring test.

Public School Group: 29 out of 33 (88%) scoring in the lower quarter on fall test scored higher on the spring test.

These facts show that the per cent of the experimental and public school students who became less cautious on the final test was approximately the same, but the per cent of those students who scored high on the fall test and scored less in the spring was much larger for the experimental class than for the status group. Again, a reasonable conclusion seems to be that students in the experimental class became more discriminating in their use of caution.

Test Results on Part II of the Generalization Test

Further evidence of improvement in students' ability to generalize is found in Table XIV which summarizes the results on Part II of the generalization test. Responses to items in this part of the test depended upon how well students could interpret graphical and tabular data as it is likely to be found in current periodicals.

TABLE XIV

Summary of Results on Interpreting Graphical and
Tabular Data (Part II of the Generalization Test)

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	32	32	120	120
Possible score	100	100	100	100
Range	11-83	22-97	10-81	10-86
First quartile	32	55	29	33
Median	46	61	37	50
Third quartile	57	72	50	59
Mean	45.59	62.47	38.63	46.79
Standard deviation	15.92	14.97	15.55	16.75
Standard error of mean	2.82	2.64	1.41	1.52
Mean difference	15.88		8.16	
Standard error	3.62		2.04	
Both differences are significant				

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 45.59 - 38.63 = 6.96$	$M_E - M_P = 62.47 - 46.79 = 15.68$
Standard error = 3.14	Standard error = 3.05
The difference is significant to the 2% level	The difference is significant

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-14 to +50	-31 to +54
Mean gain	18.63	8.13
Standard deviation	13.35	12.42
Standard error of mean	2.36	1.13
Mean difference	10.50	
Standard error	2.63	
The difference is significant		

These results indicate that while both the public school and experimental students made significant gains on their spring test, the difference between the gain of the experimental class and that of the status group was highly significant and in favor of the experimental group.

Now, just how well did the status and experimental groups do on the generalization test at the end of the year? The results on the fall test were given earlier and at that time, it will be remembered, the experimental class scored significantly higher than the status group. Results on the spring test are summarized in Table XV.

TABLE XV

Summary of Results on the Generalization Test

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	100	100	100	100
Range	33-84	41-92	14-81	25-86
First quartile	46	60	39	48
Median	57	67	47	57
Third quartile	68	78	56	65
Mean	57.00	67.30	47.55	56.25
Standard deviation	13.20	10.80	12.05	11.15
Standard error of mean	2.30	1.88	1.07	1.02
Mean difference	10.30		8.70	
Standard error	2.98		1.48	
Both differences are significant				

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 57.00 - 47.55 = \underline{9.45}$	$M_E - M_P = 67.30 - 56.25 = \underline{11.05}$
Standard error = 2.54	Standard error = 2.14
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-2 to +29	-11 to +25
Mean gain	10.44	8.38
Standard deviation	7.22	7.77
Standard error of mean	1.26	.69
Mean difference	2.06	
Standard error	1.44	
Difference is substantial but not significant		

These results show that the experimental group made greater gains than the status group, but these gains, although substantial were not significant. The fact that the experimental students did so well on Part II which dealt with graphs and tables suggests that they did not make too good a showing on the first part of the test. Results on Part I are given in Table XVI.

TABLE XVI

Summary of Results on Part I of
the Generalization Test

	Experimental Class		Public School Classes	
	Fall 1951	Spring 1952	Fall 1951	Spring 1952
Number of tests	33	33	127	127
Possible score	100	100	100	100
Range	39-92	46-89	14-82	29-85
First quartile	56	61	45	54
Median	66	71	53	63
Third quartile	74	80	63	71
Mean	64.73	70.49	53.34	62.20
Standard deviation	14.51	11.42	12.23	11.50
Standard error of mean	2.52	1.98	1.09	1.02
Mean difference	5.76		8.86	
Standard error	3.20		1.49	
	Difference is significant to the 4% level		Difference is significant	

Comparison of the Means for Each Group

Fall	Spring
$M_E - M_P = 64.73 - 53.34 = 11.39$	$M_E - M_P = 70.49 - 62.20 = 8.49$
Standard error = 2.74	Standard error = 2.23
Both differences are significant	

Comparison of the Gains for Each Group

	Experimental	Public School
Gain range	-12 to +24	-14 to +41
Mean gain	6.00	8.68
Standard deviation	8.97	9.69
Standard error of mean	1.56	.86
Mean difference	-2.68	
Standard error	1.78	
Difference is substantial but not significant		

On Part I of the generalization test the greater gain was substantially, but not significantly, in favor of the status group. One probable reason for the smaller gain of the experimental students was that on the fall test they scored markedly higher than the status group. It is interesting to note that the mean and quartile scores of the experimental class were higher in the fall than were the corresponding scores for the status group in the spring. Such high scores in the fall could have made it much more difficult for the experimental group to make gains, and the fact that they made any gain is in itself significant. Probably the primary reason for the small gain was that in answering these test items a significantly larger per cent of students in the experimental class exercised less caution than students in the public school group. Since a correct response on these items was always the most cautious possibility, a loss of cautiousness from one test to the next would almost certainly hold down gains made on the final test.

The generalizing test used in this study provided direct evidence concerning the improvement made by the experimental students. Some conclusions suggested by this evidence are that student discovery procedures, employed in the experimental class as against the textbook dominated procedures used in the status group, helped students in the following ways:

1. To make justifiable extrapolations and interpolations from data
2. To recognize situations in which certain conclusions would be unjustifiable extrapolations or interpolations
3. To use discretion in their use of caution when generalizing. (Either too much or too little caution seems to be a handicap in the discovery of generalizations.)
4. To interpret graphical and tabular data.

In concluding the discussion on the generalization test it is well again to bring to the reader's attention the question of transfer of learning. The generalization test (see Appendix A) largely consists of items that are non-mathematical in nature. Therefore, data which suggest the above four conclusions also suggest a fifth one,

5. The teaching procedures as employed in the experimental class foster the development of the ability to generalize in non-mathematical situations.

Student Discovery as a Procedure for Teaching Algebra

At the beginning of the school year both the experimental and status groups were given the Iowa Algebra Aptitude Test, and its results indicate that the status group possessed significantly more algebraic aptitude than

the experimental students. At the end of the year both groups were given an Ohio Every Pupil Algebra Test. This test was selected because it was based on the objectives and subject matter of algebra as it is taught in the ordinary school situation. Since it was assumed the objectives and subject matter of the experimental class were quite different from those in the public school classes, it was felt that the extent of the achievement of the experimental class would not be fairly measured. Therefore, another test, the Lankton First Year Algebra Test,* which would measure more accurately the algebraic achievement of the experimental class was also given to them. The results on these mathematics tests are given in Table XVII.

*Because of limitations on the use of class time in the public schools for testing, the Lankton Test was not given to the status group.

TABLE XVII

Summary of Raw and Percentile Scores
on Mathematics Tests

	Experimental Group		
	Iowa Aptitude Test Sept. 1951	Ohio Every Pupil Test June 1952	Lankton Algebra Test June 1952
Number	33	33	23
Range	25-89 (3-99)*	2-36 (1-96)	6-52 (1-99)
First quartile	35 (20)	11 (11)	23 (35)
Median	43 (32)	18 (33)	25 (37)
Third quartile	67 (61)	25 (61)	37 (68)
Mean	46.27	18.67	28.79
Standard deviation	15.69	9.16	11.00
Standard error of mean	2.73	1.62	1.91

52% of the students scored on or above the 30 percentile on the Iowa Aptitude Test.

52% of the students scored on or above the 32 percentile on the Ohio Every Pupil Test.

76% of the students scored on or above the 32 percentile on the Lankton Algebra Test.

The difference between the 52% and 76% is significant to the 2% level. (Standard error of difference 11.4%)

	Public School Group	
	Iowa Aptitude Test Sept. 1951	Ohio Every Pupil Test June 1952
Number	127	127
Range	22-78 (5-93)	2-38 (1-98)
First quartile	43 (32)	17 (30)
Median	52 (50)	24 (57)
Third quartile	62 (73)	29 (80)
Mean	51.76	22.97
Standard deviation	13.11	8.41
Standard error of mean	1.16	.75

52% of the students scored on or above the 50 percentile on the Iowa Aptitude Test.

60% of the students scored on or above the 50 percentile on the Ohio Every Pupil Test.

The difference between 52% and 60% is not significant.
(Standard error of difference 6.2%)

*Numbers in parentheses are percentile scores.

An examination of the experimental group's quartile marks on the Ohio Every Pupil Test indicate that the experimental students, with the exception of those scoring on or below the first quartile mark, made scores approximately equal to those predicted by the Iowa Algebra Aptitude Test. The status group's quartile marks on the Ohio Every Pupil Test indicate that, with the exception of those scoring on or below the first quartile, public school students made scores somewhat better (but not significantly) than was predicted. Results on the Lankton test indicate that the experimental group scored significantly higher (2 per cent level of significance) than was expected. These results, plus the fact that the experimental class devoted much less school time to the study of algebra than the status group, would seem to support the conclusion that student discovery procedures are quite satisfactory from the standpoint of teaching algebraic principles.

CHAPTER VIII

SUMMARY

In many states a study of mathematics during one year of high school* is required. Just what should be the nature of such a single mathematics course? Generally, procedures used and content studied should be of the greatest value to all students in the class. A proposed answer to this question for students in the University School of The Ohio State University is a course having as its main objective the development of the ability to generalize through student discovery of algebraic principles.

The selection of the main objective determines in a general way the teaching procedures to be used in the experimental class. Improvement in students' generalizing ability is probably most certain if the students actually practice generalizing. One way of providing this practice for students is to present the course so as to have them discover for themselves the mathematical principles to be learned.

Since this mathematics course is likely to be the terminating one for many students and for others a prepara-

*High School is assumed here to be grades nine, ten, eleven, and twelve.

tion for more advanced mathematics, care must be exercised in selecting the content to be studied by all students. Guiding principles used in the selection of content for the experimental course were as follows:

1. Students should study mathematical content which can be used to illustrate the various generalizing procedures.
2. Students should be given an overview of mathematics and become aware of the interrelatedness of its many branches.
3. Students should study topics they can make use of now and most likely will use in the future.
4. Students should study topics which review and extend their understandings of arithmetic.

The actual content studied by the experimental class can be summarized under the following headings.

- a. Number and Operation
- b. Graphs and Formulas
- c. Equations and Problem Solving
- d. Proportion and Indirect Measurement
- e. Statistics

The improvement of generalizing ability is an objective which is consistent with the aims and purposes of the University School and important from the standpoint of understanding mathematics. Improvement in both the discovery and proof of generalizations is important in further-

ing the ability to generalize, but of the two, the former is of greater concern in this experimental course. Ways by which generalizations are discovered are stated below:

Empirical generalization suggested by

- a. Simple enumeration
- b. Analogy
- c. Continuity of form
- d. Statistical procedures

Rational generalization suggested by

- a. Deduction
- b. Variation
- c. Formal analogy
- d. Inverse deduction

A superior generalizer is assumed to be a person who is able to use all these procedures in arriving at generalizations. An analysis of these procedures reveals that the superior generalizer should have the following abilities and behaviors:

1. He is able to detect likenesses and differences between situations and then to group together those situations which possess common properties.
2. He is able to correctly generalize with certainty where certainty is justified, but refrains from generalizing when it is unwarranted.
3. He is able to determine trends from available data and from them make reasonable extrapolations and interpolations.
4. He is cautious in accepting generalizations, but is willing to consider any generalization as a hypothesis.
5. He can distinguish generalizations from observable facts.

6. He uses generalizing methods independently to further his understanding of the world around him.
7. He makes use of generalizations which he senses through nonverbal means.
8. He is able to verbalize generalizations and methods used in discovering generalizations.
9. He continually searches for examples which are exceptions to formulated generalizations and then reformulates them to include the exceptions.
10. He investigates implications of generalizations under consideration.
11. He searches for explanations for generalizations discovered empirically.
12. He empirically tests or searches for an increasing number of applications to generalizations that have been suggested by rational procedures.
13. He searches for and recognizes relevant factors responsible for happenings or for behaviors of objects in certain situations.
14. He studies accepted generalizations from as many points of view as possible.
15. He uses statistical measures in making generalizations.

As an aid in determining how successful the teaching procedures are, the thirty-five students in the experimental class are compared with a group of 127 students studying algebra in several different ninth grade classes in the city of Columbus. Information collected on both the experimental and status classes at the beginning of the year indicate the following:

1. Both groups are roughly similar as to age and scores on intelligence tests.

2. The status group possessed significantly more algebraic aptitude than the experimental class (according to the Iowa Algebra Aptitude Test).

3. The experimental students were significantly better generalizers than the students in the status group (according to results on a generalization test).

The success of any experiment depends upon how well the data collected answer basic questions concerning the main objective of the experiment. The conclusions suggested by the data collected in this experiment are listed under the basic question they help to answer.

How well did the content lend itself to student discovery procedures thereby giving students practice in generalizing?

1. Teaching procedures can be organized so the average student can make use of generalizing procedures in the discovery of algebraic principles and concepts.

2. The combination of algebraic content and student discovery teaching procedures as used in the experimental class did offer excellent opportunities for students to develop the behaviors and abilities characteristic of a superior generalizer.

Did the experimental class when compared with the status group actually make significant improvement in their ability to generalize?

3. The experimental class made significantly greater improvement in their ability to draw conclusions that are justifiable extrapolations and interpolations of accepted data.

4. The experimental class made significantly more improvement in their ability to recognize conclusions that were not justifiable extrapolations and interpolations of accepted data.

5. The experimental class became noticeably more discreet in their use of caution when generalizing from data that is relatively independent of the climate of opinion.

a. Both the experimental and status groups' overall change in willingness to go beyond the data was hardly noticeable. However, in the experimental group a much larger per cent of those students most willing to go beyond the data in the fall were much less willing in the spring, and those least willing in the fall were more willing in the spring.

b. When caution was measured directly both the experimental and status groups made approximately the same gains, but a significantly (2 per cent level) larger per cent of the students in the ex-

perimental class generalized more correctly in the spring than in the fall.

c. In the experimental group a much larger per cent of those students who were most cautious in the fall were less cautious in the spring.

6. The experimental class also appeared to become more discreet in their use of caution when generalizing from data originating from the surrounding climate of opinion.

a. A significantly (2 per cent level) larger per cent of the experimental class became less cautious when generalizing from "climate of opinion" data.

b. In the experimental group a much larger per cent of those students who were most cautious in the fall were less cautious in the spring.

c. The experimental group, although generalizing significantly more correctly in the fall on the "climate of opinion" items, still indicated approximately the same improvement as the status group in the spring.

7. The experimental class made significantly greater improvement in interpreting graphical and tabular data.

8. The teaching procedures used in the experimental class did help students in developing their ability to generalize in non-mathematical situations (according to the generalization test given at the beginning and end of the year).

How well did the experimental class master algebra?

9. The experimental class achievement as measured by results on the Ohio Every Pupil Test was accurately predicted by results on the Iowa Algebra Aptitude Test. Their achievement as measured by the Lankton First Year Algebra Test was significantly greater than was predicted by the results on the Iowa Algebra Aptitude Test. This achievement was accomplished even though these students had much less school time for the study of algebra than did the students in the public school classes.

What were students' reaction to student discovery and emphasis upon generalizing?

10. At the beginning of the year student discovery procedures and the emphasis on generalizing should be introduced slowly. Apparently, students have to live with and gain confidence in such teaching procedures before they see their worth.

11. If students sense they are making progress in the discovery of a generalization, their reaction to student discovery procedures is highly favorable.

12. Successful use of student discovery gives students considerable satisfaction.

13. By the end of the year a large percent of the students liked the emphasis on student discovery, and the class at large felt the emphasis on generalizing was desirable.

14. Student discovery procedures can arouse students emotionally. This is especially noticeable whenever some generalization result contradicts a preconceived notion of what they expected to find. An example of such a generalization is the rule for multiplying two negative numbers.

15. The success of student discovery procedures on any particular day is more dependent than other procedures upon the general emotional tenor of the class.

How well do student discovery procedures take into account certain pedagogical considerations assumed to be of major importance?

16. Students in the experimental course were provided opportunities in which they, as a natural consequence, did think for themselves, and some even did seek knowledge independently.

17. The role of the teacher in student discovery procedures is an important one. He must organize materials well enough so students can see they are making progress in the discovery of generalizations, but not so well there are no discoveries of any consequence for them to make.

18. Student discovery procedures, as used in the experimental class, offered many opportunities for an

intelligent application of the principles discovered and greatly reduced the opportunities for students to apply a principle mechanically.

19. The use of student discovery procedures in the discovery of algebraic principles provides many opportunities in which students as a natural consequence think intelligently with symbols and become aware of the role of symbols in the discovery of ideas and thinking in general.

20. The successful use of student discovery in the teaching of algebraic principles requires that the teacher remain close to the understanding of his students.

21. Student discovery procedures enable the teacher to employ a wide range of desirable motivating devices and techniques.

22. The employment of student discovery procedures encourages students to strike out on their own in the discovery of new ideas thus helping to make the course flexible enough to allow for a wide range of abilities and purposes.

23. Student discovery of algebraic principles almost of a necessity requires emphases to be placed on generalizing procedures and on applications. These emphases, in turn, increase the chances that the concepts and procedures learned will transfer (see pages 300, 334).

What are other conclusions that can be made from this study?

24. Student reaction to the content studied indicated that probably less time should have been spent on factoring, special products, algebraic fractions, and probably more time given to the study of ratio and proportion, indirect measurement, and statistics.

25. Students were more generalization conscious at the end of the year than at the beginning.

Taking all the evidence into consideration it seems the conclusion can be made that the use of student discovery procedures in the experimental course significantly improved the students' ability to generalize in both mathematical and non-mathematical situations, and at the same time allowed them to gain a reasonable mastery of algebraic principles. The study would have been more conclusive if:

1. Both the experimental and status groups had been from schools with a similar overall program
2. Both groups had been more nearly equal at the beginning of the year as to algebraic aptitude and generalizing ability
3. the number of students in the experimental class had been more equal to the number in the status group.

This study suggests a number of related topics with excellent research possibilities, many of which are listed below:

1. Construct and standardize a test that will measure growth students make in all generalizing abilities and behaviors.
2. Repeat a study similar to this one in the public schools.
3. Discover when verbalizing is necessary and when not necessary in the understanding of a principle or concept.
4. Teach and evaluate a unit on signed numbers in which students learn to subtract before they learn to add.
5. Design and evaluate a course in second year algebra which emphasizes student discovery of algebraic principles.
6. Determine the effect of improved generalizing ability on drawing conclusions from situations in which opinion or emotion provide the main source of information.
7. Compile a list of significant discoveries that have been suggested by the various generalizing procedures.
8. Use student discovery procedures in teaching algebra as generalized arithmetic and then compare the students' improvement in arithmetic with that of a general mathematics course.

9. Determine whether the growth in ability to generalize made by students studying algebra in the conventional manner is largely due to the study of algebra.

10. Determine the factors in the University School program responsible for students scoring significantly higher than the students in the public schools at the beginning of the experiment.

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APPENDICES

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APPENDIX A

As an aid to the reader, the following key is provided to indicate the role of each item on the generalization test in the evaluation of generalizing abilities:

- L - Detecting likenesses and differences
- C - Generalizing with certainty in situations where justified
- U - Recognizing situations in which extrapolation and interpolation are unwarranted
- J - Extrapolating and interpolating in situations where justified
- N - Extrapolating and interpolating in situations where not justified
- O - Generalizing in situations in which climate of opinion is likely to be a strong influence
- W - Willing to go beyond data
- K - Being cautious in generalizing

GENERALIZATION TEST - FORM C

Name _____

School _____

Grade _____ Teacher _____

Part I

Answer the questions by marking them in the way the following examples are marked.

Examples:

1. Which word does not belong with the others?
☒ a. Tom b. Janet c. Ann d. Both e. Mary
2. A coward always lacks
 a. friends b. purpose ☒ c. courage d. enemies
 e. answer is not given
3. Ear is to music as nose is to
 a. face b. breath c. noise ☒ d. perfume e. tone
4. is to as is to
 a. b. c. d. ☒ e.
5. 1, 6, 11, 16, , , 31. What two numbers should be on the dotted line?
☒ a. 21 and 26 b. 17 and 25 c. 26 and 29
 d. 22 and 27 e. 20 and 25

(Key)

L,C

1. is to as is to
 a. b. c. d. e.

L,C

2. A canoe always has
 a. paddles b. canvas c. water d. paint
 e. length

Generalization Test - Form C - page 2

- (key)
L,C
3. Bad is to worse as good is to
a. better b. very bad c. medium d. worse
e. best
- L,C
4. Which word does not belong with the others?
a. pepper b. cloves c. ginger d. flour
e. cinnamon
- O,K
5. Fill in the word that fits best:
Among children, _____ boys are more emotional
than girls.
a. all b. more c. don't know d. fewer e. no
- L,C
6. A, C, F, J, _____. Which letter comes next?
a. M b. K c. L d. O e. N
- L,C
7. Automobile is to wheel as horse is to
a. leg b. tail c. gallop d. wagon e. drive
- L,C
8. A river always has
a. banks b. boats c. bridge d. ferry e. fishes
- L,C
9. is to as is to
a. b. c. d. e.
- O,K
10. Fill in the word that fits best. _____ people
of Jewish religion would try to cheat a man in a
business deal if they thought they had an oppor-
tunity to do so.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C
11. 256, 128, 64, 32, 16, __, __. What two numbers
should come next?
a. 15 and 14 b. 4 and 2 c. 12 and 8
d. 8 and 4 e. 17 and 18
- L,C
12. 64, 16, 4, 1, $1/4$, __, __. What two numbers
should come next?
a. $1/16$ and $1/64$ b. $1/8$ and $1/12$ c. $4/16$ and $16/24$
d. $4/8$ and $8/12$ e. $1/5$ and $1/7$
- O,K
13. Fill in the word that fits best. _____ Russians
are bloodthirsty and cruel people.
a. All b. Most c. Don't know d. A few or some
e. No

Generalization Test - Form C - page 3

(Key)

- L,C 14. Which word does not belong with the others?
a. goat b. cow c. snake d. dove e. robin
- L,C 15. Which word does not belong with the others?
a. coat b. jacket c. dress d. underwear
e. shirt
- O,K 16. Fill in the word that fits best. _____ labor unions are becoming too powerful.
a. All, b. Most c. Don't know d. A few or some
e. No
- L,C 17. A store always has
a. register b. stock c. windows d. fruit
e. losses
- O,K 18. Fill in the word that fits best. _____ girls are smarter than boys.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 19. 2, 4, 7, 11, 13, 16, 20, __, __. What two numbers should come next?
a. 25 and 31 b. 22 and 24 c. 24 and 28
d. 23 and 27 e. 22 and 25
- O,K 20. Fill in the word that fits best. _____ sharks are man eaters.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 21. Which word does not belong with the others?
a. shoes b. cotton c. flax d. leather
e. wool
- O,K 22. Fill in the word that fits best. _____ dogs are smarter animals than pigs.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 23. 3, 6, 10, 15, 21, 28, __, __. What two numbers come next?
a. 35 and 43 b. 36 and 44 c. 35 and 42
d. 36 and 43 e. 36 and 45
- L,C 24. A wheel always has
a. an axle b. circumference c. answer not given
d. tire e. spokes

Generalization Test - Form C - page 4

(Key)

- L,C 25. A house always has
a. a porch b. answer not given c. wallpaper
d. a basement e. an attic
- L,C 26. Which word does not belong with the others?
a. eating b. feeling c. hearing d. seeing
e. smelling
- O,K 27. Fill in the word that fits best. Hair _____ grows again on the bald spot of an older man's head.
a. always b. most always c. don't know
d. occasionally e. never
- L,C 28. 810, 270, 90, 30, _____. What number should come next.
a. 15 b. 10 c. 20 d. 0 e. 5
- Omit 29. Which figure does not belong with the others?
a. b. c. d. e.
- O,K 30. Fill in the word that fits best. _____ well-fed cats are better mousers than hungry cats.
a. All b. Most c. Don't know d. A few or some
e. No
- O,K 31. Fill in the word that fits best. _____ countries would profit greatly by doing things the way we do in the United States.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 32. Soup always has
a. spoon b. liquid c. vegetables d. salt
e. meat
- O,K 33. Fill in the word that fits best. _____ labor union leaders really try to bring improvements in wages, hours, and working conditions of the members of their unions.
a. All b. Most c. Don't know d. Few e. No
- L,C 34. Algebra always deals with
a. symbols b. equations c. problems
d. factors e. digits

Generalization Test - Form C - page 5

(Key)

- O,K 35. Fill in the word that fits best. _____ people who are getting the biggest salaries in our largest corporations today worked their way up from humble beginnings without money or influential friends to help them.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 36. Which word does not belong with the others?
a. clarinet b. cornet c. horn d. trombone
e. violin
- L,C 37. Coal is to weight as milk is to
a. cream b. volume c. bottle d. size e. height
- O,K 38. Fill in the word that fits best. _____ members of the Congress of the United States are trying to give this country the best type of government possible.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 39. Razor is to racket as shave is to
a. board b. court c. tennis d. yard e. lather
- O,K 40. Fill in the word that fits best. _____ Irishmen are more hot-tempered than Italians.
a. All b. Most c. Don't know d. Few e. No
- O,K 41. Fill in the word that fits best. _____ Eskimos have more backbone (vertebrae) than Americans.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 42. Moon is to earth as earth is to
a. ground b. Mars c. sky d. stars e. sun
- O,K 43. Fill in the word that fits best. _____ men are better car drivers than women.
a. All b. Most c. Don't know d. Fewer e. No
- L,C 44. Tree is to forest as person is to
a. crowd b. couple c. child d. men e. women
- L,C 45. (is to $\frown$ as $\smile$ is to
a. $\checkmark$ b. $<$ c. $>$ d. $\cup$ e.)

Generalization Test - Form C - page 6

(Key)

- L,C 46. 3, 7, 4, 8, 5, 9, __, __. What numbers should come next?
a. 13 and 10 b. 12 and 15 c. 12 and 8
d. 6 and 10 e. 5 and 8
- O,I 47. Fill in the word that fits best. _____ White Sox baseball players are better hitters than the Cleveland Indians.
a. All b. Most c. Don't know d. A few or some
e. No
- O,K 48. Fill in the word that fits best. _____ bulls get excited or angry when they see red objects.
a. All b. Most c. Don't know d. A few or some
e. No
- O,K 49. Fill in the word that fits best. _____ Democrats in the government are more interested in filling their pockets with money than in giving the United States good government.
a. All b. Most c. Don't know d. A few or some
e. No
- L,C 50. Which expression below says almost the same thing as, "Too many cooks spoil the broth"? _____
a. It is better to be content with little than to gamble for more.
b. A single sign is not convincing.
c. No plate can satisfy all appetites.
d. A little of a good thing is enough
e. Two captains sink the ship.
f. Give him an inch and he will take a mile.
g. No soup pleases a poor cook.
h. It pays to attend to troubles before they get worse.
i. Spend till in debt; save till solvent.
j. Many little strokes fell great oaks.
k. A burnt child dreads the fire.
- L,C 51. Which expression above says almost the same thing as: "If a camel gets his nose in the tent, his body will soon follow"?
- L,C 52. Which expression says almost the same thing as: "Drop by drop the lake is drained"?
- L,C 53. Which expression says almost the same thing as: "Excess kills more than the sword"?

Generalization Test - Form C - page 7

(key)

L,C	54.	Which expression above says almost the same thing as: "One Swallow does not make a summer"?
L,C	55.	Which expression above says almost the same thing as: "A fool that eats till he is sick must fast till he is well"?
L,C	56.	Which expression above says almost the same thing as: "A stitch in time saves nine"?

Generalization test - Form C - page 8

Part II

Mr. Brown of Hamilton, Ohio, was looking over his receipts for the paid electric bills for the year of 1938. He compared the amounts of the bills by making a monthly chart of them, such as this:

- Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

(key)

J,N,
W,K	57.	The Browns had a vacation away from home in August.	_____
U,N,
A,W	58.	In the winter the Browns spend more for electricity in order to listen to the radio.	_____
U,N,
W,K	59.	The average family in Hamilton uses more electricity in January than in July.	_____
U,N,
W,K	60.	The Browns spend too much money for electricity.	_____
U,N,
W,K	61.	The Browns use more fuel gas in January than in August.	_____
J,N,
W,K	62.	In the early part of December the Browns used more electricity than in the early part of April.	_____
J,N,
W,K	63.	On Thanksgiving, 1938, the Browns used more electricity than on the Fourth of July, 1938.	_____

Generalization test - Form C - page 9

Do Young People Drink Alcoholic Beverages?

- Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

(Key)

C	64.	Between the ages of 16 and 24 there is a steady increase in the per cent of males and females who drink.	_____
J,N,
W,K	65.	The percentage of male youth 25 years old who drink is greater than the percentage of male youth 24 years old who drink.	_____
J,N,
W,K	66.	No boys and girls of fifteen drink alcoholic beverages.	_____
J,N,
W,K	67.	The percentage of men 25 years old who drink is less than the percentage of women 25 years old who drink.	_____
U,N,
W,K	<u>68.</u>	Male and female youth over twenty smoke more than male and female youth under twenty.	_____
U,N,
W,K	69.	Alcoholic beverages should not be sold to those youth under 18.	_____

Generalization Test - Form C - page 10

The surface temperature, average distance from the sun, and the diameters of five planets are listed in the following table:

Planet	Average Temperature (Absolute)	Distance from the sun Millions of Miles	Diameter (Miles)
Mercury	650	36	3,000
Earth	290	93	7,900
Mars	250	141	4,200
Jupiter	100	483	87,000
Saturn	90	886	73,000

- Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

(Key)

- C 70. The temperature of the planets becomes less as the distance from the sun increases. _____
- U, N 71. The temperature of a planet depends solely upon the
W, K radiant energy which it receives from the sun. _____
- C 72. The temperature of a planet becomes less with increasing diameters. _____
- C 73. Planets farthest from the sun have lower temperatures than those near it. _____
- J, N 74. Uranus, a planet at a distance of 1,781 million
W, K miles from the sun, will have a lower surface temperature than Saturn. _____
- C 75. The distance from Mars to the Earth is 48 million miles. _____
- J, N 76. The temperature of Venus, at a distance from the
W, K sun of 67 million miles, is higher than that of Mercury. _____
- U, N 77. Jupiter is the largest planet of the solar
W, K system. _____
- U, N 78. A million years from now the temperature on these
W, K planets will be less than at the present time. _____

Generalization Test - Form C - page 11

How Some Maryland Youth Answered the Question:
Can Your Community Do Anything to
Keep Young People Out of Trouble?

- Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

(Key)

U,N,
W,K	79.	Youth without opinions on a problem as important as this one are not very good citizens. _____
U,N,
W,K	80.	The communities in which these young people live will not try to keep young people out of trouble. _____
J,N,
W,K	81.	A survey of youth opinion in Hamilton, Ohio, would also show that some young people have no opinion on this problem. _____
C	82.	Almost one-third of the youth who were questioned thought that their community could not do anything about keeping youth out of trouble. _____
U,N,
W,K	83.	Many of the youth were too young to have an opinion. _____
U,N,
W,K	84.	About one-fifth of the youth in the United States have no opinion as to whether their communities can help to keep young people out of trouble. _____
U,N,
W,K	85.	If these facts were known to more people, more communities would do something for young people in trouble. _____
C	86.	About one-third of the youth who were questioned had no opinion. _____

Generalization Test - Form C - page 12

Population Growth of the United States
From 1790 to 1940

- Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

(Key)

- C 87. The population of the United States in 1860 was about fifty million. _____
- J,N, 88. In 1780 the United States population was less than it was in 1790. _____
 W,K
- J,N, 89. In 1952 the population of the United States will be less than what it was in 1940. _____
 W,K
- J,N, 90. In 1775 the population of the United States was greater than in 1790. _____
 W,K
- U,N, 91. The growth in population helped to cause unemployment in the United States. _____
 W,K
- J,N, 92. In the near future the population of the United States will decrease. _____
 W,K

Scoring the Generalization Test

The generalization test was scored according to the following principles:

1. Those items dealing with each characteristic of a superior generalizer were scored separately
2. The percentage of correct responses was determined upon the basis of items attempted.

To determine per cent on the basis of the number of items attempted is a more valid measure of the testee's generalizing ability than to determine per cent on the basis of the number of items on the test. Since the tests were collected at the end of thirty-five minutes whether finished or not, the latter procedure would give the slow reader too much of a handicap.

The first principle used in scoring is a composite of procedures, depending upon the nature of the characteristic and the nature of the items involved. These procedures and also the kinds of items found in the test are discussed below.

To answer correctly such items as examples 1, 2, 3, 4, and 11 it is necessary for the testee to possess, at least to some degree, the ability to detect likenesses and differences as mentioned in the first characteristic of a superior generalizer. Scoring of these items is quite simple; every correctly marked response receives one point; every incorrect response receives no points.

Two different types of items and scoring procedures were used in determining the extent to which the testee recognized generalizations which are or are not supported by the given data. Responses on items like 64 along with 1, 2, 3, 4, and 11 give an indication of the testee's ability to recognize situations and to generalize correctly without qualifications in those situations in which certainty is justifiable. The ways of scoring items like 1, 2, 3, 4, and 11 were the same as previously described. A score of one point was given to an item like 64 if it was marked 5. Any other response received no credit. Examples of the second type of item, which tests the second characteristic of a superior generalizer, are items 68 and 69. These items are such that the given data are irrelevant and any extrapolation or interpolation is unwarranted. In scoring these, if the response was 3, the testee was given one point; any other response received no credit.

Some indication of the extent to which the testee can determine trends from available data and from them make reasonable extrapolations and interpolations, the third characteristic of a superior generalizer, is evident when responses to items like 65, 66, and 67 are examined from the point of view of going beyond the data. Going beyond the data in these examples is justifiable and one point is given if the answers are respectively 2, 4, and 4. No points are given if the responses are other than these. A

testee could justifiably go beyond the data on these items and yet not possess to any great extent the third characteristic. He may also go beyond the data in situations where it is not justifiable. In order to obtain some indication of this undesirable tendency, items like 65 through 69 were again examined, and this time each item was given one point if the responses were other than the correct ones.* In this case the lower the score the better generalizer the testee is assumed to be.

To obtain a measure of the testee's caution in generalizing, it is important not only to get some indication of the general caution displayed in his item response, but also to obtain in a direct manner some indication of his willingness to go beyond data and of his ability to generalize when the student's climate of opinion is apt to be a dominating influence. To obtain an indication of the testee's willingness to go beyond data items such as 65 through 69 were scored so that each response was given one point if in each case it was not marked 3. Items like 5 and 13 were scored separately in order to give some indication of the ability to generalize correctly when the student's climate of opinion was a strong influencing factor. The testee received one point if his response for item 5 was don't know or more and no points for any other response.

*The correct answers for items 65-69 were: 65, 1 or 3; 66, 4 or 3; 67, 4 or 3; 68, 3; 69, 3.

For item 13 responses of don't know or a few or some received one point; the other responses received no points. To score for caution alone, items such as 5 and 13 were rescored by assigning to the possible responses various values depending on the cautious nature of the response. For example, in item 5, if the response is don't know the score received for that item was 3 points; if more or fewer, one point; if all or no, no points. In question 13, if the response was don't know, the score received for that item was 3 points; a few or some, 2 points; most, 1 point; and no or all, no points. Items like 65 through 69 were also scored separately for the degree of caution displayed in the testees' responses. In scoring items 65, 66, and 67, if the response given is 3 the testee is given 2 points for each question; if the response is respectively 2, 4, and 4, he receives 1 point for each item. If the responses to items 68 and 69 are 3, the testee receives 1 point for each question. Any other responses on items 65 through 69 receive no points.

IOWA ALGEBRA APTITUDE TEST

(For copy of this test, see envelope
inside back cover.)

OHIO EVERY PUPIL TEST

(For copy of this test, see envelope
inside back cover.)

LANKTON FIRST YEAR ALGEBRA TEST

(For copy of this test, see envelope
inside back cover)

APPENDIX B

Required Lesson Sheets

The sheets included here were lessons required of all the students in the experimental class. In addition to these lesson sheets, students also were assigned problems from an algebra text. The lesson sheets included here are under the headings listed below.

	<u>Page</u>
Signed Numbers	380
Graphs and Signed Numbers	384
Graphs and Formulas	389
Formulas	395
Interpreting Data	396
Algebraic Grammar	408
Graphs and Problem Solving	418
Solving Equations	419
Algebraic Fractions	421
Exponents	423

SIGNED NUMBERS 1

(See page 105)

SIGNED NUMBERS 2

1. In the first quarter of a football game Janowicz carried the ball five times. What was his total yardage if the yardage for each try was:
 - a) 5 yds. gain, 3 yds. gain, 10 yds. gain, 7 yds. gain, 4 yds. gain? _____
 - b) 2 yds. loss, 4 yds. loss, 3 yds. loss, 1 yd. loss, 5 yds. loss? _____
 - c) 10 yds. gain, 5 yds. loss, 7 yds. gain, 3 yds. loss, 12 yds. gain? _____
 - d) 6 yds. loss, 7 yds. loss, 9 yds. gain, 3 yds. loss, 2 yds. gain? _____

2. In forecasting weather the hourly change in temperature is often more helpful than the actual temperature. What is the total change in temperature if the hourly change has been reported as:
 - a) up 2° , up 5° , up 1° , up 4° ? _____
 - b) down 1° , down 3° , down 2° , down 6° ? _____
 - c) down 2° , down 1° , up 2° , up 3° ? _____
 - d) up 3° , up 2° , down 1° , down 4° ? _____

3. What is the total change in value for the past week of one share of A. T. & T. stock if the daily changes are:
 - a) down 1, down $\frac{1}{2}$, down $\frac{1}{8}$, down $\frac{3}{8}$, down $\frac{1}{8}$, down $\frac{1}{4}$? _____
 - b) up $\frac{1}{8}$, up $\frac{3}{8}$, up 1, up $\frac{1}{2}$, up $1\frac{1}{8}$, up $\frac{5}{8}$? _____
 - c) down 1, down $1\frac{1}{2}$, up 2, down $\frac{1}{4}$, up $\frac{3}{8}$, up $\frac{1}{2}$? _____
 - d) up 1, down $\frac{1}{8}$, down $\frac{1}{2}$, up $\frac{1}{4}$, down 2, down 1? _____

4. In the middle of the 15th century, the European merchants developed the practice of marking with a positive sign (+) and a negative sign (-) barrels and boxes that weighed above or below the regulation weight. Thus, "+7" written on a barrel meant 7 pounds overweight, and "-7" meant a shortage of 7 pounds.

If a merchant bought supplies with the following marks on the boxes, what was the total amount more or less he received than he bargained for?

a) +15, +4, +6, +3	f) -3, -6, -10, -7
b) -20, -5, +3, -2	g) $+16\frac{1}{2}$, $7\frac{1}{2}$, +3
c) -10, +6, +9, +3	h) $-14\frac{3}{4}$, $15\frac{1}{2}$
d) -5, -4, +7, -6	
e) +24, -31, +42, -60	

SIGNED NUMBERS 3

Add the following signed numbers by making use of the scale above:

1. $\begin{array}{r} +6 \\ -2 \\ \hline \end{array}$	2. $+7+(-4)=$	4. $\begin{array}{r} -3 \\ -5 \\ \hline \end{array}$	5. $\begin{array}{r} -6 \\ +10 \\ \hline \end{array}$	6. $+3+(+4)=$	8. $\begin{array}{r} +5 \\ +10 \\ \hline \end{array}$
	3. $-3+(-6)=$			7. $-5+(+2)=$	

9. $+5+(-5)=$	11. $\begin{array}{r} +20 \\ -10 \\ \hline \end{array}$	12. $\begin{array}{r} 0 \\ +5 \\ \hline \end{array}$	13. $-17+(+9)=$
10. $-10+(-11)=$			14. $-12+0=$

15. $\begin{array}{r} +15 \\ -20 \\ \hline \end{array}$	16. $(-8)+(-6)=$	18. $\begin{array}{r} +8 \\ -8 \\ \hline \end{array}$	19. $\begin{array}{r} 11 \\ -20 \\ \hline \end{array}$	20. $12+(-4)=$
	17. $(-12)+(-4)=$			21. $0+(-9)=$

22. $-20 + (-5) + (+3) + (-2) =$	25. $\begin{array}{r} +15 \\ -4 \\ -6 \\ +3 \\ \hline \end{array}$	26. $\begin{array}{r} -10 \\ +6 \\ +9 \\ +3 \\ \hline \end{array}$	27. $\begin{array}{r} -8 \\ -1 \\ +5 \\ +6 \\ 0 \\ \hline +12 \end{array}$
23. $-5 + (-4) + (+7) + (-6) =$			
24. $-6 + (+6) + (-10) + (+10) =$			

28. Make up a story problem that will fit problem 27.

29. Write out a rule which will tell you how to work problems 1-27 without making use of a scale.

30.* $-7\frac{1}{5} + (-9\frac{3}{5}) + (+7\frac{1}{3}) + (+8\frac{1}{6}) + (+5\frac{1}{4}) + (-12\frac{3}{4}) + (+8\frac{5}{6}) + (+9\frac{7}{12}) =$

*Optional

SIGNED NUMBER 4

Add the following examples by using the rule we made up in class.

$$\begin{array}{llll} 1. \begin{array}{r} -3 \\ +6 \\ \hline \end{array} & 2. \begin{array}{r} +3 \\ +6 \\ \hline \end{array} & 3. -3 + (-6) = & 5. \begin{array}{r} -4 \\ +4 \\ \hline \end{array} \end{array} \quad \begin{array}{llll} 6. \begin{array}{r} -9 \\ 6 \\ \hline \end{array} & 7. 2 + (-5) = & 8. -7 + (-2) = & \end{array}$$

$$\begin{array}{llll} 9. \begin{array}{r} 7 \\ 5 \\ \hline \end{array} & 10. \begin{array}{r} -4 \\ -9 \\ \hline \end{array} & 11. -3 + (+7) = & 13. \begin{array}{r} -9 \\ 0 \\ \hline \end{array} \end{array} \quad \begin{array}{llll} 14. \begin{array}{r} +8 \\ -7 \\ \hline \end{array} & 15. \begin{array}{r} 8.6 \\ -3.7 \\ \hline \end{array} & 12. -10 + (-3) = & \end{array}$$

$$\begin{array}{llll} 16. 0 + (-9) = & 18. \begin{array}{r} -4.5 \\ 2.4 \\ \hline \end{array} & 19. 8.0 + (-3.2) = & 21. \begin{array}{r} 1.3 \\ -0.7 \\ \hline \end{array} \end{array}$$

$$17. 9 + 6 =$$

$$20. -6.0 + (-7.2) =$$

$$\begin{array}{llll} 22. \begin{array}{r} 2.4 \\ -4.5 \\ \hline \end{array} & 23. \begin{array}{r} +5 \\ -8 \\ +9 \\ -2 \\ \hline \end{array} & 24. \begin{array}{r} 5 \\ -7 \\ -8 \\ -3 \\ \hline \end{array} & 25. \begin{array}{r} -8 \\ 9 \\ 7 \\ -5 \\ \hline \end{array} \end{array} \quad \begin{array}{ll} 26. 7 + (+4) + (-9) + (+8) = & 27. -8 + (-6) + (+9) + (-8) = \\ 28. -12 + (-29) + (+37) + (-18) = & \end{array}$$

$$\begin{array}{llll} 29. \begin{array}{r} +21 \\ +64 \\ -84 \\ -91 \\ +5 \\ \hline \end{array} & 30. \begin{array}{r} -13 \\ -28 \\ +4 \\ -24 \\ +3 \\ \hline \end{array} & 31. +15 + (-18) + (-7) + (+6) = & 32. -17 + (+6) + (-8) + (-1) = \\ 33. -13 + (-28) + (+4) + (+3) + (-24) = & \end{array}$$

$$34. +3 + (-7) + (-9) + (+20) =$$

35. Make up a story problem to fit problem 26.

GRAPHS AND SIGNED NUMBERS 1

Answer the following questions and represent each question and answer in abstract form.

1. How does the amount saved by individuals for the 1st quarter 1950 compare with the 2nd quarter?
The 2nd quarter with the 3rd?
The 3rd quarter with the 4th?
The 4th quarter 1950 with the 1st quarter 1951?
The 1st quarter 1951 with the 2nd?
The 2nd quarter with the 3rd?
The 3rd quarter with the 4th?
2. How does the amount saved the 4th quarter 1951 compare with the amount saved the 1st quarter 1950?
How does the amount saved the 1st quarter 1950 compare with the amount saved the 4th quarter 1951.
3. How do you account for the larger amounts saved during 1951 than during 1950? (Give answer on back of this sheet.)
4. In what quarters did individuals borrow more money from banks than they deposited?
5. How do you interpret the negative values in each graph?

HOW AMERICANS SAVED IN 1950-1951

AMOUNT SAVED QUARTERLY BY INDIVIDUALS, 1950-1951
(Figures in billions of dollars)

PERCENTAGE OF SAVINGS IN "READY CASH" AND IN OTHER TYPES OF SAVINGS, 1951

WHERE THESE SAVINGS WERE PUT

(Figures in billions of dollars)

INVESTMENT SAVINGS
(U. S., STATE, LOCAL & CORP. SECURITIES)

READY CASH
(CURRENCY, DEPOSITS IN BANKS, SAVINGS, BUILDING & LOAN ASSOCIATIONS)

PROTECTIVE SAVINGS
(INSURANCE & PENSION RESERVES)

HOME EQUITY SAVINGS
(LIQUIDATION OF MORTGAGE DEBT)*

DEBT REDUCTION SAVINGS
(LIQUIDATION OF OTHER DEBT)

* For 1950-1951 these figures actually show a negative liquidation; that is, an increase in mortgage debts.

Total savings in 1951 amounted to \$13 billion. Of this amount \$7.9 billion, or 60.8%, was saved in currency, deposits in banks, savings, building and loan

associations. The remaining 39.2% of total savings is accounted for by insurance and pension reserves, securities, mortgage and debt liquidation.

GRAPHS AND SIGNED NUMBERS 2

(See page 145)

GRAPHS AND SIGNED NUMBERS 3

Write the answers on another sheet of paper.

1. What were the profits of the New York Transit System for each year since 1941? What are the estimated profits for 1952-1953?
2. The accompanying graph appeared in the New York Times last spring. What do you think is the purpose of the graph?
3. How do you account for the positive profit for the year 1942?
4. How do you account for the decline in profits of the New York City Transit System?
5. How many passengers were there in each year from 1948 to 1952?
6. The method of estimating for the years 1952 and 1953 is called extrapolation. Extrapolation means going beyond the given information. Describe in some detail how you think the estimates for the profits were made for the years 1952 and 1953?
7. How do you account for the rather sharp decline in number of passengers from 1948 to 1949 and then the rather steady decline since 1949?
8. What was the per cent change in profits from 1941 to 1942? From 1942 to 1943? From 1945 to 1946? From 1946 to 1947? From 1948 to 1949?

Graphs and Signed numbers 3 - continued

NEW YORK CITY TRANSIT SYSTEM'S PROFITS AND LOSSES SINCE 1941

GRAPHS AND FORMULAS

Lesson sheet 1 - See pages 153-4

2 - See pages 156-7

3 - See pages 158-9

4 - See page 161

GRAPHS AND FORMULAS 5

The height-weight, blood pressure-age, and sleep formulas which you have just studied are at best approximate formulas. The line of best fit for the data below fits much better and, therefore, its formula is much better for predicting. The data were taken from The Scientific Monthly Sept. 1927, p 261-264.

Cricket	Chirps/ min. (C)	Tempera- ture (T)	Cricket	Chirps/ min.	Tempera- ture
A	40	46	N	100	63
B	46	49	O	101	64
C	52	50	P	103	64
D	64	52	Q	108	64
E	66	54	R	110	66
F	70	55	S	115	67
G	77	56	T	120	68
H	80	59	U	120	69
I	86	59	V	126	68
J	90	59	W	130	71
K	90	61	X	137	71
L	96	61	Y	143	73
M	98	62			

1. Plot the above data and draw the line of best fit.
2. By statistical procedures the formula for the line of best fit is $T = .265C + 36.33$. Plot this line on your graph. Do the two lines fit?
3. Do the variables C and T vary in the same sense or in the opposite sense?
4. Explain why you think this data fits the above line more closely than did the data on blood pressure and on heights and weights.

GRAPHS AND FORMULAS 6

(See page 162)

GRAPHS AND FORMULAS 7

The formula for determining the distance for a car to stop traveling at a given rate of speed is given by the formula

$$d = .045 rr + 1.1 r$$

where "r" is the rate of speed in miles per hour and "d" is the distance in feet.

Give the first question a good try. If you cannot answer it, try again after working the second question.

1. Suppose the stopping distance of a car traveling at 20 m.p.h. is 40 feet. What do you think the stopping distance would be if the car were traveling 40 m.p.h.? _____
60 m.p.h.? _____ 80 m.p.h.? _____.
2. Find the distances for the rates given in the table below. (Do your calculations on another sheet of paper as instructed in class.)

r	0	10	20	30	40	50	60	70	80	90
d										

3. Plot the various points in the table and then draw a line that includes all of them.
4. How does the graph of $d = .045rr + 1.1r$ compare with the graphs of formulas $w = 5.5(h - 60) + 110$, $p = 110 + \frac{a}{2}$, and $h = 8 + \frac{18-a}{2}$?
5. Read from your graph the value for "d" when $r = 5$. _____
When $r = 15$. _____. Determine the same values for "d" by substituting the values of "r" in the formula. Do the values agree?
6. Do the values of "r" and "d" vary in the same sense or opposite sense?
7. Do negative values of "r" make sense? Explain.
8. Does the formula for stopping distance make sense if the values of "r" happen to be negative?

GRAPHS AND FORMULAS

Lesson sheet 8 - See page 163

9 - See page 168

GRAPHS AND FORMULAS 10

One filling station sells gasoline at 27¢ per gallon. A competitor advertizes the same quality gasoline at 21¢ a gallon plus 15¢ service charge.

1. Make a formula which will determine the cost of gasoline bought at the first station.
2. Make a formula which will determine the cost at the second station.
3. Make a table of values for each formula and then make a picture of these formulas on the same chart. Extrapolate the lines as far as your graph paper will allow.
4. By reading the graphs, answer the following questions:
 - a. At which station could one get the most gasoline for 50¢? For \$1? For \$2.50?
 - b. How much more gasoline could be purchased at one station for \$2.50 than at the other station?
 - c. How much could one save by buying 15 gallons at the station that offered the lower price on this amount?
 - d. With what amount of money could one purchase the same amount of gasoline at the second station as at the first?
 - e. What amount of gasoline costs the same at the second station as at the first?
5. Determine the answers to 4d and 4e by solving equations.

FORMULAS 1

The Society of Automobile Engineers uses the following formula for computing the horsepower of a gasoline engine.

$$H = \frac{D^2 N}{2.5}$$

In this formula D is the diameter of a cylinder in inches and N the number of cylinders.

1. Complete the table of values below for a 6-cylinder engine:

D	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
H							

- By extrapolation determine the values for H when $D = 5\frac{1}{2}$. _____. When $D = 6$. _____.
- What happens to the horsepower of an engine when the diameter is doubled? _____. Tripled? _____. How do you account for this peculiarity? (Determine the formula.)
- If the table of values were plotted would the points be on a curved or straight line? _____. How can you tell?

2. Complete the table of values below for engines with $3\frac{1}{2}$ in. cylinders.

N	2	4	6	8	16
H					

- By interpolation determine the value for H when $N = 12$.
- By extrapolation determine the value for H when $N = 24$.
- What happens to the horsepower of an engine as the number of cylinders are doubled? _____. Tripled? _____.
- If the table of values was plotted, would the points be on a curved or straight line? How can you tell?
- Which increases the horsepower of an engine more, doubling the diameter or doubling the number of cylinders?

INTERPRETING DATA I

Per Cent of Women Ages 15-55
Who Use Lipstick

A survey of the use of cosmetics gave the graph shown above. It indicates how the use of lipstick varies with the age of women. Assume this is a true and accurate graph. Then decide which of the 8 conclusions given below can be accurately drawn from the information shown.

1. Many more young women use lipstick than do older women.
2. There is a definite increase in the per cent of women using lipstick between the ages of 15 and 22.
3. Girls use lipstick in large numbers before high school age.
4. There is a continuous increase in the per cent of women using lipstick from age 22 to age 55.
5. If this graph were extended above age 55, there would probably be an increase in the per cent of women using lipstick.
6. This graph indicates that manufacturers of lipstick should advertise extensively in magazines read by young women.
7. Lipstick is used by more college than high school girls.
8. A larger per cent of single women than married women use lipstick.

(Taken from Mathematics for the Consumer, Schorling, Clark, and Lankford.)

INTERPRETING DATA II

(Not Included)

INTERPRETING DATA III

(See page 178)

INTERPRETING DATA IV

COST OF CONSUMER ITEMS

Work Time Equivalents for Wage Earners

	1914		1929		1939		June 1952	
	Hrs.	Min.	Hrs.	Min.	Hrs.	Min.	Hrs.	Min.
Eggs (1 doz.)	1	26		46		28		22
Bread (1 pound)		17		10		8		6
Milk (1 quart)		24		15		11		8
Coffee (1 pound)	1	20		52		22		31
Round Steak (1 pound)	1	2		46		34		40
Dress (cotton)	4	43	3	31	1	35	2	31
Slip (rayon)	4	29	2	38	2	2		53
Suit (men's worsted)	75	7	33	23	27	13	23	10
Shirt (men's cotton)	3	52	2	32	1	51	1	58
Sheets (81"x99")	3	35	2	39	1	20	1	25
Haircut (men's)	1	21	1	7		47		43

Sources N.I.C.B., E.L.S.
 Road Maps of Industry
 Oct. 31, 1952
 No. 879

Interpreting Data IV - continued

1. It takes much less work time to earn enough money to buy many items now than it did in 1914. Which items required over twice as many work hours in 1914 than in 1952?
2. How do you account for the great reduction in work time equivalents to pay for a rayon slip? A man's suit?
3. Which items require more work now than in 1939? How do you account for the reversal of trend in these items?
4. Draw any other conclusions from this table of values that you feel are justified?

INTERPRETING DATA V

The following information was taken from Time Magazine, December 31, 1951.

"Federal, state, and local taxes now take 30% of the entire national income."

"Taxes and inflation have dented the dollar to the point where a U. S. couple with two children must earn \$6,072 a year to enjoy the same standard of living that \$3,000 a year brought in 1940. And the higher the income, the higher the ratio. (See chart.)"

TIME CHART by R.M. CHAPIN JR.

Interpreting Data V - continued

Mark it 1 - if the facts given make the statement true.
 Mark it 2 - if the facts given make it probably true.
 Mark it 3 - if you cannot tell from the facts given.
 Mark it 4 - if the facts given make it probably false.
 Mark it 5 - if the facts given make the statement false.

- ___ 1. For 1951 a married couple (with 2 children) earning \$6,072 must pay over 20% of their income for taxes.
- ___ 2. For the years 1940 and 1951 the more net income a couple earned (married, 2 children) the more the amount paid in taxes.
- ___ 3. On the average a couple (with 2 children) earning a net income of \$25,000 in 1940 and a net income of \$76,322 in 1951 must pay over ten times as much in taxes for 1951 than for 1940.
- ___ 4. Taxes are higher because of inflation and because so much more must be paid for national defense.
- ___ 5. According to the four income brackets on the chart, the per cent of the 1951 net income accounted for by dollar depreciation becomes less as the net income (1951) increases.
- ___ 6. The amount Mr. and Mrs. Hava Dollar (and 2 children), whose 1951 net income was \$10,542, must pay is approximately \$1900 in taxes.
- ___ 7. A couple with two children earning \$22,850 in 1951 must pay \$6,101 in income tax.
- ___ 8. The couples (with 2 children) with a net income of \$76,322 per family pay more in taxes than the couples (with 2 children) with a net income of \$6,072 per family.
- ___ 9. The cost of government was greater in 1951 than it was in 1940.
- ___ 10. After taxes have been subtracted from the net income in each of the four income brackets, the amount each dollar would buy in 1951 was approximately the same as in 1940.

List the generalizations that this graph suggests to you:

INTERPRETING DATA VII

List the generalizations that this graph suggests to you:

CONSUMER CREDIT AND RETAIL SALES
UNITED STATES, 1929-1951

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INTERPRETING DATA VIII

Application of the Slope Concept

Values of 1 share of Sterling Drug Stock from February 17 to March 6 (1953).

	High	Low	Close	Change
Feb. 17	$33\frac{1}{2}$	$33\frac{3}{8}$	$33\frac{1}{2}$	
18	$33\frac{1}{2}$	$33\frac{1}{4}$	$33\frac{1}{4}$	$-\frac{1}{4}$
19	$33\frac{3}{8}$	$33\frac{1}{8}$	$33\frac{3}{8}$	$+\frac{1}{8}$
20	$33\frac{7}{8}$	$33\frac{1}{2}$	$33\frac{7}{8}$	$+\frac{1}{2}$
24	$34\frac{1}{4}$	$33\frac{3}{4}$	$33\frac{7}{8}$	0
25	$34\frac{1}{4}$	$33\frac{3}{4}$	$33\frac{7}{8}$	0
26	34	$33\frac{7}{8}$	34	$+\frac{1}{8}$
27	34	$33\frac{3}{4}$	34	0
Mar. 2	$33\frac{7}{8}$	$33\frac{3}{4}$	$33\frac{7}{8}$	$-\frac{1}{8}$
3	34	$33\frac{3}{4}$	$33\frac{3}{4}$	$-\frac{1}{8}$
4	$33\frac{7}{8}$	$33\frac{3}{8}$	$33\frac{3}{8}$	$-\frac{3}{8}$
5	$33\frac{1}{2}$	$33\frac{1}{8}$	$33\frac{3}{8}$	0
6	$33\frac{3}{8}$	$33\frac{1}{4}$	$33\frac{1}{4}$	$-\frac{1}{8}$

1. Make a combination bar, broken line graph of the above information.

INTERPRETING DATA IX

This graph was made from data collected in a small new England town. List the generalizations that this graph suggest to you.

INTERPRETING DATA X

Gallup Poll
MacArthur's RemovalGeneral Population

	<u>College Trained</u>	<u>High School</u>	<u>Grade School</u>
Approve	41	24	23
Disapprove	53	68	64
No Opinion	6	8	13
	<u>100</u>	<u>100</u>	<u>100</u>

	<u>Who's Who</u>	<u>General Public</u>
Approve	51	25
Disapprove	46	66
No Opinion	3	9
	<u>100</u>	<u>100</u>

Educators in Who's Who

Approve	64
Disapprove	34
No Opinion	2
	<u>100</u>

Taken from
The Columbus Citizen,
 May 16, 1951

List the generalizations these data suggest to you.

ALGEBRAIC GRAMMAR

Lesson sheet 1 - See page 182

2 - See page 186

ALGEBRAIC GRAMMAR 3

Write an equivalent algebraic expression that does not make use of the parentheses.

1. $2a + (+3b) + (+3c) =$
2. $2a - (+3b) - (+3c) =$
3. $2a + (-3a) + (-4a) =$
4. $2a + (-3b) + (-3c) =$
5. $-2d - (-4d) - (-6d) =$
6. $-2d - (-4e) - (-5f) =$
7. $-12s + (-12x) - (+b) - (-3d) =$
8. $3r^2 - (-2r) + (+3x^2) - (+2y) =$

If possible rewrite the following algebraic expression in simpler form.

9. $2a + (-6a) + (-7a) + 12a + (-2a) =$
10. $-5xy^2 + 13xy^2 + (-16xy^2) + (-2xy^2) =$
11. $3a - (-7a) - (-4a) - 6a =$
12. $(5a)(-6b)(+5c) =$
13. $-16x + (-12x) - (+4x) - (+3x) - 2y =$
14. $7a - 7b + 9c =$
15. $(-3a)(-6x)(-2y) =$

Using the formula $\underline{1 = ar^{n-1}}$

16. Find $\underline{1}$ if $a = 4$, $r = 2$, and $n = 3$.
17. Find $\underline{1}$ if $a = 5$, $r = 3$, and $n = 4$.

Using the formula $\underline{C = \frac{En}{R+nr}}$

18. Find C if $r = 2$, $n = 45$, $E = 1\frac{1}{2}$, and $R = 9$
19. Find C if $r = 2.5$, $n = 52$, $E = 1\frac{1}{2}$, and $R = 85$

ALGEBRAIC GRAMMAR 4

Express the following algebraic expressions in shorter form by using exponents and coefficients.

1. $x + x + x + x =$
2. $a \cdot a \cdot a \cdot a \cdot a =$
3. $a \cdot a \cdot a \cdot c \cdot c \cdot c \cdot c \cdot c =$
4. $ab + ab + ab =$
5. $aa + bb + aa + bb + aa =$
6. $(a + b) + (a + b) + (a + b) =$
7. $(m + n)(m + n) =$
8. $ab - cd - cd - cd =$
9. $k + l \cdot k + l \cdot k + l =$
10. $m \cdot g \cdot m \cdot g \cdot m \cdot g =$

If possible, rewrite the following algebraic expressions in a shorter form.

11. $6c + (-c) + 7d + (-2d) =$
12. $6c - c + 7d + 2d =$
13. $7ab + (-6ab) + (-ab) =$
14. $7m^2 + 6q^2 - (-3m^2) + (-6q^2) =$
15. $12k - (-6k) - k =$
16. $(4k)(-3k)(2k) =$
17. $18x + 5y - (-3z) =$
18. $z^3 + (-3y^2) + 7z^3 + 5y^2 =$
19. $-ab + 3ab + (-6a^2) + 4a^2 =$
20. $x^2 + 2x + 5 + 3x^2 - x - 4 =$

In the following formulas find Y if $a = 3$, $b = 7$, $c = 5$, $d = 8$, and $k = \frac{1}{2}$.

21. $Y = a^4 + b^2$
22. $Y = a(b^2 - c) + d$
23. $Y = 3d + k(a + b)$
24. $Y = c^2 + d^2 + k^2$

(See page 190)

ALGEBRAIC GRAMMAR 6

A. Give the answers to the following problems.

1. $9 + 4(6 + 5) =$
2. $9 + 3(6 + 5) =$
3. $9 + 2(6 + 5) =$
4. $9 + (6 + 5) =$
5. $9 + 4(6 - 5) =$
6. $9 + 3(6 - 5) =$
7. $9 + 2(6 - 5) =$
8. $9 + (6 - 5) =$

9. $9 - 4(6 + 5) =$
10. $9 - 3(6 + 5) =$
11. $9 - 2(6 + 5) =$
12. $9 - (6 + 5) =$
13. $9 - 4(6 - 5) =$
14. $9 - 3(6 - 5) =$
15. $9 - 2(6 - 5) =$
16. $9 - (6 - 5) =$

B. Rewrite the following expressions by applying the distributive law.

1. $9 + 4(x + 5) =$
2. $9 + 3(x + 5) =$
3. $9 + 2(x + 5) =$
4. $9 + (x + 5) =$
5. $9 + 4(x - 5) =$
6. $9 + 3(x - 5) =$
7. $9 + 2(x - 5) =$
8. $9 + (x - 5) =$

9. $9 - 4(x + 5) =$
10. $9 - 3(x + 5) =$
11. $9 - 2(x + 5) =$
12. $9 - (x + 5) =$
13. $9 - 4(x - 5) =$
14. $9 - 3(x - 5) =$
15. $9 - 2(x - 5) =$
16. $9 - (x - 5) =$

C. If you will examine the problems in section B you will notice that they are identical with those in section A except that x is substituted for 6. Therefore, if you substituted 6 for x in your answers to the problems under section B, you should get the same arithmetic answer as in the corresponding problems in section A. Mark those problems in section B that so agree.

ALGEBRAIC GRAMMAR 7

I. Write equivalent expressions for the following by using the distributive law.

1. $3(x + 3) =$
2. $5(a - 6) =$
3. $-7y(3x - 2) =$
4. $5y(y + 2 - 3x) =$
5. $-8b(2b^2 - 3b + 2) =$
6. $5m(2a - 3b - 2c) =$
7. $-7x^2(2a + 3x + 5b) =$
8. $5m^2n^3(1 + m) =$

II. Write equivalent expressions for the following by reversing (if possible) the above procedure.

1. $5a + 20 =$
2. $3b - 27 =$
3. $xy - xz =$
4. $10x + 12xy =$
5. $4a + 5b =$
6. $2m + 6n - 8a =$
7. $-5ab - 125ax + 10ac =$
8. $5m^3 - 40m^2n - 15m^2t =$
9. $a - ar^n =$
10. $y^3z^3 + 2xy^2z - 3xyz^2 =$

III. Solve for x.

1. $ax + 5 = bx + 20$
2. $d(x + 6) = 3(4 - x) + 7$

ALGEBRAIC GRAMMAR 8

I. Write equivalent expressions for the following by using the distributive law.

1. $a(b + 3)$	4. $ab(-a + 3)$
2. $b(b - 6)$	5. $6x^3(3x - 2)$
3. $-5x(x - 6)$	6. $6ab(a^2 - b^2)$

II. In the problems above you multiplied a monomial by a polynomial by using the distributive law. What would be the result if you multiplied the binomial, $x + 3$, by another binomial, $x + 6$? Check your result by substituting a value for x in both binomials. Keep working with the multiplication of $(x + 3)$ and $(x + 6)$ until you find a system which will give you the correct product.

III. After finding a procedure which gives you the result of $(x + 6)(x + 3)$, try it on the following. Be certain to check each multiplication.

1. $(a + 6)(b + 2)$
2. $(b - 3)(b + 6)$
3. $(x - 3)(x - 9)$
4. $(2y - 3)(5y + 2)$
5. $(a - b)(2x + 3y)$
6. $(a + 3)(a - 3)$

ALGEBRAIC GRAMMAR 9

I. What are the prime factors of:

1. 18	5. 32
2. 33	6. $6a^2$
3. 24	7. $10ab^3$
4. 45	8. 256

II. Find the products:

1. $(a - 2)(a + 2)$	7. $(a - b)(a + b)$
2. $(10 - b)(10 + b)$	8. $(4c - 7)(4c + 7)$
3. $(x - 3)(x + 3)$	9. $(9x - 4)(7 - 3x)$
4. $(b - 5)(b + 5)$	10. $(16a + 1)(10 - 10a)$
5. $(9 - y)(9 + y)$	11. $(2a + b)(c + 3d)$
6. $(z + 6)(z + 6)$	

On the back of this sheet write down any generalization you might have discovered while finding the products.

III. Find the value of:

- $5 \cdot 3\frac{1}{4} + 5 \cdot 4\frac{3}{4}$
- $3\frac{2}{7}(7\frac{8}{9}) + 3\frac{2}{7}(6\frac{1}{9})$
- $7 \cdot (3333) + 7(6667)$
- $7(96) + 3(96)$

IV. Factor

- $5 - 10x - 20y$
- $10x^2y + 15xy - 5y$
- $\frac{1}{2}a^2 + \frac{1}{2}$
- $5lmn + 9mn^2$

V. Solve and check

1. $45x - 27 + 90x = 243$	3. $5x + 4y = 40$
2. $4x + 3(x - 3) = 5(x + 6) - 10$	$x - y = -1$

V. A man invested \$16,000 in two projects. On one he made a profit of 8% and on the other a profit of 3%. If his total profit was \$1105, what was the amount of each investment?

ALGEBRAIC GRAMMAR 10

I. Find the following products.

$$1. (x - 3)(x - 3) =$$

$$2. (3 - y)^2 =$$

$$3. (a - b)(a - b) =$$

$$4. (m - 5)^2 =$$

$$5. (5 + x)(5 + x) =$$

$$6. (s + t)^2 =$$

$$7. (3c - 4)(3c - 4) =$$

$$8. (3x + 4y)^2 =$$

$$9. (3x - 1)(3x + 1) =$$

$$10. (xy - 8)^2 =$$

What are some generalizations that could be made concerning the product of a binomial by itself?

II.

1. Find the product:

$$3(a + 5) =$$

2. Factor:

$$3a + 3b =$$

3. Find the product:

$$-5(x - 2y) =$$

4. Factor:

$$5a + 10 =$$

5. Factor:

$$8d - 2 =$$

6. Find the product:

$$a(6b - 1) =$$

7. Factor:

$$cn - cp =$$

8. Factor:

$$p + prt =$$

9. Find the product:

$$7n(3n + 4) =$$

10. Factor:

$$3x^2 + x =$$

11. Find the product:

$$2x(x^2 - 5x + 6) =$$

12. Factor:

$$5x + 5y - 10z =$$

13. Factor:

$$2n^3 - 10n^2 - 4n =$$

14. Factor:

$$ac - ad + ae =$$

15. Factor:

$$2rh + 2\pi r^2 =$$

ALGEBRAIC GRAMMAR 11

Multiply the following to get rid of parentheses.

1. $(a - x)(a + x)$	6. $(\frac{1}{4}x - 3)(\frac{1}{4}x + 3)$
2. $(3x + 1)(3x + 1)$	7. $(9 + x^2)(9 - x^2)$
3. $(5 - x)(5 + x)$	8. $(R^2 + A^2)(R^2 - A^2)$
4. $(c + 3d)(2c - d)$	9. $(x^2y^2 - 13)(x^2y^2 + 13)$
5. $(2m - 3)(3m - 1)$	10. $17x^2(y^2 - 3)$

Factor:

1. $x^2 - 9$	7. $c^2 + 10cd + 25d^2$
2. $a^2 + 4a + 4$	8. $t^2 - t + \frac{1}{4}$
3. $7a - 63b$	9. $a^2 - 3a + 2$
4. $81z^2 - 1$	10. $a^2 - 64$
5. $4 - 5c + c^2$	11. $x^2 - x - 2$
6. $r^2 - 6rs - 7s^2$	*12. $-n^3 + 5n^4 + 6n^5$

*Factor completely

- $3x^3 - 27x^5 = 3x^3(1 - 9x^2) = 3x^3(1 - 3x)(1 + 3x)$
- $\frac{4}{9}k^2 - \frac{16}{25} =$
- $3(x + y) - 27(x + y)^3 =$
- $4a^2 + 8a - 5 =$

*Optional

GRAPHS AND PROBLEM SOLVING 1

A Navy jet plane and an airliner are scheduled the same day to fly the same route from Columbus to Los Angeles, 2400 air miles (by way of Chicago). The airliner which averages 225 miles per hour leaves at noon, but the jet does not leave until 4:00 p.m. If the jet averages 525 miles per hour, how far from Columbus and at what time does the jet overtake the transport plane?

<u>Air Liner</u>		<u>Jet</u>	
Hour of the Day	Distance from Columbus	Hour of the Day	Distance from Columbus
12 a.m.		4 p.m.	
1 p.m.		5 p.m.	
2 p.m.		6 p.m.	
3 p.m.		7 p.m.	
4 p.m.			
5 p.m.			
6 p.m.			
7 p.m.			
8 p.m.			

- I. Plot this information on a graph to get the answer.
- II. What formulas express the relationships between the hour of the day and the distance traveled?

SOLVING EQUATIONS 1

I. Find the value of the unknown number.

1. $x = 6 + 3$

2. $x = 7 - 6$

3. $x = 2(-5\frac{1}{2})$

4. $y = 6\frac{3}{4} + 7\frac{1}{8}$

5. $b = \frac{16\frac{1}{4}}{2}$

6. $\frac{15}{-5} = x$

7. $k - 7 = 11$

8. $7 = \frac{x}{6}$

9. $7b = 56$

10. $d - 6 = 7\frac{1}{2}$

11. $3d = 15\frac{3}{4}$

12. $10\frac{1}{2} = \frac{x}{3\frac{1}{2}}$

13. $m + 8 = 15$

14. $13d = 39$

15. $15 = 6 + c$

16. $3d + 5 = 14$

17. $25 = 32 - y$

18. $\frac{n + 4}{10} = 4$

19. $14.3k = 57.2$

20. $15.2 + x = 20.2$

21. $4 = \frac{25 - x}{5}$

22. $\frac{45}{b} = 15$

23. $18 = x + 7$

24. $75 = 3x$

SOLVING EQUATIONS 2

I. Find the value of the unknown number.

1. $x - 6 = 5$

2. $20 = 5y$

3. $\frac{1}{3}b = 11$

4. $9 = 4c - 3$

5. $4d = 18 + 10$

6. $k - 1\frac{1}{2} = 5\frac{1}{2}$

7. $2.5 = \frac{m}{4}$

8. $h + 1\frac{1}{3} = 5\frac{2}{3}$

9. $n + 9 = 17$

10. $4\frac{1}{8} = y + \frac{1}{2}$

11. $2\frac{1}{2}x = 10$

12. $7 = \frac{a + 5}{6}$

13. $2\frac{1}{2} = \frac{1}{2}x + 3$

14. $4\frac{1}{2} = \frac{1}{2}x + 3$

15. $\frac{1}{2}b - 3 = 4$

16. $12 = 3c + 5$

17. $.3d + 5 = 8$

18. $4\frac{1}{4} = \frac{x}{3}$

II. Make up equations which say the same thing as the story problems below:

a. What number minus seven and three-eighths is equal to twelve?

b. What number divided by seven and three-sixteenths is equal to six?

c. If the price of an article is $14\frac{1}{2}$, how many can be bought for $28\frac{1}{2}$?

III. On the back of this sheet make up a problem that fits the equation $x - 6 = 5$. Do the same for $20 = 5y$.

ALGEBRAIC FRACTIONS 1

Combine the following fractions into a single fraction.

$$1. \frac{1}{16} + \frac{3}{16} + \frac{7}{16} =$$

$$2. \frac{2}{n} + \frac{3}{n} + \frac{7}{n} =$$

$$3. \frac{3}{8} + \frac{5}{8} - \frac{7}{8} =$$

$$4. \frac{6}{b} - \frac{2}{b} + \frac{7}{b} =$$

$$5. \frac{a}{7} + \frac{2a}{7} + \frac{3a}{7} =$$

$$6. \frac{3a}{9} - \frac{10a}{9} + \frac{11a}{9} =$$

$$7. \frac{7m}{6} + \frac{3m}{6} - \frac{4m}{6} =$$

$$8. \frac{x}{a} + \frac{4x}{a} + \frac{5x}{a} =$$

$$9. \frac{3m}{b} - \frac{6m}{b} + \frac{8m}{b} =$$

$$10. \frac{7k}{x} + \frac{3k}{x} - \frac{3a}{x} + \frac{4a}{x} =$$

$$11. \frac{16c}{2b} + \frac{5c}{2b} - \frac{4c}{2b} =$$

$$12. \frac{18x}{3a} - \frac{6y}{3a} + \frac{5x}{3a} =$$

$$13. \frac{7b}{2a} + \frac{b}{2a} - \frac{4a}{2a} =$$

$$14. \frac{6}{a+b} + \frac{7}{a+b} - \frac{3}{a+b} =$$

$$15. \frac{c}{x-y} + \frac{3c}{x-y} - \frac{2c}{x-y} =$$

$$16. \frac{2a}{r+s} + \frac{b}{r+s} - \frac{a}{r+s} =$$

$$17. \frac{x^2}{x-1} - \frac{x}{x-1} + \frac{x}{x-1} =$$

$$18. \frac{a}{a-b} - \frac{2b}{a-b} + \frac{b}{a-b} =$$

ALGEBRAIC FRACTIONS 2

Combine the following fractions into a single fraction.

$$1. \quad \frac{5}{12} - \frac{1}{3} + \frac{3}{4} =$$

$$2. \quad \frac{3}{5} - \frac{1}{6} + \frac{2}{3} =$$

$$3. \quad \frac{1}{2} + \frac{2}{5} - \frac{1}{3} =$$

$$4. \quad \frac{1}{6} + \frac{1}{4} + \frac{4}{5} =$$

$$5. \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{7} =$$

$$6. \quad \frac{2}{5} - \frac{1}{7} + \frac{1}{3} =$$

$$7. \quad \frac{x}{4} - \frac{x}{2} + \frac{x}{3} =$$

$$8. \quad \frac{3a}{8} + \frac{a}{3} - \frac{a}{3} =$$

$$9. \quad \frac{6a}{8} + \frac{2b}{7} - \frac{3a}{5} =$$

$$10. \quad \frac{a}{x} - \frac{2}{x} + \frac{3a}{2} =$$

$$11. \quad \frac{8}{3x} + \frac{11}{30x} =$$

$$12. \quad \frac{2}{x} - \frac{3}{2x} + \frac{1}{4} =$$

$$13. \quad \frac{2x}{3} + \frac{5x}{6} + \frac{x}{2} =$$

$$14. \quad \frac{2xy}{3} + \frac{xy}{8} - \frac{3xy}{4} =$$

$$15. \quad \frac{a}{3} - \frac{b}{4} + \frac{b}{5} =$$

$$16. \quad \frac{2x}{3} + \frac{3x}{6} - \frac{4x}{7} =$$

$$17. \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} =$$

$$18. \quad \frac{1}{x} + \frac{1}{y} - \frac{1}{z} =$$

$$19. \quad \frac{3}{x} + \frac{4}{y} - \frac{1}{x} =$$

EXPONENTS 1

I. Find the product.

1. $x^3 \cdot x^4 =$

5. $y^{15} \cdot y^6 =$

9. $a^3 \cdot a^4 \cdot a^6 =$

2. $z^2 \cdot z^3 =$

6. $2^4 \cdot 2^5 =$

10. $p^5 \cdot p \cdot p^4 =$

3. $a^6 \cdot a^4 =$

7. $c \cdot c^5 =$

11. $a^b \cdot a^2 =$

4. $b^7 \cdot b^2 =$

8. $3^4 \cdot 3^3 =$

12. $a^m \cdot a^n =$

Write a rule telling how to work problems like those above.

II. Find the quotient.

1. $\frac{x^3}{x} =$

5. $a^7 \div a^4 =$

9. $\frac{z^c}{z^3} =$

2. $\frac{a^6}{a^4} =$

6. $\frac{3^5}{3^3} =$

10. $2^m \div 2^4 =$

3. $\frac{b^{15}}{b^{10}} =$

7. $4^7 \div 4^2 =$

11. $\frac{a^m}{a^n} =$

4. $\frac{c^6}{c^4} =$

8. $y^m \div y^2 =$

12. $\frac{x^5}{x^7} =$

Write a rule telling how to work the problems you have just worked. (if you were to choose one of the problems above as being most different from the rest, which one would you select? Why?)

III. Find the products. Before working the problems, note the difference between them and those problems in Part I.

1. $(a^3)^5 =$

4. $(a^5)^4 =$

7. $(b^6)^2 =$

10. $(b^m)^n =$

2. $(2^3)^3 =$

5. $(2^1 a^1)^3 =$

8. $(b^2)^m =$

11. $(2x^3)^2 =$

3. $(b^7)^2 =$

6. $(5^1 x^4)^3 =$

9. $2(x^3)^2 =$

12. $(2x^n)^m =$

Write a rule telling how to work the problems you just worked.

EXPONENTS 2

I. Express the answers in two different ways.

$$1. \frac{a^2}{a^2} =$$

$$2. \frac{2^3}{2^4} =$$

$$3. \frac{x^{15}}{x^{17}} =$$

$$4. \frac{3^5}{3^5} =$$

$$5. \frac{b^5}{b^7} =$$

II. Rewrite the following in a different way.

$$1. a^{-6} =$$

$$4. c^{-1} =$$

$$7. 3^{-2} =$$

$$2. a^0 =$$

$$5. \frac{1}{x^3} =$$

$$8. 5^0 =$$

$$3. \frac{1}{a^2} =$$

$$6. b^{-3} =$$

$$9. 1 =$$

$$*10. \frac{1}{a^{-3}} =$$

III. Perform the operation indicated.

$$1. a^7 \cdot a^{-3} =$$

$$3. \frac{b^4}{b^{-5}} =$$

$$5. \frac{3^{-4}}{3^{-4}} =$$

$$2. 5^5 \cdot 5^{-6} =$$

$$4. \frac{k^5}{k^5} =$$

IV. Perform the indicated operations.

$$1. (x^4)^5 =$$

$$6. -4mn(4m^2n^2) =$$

$$2. h^5 \cdot h^5 =$$

$$7. (4b^2)^2 =$$

$$3. (-r^2)^3 =$$

$$8. 4(b^2)^2 =$$

$$4. 2x^3(y^2) =$$

$$9. \frac{5}{3}p(-p^2r^2) =$$

$$5. (2a^2b)^2 =$$

$$10. (a^5)^2(-a^3)^4(-a^2)^5(a^3) =$$

APPENDIX C

Optional Lesson Sheets

OPTIONAL ASSIGNMENT
for
ALGEBRAIC GRAMMAR 8*

1. In track meets it is often necessary to make lanes 1 yd. wide for the runners. If the runner is using an outside lane he is given a head start. How much head start must a runner running a 440 yd. dash (once around the track) be given at the starting point if he is to run the same distance as the runner immediately to his inside?
2. How much would this head start be if the track were a half mile track and if the race were the 380 yd. run?
3. Could you find the distance in question 1 if the dimensions of the track were not given?

*There are no optional lessons in connection with lesson sheets Algebraic Grammar 1 through 7.

OPTIONAL ASSIGNMENT
for
ALGEBRAIC GRAMMER 10

Find the products:

1. $(a + b)(a + b)$
2. $(a + b)(a + b)(a + b)$
3. $(a + b)(a + b)(a + b)(a + b)$
4. $(a + b)(a + b)(a + b)(a + b)(a + b)$
5. From studying the products above predict what the product of $(a + b)^6$ will be. Check your prediction.

If your prediction is correct on $(a + b)^6$, what do you believe the product will be for:

6. $(a + b)^7$

7. $(a + b)^8$

OPTIONAL ASSIGNMENT
for
GRAPHS AND FORMULAS 11

If a projectile is fired directly upward with a muzzle velocity of $\underline{v}$ feet per second from a height of $\underline{d}$ feet, its height ($\underline{h}$ feet) above the ground at any given time ($\underline{t}$ seconds) is given by the 2nd degree equation

$$h = -16t^2 + vt + d$$

1. If $v = 320$ feet and $d = 500$ ft., what is the equation for finding h .
2. Graph this equation. Let $t = 1$, then 2, etc., to $t = 25$.
3. What was the maximum height attained by the projectile?
4. How long after the gun was fired did the projectile reach the ground.

ALGEBRAIC GRAMMAR 12

You have learned how to multiply a polynomial by a polynomial. See if you can divide one polynomial by another polynomial.

$$1. \quad \overline{x + 1/x^2 + 3x + 2}$$

$$2. \quad \overline{x + 6/3x^2 + 16x - 12}$$

$$3. \quad (1 + 2a + a^2) \div (a + 1)$$

$$4. \quad (12x^3 - 30x^2 + 2x - 5) \div 2x - 5$$

$$5. \quad (a^3 + 1) \div (a + 1)$$

$$6. \quad \frac{6x^4 + 35x^3 + 39x^2 - 2x + 28}{2x + 7}$$

ALGEBRAIC GRAMMAR 13

- I. Suppose that a metal tape could be fitted firmly around the earth at the equator. Then imagine this tape increased in length by one foot. If this enlarged tape could be held in a position such that at all points it would be the same distance away from the equator, would it be possible for an animal as large as a mouse to pass under the tape, from one side to the other, without touching it? (Support your answer with mathematical calculations.)

II. Factor

1. $r^3s + 10r^2s^2 + 25rs^3 =$
2. $3x^2 + 19xy + 20y^2 =$
3. $3(x + y) - 27(x + y)^3 =$
4. $(a + b)^4 - (x - y)^2 =$
5. $(x^2r^2 - 1)^2 - 9 =$
6. $(xa + 1)(xa + 2) =$
7. $ax - ay + bx - by =$
8. $cd + cb + 2d + 2b =$

III. Solve for x (On another sheet of paper)

1. $(b + c)x = b^2 + 2bc + c^2$
2. $ax - dx = a^2 - 2ad + d^2$
3. $rx + 8r = 4x + r^2 + 16$
4. $5ax - x = 25a^2 - 10a + 1$

QUADRATIC EQUATIONS 1

So far, you have learned to solve linear equations in one unknown and linear equations with two unknowns. The next type to solve is the quadratic equation. Quadratic equations are usually thought of as second degree equations in one unknown. There are several methods that can be used to solve such equations. One can always use a guess-and-check method or the graphical method. As a review, solve the following by the graphical method.

- 1.* $x^2 + 7x + 6 = 0$
2. $x^2 + 2x - 24 = 0$
3. $x^2 - 3x - 18 = 0$
4. $x^2 - 4x - 15 = 0$
5. $x^2 - 4x + 8 = 0$

*To solve for x

1. Let $x^2 + 7x + 6 = y$
2. Make a table of values for x and y
3. Graph these values and draw a curve. Be certain the general trend of graph is determined.
4. The desired values for x can be read from the graph (the place where curve crosses the x axis).

QUADRATIC EQUATIONS 2

- I. Find the arithmetic value or values of the letter number in each of the quadratics below. (Use guess-and-check method). In working these equations continually be alert for some method that is more systematic than trial and error.

1. $r^2 - 13r + 40 = 0$

2. $n^2 + 5n + 6 = 0$

3. $x^2 - 9x - 36 = 0$

4. $b^2 + 6x - 72 = 0$

5. $2x^2 - 5x + 3 = 0$

- II. Describe below any more systematic methods you have discovered. Also state any generalization you may have discovered while working the problems.

QUADRATIC EQUATIONS 3

1. Solve the following equations by the factoring method (if possible):

$$x^2 - 6x + 9 = 0$$

$$x^2 - 6x + 12 = 0$$

$$x^2 - 6x + 7 = 0$$

$$x^2 - 6x - 12 = 0$$

$$x^2 - 6x + 4 = 0$$

$$x^2 - 6x + 8 = 0$$

$$x^2 - 6x - 7 = 0$$

$$x^2 - 6x + 2 = 0$$

$$x^2 - 6x - 16 = 0$$

2. Which equations above are difficult to solve by the factoring method? Why?
3. Solve the above equations by the graphical method (if possible). Plot all the equations on the same chart.
4. Which problems cannot be solved by the graphical method? Why?
5. What generalizations can you make from your experience in working on this lesson sheet?

QUADRATIC EQUATIONS 4

So far, you have learned to solve quadratic equations by three methods:

1. guess-and-check method
2. factoring method
3. graphical method

The graphical method has the advantage of being able to solve quadratics which are extremely difficult to solve by the other two methods. The disadvantages of the method are:

1. answers (or roots as they are called in algebra) are often very approximate
2. method is too time consuming

There is still another method of solving the quadratic. It is based on knowing the properties of a trinomial perfect square. The purpose of this next exercise is for you to discover properties of the trinomial square.

I. Actually multiply the following perfect squares so they are trinomials.

1. $(x - 3)^2$	10. $(b - \frac{1}{3})(b - \frac{1}{3})$
2. $(5 + m)(5 + m)$	11. $(3x + 4y)^2$
3. $(a - b)^2$	12. $(2 - .5ab)^2$
4. $(m - 5)(m + 5)$	13. $(2 - \frac{3m}{5})(2 - \frac{3m}{5})$
5. $(y + 1)^2$	14. $(-x + y)^2$
6. $(m + n)^2$	15. $(\frac{x}{y} + \frac{y}{x})^2$
7. $(2x - y)(2x - y)$	16. $(2a^2 - b^2c^2)^2$
8. $(5x - 1)^2$	17. $(-x - 2y)(-x - 2y)$
9. $(ab - 3)^2$	18. $(a + .01b)^2$

II. If you have worked the above examples correctly your answers are all perfect trinomial squares. Study the answers; compare them with the original problem and with each other. List below the generalizations you are willing to make.

QUADRATIC EQUATIONS 5

I. In each of the following exercises, find the term needed to make the expression a trinomial square, and state the square root of the resulting trinomial.

1. $x^2 + 2xy + (\quad)$

2. $s^2 - 10st + (\quad)$

3. $y^2 + 6y + (\quad)$

4. $h^2 - 12h + (\quad)$

5. $c^2 - c + (\quad)$

6. $d^2 + 7d + (\quad)$

7. $r^2 + 3r + (\quad)$

8. $x^2 - .4x + (\quad)$

9. $g^2 - \frac{4}{3}g + (\quad)$

10. $z^2 - \frac{1}{2}z + (\quad)$

11. $y^2 - \frac{3}{8}y + (\quad)$

II. Write a general statement as to how you found the last term in the above expressions.

III. Does the generalization made in II hold for the following?

1. $2x^2 + 8x + \underline{\hspace{2cm}}$

2. $4x^2 - 12x + \underline{\hspace{2cm}}$

3. $9y^2 + 30y + \underline{\hspace{2cm}}$

If not, reword the generalization in II so it will hold.

QUADRATIC EQUATIONS 6

Solve and check:

1. $x^2 = 49$
2. $x^2 = 56$
3. $x^2 + 2x = 3$
4. $a^2 + 8a = 20$
5. $t^2 + 14t - 51 = 0$
6. $x^2 - 6x + 3 = 0$
7. $3t^2 - 10t = -3$
8. $x^2 - 2x - 1 = 0$
9. $10x^2 + 21x - 10 = 0$
10. $a^2 + 8a + 9 = 0$

- a. Is the left member of the equation a perfect square?
- b. If not, what would you add to that member to make it a perfect square?
- c. Can this be added to the left member only and the value of x in the final equation be the same as the value of x in the first equation?
- d. With the above hints can you now solve for x?

QUADRATIC EQUATIONS 7

Work all problems on another sheet of paper.

Solve and check:

$$1. \quad 12x^2 - 5x = 3$$

$$2. \quad 2t^2 - 9t + 10 = 0$$

$$3. \quad 12c^2 = c + 1$$

$$4. \quad 4x^2 + 20x = 11$$

$$5. \quad 2n^2 + 5n + 3 = 0$$

$$6. \quad \frac{4}{a+3} = a$$

Find the answer to nearest tenth:

$$7. \quad y^2 = 8 + 4y$$

$$8. \quad 2m^2 + m - 5 = 0$$

$$9. \quad x^2 - 6x = 3$$

Solve for x : (These last two problems are a generalization of the first nine you have worked.)

$$10. \quad 8x^2 + 2ax = 3a^2$$

$$11. \quad ax^2 + bx + c = 0$$

12. Which results, 10 or 11, are the most general?

13. Could either result be used as a formula for finding the values of a quadratic in one unknown? Explain.

OPTIONAL ASSIGNMENT

Geometry and Algebra

A	B	C	D
s	d_1	d_2	L

1. Draw a quadrilateral. In Column A record the number of sides of the quadrilateral. In Column B record the number of diagonals that can be drawn from any single vertex. In Column C record the sum of all diagonals that can be drawn from all vertices. In Column D record the number of straight lines you have drawn altogether (including the sides and the diagonals).
2. Repeat the process for a pentagon, hexagon, and heptagon.
3. By extrapolation determine s, d_1 , d_2 , and L for a triangle, octagon, nonagon, and decagon. (Check the extrapolations by actually counting d_1 , d_2 , and L for the triangle and the decagon.)
4. Write an equation that fits the data in Column A and B. (In other words, express the relationship mathematically by using the variables S and d_1)
5. Examine the data in Columns A, B, and C. Do you see any relationship between s, d_1 , and d_2 that will hold in all cases? If so, express this relationship mathematically using the variables s, d_1 , and d_2 .

OPTIONAL ASSIGNMENT - Geometry and Algebra - continued

6. Express the relationship by using only the letters d_2 and s .
7. Express the relationship between s and L in each case by using only the variables s and L .
8. Graph on a single chart all the values in the above table.

AUTOBIOGRAPHY

I, Oscar Frederick Schaaf, was born in Kearney County, Kansas, June 18, 1918. I received my elementary and secondary school education in a parochial (Lutheran) and in the public schools in the city of Wichita, Kansas. My undergraduate training was obtained at the University of Wichita, Wichita, Kansas, from which I received a Bachelor of Arts in Education in 1942. As an undergraduate I was an assistant to Dr. Laurence Morehouse in the Department of Physical Education. I began the school year of 1942-43 as an instructor in aeronautics and mathematics in the Wichita County Community High School in Leoti, Kansas. In November, 1942 I was called into the Air Corps to commence my training as a meteorologist. In September, 1943 I received a Certificate of Competence in Meteorology from the University of Chicago and was commissioned as an officer. While in the Air Corps I served as a weather forecaster in California and England. Immediately upon being discharged from the Air Corps I began my graduate study at the University of Chicago and received a Master of Arts degree in September, 1946. I held a position as an instructor in mathematics, science, and physical education at Anthony High School, Anthony, Kansas, from 1946-48. I began my graduate study for the degree Doctor of Philosophy at The Ohio State University in the summer of 1947 and received an appointment as instructor in the Department of Education in October, 1948 where I acted as an advisor to undergraduate students and supervisor of student teachers in mathematics. In 1951 I accepted a position as instructor in mathematics at the University School, The Ohio State University, and am holding that position at the present time.

Wartime and Postwar Shifts in U.S. Population For Nine Geographical Regions

U. S. POPULATION BY REGIONS 1940-1948 (Excluding Armed Forces Overseas)

	1940 (Census)	1944 (July)	% Change 1944 Over 1940	1948 (July)	% Change 1948 Over 1940
Total U. S.	131,669,775	132,552,000	0.7%	146,114,000	+ 11.0%
New England	8,437,290	8,208,000	- 2.7%	9,300,000	+10.2%
Middle Atlantic	27,539,487	26,321,000	- 4.4%	29,805,000	+ 8.2%
East North Central	26,626,342	26,253,000	- 1.4%	29,883,000	+12.2%
West North Central	13,516,990	12,742,000	- 5.7%	13,963,000	+ 3.3%
South Atlantic	17,823,151	18,895,000	+ 6.0%	19,478,000	+ 9.3%
East South Central	10,778,225	10,510,000	- 2.5%	10,937,000	+ 1.5%
West South Central	13,064,525	13,272,000	+ 1.6%	14,092,000	+ 7.9%
Mountain	4,150,003	4,248,000	+ 2.4%	4,512,000	+ 8.7%
Pacific	9,733,262	12,103,000	+24.3%	14,144,000	+45.3%

SOURCES: U. S. Bureau of the Census—Current Population Reports.

U. S. BUSINESSES

Total U. S.	
New England	
Middle Atlantic	
East North Central	
West North Central	
South Atlantic	
East South Central	
West South Central	
Mountain	
Pacific	

SOURCE: Dun & Bradstreet, Inc.

Population and Businesses Regions

sources: Dun & Bradstreet, Inc. and U. S. Bureau of the Census

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POPULATION BY REGIONS 1940-1948

	July 1940	July 1944	% Change 1944 Over 1940	January 1949	% Change 1948 Over 1940
.....	2,156,450	1,855,033	-14.0%	2,635,939	+22.2%
.....	158,512	136,181	-14.1%	181,180	+14.3%
.....	508,868	438,582	-13.8%	586,151	+15.2%
.....	459,856	404,474	-12.0%	545,389	+18.0%
.....	254,790	210,407	-17.4%	284,571	+11.7%
.....	212,008	191,435	-9.7%	293,767	+38.6%
.....	109,089	95,800	-12.2%	150,866	+38.3%
.....	184,583	153,519	-16.8%	241,176	+30.7%
.....	68,443	56,315	-17.7%	88,650	+29.5%
.....	200,301	168,320	-16.0%	264,189	+31.9%

Inc. Reference Book figures for issues indicated.

America's civilian population has moved freely during the past eight years, and this tidal surge of migrants has had an important influence on the vital statistics of the nation and the business welfare of our economy.

Some of the temporary shifts of population were due to the high and low tide of the war effort. Some regions have benefited considerably, such as the Pacific States with a net gain of 45.3 per cent in 1948 over 1940, which is about four times the national average. Some of the 1944 population losses were due to the transfer of workers to war jobs, but a large portion of the loss was due to the absence of men in the military service overseas. While a few States have net losses for the eight-year period, all of the nine regions report a population gain in 1948, the smallest being in the East South Central States.

The gain in the number of businesses, as reflected by the Dun & Bradstreet Reference Books, reveals several interesting contrasts with the population percentages of change. In the Pacific Coast States the business population gain is 31.9 per cent since 1940, or 13.4 percentage points less than the apparent consumer increase since 1940. This business population gain is 9.7 per cent above the national average and actually represents a 57.0 per cent gain since 1944.

Percentage-wise, the South Atlantic and East South Central States showed the greatest increase in business establishments in the eight year period. Numerically, the Middle Atlantic States gained 147,569 concerns since 1944 or 33.6 per cent, or 15.2 per cent since 1940.

The year 1944 was a low point in our commercial population due to the effects of a war economy on new concerns. This accounted for the shrinkage between 1940 and 1944. The shortage of merchandise, the better pay opportunities, and the necessity of manpower employment in essential industry accounted for the lack of new concerns. The post-war rush of ex-servicemen and war workers into business is evident in the rapid expansion of commercial enterprises. There was a 22.2 per cent gain over 1940 and a gain of 42.1 per cent over 1944.

Dividing the Reference Book figures into population totals, there were 61 people to each enterprise in 1940, 71 in 1944, and 55 in 1948. The 1948 figures range from a low of 49 in the West North Central to a high of 72 in the East South Central. The Pacific Coast average is 53 to New England's 51.

Discounting the freedom with which a substantial nomadic group moves about the nation, the population shifts reflect several important trends, as industry moves to the Far West and to the Southern States and changes the economic relationship of several industrial and agricultural States.

Prepared by
Dun & Bradstreet, Inc.

IOWA ALGEBRA APTITUDE TESTRevised
edition

By

H. A. GREENE AND A. H. PIPER

Name _____ Age _____
last first years mos.

Name of School _____ Grade _____

Name of City _____ State _____

Have you ever taken first year algebra before? Yes _____ No _____

TEST RECORD

	Part 1	Part 2	Part 3	Part 4	Total
Perfect Scores	30	25	40	10	105
Pupil's Scores					
Percentile Scores					

Part 1. ARITHMETIC

Time allowance—12 minutes

Directions: Work each arithmetic example as directed. Do your work on this sheet or on other paper. Compare your answers with those given in the numbered answer spaces at the right. For each example, three possible answers are given, only one of which is right. Place a cross (X) in the circle directly over the answer that agrees with yours. If no answer agrees with yours, place the X in the circle over "Not Given." You will receive no credit for a correct answer unless it is marked in the correct answer space. The samples at the right are answered correctly.

Samples**Answers**

A. Add: $\begin{array}{r} 286 \\ 475 \\ \hline 761 \end{array}$

A. ☐ 751 ☒ 761 ☐ 771 Not Given

B. $\frac{1}{2} + 2\frac{1}{2} = 3$

B. ☐ 2 ☐ 4 ☐ 5 ☒ Not Given

1. Add

$$\begin{array}{r} 7 \\ 9 \\ 8 \\ 4 \\ 3 \\ 5 \\ 6 \\ \hline \end{array}$$

Check your answer here $\Rightarrow$

2. Subtract

$$\begin{array}{r} 976 \\ 249 \\ \hline \end{array}$$

3. Divide: $8 \overline{) 424}$ **ANSWERS**

1. ☐ 40 ☐ 41 ☐ 42 Not Given

2. ☐ 729 ☐ 739 ☐ 1227 Not Given

3. ☐ 52 ☐ 54 ☐ 56 Not Given

Turn to page 2 and go right on working.

4. Multiply

934
29

Check your answer here

ANSWERS

4. ☐ 27,086 ☐ 28,996 ☐ 27,086 ☐ Not Given

5. Add

75
98
488
976

5. ☐ 1507 ☐ 1607 ☐ 1617 ☐ Not Given

6. Write the fraction $\frac{1}{5}$ as a per cent. _____

6. ☐ .3 ☐ 3% ☐ 30% ☐ Not Given

7. What is the product of seventy-two and zero? _____

7. ☐ 0 ☐ 1 ☐ 72 ☐ Not Given

8. Subtract

15434
7970

8. ☐ 8564 ☐ 7484 ☐ 7480 ☐ Not Given

9. Divide: $9/34704$ _____

9. ☐ 3806 ☐ 3850 ☐ 3856 ☐ Not Given

10. Add

743
594
764
436

10. ☐ 2336 ☐ 2526 ☐ 2535 ☐ Not Given

11. Solve

$\frac{1}{2} + \frac{1}{3} =$ _____

11. ☐ $1\frac{1}{5}$ ☐ $\frac{5}{7}$ ☐ $\frac{15}{38}$ ☐ Not Given

12. Write the decimal .34 as a per cent. _____

12. ☐ .34% ☐ 3.4% ☐ 34% ☐ Not Given

13. Change this per cent to a fraction: 120% _____

13. ☐ $1\frac{1}{5}$ ☐ $1\frac{1}{2}$ ☐ $1\frac{1}{3}$ ☐ Not Given

14. Five Hundred Twelve Dollars less \$104.76 = _____

14. ☐ 407.24 ☐ 408.34 ☐ 616.76 ☐ Not Given

15. What must be added to 537 to make 1195? _____

15. ☐ 537 ☐ 668 ☐ 1732 ☐ Not Given

16. Multiply

427
306

16. ☐ 15,372 ☐ 130,562 ☐ 130,662 ☐ Not Given

17. What is 15% of 80? _____

17. ☐ 8 ☐ 12 ☐ 15 ☐ Not Given

Turn to Page 3 and go right on working.

Part 1. ARITHMETIC—(Continued)

ANSWERS

18. Subtract $\begin{array}{r} 15 \\ 9\frac{1}{2} \\ \hline \end{array}$ _____ 18. ☐ $6\frac{1}{2}$ ☐ $5\frac{1}{2}$ ☐ $5\frac{1}{2}$ Not ☐ Given
19. Divide and point off: $34 \overline{) 205.02}$ _____ 19. ☐ 6.03 ☐ 6.3 ☐ 60.3 Not ☐ Given
20. Solve $\frac{3}{2} \times \frac{5}{12} =$ _____ 20. ☐ $3\frac{1}{4}$ ☐ $1\frac{1}{4}$ ☐ $\frac{1}{4}$ Not ☐ Given
21. Add $\begin{array}{r} 3\frac{1}{4} \\ 9 \\ 5\frac{1}{2} \\ \hline \end{array}$ _____ 21. ☐ $18\frac{1}{2}$ ☐ $17\frac{1}{2}$ ☐ $17\frac{1}{2}$ Not ☐ Given
22. What is 120% of 15? _____ 22. ☐ $1\frac{1}{2}$ ☐ 18 ☐ 60 Not ☐ Given
23. Divide and point off: $24 \overline{) 514.32}$ _____ 23. ☐ 21.43 ☐ 2.143 ☐ 2143 Not ☐ Given
24. Solve: $8\frac{1}{2} - 3\frac{1}{2} =$ _____ 24. ☐ $5\frac{1}{2}$ ☐ $4\frac{1}{2}$ ☐ $3\frac{1}{2}$ Not ☐ Given
25. What per cent of 20 is 4? _____ 25. ☐ 5 ☐ 8 ☐ 25 Not ☐ Given
26. Solve: $\frac{1}{2} + 1.2 =$ _____ 26. ☐ 3 ☐ $2\frac{1}{4}$ ☐ $\frac{1}{4}$ Not ☐ Given
27. Solve: $\frac{1}{2} \times 15 =$ _____ 27. ☐ $11\frac{1}{2}$ ☐ 20 ☐ $\frac{1}{20}$ Not ☐ Given
28. Multiply and point off: $\begin{array}{r} 20.73 \\ 40.3 \\ \hline \end{array}$ _____ 28. ☐ 835.419 ☐ 8354.19 ☐ 89.049 Not ☐ Given
29. Copy and add
.003 + 4.2 + 7.65 + .17

Copy Example 29 here:

- _____ 29. ☐ 8.27 ☐ 12.023 ☐ 12.23 Not ☐ Given

30. Goods marked at 125% of cost sell at \$7.50. What is the cost? _____ 30. ☐ 6.00 ☐ 5.83 ☐ 5.62 Not ☐ Given

Score on Part 1 = Number right = _____

End of Part 1.

Do not turn to the next page until told to do so.

Part 2. ABSTRACT COMPUTATION

Time allowance—8 minutes

Directions: The answers to these problems are obtained by adding, subtracting, multiplying, dividing, or by making combinations of these processes. Study each problem carefully and make up your mind what the answer is to the question asked in each problem. Find this answer in the answers at the right of the question and make a cross (X) in the circle above the correct answer. If you find no answer that agrees with yours place the X in the circle over "Not Given." The sample is answered correctly.

Sample. A. How many pencils would I have if I bought k pencils and some one gave me m pencils more?

Answers

A. $k-m$ ☒ $k+m$ $k \div m$ Not Given

The correct answer is found by adding the k and the m . The answer is $k+m$, so the space over this response is marked with an X.

		ANSWERS			
1.	Monday I read a pages of a book, Tuesday I read b pages, Wednesday I read c pages. How many pages did I read altogether?	1.	☐ abc	☐ $a+b+c$	☐ $3abc$ Not ☐ Given
2.	Ray wants to buy a gun which costs x dollars. He has y dollars, but that is not enough. How many more dollars does he need?	2.	☐ $x+y$	☐ xy	☐ $x-y$ Not ☐ Given
3.	What is the volume of a room which is r feet wide, s feet long and t feet high?	3.	☐ rst	☐ $r+s+t$	☐ $r+st$ Not ☐ Given
4.	John and James have x marbles between them, and each has as many as the other. How many has each?	4.	☐ $2x$	☐ $\frac{2}{x}$	☐ $\frac{x}{2}$ Not ☐ Given
5.	If one load of coal costs c dollars, what will be the cost of b loads of such coal?	5.	☐ $b-c$	☐ $b+c$	☐ $\frac{b}{c}$ Not ☐ Given
6.	A man 45 years old has a son who is y years younger. Indicate the age of the son.	6.	☐ $y-45$	☐ $45-y$	☐ $45y$ Not ☐ Given
7.	Indicate the number of days in r weeks.	7.	☐ $7r$	☐ $\frac{r}{7}$	☐ $r+7$ Not ☐ Given
8.	How much money in Joe's bank account when y dollars are deposited and x dollars withdrawn if the previous balance was k dollars?	8.	☐ $y-xk$	☐ $k+y-x$	☐ $y-k+x$ Not ☐ Given
9.	If a certain number be represented by x and we give x a value of 5, what would be the value of $x+2$?	9.	☐ 10	☐ $5x+2$	☐ 7 Not ☐ Given
10.	The interest on s dollars for one year amounts to d dollars. How much will be the simple interest at the end of y years?	10.	☐ sd	☐ dy	☐ ys Not ☐ Given
11.	Indicate the number of pennies in m dimes and 2 pennies.....	11.	☐ 12	☐ $m+2$	☐ $10m$ Not ☐ Given
12.	One day a boy caught f fish, then gave away a of them. The next day he caught b more, but sold half of them. How many fish had he left?	12.	☐ $f-a-b$	☐ $f+ab$	☐ $f-a+\frac{1}{2}b$ Not ☐ Given

Turn to Page 5 and go right on working.

Part 2. ABSTRACT COMPUTATION—(Continued)

Answers

13. If the product of two numbers is 24, and one of them is m , indicate the other number.	13.	$\frac{24}{m}$	$24m$	$24-m$	Not Given
14. If the quotient of two numbers is y and the divisor is m , represent the dividend.	14.	$y \div m$	my	$m \div y$	Not Given
15. How many hours will it take a person to go m miles, if he goes r miles in one hour?	15.	rm	$r \div m$	$m \div r$	Not Given
16. Represent the integer next larger than the one represented by m	16.	m	$m+1$	$1-m$	Not Given
17. A man has y dollars. Indicate how much he has left after spending one-third of his money.	17.	$y + \frac{1}{3}$	$\frac{1}{3}y$	y	Not Given
18. I bought m 2-cent stamps, and three times as many 1-cent stamps. How many stamps did I buy altogether?	18.	$3m$	$m+3$	$4m$	Not Given
19. If y represents an even number, represent the next larger even number.	19.	$y+2$	$y+1$	$2y+1$	Not Given
20. A book, a pencil, and a tablet together cost y cents. If the pencil cost m cents, and the tablet twice as much as the pencil, what did the book cost?	20.	$y-3m$	$2m-y$	$3m$	Not Given
21. One roll of telephone wire will reach t feet. How many rolls of such wire are needed to reach t feet?.....	21.	$t-t$	$t \div t$	tf	Not Given
22. Three boys decide to run a refreshment stand. The equipment costs r dollars. The father of one of the boys gave them \$20.00. Indicate how much each boy will have to pay, if the remaining expense be divided evenly among them.	22.	$r+20 \div 3$	$20r \div 3$	$\frac{r-20}{3}$	Not Given
23. A horse runs m miles in c minutes. How many miles will it run in one-fifth of a minute?	23.	$m \div 5c$	$m \div \frac{1}{5}c$	$5mc$	Not Given
24. A man has 4 bins each capable of holding a bushels of grain. These bins are empty. He buys b loads of grain, each load containing c bushels of grain, which his present bins will not hold. Indicate the capacity of the new bin he will need to build in order to have sufficient room for the grain.....	24.	$bc-4a$	$4a-bc$	$4a+bc$	Not Given
25. A newspaper charged \$8.00 for the first insertion of an advertisement, and \$4.00 for each following insertion. Express the cost of n insertions.	25.	$4(n-2)$	$4(n-1)$	$4(n+1)$	Not Given

Score on Part 2 = Number right = _____

End of Part 2.

Do not turn to the next page until told to do so.

Part 3. NUMERICAL SERIES

Time allowance—12 minutes.

Directions: Each of the following number series is made up according to some rule. Addition, subtraction, multiplication, and division, and various combinations of these processes are used in forming the different series. Discover the rule for each example, decide what the next term would be, and write it on the blank line following the series. Then place a cross (X) in the circle directly over the answer that agrees with yours. If no answer agrees with yours place the X in the circle over "Not Given." You will receive no credit for a correct answer unless it is marked in the correct answer space. The sample is answered correctly.

Sample:										Answers				
1	2	3	4	5	6	7	8			7	☒	8	9	Not Given
ANSWERS														
1.	2	4	6	8	10	_____	1.			☐ 11	☐ 12	☐ 13	☐ Not Given	
2.	9	8	7	6	5	_____	2.			☐ 5	☐ 3	☐ 2	☐ Not Given	
3.	1	1	5	5	9	9 _____	3.			☐ 11	☐ 12	☐ 13	☐ Not Given	
4.	2	4	8	16	32	_____	4.			☐ 48	☐ 64	☐ 96	☐ Not Given	
5.	5	8	11	14	17	_____	5.			☐ 20	☐ 21	☐ 23	☐ Not Given	
6.	12.9	11.8	10.7	9.6	8.5	_____	6.			☐ 8.3	☐ 7.4	☐ 6.5	☐ Not Given	
7.	1	4	7	10	13	_____	7.			☐ 19	☐ 17	☐ 15	☐ Not Given	
8.	r^2	$6r$	r^3	$5r$	r^4	_____	8.			☐ r^5	☐ $4r^2$	☐ $4r$	☐ Not Given	
9.	12	0	10	0	8	0 _____	9.			☐ 6	☐ 4	☐ 0	☐ Not Given	
10.	1	3	6	10	15	_____	10.			☐ 20	☐ 21	☐ 22	☐ Not Given	
11.	42	37	32	27	_____	_____	11.			☐ 22	☐ 23	☐ 24	☐ Not Given	
12.	128	64	32	16	8	_____	12.			☐ 0	☐ 6	☐ 4	☐ Not Given	
13.	4	5	7	8	10	_____	13.			☐ 12	☐ 10	☐ 11	☐ Not Given	
14.	$\frac{9}{9}$	$\frac{8}{10}$	$\frac{7}{11}$	$\frac{6}{12}$	_____	_____	14.			☐ $\frac{5}{12}$	☐ $\frac{4}{15}$	☐ $\frac{5}{13}$	☐ Not Given	
15.	3	2	4	3	5	4 _____	15.			☐ 6	☐ 5	☐ 4	☐ Not Given	
16.	2	5	9	14	20	_____	16.			☐ 27	☐ 26	☐ 25	☐ Not Given	
17.	17	17	13	9	9	_____	17.			☐ 4	☐ 5	☐ 6	☐ Not Given	

Turn to page 7 and go right on working.

Part 3. NUMERICAL SERIES—(Continued)

ANSWERS

18.	2	4	6	7	9	11	_____	18.	☐ 14	☐ 12	☐ 11	Not ☐ Given	
19.	ab	4589	2ab	458	3ab	_____	19.	☐ 45	☐ 58	☐ 4ab	Not ☐ Given		
20.	24	23	22	20	19	18	_____	20.	☐ 14	☐ 15	☐ 16	Not ☐ Given	
21.	$\frac{a+4}{2a}$	$\frac{a+10}{5a}$	$\frac{a+16}{8a}$	_____	21.	$\frac{a+18}{9a}$	$\frac{a+20}{10a}$	$\frac{a+22}{11a}$	Not ☐ Given				
22.	243	81	27	9	_____	22.	☐ 6	☐ 4	☐ 2	Not ☐ Given			
23.	2	6	10	11	15	19	_____	23.	☐ 20	☐ 21	☐ 24	Not ☐ Given	
24.	25	28	29	33	36	_____	24.	☐ 37	☐ 41	☐ 43	Not ☐ Given		
25.	56342	xy'	5634	x'y'	563	x'y'	_____	25.	☐ x'y'	☐ 6	☐ 3	Not ☐ Given	
26.	$\frac{y''}{a+b+c+d}$	$\frac{y''}{b+c+d}$	$\frac{y''}{c+d}$	_____	26.	$\frac{y''}{d}$	$\frac{y''}{d}$	y	Not ☐ Given				
27.	0	$\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	_____	27.	☐ $2\frac{1}{2}$	☐ $2\frac{1}{2}$	☐ 3	Not ☐ Given		
28.	4	5	10	11	22	23	_____	28.	☐ 24	☐ 46	☐ 47	Not ☐ Given	
29.	2	4	8	14	22	_____	29.	☐ 44	☐ 34	☐ 32	Not ☐ Given		
30.	4y'	5y'	4y'	7y'	4y'	_____	30.	☐ 4y'	☐ 9y'	☐ 11y'	Not ☐ Given		
31.	1	3	7	15	_____	31.	☐ 31	☐ 30	☐ 29	Not ☐ Given			
32.	3	8	13	18	23	_____	32.	☐ 31	☐ 29	☐ 27	Not ☐ Given		
33.	48.08	24.04	12.02	_____	33.	☐ 6.2	☐ 6.1	☐ 6.01	Not ☐ Given				
34.	1	3	7	8	11	18	26	_____	34.	☐ 33	☐ 37	☐ 39	Not ☐ Given
35.	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{8}$	$1\frac{1}{2}$	$2\frac{1}{2}$	_____	35.	☐ 3	☐ $3\frac{1}{2}$	☐ 5	Not ☐ Given		
36.	1	1	2	6	24	_____	36.	☐ 72	☐ 120	☐ 144	Not ☐ Given		
37.	2y'-10	6y'-16	10y''-22	_____	37.	14y'''-28	15y''-26	16y''-24	Not ☐ Given				
38.	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{1}{3}$	_____	38.	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{7}{15}$	Not ☐ Given				
39.	1	2	4	12	36	_____	39.	☐ 72	☐ 108	☐ 144	Not ☐ Given		
40.	$\frac{5}{108}$	$\frac{5}{36}$	$\frac{5}{12}$	$\frac{5}{4}$	$3\frac{3}{4}$	_____	40.	☐ $9\frac{1}{2}$	☐ $11\frac{1}{2}$	☐ 12	Not ☐ Given		

Score on Part 3 = Number right = _____

End of Part 3. Do not turn to next page until told to do so.

Part 4. DEPENDENCE AND VARIATION

Time allowance—3 minutes

Directions: Each of the following exercises can be answered by one of the four numbered phrases given below each question. Study the example and the question based upon the example. Decide which one of the numbered phrases below the question is the correct answer. Note the number of this phrase. Find this number among the answer spaces at the right of the question and make a cross (X) in the correct circle. The sample is answered correctly.

Sample: $X - 7 = 10$ If the 7 were made larger, what change would need to be made in the value of X so that the answer 10 would still be correct?

(1) remain the same (2) become larger (3) become smaller (4) cannot tell.

(1) (2) (3) (4)
○ ⊗ ○ ○

1. $X = y$	If y were made larger, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	(1) (2) (3) (4)
○ ○ ○ ○
2. $X = y + m$	If y were made smaller, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	1. ○ ○ ○ ○
3. $X = y + \frac{m}{s}$	If m were made smaller, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	2. ○ ○ ○ ○
4. $X = y + m$	If y were to be made 1 larger, and m were to be made 1 smaller, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	3. ○ ○ ○ ○
5. $X = y - \frac{1}{y}$	If the value of y were made larger, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	4. ○ ○ ○ ○
6. $X = \frac{y}{m}$	Given that y and m are always equal. What change would be made in the value of X if y and m would be made larger in the same proportion?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	5. ○ ○ ○ ○
7. $X = y - m$	If m were made larger, what change would be produced in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	6. ○ ○ ○ ○
8. $X = y + m - n$	If y, m, and n were all given the same value, what change would be produced in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	7. ○ ○ ○ ○
9. $X = y + \frac{m}{s}$	If s were made smaller, what change would be made in the value of X?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	8. ○ ○ ○ ○
10. $X = y + m$	If m were made larger, what change would need to be made in the value of y so that X would not change in value?
(1) remain the same (2) become larger (3) become smaller (4) cannot tell	9. ○ ○ ○ ○
		10. ○ ○ ○ ○

Score on Part 4 = Number right = _____

End of test; Close your paper.

DIRECTIONS

To the Teacher:

Please return Forms 2 and 4
 for each class to the OHIO
 SCHOLARSHIP TESTS by April 9.
 The test papers nearest the 1st,
 2nd, and 3rd quartiles may be
 sent later.

To the Pupil:

You will be given 40 minutes
 for the entire test. If you
 complete the test before the
 time limit, you may go back and
 check your work.

ELEMENTARY ALGEBRA

EVERY PUPIL TEST

April, 1945

Ohio Scholarship Tests
 State Department of Education
 Columbus, Ohio

Constructed by
 Wade H. Carpenter, Principal
 Wellsville High School

Directed by
 H. C. Christefferson, Miami University

Score = Rights (40)

Score = _____

Name _____ Age _____ Grade _____ Date _____

School _____ Town _____ State _____

DIRECTIONS: Perform the indicated operations and place your answer on the line to the left of the item number. Use the blank spaces for your computation.

_____ 1. $4a + 2 = 10$ $a = ?$

_____ 2. A man is x years old. How old will he be in y years?

_____ 3. If $a = 5$ and $b = 8$, then $a - b = ?$

_____ 4. Add: $3x^2 + 2x - 8$
 $2x^2 - 5x + 3$

_____ 5. $6x - 8x + 5x = ?$

_____ 6. $x^6 + x^2 = ?$

_____ 7. Factor: $a^2 - 9b^2$

_____ 8. 8 is how much more than -3 ?

_____ 9. $(5a)^2 (2a) = ?$

_____ 10. $(2x + 1)(2x + 1) = ?$

_____ 11. Factor: $x^2 - 3x - 4$

_____ 12. Factor: $ad + d$

_____ 13. In $5a^2$ the exponent is (?)

_____ 14. $\sqrt{36a^2b^4} = ?$

_____ 15. In the formula, $y = \frac{6}{x}$, if x increases, will y increase or decrease?

_____ 16. Remove parentheses and collect terms: $5a + (a - b) - (2a + b) = ?$

_____ 17. $9x + .01 = 1$ $x = ?$

_____ 18. $ay - (-ay) = ?$

_____ 19. $\sqrt{50} - \sqrt{18} + 2\sqrt{2} = ?$ (Leave answer in radical form.)

_____ 20. $ax - b = c$ $x = ?$

21. A basketball player in three games scored a, b, and c points respectively. Represent the player's average score.

22. If x tops cost c cents, what is the cost of one top?

23. Reduce to lowest terms: $\frac{16a^2b}{4a^2b^2} = ?$

24. Divide: $(a^2 - a - 6)$ by $(a + 2)$

25. $\frac{a}{2} + \frac{2a}{3} = 14$ $a = ?$

26. Subtract $2a^2 - 3a + 7$ from $3a^2 - a - 9$.

x = _____ 27. Solve for x and y:

y = _____ 28. $\begin{cases} x + 2y = 5 \\ 3x - y = 1 \end{cases}$

29. John can mow the lawn in m minutes. What part of the lawn can he mow in x minutes?

30. $x^2 - 5x = 24$ $x = ?$

31. Separate 25 into two parts so that their product will be 120.

32. Multiply $a^2 + ab + b^2$ by $a - b$.

33. A tree casts a shadow 60 feet long at the same time a boy 5 feet tall casts a shadow 4 feet long. How high is the tree?

34. $\sqrt{x + 2} = 5$ $x = ?$

35. $\frac{a^2 - a - 6}{a} \cdot \frac{a^2}{a + 2} = ?$

36. $\frac{2a - 5}{5} + \frac{a}{2} = ?$

37. $6a^2 - 51 = (2a + 1)(3a - 6)$ $a = ?$

38. The legs of a right triangle are 5 in. and 12 in. What is the perimeter?

39. $N = \frac{E}{R + r}$ $r = ?$

40. An automobile traveling at the rate of 25 miles per hour is 20 miles ahead of a second car. At what rate must the second car travel to overtake the former in 4 hours?

LANKTON FIRST-YEAR ALGEBRA TEST

BY ROBERT LANKTON

TEACHERS COLLEGE, CEDAR FALLS, IOWA

FORM

Am

DIRECTIONS:*Do not open this booklet until you are told to do so.*

This is a test of your competence in first-year algebra. For each question there are five possible answers. You are to work each question and determine which answer is correct. You may answer a question even when you are not perfectly sure that your answer is correct, but you should avoid wild guessing. Do not spend too much time on any one question.

Study the sample questions below, and notice how the answers are to be marked on the separate answer sheet.

Sample A. If $3x = 15$, then x equals

- a. 45
- b. 18
- c. 12
- d. 5
- e. none of the above

For Sample A the answer, of course, is "5," which is answer d. Now look at your answer sheet. At the top of the page in the left-hand column is a box marked SAMPLES. In the five answer spaces after Sample A, a heavy mark has been made filling the space (the pair of dotted lines) marked d.

Sample B. The equation which states that n is 2 less than 10 is

- f. $\frac{n}{2} = 10$
- g. $n - 2 = 10$
- h. $n + 2 = 10$
- i. $n = \frac{10}{2}$
- j. none of the above

The correct answer for Sample B is " $n + 2 = 10$," which is answer h; so you would answer Sample B by making a heavy black mark that fills the space under the letter h. Do this now. If the correct answer had not been given, you would have chosen answer j, "none of the above."

Read each question carefully and decide which one of the answers is best. Notice what letter your choice is. Then, on the separate answer sheet, make a heavy black mark in the space under that letter. In marking your answers, always be sure that the question number in the test booklet is the same as the question number on the answer sheet. Erase completely any answer you wish to change, and be careful not to make stray marks of any kind on your answer sheet or on your test booklet. When you finish a page, go on to the next page. If you finish the entire test before the time is up, go back and check your answers. Work as rapidly and as accurately as you can.

When you are told to do so, open your booklet to page 2 and begin. The working time for this test is 40 minutes.

1. n^3 has the same meaning as
 - a. $n \times 3$
 - b. $n + 3$
 - c. $n \times n \times n$
 - d. $n + n + n$
 - e. none of the above
2. In the expression $4x^2 + 3x^2 + 4x + 3$ the like (or similar) terms are
 - f. $4x^2$ and $3x^2$
 - g. $4x^2$ and $4x$
 - h. $3x^2$ and $4x$
 - i. $3x^2$ and 3
 - j. $4x$ and 3
3. In the expression $27x^3$, the 27 is called the
 - a. root.
 - b. coefficient.
 - c. exponent.
 - d. power.
 - e. radical.
4. $\frac{6r^3}{4r^2t}$ equals
 - f. $\frac{3}{2}$
 - g. $\frac{3t}{2r}$
 - h. $\frac{3t^2}{2r^2}$
 - i. $\frac{3t^3}{2r^3}$
 - j. $\frac{3t^4}{2r^4}$
5. $(5m)(3m^4)$ equals
 - a. $8m^4$
 - b. $8m^5$
 - c. $12m^2$
 - d. $15m^4$
 - e. $15m^6$
6. The cost in dollars of n shirts at d dollars each is
 - f. dn
 - g. $\$1.00n$
 - h. $\frac{dn}{100}$
 - i. $\frac{d}{100}$
 - j. $\$1.00d$
7. A number which is 7 less than n is
 - a. $7 - n$
 - b. $n - 7$
 - c. $n + 7$
 - d. $\frac{n}{7}$
 - e. none of the above
8. If 13 is subtracted from 5, the answer is
 - f. 8
 - g. -8
 - h. 18
 - i. -18
 - j. none of the above
9. The sum of $-5a$ and $+6a$ is
 - a. 1
 - b. a
 - c. $-11a$
 - d. $11a$
 - e. none of the above
10. Which of the following numbers is less than -5 ?
 - f. -4
 - g. -3
 - h. -2
 - i. 0
 - j. none of the above
11. If -9 is added to a certain number n , the new number is
 - a. $n - 9$
 - b. $9 - n$
 - c. $n + 9$
 - d. $-9n$
 - e. $9n$
12. If $n + 2n = 12$, then the value of n is
 - f. 1
 - g. 2
 - h. 3
 - i. 4
 - j. 6
13. The temperature of a gas is -31 degrees. If its temperature drops 44 degrees, its new temperature is
 - a. 75 degrees.
 - b. 13 degrees.
 - c. -13 degrees.
 - d. -15 degrees.
 - e. -75 degrees.
14. One of the factors of $a^2 - 18a + 80$ is
 - f. $a + 4$
 - g. $a + 5$
 - h. $a - 10$
 - i. $a - 16$
 - j. $a - 20$
15. The statement, "If a certain number is multiplied by 3 and the product is decreased by 5, the result is 21," may be written in equation form as
 - a. $\frac{n - 5}{3} = 21$
 - b. $\frac{n + 5}{3} = 21$
 - c. $3n - 5 = 21$
 - d. $n(3 - 5) = 21$
 - e. $3(n - 5) = 21$

16. $-4x^3y(3y - x - 3)$ equals

- f. $-4x^3y$
- g. $-x^3 - 4x^3y - 7x^3y$
- h. $-7x^3y^2 - 5x^3y - 7x^3y$
- i. $-12x^3y^2 - 4x^3y + 12x^3y$
- j. $-12x^3y^2 + 4x^3y + 12x^3y$

17. If $g = -3$, then $g^2 + g$ equals

- a. -9
- b. -3
- c. 3
- d. 6
- e. 12

18. $A = \frac{1}{2}ab$ is a formula for the area of a triangle. In order to use this formula, what restrictions must be observed for the values of a and b ?

- f. a must be larger than b
- g. a must be less than b
- h. a must equal $\frac{1}{2}$ of b
- i. a must equal 2 times b
- j. none of the above

19. If the sides of a triangle are doubled, the perimeter of the triangle is multiplied by

- a. 2
- b. 3
- c. 4
- d. 6
- e. 8

20. Houghton School, on three successive plays in a football game, advanced the ball 90 yards for a touchdown. If the team gained 4 yards on the first play and lost 7 yards on the second play, how long was the final run for the touchdown?

- f. 79 yd.
- g. 83 yd.
- h. 87 yd.
- i. 93 yd.
- j. none of the above

21. Which of the following formulas gives the area of the rectangular side of the building shown above, not including the doors?

- a. $A = HL - hw$
- b. $A = (HL)(-hw)$
- c. $A = (H - h)(L - w)$
- d. $A = (H + L)(-h - w)$
- e. $A = 2(H + L) - 2(h + w)$

22. If $x = \frac{2}{3}$ and $6x - 5 = k$, then the value of k is

- f. $-\frac{2}{3}$
- g. 1
- h. 2
- i. 3
- j. none of the above

23. The value of $r + 5$ is negative for all values of r less than

- a. -5
- b. -4
- c. 0
- d. 4
- e. 5

24. $\frac{a - b}{a + b} - \frac{b}{a + b}$ equals

- f. $\frac{1}{b}$
- g. $\frac{a - 1}{a}$
- h. $\frac{a}{2(a + b)}$
- i. $\frac{a - 2b}{a + b}$
- j. $\frac{a - 2b}{a(a + b)}$

25. If $10x$ equals some number, then x is

- a. 10 times the number.
- b. 10 more than the number.
- c. the number divided by 10.
- d. 10 less than the number.
- e. equal to the number plus 10 times the number.

26. If $2x - 16 = 3x - 9$, then x equals

- f. $-\frac{7}{3}$
- g. -7
- h. 5
- i. 25
- j. none of the above

27. If $\frac{1}{3}x = 12$, then x equals

- a. $\frac{3}{4}$
- b. $\frac{4}{3}$
- c. 3
- d. 108
- e. none of the above

28. $x = 5$ may be obtained from the equation $x + 5 = 10$ by

- f. dividing both sides of the equation by 5.
- g. dividing both sides of the equation by 2.
- h. subtracting 5 from both sides of the equation.
- i. subtracting 2 from both sides of the equation.
- j. adding 5 to both sides of the equation.

29. If $\frac{x}{4} = \frac{5}{9}$, then $9x = 20$ may be obtained by multiplying both sides of the equation by
- 4
 - 5
 - 9
 - 13
 - 36

30. Correct to one significant figure, $\sqrt{5000}$ equals
- 20
 - 50
 - 70
 - 200
 - 700

31. If $x = 7$, $y = 3$, and $z = 2$, then $x + yz$ equals
- 12
 - 13
 - 20
 - 27
 - 35

32. In the formula $p = 8 + 2w$, if w varies from 1 to 5, p varies from
- 1 to 5
 - 9 to 13
 - 11 to 15
 - 10 to 18
 - 10 to 50

33. In the graph above, if x changes 1 unit, then y changes
- 1 unit.
 - 2 units.
 - 3 units.
 - 4 units.
 - 5 units.

34. $\frac{2x}{5} + \frac{5}{2x}$ equals
- 1
 - 2
 - $\frac{4x^2}{25}$
 - $\frac{2x+5}{10x}$
 - $\frac{4x^2+25}{10x}$

35. If $E = IR$, then the formula for I is

- $I = R - E$
- $I = \frac{E}{R}$
- $I = ER$
- $I = \frac{R}{E}$
- $I = E - R$

36. If n is 3 more than y , the equation relating n and y is

- $\frac{n}{3} = y$
- $3n = y$
- $n + 3 = y$
- $n - 3 = y$
- $3 - n = y$

37. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. The number of cubic yards of sand in the pile shown in the figure above is most nearly

- 100
- 600
- 1100
- 1600
- 2500

38. The weight of one boy on the seesaw is W and the weight of his big brother is H . The seesaw will balance if the product of the smaller weight (W) and the distance (r) equals the product of the larger weight (H) and the distance (s). How may this relationship be expressed?

- $W + r = H + s$
- $(W + r)(H + s) = 0$
- $WH = rs$
- $Wr = Hs$
- $\frac{W}{r} = \frac{H}{s}$

39. $\frac{30x+12}{5x+2}$ equals

- 6
- $5x + 2$
- $6x + 2$
- $6x + 6$
- $\frac{30x+6}{5x}$

40. If $.16 = \frac{82}{x}$, then x equals

- f. .005
- g. .05
- h. .5
- i. 2
- j. 200

41. If $\frac{3}{m} + \frac{8}{15} = \frac{8}{3m}$, then m equals

- a. $-1\frac{1}{8}$
- b. $-\frac{8}{15}$
- c. $-\frac{15}{8}$
- d. $\frac{8}{15}$
- e. 15

42. Which of the following graphs shows the same relationship between C and d as that expressed by the formula $C = \pi d$?

j. none of the above

43. The distance (s) in the figure above may be found by using the equation

- a. $\sin 25^\circ = \frac{s}{96}$
- b. $\sin 25^\circ = \frac{96}{s}$
- c. $\cos 25^\circ = \frac{96}{s}$
- d. $\tan 25^\circ = \frac{s}{96}$
- e. $\tan 25^\circ = \frac{96}{s}$

44. $\sqrt{12x^3}$ equals

- f. $2x\sqrt{3x}$
- g. $2x^1\sqrt{3x}$
- h. $2x^1\sqrt{3x}$
- i. $2x^2\sqrt{3x^3}$
- j. $6x^2\sqrt{2x}$

45. The value of x which satisfies both of the equations

$$\begin{cases} 3x + y = 7 \\ x - 2y = 1 \end{cases} \text{ is}$$

- a. $1\frac{1}{3}$
- b. $1\frac{1}{5}$
- c. $2\frac{1}{3}$
- d. $2\frac{1}{5}$
- e. none of the above

46. Which of the following equations may be used to solve the problem: If 1 gallon of paint will cover 425 square feet, how many gallons of paint will be needed for 1300 square feet?

- f. $\frac{1}{425} = \frac{x}{1300}$
- g. $\frac{1}{425} = \frac{1300}{x}$
- h. $\frac{x}{425} = \frac{1300}{1}$
- i. $\frac{425}{1300} = \frac{x}{1}$
- j. none of the above

47. To solve the equation $2x + 3 = 13$, a good first step is to subtract 3 from both sides of the equation because

- a. the term involving the unknown will be obtained alone on one side.
- b. the coefficient of x and the constant term will both be even numbers.
- c. subtracting 3 from 13 leaves an easier number to work with.
- d. the question is a linear equation in only one variable.
- e. it is impossible to divide 13 and 3 by 2.

48. A tennis racket was purchased during a sale for \$7.50. This was 20% less than the original price. By which of the following equations may the original price of the racket be found?

- f. $.20x = 7.50$
- g. $x - .20(7.50) = 7.50$
- h. $x - .20x = 7.50$
- i. $\frac{x}{.20} = 7.50$
- j. $\frac{.20}{x} = 7.50$

**ENROLLMENT IN PUBLIC HIGH SCHOOLS
IN UNITED STATES**

YEAR	NUMBER OF STUDENTS
1890	202,963
1900	519,251
1910	915,061
1920	2,131,655
1930	4,399,422
1940	6,601,444

Questions 49 and 50 are based on the table above.

49. During every ten-year period, except from 1930 to 1940, the number of students enrolled in public high schools has been multiplied by approximately

a. 2
b. 3
c. 4
d. 5
e. 6

50. The ratio of the number of students in public high schools in the United States in 1940 to the number in 1910 is approximately

f. 7 to 1
g. 1 to 7
h. 9 to 7
i. 7 to 9
j. none of the above

51. How many pairs of values of x and y will satisfy the linear equation $x + 2y = 10$?

a. one
b. two
c. three
d. four
e. more than four

52. Business profits of \$10,400 are to be divided among three partners in proportion to the amount each one originally invested. Mr. Adams invested twice as much as Mr. Crane, while Mr. Daniels invested $\frac{1}{3}$ as much as Mr. Crane. How large a share of the \$10,400 will Mr. Adams receive?

f. \$1040
g. \$2080
h. \$4160
i. \$5200
j. \$6240

53. The values of x which satisfy the equation $x^2 + 4x = 5$ are

a. -1 and 1
b. -3 and 3
c. -1 and 5
d. 1 and -5
e. 1 and 5

A.

B.

C.

D.

SCALE: 1 space equals 1 unit.

Questions 54 and 55 are based on the graphs above.

54. On which of the graphs above is y negative for all values of x less than 2?

f. A
g. B
h. C
i. D
j. none of the above

55. Which of the graphs above passes through the point $(-2, 1)$?

a. A
b. B
c. C
d. D
e. none of the above