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A STUDENT-EXPERIENCE-DISCOVERY

APPROACH

TO THE

TEACHING OF THE CALCULUS

DISSERTATION

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KENNETH BURDETTE CUMMINS, B. A., M. A.

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Approved by:

Adviser

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CHAPTER ONE

INTRODUCTION

A Growing Emphasis on Meaning

In the last several decades there has arisen much concern over the importance of understanding and of meaning in mathematics as developed by the student himself. These meanings are developed or formulated and nurtured as much as possible through his own experiences and through somewhat self-directed organization of material as opposed to meanings received by direct information. The latter frequently result not only in incomplete understanding but also in mere manipulation of symbols. Evidence of this concern is found in the various types of student-experience programs in which natural or contrived activities help him to arrive at meaning. Also there is an increasing number of studies and suggestions which emphasize meanings and their formulation by the student even in fields where the teaching, for the most part, is very traditional.

Various types of student-experience methods are now found in practically the entire school program. Historically, such work began in the elementary school with the use of concrete experiences, activity movements, unit studies, and the use of other such plans which were more or less student-centered. High school teachers became concerned also, and now there are a number of significant studies concerning the organization of classroom activities through which students develop

understandings and insights on this level. This development has led many high school teachers to attempt to improve their teaching methods, and, although some attempts toward a more student-centered approach have been mishandled, and some innovations have been regarded with ridicule, yet the steady tendency toward an increasing emphasis on meaning has exercised wide influence. Colleges and universities have not been immune to this spreading concern for the importance of meaning, and the value of emphasizing understanding has had an effect on ideas concerning the teaching of the calculus. Murnaghan, for one, gives evidence of this when he says (on the calculus)

. . . teach your young students understanding of fundamental principles rather than manipulative detail. The latter they can gain for themselves, but they will never learn the former unless you teach it to them. It is my firm conviction . . . that one can teach calculus correctly from the very beginning without losing the ordinary run-of-the-mine student and without sacrificing manipulative technique.¹

Origin and Background of the Problem

In attempting to encourage advanced mathematical study by interested and able students, the writer devised courses in elementary calculus as part of a four-year program in secondary mathematics. At first, as might be expected, these courses followed closely a simplified content and approach of those found on the college level; it did not take long, however, to observe that although the students could manipulate symbols they did not really understand what they were doing.

¹F. D. Murnaghan, "The Teaching of College Mathematics," The American Mathematical Monthly, LIII (October, 1946), 422.

Believing deeply that instruction should be more than training in manipulation of symbols and more than developing blind obedience to rules, important as these are, it seemed desirable, if not imperative, to use study procedures which would encourage the student, not only to understand more fully the nature of the calculus, but also to help him to develop and organize his own basic principles. To write fruitful study-guides and exercises to help the student "generate" some of "his own" calculus was itself a challenging undertaking which required thoughtful planning and preparation. Nor would one expect such study-guides to have been written well during the first experience; rather, it was found that each year brought a slightly different point of view and the use of improved exercises enabled the student to see elementary calculus in its unveiled beauty and clarity. By means of a numerical approach which emphasized the reasonableness and naturalness of certain procedures of the calculus, the student met none of the mystery which so often shrouds this subject.

The results of these attempts have been very satisfying. There seemed to develop a naturalness and ease of transition with which students progressed from one stage to another while formulating and learning elementary concepts of the calculus. Teacher-made tests indicated a noteworthy level of good understanding. Too, there was not infrequently observed a sense of satisfaction and thrill experienced by the student in sometimes inventing his own mathematics. Some of these students have continued their study of mathematics on the university level and they have related how greatly these humble beginnings have helped them in their elementary studies, including the calculus.

Out of this suggestive experience, which seemed to divest the calculus of much of its mystery as viewed by the students, it was not long before the question arose concerning the possibility of extending these procedures to the study of calculus at the university level.

It was a timely and fortunate opportunity, therefore, when there was organized at The Ohio State University a seminar in the teaching of the calculus.² In the progress of this work, as topics in the calculus were critically examined, as methods of teaching applicable to the calculus were discussed, as it was noted that the few writers on the teaching of the calculus generally decried present tactics, and as it was observed that text-writers professed to eradicate this or that evil which had crept into the calculus classrooms, the problem of investigating more inductive experiential approaches on the university level became more pressing and more sharply focused. It was therefore proposed to suggest and test the efficiency of a method which utilizes numerical and experiential activities with an emphasis on student discovery and student organization of materials. This has been called a student-experience-discovery approach.

Need for the Study

The problem of helping students understand the calculus has been long recognized but it seems that little has been done. As long ago as 1907, Saxelby, in writing about his approach to differentiation, suggested that "it too often happens that a student . . . acquires a merely

²Mathematics 896. Seminar in the Teaching of the Calculus. This was conducted jointly by the Department of Mathematics and the Department of Mathematics Education during the fall and winter quarters in 1955-1956. Dr. Nathan Lazar and Dr. Paul V. Reichelderfer were in charge.

fatal facility (*italics mine*) in differentiation, regarding it as a mechanical juggling with symbols but having no conception of its relation to experience."³ It is such "fatal facility" which, indeed, has often been encouraged by the use of methods advocating the memorization of techniques and routine applications of rules which strangely originated from the text, or, most often, from the teacher. Present writers frequently imply that there is need for improvement in the teaching of the calculus when, in prefaces to their texts, they make the claim to an unique approach that will foster deeper understanding and hence more efficient performance on the part of the student. Murnaghan, in the preface of his text, makes the very inviting statement "I want all the students, not merely . . . (the) . . . privileged better ones to have a chance to know what it is all about."⁴ But the problem of knowing "what it is all about" is still with us and one can surely say that not too much has been done toward its solution.

Review of Related Literature

Some of the early scholar-teachers believed that the secret of developing understanding was to start with "things" through an experimental approach. Archimedes is said to have "used anything and everything that suggested itself as a weapon to attack his problem"⁵--he,

³F. M. Saxelby, A Course in Practical Mathematics (London: Longmans, Green, and Company, 1907), p. v.

⁴F. D. Murnaghan, Differential and Integral Calculus (Brooklyn: Rensselaer Press, 1947), p. iv.

⁵E. T. Bell, Men of Mathematics (New York: Simon and Schuster, 1937), p. 31.

". . . the greatest mathematician of antiquity, possibly the greatest mathematician of all time. . . ." ⁶ suggests to us the productive results obtained by the use of a practical approach to problems although he was, as well, a master at deduction. Archimedes, surely a member of the so-called "modern" school in mathematics, challenges us to use intuitive, experimental, practical approaches to problems. Comenius, "the father of modern educational practice," ⁷ urges that learning is helped much when we use "non verba, sed res." ⁸ Emmanuel Kant urges that we must help build ideas by the use of specific examples either in mental imagery or by means of physicalization of ideas and processes for he writes: "Begriffe ohne Anschauung sind leer." ⁹ C. S. Peirce, son of B. O. Peirce of integral table fame, in 1878, wrote an article through which he expounded his original ideas on what is now called "operationalism." Peirce writes that he desires only "to point out how impossible it is that we should have an idea in our minds which relates to anything but conceived sensible effects of things . . . our idea of anything is our idea of its sensible effects." ¹⁰ At present it seems that studies in the field of learning concur that

⁶Alfred Hooper, Makers of Mathematics (New York: Random House, 1948), p. 64.

⁷New Century Encyclopedia of Names (New York: Appleton-Century-Crofts, 1954), p. 1043.

⁸National Council of Teachers of Mathematics. The Teaching of Mathematics in the Secondary School. Eighth Yearbook (Washington, D.C.: the Council, 1933), p. 226.

⁹C. S. Peirce, "How to Make Ideas Clear," Popular Science Monthly, XII (January, 1878), 293.

¹⁰Ibid., p. 293.

. . . initial learnings come from experience (physical and mental experiments), constructive methods, not from definitions. It is the dynamic aspects of events that aid learning. Whatever is to be learned must have its roots in some challenging problem-presenting situation.¹¹

At present also modern thought has produced various "dimensions of meaning": (a) the syntactic dimension, or meaning given to a symbol by the way it is used in a formula or a sentence, (b) the pragmatic dimension, or meaning given to a word according to purpose, and (c) semantic dimension, or meaning given to a symbol or word by the nature of its referent which may be a concrete object, an abstraction, or a mental construct. The modern mathematics educator is much interested in the semantic dimension and somewhat concerned with the syntactical dimension.

The modern educator, too, attempts to make a distinction between "meaning" and "understanding" although sometimes they are used interchangeably. Meaning, VanEngen writes, "is that which is 'read into' a symbol by the pupil. . . the pupil realizes that the symbol is a substitute for an object."¹² Understanding, on the other hand,

. . . refers to something that is in possession of an individual. The individual who understands is aware of a satisfying feeling, a psychological closure, which results from having fitted everything in its proper place. . . (he is) in possession of the cause and effect relationship--the logical implication and sequences of thought that unite two or more statements by means of the bonds of logic . . . understanding is an organizational process.¹³

¹¹National Council of Teachers of Mathematics. The Learning of Mathematics; Its Theory and Practice. Twenty-First Yearbook (Washington, D.C.: The Council, 1953), p. 70.

¹²Ibid., p. 75.

¹³Ibid., pp. 74-76.

Some writers have suggested the use of numerical situations to help build ideas. As long ago as 1902 John Perry urged that computational or arithmetical methods would help establish referents so that the calculus symbols would be better understood. He gives this advice: "However good a mathematician a student thinks himself to be, he ought to practice working out numerical values. . . . He must not only think he knows; he must really do the numerical work."¹⁴ Even earlier, in 1894, Fletcher Durell must have caused some repercussions when, in applying the "new education" to the teaching of the calculus, he explained that "the student in each new advance is to begin with the concrete object, something which he can see and handle and perhaps make, and go on to abstractions only for the sake of realized advantages."¹⁵ In emphasizing this point Durell continues by explaining how Gauss and Cauchy conceived the idea of representing a complex quantity by a pair of numbers and by a point in a plane and then argues his case further:

If the greatest mathematical minds feel such an aid necessary in their work, why should not the corresponding aid be offered to the farmer's boy, who is engaged, year after year, struggling with numbers whose sensible relation to sensible objects he can have clear conception?¹⁶

In like manner there comes from the influential pen of E. H. Moore an expression of his concern for meaning. He proposes:

¹⁴John Perry, Calculus for Engineers (London: Edward Arnold, 1897), p. 2.

¹⁵Fletcher Durell, "Applications of the New Education to the Differential and Integral Calculus," American Mathematical Monthly, I (January, 1894), 15.

¹⁶Ibid., p. 41.

. . . the teacher should lead up to an important theorem gradually in such a way that the precise meaning of the statement in question, and further, the practical--i.e. the computational or graphical or experimental--truth of the theorem is fully appreciated; and furthermore, the importance of the theorem is understood, and indeed the desire for the formal proof is awakened, before the formal proof itself is developed. Indeed, much of the proof should be secured by the research of the students themselves.¹⁷

It is urged also that opportunity to contribute and to organize one's own materials may increase understanding. C. A. Hutchinson writes: "May it be that we have overlooked our best allies in a fight for sound education based on fundamentals--the students themselves?"¹⁸ Then, from this powerful source comes the challenge of a college student: "As a matter of fact we are anxious to learn. All we ask is that there be some pleasure, some intellectual excitement in the process. . . ."¹⁹ In fact, one notable experiment in determining the value of having students formulate and organize much of their own knowledge was carried on by Dr. Harold Fawcett²⁰ in the study of geometry at the University School at The Ohio State University. In this research it was convincingly demonstrated that through emphasis on the "development of geometry in

¹⁷E. H. Moore, "On the Foundations of Mathematics," Bulletin of the American Mathematical Society, IX (May, 1903), 419. This is the presidential address given before the American Mathematical Society at its ninth annual meeting December 29, 1902.

¹⁸C. A. Hutchinson (editor of a section on mathematics education), American Mathematical Monthly, XLVI (May, 1939), 280.

¹⁹Ibid., p. 280.

²⁰Harold P. Fawcett, The Nature of Proof. Thirteenth Yearbook (Washington, D.C.: National Council of Teachers of Mathematics, 1938).

his own way" the student suffered no loss in content of subject matter, but, in addition, improved his ability in reflective thinking not only in geometry, but in "a variety of situations."²¹ Another study which should be mentioned is that of Max A. Sobel²² who found similar results in the teaching of algebra. In this investigation a four-week program was carried on under two teaching methods: one "an abstract, verbalized, deductive method with concepts defined and presented by the teacher, followed by practice exercises,"²³ and the other "a concrete, non-verbalized, inductive method, with students guided, through experiences involving applications, to discover and verbalize concepts."²⁴ Sobel found that the bright students learned and retained better through the inductive approach (contrary to the way by which these concepts are presented in texts) and the performance of the average student was just as good under this approach. In addition, under the inductive approach there were developed certain attitudes and appreciations in the meaningful learning of mathematics. A study by Oscar Schaaf demonstrates also that "student discovery procedures are quite satisfactory from the standpoint of teaching algebraic principles."²⁵ The challenge of the seminar on

²¹Ibid., p. 119.

²²Max A. Sobel, "Concept Learning in Mathematics," The Mathematics Teacher, XLIX (October, 1956), 425-430.

²³Ibid., p. 425.

²⁴Ibid., p. 426.

²⁵Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," p. 337. Unpublished doctoral dissertation, The Ohio State University, 1954.

teaching the calculus to devise means of increasing and testing the deepening of meaning and understanding on the university level has been largely strengthened by the results of the above studies.

Indeed, there have been a few investigations on the efficiency of various methods and emphases in the teaching of college mathematics. One is that of Albert E. Filano²⁶ in which he examined the effectiveness of a "student participation method" in college algebra and in analytic geometry. In the experimental group in college algebra, in which there was a minimum of one-half the class time given to student participation (as contrasted with listening and watching in the control group), there were significant gains in performance; results in analytic geometry groups were not significant. Filano also laments the lack of research in the teaching of college mathematics, which, he points out, is under more adverse criticism than the teaching of any other subject on the college level. Another more recent study is that of Sarah Burkhardt,²⁷ which not only indicates a rising interest in questions on learning the calculus, but the dissertation itself emphasizes again the need for teaching procedures which promote understanding. Dr. Burkhardt found that students gained in understanding and competence and in applications of calculus to related areas when more precision was employed in definitions and more care was given to the formation of

²⁶Albert E. Filano, "Study of Effects of Various Teaching Methods on Achievement in College Algebra and Analytic Geometry with Special Inquiry into Student Participation," (Abstract), Pennsylvania State Review of Educational Research, VI (May, 1955), No. 2, 13-16.

²⁷Sarah Maybelle Burkhardt, "A Study of Concept Learning in Differential Calculus," Dissertation Abstracts, XVI, No. 11 (Ann Arbor: University Microfilms, 1956), 2102.

concepts--also that "advanced calculus definitions" were well within the grasp of beginning students.

Support for a Student-Centered Plan

Such a method as that proposed does not seem to be without support. Saxelby, even before our modern ideas on meaning, pointed out the value of experimental or numerical or arithmetical and graphical methods. He says, "It is only after such a course that most students are ripe for the analytical methods of differentiation. . . (and) . . . analytical methods of integration."²⁸ Saxelby drives hard at this point when he defends his approach by saying that "this intuitional direct vision method is intended, not to take the place of, but to prepare the way for a more rigorous analytical study of the subject."²⁹ It is difficult to keep from quoting Saxelby further; he writes:

The logical order of a mathematical subject may not be the same as the best educational order, or as the order of its historical development.

The most natural method of advance is by a series of successive approximations to logical rigor, and, in fact, this is the way in which the subject has actually grown up. . . . The process by which the science itself was formed is also the most natural for the mind of the student.³⁰

Courant seeks to help the student gain understanding by giving " . . . due credit to intuition as the source of mathematical truth. . . ."³¹

²⁸Saxelby, op. cit., pp. v, vi.

²⁹Ibid.

³⁰Ibid., p. vi.

³¹Richard Courant, Differential and Integral Calculus (translated by E. J. McShane), New York: Interscience Publishers, Inc., 1937, Vol. I, p. vi.

Kells writes that in his text "a strong emphasis has been placed on the role played by intuition because of its power to suggest the basic concepts of mathematics, to give the driving force for investigation, and to furnish the idealization used in applications."³² Bernard Meland urges us to take steps toward getting "inside" the student when, writing on higher education, he asserts that much of our teaching has resulted in ". . . almost wholly a knowledge about with little or no acquaintance with the creative thrust (*italics mine*) which inheres in every event or act to which the intellect attends."³³ He compellingly advises that we should encourage the "culture of the human spirit" by which he means "the stimulus of the imagination and the cultivation of the individual's powers of appreciation."³⁴ "Learning," Meland writes, "can be deepened and be made more genuinely human as well as beneficial as a human act if it appropriates the procedures of both creative artist and critic."³⁵ Can the calculus be used as a tool to develop such a "human spirit" and will a method which "appropriates the procedures of both creative artist and critic" be effective in producing more understanding and improved competence among calculus students?

³²Lyman M. Kells, Calculus (New York: Prentice-Hall, Inc., 1944), p. v.

³³Bernard E. Meland, Higher Education and the Human Spirit (Chicago: The University of Chicago Press, 1953), p. 80.

³⁴Ibid.

³⁵Ibid., p. 81.

To recognize an urgent need for improvement is not difficult; to be inspired to think in this direction by such distinguished writers as these mentioned is the inevitable result of their challenging ideas; but to develop plans, procedures, and teaching materials which will thus improve the teaching of the calculus is an entirely different matter. This, however, is the proposal.

The Student-Experience-Discovery Approach

Just as one would never think of defining a derivative unless the student understood the meaning of "change," so, perhaps this proposed approach should not be enlarged until one has clarified some of the methods of teaching which are considered classical. This will not only furnish some "blocks" upon which variations of method are erected but it will help also to place the so-called student-experience-approach in proper perspective. In this discussion a method shall indicate a means by which the student learns a concept or develops a skill or an appreciation. If, for example, a student is told, this is called the lecture method; if he is provided with materials whereby he is led experimentally to perceive a result which he accepts or which he proposes as a hypothesis subject to investigation by other methods, then he is said to be using a laboratory method.

A Background of Methods

Arthur Schultze very succinctly summarizes methods of teaching mathematics and indicates their merits and disadvantages in his book

on teaching mathematics in secondary schools. It is true that this book is forty-four years old and, although names have changed and few references are made now to all the methods as he presents them, yet Schultze is still quoted by modern writers as a reliable source of ideas on the teaching of mathematics. The Schultze classifications follow:

- A. synthetic and analytic methods
- B. inductive and deductive methods
- C. dogmatic and psychological methods
- D. lecture and heuristic methods
- E. the laboratory method

Each of these methods refers to a different phase of presentation of the subject, and consequently they do not exclude one another,³⁶

that is, a synthetic method can be "inductively approached" as it is in high school geometry when the usual argument is used to show that the sum of the interior angles of a convex polygon is $(n-2)180^\circ$. This may be a psychological method also or the argument could be made by heuristic and lecture methods. It is not difficult to agree with Schultze that many times we use combinations of these methods. We now examine some of these more closely.

A. Synthetic and analytic methods. The synthetic method, as it is commonly understood, is one in which reasoning is done from the known to the unknown. It is a method with which we first become

³⁶Arthur Schultze, The Teaching of Mathematics in Secondary Schools (New York: The Macmillan Company, 1914), p. 30.

acquainted, in ordinary geometry, where we argue that given A, then it follows that B is true, then C is true, and so on. Synthesis shows every step but it does not explain why the step is taken; it convinces but it does not reveal the plan of the demonstration. The synthetic way is not the method by which proofs are discovered and if they are forgotten they are difficult to construct. Proofs which are synthetic, however, are usually short and very elegant. They are demonstrations of mathematical beauty, and one is often thrilled and amazed at the power of man to create such (sometimes) logical beauty.

The analytic method, on the other hand, proceeds from the unknown to the known. Its general pattern is: I want to show that C is true. Now C is true if I can show that B is true, and B is true if I can show that A is true. Oh yes! A is true for _____--hence C is true. This is the method of analysis and, in general, it is the way that mathematicians work. The method of analysis is lengthy and it is not elegant but it is the only method which accounts fully for each step of the demonstration and it is a good method by which to discover proofs, or to rediscover them when forgotten. Schultze says: "Analysis is the method of discovery, synthesis the method of concise and elegant presentation."³⁷ It is often recommended that teachers help students to discover proofs by analysis and then to organize them synthetically. Schultze comments that "exclusive synthetic teaching may be excusable

³⁷Ibid., p. 36.

when used with university students, who are able to analyze for themselves, but even there bad results will frequently follow."³⁸

B. Inductive and deductive methods. Induction as a method of teaching makes use of examples to help the student suggest hypotheses which can then be tested by whatever methods seem appropriate. Polya calls these hypotheses "conjectures."³⁹ Some advantages of this method are that the examples are often concrete, most students understand readily what is being done, and, with enough examples, most students can make a generalization. From the standpoint of teaching one disadvantage is that several examples are required and that the progress may be slow. Moreover, simply making generalizations from the study of examples and doing nothing more, does not give the student real mathematical experience.

Deductive methods, on the other hand, give results based upon previous assumptions and theorems, and the student is aware of the logical dependence of the conclusion upon the foregoing structure; furthermore, such methods are relatively quick. It is true, however, that to make acceptable deductions in mathematics requires a rather specialized type of skill which is sometimes difficult for students.⁴⁰

³⁸Ibid.

³⁹B. Polya, Induction and Analogy in Mathematics (Princeton: University Press, 1954), p. 7.

⁴⁰Harold P. Fawcett. Lectures. Education 840. The Ohio State University. Summer, 1953.

C. Dogmatic and psychological methods. In the so-called dogmatic method, rule and rigor are the chief concern; it utilizes models of proofs and examples which the student follows in doing problems. It is noted that even in this supposedly educationally enlightened day many texts and many teachers essentially still say to the student "this is the way it is done" or "here are the rules" and the student has little or no idea of how these rules are formulated, let alone having had a part in their development. This should not be confused with the lecture method to be discussed later. Lectures are not necessarily dogmatic in character. Psychological methods consider not only the scientific aspect of the subject but also the mental development of the student himself. Rigor is advocated only as far as the capacity of the student justifies it. As to rigor Schultze comments that "an exactness not understood by the student is not an exactness for the student" and that "over-rigorous teaching will not lead to rigorous thinking."⁴¹

D. Lecture and heuristic methods. In the lecture method the student listens to a logically organized exposition. The teacher gives; the student receives. Lectures, however, have the advantage that a large amount of material can be presented in a given time, the method can be used for large audiences, the logical sequence of ideas is not interrupted, and it is a comparatively easy method for the teacher to use. Lectures are sometimes very inspirational, and, when the material is

⁴¹Schultze, op. cit., p. 41.

skillfully organized and presented, the student can be shown much and he can be motivated deeply.

The lecture method also has limitations and some of its drawbacks merit consideration. One who is receiving information about mathematics is not really studying mathematics in which he comes to certain conclusions from certain given facts, for learning is an active process and the student must become involved in "creation" rather than passively accept unexamined ideas. Too, the lecture may proceed so rapidly (for any one student) that there is little comprehension and the student may not really understand the full significance of its content. Essential points not understood may make the remaining matter unintelligible. A lecturer on mathematics is often not in contact with the members of the class as a teacher should at all times be.

The term "heuristic" originates from the Greek H E U R I S K E I N which means "to discern." Heuristic means "to discover or to reveal."⁴² The heuristic method, then, attempts to guide the student so that he discovers as much as possible for himself and the provision of direct information is reduced to a minimum. It uses leading questions to guide the student so that he discovers as much as possible for himself and the provision of direct information is reduced to a minimum. It uses leading questions to guide individual student thought. If these questions are directed to the class as a whole then this is called the genetic method. Among the advantages claimed for the heuristic method are that

⁴²Webster's Dictionary. New International Second Edition.

- a. students think for themselves and they are not merely listening or taking notes.
- b. students acquire a real understanding of the subject under study.
- c. interest of the students and resulting willingness to work are greater than when direct methods are used.
- d. teachers are in quite complete touch with the thinking and progress of the students.

Of course, there are some disadvantages, also, in that

- a. it is slow, especially at the beginning.
- b. sometimes it is difficult to provide proper experiences from which students are guided to discover certain facts.
- c. it is difficult to maintain the true spirit of the heuristic method.

Recently Polya has written on the heuristic method and the reader is informed that his "present booklet is an attempt to revive (the) heuristic (method) in a modern and modest form."⁴³

E. The laboratory method. The laboratory method bears some resemblance to the heuristic in that it proposes also to lead students to the discovery of mathematical facts. However, this is not done by questions, as in the truly heuristic method, but through experiments performed by the pupil. Some good features are that the laboratory method

⁴³G. Polya, How to Solve It (Princeton: Princeton University Press, 1946), p. 102.

proceeds from the concrete to the abstract which is the way by which the human race has learned; the concreteness makes mathematics interesting to more students--and especially to younger students; it emphasizes "doing" rather than merely "following." As in the heuristic method, there are some weakness in the laboratory method also: it is not easy to "make" the student discover by experimental methods and the work may progress exceedingly slowly. Further, the method must be used cautiously for there is danger in laboratory work degenerating into a type of busy work, and, unless working with "things" helps the student to make generalizations or to establish mathematical principles, which may be tested later by other means, there has been no training in actual mathematical thinking.

Student-Experience-Discovery Method

The student-experience-discovery method, therefore, is characterized as a "pattern of attack" rather than as another method co-ordinate with the so-called classical methods. It is a philosophy of teaching which will permit the use of any or all of the classical methods to provide the student with exploratory and instructive experiences, to furnish means by which he can make discoveries, or to assist him to organize material in his own way. That is, this proposed approach in the teaching of the calculus may use the laboratory method to help formulate some of the problems of the calculus; it may use the genetic or the heuristic method to explore the algebra of sequences; it may employ deduction to connect hypotheses suggested on derivatives with theorems

on limits; it may include the use of the lecture method to emphasize various parts of the calculus as a unified whole. This pattern of attack is characterized by an emphasis on beginning in areas in which fruitful exploration can take place and then formulating generalizations and abstractions later. It keeps in mind that ". . . progress is not made along a straight line from the fundamentals to the generalized theorems. This is the way the final learning can be organized, and it is frequently so organized in texts, but it is not the way most students grow in wisdom."⁴⁴ Furthermore, a teacher in whose work this spirit prevails must be described as

(one) who has a balanced emphasis on the various phases of learning. He uses the heuristic method only so far as pupils need directed questions in the quest of their learning. He has an experimental attitude that allows freedom of approach in learning to the degree that the experiment is headed toward a desired concept, or relationship, and directs the experiment back to fruitful approaches when the student is adrift. He recognizes that drill can be dangerous and boring as well as good habit formation procedure. He balances each step from concrete experience to semiconcrete representations, to words and symbols, to generalized abstract theorems through proper evaluation and by appropriate attainable challenges. He also recognizes that not all learning begins in concrete material objects, but that much new learning may start in already learned abstractions.⁴⁵

It must be emphasized that this method makes use of the contribution of students--that the subject grows by efforts of the teacher and student alike and that the student shares the development of the subject as an exciting adventure.

⁴⁴The Learning of Mathematics; Its Theory and Practice, op. cit.,
p. 347.

⁴⁵Ibid.

Polya helps to describe this approach by adding another facet to its general nature when he suggests a "second aspect" of mathematics:

Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself; mathematics 'in statu nascendi,' in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public.⁴⁶

He further proposes a plan for developing new ideas which is close to the subject of this study when he says ". . . we have discovered an interesting result but the reasoning that led to it was merely plausible, experimental, provisional, heuristic; let us try (now) to establish it definitely by a rigorous proof."⁴⁷ Polya further quotes Leibnitz as saying: "Nothing is more important than to see the sources of invention which are, in my opinion, more interesting than the inventions themselves."⁴⁸

Perhaps the above discussion has helped to develop a frame of mind for an attempted more formal definition of a student-experience-discovery approach. Such a method has these characteristics:

- a. Important ideas of the calculus are introduced through numerical problems and are slowly nurtured toward generalized results.

⁴⁶Polya, How to Solve It, p. vii.

⁴⁷Ibid., pp. 106-107.

⁴⁸Ibid., p. 112.

- b. These generalized results are also reached through developmental procedures carefully planned and timed to suit the growing insight of the students.
- c. From these results are suggested assumptions, definitions, and hypotheses to be tested.
- d. On the basis of student explorations, student contributions, and teacher-student discussion, a logical structure is built with the origin and role of assumptions, definitions, and theorems, clearly demarcated and recognized by the student.

Aims of this Study

There are several objectives to this study. Simply out of respect to the student as a person and to his potentialities, one purpose is to test and discuss the possibility that teaching can be so geared that the student is not denied some of the thrill of discovery and creativity which it is the deep pleasure of those who create mathematics to experience. To deny this, it is held by the writer, denies them the essence of a real education. With regard to competence, the aim is to present data on the effectiveness of using the student's proposals, the ensuing discussion, the subsequent testing of hypotheses, and the student's own organization of his materials, on deepening his understanding and elevating his level of accomplishment and making his calculus more useful. From the standpoint of the teacher, this study reviews the

productivity of methods which emphasize taking the student from where he is, giving him opportunity to think about the calculus, and helping him to organize in into a workable system of knowledge under conditions which encourage the student to do much of this himself. Another aim is to furnish results on the use of the seemingly promising advice of J. W. Young on the teaching of algebra and geometry, but which could apply to the calculus as well:

The trouble in brief is that authors of practically all our current textbooks lay all emphasis on the formal logical side, to the almost complete exclusion of the psychological, which latter is without doubt far more important at the beginning of a first course in algebra or geometry. . . . it is necessary to arouse his (the student's) interest and then let him think about the subject in his own way.⁴⁹

or, as Young continues,

Let the teacher once fully realize that his science . . . is alive and growing. . . and he will bring to his daily teaching a new enthusiasm which will greatly enhance the pleasure of his labors and prove an inspiration to his pupils.⁵⁰

or, as Polya puts it: "The mathematical experience of a student is incomplete if he never had an opportunity to solve a problem invented by himself."⁵¹ This research gives results of putting to test a plan which makes provisions for translating into practice the principle suggested by Thorndike when he says: "Knowledge which is useful for problem-solving is organized knowledge, organized on the basis of the structure of the field and in terms of significant generalizations,

⁴⁹J. W. Young, Fundamental Concepts of Algebra and Geometry (New York: MacMillan Company, 1911), p. 5.

⁵⁰Ibid., p. 7.

⁵¹Polya, op. cit., p. 66.

organized by the learner and applied by him in a variety of contexts."⁵²

To summarize the above objectives in a more explicit manner, it is aimed in this report to present research methods and results on a student-experience-discovery method of teaching calculus which tests the assumptions that

- a. experiencing the growth of the idea of the derivative and the integral from numerical considerations fosters better understanding of the meaning and use of these concepts than if they are presented in more traditional ways;
- b. student formulation of hypotheses and theorems with only guidance from the teacher insures a better understanding of logical relations and the appreciation of calculus as a logical structure than if the theorems were presented by lecture or simply referred to in the text;
- c. the student-centered approach encourages a better understanding of the nature and growth of mathematics than if the material is accepted passively from a text;
- d. through these procedures the student is stimulated to feel the thrill of discovery.

⁵²R. L. Thorndike, "How Children Learn the Principles and Techniques of Problem-Solving." Learning and Instruction. Forty-ninth Yearbook, Part I (Chicago: National Society for the Study of Education), 1950, p. 212.

Detailed Analysis

It now becomes evident that the general problem resolves itself into more detailed problems:

- a. selecting ideas in the calculus to encourage student discovery as well as to insure the proficient use of calculus skills;
- b. providing a sequential development most conducive to helping the student become aware of a logical organization of the calculus;
- c. preparing study-guides designed to direct student experiences which lead to discovery and which deepen understanding of the calculus;
- d. making selections, provisions, and preparations as mentioned above to cope with the time-pressure and the pace of university courses;
- e. securing evidence concerning the extent to which these objectives have been realized.

General Plan of the Dissertation

The presentation of the nature, the study, and the results of this problem has been divided into several parts or chapters.

Chapter II includes a summary of the experimental conditions and data concerning the students involved in this experiment. This chapter also contains a discussion on the content which was planned for the experimental course, suggestions on possible "timing" or topics, and data on methods and procedures.

Chapter III reviews in some detail teaching procedures and classroom experiences. An attempt is made to convey a word-picture to describe the atmosphere in which the teaching was done: the discussion of details or new ideas, the origin of suggestions, and communications of students on their conquest over a problem in understanding the calculus.

In Chapter IV is described in some detail the content which resulted in the experimental course. It presents a formal outline of the topics studied along with skills developed in the course. There are included also some methods of argument, some interesting discoveries or contributions on the part of students, and some comments on noteworthy items related to content which are found in notebooks.

There comes the important discussion of evaluation in Chapter V. This includes the results of a statistical study and cross-testing of sections based upon the giving of the same teacher-made tests to experimentally- and traditionally-taught groups. Furthermore, there is an evaluation by students as expressed by their comments on this method of teaching calculus as well as a few comments by the investigator. Evidence of creative and original work and thinking on the part of the students will be presented as an aid to weighing the effectiveness of this approach.

Although it may appear at a superficial glance that there is duplication in Chapters II, III, IV, and V, it must be explained that the items of content found in three of these chapters are there merely

to illustrate or support claims made therein. Only Chapter III is directed to formal content.

A general summary of this investigation is made in Chapter VI. In this are included general conclusions as to the performance of students and the content of the course as developed by this method. As gained from this study there are mentioned problems which have come to light and some precautions which a teacher using this method must take. Some recommendations and suggestions conclude the discussion.

CHAPTER TWO

EXPERIMENTAL DATA AND CONDITIONS

The Calculus Sections

There was a time when an experiment of this kind required an attempt to establish two sections in which students were paired so they would be alike with respect to as many characteristics as possible and so that the only variable (it was usually claimed) was the way in which the calculus would be handled in each of the groups. Any difference in depth of understanding or in performance of the techniques of the calculus would then be attributed to the difference in the teaching method used in the experimental and the control groups.

This study was carried on with experimental and control groups but the sections were not matched for several reasons. First, it is very questionable if it is possible to set up two sections equally matched with respect to measurable features and to have the groups identical with respect to mathematical ability, mathematical interest, and equally appreciative of things and processes mathematical. Because of his inner convictions, whatever they may be, it would be difficult for the same experimenter to teach both sections in keeping with the spirit of this experiment. Indeed, so many factors enter that two instructors with the same method on matched sections might be expected to have different results. These complexities, however, according to

Helen M. Walker, seem not to be a problem anymore, for newer methods of analysis have been developed so that the "era of exhaustive person-to-person matching appears now to be over."¹ Second, at Kent State University, where this research was done, it was impossible, even if it had been desirable, to attempt to construct such matched sections, for the students were enrolled in the calculus sections under different instructors for reasons usually existent on a university campus. Third, even if it were possible to have matched sections, it is believed by the investigator that such a classification of students is not of primary importance for the purpose of this study. In this experiment one is interested not only in proficiency but also in learning whether or not it is possible for the student or a group of students to establish a common-sense basis and structure of the calculus much by their common effort and in helping them feel the joy and thrill of discovery and invention. That is, it is an important aim of this study to see if more permanent values can be realized.

In this research the same teacher taught all the experimental groups but there were different teachers for the control sections; all were un-matched and there were no special circumstances imposed. There are advantages to such an arrangement. Very often, if there is an attempt to keep all variables under control except one, educational studies must be carried on under the direction of statistical experts

¹Helen M. Walker, "Methods of Research," Review of Educational Research, XXVI (June, 1956), 334.

of such high proficiency that the study becomes detached from the person for whom the study will do the most good--the teacher in the classroom. In fact, it has been long recognized that research for teachers carried on by teachers is a very valuable and applicable type of study. This kind of investigation, called "action research," according to Walker, "is currently a 'frontier movement.'"² Action research is well introduced and described by Stephen M. Corey when he suggests that those whose practices which will be affected should carry on the studies. He writes:

I have lost much of the faith I once had in the consequence of asking only the professional educational investigator to study the schools The studies must be undertaken by those who may have to change the way they do things as a result of the studies. Our schools cannot keep up with the life they are supposed to sustain and improve unless teachers . . . continuously examine the work they are doing. . . they must use their imaginations creatively and constructively to identify the practices that must be changed to meet the needs and demands of modern life, courageously try out those practices that give better promise, and methodically and systematically gather evidence to test their worth.

This process I call action research . . . (it is) . . . research that is undertaken by educational practitioners because they believe that by doing so they can make better decisions and engage in better actions.³

Hence the physical conditions of this study seemed to lead to a type of action research. As such there was no attempt to set situations

²Ibid., p. 335.

³Stephen M. Corey, Action Research to Improve School Practices (New York: Bureau of Publications, Teachers College, Columbia University, 1953), p. viii.

whereby the only variables were those in teaching methods, but this investigation progressed "in the complicated psycho-sociological climate of on-going activities."⁴ Such procedures may be "lacking in precision (but) the results . . . have meaning for practice because they derive from an inquiry carried out in a real situation."⁵

This student-experience-discovery approach was tried on sections of the first-quarter calculus, Mathematics 263, at Kent State University, during the academic year 1957-1958.

There were seven students who first attempted the study of calculus by this student-experience-discovery approach during the fall quarter of 1957; all these volunteered from another section of Mathematics 263 to comprise a pilot group for this investigation. These students came into this section for various reasons: one volunteered because she planned to become a teacher and was therefore interested in this approach; another had failed calculus previously and was looking for a "different method." Some were interested in this experiment as something new; still others, it seemed, were willing to be helpful to one wishing to conduct an experiment. This group was used to help test the effectiveness of the study-guides which had been previously prepared and to gain information on the rate at which the students could progress through the course as written--in fact, this was a study in miniature. Also, to be sure, there was interest in the effect of this approach

⁴Ibid., p. 148.

⁵Ibid.

upon these students and the total evaluation includes some results from this group, although data from this section are not used in the formal statistical study.

This research was undertaken for three quarters and with three experimental classes including the small pilot group. In the fall quarter the pilot group was used and modest studies and comparisons were made; during the winter quarter the formal experimental work began with an experimental Section One and a traditional Section A; through the spring quarter there were an experimental Section Two and a traditional Section B. The students in these classes were not selected in an extraordinary way. They were in the respective sections either by choice or because they could not schedule calculus at any other time. In the spring quarter, however, one large section of the first-quarter calculus was divided and a few students were placed in the experimental section by assignment. It should be commented that all these sections, except the pilot group, were "off quarter" in the sense that the normal mathematical program enables students who have no preliminary difficulties to begin the first-quarter calculus in the fall of the sophomore year. A few of the students in these sections, however, had previous contact with the calculus for a period of only several weeks or were failures from preceding quarters.

The students were distributed throughout several class ranks, as shown in Table 1.

TABLE 1
CLASS STANDING OF STUDENTS IN VARIOUS SECTIONS

	Experimental Sections		Traditional Sections	
	1 (winter)	2 (spring)	A (winter)	B (spring)
Freshmen	1	1		
Sophomore	14	10	10	5
Junior		6	1	4
Senior	5	1	1	3
Special	1*			1**
Total	21	18	12	13

*An ROTC instructor.

**A University High School student.

The average age of the students in the various sections differed only slightly, although there was considerable spread of ages. Too, some of the students had the additional experience of military service; this may have produced various effects: more student maturity, a greater number of experiences with which to associate mathematical ideas, knowledge of situations where mathematics applies, and, of course, in some cases, more time for disuse to produce less proficiency in mathematics. The number of girls in the classes was very small. A summary of information on these items is given in Table 2.

TABLE 2

AGE DISTRIBUTIONS IN VARIOUS SECTIONS, MILITARY
SERVICE, SEX OF STUDENTS

Age	Experimental Sections		Traditional Sections	
	1	2	A	B
17				1
18				
19	6	7	2	
20		4	3	3
21	4	1	1	1
22	3		1	
23				4
24	1	2	1	1
25	3	1		2
26	3	1	1	
27	1			1
28			2	
29			1	
30				
32		1		
37		1		
Average	22	22	23	22
Number in military service	9	4	6	6
Number of girls	2	2	2	0
Number of boys	19	16	10	13
Total	21	18	12	13

Another item of data which might influence progress in beginning calculus is the presence or nonpresence of a "gap" between the work in analytic geometry and that in the calculus. Table 3 shows the situation in the various sections in this regard.

TABLE 3
PRESENCE AND LENGTH OF TIME GAP BETWEEN THE STUDY OF
ANALYTIC GEOMETRY AND THE CALCULUS

	Experimental Sections		Traditional Sections	
	1	2	A	B
No gap	15	14	11	10
Number of quarters gap:				
One		3	1	2
Two		1		
Three	1			1
Six	1			
Eight	1			
Ten	3			

The American Council of Education (ACE) Psychological Test scores were available for most of the students and the range, the median, and the first and third quartiles for the various groups are given in Table 4. These data are used later in a statistical study to see if

differences in results in the experimental and the traditional sections are significant.

TABLE 4
ACE PSYCHOLOGICAL TEST SCORES FOR VARIOUS SECTIONS

	Experimental Sections		Traditional Sections	
	1	2	A	B
Range	75	83	37	77
Q ₁	42	70	54	20
Q ₂	71	80	78	70
Q ₃	78	86	82	73

Digression on Attitude Testing

It was thought that there might be value in having some measure of "attitude toward mathematics" to assist in further characterizing the calculus sections under study.

Since no attitudinal tests for university age students were found, efforts were made to construct one. Experimental work on attitudes toward mathematics in the elementary school has been done by Corey, in which he was interested in change of attitude during a year's contact with mathematics, and, although he admits that "to determine whether or not real changes have occurred is very difficult, especially when one is forced to resort to pencil-and-paper tests,"⁷ it was decided to follow

⁷Stephen M. Corey, "Measuring Attitudes in the Class Room," Elementary School Journal, XLIII (April, 1943), 458.

techniques used by him to construct a test for use in this investigation. Accordingly, the following procedure was used:

1. Students enrolled in classes in college algebra and trigonometry at Kent State University and at The Ohio State University during the summer of 1957 were asked to write statements either on their own attitudes and thoughts about the study of mathematics or expressions which they had heard others make. Some opinions were also collected from other sources.⁸ These expressions were read and assembled into some one hundred different items.

2. Several teachers, college students, and capable upper classmen drawn from a variety of walks of life and interests in the writer's local area then examined these statements and classified them into "favoring the study of mathematics" or "opposing the study of mathematics" or "ambiguous." These items which were ambiguous concerning the kind of attitude toward mathematics which was expressed, or, on which there was not eighty per cent agreement among members of the committee, were rejected. The result was a list of forty items each of which expressed a reaction to the study of mathematics. Some samples appear in Table 5 and the complete test is included in Appendix A. Those which the committee labeled as favorable to the study of mathematics are indicated with an asterisk. In order to furnish practice in scoring

⁸National Council of Teachers of Mathematics. Emerging Practices in Mathematics Education. Twenty-Second Yearbook (Washington, D.C.: the Council, 1954), pp. 408-09.

and simply to see what results would be obtained the test was tried on a group of upper classmen of the local high school.

TABLE 5
A SELECTION OF ITEMS FROM THE ATTITUDE TEST

_____	5. Mathematics is not one of my favorite subjects.
_____	8. The study of mathematics is one of the hardest chores I have ever undertaken.
* _____	12. It is one of the finest exercises for one's mind.
_____	16. I hate the stuff.
* _____	17. Mathematics is great fun; I love it, really.
_____	23. I have something of a fear of mathematics.
* _____	34. Sometimes I enjoy the challenge presented by a mathematics problem.
_____	39. I avoid using mathematics because I am not very good at figures.

3. By the use of the above sampling one can explain how the test was scored. The directions for taking the test were as follows:

This is not a test in the sense that any particular statement is right or wrong. All these statements represent opinions which people hold about mathematics or the study of mathematics. Indicate whether you agree or disagree with the statement by placing a + sign before all the opinions with which you agree and by placing a - sign before those with which you disagree. If you are uncertain if a statement represents your attitude make a question mark (?) before the statement. Now go back and draw a circle (○) around the + sign with which you agree strongly and a circle (○) around the - sign where you disagree strongly.

In scoring the tests, each item which favored the study of mathematics was identified and each of these was scored as follows: five points for $\oplus$, four points for +, three points for ?, two points for -, and one point for $\ominus$. The remaining items whose expression indicated an attitude unfavorable toward mathematics were therefore scored in this manner: five points for $\ominus$, four points for -, three points for ?, two points for +, and one point for $\oplus$. The total score was then regarded as a measure of one's "favorableness" to the study of mathematics. Since the lowest possible score is $40 \times 1 = 40$ and the highest is $40 \times 5 = 200$ it was decided to break the range of $200 - 40 = 160$ into intervals of 32. The intervals of division were characterized in the following way:

168 - 200	-	very favorable
136 - 167	-	favorable
104 - 135	-	indifferent
72 - 103	-	unfavorable
40 - 71	-	very unfavorable.

This test was given to all the calculus classes used in this study at the beginning of their respective quarters. Results are shown in Table 6, and they indicate a generally favorable reaction to the study of mathematics among the calculus students. To see if the test demonstrated any real discriminating power it was decided to go further and ask students in some of the freshmen college and pre-college courses to take this test, for it seemed reasonable to expect more contrast in

in attitude among students in these courses. It might have proved interesting to give this test to nonmathematics classes also but this was not done. Results of this further study are also shown in Table 6.

TABLE 6
RESULTS ON ATTITUDE TESTING

Groups Tested	Range	Median Score	Interpretation
Calculus Sections			
Section 1 (experimental)	127-173	154	favorable
Section 2 (experimental)	133-176	148	favorable
Section A (traditional)	139-185	159	high favorable
Section B (traditional)	124-180	149	favorable
Other Sections in Mathematics			
Mathematics 15 (precollege)	106-162	131	indifferent
Mathematics 26 (precollege)	106-174	141	low favorable
Mathematics 172 (commercial algebra)	91-185	127	indifferent
Mathematics 165 (fundamentals, terminal)	94-160	130	indifferent
Mathematics 140 (business mathematics)	91-168	137	low favorable
Mathematics 160 (college algebra)	113-179	138	low favorable
Mathematics (cadet teachers evening course)	99-172	152	favorable

Digression on the Calculus Pretest

To provide further comparison of the classes which participated in this study there was prepared a preliminary test on some of the mathematical skills used in first-quarter calculus. Items for use in this test were selected from an examination of texts, knowledge of the exercises to be used in this course, and by reference to a classic survey by Fagerstrom⁹ on topics in elementary mathematics used in the calculus. The students were asked to take this as well as the attitude test outside class during the first week of the quarter. A copy of the pre-calculus test is included in Appendix B.

The information thus secured revealed many things. In addition to comparing the calculus groups as to preparation, certain weaknesses¹⁰ in mathematical skill and understanding were revealed. The results of these pretests for the various sections are shown in Table 7.

The sections were also compared as to average grades on the three previous courses in mathematics: college algebra, trigonometry, and analytic geometry. To average these grades a correspondence was set up as follows:

A	11	B-	7	D+	3
A-	10	C+	6	D	2
B+	9	C	5	D-	1
B	8	C-	4	F	0

⁹William Henry Fagerstrom. Mathematical Facts and Processes Prerequisite to the Study of the Calculus. Contributions to Education, No. 572 (New York: Bureau of Publications, Teachers College, Columbia University, 1933).

¹⁰One of these seems to be that of interpreting and handling of negative exponents.

TABLE 7

A SUMMARY OF PERFORMANCE ON THE CALCULUS
PRETEST BY THE VARIOUS SECTIONS

	Experimental Sections		Traditional Sections	
	1	2	A	B
Range	70-234	56-189	100-233	96-227
Q ₁	108	150	139	119
Q ₂	168	166	163	153
Q ₃	190	176	176	209

The three letter-grades found in the records for each student at the mathematics office at Kent State University were averaged by computing the mean of the three corresponding numbers as gained from the matching on the preceding page. The mean of all these student averages was found in a similar manner. The results for the different sections are shown in Table 8. It is noted that these results are not consistent with those which are shown in Table 7.

This completes the study of characterizing features of the various sections made in an attempt to gain a picture of the type of students, their preparation, their attitude, their innate ability, and their previous experiences which could influence, to some extent, their progress in the study of the calculus.

TABLE 8
AVERAGE GRADES IN COURSES PREREQUISITE
TO THE CALCULUS

	Experimental Sections		Traditional Sections	
	1	2	A	B
Average of the Section	B-	B-	C+	C+

Data on Planned Content

In some chemical research the experimenter will have materials on which he proposes to impose certain conditions; the conditions and the materials are therefore both parts of the investigation; also, in this study, the given data include not only the student groups and their characteristics but also the intended materials in calculus, as well as other items to be described later. In this experiment it was planned to incorporate certain main topics into the calculus study, but, of course, it was expected that the inductive and discovery approach would sometimes lead the student into different, more specialized, areas as he pursued a "lead" or a topic which appeared inviting and which he wished to analyze further. In fact, provision was made for the possibility of such "excursions" into areas where future developments were planned.

Several representative texts were examined for ideas which might assist in the selection of desirable content for first-quarter calculus. These texts were selected with several aims in view;¹¹ it was thought that it would be well to have some of these in the sampling:

- a. earlier treatments of the calculus,
- b. books for practical training courses,
- c. texts encouraging more rigorous study,
- d. books which yield thought of foreign writers on the subject.

¹¹These texts are listed here for convenience:

Hollis R. Cooley, First Course in Calculus (New York: John Wiley and Sons, Inc., 1954).

Richard Courant, Differential and Integral Calculus (New York: Interscience Publishing Company, 1937).

Paul L. Evans, Mathematics for Technical Training. Calculus (New York: Ginn and Company, 1942).

Henry B. Fine, Calculus (New York: MacMillan Company, 1929).

William A. Granville and Percy F. Smith, Elements of the Differential and Integral Calculus (New York: Ginn and Company, 1911).

Lyman M. Kells, Calculus (New York: Prentice-Hall, Inc., 1944).

Gerhard Kowalewski, Grundzüge der Differential-und Integralrechnung (Berlin: B. G. Teubner, 1928).

Edmund G. H. Landau, Differential and Integral Calculus (New York: Chelsea Publishing Company, 1951).

E. A. Maxwell, An Analytical Calculus for School and University. Volume I (Cambridge: University Press, 1954).

Ross R. Middlemiss, Differential and Integral Calculus (New York: McGraw-Hill Book Company, 1946).

F. D. Murnaghan, Differential and Integral Calculus (Brooklyn: Remsen Press, 1947).

William R. Ransom, The Calculus (According to a New Plan) (Medford, Mass.: Tufts College Bookstore, 1949).

F. M. Saxelby, A Course In Practical Mathematics (London: Longmans, Green, and Company, 1907).

G. E. E. Sherwood and Agnus E. Taylor, Calculus (New York: Prentice-Hall, Inc., 1946).

Frederick S. Woods and Frederick H. Bailey, Elementary Calculus (New York: Ginn and Company, 1928).

There was an attempt made to determine the features of these texts with respect to the following points:

- A. What the initial concepts are and how they are introduced
- B. What meaning is given to the derivative
- C. What meaning is given to the integral
- D. The statement of the fundamental theorem of the calculus.

These texts and other sources were studied also to learn of possible devices or approaches helpful in clarifying the above ideas and in suggesting developments by which a student might be led to discover some of the calculus facts by himself.

The results of the study of even a few of the many, many tests show that there are different approaches to the calculus itself. To one who is interested in the logical thread which unifies the content of the calculus, this is of utmost importance. A summary of these studies relative to the above features follows:

- A. There seem to be essentially three approaches to introducing initial concepts of the calculus. These are
 - 1. a deductive approach based upon assumptions concerning the existence and properties of real numbers with a discussion of variables, functions, sequences, and limits. Fine, Granville and Smith, Middlemiss, and Kells wrote texts of this type.

2. a deductive approach based upon fewer assumptions.

These writers, as Kowalewski, Murnaghan, and Courant, begin with integers or rationals, define the irrational number by means of a sequence of nested intervals or a Dedekind cut, and include a much more rigorous treatment of preliminary ideas.

3. an inductive (arithmetical) approach. Texts by Woods and Bailey, Saxelby, and Evans, are examples of those which employ numerical methods to help build ideas.

In this experimental course the existence and properties of real numbers were assumed and arithmetical methods were used to develop ideas of sequence and limit and related topics. After these concepts were understood, however, deductive processes were employed to systematize the knowledge thus gained and to establish a logical basis for further work.

B. The few texts examined revealed general agreement on the definition of the derivative, although there were some variations of approach and expression. Some texts used more than one of these variations to help make this concept clear. Examples follow:

1. $\Delta y, \Delta x$ -four-step rule. This was used by such writers as Woods and Bailey, Middlemiss, and Kells.
2. $D_x y \equiv \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$. This was implied by all writers but this variation in expression

(or notation) was presented by Landau, Cooley, and Murnaghan. Some texts used a combination of 1. and 2.

3. inductive approach through speed, slope, rates. Wood and Bailey and Saxelby make much use of this method.
4. first defining a differentiable function. Landau¹² and Murnaghan¹³ do this and define the derivative at a point in terms of the condition that the function be differentiable at that point.
5. common limit of a sequence of quotients as the change in the independent variable approaches zero in any manner. Kowalewski¹⁴ emphasizes this aspect of the derivative.

The planned approach to the concept of the derivative in the experimental course has these features:

1. Use real or imagined physical situations treated arithmetically to develop the idea of rate of change. Gradually generalize these computations by the use of algebra to arrive at a more abstract formulation.
2. Use geometric situations treated in the same manner to develop the idea of the slope of a curve, $y = f(x)$,

¹²Op. cit., p. 72.

¹³Op. cit., p. 56.

¹⁴Op. cit., p. 55.

at a point, or, to develop the idea of rate of change of y with respect to x at a given value of x . Develop the idea that this slope of a curve at a point can be approximated to any desired degree of accuracy by the slope of a proper secant.

3. Stress the aspects discussed in 2. and 5. of the last paragraph.

All texts and courses meet functions $f(x)$ related to others such as $F(x) + C$ where $D_x [F(x) + C] = f(x)$. In the planned experimental program the $F(x) + C$, where C is an arbitrary constant, was called the antiderivative, and denoted by $D_x^{-1}f$, but authors differ here also: by some, this antiderivative is called an inverse derivative; by one, a converse derivative; some, as Fine, Saxelby, Granville and Smith, Sherwood and Taylor, and Landau, do not name this at all with reference to the derivative, but rather call this the indefinite integral.

C. There is perhaps the widest variance in developing the idea of the definite integral for the several viewpoints are conceptually quite different. The definition for the definite integral which was used in the experimental course is as follows:

Given a function $f(x)$ continuous over an interval $[a, b]$, then the definite integral

$$\int_a^b f(x)dx \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)h_i ,$$

where $h_i = h$ and $nh = b-a$. (In discussion it was brought out that the h_i need not be equal, the x_i

may be anywhere in the small subinterval whose width is h_i , and that piecewise continuity would suffice; furthermore, it was assumed that the above limit exists.)

There is also essentially the definition given by such writers as Cooley, Sherwood and Taylor, Murnaghan, Courant, Maxwell, and others: the definite integral is the limit of a sequence of sums of products.

Another prevalent approach to the definite integral is through the use of what is called the antiderivative in the experimental course. Texts as Woods and Bailey, Middlemiss, and Granville and Smith define the definite integral

$$\int_a^b f(x)dx \equiv F(a) - F(b) ,$$

where $F(x)$ is an antiderivative of $f(x)$. It should be noted that this definition of a definite integral has a meaning entirely different from that in the preceding paragraph.

Given $y = y(x)$ bounded on $[a, b]$ Murnaghan¹⁵ defines the area under the curve and then defines this to be the integral, while Landau¹⁶ defines the definite integral in terms of an "integrable function."

Some texts, as Middlemiss, Kells, Woods and Bailey, use both the antiderivative and the limit of a sum approach. They argue that the rate of change of area under a curve $y = y(x)$ from $x = a$ to $x = x$ is equal to the ordinate for any x , that is, that $D_x A = y$, whence $A = \int_a^x y dx$. This same area, they claim, can be evaluated by a method

¹⁵Op. cit., p. 250.

¹⁶Op. cit., pp. 281-282.

of taking the limit of a sequence of sums of areas of rectangles and therefore this limit is also an integral.

D. The fundamental theorem of the calculus in the experimental course is given as

$$D_t \int_a^t f(x) dx = f(t) \quad (a \leq t \leq b)$$

provided the function $f(x)$ is continuous on the closed interval $[a, b]$; from this the statement

$$\int_a^b f(x) dx = F(b) - F(a),$$

where $D_x F(x) = f(x)$, follows as a corollary.

In the tests surveyed, however, there were various presentations of the fundamental theorem. These may be summarized as follows:

1. the fundamental theorem of the experimental course.

Cooley and Murnaghan are examples of writers who do this.

2. $\int_a^b f(t) dt = F(b) - F(a)$. This is stated by Sherwood and Taylor; Maxwell calls it a "fundamental link."

3. $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i = \int_a^b f(x) dx$. Middlemiss represents writers who use this as the fundamental theorem.

4. no theorem has the name "fundamental" attached to it. Woods and Bailey, and Landau are examples of writers of texts of this type.

Since this study is not primarily an investigation of texts on calculus the above discussion has been restricted to a brief summary merely to indicate the diversity of approach and to emphasize that in

planning a course for experimentation such as this, it was not a simple task to decide upon logical structure and point of view concerning the basic ideas.

To resolve the dilemma which would have resulted in attempting to follow various treatments of the calculus the writer made his own choice of topics and organization. Since the derivative and the integral are of equal importance and their historical origins are conceptually different and the writer feels that these features should be emphasized, both these terms were defined independently of each other. These definitions are found on pp. 49 and 50 of this study. These two great disciplines--"derivatives" and "integrals"--are then connected through a "fundamental theorem,"

$$D_t \int_a^t f(x) dx = f(t),$$

which climaxes the logical development in this experimental first-quarter calculus. Other topics, problems, and applications were included to make the course as rich as possible. This was not only desirable on the part of the investigator but imperative as well, for permission was given to carry on this study only if it would not endanger progress of the students who might find themselves in second-quarter sections with different instructors.

The outline of the planned course can be presented best by including here the Table of Contents of the manual of study-guides which were prepared for teacher and student use. These contents appear in Table 9.

TABLE 9

TABLE OF CONTENTS OF EXPERIMENTAL COURSE

I. Problems of the Calculus**A. The General Rate Problem**

1. Estimating experimentally the speed of a falling body. Exercises.
2. Estimating the speed when a formula for distance is known. Exercises. Fundamental terms. Exercises.
3. Estimating speed with less numerical computation. Functions. Limits.
4. Estimating speed at any time t_b . Exercises.
5. Finding the slope of a curve. Graphs. Maximima and minima. Exercises.
6. General rate of change. Functions. General picture. Exercises.
7. Uses of changing rates. Related rates. Errors. Small changes.
8. General review exercises.

B. The General Area Problem

1. Approximating area (intuitive) under a curve. Numerical methods. Other methods. Exercises.
2. More work on areas. Exercises.
3. Approximation of work done on a spring. Exercises.
4. Use of approximating functions. Exercises.

C. The Fundamental Problems Recapitulated**II. A Systematic Look at the Calculus**

1. Sequences. Limits. Theorems.
2. Limits of functions.
3. Continuous functions.
4. Review.
5. General review exercises.

(continued)

TABLE 9 (concluded)

III. The General Rate Problem Treated

A. The Derivative

1. Definition developed from a previous process. Some Theorems. Exercises.
2. Making differentiation easier. Differentiation of products, quotients.
3. Differentiation of a function of a function. Exercises.
4. Inverse functions. Parametric functions.
5. Implicit functions.
6. Operation inverse to differentiation.
7. General review exercises.

B. Second and Higher Derivatives

1. Derivatives of derivatives and their interpretation. Use.
2. Derivatives and the study of rectilinear motion. Exercises.
3. Review exercises.
4. Higher derivatives of parametric functions.
5. Higher derivatives of implicit functions.

C. Inverse Rate Problem Continued

1. How can inverse differentiation be used?
2. Given slope to find equation of a curve. Finding the length of a curved segment.
3. A general formula for antidifferentiation of ax^n

IV. The General Area Problem Treated

A. The Definite Integral

1. Defined as a name for something we have developed before. Theorems. Refining our thinking. Fundamental terms.
2. Fundamentals, theorems, applications.

B. Evaluation of Integrals.

The Mean value theorem. Applications of integrals. The fundamental theorem of the calculus.

C. Some Applications of Integrals.

Areas. Work. Volumes. Force of liquids on a vertical wall.

V. General Review

The fact that the greater part of the outline concerns the general rate problem may cause the reader to think that a disproportionate amount of time was spent on derivatives with the consequent neglect of integrals. The writer submits these comments concerning this situation.

1. Many of the fundamental ideas as those of sequences and limits were developed under the general rate problem but they apply equally well to basic work on integrals.
2. Ideas and applications which were to lead to the integral were nurtured from a point early in the course but they received more systematic treatment near the latter third of the quarter.
3. It is customary to do much work on differentiation in the first-quarter calculus and more study of integration during the second quarter.

Data on Timing of Topics

This experiment was carried on under the assumption that the outline to be followed is not necessarily an absolute chronological schedule of topics and that the outline in Table 9 may not represent wholly a psychological order of presentation. The experience-discovery approach invites discussion and encourages exploration of promising avenues; in fact, a course so given could take different "minor turns" for each group of students studying it. Indeed, some deviations did occur; one of these is described in Chapter Three, pages 80-81.

The timing of remarks or topics or "sowing a seed" which may develop later is very important. Concepts then have time to grow, new ideas are not so strange to the student, and he has time to explore and experiment before he is required to generalize, which is often quickly expected in the traditional programs. The student, himself, should see the subject grow through his own contribution and his own effort.

More detailed general principles, then, which guided the teacher in the conduct of this experiment, follow:

1. Numerical exercises of an introductory nature were closely examined for as much information as possible to gain ideas to be used later.
2. Questions for study were suggested at timely places to help "sow seeds" which would develop later into big ideas.
3. Queries on validity of processes, which were really answered later, were introduced at opportune times to encourage student thinking about such items.
4. A discovery by the student was capitalized upon immediately no matter which way, for the moment, the course turned, as long as it furthered general progress in the outline.
5. Suggestions made by students were considered as working assumptions until proved or disproved or until it was decided to accept them as hypotheses.

Assumptions Concerning Methods and Procedures

For this study certain assumptions and conditions were applied to the experimental classes and the planned content. Precise procedures differed with situations but here are some guiding principles which the teacher tried to follow:

1. It should not be difficult for a student to grasp the central ideas of the calculus.
2. The student should be provided with experiences so he can draw conclusions about the calculus as independently as possible from other sources.
3. Student discoveries and suggestions should be used to help build the logical structure of the calculus.
4. The classes should be a place for review, for explanation, for discussion, for introduction to new material, and an investigation center.

The above discussion has indicated data about students, data about calculus content, and guides to planned procedures in the class room. In the next chapter an attempt is made to show how all these worked together while this experiment was being carried on. In other words, the next chapter describes the experiment in operation.

CHAPTER THREE

CLASSROOM EXPERIENCES

Introductory Statement

Before describing the actual classroom procedures and responses made by the students in this approach to the calculus, it is desirable to create a general picture to indicate how various instructional materials were used. It may be difficult, at times, to keep separate the procedures from the general results and content, for often the latter help to determine the former. Hence in this chapter, reactions will be described which help to reinforce or supplement the description of the classroom activity and the more formal content will be reserved for later discussion. Furthermore, since it is impractical to attempt to describe everything which took place, those features will be emphasized which show the nature of the student-experience-discovery approach.

Perhaps the most important means of instruction other than the classroom discussion were the study-guides in the calculus which had been prepared previously. These are found in Appendix C. Portions of this manual were distributed at various times; each subsection, as III. A. 1. or III. A. 2. or IV. B. 2. was considered as one day's assignment and the sheets were written with that in mind. Some students worked on these study-guides in class and frequently used them to direct

their pursuit into a new topic while difficulties of interest only to a few students were being discussed. Frequently these guides were used in class as a basis for the genetic development of new ideas and they always directed the students' study in preparation for the next class. These study-guides were sometimes checked in class, points were discussed, individual contributions presented, or the more difficult problems were worked. Answer-sheets for the problems were constructed and given to the students along with the study-guides. It was suggested that two hours a day be spent on these sheets and that they be handed in regularly with the name, the time spent, and the course identification written on each assignment. Some of the students followed this instruction and some did not; all sheets were handed in eventually, however. Upon receiving the completed study-guides, the teacher checked that the work was done, examined solutions to some of the problems, made instructive comments in places, and promptly returned the work to the students. A record of the daily time spent on the assignment sheets was kept and the average time for each section is shown in Table 10.

TABLE 10
AVERAGE TIME SPENT ON EACH ASSIGNMENT

	Section 1	Section 2
Average time	2.0 hours	2.3 hours

The text (Hollis R. Cooley's First Course in Calculus) was used for problem material when these exercises served the purpose of the course. The student was sometimes asked also to compare his generalized results with those of the text to give him confidence and to help him feel that he, too, was investigating and writing in the field of calculus. Also the use of the text gave the student assurance that although this was an experimental course, he was not being deprived of experience in the regular university calculus. Some students, no doubt, used the text for reference on theory and on how to work problems; this was not discouraged, for one purpose of this approach was to encourage the student to be an independent thinker and investigator.

The role of the teacher was very important and at the same time difficult. It was difficult, for the teacher had to decide when to say nothing, when to drop a hint, when to make an appropriate remark which might germinate into an idea, when to be a drill-master, when to be a guide in an investigation, or when to be a lecturer and tell the student what he must know. Along with the above there was the matter of maintaining a vigorous classroom spirit, and of being patient with the slow, firm with the indolent, and at all times encouraging to each student.

At the beginning of the class there was often a short quiz which required about ten minutes and which contained largely review and diagnostic items. The purpose of these quizzes was to help the student find weaknesses, to see what was expected, and to assist him in "checking off" items which he now understood well. These quizzes also enabled

the teacher to gain immediate information on class progress--rather than to learn of deficiencies at (the too late) testing time. Each student checked his answers in class as the questions were discussed; the quizzes were then handed in for teacher-checking and recording and were promptly returned. Samples of some of these quizzes follow:

Quiz No. 1.

1. Compute by the h-process what we call the speed at the end of the 2nd second if the distance $s = 4t^2$.
2. What is a sequence?
3. In our computations we encounter expressions as

$$\frac{16h + 4h^2}{h} = 16 + 4h$$

What restriction is placed on the h for this identity to hold?

Quiz No. 2.

1. Define the speed at the end of any second if the motion is accelerated.
2. What is the approximate slope of this curve at the point shown?
3. Draw a figure of a curve which has positive slope at every point.
4. Draw a figure of a curve which has negative slope at every point.

Quiz No. 3.

1. Given $y = x^2 - 4x$ what is $R_x y$?
2. For $x = 6$ in No. 1 what is the rate of change of y with respect to x ?
3. At $x = -1$ what is the slope of the curve whose formula is given in No. 1.?
4. Is there a value of x such that the slope of the curve is equal to zero? If so, what is the value of x ?
5. Write what you regard as an acceptable definition of the slope of a curve at a given point.

Quiz No. 9.

Find $D_x y$ given the following expressions:

1. $y = (x^2 + 3x + 2)^3$
2. $y = 3u^2 + 2u$ but $u = 3x^2$
3. $\begin{cases} y = 3t^2 + 2t \\ x = 4t^3 \end{cases}$
4. $y = \sqrt{1 + x^2}$
5. $x = 3y^2 + 2y$
6. $3y^2 + 10x = 14$

During the quarter there were six teacher-made tests of one-period duration and a teacher-made final examination. Each of the tests was preceded by a special review guide which served as a check-list to help the student review terms, ideas, processes, and typical problems. The review guides always covered all items studied since the

last test plus other special bits for review. The examinations covered the entire course and review guides were again used to assist the student.

The resulting general atmosphere of the class was one of informality after the students saw that this method was not one of formal-lecture-discussion. Students felt free to ask questions and to press for an answer if that became necessary; they were encouraged to make suggestions and to comment on another's remarks and results. Contributions of discovery and students' initiative were usually received by the class as a unit and it seemed that the whole group was pleased when one of its number announced a finding. There was an air of earnestness and interest in the classroom even from those who did not seem to spend quite enough time on calculus. The students knew that this was an experiment and some of them gave generously of their time and energy to talk and to share their enthusiasm and to make helpful suggestions. In fact, some of these comments helped, in a minor way, to shape certain features of the course. Some indication of the spirit in the classroom may be gained from sample expressions of a few of the students as they evaluated certain features of this approach. One writes:

I felt that I had a little more to say. I was interested in going to class. I felt that if I didn't understand something the first time it was presented I could get the second. My last course in mathematics was pure lecture. . . few ventured to ask questions and in this course I wasn't ashamed to ask questions or to admit that I did not understand.

Another says "complete class participation was a good point in this course" while another student writes "in this course the approach is different because the students enter into the discussion." Another student says:

I was also amazed at the speed at which the time passed in the class. Sometimes, I found that it was hard to leave the class, and when I did leave, I found a certain stimulation after the class as the feeling of something well done.

In this introductory section have thus been shown the use of the study-guides and quizzes and a general picture of how the class was conducted. To make more clear the nature of this student-discovery approach, classroom experiences relative to topics in the course are described. The order of these descriptions will follow the main outline of the quarter's work.

The Rate of Change Problem

This problem was introduced by means of a physical device, namely, an improvised accelerated motion apparatus as shown in Figure 1. It consisted of two weights, m and $m + f$, a bakelite pulley, a string, a support, a seconds pendulum, and a yardstick. As the heavier weight fell from the top to the floor there was set up a motion, which, it was pointed out, would introduce the group to a simple rate problem. Two boys volunteered to "call time" and to mark the position of the descending weight at the end of each second. Data taken are

shown in Table 11. The students observed that the weight was falling faster and faster and that to come to some conclusion as to the speed at the end of the second second (say) would be a challenging problem. In fact, if confusion in responses is any indication, several students found that the computation of even average speeds put their arithmetic to work in a hitherto new way. To make an approximation to the speed at the end of the second second one student suggested finding the distance by the time $t = 2 \frac{1}{2}$ sec. and computing the average speed during the one-half second from 2 seconds to $2 \frac{1}{2}$ seconds. There was objection here for others claimed that there could be computed average speeds which more closely approximate what the speed must be at the end of the second second by taking the time interval smaller. Practical difficulties were discussed and finally it was suggested by a student to use the distance at the end of the second second, find distances for a

Figure 1
Accelerated Motion Apparatus

further 1 second, 1/2 second, 1/4 second, . . ., compute the respective average speeds, and then see if one could "guess" at a number which would be the speed at the end of the second second. This was attempted but the marker and the pendulum had difficulty keeping together! The class saw the point that a more powerful tool was

TABLE 11
DISTANCE-TIME DATA FROM ACCELERATED MOTION APPARATUS¹

Time Interval	Distance the Weight Has Descended (inches)		
	Pilot Section	Section 1	Section 2
0 sec.	0.0	0.0	0.0
1 sec.	7.8	8.3	5.5
2 sec.	16.9	25.0	17.8
3 sec.	31.0	52.0	34.5
4 sec.	46.0		66.3

needed; one by which changing rates could be studied. The first day's class closed with a perception of the problem at hand and the necessity for some other method to tackle it. Too, groundwork was laid in developing the idea of a sequence.

¹Different weights were used for this experiment in the different sections; this fact accounts for different results in the various sections. The acceleration, a , of the motion of such a system is given by the law $f = k(m + m)a$.

Use of Idealized Expressions

This led naturally to the use of an idealized expression for an accelerated motion; the commonly known $s = 16 t^2$ was chosen. There was some discussion on the idealization of physical phenomena into mathematical formulae. The use of $s = 16 t^2$ to arrive at some conclusion about the speed at the end of the second second was then considered; students were ready to suggest that one could use $t = 2$, $t = 2 \frac{1}{2}$, $t = 2 \frac{1}{4}$, $t = 2 \frac{1}{8}$, . . . to compute distances through which the body has fallen at the end of these times and from these distances to arrive at average speeds during various time intervals. This work is summarized on the table below.

SUMMARY OF WORK ON AVERAGE SPEEDS
($s = 16t^2$)

Time Body Has Fallen (sec.)	Distance Body Has Fallen (ft.)	Time Interval (sec.)	Average Speed (ft./sec.)
2	64		
$2 \frac{1}{2}$	100	$\frac{1}{2}$	$(100-64)/\frac{1}{2} = 72$
$2 \frac{1}{4}$	81	$\frac{1}{4}$	$(81-64)/\frac{1}{4} = 68$
$2 \frac{1}{8}$	$72 \frac{1}{4}$	$\frac{1}{8}$	$(72 \frac{1}{4}-64)/\frac{1}{8} = 66$

This list of average speeds immediately brought forth a discussion on sequences and limits and such ideas as hypotheses and definitions. The tentative hypothesis suggested was that as the time intervals were made smaller the average speeds would go through the sequence 72, 68, 66, 65, $64 \frac{1}{2}$, $64 \frac{1}{2}$, . . . and hence one could suggest 64 feet per second as the speed at the end of the second second.

One student raised the query: "Does it make any difference on the proposed answer how the times taken approach $t = 2$ --that is, whether $t = 2$ is approached from "above" or from "below," or, would the actual fractions of seconds used make any difference?" These questions had been anticipated in the exercises (see Appendix C, p. 198) so, for a part of the next day's assignment, the students in one row were asked to find a number which they would present as the speed at $t = 2$ seconds by using 3, $2 \frac{1}{2}$, $2 \frac{1}{4}$, $2 \frac{1}{8}$. . . seconds; those of another row by the use of $t = 1$, $t = 1 \frac{1}{2}$, $t = 1 \frac{3}{4}$, . . . seconds; still another group was asked to use $t = 2.1$, $t = 2.01$, $t = 2.001$, . . . seconds to compute an average speed sequence. In each case the number proposed as the answer was the same.

Formulation of Hypotheses and Definitions

By this time informal discussion produced several hypotheses which are mentioned here to assist in describing this approach at work:

H_1 : The sequence of average speeds approaches a limit
("limit" and "approaches" accepted as undefined).

H_2 : This limit is the speed we seek.

H₃: One may reduce the time interval in any regular manner whatsoever, and he will obtain different sequences of average speeds, but the limit will be the same.

H₄: One may approach (say) $t = 2$ from times less than $t = 2$ or times greater than $t = 2$ and although he will obtain different sequences, the limit will be the same.

In response to the question, "Is anyone ready to set up a definition of speed?" one student offered

$$D_1: \text{speed at any time } t \quad \equiv \quad \text{limit (as time interval approaches zero) of } \left(\begin{array}{l} \text{an appropriate} \\ \text{average speed} \\ \text{sequence} \end{array} \right)$$

Throughout the course this definition, D_1 , was hereafter also known as the (student's name) definition of speed; this custom was used with several other theorems to help the students realize how names applied to functions, theorems, processes, or definitions, arise in mathematics. All students were asked at various times in class and in the study guides to phrase definitions of their own. Some of these were assembled and given to the classes for their comment and criticism. Note Appendix C, p. 230.

Using Algebra

Realizing that the above method for finding speed required much computation the teacher introduced the idea of using $t = 2 + h$ as a symbolizing time which departs by h seconds from $t = 2$. One student

continued from this beginning and very nicely presented in class the development that if $s = 16 t^2$ then (for $t = 2$) the average speed, where h is the time interval, would be $64 + 16h$. Another student suggested now that if h were regarded as very small, one could conclude that the speed at $t = 2$ seconds would be very close to 64 feet per second. Here, again, there was confirmed for some, born for others, the idea of a sequence approaching a limit, and, as the time sequence approaches zero, the average speed sequence approaches a definite number which one has defined to be the speed sought. In fact, here, by assigning values of h to be $1, 1/2, 1/4, 1/8, \dots$ one student reported that he could have used $64 + 16h$ to get arithmetical results too. This is a good example of exploration and comparison of relationships.

A semiclimax came when another student wondered if, instead of using $2 + h, 3 + h$, or $10 + h$ as required by the problem, one could not use a general time a , use $a + h$, find a general average speed expression for the time interval h , and get a formula "once and for all" for the speed at the end of the a th second. Of course this proposed problem was welcomed and one of the members of the class carried the mathematical manipulation through to a successful completion. In the study-guides (Appendix C) this is found on page 203.

It is of interest to note that in all sections rather philosophical questions of fundamental nature arose, such as: "Is calculus based upon things 'approaching'--if so, isn't this a rather weak foundation?" Here was an opportune time to discuss again the structure of mathematics: the place of undefined terms, defined terms, postulates, and theorems.

And, as one student expressed it, "our faith that a certain number is the answer to a speed problem is just as firm as our faith that a certain sequence approaches a certain limit."

Slope of a Curve

What definition would you suggest as the slope of a given curve at a given point? One student used a practical approach when he outlined his method as (1) locating the point on the curve, (2) drawing a tangent to the curve by using the "eye," (3) measuring the slope of the tangent, and (4) calling this the slope of the curve at the given point. Of course some discussion followed, with the conclusion that this definition "was no stronger than the student's eye," and the further suggestion came that one would perhaps need an equation of the curve. The class and the teacher then began work on Study-Guide I.A.5. (Appendix C). Discussion of this material led to an acceptable definition of the slope of a curve at any point which was stated in this manner: the slope of a curve at any point is the limit of the sequence of slopes of secants going through that point as the x-interval length approaches zero.

During class discussion it was emphasized that both speed and slope were actually rates of change and so, at a psychologically appropriate time, the symbol $\mathcal{R}$ was introduced to indicate "rate of change." $\mathcal{R}_{xy}$ means the rate of change of y with respect to x; sometimes the printed R was used.

At the suggestion that students look for "short cuts" to write $\mathcal{R}_x y$ immediately one-half of the class on the next day reported success

in discovering a quick method; one student learned to write $R_x y$ for $y = 1/x$ by the use of negative exponents.

Along with slope were considered values of x for which the slope was equal to zero, which introduced the ideas of maximum and minimum values of a function. Hence important applications of the calculus came early in the course. Some of these beginning ideas were developed by means of what the writer calls "inductive quizzes" which insure student participation and which encourage student growth. An example of one follows:

1. Given $y = 3x^2 - 12x$, what is $R_x y$?
2. What is the value of $R_x y$ at $x = 5$?
3. For what value of x , if any, is $R_x y = 0$?
4. What is $R_x y$ at $x = 3$?
5. What is $R_x y$ at $x = 1$?
6. Does (a)

or (b)

more nearly represent the situation learned from Nos. 3, 4, and 5?

7. Hence does the function have a point of minimum or maximum at $x = 2$?
8. Find the value of x for which $y = x^2 - 4x$ has zero slope and determine if it is a point of minimum or maximum. Show argument.

After the student learned to write $R_x y$ at sight the question was raised as to the validity of taking $R_x(x^2 + 3x)$ by using $R_x x^2$ and $R_x 3x$ and then adding--that is, is the R_x -process distributive with respect

to addition and subtraction? There was an attempt at all times to sprinkle into the course some of these questions to set the stage for a later systematic discussion.

Related Rates

This is an important topic, and in this section (I. A. Rates. 7. in Appendix C) was developed an idea which became a very useful concept in the differential calculus. The relation referred to is

$$y_f - y_b = R_{x,y} \cdot (x_f - x_b) + E$$

(at
x = x_b)

where E is the error introduced by using the rate of change, $R_{x,y}$, to approximate the actual change in y due to the change in x from x_b to x_f .² A geometrical interpretation of this relation follows in

Figure 2.

Figure 2

Actual Vs. Approximate Change

²The subscripts "f" and "b" mean "final" and "beginning" respectively.

Another form of this expression is

$$\frac{y_f - y_b}{x_f - x_b} = R_x y + \eta,$$

where $\eta \equiv \frac{E}{x_f - x_b} \quad (x_f \neq x_b),$

$$\eta \rightarrow 0 \text{ as } x_b \rightarrow x_f,$$

and $\eta \equiv 0 \text{ for } x_b = x_f.$

Summary

From the above discussion on the teaching of the general rate problem several features of this approach are apparent:

1. The study of topics began with the concrete and the tangible.
2. The beginnings of fundamental ideas were largely arithmetical.
3. Steps forward came as a result of students' experiences and as a result of encouraging students to think about the generalization of processes.
4. The general attitude of students was one of active participation in the growth of ideas rather than passively receiving knowledge.
5. Many ideas: sequence, function, rate of change, slope, points of zero slope, maximum values, minimum values, related rates, all developed somewhat simultaneously. Each, it seems, had a better "setting" than if the idea had been developed alone.

6. Students were encouraged to ask questions and these questions were utilized in teaching.
7. Students' names were used to identify definitions which not only helped to maintain interest, but more important, it helped the student to sense how mathematics grows.
8. The use of quizzes as learning devices and the so-called inductive quiz were illustrated.

The Area Problem

Finding area under a curve was discussed as a historical problem with the emphasis that the method for computing the area of rectilinear figures was not directly applicable. The development followed somewhat the study-guides I. B. Areas. 1. which are found in Appendix C.

Area Under a Curve

Having had previous experience in approximating speed and slope by the use of sequences the students needed merely a hint that perhaps a similar method might show promise in attacking the problem of finding the area under a curve. Hence, on the following day, there came several fruitful suggestions on finding the area under the curve $y = x^2$ and between the lines $x = 1$ and $x = 3$ and the x -axis (see figure 3). One student divided the x -interval into ten parts, constructed ten rectangles, and presented the sum of their areas as an approximation to the area under the curve, with the comment that had he used twenty rectangles, then forty, then eighty, . . . he would have obtained successively better approximations. This suggestion was discussed

further and with the help of students the teacher set up the sum of n such rectangles to show the group how such a sum could be represented. Furthermore, by

demonstrating that there could be both rectangles entirely under the

curve or rectangles which projected

above the curve, it was shown that one could obtain upper and lower

bounds on the area. Another student contributed a method which made use of a sequence of sums of trapezoids. In Table 12 are shown the results which he presented.

Figure 3

Area Under Curve $y = x^2$

TABLE 12

SUMS OF AREAS OF TRAPEZOIDS IN APPROXIMATING
(INTUITIVE) AREA IN FIGURE 3

Number of Trapezoids Used	Sum of Areas of These Trapezoids
1	10
2	9
4	$8 \frac{3}{4}$
8	$8 \frac{11}{16}$
16	$8 \frac{43}{64}$

As a class exercise it was noted that the sum for any number of trapezoids could be expressed as

$$S = 10 - (1 + 1/4 + 1/16 + 1/64 + \dots),$$

and that this could be evaluated by the use of the well-known formula for the sum of an infinite geometric progression, $S_{\infty} = \frac{a}{1 - r}$, whence $S = 8 \frac{2}{3}$ square units. The students of the class were pleased at this utilization of their previous mathematical experience and how beautifully the result came. One student asked, however: "Isn't there a shorter method or a formula which would give the answer?" Such questions provided an excellent opportunity to suggest what might be ahead and to convince the students that mathematics is a powerful tool. It should be noted that Exercise 3 in the study-guides I. B. Areas. 1. suggests another method--the use of graph paper.

Computing Work

Next was considered the problem of work.³ When the force is constant this is simple; but when the force is variable "troubles begin." The teacher set up an apparatus to show that as a spring is stretched more force is required and hence force times distance at one place is not the same as the force times distance at another (with equal distances). Given $F = 32x$, in which x is distance in feet and F is force in pounds, it was proposed to find somehow a number which could be presented as the work done in stretching the spring from $x = 2$ to $x = 3$ feet. Some of the students seemed to fall quite readily into the pattern of estimating the work done by means of a sequence of approximations--the idea was catching on!

³It was surprising to note the number of students who had not had high-school physics; this made it necessary to explain the meaning of the term.

After discussion the class was well aware that $F = 32 \times 2 = 64$ lbs. would not represent the force throughout the whole interval from $x = 2$ to $x = 3$ feet and that therefore the quantity $64 \times (3-2) = 64$ ft. lbs. could be only a first approximation to the work done. Computing the work done over two subintervals as

$$64 \times (2 \frac{1}{2} - 2) + 80(3 - 2 \frac{1}{2}) = 32 + 40 = 72 \text{ ft. lbs.}$$

have a better approximation. Next four subintervals were used and an approximation of 76 ft. lbs. was obtained. Some of the students became "lost" here until they understood, through give-and-take discussion, that it was being assumed that the force on the spring did not change for "little bits" of the distance or throughout any one subinterval. The sequence of "approximate works" was

$$64, 72, 76, 78 \text{ ft. lbs.}$$

from which it was assumed that one could write

$$64, 72, 76, 78, 79, 79 \frac{1}{2}, 79 \frac{3}{4}, 79 \frac{7}{8}, \dots$$

which caused the formulation of the hypothesis that the number 80 ft. lbs. could represent the work done in the interval from $x = 2$ to $x = 3$ feet. For a more detailed explanation of this development one is referred to the Study-guides I. B. Work. 3. in the Appendix C.

Now was the time for analysis and abstraction. Encouraging students to express themselves, as was always done, brought forth the statement that area under a curve is the limit of a sequence of sums of areas of rectangles as the number of rectangles increases and as the widths of the rectangles approach zero. Work is the limit of a sequence of sums of "little works" as the interval of subdivision approaches zero. Do these proposed definitions have anything in common?

What a thrill it was for these students when one of them suggested, and the whole class perceived, that both were cases of the limit of a sequence of sums of products! Here was an abstraction of the idea of work and area for which there was no name--this was emphasized. It was dramatically sensed by the group that here was a "big idea" which embraced much--and they had experienced its growth. Several students immediately perceived that area and work problems were really identical. It seems that, at this moment, some of the students realized that abstractions are not found exclusively in the realm of the mathematical theorist, but that they arise naturally and that they become a principal means of gathering several items of different origin into one big law or principle; this great principle can then be interpreted or translated to fit special or subordinate situations. At this time the idea of R_{xy} was reviewed and it was now seen abstractly to be a limit of a sequence of quotients.

The area problem was now reviewed and functions were developed which would yield the area for any number, n , of rectangles used in the approximation. This work is classic and for its development one is referred to the Study-guides I. B. Area. 4. Again the students were asked to observe results carefully to see if they could discover methods to arrive at the area under a given curve "at sight."

An interesting discovery came to light here. One student did some experimenting and found that the area under $y = x^2$ from $x = 0$ to $x = 1$ is $1/3$; under $y = x^3$ from $x = 0$ to $x = 1$ is $1/4$. She conjectured that under $y = x^4$ the area would be $1/5$, and, in general, under $y = x^n$,

it would be $1/n$. Not only this, but on the next day, encouraged to study this further, she came back with these results (stated in general):

1. if the curve is $y = x^2$ and the interval is from $x = 0$ to $x = a$ then the area $A = \frac{a^3}{3}$;
2. if the curve is $y = x^3$ and the interval is from $x = 0$ to $x = a$ then the area $A = \frac{a^4}{4}$.

She then hypothesized that

3. if the curve is $y = x^4$ and the interval is from $x = 0$ to $x = a$ then the area $A = \frac{a^5}{5}$.

Here was a method, which, at sight, would give the same result as that found by the tedious process of finding the limit of a sequence of sums of products!

Summary

As to procedures and classroom experiences in this section perhaps these are the most significant:

1. the use of physical devices to introduce the problem of work and area and to emphasize further that much mathematics has grown fundamentally out of man's experience with concrete things
2. the inductive development of the rather dramatic abstraction of the common features in solutions to work and area problems: that they were both limits of sequences of sums of products--an abstraction as yet unnamed
3. further evidences of the capabilities of students to do much for themselves and to make helpful suggestions in teaching.

The work of this section required about four or five days.

The Notebook

At this juncture in the course was mentioned the organization of materials into a notebook. It was suggested that it should contain perhaps one section devoted entirely to the logical structure of the calculus: definitions, hypotheses, indicated hypotheses which later became theorems, and any other items pertinent to the flow of deductive thinking. Another part, it was commented, might consist of a "recipe book" of directions for doing certain things with the calculus. Class notes and study-guides would also be included.

It was emphasized, of course, that the student could actually make use of as much originality as he pleased in the organization of his notebook.

A Systematic Approach to the Calculus

The aim of this section was to attempt to extract or erect a logical structure from the experiences in the calculus thus far and to formulate a basis upon which further structure could be built.

Students Help to Formulate Definition of Limit

Up to this time the students had worked with many sequences⁴ and frequently the question had been raised: "What is meant when it is said

⁴Monotone sequences were used in this course. It is natural to use this type on this level; intuitively monotone sequences either have a limit or go to infinity.

that a sequence a_n approaches A as a limit?" It was now the purpose to have the group give full attention to this question. This required time. One girl suggested: you choose a term a_i ; then for any other term $a_i + x$ it is true that $a_i + x$ is "closer" to A than a_i ; or, if this is true, then a_n approaches A . At the suggestion of the use of the traditional ϵ and with other help by the teacher during class discussion another student formulated this definition: you choose a number $\epsilon > 0$; then there exists a term numbered N such that for all terms beyond this N th one it is true that A minus any of them is less than ϵ . There was objection to this until the proposer used "absolute values" in the definition. The definition accepted by the group for later deductions assumed that all convergent sequences (which would be used) were monotonic. It was pointed out with the aid of blackboard drawings that this could mean approach from the "left" or approach from the "right." The "approach from the left" definition was formulated by the students as follows: to say that $\{a_n\} \rightarrow A$ means that for any $\epsilon > 0$ there exists an N such that for all $n > N$ it is true that $a_n > a - \epsilon$.⁵ Study-guides II. 1. (Appendix C) show the exercises which were used to assist in this exploration.

⁵It seemed that a choice had to be made here. In the pilot section it was found that usual deductions of theorems on sequences were difficult to understand because such laws as $|a| + |b| \geq |a + b|$ were not a well-established part of the students' mathematical experience. Time was at such a premium that to insist upon the presence and consequent use of absolute values in deductions would have denied a good number of students a deep appreciation of the dependency of theorems and operations with convergent sequences upon the definition of limit. Hence the definition given above was used. The students were aware of this restriction and they further realized that, at any time necessary, proofs could be constructed for sequences which were "approaching from the right."

An Algebra of Sequences

In the study-guides (Appendix C, p.241) the idea of adding, subtracting, multiplying, and dividing the corresponding terms of two convergent sequences to get a new sequence is introduced by means of exercises. This paved the way for the students, utilizing the heuristic method, to formulate hypotheses on the termwise sum, difference, produce, and quotient of two sequences and to test them by deduction. The hypotheses were well formulated but help was required in making deductions. Only the conclusion concerning the sum of two convergent sequences was deduced.

Limits of Functions

The concept that $\lim_{x \rightarrow a} f(x) = K$ was developed by a numerical example such as $y = f(x) = 3x^2 + 6x + 2$ by computing the values of $y = f(x)$ for $x = 1$, $x = 1/2$, $x = 1/4$, $x = 1/8$, . . . to learn that the corresponding values of the function: 11, $5 \frac{3}{4}$, $3 \frac{13}{16}$, . . . approached 2 as x approached zero. It was emphasized, however, that it does not generally follow that $f(x) = 2$ when $x = 0$.⁶ Study-guides II. 2. helped with this. The δ, ϵ definition was found to be quite difficult to repeat with understanding so this idea of the "limit of a function" was left as one to be developed intuitively and the definitions are so stated.

⁶It was brought out that a discussion of the limit of a function $f(x)$ as x approaches a demands the requirement that the function be defined for all values of x sufficiently close to a , and the restriction that x never equal a even though the function may be defined there.

Exploration went forward. What hypotheses can be made on limits of the sum, difference, product, and quotient of two different functions of x as $x \rightarrow a$? Some were formulated and it was deduced from previous theorems that if $\lim_{x \rightarrow a} f(x) = K$ and $\lim_{x \rightarrow a} g(x) = L$ then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = K + L.$$

One student asked, "Doesn't $\lim_{x \rightarrow 2} (2x + 3) = 7$ come from $\lim (a_n + b_n) = A + B$?" She further argued that

$$\begin{aligned} \lim_{x \rightarrow 2} (2x + 3) &= \lim_{x \rightarrow 2} (x + x + 3) \\ &= \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 3 \\ &= 2 + 2 + 3 = 7 \end{aligned}$$

Yes! This was a splendid question and the discussion of it was a convincing way to emphasize logical dependence of limits of functions upon limits of sequences which, in turn, had beginnings in the definition of limits. This student's example helped the class to perceive more deeply the beginning of a logical pattern.

Continuous Functions

The idea of these was developed largely with the aid of the study-guides in Appendix C: II. 3. The approach was first intuitive and then the usual analytic definition followed. Exercises strengthened the students' understanding of continuous functions.

Summary

Some of the same "methods" were used as those mentioned in the last summary but here the heuristic and genetic approach yielded

particularly good results for the topics on limits and on the algebra of sequences. Much experiential activity led to formulation and discussion of good questions which nurtured understanding on limits of functions. It is seen that, here again, the contributions of students sometimes influence the turn which a course takes.

The Derivative. Higher Derivatives.
Inverse Derivatives.

$R_x y$ to $D_x y$

It was an easy matter to change the symbol for rate of change from $R_x y$ to $D_x y$. The student, therefore, already knew the meaning of $D_x y$, could compute it at sight for polynomial functions, and were continuously using the derivative concepts to work maximum-minimum problems, various types of problems involving slopes of curves, and exercises on related rates. Furthermore, the question of the "rate of change" had introduced students to the concept of a "second" derivative.

The concepts of right- and left-hand derivatives were developed and the student himself experienced the fact that one could have a function continuous at a point which has no derivative at that point. It was brought out in the discussion that the right- and left-hand derivatives had to exist and be equal at a point for the derivative to exist at that point. Study-guides III. The Derivative. A. 1. illustrate the method of approach.

Theorems on Derivatives

Up to this time the student differentiated the function $y = ax^n$ to get $D_x y = n ax^{n-1}$ merely at sight; the method had been established by induction⁷ and it had been hypothesized that this process would give the derivative. Now this conclusion was deduced from the definition of the derivative and the binomial expansion $(x + h)^n$. With several students from the class contributing, this development or argument was taken through to a successful conclusion; on the other hand, some students barely remembered anything about binomial expansions. This topic seemed to be a good tool whereby one could distinguish again in the minds of the students the difference between establishing a hypothesis by induction and a theorem by deduction-- and this was emphasized.

The product and quotient rules were presented as ways of making differentiation easier. The student did not discover these, but by the use of simple examples he was led to see that they work. They were immediately accepted as hypotheses, put to work in problems, and as time permitted they were deduced as theorems. The expression on page 74 of this study was used as the starting point in these deductions.

The chain rule (Study-guide III. A. 3), or the method of differentiating a function of a function or a composite function⁸

⁷By the term "induction" in this study is meant the process of making a generalization from the study of examples and not the more rigorous mathematical induction.

⁸One student suggested the name "combination function."

was developed inductively and in one class period the students had it well in hand. One student employed a better and smoother approach than that suggested in the study-guides. Given $y = 3t^2$ but $t = 6x^3$ he wrote $D_t y = 6t$ and $D_x t = 18x^2$. "Now," he says, " $D_t y$ means that y changes $6t$ times as fast as t , but $D_x t$ means that t changes $18x^2$ times as fast as x , therefore y changes $6t \cdot 18x^2$ times as fast as x , and $D_x y$ is therefore $6t \cdot 18x^2$." He then postulated, subject to possible confirmation by deduction, that $D_x y = D_t y \cdot D_x t$. This was later deduced and designated as a theorem. The problems in the text, the quizzes in class, and the exercises in the study-guides, used this rule often to make the student proficient in its use.

Derivatives of Inverse Functions

The study-guides very successfully helped the student to develop inductively the relation $D_x y = \frac{1}{D_y x}$ and then, in Exercise 7, Appendix C, p. 266 attention was called to establishing this result by deduction.

Derivatives of Parametric Functions

It was necessary first to help the students understand what parametric functions are. They understood well after the use of this device: given $x = 3t$ and $y = 2t^2$ one student (called the t -student) suggested values of t after which another student gave his computed value of x and still another told his computed value of y ; the x - and y -students had to wait until the t -student assigned a value to t --the t -student controlled the situation. Also this was all represented graphically

on the blackboard with a t -axis and the usual x - and y -axes. Work on differentiation to compute $D_x y$ followed the Study-guide III. A. 4.

After this class one student remarked, "I am now convinced that a teacher must be a showman also--the 'show' on parametric functions helped me to understand what they really are--for the first time." Another student commented, "mathematics is not alivelike this in the book!"

The rule for computing $D_x y$ given $y = f(t)$ and $x = g(t)$ was later deduced.

Developing Work on Areas

Back in the section on A Systematic Look at the Calculus began the development of a process for finding the area under a curve. Intuition made reasonable the assumption of the existence of this area, and, throughout all the work on derivatives, a little was added each day to this growing body of skills, hypotheses, and results. There was a serious attempt made to keep this work alive and developing so that the student might have sufficient background to perceive a nearly coincidental relation between the process of finding area under a curve and the process of inverse differentiation.

Inverse Differentiation

Study-guide III. A. 6. was designed to use a largely heuristic method to assist the student in understanding the meaning and process of inverse differentiation. In one section, about one-third of the students thought that they had come to some successful conclusions on

this topic, while working out the study-guides. This subject was introduced by the use of questions relating to the general meaning of an inverse operation.

From all sections came this observation: "It seems that there is a connection between the fact that the area under the curve $y = x^2$ from $x = 0$ to $x = a$ is $\frac{a^3}{3}$ and that the fact that an antiderivative of x^2 is $x^3/3$!" This was tried on other curves on which the student had worked. Was this an accident or was there a logical connection?

The above led to another question of fundamental importance. The student was already antidifferentiating polynomials so that if there is a connection between antiderivatives and areas does this mean that⁹ $A(x^2) + A(x^3) = A(x^2 + x^3)$? This seemed to astound some students and some said yes and some said no. At the next class some students expressed confidence that it is really true that

$$A(x^2 + x^3) = A(x^2) + A(x^2) + A(x^3),$$

or that the "area-computing process" is distributive with respect to addition. One student's argument follows:

1. $A(x^2 + x^3) =$ limit of a sequence of sums of areas of rectangles under $y = x^2 + x^3$.

$$= \text{limit } [S(x^2 + x^3)].^{10}$$

2. But by a theorem on limits of functions

$$\text{limit } [S(x^2 + x^3)] = \lim [S(x^2)] + \lim [S(x^3)].$$

⁹ $A(x^2)$ is to be read "area under the curve $y = x^2$ between $x = a$ and $x = b$."

¹⁰ $S(x^2 + x^3)$ means the sum of areas of rectangles whose bases form a partition of the interval between $x = a$ and $x = b$.

3. Now by definition

$$\lim [S(x^2)] = A(x^2) \text{ and } \lim [S(x^3)] = A(x^3)$$

hence

$$\lim [S(x^2 + x^3)] = A(x^2) + A(x^3)$$

4. and therefore

$$A(x^2 + x^3) = A(x^2) + A(x^3)$$

5. and the "area-process" is distributive with respect to addition!

This argument pleased many students and there was evident a sense of deeper understanding of the use and power of deduction. It was seen clearly here that the term distributivity is merely not one which mathematicians just talk about! This was another big day in the group's mathematical adventure and development.

Higher Derivatives or Derivatives of Higher Order

This is a big section and the study-guides use some eleven pages of exercises, problems, and review items to help the student through these topics and applications.

The teaching methods used were largely inductive to show the use of especially the second derivative in helping to determine maximum, minimum, and inflection points of a curve. It seems that the statement "a student once said that ' $D_x^2 y$ ' would tell about $D_x y$ the same that $D_x y$ tells about y ' " was very helpful.

Inductive quizzes helped here to develop ideas on geometrical interpretations of higher derivatives and applications to the distance-time function.

One should note also in the Study-guide III. B. that there was not only much review of former material but also exercises on areas under curves to help prepare the student for the succeeding work on integration.

Summary

This section brings into sharper focus several aspects of this student-experience-discovery approach. It should be noted especially how:

1. the use of numerical and algebraic work insures ease of transition in passing from the idea of $R_x y$ to $D_x y$;
2. the growth of the student in the subject comes as a product of individual and class effort in creating, at times, some peaks of mathematical excitement;
3. experiential activities on areas under curves were timed and kept alive to later culminate in generalized results;
4. problems, such as those involving rules of differentiation, are worked using these rules as hypotheses, and then deductions follow as the student understands better exactly what he is attempting to deduce.

The Area Problem and Integrals

Integrals

In this section particularly one could reap the benefit of introducing early the ideas of area under a curve, work, the setting up of sequences of sums of products, and the later hypotheses and theorems that from $x = a$ to $x = b$ ($a < b$)

$$\begin{aligned} 1. \quad A(x) &= \frac{b^2}{2} - \frac{a^2}{2} \\ 2. \quad A(x^2) &= \frac{b^3}{3} - \frac{a^3}{3} \\ 3. \quad A(x^3) &= \frac{b^4}{4} - \frac{a^4}{4} \end{aligned}$$

and that

$$4. \quad A(x^3 + x^2) = A(x^3) + A(x^2).$$

One important item yet desirable was a name for the limit of a sequence of sums of products. This name was now furnished: the term integral! Notational explanations and inductive exercises as noted in Appendix C, p. 305 show how this was handled and how the definition of the definite integral was built from previous experiences.

It was emphasized repeatedly that through concepts of work and area this group had abstracted a definition of something which was now named the definite integral.

By this time the students were quite adept at proposing hypotheses or possible theorems and the standard theorems on the sum and difference of integrals came quickly. Geometric figures helped students to formulate other hypotheses; these are found on page 106 of this study.

In all this there remained one fundamental question. As one student phrased it: "The area, A, under, say, $y = x^2$ from $x = a$ to $x = b$ is defined to be $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 h_i$ ($nh = b-a$). Now $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 h_i$ is defined to be $\int_a^b x^2 dx$. But it has been deduced that for $y = x^2$, from $x = a$ to $x = b$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 h_i = \frac{b^3}{3} - \frac{a^3}{3}$$

which one observes is obtainable from $D_x^{-1} x^2$ by subtracting the value of $D_x^{-1} x^2$ at $x = a$ from the value of $D_x^{-1} x^2$ at $x = b$. Is it valid to evaluate a definite integral, which means the limit of a sequence of sums of products, by the process of antidifferentiation, which means something entirely different?" This created some mental searching in the group. For several days or more it was hypothesized that the above proposal was valid and practical applications of the integral were studied while, day by day, the teacher and the groups slowly developed bits of theory which lead to the Fundamental Theorem.

The Mean Value Theorem for Integrals

By means of exercises (see Appendix C, Exercise 4, p. 314 for the first one) the students were actually finding numerical values of x for given functions and intervals which satisfied the mean value theorem before they knew what it was. Later, the generalization which had been made was even applied to integrals such as $\int_a^b f(x) dx$ and $\int_a^{a+h} f(x) dx$ to set the stage for the proof of the Fundamental Theorem. Instead of the usual ξ or x' of the mean value theorem one group used the letter "w" for this was the symbol suggested by the first student who phrased the mean value theorem. Hence for this group the statement of

the mean value theorem was: given the integral $\int_u^{u+h} f(x) dx$, where $f(x)$ is continuous on the interval $[u, u+h]$, there exists a value of x , say " w " where $u \leq w \leq u+h$ such that $\int_u^{u+h} f(x) dx = h \cdot f(w)$. The mean value theorem was not deduced in these classes but it was assumed to be true; it was postulated or accepted as an "unproved theorem."

Fundamental Theorem of the Calculus

The stage was set for this most wonderful theorem and for several days there was an expectant air of growing anticipation in the classroom to learn if one could logically validate the process of antidifferentiation for replacing the tedious process of evaluating definite integrals by definition.

The Fundamental Theorem

Finally (see Appendix C, p. 318) the question was raised as to the value of $D_t \int_a^t f(x) dx$. Knowing that $\int_a^t f(x) dx$ defined a function of t , the students, with help from the teacher, suggested steps to be written on the blackboard, and, applying the mean value theorem, deduced that

$$D_t \int_a^t f(x) dx = f(t).$$

This was called the Fundamental Theorem of the Calculus. Several examples strengthened the students in their understanding of just what this theorem meant.

An Important Corollary

It follows operationally, with the proper choice of a constant, that

$$D_t^{-1} D_t \int_a^t f(x) dx = D_t^{-1} f(t)$$

and

$$\int_a^t f(x) dx = D_t^{-1} f(t),$$

or one can evaluate $\int_a^x f(x) dx$ by antidifferentiation!

Much was made of this great theorem and its corollary in actual practice. "Shall we evaluate this integral by definition or by antidifferentiation?" "May we evaluate by antidifferentiation? Is the integrand continuous in the closed interval defined by the limits of integration?" These were questions which were often raised in working problems. All this served to emphasize the power of this corollary, limitations of its usefulness, and, in practice, the hypothesis of a theorem must be satisfied before it can be used.

The students felt that their experiences had come to a fitting climax: two great disciplines, those of differentiation and integration, had been logically connected. It seemed that there was a sense of completeness--and just at the right time: the end of the quarter.

Summary

It seems that the last two sections demonstrate well the planning and timing of experiences and keeping them alive and growing until the psychological moment comes to have some new idea or some great theorem make dramatic entry to climax the whole calculus experience.

CHAPTER FOUR

CALCULUS CONTENT AND ORGANIZATION

Introduction

Reference has already been made in Chapter Two to the planned content in this experimental course and, in the last chapter, experiences have been described during and by means of which an understanding of these topics was nurtured and developed in the students.

Since this is a study on the teaching of the calculus, the materials in this chapter, in contrast to what the title suggests, may yet, and perhaps should, provide another facet to the general nature of this approach and show the possibilities of richness of results when students are encouraged to think about the calculus in their own way.

The calculus concepts, skills, logical organization, and other materials which will be described in this chapter are a composite of the pilot section and experimental Sections One and Two; individual notebooks and organizations departed from this somewhat in details but the mathematical sequence was essentially what will be given.

An Outline of the Calculus

This will be a formal outline with little mention of procedure but at times the classroom techniques can be inferred from the arrangement of the content.

It should be emphasized that these hypotheses and theorems were stated largely by the students, and, with guidance by the teacher, these expressions were refined by the students to appear as they are written here. Furthermore, although in general these definitions and theorems appear in "chronological order" also, yet there were several instances where further study caused the groups to reconsider definitions and the numbering of them. In such cases the D or the T or the H will be underlined; these letters stand for definition, theorem, hypothesis, respectively, and the letter R will represent remark.

Some Beginning Hypotheses

- H₁: Each of the sequences of average speeds (in problems we are working) approaches a limit ("limit" undefined).
- H₂: This limit of average speeds is the speed we seek.
- H₃: One can reduce the length of the time intervals in any regular manner whatsoever; one will get different sequences, but the limit of each will be the same.
- H₄: One can approach $t = a$ from above or from below; he will get different sequences, but the limit of each will be the same.

An Hypothesis Becomes the First Definition

The definition of speed which was proposed and accepted by the class was really a restatement of Hypothesis H₂, hence H₂ was deleted and the logical outline proceeded with the addition of the first definition, D₁. The group now had

- D₁: Speed at any time $t = a$ is defined to be the limit of a sequence of average speeds, with each time interval beginning at $t = a$, as the time interval approaches zero.

D₂: The slope of a curve at any point $x = a$ is defined to be the limit of a sequence of secants, all going through $x = a$, as the x -interval approaches zero.

D₃: The rate of change, $R_x y$, of a function $y = f(x)$ at $x = x_b$ is defined to be the limit of a sequence of quotients

$$\frac{y_f - y_b}{x_f - x_b}$$

as $x_f \rightarrow x_b$.*

H₅: Given $y = f(x)$ continuous in the vicinity of $x = x_b$ and with a rate of change, $R_x y$, at x_b , then

$$y_f - y_b = R_x y (x_f - x_b) + E$$

where E denotes the error caused by using $R_x y$ to compute the change in y_b to y_f , as x_b changes to x_f rather than using $y = f(x)$ itself to compute the change.

D₄: The area under a curve $y = f(x)$ from $x = a$ to $x = b$ is defined to be the limit of a sequence of sums of areas of rectangles whose bases are intervals which add up to the interval $[a, b]$.⁺

D₅: The work done by a force $F = f(x)$ from $x = a$ to $x = b$ is defined to be the limit of a sequence of sums of "little works" as the distances for each "little work" approaches zero.

D₆: Abstractly D₄ and D₅ imply the existence of a limit of a sequence of sums of "little products" for which as yet we have no name. (This was regarded as a definition for something not yet named--a parallel to "rate of change.")

H₆: In addition to the sequences of average speeds mentioned in H₁, it is further hypothesized that sequences of slopes of secants, sequences of sums of products all have limits. Statements analogous to H₃ and H₄ hold for these sequences also.

*See the footnote on p. 74 if the meanings of the subscripts "f" and "b" are not recalled.

⁺Useful relations such as $1^2 + 2^2 + 3^2 + 4^2 \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$ which are borrowed from other subjects are not included in this outline.

D₇: Assuming that $a_n \leq A$, then to say that $\{a_n\} \rightarrow A$ means that given any $\epsilon > 0$ there exists an Nth term such that for all terms, a_n , of the sequence beyond this Nth one, it is true that $a_n > A - \epsilon$.

Some theorems which were deduced from definition D₇ are:

T₁: If $\{a_n\} \rightarrow A$ and $\{b_n\} \rightarrow B$ then $\{a_n + b_n\} \rightarrow A + B$, or,

if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$ then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = A + B.$$

T₂: If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$

then $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$.^{† §}

T₃: If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$

then $\lim_{n \rightarrow \infty} a_n b_n = A \cdot B$.[‡]

T₄: If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$

then $\lim_{n \rightarrow \infty} a_n / b_n = A/B$. ($B \neq 0$)[§]

Definitions and Theorems on Functions

D₈: y is said to be a function of x if for each value of x over a certain given domain there exists a single corresponding value of y .

[†]This and T₁ are important theorems for they show the distributivity of the limit process with respect to addition and subtraction.

[§] Class time was not spent on formal deductions of these theorems from the definition. They were accepted as unproved theorems; they, of course, could be called postulates, or hypotheses.

D₉: To say that $\lim_{x \rightarrow a} f(x) = K$ means that

as the sequence of x 's approaches a , the corresponding sequence of $f(x)$ -values approaches K .[¶]

T₅: If $\lim_{x \rightarrow a} f(x) = K$ and $\lim_{x \rightarrow a} g(x) = L$ then

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) + g(x)] &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \\ &= K + L. \end{aligned}$$

A formal argument or proof was not written for the preceding theorem; student suggestions indicated that they were aware that this was a logical consequence of Theorem 1.

T₆: If $\lim_{x \rightarrow a} f(x) = K$ and $\lim_{x \rightarrow a} g(x) = L$

$$\begin{aligned} \text{then } \lim_{x \rightarrow a} [f(x) - g(x)] &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) \\ &= K - L. \end{aligned} \quad \text{¶}$$

T₇: If $\lim_{x \rightarrow a} f(x) = K$ and $\lim_{x \rightarrow a} g(x) = L$

$$\begin{aligned} \text{then } \lim_{x \rightarrow a} [f(x) \cdot g(x)] &= \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) \\ &= K \cdot L. \end{aligned} \quad \text{¶}$$

T₈: If $\lim_{x \rightarrow a} f(x) = K$ and $\lim_{x \rightarrow a} g(x) = L$

$$\begin{aligned} \text{then } \lim_{x \rightarrow a} [f(x)/g(x)] &= \lim_{x \rightarrow a} f(x) / \lim_{x \rightarrow a} g(x) \\ &= K/L \end{aligned}$$

provided $L \neq 0$.[¶]

[¶]This is not the usual ϵ, δ definition; a few of the students understood this well, but for many of them the classical definition concealed the real meaning.

[¶]Arguments were not written for these theorems but the students perceived that T₆, T₇, and T₈ were logical consequences of T₂, T₃, and T₄ respectively.

D₁₀: $f(x)$ is said to be a continuous function at the value $x = a$ if

- i) the function $f(x)$ is defined at $x = a$,
- ii) the $\lim_{x \rightarrow a} f(x)$ exists at $x = a$, and
- iii) the $\lim_{x \rightarrow a} f(x) = f(a)$.

Exercises were given to emphasize the necessity of each part of the definition.

T₉: If $f(x)$ is continuous at $x = a$ and if $g(x)$ is continuous at $x = a$ then the function obtained by adding them, $f(x) + g(x)$, is continuous at $x = a$.

T₁₀: If $f(x)$ is continuous at $x = a$ and if $g(x)$ is continuous at $x = a$ then the function obtained by subtracting these, $f(x) - g(x)$, is continuous at $x = a$.**

T₁₁: If $f(x)$ is continuous at $x = a$ and if $g(x)$ is continuous at $x = a$ then the function obtained by multiplying these, $f(x) \cdot g(x)$, is continuous at $x = a$.**

T₁₂: If $f(x)$ is continuous at $x = a$ and if $g(x)$ is continuous at $x = a$ and the function obtained by dividing these, $f(x)/g(x)$, is continuous at $x = a$, provided $g(a) \neq 0$.**

To emphasize logical relations in the calculus there were frequent reviews in the study-guides, in the quizzes, and during the class discussion with results such as these (where $\longleftarrow$ means "is a consequence of"):

T₁₂ $\longleftarrow$ T₈ $\longleftarrow$ T₄ $\longleftarrow$ D₇ ,

**Theorems T₁₀, T₁₁, T₁₂ were not deduced formally but the groups sensed that they are logical consequences of T₆, T₇, T₈.

$$\begin{array}{llll}
T_{11} & \longleftarrow & T_7 & \longleftarrow & T_3 & \longleftarrow & D_7 & , \\
T_{10} & \longleftarrow & T_6 & \longleftarrow & T_2 & \longleftarrow & D_7 & , \\
T_9 & \longleftarrow & T_5 & \longleftarrow & T_1 & \longleftarrow & D_7 & .
\end{array}$$

This was very helpful and it kept the logical structure alive and growing.

Theorems on Derivatives^{††}

D₁₁: D₃ is now accepted as the definition of the derivative of the function $y = f(x)$ at the point $x = x_b$. This can be rephrased to mean

$$D_x y = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad h \neq 0$$

or, from H₅,

$$D_x y = \frac{y_f - y_b}{x_f - x_b} - \frac{E}{x_f - x_b}$$

or

$$D_x y = \frac{y_f - y_b}{x_f - x_b} - \eta$$

where $\eta = \frac{E}{x_f - x_b}$ for $x_f \neq x_b$.

R₁: A derivative $D_x y$ exists if both right- and left-hand derivatives exist and are equal. A right-hand derivative arises when $x_f > x_b$ and $x_f \rightarrow x_b$; a left-hand derivative arises when $x_f < x_b$ and $x_f \rightarrow x_b$. Continuity of a function at a point does not insure the existence of a derivative there, but the existence of a derivative implies that the function is continuous.

T₁₃: If $y = ax^n$ then $D_x y = n ax^{n-1}$.

^{††} We are assuming that these derivatives exist by H₆.

T₁₄: If $D_x f$ and $D_x g$ exist then $D_x(f + g)$ exists and

$$D_x(f + g) = D_x f + D_x g.$$

T₁₅: If $D_x f$ and $D_x g$ exist then $D_x(f - g)$ exists and $D_x(f - g)$

$$= D_x f - D_x g.$$

T₁₆: Assuming that the derivatives of $f(x)$ and $g(x)$ exist then

$$D_x f(x) \cdot g(x) = f(x) \cdot D_x g + g(x) \cdot D_x f.$$

T₁₇: Assuming that the derivatives of $f(x)$ and $g(x)$ exist and that $g(x) \neq 0$, then

$$D_x \frac{f(x)}{g(x)} = \frac{g(x) \cdot D_x f - f(x) \cdot D_x g}{[g(x)]^2}$$

T₁₈: If $D_x y$ and $D_t x$ exist then so does $D_t y$ and

$$D_t y = D_x y \cdot D_t x.$$

This is called the chain rule.

T₁₉: Given $y = f(x)$ with the inverse function $x = g(y)$; if $D_y x$ exists and $D_y x \neq 0$, then $D_x y$ exists and

$$D_x y = \frac{1}{D_y x}.$$

T₂₀: Given $y = f(t)$ and $x = g(t)$; if $D_t y$ and $D_t x$ exist and $D_t x \neq 0$, then

$$D_x y = \frac{D_t y}{D_t x}.$$

Inverse Derivatives

D₁₂: If $D_x F(x) = f(x)$ then $D_x^{-1}(x) = F(x) + C.$

T₂₁: If $D_x^{-1} f = F$ and $D_x^{-1} g = G$

$$\text{then } D_x^{-1}(f + g) = D_x^{-1} f + D_x^{-1} g = F + G. \#$$

It was left optional to the student to deduce this and the next theorem. It was suggested that he do it but class time was not taken for a formal demonstration.

T_{22} : If $D_x^{-1}f = F$ and $D_x^{-1}g = G$
 then $D_x^{-1}(f - g) = D_x^{-1}f - D_x^{-1}g = F - G$.

Definitions and Theorems and Hypotheses on Integrals

T_{23} : Areas, A , bounded by various curves, $y = kx^n$, between $x = a$, $x = b$, and the x -axis are as follows:

$$1. \quad y = kx^0, \quad A = \frac{kb}{1} - \frac{ka}{1} ;$$

$$2. \quad y = kx, \quad A = \frac{kb^2}{2} - \frac{ka^2}{2} ;$$

$$3. \quad y = kx^2, \quad A = \frac{kb^3}{3} - \frac{ka^3}{3} ;$$

$$4. \quad y = kx^3, \quad A = \frac{kb^4}{4} - \frac{ka^4}{4} .$$

H_7 : With the same conditions as those mentioned in T_{23}

$$y = kx^n, \quad A = \frac{kb^{n+1}}{n+1} - \frac{ka^{n+1}}{n+1} .$$

R_2 : The limit of a sequence of sums of products such as those expressed in D_6 may be indicated notationally by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)h_i .$$

D_{13} : The definite integral $\int_a^b f(x)dx$ is defined to be this

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)h_i \quad (h = h_i, i = 1, 2, \dots, n, \quad b-a = nh)$$

$$T_{24}: \quad \int_a^b (f + g) = \int_a^b f + \int_a^b g. \quad \text{||}$$

$$T_{25}: \quad \int_a^b (f - g) = \int_a^b f - \int_a^b g. \quad \text{||}$$

$$T_{26}: \quad \int_a^b f = - \int_b^a f.$$

^{§§} All the integrals mentioned in the remainder of the outline will be assumed to exist.

^{||} These theorems were not formally deduced but were perceived to be consequences of D_{13} and T_1 and T_2 respectively.

$$H_8: \int_a^b k f = k \int_a^b f.$$

$$H_9: \int_a^a f = 0.$$

$$H_{10}: \int_a^b f + \int_b^c f = \int_a^c f. \quad (a < b < c)$$

$$H_{11}: \int_a^c f - \int_a^b f = \int_b^c f. \quad (a < b < c)$$

H_{12} : (Mean value theorem.) If $f(x)$ is continuous in the closed interval $[a, b]$ then there exists a value of x , $x = w$, such that

$$\int_a^b f(x) dx = (b-a) \cdot f(w)$$

where $a \leq w \leq b$.

The Fundamental Theorem

T_{27} : If $f(x)$ is continuous in the closed interval $[a, b]$ then

$$D_t \int_a^t f(x) dx = f(t). \quad (a \leq t \leq b)$$

T_{28} : If $f(x)$ is continuous in the closed interval $[a, b]$ then

$$\int_a^b f(x) dx = \left. D^{-1} f(x) \right|_a^b$$

or

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$\text{where } D_x^{-1} f = F(x) + c$$

Skills and Concepts Developed in the Course

It is certain that there are some mathematical concepts and skills which the student reviewed and developed for the first time which more properly belonged to algebra and arithmetic. In fact, in the beginning

of this course, these were heavily relied upon to compute rates of change. These will not be reviewed in detail.

It is proposed to follow the main subdivisions of the course in listing skills and concepts which were developed in both the study-guides and the exercises. This plan will assist in creating another point of view (especially on arrangement of materials) from which to note features of this student-experience-discovery approach. Skills and concepts do merge and so there may be some duplication but an attempt will be made to keep this at a minimum. Any skill or concept mentioned is listed where it first appears in the course; of course, review and further use kept many of these alive throughout the remainder of the quarter.

The General Rate Problem

Skills	Concepts
computing average speed arithmetically	
computing average speed algebraically	
given a distance-time formula	
use of the formula for finding the sum	uniform motion
of an infinite geometric progression	accelerated motion
use of binomial expansions	average speed
analysis of functions to see if they	speed
are defined	the difficulty of measur-
finding limits of sequences	ing speed when the motion
constructing graphs of functions	is accelerated
using the quadratic formula	idealization of physical
solving equations of order higher than	phenomena
two	hypothesis
computing slopes of lines	variable
computing slopes of curves at a given	constant
point	function
finding values of x such that the slope	sequence
of a curve is zero	slope
algebraic computation of $R_x y$ (later to	(relative) minimum value
become $D_x y$)	of a function

finding $R_x y$ at sight
 related rate problems: approximations
 of errors, expansion, contraction,
 change
 simple geometric maximum and minimum
 problems
 simple minimum cost problems
 using $R_x y$ to approximate the change
 in y due to a given change in x

(relative) maximum value
 of a function
 rate of change, $R_x y$, of
 y with respect to x

The General Area Problem

Skills

approximating area under a curve by
 setting up a sequence of sums of
 areas of rectangles
 obtaining "bounds" for the area under
 a curve
 approximating the area under a curve
 by setting up a sequence of sums of
 areas of trapezoids
 approximating work when the force is
 variable by the use of a sequence of
 sums of "little works"
 setting up an expression for the sum of
 rectangles in finding the area under
 a curve
 using expressions such as
 $1^2 + 2^2 + 3^2 + \dots + n^2$

$$= \frac{n(2n + 1)(n + 1)}{6}$$

Concepts

"intuitive" area under a
 curve
 work
 work when the force is
 variable
 "regular" subdivision versus
 "random" subdivision

Fundamental Problems Recapitulated. Systematic Approach

Skills

finding limits of sequences
 performing operations termwise on
 sequences
 finding limits of functions
 examining functions for continuity
 examining functions for points and
 types of discontinuity
 carefully defining a function
 making deductions on sequences
 finding the equation of a tangent
 to a curve

Concepts

the generalized rate problem
 the generalized area problem
 limit (defined)
 real number
 convergence
 divergence
 convergent sequence
 divergent sequence
 limit of a function
 continuous function
 discontinuous function

computing area under $y = x^2$ from
 $x = 0$ to $x = a$ by setting up an
 expression for the sum of n rec-
 tangles and then letting $n \rightarrow \infty$

infinite discontinuity
 finite discontinuity
 deduction vs. induction

The Derivative

Skills

differentiation of polynomials
 area under curve $y = x^2$ from
 $x = a$ to $x = b$
 differentiation of products of
 two or more functions
 differentiation of the quotient
 of two or more functions
 differentiation of a function of
 a function (chain rule)
 differentiation of a power of a
 function
 differentiation of implicit
 functions
 area under a curve $y = x^3$ from
 $x = a$ to $x = b$
 inverse differentiation
 checking rules for differentiation
 by doing them the "long way"
 given the slope of a curve and a
 point on it to find the equation
 of the curve
 computing $D_x y$ when y and x are given
 in terms of a parameter

Concepts

derivative
 differentiation
 notations for the deriva-
 tive
 functions defined by
 derivatives
 inverse functions
 parametric functions
 functional expression
 independent of notation
 implicit function
 "inverse" operations
 inverse derivative
 antidifferentiation
 "constant" coming from
 inverse differentiation

Derivatives of Higher Order

Skills

finding second and higher deriva-
 tives of implicit functions
 using $D_x^2 y$ to give certain further
 information about curves
 using $D_x^2 y$ to test for points of
 maximum and minimum
 using $D_x^2 y$ in regard to points of
 inflection
 given acceleration as a function of
 time t to find expressions for
 velocity and distance
 formulation of hypotheses and working
 problems on area under the curve
 $y = x^n$ between $x = a$ and $x = b$

Concepts

derivatives of higher
 order
 notations for derivatives
 of higher order
 geometric significance of
 the second derivative
 inflection point
 necessary and sufficient
 conditions
 rectilinear motion
 positive velocity
 negative velocity

finding area under the curve
 $y = kx^n$ between $x = a$ and $x = b$
 computing higher derivatives of
 parametric functions
 computing higher derivatives of
 implicit functions

positive acceleration
 negative acceleration

Inverse Rate Problem Continued

Skills

finding the length of very special
 curves
 antidifferentiating algebraic expres-
 sions with assistance of the chain
 rule
 constructing without computation an
 approximate graph of the first
 derived function of a function

Concepts

Pythagorean Theorem
 "length" of a curve
 "first derived" curve

The Definite Integral. Evaluation. Applications.

Skills

computing simple integrals by the
 use of theorems and hypotheses
 computing integrals by antidifferentia-
 tion
 finding areas under curves
 finding areas between curves
 differentiation of function defined by
 integrals
 finding volumes of revolution
 finding the force of a liquid on the
 sides of a container
 finding volumes of solids which are
 not solids of revolution
 finding work done when the force is
 variable
 deduction of the fundamental theorem
 deduction of a corollary from the
 fundamental theorem

Concepts

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)h_i$$

definite integral
 integration
 integrand
 limits of integration
 notations for integrals
 functions defined by
 integrals
 "element" of area
 "element" of volume
 "element" of work

Some Methods of Argument

In addition to the content relating to logical organization of materials and the calculus facts and skills, there was frequent emphasis on the nature of arguments which were being used. Hence, whenever possible, differences were considered between arriving at a hypothesis by induction and establishing a result by deduction. The students were aware that, at times, problems were being "worked" and theory was being extended with the knowledge that some of the principles involved were still hypotheses which were yet subject to test by deduction. Hence frequently one heard such expressions as "it appears as though. . . ." or "assuming this is true, then" It is believed that the students understood these differences very well.

Deduction

To help confirm the fact that part of the content of this course was concerned with making deductions there are here included several examples taken from notebooks or tests. It may be true that professional mathematicians could improve on these but the results of this teaching method would not be honestly presented if these examples appeared in any way other than that in which they were originally written. Of course, as one might expect in any class, there were some students who either did these not at all or who wrote the proofs rather poorly.

1. Given $y = y(u)$ but $u = u(x)$, to deduce that

$$D_x y = D_u y \cdot D_x u.$$

Deduction:

$$\text{From } D_{11} \quad D_x y = \frac{y_f - y_b}{x_f - x_b} = \varepsilon.$$

$$\text{Now} \quad \frac{y_f - y_b}{x_f - x_b} = \frac{y_f - y_b}{u_f - u_b} \cdot \frac{u_f - u_b}{x_b - x_b} = D_x y + \varepsilon$$

$$\text{But} \quad \frac{y_f - y_b}{u_f - u_b} = D_u y - \eta$$

$$\text{and} \quad \frac{u_f - u_b}{x_f - x_b} = D_x u - \delta.$$

$$\text{Hence } D_x y + \varepsilon = (D_u y - \eta) (D_x u - \delta).$$

Now as the interval or difference $x_f - x_b \rightarrow 0$

then $\delta \rightarrow 0$, $\eta \rightarrow 0$, and $\varepsilon \rightarrow 0$ and one has

$$D_x y = D_u y \cdot D_x u.$$

Q.E.D.

2. Given $f(x) = u(x) \cdot v(x)$, to deduce that

$$D_x(uv) = u \cdot D_x v + v \cdot D_x u$$

Some students used notation as in No. 1 above and began their deductions with something like this:

$$D_x(uv) = \frac{u_f v_f - u_b v_b}{x_f - x_b} = \varepsilon$$

and then placed in the numerator the expression $u_f v_b$ and finally came out with the desired result.

Several students employed this method:

$$\begin{aligned} f(x+h) - f(x) &= u(x+h) \cdot v(x+h) - u(x) \cdot v(x) \\ &= u(x+h) \cdot v(x+h) - u(x) \cdot v(x+h) \\ &\quad + u(x) \cdot v(x+h) - u(x) \cdot v(x). \end{aligned}$$

Hence

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \\ &= \frac{v(x+h) \cdot [u(x+h) - u(x)] + u(x) [v(x+h) - v(x)]}{h} \end{aligned}$$

and

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \left[\lim_{h \rightarrow 0} v(x+h) \right] \cdot \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} \right] \\ &\quad + u(x) \cdot \lim_{h \rightarrow 0} \left[\frac{v(x+h) - v(x)}{h} \right] \end{aligned}$$

or

$$D_x y = v \cdot D_x u + u \cdot D_x v.$$

Q.E.D.

3. The fundamental theorem.

To show that

$$D_b \int_a^b f(x) dx = f(b).$$

One student had this deduction recorded in the following manner

in his notebook:

$$\int_a^{b+h} f(x) dx - \int_a^b f(x) dx = \int_b^{b+h} f(x) dx \xrightarrow[b \leq w \leq b+h]{\text{Mean Value Theorem}} f(w)(b+h-b).$$

Hence

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\int_a^{b+h} f(x) dx - \int_a^b f(x) dx}{h} &= \\ \lim_{h \rightarrow 0} \frac{f(w) h}{h} &= \lim_{h \rightarrow 0} f(w) \xrightarrow[b \leftarrow w \leftarrow b]{\text{since, as } h \rightarrow 0} f(b). \end{aligned}$$

Q.E.D.

Induction

Some examples of the use of induction follow:

1. After finding $R_x y$ ($D_x y$) for many polynomials the student observed that one could compute $R_x y$ at sight by the use of the rule that if $y = ax^n$ then $D_x y = nax^{n-1}$. Later the result was deduced in the traditional manner.
2. In the course of exercises the students found the area under $y = x^2$ bounded by the line $x = 1$ and the x-axis; this was done by arithmetical computation--that is, the setting up of a sequence of sums of areas of trapezoids. For $x = 1$ the area was $A = 1/3$; when $x = 2$, $A = 8/3$; when $x = 3$, $A = 27/3$. It was then hypothesized that for $x = a$, $A = a^3/3$. Problems were worked by the use of this hypothesis and its validity was later established by deduction.

Generalization from Analogous Cases

This differs somewhat from the method of arguing by analogy. In analogy two situations A and B are shown to have identical elements; then a conclusion known to be true for A is suggested to be true for B. In this course, however, common elements from two situations A and B were extracted to form an abstract discipline which embraced them both. Such was the case in studying

1. slope and speed to develop the idea of rate of change and the limit of a sequence of quotients--later to be known as the derivative,

2. area under a curve and work done when the force is variable
to develop the idea of a limit of a sequence of sums of
products--later to be called the integral.

That the students, in general, noted these analogies is supported by results taken from tests. In the first test there was this problem: Given $s = 4t^2$, as an equation of motion, make at least four numerical approximations to the speed at the end of the first second and then suggest a number which you would present as the speed at the end of the first second. Another question (later) was: If you were given a curve, $s = 4t^2$, and you were asked to find the slope of this curve at $t = 1$ by numerical approximations (a) would your computations need to be different, (b) could they be exactly the same, (c) could they be only similar to those in the previous problem? In the Sections One and Two, twenty-six of the thirty-eight students said that the computations could be exactly the same; nine agreed that they could be similar; three write that they would need to be different.

Similarly the responses to a test item on area and work are interesting. In thirty out of thirty-eight cases the students identified the finding of area under the curve $y = 3x^2$ from $x = 2$ to $x = 4$ to be the same as computing work from $x = 2$ to $x = 4$ when the variable force is given by $F = 3x^2$. In most cases the student said, "The computation and the numerical answer would be the same, but the answer would have a different interpretation."

Interplay of Induction and Deduction

The alternate prominence of induction and deduction was very much in evidence during the latter half of the course. First the results of areas under curves from $x = a$ to $x = b$

$$1. \quad A(x) = \frac{b^2}{2} - \frac{a^2}{2},$$

$$2. \quad A(x^2) = \frac{b^3}{3} - \frac{a^3}{3},$$

and $3. \quad A(x^3) = \frac{b^4}{4} - \frac{a^4}{4},$

were deduced from theorems in algebra as indicated on page 100. Then by induction was suggested the hypothesis that

$$4. \quad A(x^n) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

Using this, special numerical cases and problems such as

$$A(x^3 + x^2 + 3x) = \frac{b^4}{4} - \frac{a^4}{4} + \frac{b^3}{3} - \frac{a^3}{3} + \frac{3b^2}{2} - \frac{3a^2}{2}$$

were deduced from the hypothesis. Later this hypothesis was changed into a theorem by deducing it from new definitions, new theorems, and the fundamental theorem.

Interesting Variations in Content

Under this method of teaching which is being tested, students possess a core of hypotheses, definitions, theorems, concepts and skills in common, although they attacked problems differently and regarded concepts of the calculus from slightly different points of view. Indeed, during grading of tests and studying of notebooks prepared by the students, and in the course of classroom discussion,

several bits of evidence came to light to show that the content in toto for any one student was often an individual product. Some of these interesting discoveries have already been mentioned in this study; one need merely to refer to pages 80, 85, or 89 for examples of different results from this emphasis on an encouragement of the use of student initiative. Some others follow.

Ways of Estimating Speed

Although it may be thought that the following illustrations merely exhibit several ways to work the same problem, they also show that the students themselves used originality to devise different methods to arrive at the desired result. Moreover, some of the examples to follow were taken from tests; this fact indicates that the students did not have the greater amount of time which one might have in homework assignments. Hence this is even more convincing evidence that these methods were an intrinsic part of the individual student's own conception of the calculus.

In response to the test question, "A body is moving such that the distance-time formula is given by

$$s = 8t^2 + 4.$$

By numerical methods arrive at a number which you would present as the speed at the end of the second second ($t = 2$)."

several solutions were presented as follows:

1. One student arranged his arithmetical data in this way:

t	s			
2 sec.	36 feet			
2 1/4 sec.	44 1/2 feet	↓ 40 ft.	↓ 18 ft.	↓ 8 1/2 ft.
2 1/2 sec.	54 feet			
3 sec.	76 feet	↓		

His first approximation was the average speed

$$A_1 = \frac{76 - 36}{1} = 40 \text{ feet per second;}$$

then, his second approximation was

$$A_2 = \frac{54 - 36}{1/2} = 36 \text{ feet per second;}$$

the third was

$$A_3 = \frac{44 \frac{1}{2} - 36}{1/4} = 34 \text{ feet per second.}$$

He then hypothesized that the sequence of average speeds would be 40, 36, 34, 33, 32 1/2, . . . and hence "presented" the number 32 feet per second as the answer.

2. Another student viewed the problem differently. His first approximation to the speed at $t = 2$ sec., $A_1 = 18$ feet per second was obtained by dividing

36 feet by 2 seconds to find the average over the first two seconds.

His second approximation gives the average speed over the last second (i.e. from $t = 1$ to $t = 2$), or

$$A_2 = \frac{36 - 12}{1} = 24 \text{ feet per second}$$

In a similar manner

$$A_3 = \frac{36 - 22}{1/2} = 28 \text{ feet per second}$$

and

$$A_4 = \frac{36 - 28 1/2}{1/4} = 30 \text{ feet per second.}$$

A promising sequence 24, 28, 30 feet per second seems to form and he hypothesized it to continue as

$$24, 28, 30, 31, 31 \frac{1}{2}, \dots$$

and presented 32 feet per second as the speed at the end of the second second.

Ways of Estimating Slope

There were two methods used in arriving at the slope of a given curve at a given point. These methods differed in the same way as the solutions for the "speed" problem; several students approached the given abscissa from the left and others from the right. To be more specific, to "present" a number as the slope of $y = 2x^2$ at $x = 3$ some worked out the slope-sequence 10, 11, 11 $\frac{1}{2}$, 11 $\frac{3}{4}$, . . . $\rightarrow$ 12, while others obtained 16, 14, 13, 12 $\frac{1}{2}$, 12 $\frac{1}{4}$, . . . $\rightarrow$ 12.

Ways of Estimating Work

Given $F = 3x$ there were three ways used by students to compute the work done in exerting this force from $x = 1$ to $x = 3$. These are outlined here.

1. Most students used the force at the beginning of each distance interval, multiplied this force by the distance, and obtained a sequence as follows (A_i means the i th approximation):

$$A_1 = 3(3 - 1) = 6 \text{ foot-pounds.}$$

$$A_2 = 3(2 - 1) + 6(3 - 2) = 9 \text{ foot-pounds}$$

$$A_3 = 3(1\frac{1}{2} - 1) + 9/2(2 - 1/2) \\ + 6(2\frac{1}{2} - 2) + 15/2(3 - 2\frac{1}{2}) = 10\frac{1}{2} \text{ foot-pounds.}$$

This sequence was assumed to continue as the distance intervals became smaller and the number 12 foot-pounds was given as the answer.

2. Several responses were found in which the force used was that at the end of each distance interval. In these cases the approximating sequence developed as follows:

$$A_1 = 9(3 - 1) = 18 \text{ foot-pounds.}$$

$$A_2 = 6(2 - 1) + 9(3 - 2) = 15 \text{ foot-pounds.}$$

$$A_3 = 9/2(1\frac{1}{2} - 1) + 6(2 - 1\frac{1}{2}) + 15/2(2\frac{1}{2} - 2) \\ + 9(3 - 2\frac{1}{2}) = 13\frac{1}{2} \text{ foot-pounds.}$$

The sequence of "works" which is assumed to exist is therefore 18, 15, $13\frac{1}{2}$, $12\frac{3}{4}$, . . . ; these students hence presented the hypothesized limit, 12 foot-pounds, to be the answer.

3. Another student made use of his understanding that work and area were abstractly the same. He merely drew the figure which is shown at the right and computed the area of the trapezoid determined by the curve $F = 3x$, the lines $x = 1$ and $x = 3$, and the x -axis. He gave as the answer $A = \frac{2(3+9)}{2} = 12$ foot-pounds.

Expressions in Notebooks

Earlier it was mentioned that in details the calculus content would be expected to differ for each student. This was easily confirmed by an examination of the notebooks of the students;⁹⁹ in many cases the formal definitions were more numerous than those in the summary list given in this chapter. Some of these were found worded in such a way that they could be offered as strong evidence that they were products of students' own thought; this again helps to describe features of this approach to the calculus. A few examples of student expressions are given here.

- D. A sequence which approaches a limit is a convergent sequence.
- D. A sequence which doesn't approach a limit is called a divergent sequence.
- D. Functional relations employing a third variable, where two of the variables are both functions of the third, are called parametric relations.
- D. A point where the "y" value ceases to increase, but momentarily stops, and then decreases as the "x" constantly increases, is termed a point of (relative) maximum.
- D. A point where the "y" value ceases to move and then resumes motion in the same general direction is called a minimax or a point of inflection.
- H. Such a thing as a limit of a sequence exists.
- H. Two convergent sequences multiplied termwise will give a third convergent sequence, the limit of which is the product of the limits of the two given sequences.

⁹⁹ Instructions and suggestions given for the organization of the notebooks are found in Chapter Three, p. 82.

- R. A method by which expression for slope could be written at sight (given $f(x)$) is to use the exponent as a number in front of the x -term and reduce the exponent by 1. The last term is omitted unless it is an x -term in which case the whole number before it is used.
- D. Derivative. Given $f(x)$ then
 - 1) replace x by $x + h$
 - 2) compute the change in y
 - 3) divide by h
 - 4) find the limit as h approaches 0.
- D. The function $f(x)$ is said to have an inflection point when for a value of x which makes $f'(x) = 0$ all values to the right are greater than $f(x)$ and all values to the left are less than $f(x)$ or vica versa.
- T. (Mean Value Theorem)
Somewhere between a and b there exists an ordinate y which, when multiplied by $(b-a)$, gives the area under the curve.
- D. A function is made up of two sets of numbers which are somehow related.
- R. Assumptions:
 - 1) $h \rightarrow 0$ in any manner.
 - 2) the limits of the approximating functions exist
 - 3) the limits of sequences of sums of products exist; that is, integrals exist.

Organizations of Notebooks

As hinted above, different students arranged variously detailed organizations of their calculus. Of course, "in the large" they followed the outline of the quarter's work, but "in the small" different ways of thinking about it came into evidence. There are made here a few comments on the various organizations of content.

1. About ten students had their work organized roughly into sections such as these: I. Definitions, II. Theorems. Six added a section on assumptions.

2. Approximately fourteen of the students used the style of stating a definition with theorems concerning it and then moving to a next definition followed by its related theorems.

3. Others had methods of arranging their material which did not fall into either of the above groups. Not all these books were well-organized, but some were quite distinctive. One, in particular, summarized the content by the use of such topics as "Derivatives--what one can do with them" and "Integrals--their use" and others. One student used his definition list also as an index and had carefully entered page numbers of his outline and the study-guides. Another called her quite complete word-list a "calculus dictionary," followed this by a topical outline of the calculus with deductions of theorems, and concluded with "procedures for solutions." Another's notebook was simply a "recipe book" to guide him in the solution of problems. Several used colored inks and various types of lettering to emphasize topics to bring out different constituents of content.

Nor were all the notebooks arranged in the formality of an outline. One student wrote a resume' of the first several day's thinking on calculus and in a somewhat story-form led his reader into thinking on this subject. Several others, at appropriate places in their work, interjected brief discussions to furnish more smooth continuity in the logical development of the calculus. At the ends of various sections in some books were found summaries with outlines as follows:

4. Approximately one-third of the students included from several to a good number of deductions in their logical organizations; five included examples and illustrations.

CHAPTER FIVE

EVALUATION

Introduction

In Chapter One have been suggested four hypotheses¹ which are under test in this research. These concern the understanding and use of the calculus, an appreciation of the calculus as a logical structure, an acquaintance with the nature and process of growth of mathematics, and experiencing the thrill of discovery. Somehow the extent to which each of these four items has been achieved must be appraised. The fact that the evaluation of the results of this experiment is not a simple task, and therefore is challenging, is stated well in the Twenty-second Yearbook where it is written

It is relatively easy for teachers of mathematics to evaluate the ability to recall information and to perform certain skills. It is not so easy, however, to assess the growth of concepts and the development of understanding. It is still more difficult to gather evidence of the improvement in the ability to apply what is learned in situations beyond those in which the learning took place. Finally, it is a real challenge to be able to judge whether students are gaining certain attitudes and appreciations.²

¹See p. 26.

²Emerging Practices in Mathematics Education, Twenty-second Yearbook of the National Council of Teachers of Mathematics, p. 339.

Various means have been devised to test the hypotheses³ proposed in this study and to grapple necessarily with the difficulties proposed in the Yearbook. The items used to evaluate the effectiveness of this method of teaching are:

1. Scores made by students on teacher-made cross-tests on proficiency and understanding.
2. Responses made on queries concerned with logical organization of the calculus.
3. Illustrations of perceptions of the nature of mathematics and insights to how mathematics grows.
4. Short papers by students on "What I Conceive the Calculus to Be."
5. Evidences of change of attitude toward mathematics.
6. Responses by students on an "evaluation sheet" and other sources.
7. Evidences of the effect of this student-experience-discovery approach in second-quarter calculus.

The Cross-Testing Program

Lack of suitable standardized tests made it necessary to use teacher-made instruments in cross-testing to compare understanding and proficiency in the experimental and traditional sections of first-quarter calculus.

³These four hypotheses are presented in Chapter One, p. 26.

A "cross-test" is a situation in which the writer and the teacher of a traditional section each prepared tests on the same materials primarily for his own section. Then both of these tests were administered to the students in both of the sections. Each teacher used these tests in whatever manner he wished for his own course and then they were mixed into a packet and graded uniformly by the writer. The resulting scores are the scores which appear in this study.

Winter Quarter Cross-Testing Results

At the close of the winter quarter for Section One⁴ and Section A there were two tests given. One of these, Test One, was made out by the writer on the theory of integrals and simple applications and a general review of the theory of the calculus. This was a two-hour test. The other one, Test A, also a two-hour test, covered the same material and it was arranged by the teacher of Section A. These tests are included in Appendix D. The results are shown in Table 13.

TABLE 13
ARITHMETIC MEANS OF SCORES ON TESTS

	Score on Test One Made for Experimental Section One (Possible Score = 272)	Score on Test A Made for Traditional Section A (Possible Score = 180)
Section One (experimental)	217	130
Section A (traditional)	131	130

⁴The reader is referred to Chapter Two, p. 34 for the identification of the various classes.

The test made for Section One, it will be noted, consisted of items which emphasize understanding of the calculus while that made for Section A was essentially a list of problems.⁵ Hence at a glance it is seen that the experimental section surpassed the traditional one in understanding of concepts and logical organization as measured by the test and they were just as proficient in working problems.

Winter Quarter Results Studied Statistically

It seemed advisable to study further these final test results and initial data on previous grades, the pretest , and the A C E scores to make some conclusions on performance independent of the initial data.

Accordingly, letting

x_1 be the student's average score in previous mathematics courses,

x_2 be the student's calculus pretest score,

x_3 be the student's A C E score,

y_1 be the student's score on Test One,

y_A be the student's score on Test A,

the data of the experimental and traditional groups of twenty-one and twelve students respectively were combined and there were developed regression equations which would predict an expected score on Test One and Test A on the basis of scores x_1 , x_2 , and x_3 of both sections. The data are shown in Table 14.

⁵Appendix D, p. 330.

TABLE 14

INITIAL DATA AND SCORES ON TEST ONE (y_1)
AND TEST A (y_A) FOR EACH SECTION

Identification	Initial Data			Scores on Final Tests	
	x_1	x_2	x_3	y_1 (Test One)	y_A (Test A)
<u>Experimental Group</u>					
101	6	170	80	223	130
102	3	104	53	196	83
103	4	70	57	153	61
104	8	199	35	233	140
105	5	147	56	215	100
106	9	168	91	221	128
107	9	179	76	268	162
108	9	190	88	250	162
109	8	187	71	250	159
110	5	96	59	153	79
111	7	150	34	207	153
112	7	233	20	232	147
113	10	121	95	262	163
114	5	122	74	203	140
115	4	71	72	214	143
116	5	167	91	201	97
117	11	195	77	224	143
118	8	93	20	219	142
119	9	169	66	185	83
120	10	234	37	225	163
121	6	188	92	212	156
<u>Traditional Group</u>					
201	3	150	83	133	145
202	5	56	78	159	116
203	9	172	70*	156	147
204	5	139	87	177	165
205	7	178	42	151	136
206	8	162	65	142	149
207	9	176	69	117	147
208	7	150	40*	108	92
209	8	189	70*	131	156
210	4	171	78	94	105
211	6	163	83	124	145
212	2	168	81	82	58

*Estimated.

These regression equations are

$$y_1 = 130.4 + 11.02^* x_1 - 0.119 x_2 - 0.0026 x_3,$$

$$y_A = 53.4 + 6.75^{**} x_1 - 0.115 x_2 + 0.2046 x_3.$$

Only the coefficients of x_1 are significantly different from zero. Thus the calculus pretest score and the ACE score, at least as they are used here, have no significant effect on the scores on Test One and Test A.

These regression equations were then used to compute expected scores $\hat{y}_1$ on Test One and $\hat{y}_A$ on Test A for each student. Next the differences between the expected results and the actual scores, $y_A - \hat{y}_A$ and $y_1 - \hat{y}_1$, were computed. The final test scores, y_1 and y_A , the expected test scores, $\hat{y}_1$ and $\hat{y}_A$, and the differences, $y_A - \hat{y}_A$ and $y_1 - \hat{y}_1$, for each student are found in Table 15.

The arithmetic means, $\overline{y_1 - \hat{y}_1}$ and $\overline{y_A - \hat{y}_A}$, of the differences $y_1 - \hat{y}_1$ and $y_A - \hat{y}_A$ are shown in Table 16. The positive value of $\overline{y_1 - \hat{y}_1} = 27.10$ means that in the experimental group the students scored on the average 27.10 points higher on Test One than would be expected on the basis of their x_1 , x_2 , x_3 scores. The difference between the means of Section One and Section A is significant for Test One at the 1% level.⁶ The difference in their means (-1.79 and 3.14) is not significant for Test A.

*This coefficient is significantly different from zero at the 5% level. This means that under the assumption that the real value of this coefficient is zero, the probability that in an experiment of this kind, the sample value of the coefficient would be greater than or equal to 11.02 is 5% or less.

**This coefficient is significantly different from zero at the 1% level.

⁶This means that under the assumption that the difference between the true means (whose sample values are 27.10 and -51.59) is zero, the probability that in this experiment this difference is equal to or greater than 78.69 is 1% or less.

TABLE 15

ACTUAL FINAL TEST SCORES, EXPECTED
TEST SCORES, AND THEIR DIFFERENCES

Identification	y_1	$\hat{y}_1$	$y_1 - \hat{y}_1$	y_A	$\hat{y}_A$	$y_A - \hat{y}_A$
Experimental Section One						
101	223	176.00	+47.00	130	129.92	+ .08
102	196	150.89	+45.11	83	96.53	-13.53
103	153	165.95	-12.95	61	100.18	-39.18
104	233	194.68	+38.32	140	137.57	+ 2.43
105	215	167.79	+47.21	100	115.61	-15.61
106	221	209.25	+11.75	128	152.20	-24.20
107	268	207.98	+60.02	162	150.40	+11.60
108	250	206.64	+43.36	162	154.12	+ 7.88
109	250	196.02	+53.98	159	143.55	+15.45
110	153	173.86	-20.86	79	110.34	-31.34
111	207	189.52	+17.48	153	124.96	+28.04
112	232	179.65	+52.35	147	131.67	+15.33
113	262	225.87	+36.13	163	154.35	+ 8.65
114	203	170.72	+32.28	140	116.41	+23.59
115	214	165.80	+48.20	143	103.36	+39.64
116	201	165.31	+35.69	97	125.07	-28.07
117	224	228.10	- 4.10	143	165.96	-22.96
118	219	207.37	+11.63	142	122.27	+19.73
119	185	209.20	-24.20	83	147.20	-64.20
120	225	212.53	+12.47	163	155.52	+ 7.48
121	212	173.82	+38.18	156	134.45	+21.55
Traditional Section A						
201	133	195.33	-62.33	145	107.97	+37.03
202	159	178.59	-19.59	116	109.61	+ 6.39
203	156	208.83	-52.83	147	148.37	- 1.37
204	177	168.66	+ 8.34	165	121.03	+43.97
205	151	186.15	-35.15	136	129.83	+ 6.18
206	142	199.02	-57.02	149	139.44	+ 9.56
207	117	208.36	-91.36	147	148.62	- 1.62
208	108	189.50	-81.50	92	126.19	-34.19
209	131	195.79	-64.79	156	143.57	+12.43
210	94	153.85	-59.85	105	116.12	-11.12
211	124	176.82	-52.82	145	129.73	+15.27
211	124	176.82	-52.82	145	129.73	+15.27
212	82	132.17	-50.17	58	102.89	-44.89

TABLE 16
ARITHMETIC MEANS OF THE DIFFERENCES
BETWEEN ACTUAL AND EXPECTED RESULTS
ON TEST ONE AND TEST A

	Experimental Section One	Traditional Section A
$\overline{y_1 - \hat{y}_1}$	27.10	-51.59
$\overline{y_A - \hat{y}_A}$	- 1.79	3.14

The results shown above indicate that the method of teaching under study was more effective especially in promoting a deeper understanding of the calculus and that this was not done at the sacrifice of proficiency in manipulations and applications. It is possible that these differences in achievement may be due to certain factors of which the experimenter is not aware, but the statistical analysis indicates that they are not due to student variations in previous grades, mathematical preparation as measured by the calculus pretest, and innate ability as indicated by the A C E scores.

The experimenter must suggest, however, that there is one factor which may have had a small effect upon the results of the tests. Some of the short test items may have been phrased in a "special language" which the students in the traditional sections found difficult to understand. It is thought that there were not very many of these.

Spring Quarter Results

Scores from testing at the close of the spring quarter were not used to make any statistical study as presented in the previous section. The same test (Part II) made by the writer which was used at the end of the winter quarter was given to both Sections Two and B; this test was a final examination which emphasized largely the fundamental concepts of the calculus. The teacher of Section B prepared his final test consisting solely of problems, and both sections took it.

The results are interesting but they must be treated with reservations. The teacher of Section B remarked that he had told his students earlier in the quarter that they would not be "held" for definitions or theory. On the other hand, the experimental section, through interest in questions in understanding the calculus, almost unnoticeably fell behind the planned schedule by several days, which, plus the time taken for cross-testing with still a third section⁷ and other necessary experimental activities, gave this section about a week's less time to work on applications of integration.⁸ The results are shown in Table 17.

Section B was more proficient in working problems involving integrals and Section Two demonstrated a higher level of understanding of concepts and structure of the calculus--perhaps at the expense of

⁷The cross-testing program with a second traditional section in the spring quarter did not come to a state of completion sufficient for the data to be used.

⁸This will be discussed later as a precaution to be taken in the use of this student-experience-discovery method.

proficiency in problem-solving. A week's practice on integrals for Section Two and perhaps a review of fundamental ideas for Section B might have changed the picture somewhat. If the two tests were considered as one, then Section Two's mean score is 102 compared to 90 for Section B. These tests are found in Appendix D.

TABLE 17

ARITHMETIC MEANS OF SCORES ON TESTS MADE
BY SECTION TWO AND SECTION B

	Test Made for Section Two	Test Made for Section B
Section Two (experimental)	59	43
Section B (traditional)	25	65
Total Score Possible on Test	82	86

Responses on Logical Organization

It is more difficult to gather evidence on the student's understanding of foundations and logical relations in the calculus than it is to study proficiency in problem-solving. In response to short test-items, if the student doesn't know the answer, he may not understand what is meant by the item, or, if the answer is correct, there is danger that it might have been memorized. However, with these

shortcomings in mind, there were used short test items and some discussion questions on tests which requested answers concerning logical organization. Study of the evaluation questionnaires, evidences in class, and an examination of the notebooks, would be expected to furnish material on student responses to queries on logical relations.

Test Responses: Awareness of Logical Organization

Throughout the tests given at the close of both the winter and spring quarters were interspersed items which concerned the relation of ideas in the calculus.⁹ Items have been taken from these tests and the number of satisfactory responses in each section has been indicated in Table 18. The questions or items are stated in the left-hand column of the table and expected responses appear in the next column; a somewhat longer discussion than the brief answer given was expected for the last item. For the next to the last item a correct deduction was expected.

It is difficult to refrain from quoting some of the students in Section One and Two on the last test items of the table to give evidence of the relation of various parts of the calculus to each other. Some expressions follow:

1. "They (differentiation and integration) meet on the common ground of antidifferentiation which is the inverse of differentiation; antidifferentiation in turn becomes involved with the integration problem through the fundamental theorem of the calculus."

⁹Of course all tests given to Sections One and Two throughout the quarter contained items on logical relations.

TABLE 18

NUMBER OF SATISFACTORY RESPONSES TO
SOME TEST ITEMS ON LOGICAL RELATIONS

Test Item	Nature of Expected Responses	Tests from Section			
		One	A	Two	B
$D_x y = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$	$h \neq 0$				
What is the limitation placed on h ?		20	4	16	6
The relation $D_x(f+g)$ comes from what basic theorem on limits?	$\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$	14	6	10	1
Given $A = 3r^2$ explain how you could find the rate of change of A with respect to r when r equals 4 <u>if you did not know any calculus.</u>	Explanation in terms of arithmetical sequences or algebraic "h" method	12	1	8	0
What would you suggest as the concept most basic to the calculus	Sequences or limits	12	3	12	2
What would you suggest would be the first necessary term to be defined?	Limit of a sequence			16	0
Deduce the quotient rule, product rule, or the chain rule.	(A correct deduction)	13	1	12	0
Explain briefly or describe the logical thread which connects the process of differentiation with that of integration.	Antidifferentiation and the fundamental theorem	11	0	7	0
Total number of responses		20	12	18	12

2. DifferentiationIntegration

3. "They both have the same beginning in the theories of limits. They are both limits only the derivative is the limit of a sequence of quotients and the integral is the limit of a sequence of sums of products."
4. "We start with sequences and go from there to the rate of change, and then to the derivative and the anti-derivative. On the other side of the picture we go from sequences up through areas to the integral. Here we go through the mean value theorem to the fundamental theorem. Here the integral and the antiderivative come together. The fundamental theorem, therefore, is the link which holds the two chains together."
5. "The fundamental theorem is the missing link which connects the integral calculus to the differential calculus."
6. "The derivative is a limit of a sequence of quotients while the integral is a limit of a sequence of sums of products. Therefore the concepts are different. They are connected through the fundamental theorem and the inverse derivative and the mean value theorem (therefore the function must be continuous). The real connection occurs by the fundamental theorem

$$D_u \int_a^u f(x) dx = f(u)."$$

Responses from Other Sources

Both Chapters Three and Four furnish evidence of the effort and results on the part of the teacher and students to understand the structure of the calculus, and the responses given there to help realize the aims of these chapters will not be repeated. Student responses which become significant contributions to the development of the quarter's calculus are found in Chapter Three on pages 69-72, 76-78, 80-81, 85, 88, 90-91, 94, and some responses in Chapter Four are found on pages 97-106 and 111-122.

Some of the papers on "What I Conceive the Calculus to Be" revealed interesting statements which indicated that the writers realized something about the logical structure of the calculus. The following statements have been taken from the student papers:

1. "From the integrals we develop a mean value theorem and from derivatives we go to antiderivatives and the two branches meet in one culminating theorem."
2. ". . . therefore, the calculus to me is a study of sequences which approach limits and the methods of finding these limits."
3. "Calculus is a mathematics built upon the foundation of one hypothesis: the existence of limits. From the foundation, the calculus rises in two columns. One column arises through speed and slope to the development of differentiation. The second column rises through area and work computations to the development of integration. . . . At the top there is an arch connecting these two columns. The arch is made up of three sections: antidifferentiation, the fundamental theorem, and the mean value theorem." (There follows in the paper a figure of this arch.)
4. "In the initial phase or first few days of this course I had no idea what calculus was. After a few days I began to feel that calculus was rather 'weak' compared

to other more precise aspects of mathematics. This was rather difficult to digest mentally. However, after coming to the concept of limits and the definition we accepted for it, I soon began to realize that calculus was only as weak as we wanted to believe it was."

5. "The calculus treats the actual situation by generalization and idealization."
6. "To me calculus is a quick way to solve a hard problem and the answer to the problem is either the 'limit of a sequence of quotients' or the 'limit of a sequence of sums of products.' This is calculus."
7. "Calculus is a method of investigation using the applications of limits of sequences."
8. "Calculus depends upon the theory and application of limits."

The evaluation questionnaires¹⁰ contained several items related to logical structure. One of these, 3(d), asked, "Do you feel that you perceive any logical connections in the calculus? If so, name some." Various responses were obtained from this question. Results are shown in Table 19. No. 7(d) asks, "Do you sense a 'logical thread' running through our treatment? What do you feel all calculus is based upon?" Answers to the latter question are summarized in Table 20.

It is evident that many students in the experimental sections possessed a keen insight into the logical structure of the calculus. Some of this understanding, no doubt, came from emphasis in class, but it is believed that much of it is due to the method of teaching the calculus. The responses recorded in Tables 19 and 20 and all the

¹⁰A copy of the evaluation questionnaire is found in Appendix E.

TABLE 19
SUMMARY OF THIRTY-FOUR COMMENTS
ON LOGICAL RELATIONS

Kind of Comment Made	Number Making Comment
Speed, slope, rate of change $\longrightarrow$ derivatives	16
Areas, work, sequence of sums of products $\longrightarrow$ integrals	10
Relation between integrals and antiderivatives	5
Integrals and derivatives all stem from sequences	3

TABLE 20
SUMMARY OF THIRTY-FIVE RESPONSES TO
QUERY ON THE BASIS OF THE CALCULUS

Kind of Comment Made	Number Making Comment
Limits of sequences	20
Theory and application of limits	5
Rates of change	4
Functional relations and changes	2
Based upon faith (i.e. that a sequence converges)	2
Hypothesis that a limit exists	1
Idealization of phenomena and hypotheses	1

quotations were direct expressions of the students and not alternatives to be checked from lists of possible answers.

Nature and Growth of Mathematics

Just how one can determine the extent of perception by the student of some of the characterizing features of mathematics and how mathematics develops is very difficult. It is evident from the foregoing discussion that the students were generally aware that mathematics is a deductive science--that it is built upon a few premises and that each succeeding part is a logical consequence of a previous one.

Student Responses on Mathematics Growing

As to how mathematics grows it is certain that the students in the experimental sections One and Two have experienced an inductive-deductive interplay in the growth of first-quarter calculus. Furthermore, some of the students have been part-producers in the development of this mathematical discipline. One need only to refer to the chapter on classroom experiences to substantiate this.

A study of the evaluation sheets reveals some comments by students to the question: "Did you at any time feel that you began to see 'how mathematics grows'? If so, name an instance."¹¹ These are summarized in Table 21.

¹¹See No. 7(c) of the evaluation questionnaire in Appendix E.

TABLE 21

STUDENT CONSCIOUSNESS OF MATHEMATICS GROWTH;
A SUMMARY OF TWENTY-EIGHT RESPONSES

Instance Mentioned by Student	Number of Times Indicated
Arithmetic sequences to derivatives	10
The whole quarter's work grew	7
Arithmetic sequences on areas to integrals	5
Sequences to limits of functions then integrals	2
Chain rule and product rule as consequences of the fundamental error relation	2
Topics just fell in order	1
Problem proposed, then methods of solving it became progressively refined	1

Other Expressions

Other responses from the evaluation sheets indicate awareness of students that mathematics grows essentially from physical and numerical situations, then through discovery, perhaps with some trial and error, generalizations are made, and, finally, a logical system is set up. Some statements of students are given here.

1. "I could see the calculus grow from the beginning arithmetic methods to the more involved theorems."
2. "The h-process and its improvement over actual measurement taught me to see how mathematics grows."

3. "I saw the development of the calculus."
4. "We saw how new ideas came about."
5. "During the quarter I have seen mathematics grow from simple to complex ideas."
6. "We saw the calculus start from experience and develop into a study of idealized situations."
7. "We saw a general problem proposed and then found simpler methods of finding the answer."
8. "Our ideas started on a 'man-on-the-street level' and then grew."
9. "It does away with the old, lengthy numerical form upon which it is based and comes forth with the new, speedy, streamlined form."

Attitude Toward Mathematics

An attempt will be made to draw some conclusions as to changes in attitude of students during the calculus course by the use of two sources of data: (a) any change on scores on the attitude tests and (b) statements made by the students on evaluation sheets and elsewhere.

A Study on Attitude Tests

The construction and scoring of the attitude test are described on pp. 38-41 of this study; the attitude test is found in Appendix A.

The median scores on the attitude tests both at the beginning and at the close of each of the two experimental sections are shown in Table 22. So few attitude tests were returned at the close of the traditional section courses that these results were not included in the study.

TABLE 22

MEDIANS OF SCORES ON ATTITUDE TESTS AT
BEGINNING AND CLOSE OF EXPERIMENTAL COURSE¹²

Experimental Section	Median Change	Range Change	Interpretation
One	154 → 160	127-173 → 119-171	favorable → favorable
Two	148 → 154	133-176 → 109-196	favorable → favorable

Apparently the median scores increased somewhat and some students in the second experimental section developed more marked feelings with respect to mathematics, for the range increased considerably.

Written Responses of Students

Perhaps a more reliable indication of students' changes of attitude during the experimental courses came from their writings on various items on the evaluation sheet and from the short paper on the calculus. The writer is not unaware that students do have a change of heart toward mathematics as they study the calculus--no matter how it is taught--and so it is not intended to make a bold claim that all the changes in feeling with regard to mathematics have come from this

¹²Perhaps these scores should be read with reservation. One student remarked to the writer that from observations which he had made it might have been better for no name to have been entered on the attitude test.

approach. Some comments taken from student papers to indicate changes in attitude are given as follows:

1. "Calculus really isn't as bad as I had thought, and I can see practical value in studying it."
2. "I believe that the person with a knowledge of calculus has a very definite advantage over the person without such knowledge. . . by this I mean knowing things which can be done which most people don't know about."
3. "My experience in calculus has been more than just a mathematics course. The calculus course opened my eyes to mathematics . . . in this course I found my future in teaching (mathematics)."
4. "To me, calculus is the most fascinating mathematics I have ever studied."
5. "I enjoyed calculus more than any other course in mathematics. When I signed up for this course I was skeptical about the course and wondered if I could learn about its applications."
6. "Yes, I enjoyed calculus. I must say that it was much different from what I had been told. It really isn't as difficult as I was led to believe especially after you see it grow."
7. "Before I took this course I had been told that calculus was a very 'hard' mathematics course. But I have found calculus to be more interesting than most mathematics courses I have taken."
8. "Calculus has played an important part in the field of science. It makes the solving of very difficult problems relatively easy for the individual with an understanding of calculus. To me it has proved to be much more than a bunch of jumbled figures. These figures now have meaning."
9. "What is calculus to the layman? It is a very deep and complicated course in some kind of mathematics. Even to most college students it is a word and a course to avoid. . . . Calculus is a wonderful, enlightening subject, and once you realize what a tremendous field it opens up, you eagerly wait for each new adventure."

10. ". . . All this is just a beginning to calculus. From what we have seen from this introduction, one cannot help but realize that calculus is the open doorway to many and various areas of investigation and practical applications.

More examples of change of attitude will appear in several of the student papers which are quoted in toto in a later section.

Enjoying Mathematics

One of the essential elements which helps to form a student's attitude toward mathematics is that of whether or not he enjoys mathematics. On the evaluation sheet was asked the question: "Did you enjoy the study of the calculus?"¹³ Thirty-two of the thirty-four responses indicated that they enjoyed the study of the calculus. Others, in wishing to supplement their answers, wrote more. Some of their additional statements are

"It was a lot of work but I enjoyed it."

"Yes, very much. It was my kind of course."

"Yes, it is the first mathematics course I have enjoyed."

"By all means, I did."

"Yes, very much. If I could I would like to take the next two quarters in a similar way."

"It certainly did not appear to be the bad course I was told it would be."

Two of the students made no response; one said that he did not enjoy it as much as he should have.

Other Aspects of Attitude

Attitude is determined by emotional elements. One who can experience a thrill of discovery or who can feel a deep sense of satisfaction in the study of the calculus, or who possesses an

¹³See Appendix E, p. 351, No. 7(g).

appreciation for the immense power of the subject, has a favorable attitude toward it. The students in the experimental sections were given an opportunity to make comments on these three aspects of the course.

In response to the query: "Did you experience any 'thrill of discovery' large or small? If so, mention when or in what topic,"¹⁴ there were various expressions by the students. It is impractical to quote them all but an attempt to summarize them is made in Table 23.

TABLE 23
INSTANCES OF FEELING OF DISCOVERY;
A SUMMARY OF THIRTY-FOUR RESPONSES

Instances or Topics Related	Number of Students
Methods of finding area under a curve	5
Areas by integration; integration by anti-differentiation	5
Discovering how to differentiate	5
That I could understand calculus	2
Maximum and minimum problems	2
Other responses	7
No responses	8

¹⁴See Appendix E, p. 351, No. 7(b).

Some of the "other responses" were that students felt a thrill of discovery when they were developing theory, the chain rule, and expressions for work; others suggested were: practical applications to problems, the realization that speed and slope were "derivative" problems, and being able to get the "most" for the "least."

Another question asked the students was, "Did you feel any great sense of satisfaction in 'understanding something,' and, if so, what was the occasion for this feeling?"¹⁵ Students registered various reactions to this query. These are summarized in Table 24.

TABLE 24
INSTANCES OF SENSE OF SATISFACTION IN
UNDERSTANDING; A SUMMARY OF FORTY RESPONSES

Instances Given for Sources of Satisfaction in Understanding	Number of Students
Understanding how calculus developed	7
Being able to find areas under curves	5
Being able to work what heretofore seemed to be very difficult problems	2
Using "h" to find rate of change	2
Speed, acceleration, work, are not such "demons" after all	2
Maximum and minimum problem work	2
At the end when all the "ends" seemed to tie together	2
Other responses	11
No response on this item	7

¹⁵See the Appendix E, p. 350, No. 7(a).

Some of the "other responses" were

"When ideas of limits and derivatives became clear."
 "When I found that a 'discovered principle' worked."
 "Understanding integration and antidifferentiation."
 "Deducing relations."
 "Using algebra expressions to sum sequences."
 "When former mathematics was put to use."
 "Seeing that $A(x^2 + x^3) = A(x^2) + A(x^3)$."
 "Minimum cost problems."
 "Sensing that mathematics was logical rather than a product of memorization."
 "Relations between D and $\int$."
 "Realization of the essence of the word 'limit' and its role in the calculus."

Responses to the question, "Did you at any time sense the immense power of the calculus? When was this particularly evident?" have also been studied.¹⁶ These are summarized in Table 25.

TABLE 25

RESPONSES ON THE POWER OF THE CALCULUS;
 A SUMMARY OF THIRTY-NINE RESPONSES

Instances Related	Number of Students
Working with areas, volumes, antidifferentiation	15
Working with maximum and minimum problems	11
Relation between rate of change and differentiation	5
Other responses	8

¹⁶See Appendix E, p. 351, No. 7(f).

One of the "other responses" was that one student sensed the power of the calculus when he realized that the "limit of a certain sequence was the 'answer' to a problem." Another said that he sensed the power of the calculus "from the second day on!" A third student mentioned that the power of the calculus which he sensed was the completeness of its structure; he writes: "it is to me like after building a house--I don't just look at the roof, but at all the beautiful parts of the house under the roof."

Summary

An attempt to study the change in attitude of the students toward mathematics has been done as objectively as possible with data from several sources. Using the best methods at hand, though they are perhaps crude, it seems that there was increasing interest and deep-seated satisfaction in the study of the calculus by the students in these experimental sections.

Student Evaluation of this Approach

The students were asked to express their own ideas concerning the effectiveness of this approach to the calculus. First, the investigator asked if the students thought that this approach really is different than that which they had previously experienced in previous mathematics courses; secondly, they were given an opportunity to mention good features of this method and to offer adverse criticism. A few other items furnished them with further means of expression.

This type of evaluation has its limitations. Yet it is believed that, for the most part, the statements by the students were sincere, for it was explained to them on the first day that they were in an experimental course and that it would be helpful if they would watch for strong and weak points. In fact, they were told that their suggestions during the progress of the course would be welcome.

Is This Approach Really Different?

A summary of responses to "Do you think that this approach is really different?"¹⁷ is given in Table 26.

TABLE 26

A SUMMARY OF FIFTY-THREE RESPONSES
ON DIFFERENCE OF APPROACH

Responses by Students	Number of These
I understand reasons for doing rather than doing these mechanically	22
New work is developed by the student from knowledge of his own	9
Students helped to develop the course; not merely passive acceptance	5
Class more 'human'; better student-teacher relation	4
We saw how things develop and cooperated to bring out a new idea	4
We saw a problem then found progressively simpler methods to find the answer	4
This approach gives one a good beginning	3
My interest has been aroused more than ever before	2

¹⁷See Appendix E, p. 349, No. 3(a).

Some single responses were: "not too different except for the use of the study-guides," "much more work for the teacher," and "in this approach one must know the theory more than the applications."

Some statements which seem to convey more than is possible when they are placed in a summary table follow:

1. "It gives the student more of an insight as to how the calculus is built up in a step-by-step process."
2. (This approach is different) "because we really got to the very basic beginning which is something I have really not done before; most other courses are such that we work a lot of problems with very little theory."
3. "It stresses the reasons and theory behind the problems. This is something I have never seen and have always wanted in a mathematics course."
4. "Mainly because it is the first time in all my mathematics that the very basic material was taught so for once I knew what I was really studying."
5. "It is different in many ways. The most significant is the fact that the student is helped to discover the principles for himself, thus giving a clearer understanding."
6. "The method leaves you with a feeling that a very sincere effort is being made to teach you the calculus."
7. "I found it not a course that was presented in a manner to confuse but with an honest effort to see the logical connections in the development of the fundamental theory."
8. "It has aroused my interest, and, to my knowledge, more so than any classes I have been in until now. It has become a little more human."
9. "This method starts more with the 'man-on-the-street' level and then picks up there. This is good."
10. "This method seems to devote more time to carefully developing the logical principles of mathematics rather than working a multitude of stereotyped problems."

11. "More class participation--to a much greater degree than in most college mathematics courses."

Some Good Points About This Approach

Student responses to this query¹⁸ have been classified roughly in Table 27.

TABLE 27

A SUMMARY OF FIFTY-NINE RESPONSES TO
THE QUERY OF GOOD POINTS

Nature of Response Given by Student	Number
More student understanding	24
More personal approach	9
Students helped to develop the course	9
Creates incentive and interest	7
Calculus seems more reasonable, more real	6
No gaps, student not left in confusion	2
Emphasis on learning rather than maintaining a pace	1
Elementary concepts presented in actual physical situations, developed, then relabeled	1

Some expressions by students which perhaps lose their descriptive power when they are submerged in classifications in the table follow:

¹⁸See the Appendix E, p. 350, No. 3(1).

1. "From this approach I can see now how ideas are discovered just as I discovered some myself."
2. "I think the method is terrific. I only wish the rest of my courses could have been of the same calibre."
3. "It makes sense when you can actually see what you are doing and where you have been. This approach does that."
4. "It is a more human and understanding-promoting approach. . . it created an incentive which other mathematics courses seldom give."
5. "It gives a better background and a deeper understanding of the uses of the calculus."
6. (This method helped me to) "understand the logical basis very, very well. It all connects and makes a nice pattern."
7. ". . . this makes (the calculus) a part of him (the student) and it stays alive for him. . . this is the best understood course, the finest, and the most clear course of any kind that I have ever taken."
8. "This approach should have more wide-spread use for better understanding. . . ."
9. "It forms a good foundation; it makes the work seem easier and the class very interesting."

Some Adverse Criticism

Investigations on an approach to teaching should also entertain comments which express disfavor toward the method. This will be mentioned in the next chapter but student reaction is presented and summarized here in Table 28.¹⁹

¹⁹See Appendix E, p. 350, No. 3(m).

TABLE 28
A SUMMARY OF FORTY-ONE ADVERSE COMMENTS

Nature of Response Given by Student	Number
No response	12
Too much homework at times	6
Pace fast at times	5
Lack of time (this method takes longer)	5
Sometimes expressions on sheets were hard to understand	3
Other responses	10

"Other responses" as indicated in the table are as follows:

"Too little time to carry out program completely."
 "Too much work for the instructor."
 "Sometimes different symbols were used."
 "Too much testing near the close."
 "Should have had more answers to check out work."
 "Too much numerical work at the beginning."
 "Perhaps placing of many problems on board would help students to catch mistakes."
 "More problems should be set up in class."
 "Too much theory."

The two responses on "lack of time" and "pace too fast at times" perhaps stem from the fact that in one section (Section Two) the group had lost approximately a week from various causes, and so, of course, the students (and teacher) quite naturally felt a deficiency of time. The "no response" item means that either the student wrote nothing or that he merely said "none."

Other Comments

In response to the question, "Had you known that the course would be conducted in this manner would you have enrolled in it?" Thirty-eight of the students made affirmative replies while one said he would not have enrolled. Three students suggested that they would have been skeptical since it was an experiment.

Every student thought that this approach gave more opportunity for self-growth and various instances were given.

Reactions from Other Sources

Some opinions about this approach came from sources other than the evaluation questionnaire. Conversations with students often gave an opportunity to discuss this method freely. Some of these comments are interesting:

1. "I like this approach. . . . I understand. . . ."
2. "When I looked into the text it all seemed like looking into a black tube--this method illuminates the calculus."
3. "This is the only course I am taking in which it is clear to me what we are doing."
4. "Several of my buddies wish they were studying calculus this way."
5. "It used to be that I did not know how they got what they got in mathematics but now I do."
6. "This approach makes the mathematics 'alive.' It is living and growing! Some of the students in other classes just differentiate--they do not know what they are doing nor do they understand what a derivative is."
7. "We have seen nothing like this approach! Yes, at first it was slow--but what are we in classes for? In others it seems that we do or die. We, even behind others, have a greater understanding. We know this by conversations with others."

Teacher Evaluation of this Approach

It is difficult to be objective concerning an approach in which one is intensely interested. At the same time care must be taken to refrain from repeating what has already been presented in the tables and from including what more properly belongs to the general conclusions of this experiment.

Teacher Work Required

Readers interested in this approach may rightfully inquire concerning the necessary teacher activity. Some aspects of the work of the teacher have already been noted by some of the students. One says, "I am wondering if such an approach wouldn't require too much work for most teachers." Another remarks, "This is one course in which I believe the teacher is working as hard as the students. This is gratifying because it makes us feel that the teacher is concerned and can assume the role of a student too." It would be expected that this method requires teacher-time and energy.

Since this course was being given as a part of a doctoral study there were some details which probably would not have received attention under other circumstances. One of these is the matter of recording the time spent on assignments; another is the keeping of a day-by-day diary of experiences in and out of the experimental class sections.

The remainder of the work connected with this course is part of the usual activity of this teacher. Experience, thought, and study direct the teacher as to content, and the study-guides are prepared

daily (ideally) in the light of student need and the progress which is planned. Most of the guides, in this case, however, were prepared during the preceding summer. Once they are assembled they are only daily carefully scrutinized and changes, deletions, or additions, are made for the next day's activity. These sheets are handed in for checking and the reading of certain items; the quizzes, although graded in class, are also quickly read and recorded. Tests, which are given about every two weeks, and which consist of more than problems, are carefully graded and recorded. If study, writing, checking, preparation of exhibits, and detailed grading are included, the teacher may plan on spending an average of two hours each day. After the course had been given a few times the work may become appreciably reduced for there are days for which only paper-checking is required.

The investigator wishes to suggest that, except for the grading of papers which a teacher has under any approach, a greater number of students than the approximately twenty in these sections would not involve much more work. Perhaps a lower percentage of student suggestions could be entertained but the spirit of discovery and the sharing of results would be present just the same.

Rewards for Effort

The preceding test results and comments found in the evaluations would be sufficient rewards in themselves. However, by the use of this approach, it is possible for the teacher to experience as much or more "thrill of discovery" than the students. Furthermore, there is keen

satisfaction derived from observing expressions of joy of understanding and success on the faces of students as they think imaginatively and as the classroom becomes a center for investigation. If one wishes to be practical, the teacher learns much also, for under the encouragement of this approach, students sometimes originate ingenious methods and can propose searching questions.

Contributions to Evaluation from Second-Quarter Calculus

Most students of Section One remained with the experimenter for second-quarter calculus. Six new students entered with various backgrounds. Although study-guides had not been prepared previously, lesson sheets were used, and the classroom approach was similar to that used in the first quarter.

Break-Throughs of This Pattern of Attack

At several places in the second-quarter calculus student suggestions concerning how to investigate a proposed problem reflected clearly their "conditioning" to this pattern of attack which characterized the previous work.

The first of these occurred in computing the derivative of $\sin x$. It came from the class that one might estimate the slopes of $y = \sin x$ at various points on the curve, plot these slopes against x , and hence perhaps recognize $D_x \sin x$ as a standard curve. This was done, and the first hypothesis in Mathematics 264 was

$$H_1: D_x \sin x = \cos x !$$

Several (especially new) students were amazed at this. This hypothesis was immediately put to work in the derivation of the derivatives of all the other trigonometric functions. $D_x \sin x = \cos x$ was finally established as a theorem.

Another point at which experience in this method caused students to make fruitful suggestions was in the development of $D_x \log x$. The evaluation of $\lim_{h \rightarrow 0} (1 + h)^{1/h}$ was being discussed. Acquaintance with numerical approximations early in first-quarter calculus seemed to decrease the difficulty of the problem, and with surprising readiness, there came a response which appeared to be considered as an accepted procedure: "Let us set up a numerical sequence." The number "e" was hence born from this arithmetical work and this limit was accepted as a hypothesis. In fact, that the function $(1 + h)^{1/h}$ approaches a limit, denoted by e, as h approaches zero, is still a hypothesis for these students.

The preceding constitute just two examples which show the effect of this approach at work. There were others also. Not only the method of attack but also the well-understood content of the experimental course made the study of some of the later topics easier. One in particular is that of differentials, a treatment for which comes directly from the rate of change-error-relation of pp. 213-214 of the study-guides.

Student Evaluation

The new students entered into the spirit of this approach quite well and through the use of planned review on the study-guides they

became stronger in understanding of the content which had been treated in the first-quarter course.

As in the previous quarter the group was asked to complete an evaluation questionnaire and, although there were no names attached, it was not difficult to select the papers of the new students. It is interesting to note their comments on the ("active") approach under test as compared to a more ("passive") traditional method. Some expressions follow:

1. "I would rather have this more active method because it is more stimulating. The approach was at its best when we were trying to find $D_x \sin x$. In most other mathematics courses the student is asked to merely read the theorems and applications."
2. "I think (this method). . . is better. . . . When I get a new equation I like to know where it came from or how it was derived."
3. "This method seems to take the magic or mystery out of mathematics."
4. "I would rather have a professor guide me to make hypotheses instead of his just spoon-feeding me with them. I think I learn a lot more that way. I'm in favor of the method used in this course."
5. "Favorable, I am going to teach myself and I believe that understanding is the think principally to be desired--not manipulative skill. This approach hit me most forcibly when we were working on the derivatives of the trigonometric functions."

While, to some, this type of evaluation may be wanting in objectivity, yet these statements are interesting points of view of students who have come from traditional first-quarter calculus sections.

Students' Conception of the Calculus

Some of the papers on "What I Conceive the Calculus to Be" revealed keen insight into the nature of the calculus. There were nearly forty of these and some of them merely constituted a summary of the course while others were more personal and indicated how the calculus affected them as it developed. A few very succinctly indicated the logical structure or bases of the calculus.

It is agreed that these papers may furnish no direct evaluation of this method of teaching and it would be difficult to compare these discussions with similar efforts written by students of the traditional sections had they been requested to do so. Yet one cannot but feel that the quality of these essays does, in a way, reflect the genuine earnestness and deep understanding of these students during their study of the calculus.

Exerpts from Papers Used

Exerpts of some of these papers have been used in the preceding sections on logical organization and on attitude toward mathematics.

A few more expressions follow:

1. "I conceive calculus as a massive structure. . . with two rather large areas built on the rock (which might be called) sequences. . . . From this firm base we go into two directions. . . the derivative. . . the integral."
2. "Calculus provides man with a means of solving problems. . . where there is change (involved)."

Papers in Toto

Several papers, though admittedly not cold scientific statements of the students' concepts of the calculus, have described in such a warm personal style reviews of this course that it seems desirable to present them completely. Not only do they present the student's ideas on calculus but they also give the reader another view of this student-experience-discovery approach.

1. "Just ten weeks ago, the only thing I knew about calculus was what I had heard from other students who had already taken it. They used words like 'derivative' and 'integral,' and they said that calculus was a very difficult subject with almost no practical value. "So when I signed up for calculus, I was rather worried about what I might be getting into. But after the first two weeks, I thought I understood everything we were doing, although I didn't see what it had to do with calculus. It seemed that everything we were doing was just simple numerical exercises like finding the speed of a freely falling body at the end of a certain time. We called this the rate of change of the distance with respect to time, and found that this expression would give velocity. "Then we started doing problems on slope, and found that slope is merely the rate of change of the "y" with respect to "x," and so it is closely related to the speed problem. "As I look back now, I can see that this is the real basis for the study of calculus. Everything we have done from the beginning has had to do with the rate of change of the velocity with respect to time. "Then we learned another way of saying the same thing. 'The derivative of the expression giving velocity will give the expression for acceleration.' I realized that I had been working with derivatives all the time and never knew it because it was disguised under the name rate of change. In the following weeks we developed this idea into other types of problems such as finding maximum volumes and minimum costs. "We learned to find area under a curve by numerical methods. To do this, we found the limit of a sequence of sums of little areas as the number of subdivisions approached infinity. Then I learned that this can be applied to the definition of an integral.

"Everything seemed to tie together and seemed to be related. Derivatives and integrals are related to rates of change and limits of sequences, and the whole course of calculus is based upon the exact study of these relationships.

"Calculus really isn't as bad as I had thought, and I can see practical value in studying it."

2. "By talking to calculus students before I had enrolled in it myself, I had obtained a great variety of opinions concerning calculus. Some would say it is hard, others would say that it is relatively easy. Then when I first looked at a calculus book, the subject appeared to be a collection of symbolisms almost impossible to translate. This alone was enough to make me leery of the subject, calculus.
 "Upon starting calculus, I found it to be very interesting, and a subject which opens the doors to a wider use of mathematics. I don't believe that the calculus itself is so exceptionally hard, but the difficult part is the use of the algebra and trigonometry from previous courses.
 "I liked the way the course was taught in as much as we were able to see the way calculus was brought about from the numerical methods. This was very interesting.
 "To me calculus may be defined as the theory and applications of limits of sequences.
 "Throughout the course it was shown that the speed of a moving body, the slope of a curve, and the rate of change in volume, are all related to the derivative. Later it was shown how the derivative and the antiderivative are related.
 "Also shown was the relation between areas and volumes with respect to the integral. The integral is then related to the mean value theorem.
 "From these two conceptions, differential calculus and integral calculus, comes the Fundamental Theorem which is formed by the mean value theorem and the antiderivative. This, the Fundamental Theorem, is the missing link connecting integration to differentiation."
3. "Before my introduction to calculus, my only understanding was that a new concept or idea had been used as a basis for the study. I had heard all manners of reports from students who had taken or who were taking the course, such as: it's a new concept of mathematics, a tough course, or (a measure) of a person's ability to place component parts (of mathematics) into a relationship usable for calculus methods.

"From the point of view of our approach, some of the ideas were new, some were ideas from previous mathematics courses, and some were combinations of these forming a pattern to help solidify the foundations of the calculus. Beginning with numerical calculations of sequences of speed and slope there were then developed more general processes to find speed and slope. These finally appeared as the rate of change or derivative and their applications to all kinds of situations. These, along with sequences were developed and noticed or 'discovered' first. This definite relationship of sequences and rate of change is one of the basic ideas of the calculus. The relationship of small changes of units such as volume, area, cost, and so on, relative to some other variable, enable us to work problems in maximum and minimum, and, in industry, become matters of economy. Rate of change of idealized physical situations can be studied also. Inverse derivatives, higher derivatives, parameters, implicit functions, are all matters of rate of change, or can be.

"Finally the integral shows itself in the form of numerical exercises or area and work problems, and, abstractly, integrals are limits of sequences of sums of products. This abstract sequence is so refined that finally one need set up only one product and from this develop an 'unseen sum of products' which approach a limit as the number of products becomes infinite.

"The study of calculus, to my way of thinking, is based upon sequences approaching a limit as a related sequence approaches either zero or infinity, and also that for a small change of a unit there is a definite related change of something else. From these stem theorems on limits, sequences, rate of change or derivative, approximation of area under curves, integrals."

4. "I would like to express my concept of the calculus by first giving the reader of this paper an idea of what I had first expected before entering the course and then comparing this with my attitude after completion of the course. "Prior to entering the course I had looked upon calculus with an air of dread. To me the calculus was in a class of mathematics which could be the 'making' or 'breaking' of a mathematician and I was particularly fearful of the latter. I assumed that calculus was a branch of mathematics that my capability would be taxed to the utmost. Perhaps this was true because of the reverent and awed respect given to those taking this 'giant of a mathematical

course.' Whatever it was, it certainly had given me a feeling of inadequacy. I also looked upon calculus as being based on near mystical concepts. I thought that one would have to be a near-genius to be able to understand it with all its complicated theorems and its many symbols. I also felt sure that I would not be able to see its practical applications. All these added together put me in a state of quandary preparatory to entering the course.

"Today I can look back and smile at all my fears and misgivings. Having completed my first course, I am looking forward to the next. One might ask: why so remarkable a change of face? Herein lies the answer.

"I have seen and I have developed this course through the method of teaching of this course. It is, of course, the intuitive or self-discovery method of which I speak. By this method I have seen calculus grow from a seed into a sapling with the full-grown tree of calculus yet awaiting me. I have seen the logical premises and deductions, taking place in a step-by-step manner. I now hope to justify my feeling that with this background I shall see the remaining quarters of calculus evolve in a similar manner.

"I should add that I have also derived a keen satisfaction from being able to apply the calculus to some of the problems in everyday business and industry. Right now I am thinking of the maximum-minimum applications by the use of the second derivative and, too, the application of calculus to area and volume problems. But why name more? They are to be seen by any who look for them and who have been taught how and why they are using this tool of mathematics."

CHAPTER SIX

GENERAL SUMMARY

A Review of this Experiment

In spite of the emphasis on meaning and understanding in the teaching of mathematics little has been done to alter methods of teaching on the undergraduate university level to improve student understanding and particularly in the field of calculus. It is generally agreed that concern for understanding, rather than manipulative skill alone, is a matter of utmost importance to encourage the attainment of the richness which a mathematical education should provide.

In fact, this has been re-emphasized again only recently in a report "Guidelines for Science and Mathematics" on a two-day conference in Washington, D.C., held in February, 1958, and called by Commissioner of Education, Lawrence G. Derthick. Some excerpts from this report relative to mathematics are:

1. "Science and mathematics are methods of inquiry as much as they are bodies of knowledge. . . mathematics teachers must develop skills in using mathematics as a way of thinking. . . ."
2. ". . . (students must develop ability in) proposing hypotheses, identifying assumptions, interpreting data, and drawing sound conclusions."

3. "Improvement in quality. . . is more important than increase in quantity."¹

The Study Proposed

Hence there is proposed in this study a student-experience-discovery approach, whose features are contained in the title, but characteristics of which may be further defined as follows:

1. Important ideas of the calculus are introduced through numerical problems and are slowly nurtured toward generalized results.
2. These generalized results are also reached through developmental procedures carefully planned and timed to suit the growing insight of the students.
3. From these results are suggested assumptions, definitions, and hypotheses to be tested.
4. On the basis of student explorations, student contributions, and teacher-student discussions a logical structure is built up with the origin and role of assumptions, definitions, and theorems, clearly demarcated and recognized by the students.

Features of the Approach in Use

The above definition has been used as much as possible as a guide in the formulation and exercise of classroom procedures. Chapter

¹U. S. Department of Health, Education, and Welfare. Guidelines for Science and Mathematics (Washington, D.C.: Office of Education, May, 1958), p. 1.

Three gives instances of this approach "put into practice." Some features noted follow:

1. General problems find their initial setting in natural situations.
2. The students attack these problems with the means they know--often arithmetic.
3. Results are studied, examined, discussed; definitions and hypotheses are made.
4. Abstract considerations flow naturally out of the concrete.
5. The teacher guides the student to the consideration of more profound questions.
6. Statements accepted as hypotheses remain so until deduced as theorems.
7. Student initiative, discovery, and suggestions are encouraged and utilized to help develop the calculus.
8. What is being assumed is clearly delineated and set apart from what is being deduced.
9. The students experience how mathematics grows.

Results in Content

The resulting calculus content, some of which is implied in Chapter Three, is summarized in Chapter Four. This summary contains not only hypotheses and theorems which emphasize the logical structure which was erected, but also lists of concepts and skills which were developed. Furthermore, Chapter Four describes types of argument

used and includes a discussion of some of the notebooks which students organized.

Generalizations from this Experiment

From the results of the evaluation described in Chapter Five there are some educationally (and mathematically) vital generalizations which concern the student and some important and compelling facts which should gain the attention of the teacher. Of course, those which concern the student should be of intense interest to the teacher also.

Results from the Viewpoint of the Student

This study has some tentative "good news" for the learner! The writer feels that on the basis of this experiment, which set out to test the assumptions listed on page 26, he can propose several working hypotheses; the students taught by this "discovery" approach have these advantages:

1. They do as well on problems and manipulative skills as those with traditional instruction but, in addition, they have increased understanding. That is, the methods of this approach equate performance with other sections in quantity but increase the quality of understanding. In fact, this experiment does exactly what the "Guidelines" suggests when it says: "Improvement in quality in . . . mathematics education is more important than increases in quantity."²
2. They have a superior knowledge of the fundamental theory and logical relations among parts of the calculus. They

²Ibid., p. 1.

are "aware of a satisfying feeling, a psychological closure, which results from having everything fitted in its proper place. . . ." ³ The students see calculus as a doctrine of well-organized material rather than as an assemblage of unrelated skills.

3. They have experienced the thrill of discovery and the satisfaction of producing results through creative effort as the use of intellectual powers has been encouraged--all of which lead to greater enjoyment of mathematics and a deeper understanding of its nature and use.
4. They have expressed ideas of the calculus in their own language and they have had the stimulating and disciplinary experience of having their expressions and ideas sharpened by examination by other students as well as by the teacher.

Results from the Viewpoint of the Teacher

The teacher, too, gains much from the use of this approach. First of all, there is the stimulating, yes, invigorating growth on the part of the teacher as students bring forth new ideas and searching questions. Secondly, the much closer contact with student thinking, the informality of the classroom, and the expressions and welcomed questions of the students, give the teacher an unequalled opportunity to gain insight into methods of learning, and therefore, teaching.

³National Council of Teachers of Mathematics. The Learning of Mathematics; Its Theory and Practice, p. 75.

Need for Further Evidence

The writer suggests that the stated results of this experiment be regarded as tentative, or working, hypotheses to be studied further for, on the basis of one experiment, he can hardly present them as established conclusions. Furthermore, as the experimenter forgets now the thrilling moments in the classroom, the expressions of satisfaction and enthusiasm on the faces of his students, and the demonstrations of insight into the calculus as indicated in the student papers, the scientific aspects of this study immediately make him aware of need for more experimentation to see if these proposed working hypotheses can be validated. There is need for further evidence as indicated, at least partially, by these questions:

1. How and to what extent do the personalities of the teachers involved enter into the results of this experiment on comparing methods of teaching?
2. How and to what extent does the enthusiasm of teachers for mathematics and their ability to communicate this enthusiasm have bearing upon the results of such an experiment?
3. What kind of results would be obtained by a teacher who is interested in trying this approach under practical working conditions--that is, with little special critical study of the calculus and with

little previous experience with this method of teaching?

4. Could this experimenter have obtained similar results by another method of teaching?
5. Would the results have been similar had comparison tests on understanding and proficiency in the calculus prepared by a capable, interested, but impartial third teacher been used rather than the cross-tests used in this experiment?

Need for Further Experimentation

Each of the questions suggestive of further evidence needed to validate the tentative results as proposed on pp. 170-71 indicates a desirable step toward further experimentation. Some suggested plans for continued study are:

1. Have other teachers use this experience-discovery-approach and compare results with those found by the use of traditional methods so the number of "samplings" is much greater.
2. Use impartial sources for tests on understanding and proficiency to eliminate possible differences in performance due to language variations and bias of instructors involved.

3. Learn the attitude of other instructors toward this method of teaching both while they are using it and as they teach students who have studied under this approach.
4. Collect information on what students who have studied under this method think of this approach after the enthusiasm observed during its use wears off; collect expressions from such students several years from now to see if the use of this approach left impressions different from those of traditional methods.
5. Study the progress of students who learned calculus under this approach to see if a greater percentage take further mathematics courses and if they are more successful than students normally have been.

It is seen that an experiment which began as a comparison of two methods of teaching with two experimental and two traditional sections must really expand very much before one can make claim to those interested that the results proposed as tentative hypotheses are, or are not, well-established conclusions.

There are, of course, some topics for investigation which might provide better tools for carrying on such an experiment and for enabling teachers to have a better understanding of their students and to work toward more effective instruction in the calculus. These are improved

precalculus tests, tests for diagnosis of difficulties in calculus, tests on understanding in calculus, and mathematical attitude tests. At present practically no such tests are available.

Limitations and Precautions

Although the writer is very enthusiastic about this student-experience-discovery method, he is aware that there are some limitations to its use and there are some precautions which must be taken by anyone who desires to employ it. Some of these are briefly expressed as follows:

1. There must be a judicious balance of time spent on inductive procedures to give the students an understanding of the variety of topics included in a first course of calculus at the university level. Often several ideas must be developed simultaneously.
2. The teacher must have some firm convictions relative to the extent of understanding logical relations in the calculus versus the emphasis on manipulatory skills.
3. With the student-discovery-experience approach the teacher has a continuing responsibility of examining papers to note the progress of students and of studying their suggestions for possible use in the classroom. At times this approach requires more than the average amount of student-conference work for the student is often anxious to discuss his ideas with the teacher.

Recommendations and Suggestions

The data and results of this experiment, in this particular case, justify the assertion that this experience-discovery-approach yields

depth of meaning and understanding of the calculus to a far greater extent than do traditional methods without loss of proficiency in calculus. In fact, the writer claims that this experiment has elevated all the assumptions on page 26 to the following more respectable position: they can now be accepted as more firmly entrenched working hypotheses. Of course, these are still subject to test by further experimentation. Through this approach the experimenter claims that there is much promise that "organizational adaptation. . . (can be) made for the maximum development of both the slow and the rapid learner."⁴ It is recommended therefore that teachers of the calculus set their minds steadfastly toward the use of this approach to see if they too can experience the same satisfaction and results among students as this experiment has indicated and to test further the hypothesis that this method shows promise for more generally effective calculus teaching. To continue the teaching of calculus by methods which are antithetical to those of this experience-discovery approach may be depriving our students of a richness of mathematical education which they deserve and which it is so imperative that they have.

⁴U. S. Department of Health, Education, and Welfare, op. cit., p. 4.

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APPENDIX A

ATTITUDE TEST

ATTITUDE TEST

Name.....

Date.....

This is not a test in the sense that any particular statement is right or wrong. All these statements represent opinions which people hold about mathematics or about the study of mathematics. Indicate whether you agree or disagree with the statement by placing a + sign before all the opinions with which you agree and by placing a - sign before those with which you disagree. If you are uncertain if a statement represents your attitude make a question mark (?) before the statement. Now go back and draw a circle (○) around the plus sign with which you agree strongly and also a circle (○) around the minus (-) sign where you disagree strongly.

Be sure to be sincere about your markings.

- * 1. I have always liked mathematics of any kind and I think it is an interesting field of study.
- * 2. Mathematics stimulates the mind more than any other study.
- * 3. I think more importance should be placed on mathematics in the grade school and in the high school.
- 4. Mathematics is extremely difficult and in beginning courses it offers few practical problems or examples.
- 5. Mathematics is not one of my favorite subjects.
- * 6. I am very much in favor of mathematics.
- * 7. I feel that the study of mathematics is an exceedingly important and necessary part of education.
- 8. The study of mathematics is one of the hardest chores I have ever undertaken.
- * 9. Mathematics, to me, is very interesting especially when practical applications are involved.
- 10. I have a severe dislike for mathematics--this topic has been my continual stumbling-block.
- * 11. To me mathematics is a sound way of reasoning--a wonderful tool for advancement in every field.
- * 12. It is one of the finest exercises for one's mind.

- _____ 13. I have a fearful approach of inability toward solving any mathematical problem.
- * _____ 14. Mathematics is a very handy tool.
- * _____ 15. Mathematics is one of the basic elements of an education.
- _____ 16. I hate the stuff.
- * _____ 17. Mathematics is great fun; I love it, really.
- * _____ 18. When I finally figure out a hard problem it gives me a feeling of self-accomplishment which is incomparable to the satisfaction derived from any other source.
- * _____ 19. Mathematics is a useful tool which I find increasingly necessary in every day life.
- * _____ 20. Mathematics, as it becomes more difficult, presents a greater challenge and breeds increased interest.
- * _____ 21. The study of mathematics is a fascinating study and it is a rewarding endeavor.
- * _____ 22. I still can't resist taking a mathematics course whenever I have a vacancy in my schedule.
- _____ 23. I have something of a fear of mathematics.
- * _____ 24. Mathematics is an exact science and definitely should be a required subject.
- _____ 25. In general I dislike the study of mathematics.
- * _____ 26. I like mathematics and enjoy solving problems.
- * _____ 27. I find mathematics more interesting as I go more deeply into the subject.
- * _____ 28. I think the study of mathematics is necessary to think and reason soundly.
- _____ 29. Mathematics is the basis of many other sciences but to me it seems a necessary evil.
- _____ 30. I detest mathematics and avoid using it at all times.
- _____ 31. I have never liked mathematics.
- _____ 32. I am afraid of doing word problems.
- _____ 33. I have always been afraid of mathematics.

- * 34. Sometimes I enjoy the challenge presented by a mathematics problem.
- * 35. Mathematics thrills me and I like it better than any other study.
- * 36. I think about mathematics problems outside of school and I like to work them out.
37. I can't see much value in mathematics.
- * 38. I never get tired working in mathematics.
39. I avoid using mathematics because I am not very good at figures.
40. Mathematics is an abstract science which fascinates many minds but not mine.

APPENDIX B

PRECALCULUS TEST

Precalculus Test

Name..... Date

NOTE: This test is constructed of some items which involve skills and understandings which are useful in the calculus. What you do on this test will in no way influence your grade in the calculus course; rather, it may help your instructor to sense the adequacy of your ability on this first day of the quarter and to help him arrange work to help you. Hence we ask two things: (a) note the time which is required by you to complete this test and (b) do not receive help from any other students or texts in mathematics. Really it will be to your advantage to make this a product of your own effort. When you have finished the test complete these statements and questions:

(a) Time spent: _____

(b) Did you receive help from any student? _____

(c) Did you consult any text on mathematics? _____

Part I. Algebra. Arithmetic.

DIRECTIONS: Do all the work on this sheet in the space provided near each item.

1. Collect terms, remove parentheses, make expressions more simple, expand, or do as requested. These problems are taken from algebra and arithmetic.

(a) $3a - 6b - 10a + 3b =$

(b) $-4(a - 3b) + 2(-a - 6b) - 4(-2a + 3) =$

(c)
$$\frac{6a + 2b^2}{4a - 3b^2}$$

(-)

(d)
$$\frac{-x^2 - 2xy + y^2}{-3x - 2y}$$

(X)

(e) $4 \cdot 0 =$

(f) $(x - h)^4 =$

(g) $(2 + h)^3 =$

Precalculus Test

Name..... Date.....

Part I. Algebra. Arithmetic (continued).

(h) $\frac{x^2 - 3x}{-x} =$

(i) $\frac{x^2 - 6x + 10}{x} =$

(j) $a \div 2/3 =$

(k) $\frac{(x + h)^3 - x^3}{h} =$

(l) $\frac{1}{x+h} - \frac{1}{x} =$

(m) $\frac{(x + h) + 2}{4(x + h)} - \frac{x + 2}{4x} =$

2. Write as single positive powers or take roots where you can.

(a) $(x^2)^3 =$

(b) $(-A^2)^5 =$

(c) $(2a^2)^{-3} =$

(d) $(4a^8)^{-1/2} =$

(e) $(9a^{12})^{-1/2} =$

(f) $8^{1/3} =$

(g) $(.01)^{-3} =$

(h) $\left(\frac{15a^3b^2}{ab^2}\right)^0 =$

3. Simplify or write as one radical if possible.

(a) $\sqrt{25a^2b^4} =$

(b) $\sqrt{5a}\sqrt{12a} =$

(c) $\sqrt[3]{24x^6} =$

(d) $\sqrt{8a^4} =$

(e) $\sqrt{18} + \sqrt{50} + \sqrt{200} =$

Precalculus Test

Name..... Date.....

Part I. Algebra. Arithmetic (continued).

4. Solving equations. Solve each of these for the letters present. In case there are several unknowns be sure to pair the values properly.

$$(a) \quad \frac{5}{x+1} = \frac{7}{x-1} + \frac{2}{x+1}$$

$$(b) \quad 3y^2 + 3y + 1 = 0$$

$$(c) \quad x^3 - x^2 - 30x = 0$$

$$(d) \quad \begin{cases} 4x + 3y = -7 \\ 2x - 5y = 16 \end{cases}$$

$$(e) \quad \begin{cases} x^2 + y^2 = 25 \\ 3x - 4y = 25 \end{cases}$$

- (f) Solve for x (only).

$$\begin{cases} 3x + 2y + 5z = 7 \\ 2x - 3y - 2z = -3 \\ x - 2y + 3z = 5 \end{cases}$$

- (g) If $y = 3x + 8$ what is x in terms of y ?

$$(h) \quad \text{Solve } s = \frac{rl - a}{r - 1} \text{ for } r.$$

Precalculus Test

Name..... Date.....

Part I. Algebra. Arithmetic (continued).

5. Miscellaneous Items.

- (a) A rectangular solid has a base with dimensions a and b ; the volume is V . What is the height in terms of a , b , and V ?
- (b) If $f(t) = t^2 + 3t + 4$ what is $f(1.01)$?
- (c) Continue this geometric progression through several more terms and then write the n th term and the $(n + 1)$ st term:
 $4, 2, 1, 1/2, \dots$
- (d) What is the value of $y = x^4 - 3x^3 + 2x^2 + x + 10$ when $x = 3$?

Show the use of the remainder theorem.

- (e) A box has dimensions as shown. The box is heated and it expands. The 4" edge becomes 4.02" long and the other edges become longer in the same ratio. What is the new volume?

What is the increase in volume?

- (f) Set up relations which a, b, c must satisfy if the curve whose equation is $y = ax^2 + bx + c$ is to go through the points $(1, 3)$ $(3, 13)$ $(-1, 9)$.

Precalculus Test

Name..... Date.....

Part I. Arithmetic. Algebra (continued).

- (g) A curve is given by $\begin{cases} x = 2t \\ y = 3t^2 \end{cases}$.

Construct table of values and graph for $-2 \leq t \leq 4$.

- (h) Consider $y = x^2 - 9x + 10$.

- (a) Construct a graph over $-3 \leq x \leq 13$. To save time let the x range by "two's."

- (b) How many roots do you expect? _____
 (c) What do you suggest the roots are? _____
 (d) Is the slope positive negative for $x \geq 6$? _____
 (e) Is the slope positive or negative for $2 \leq x \leq 4$? _____
 (f) What is the y-intercept? _____
- (i) Find the tenth term and the sum of the first ten terms of this arithmetic progression: $-6, 3, 12, \dots$. See formula below.
- (j) What is the sum of an infinite number of terms of this geometric progression: $9, 6, 4, 8/3, \dots$? See formulae.

Formulae you may wish to use:

$$S = a + (n-1)d \quad S = \frac{n}{2} (a + l)$$

$$l = ar^{n-1} \quad S = \frac{ar^n - a}{r - 1}$$

$$S_{\infty} = \frac{a}{1-r} \quad (r < 1)$$

Precalculus Test

Name..... Date.....

Part I. Algebra. Arithmetic (continued).

(k) What is the value of $\lim_{n \rightarrow \infty} \left(4 + \frac{1}{n} + \frac{1}{n^2} \right)$

(l) What is the value of $\lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{n}}{2 + \frac{3}{n}} \right)$

Part II. Geometry.

1. What is the value of the letter used in these figures?

(a) (Similar Δ) (c) $a \parallel b$
 $x =$

(e)

(b)

(d)

 $t \parallel l$
 $x =$

(f) Find the area of the ring.

2. (a) What is the area?

(b) Find the volume.

Precalculus Test

Name..... Date.....

Part II. Geometry (continued).

- (c) Find the area of the cylindrical surface in this figure.

- (d) If the volume is V and the radius is r what is the height in terms of V and r ?

3. What are some essential parts to the logical structure of mathematics? These are often first learned in geometry.
4. Find the distance between the points $(-3, 2)$ and $(4, -6)$.
5. What is the equation of the line which joins the two points in No. 4?
6. What is the slope of the line in Nos, 4, 5?
7. What type of curves (circles, ellipses, etc.) does each of these equations or relations represent?
(a) $3x^2 + 3y^2 = 10$	(d) $4y = -3x + 10$
(b) $3x^2 - y^2 = 8$	(e) $y^2 = x^2 + 10$
(c) $x^2 = 16 - 4y^2$	(f) $y = x^3$

APPENDIX C

MANUAL OF STUDY-GUIDES
FOR THE
EXPERIMENTAL COURSE

(MATHEMATICS 263)

Mathematics 263
First-Quarter Calculus

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I. A. Rates. 1.

One of the important problems of the calculus is to determine how fast a quantity changes with respect to another quantity. For example, in motion, which is one of the most interesting and challenging phenomena in our lives, we are often interested in how fast distance changes with respect to time; this is called speed; the speedometer on our automobile tells us this rate at any time. One who invests money may be interested in placing his money in a project in which the amount earned is greatest with respect to time, while a scientific farmer may be concerned with the rate of increase of yield relative to changes in water furnished, or to variations in the CO₂ - content of the air, or to a change in the percent nitrogen in the fertilizer which he uses. The manufacturer is interested in the change in the volume of business due to a change in the price of the article. The field of calculus attempts to build a system of ideas and techniques which will help us to study changes.

EXERCISES. Write carefully one or more rate studies in which these people might be interested. Use reverse side if needed.

1. bacteriologist. A bacteriologist might be interested in the change in the mortality rate of bacteria due to a change in the concentration of an antiseptic.
2. farmer
3. auto-tire manufacturer
4. a student in mathematics
5. furnace tender for heating
6. one you might suggest

I. A. Rates. 1.
Can rate be determined by measurement?

To investigate a typical rate problem we consider a body which is constantly being "pushed," or, which is being acted upon by a constant force. In physics this is the origin of what is known as accelerated motion.

- (a) Arrange a table with data on the distance traversed by the end of the 0th, the 1st, the 2nd, the 3rd, ..., seconds.

time	distance

- (b) Place the above data on the scaled line on this sheet. What visual evidence (other than that obtained by watching the apparatus) does this give about the distance traveled each second?
- (c) Apparently how does the speed change with time (remain the same, become slower, faster)?
- (d) Either from the scaled line or from the table in (a) consider the following questions
- i) What is the distance traveled during the first second? _____;
what is the average speed during the first second? _____
 - ii) What is the distance traversed during the second second? _____;
what is the average speed during the second second? _____

I. A. Rates. 1.
(continued)

- iii) What is the average speed during the third second? _____
- iv) What is happening to the average speed as the time increases? _____

- (e) Is the average speed during the second second the same as that speed which we would be inclined to call the speed at the end of the second second? _____
- (f) How could you find an average rate closer to the rate or speed at the end of the second second?
Describe your suggestion.
- (g) What practical difficulties would you encounter in doing what you suggested in (f)?

EXERCISES.

1. An auto started from rest and during the second between $t = 1$ sec. and $t = 2$ sec. it went 30 feet.
During the first half-second between $t = 1$ and $t = 2$ it went 10 feet and during the first fourth-second between $t = 1$ and $t = 2$ it went 4 feet.
- (a) What was the average speed between $t = 1$ and $t = 2$ sec.? _____; between $t = 1$ and $t = 1 \frac{1}{2}$ sec.? _____; between $t = 1$ and $t = 1 \frac{1}{4}$ sec.? _____.
- (b) Which of these is the "best approximation" to the speed at the end of the first second ($t = 1$)? _____
- (c) How could one obtain a better approximation to the speed at the end of the first second ($t = 1$)?
- (d) Are there any difficulties which need to be overcome in making very good approximations to the speed of the auto at the end of the first ($t = 1$) second? If so, what?

I. A. Rates. 1.
(continued)

2. A wire is bent in the form of a circle with a diameter of 14 inches. A bead on the wire is released from the top of the circle and it slides to the bottom in 3 seconds.
- (a) What is the average speed in inches per second?
 - (b) Would the speed be a "changing" speed or would it be a "constant" speed?
 - (c) How could one attempt to find the speed at the end of the third second?
3. Explain in words how one finds average speed.

I. A. Rates. 2.
How can one make approximations
from formulae?

In our last exercise we noted that determination of speed by making measurements presented difficulties. Among others, perhaps, the main difficulty is _____

We can be helped out of this difficulty by making use of a very powerful concept in mathematics--that of a formula or a statement of a law. Formulae or mathematical expressions of law are not given to man; they are discovered and developed by him. For example, after making many measurements of distances through which a freely falling body falls in corresponding times (similar to that which we assembled on p. 2) the scientist makes a generalization by the use of these data and forms the hypothesis that they obey the relationship expressed by

$$s = 16 t^2.$$

This hypothesis is then checked by further measurements on time and distance and also predictions made by its use are checked against actual results. Sometimes these formulae need changing with further study.

Assuming now that the "law of falling bodies" is expressed by

$$s = 16 t^2$$

let us again center our attention on approximating the speed at the end of the second second. That is, by the study of average speeds during time intervals near $t = 2$ sec., what speed would you be willing to suggest as the speed when $t = 2$ sec.? To find this is our problem.

- (a) Suggest and carry out some computational plan to arrive at some conclusion about the speed as the time interval about $t = 2$ becomes smaller and smaller.
- (b) It is often desirable to use what we call a "method of successive approximation" in which the time intervals are taken shorter and shorter in the manner of a geometric progression: as, intervals of 1 sec., 1/2 sec., 1/4 sec., etc. . . . or 1 sec., 0.1 sec., 0.01 sec., . . . etc.
If you did not use such a method in (a) do it now.
What is the speed (which you suggest) at $t = 2$ as approximated by this method?

I. A. Rates. 2.
(continued)

- (c) Arrange the "average speeds" in the form of a table to exhibit the "sequence nature" of the data obtained.

What speed do you suggest or hypothesize the average speeds approach as the time intervals become smaller and smaller? _____

See if you can confirm this if you assume the truth of the formula for the sum of an infinite number of terms of a geometric progression whose ratio is less than one:

$$S_{\infty} = \frac{a}{1-r}$$

- (d) Why is it suggested that the result is a hypothesis rather than a final conclusion?

EXERCISES.

1. Apply the method which you have just developed to estimate the speed of a falling body at the end of the third second. Show all the work neatly arranged.
2. Apply the method developed above to estimate the speed of a falling body at the end of the fourth second. Arrange work neatly.
3. A body is traversing distance in such a manner that $s = t^2 + 3t$. What is the speed (which you would suggest) at the end of the third second?
4. A body moves such that the distance s and the time t are related by this expression: $s = 3t^2 - 2t$. What speed would you suggest describes the behavior of the body at the end of the second second?
- *5. Reconsider No. 1 but this time see if it makes any difference if you use another method to make the time interval smaller. Show the work in good form.

I. A. Rates. 2
(continued)

EXERCISES (continued).

- *6. Use another method to make the time interval smaller in No. 1.
- *7. Try to use a method which approaches $t = 3$ from "beneath" it. E.g. use the interval from $t = 2$ to $t = 3$, the interval from $t = 2 \frac{1}{2}$ to $t = 3$, the interval from $t = 2 \frac{3}{4}$ to $t = 3$, etc.
- *8. What do you conclude from your work in Nos. 1,5,6,7 about the effect of making the interval smaller? Make a complete statement on this.
- 9. The expressions of s in terms of t reintroduce us to terms such as variable, constant, function, sequence, limit. Explain what you think these words mean. Consult texts if you wish.
- 10. What is an advantage of making approximations to speed by formula rather than by measurement?
- *11. Using your generalization from No. 8 and terms from No. 9 what hypothesis can you suggest about sequences and limits?
- 12. In using $s = 16 t^2$ and other expressions for s we have been using idealized situations. What does this mean?
- *13. When one arrives at a conclusion by the use of examples is he using a deductive or an inductive method? What are some features of this method?

I. A. Rates. 3.

Can we estimate speed with less numerical computation?

- (a) Review. Suggest two ways which we have used so far to estimate the speed of a falling body (whose distance-time formula is $s = 16 t^2$) and explain the mechanics of the method.

i)

ii)

- (b) Review. What two hypotheses have we made on estimating speed by the use of a sequence of approximations?

H_1 :

H_2 :

- (c) Consider again $s = 16 t^2$. We wish to make an estimate of the speed at the end of the second second ($t = 2$) by a method which requires less numerical work.

- i) How far has this body fallen in $t = 2$ sec.? _____
- ii) Instead of taking numerical values for times such as $t = 2 \frac{1}{2}$, $t = 2 \frac{1}{4}$, $t = 2 \frac{1}{8}$, etc., in which cases we are "working down" to $t = 2$, or $t = 1 \frac{1}{2}$, $t = 1 \frac{3}{4}$, $t = 1 \frac{7}{8}$ etc. in which cases we are "working up" to $t = 2$, let us now g e n e r a l i z e this process by using as our "second" or "final" time $t = 2 + h$.
- iii) Find the distance traversed, the average speed during the h -time interval.
Show work here.
- iv) From a study of this "generalized average speed" what hypothesis would you make on the speed at the end of the second second, or, when $t = 2$?

I. A. Rates. 3.
(continued)

- v) How does this result compare with that obtained in (b) p. 5?
- vi) What advantage does this method have over that used in A. Rates. 2.?

EXERCISES.

1. By the method just developed estimate the speed of a falling body at the end of the third second. Show work. Use a figure if you wish.
2. Find the speed of a falling body at the end of the fourth second. Show work.
3. Find the speed at the end of the second second by the method developed in this section if
 - (a) $s = t^2 + 3t + 2.$
 - (b) $s = t^3 - 2t^2.$
 - (c) $s = 2t^2 + 4t.$
4. Find the speed at the end of the third second if $s = \frac{1}{t}.$
5. Find the speed at the end of the fourth second if

$$s = \frac{t - 4}{t}.$$
6. In No. 3(a) we note that for each value of t we get a distance s . That is, each value of t determines a distance s , or, to each value of t there corresponds a value of s . We recall from previous courses that we say that s is a function of t . The idea of function is one of the most important concepts in mathematics and we must have an accurate understanding of it.

Write carefully meanings or definitions of variable, constant, function, independent variable, dependent variable, range. The text may help (pp. 1 to 8).

I. A. Rates. 3.
(continued)

EXERCISES (continued).

7. What two things are necessary for y to be defined as a function of x ? (p. 4). (Text)
Hence is $y = x^2 - 3x$ clearly defined?
Hence is $y = 2/x$ clearly defined?
8. Text. p. 8. Recall functional notation. Nos. 6 (first eight parts).
9. Review. Do No. 2 above by an arithmetical method.
10. Review. What does it mean to say that the relation $s = 16t^2$ is accepted as an idealized relation by the mathematician and physicist?
- *11. Note that the expression which you obtained from (c-iii) p. 8 for the average speed over the time interval from $t = 2$ to $t = 2 + h$ is
- $$\text{Ave. sp.} = \frac{s_f - s_b}{h} = \frac{64h + 16h^2}{h} = 64 + 16h \text{ for any } h \text{ except } h = 0.$$
- Hence
- $$\frac{s_f - s_b}{t_f - t_b} = \frac{s_f - s_b}{2+h - 2} = \frac{s_f - s_b}{h} = 64 + 16h$$
- is a function of h . What would you suggest as a good name for this function?
- *12. Now on p. 5, (b) in order to estimate the speed at the end of the second second ($t = 2$) we made arithmetic computations of the average speeds for different time intervals: 1 sec., 1/2 sec., 1/4 sec., 1/8 sec., etc. Use the function above by replacing h by each of these time intervals and see if you get results which agree with those on p. 5.
- What conclusion can you draw from this?

I. A. Rates. 4.

How can we estimate the speed at any time t_b ?

- (a) Review. In our last section in order to make an estimate on the speed at $t = 2$ we used as values of the independent variable 2 and $2 + h$; for making an estimate of speed when $t = 3$ we used as values of the independent variable _____ and _____; to find the speed when $t = 4$ we used _____ and _____. For each different time in question we require a _____.

We attempt now to develop a method for estimating the speed at any time t_b . (We establish a convention here: in our discussion t_b shall mean a fixed time t , as $t = 2$, $t = 3$, $t = 4.5$ sec., etc.; the symbol t will mean an arbitrary t . This distinction is not always made in texts so the student should use care in reading them.)

- (b) Consider again $s = 16 t^2$ (our idealized expression). With experience gained in the use of the h -method for finding the speed at the end of the second second, or at the end of the third second, you develop an expression to give the speed at the end of any time t_b .
- (c) State your final argument in "limit language."
- (d) What is the advantage of the method just developed over those used previously?

EXERCISES.

1. By the method developed above estimate the speed at the end of the times indicated if the distance functions s of the time t are given as follows:

(a) $s = 3t^2 + 6t - 5$ at $t = 2$ sec. (d) $s = \frac{t+3}{t-4}$ for $t = 5$ sec.

(b) $s = 1/t^2$ at $t = 3$ sec.

(c) $s = \frac{t^2+4}{t}$ at $t = 2$ sec.

(e) $s = \frac{t^2+4}{t+1}$ for $t = 3$ sec.

2. On re-examining (b) above what things or facts about limits do you think we should have to be "more sure" that our work is on a firm foundation? What assumptions do we seem to be making?

I. A. Rates. 4.
(continued)

EXERCISES (continued).

3. We have made an estimate of the speed at the end of the second second by four methods. List these and add a descriptive phrase about each. What features do these have in common? Again, what assumptions do we seem to be making?
4. Text. p. 23. Nos. 1 and 2. Do half in each list. What is the meaning of the term "sequence?" Is a sequence a function? What notation do we adopt to indicate a sequence?

I. A. Rates. 5.

How can we find the "slope" of a curve at any point?

- (a) In analytical geometry we became acquainted with the idea of the slope of a line segment which is a measure of _____. In fact, the term slope was defined only for straight line segments and not for curves.
- (b) Curves also demonstrate or exhibit a "change in y " relative to a certain "change in x " and hence it is conceivable that one might wish to use the term "slope" with reference to curves. Note the figure at the right.

What definition would you suggest as the slope of a curve at any point?

What would be a difference between the slope of a curve as x changes and the slope of a straight line segment? _____

- (c) Hence what would you suggest as the slope of the curve shown at the right at the point A? _____
 at the point B? _____;
 at the point C? _____;
 at the point D? _____;
 at the point E? _____.

Express all these in "rate language."

In the vicinity of A _____
 In the vicinity of B _____
 In the vicinity of C _____
 In the vicinity of D _____
 In the vicinity of E _____

- (d) In the figure at the right what is the slope of the No. 1 secant shown? _____;
 What is the slope of the No. 2 secant?

- (e) Suggest and carry out a method for estimating the slope of the tangent at $(1,3)$, given that the equation of the curve is $y = x^2 + 2$.

I. A. Rates. 5.
(continued)

- (f) Express your result obtained in (e) in "rate language."
- (g) Here are graphs or curves of functions which have points of relative maxima and minima.

What hypotheses or statements would you be willing to make on relations between the slope of a curve and a point of maximum? _____; the slope of a curve and points of minimum? _____

EXERCISES.

1. Find the slopes of these curves at any point x_1 by the use of the method developed above (h-method). Could you use $x_1 - h$ as well as $x_1 + h$? Try it on several.

(a) $y = 3x^2$.

(d) $y = x^3 - 3x^2 - 24x$.

(b) $y = x^2 - 4x$.

(e) $y = x^3 - 12x + 10$.

(c) $y = ax^2 + bx + c$.

(f) $y = \frac{1}{x}$.

Look at your answers. Do you see any "scheme" or "method" by which you could write expressions for slope merely at sight? If you note a scheme or method what is it?

2. In No. 1 (a) what is the slope when $x = -1$? $x = 1$? $x = 2$? $x = 4$? Is there a value of x for which the slope is 0? If so, what? Is this a point of maximum or a point of minimum? How can you tell?
3. In No. 1 (d) are there values of x for which the slope is zero? What, if any? Analyze to see if at these values of x the function has a maximum or a minimum?
4. If the slope of a curve $y = f(x)$ at a certain point is three then this means that in the vicinity of this point what approximate relation exists between the x -changes and the y -changes?

I. A. Rates. 5.
(continued)

EXERCISES (continued).

5. Consider these two expressions:

$$y = x^3 - 5x^2 - 30x$$

(a) We have been thinking of this expression merely as _____

(b) If we apply the h-method to this function as we have at any point x_1 we get an expression for _____ at the point x_1 .

$$s = t^3 - 5t^2 - 30t$$

(a) We have been thinking of this expression merely as _____

(b) If we apply the h-method to this expression (function) at any time t_b then we get an expression for _____ at the point t_b .

(c) Is there a "relation" between slope and speed? What?

6. Text. p. 11. No. 1. Show work. Why is our "h-method" better than this?

7. Quick review:

(a) a sequence is _____

(b) to say that $a(b + c) = ab + ac$ is to say that multiplication is _____ with respect to addition.

(c) average speed is _____

(d) $(2 + h)^2 =$

(e) two assumptions or hypotheses we are making on our speed problems (or methods):

i)

ii)

(f) draw a curve whose slope is always positive.

(g) draw a curve whose slope is always negative.

(h) draw a curve with a point at which the slope is zero.

I. A. Rates. 5.
(continued)

EXERCISES (continued).

- (i) a line has a slope of 3. Express this in "rate language."
- (j) at a certain point a a curve has a slope of 5. Express this in "rate language."
- (k) what a curve may have at a point of zero slope?

I. A. Rates. 6.
How are the problems studied thus
far alike?

- (a) Let us consider these functions as we did in No. 5 p. 15.

$$s = t^2 + 4t + 3$$

- i) the application of our
"h-method" at any time
t gives us what we call

- ii) but what is the meaning
of "speed" (in terms of
distance and time)?

$$y = x^2 + 4x + 3$$

- i) the application of our
"h-method" at any value
of x gives us what we call
the _____

- ii) but what, really, is the
meaning of the word "slope?"

- iii) Hence what do the two meanings (speed and slope) have in
common? _____

- iv) How can each be expressed as a "rate of change?" _____

- (b) What would be the meaning of the expression which would be obtained
by an application of the "h-method" to these rather well-known
functions where the independent variable is considered as the
letter in parentheses. Write.

i) $V = e^3$ (e)

ii) $A = 3r^2$ (r)

iii) $A = 2\pi r^2 + 2\pi rh$ (r)

iv) $V = \frac{4\pi r^2}{3}$ (r)

- (c) Here is a function $y = f(x)$. If we apply the "h-method" to this
function explain what we get.

- (d) We now have "several ways of saying things" which we have obtained
by the application of the "h-method." What are these? Are they, in
a sense, synonymous? List these ideas.

i)

ii)

iii)

Does one of these include the other two? Which?

I. A. Rates. 6.
(continued)

- (e) If a curve has a point such that the rate of change of y with respect to x is zero what might be the significance of such a point?

EXERCISES

1. A box is such that it is twice as wide as it is high and it is three times as long as it is high. (a) What is the volume of such a box? (b) What is the rate of change of the volume of the box with respect to the height x ? (c) What does this mean when the height is 4 inches? (d) When the height is 5 inches? When the height is 8 inches?

2. A box has dimensions which are related as shown. (a) What is the volume of this box? (b) What is the rate of change of the volume of this box when the height is x ? (c) What is the meaning of this rate of change of the volume with respect to the height x when x (or the height) is 10 inches? (d) When the height is 12 inches?

3. The volume of a box is given by $V = x^3 - 9x^2 + 24x$. (a) Determine by graphical methods at what value of x this volume has a maximum value. (b) What is the slope of the curve at this value of x ? (c) How could we use our "rate of change" idea to find the value of x at which the volume is a maximum? (d) Do it. (e) How do the results agree with those obtained by using a graph?
4. A certain rectangular plot has a perimeter of 20 feet. Suppose we call the width x . (a) What is the expression which gives the area A ? (b) What is the rate of change of area A with respect to the width x ? (c) What does this mean when $x = 4$? (d) Is there a value of x for which the rate of change of area with respect to x is zero? What? (e) What is the significance of this value of x ? (f) Confirm by means of a graph.

5. Consider all the pairs of numbers which add to make 16. What pair gives the largest product?
6. See if you get the same result by the use of our "rate of change" idea. If x is one of the numbers, $16 - x$ is the other; their product is $P = x(16 - x)$. Study this function.

I. A. Rates. 6.
(continued)

EXERCISES (continued).

7. A stone is dropped into a quiet pool. Ever-expanding ripples appear. As the "radius" gets bigger the area enclosed becomes larger also. How does a change in the radius affect the area? Study by the use of $A = \pi r^2$ (a) When the radius is five feet what can you say about the area changes? (b) When the radius is 10 feet what can be said?
8. Text. p. 16. No. 13.
9. Text. p. 16. No. 14.
10. Draw a curve whose slope is decreasing as x increases; is this curve concave up or down? Draw a curve whose slope is increasing as x increases; is this curve concave up or down?
11. A bin is constructed so it has a square base and an open top. If the base is 5 feet on each side and the volume is 200 cu. ft. what is the height? What is the height if the base is 2 feet on a side? 10 feet on a side? b feet on a side?
12. If the base in No. 11 is 5 feet on a side what is the total surface area (or square feet of material required, without a top)? How much material is required to make the open box if the base is b feet on a side?
13. In No. 12 (a) what is the rate of change of area A with respect to the base b ? (b) What is the meaning of this when b is 10 feet? (c) When b is five feet? (d) Is there a value of b such that the rate of change of volume with respect to the base edge b is zero? If so, what? (e) If the area function were graphed what would be the significance of the values of b for which the rate of change of A with respect to b is zero? (f) Make a graph of the A function with respect to x . Do you obtain results which you would expect?
14. Suppose we now consider the bin in No. 11 to be constructed of material such that the sides cost 15 cents per square foot and the bottom costs 25 cents per square foot. (a) What will the base cost? (b) What will the sides cost? (c) What expression will give the cost C as a function of b ? (d) For what value of b is the cost a minimum? Show work and argument.
15. A bin is to be constructed so that the length is twice the width. The volume is to be 240 cu. ft. The bottom material costs 30 cents per square foot and the side material costs 10 cents per square foot. What width will give the minimum cost? What will be the other dimensions of the box? Show work and argument. How could you confirm your results?

I. A. Rates. 6.
(continued)

EXERCISES (continued).

16. Rework No. 15 but this time place a top on the box at a cost of 50 cents per square foot.
17. Rework No. 15 but this time place a top on the box at a cost of \$1.50 per square foot. Do you obtain different dimensions for minimum cost? Why?
18. A certain firm makes x radio sets per week at a (variable) cost given by

$$C = 3x + \frac{x^2}{25}.$$

What is the rate of change of the cost C with respect to the number of sets made (x)? This is called the marginal cost.

I. A. Rates. 7.

What use can be made of changing rates?

Related Rates.

- (a) Review. Consider an expanding soap bubble or an expanding balloon (each of which is spherical) where $V = \frac{4\pi r^3}{3}$, or by approximation of π to the nearest integer, $V = 4r^3$.
- i) What is the rate of change of V with respect to r , or what is $R_r V$?
 - ii) What is the meaning of this $R_r V$ (rate of change of V with respect to r) when

$r = 4$?

$r = 10$?

$r = r$?
 - iii) What graphical interpretation can you give to your statements in ii) above? Illustrate on one or two.
- (b) A balloon with a radius of 4 inches expands so that its new radius is 4.1 inches. Is the volume change an increase or decrease?
- i) What is the volume increase?
Show work. We shall help you illustrate what you are doing graphically.
 - ii) Could you have used (a-ii) above to predict an approximate answer? How? What?
- (c) A spherical bearing is known to be "exactly" 4 cm. in radius. An experimenter reported the radius as 3.8 cm. By how much (therefore) is the computed volume in error?
- ii) Use (a-ii) to give an approximate answer.
- (d) Illustrate graphically the difference between an exact change and an approximate change (approximated by the use of $R_r V$).

I. A. Rates. 7.
(continued)

- (e) Which method of determining volume change gives the exact change? _____

Which method gives the approximate change? _____

In what ways are approximate changes often sufficient? _____

What feature or property of the volume function (or graph) would tend to make the approximate change quite close to the exact change? _____

- (f) Of course, the exact change differs from the approximate change. This is the error in the approximation. What relation exists among these three quantities? _____

- (g) By the use of this figure and suitable notation write an expression involving error E , exact volume change, and the approximate volume change as computed by the use of $R_n V$.

What do you expect to happen as $h \rightarrow 0$? _____

- (h) Consider this figure (which is more general) and write a relation between y_f , y_b , $R_x y$, x_f , x_b , E . We shall make much use of this relation.

You ought to reproduce this figure and its idea often.

EXERCISES.

1. A rectangular solid bar has a volume of $V = x^3 + 18x^2 - 40x$ in terms of the height x . This bar expands when it is heated.

- (a) What is the rate of change of the volume of the bar with respect to x : $R_x V$?
- (b) What is the rate of change of the volume with respect to x when $x = 4$ inches.

I. A. Rates. 7.
(continued)

EXERCISES (continued).

- (c) Hence when the height is 4 inches if there is an expansion of the height to 4.02 inches there will be approximately what volume change?

2. A body is moving such that its distance s at any time t is given by

$$s = t^2 + 3t - 10.$$

If, when $t = 3$ sec., an error of $+0.1$ sec. is made, what is the approximate error in the distance? If the error in the time is -0.1 sec. what is the approximate error in the distance?

3. So far we see that the "h-method" applied to functions gives us a rate of change which
- i) interpreted for a distance-time function gives _____;
 - ii) interpreted for functions which are graphed, such as the graph of y against x , gives _____;
 - iii) interpreted for functions in general, gives the change produced in the _____ variable when the change in the _____ variable is known (and the value if the _____ variable is known).
 - iv) in all cases the rate of change can be interpreted geometrically or graphically as _____.

4. Often in our computations thus far we have encountered expressions as

$$\frac{2xh + h^2}{h} = 2x + h.$$

What restrictions are placed on the h for this identity to hold?

5. Text. p. 55. Nos. 1,2,3,4.
6. A cylindrical solid tube has a volume of $V = 100\pi r^3$. This solid piece expands when it is heated. When the radius is 1 inch an increase of 0.01 in. in the radius produces what approximate change in volume? Describe completely; i.e. state the amount of change and tell whether it is an increase or a decrease.
7. In No. 6 the radius (let us suppose) is 2 inches and the error in the radius is $+1\%$. What is the approximate error in the volume?

I. A. Rates. 7.
(continued)

EXERCISES (continued).

8. A rectangular piece of tin 20" by 16" has square corners cut out and a box is made by folding up the sides. What is the length of the box? the width of the box? the height of the box? the volume of the box?

9. From no. 8
- What is the rate of change of the volume of the box with respect to the height x ?
 - This means that when the height is within 0.1 in. of 2 inches the error in the volume will have approximately what range?
 - This means that if the height expands from 2.0 inches to 2.13 inches what will be the approximate volume change?
 - This means that if the height changes from 2.0 to 1.95 inches what will be the approximate volume change?
 - This means that if the height changes from 1.93 to 2.0 what will be the approximate volume change?
 - Is there a value of the height x for which the volume is a maximum? If so, what is the value, and show your argument?
 - What are the other dimensions (length and width) for the volume of the box to be a maximum?
 - What is the maximum volume?
10. A square piece of material 30" by 30" has squares cut from the corners and the resulting rectangles are then bent up to make a box.
- What is the expression for the volume of the box in terms of the height x ?
 - For what value of x does the box have a maximum volume?
 - What are the other dimensions?
 - What is the volume when the volume is a maximum?

I. A. Rates. 7.
(continued)

EXERCISES (continued).

14. Suppose that $s = f(t) = t^2 - 3t$.

- (a) Construct a graph of the function.
- (b) What is the rate of change of s with respect to t ?
- (c) What is the rate of change of s with respect to t when $t = 5$?
- (d) What does this rate of change mean if s is distance and t is time?
- (e) What does this rate of change mean graphically? Show this on the graph.
- (f) What "three ideas" have we been using which mean essentially the same thing?

15. As x increases what can you say about the slope of a curve with this shape?

As x increases what can you say about the slope of a curve with this shape?

I. A. Rates. 8.

GENERAL REVIEW EXERCISES.

1. Explain meanings of these terms or answer briefly:
 - (a) average speed
 - (b) speed
 - (c) difficulty in determining speed by measurement
 - (d) "successive approximations" to speed
 - (e) hypothesis
 - (f) sequence
 - (g) limit
 - (h) constant
 - (i) variable
 - (j) function
 - (k) "approximating function" for speed
 - (l) steps in the "h-method"
 - (m) slope of a line segment
 - (n) slope related to trigonometric ratios
 - (o) "slope" of a curve at any point
 - (p) how one can approximate the slope of a curve
 - (q) what "rate of change" means
 - (r) "maximum" of a function (local)
 - (s) "minimum" of a function (local)
 - (t) error in using rate of change to compute exact change.
2.
 - (a) What process and what ideas are common to finding speed and slope?
 - (b) What general idea envelopes both these?

I. A. Rates. 8.
(continued)

GENERAL REVIEW EXERCISES.

3. (a) Suggest several terms which you think should be clearly defined or any "theorems" or hypotheses which we have made either explicitly or tacitly which you think need examination and attempts at proof to place our work on a firm logical basis.
(b) What ideas do you feel occur as "weak links" in our development and which therefore need careful examination?
4. Find the rate of change of the volume of a cube with respect to its edge when the edge is 5 inches. This and the following problems will give some practice in setting up functions and then determining or computing the rates of change.
5. Find the rate of change of the area of a circle with respect to its
(a) radius (b) diameter.
6. (a) Show that the rate of change of the volume of a sphere with respect to the radius is equal to the area of the sphere.
(b) What similar statement can be made with respect to circles? Show it.
7. Find the rate of change of the area of a sector of a circle with respect to its central angle. Let the radius of the circle be some constant as a . How fast is the area of the sector changing when the central angle is 40° ?
8. The altitude of a right circular cone is always equal to the diameter of the base. When heated this expands to make a slightly larger cone similar to the original. What is the rate of change of the volume with respect to the radius of the base? What is the meaning of the "rate of change" when $r = 10$?
9. In No. 8 suppose the radius is 10". (a) If heat causes the cone to expand so that the new radius is 10.01" what is the corresponding approximate volume change? (b) If an experimenter measures the radius to be 10.12" what is the corresponding error in volume?
10. The area of a rectangle is 25 square inches. If one side is represented by x what is the perimeter? For what value of x is the perimeter a minimum?
11. Find the coordinates of the points on the curve

$$y = x^4 - 4x^3 + 4x^2 + 10$$
at which the slope is neither decreasing or increasing.

I. A. Rates. 8.
(continued)

GENERAL REVIEW EXERCISES (continued).

12. Are the points found in No. 11 points of maximum or minimum or neither? Show your argument.
13. Find two numbers whose sum is 12 so that the square of the one plus twice the other is a minimum. Write argument also.
14. The path of a punted football is given by the equation $y = 160x - x^2$ where the x-axis extends along the direction of the ground in the direction of the path of the ball. (a) How far from the starting point does the ball reach its maximum height? (b) How far from the starting point does the ball reach the ground? (c) What is the slope of the curve at this point? (d) With what angle does the ball hit the ground? (e) What is the slope of the curve of the ball when it starts? (f) How does this compare with the angle at which it hits?
15. Text. p. 16. Nos. 3,4,5,6,7.

Mathematics 263
Study-helps for Test

I. You should be able to explain meanings of

sequence (two aspects)	relative minimum of a function
well-defined function	relative maximum of a function
variable	saying the slope of a <u>line</u> is <u>four</u>
independent variable	saying the slope of a <u>curve</u> is
dependent variable	<u>4</u> at a certain point
slope	to say that multiplication is
slope positive	<u>distributive</u> with respect to
slope negative	addition
zero slope	<u>EXACT</u> change in a function due
average speed	to a change in x
formulae as "idealized"	<u>APPROXIMATE</u> (as approximated by
situations	"rate of change") change in a
range of a variable	function due to a change in x
hypothesis	
"approximating function" for	
average speed	
"approximating function" for	
average slope	

II. You should be able to state well definitions on sequence, slope of a curve at any point, speed of a moving (non-uniform motion) body.

You should be able to suggest several fundamental assumptions or hypotheses we have been making on speed, slope.

You should be able to draw the figure relating exact change, approximate change, error.

You should be able to discuss, show, the two sequences which seem to arise in each speed, slope, rate of change problem. What limitations are placed on "h"?

III. Problems. You should be able to work problems on

speed, slope, rate of change by numerical methods (pp. 5,6,7)
same by the use of "h" (p. 9) (pp. 11, 14)

maximum and minimum problems (boxes, rectangles, etc.)

expansion, error, rate of change problems

computation of $R_x y$ by the use of the h-method in cases as $y = \frac{x+4}{x}$.

IV. Suggestions on sheets 26, 27, 28.

Mathematics 263

Name..... Date.....

I. Short Answers.

1. To say that $\{a_n\}$ is a sequence means
 (1) _____
 (2) _____
3. A function is said to be well-defined
 when (3) _____, (4) _____
5. Is this function well-defined? (5) _____

$$F = \begin{cases} \frac{x+1}{x-2} & \text{for } x = 2 \\ 1 & \text{for } x = 2 \end{cases}$$
 (6) _____
 Why? (6) _____
7. Use $y = f(x) = x^2 - 4$. What is $f(3)$?
 (7) _____; what is $f(0)$? (8) _____
9. What two sequences always seem to be
 involved in arriving at an answer to the
 speed of a falling body at, say, the end
 of the second second? (9, 10) _____
 (11) _____
11. To say that the rate of change of y
 with respect to x is 5 means that (11) _____
 (12) _____
12. The identity

$$\frac{3xh + h^2}{h} = 3x + h$$
 (13) _____
 holds under what limitations? (12) _____
 (14) _____
13. 10, 5, $2\frac{1}{2}$, What is the sum of
 this infinite progression? (13) _____
 (14) _____
14. If $y = 3x^2 - 6x - 10$ what expression
 gives $R_x y$? (14) _____
 (15) _____
15. What interpretation can you place on
 $R_x y$ in No. 14 if $x = 4$? (15) _____
 (16) _____
16. What is the slope of the line which
 joins the points $(-4, 6)$ and $(0, 12)$?
 (16) _____

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17. At a point of maximum or minimum of a curve the slope (17) . What can be said about $R_x y$ at such points? (18) .
19. To say that multiplication is distributive with respect to addition means that (19) .
20. By a hypothesis is meant (20) .
21. If $y = x^2 - 8x + 10$ for what value of x is the slope zero? (21) .
22. In No. 21 for what value of x is the slope equal to +4? (22) .

II. Problems. Discussion.

1. A rectangular solid has a length which is three more than twice the height. The width is two less than the height.
- What is the rate of change of the volume with respect to the height?
 - When the height is 3 inches the bar is warmed so that the height expands to 3.02 inches. What is the corresponding approximate volume change?
 - How would one find the exact volume change? (tell how only)
 - What is the value of x for which the volume is the greatest?
2. Draw a figure to show the meaning of the actual change (or exact change) and also approximate change when the rate of change is used to make the approximation. Show error also. Use any kind of function you wish.
3. Given $y = \frac{4}{x+2}$. By the "h-method" find $R_x y$. Show work.

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III. Problems. Discussion.

1. Suppose that $s = 4t^2$ is the distance s which a body moves in t seconds. Make at least four numerical approximations to the speed at the end of the first second ($t = 1$) and then suggest a number which you would present as the speed at the end of the first second.

2. If you were given the above expression, $s = 4t^2$ as the equation of a curve and you were asked to find the slope at $t = 1$ by numerical approximation would your computations be (a) different (b) exactly the same (c) similar? Answer: _____

3. How have we defined speed at the end of a certain time (say $t = t_b$) if the motion is nonuniform? Write well.

4. How do we define the slope of a curve at, say, $x = x_b$? Write well.

5. Suggest several fundamental assumptions we have been making on speed and slope.
 - a)
 - b)
 - c)

I. B. Areas. 1.

Historically speaking, this section might have preceded our first work on the calculus, for methods of the calculus had their origin in an attempt to find areas of figures with at least one side being a curve. Archimedes, it was, who tackled this problem in a way similar to that which you might suggest. The conventional geometry of Archimedes' day and that of the present (ordinary Euclidean geometry) are, of course, limited to methods of finding areas of rectilinear figures such as triangles, trapezoids, other quadrilaterals, and other polygons.

The application of an "unit area" as is done in the computation of rectilinear figure areas, does not work when one side of a figure is curved, so the "area" under a curve is really not "defined." Intuitively, however, we feel that there is an area under a curve as that shown in the figure at the right.

At the moment this presents a problem; yet definitions do not come automatically or are they given by edict. As we work with this intuitive "area" under a curve some ideas may develop which will lead us to formulate a definition on a very natural manner, and this, in general, is the way by which definitions come into being. Formal statements of definitions are often held in abeyance until experience, intuition, and natural processes, enable one to express an idea with clarity and with a minimum of necessary terms.

Let us consider a curve as shown, and, to make our discussion more definite, let the equation of the curve be

$$y = x^2$$

and let us suppose, further, that we are interested in finding the "area" bounded by the curve, the x-axis, and the lines $x = 1$ and $x = 3$.

- (a) Why is it not possible to use the ordinary definition of area as applied to rectilinear figures to the "area" under the curve?
- (b) Could you use the definition of area as applied to rectilinear figures in any way at all to come to some conclusion about the "area" under the curve? How? How about the accuracy of your conclusion?

I. B. Areas. 1.
(continued)

- (c) Suggest a method or methods for getting a close approximation to what we feel we could call the "area" designated in the accompanying figure. Describe (merely) what methods you suggest. Use figures to help.

- i)
 - ii)
 - iii)
 - iv)
- (d) "Do" one of the methods which you suggest. Show work or evidence of the use of this method. Show work on the reverse side or on other paper.
- (e) Try another of the suggested methods if you wish.
- (f) From either of the above methods see if you can get a "sequence of approximations" to that which you would be willing intuitively to call "area." Show your work.
- (g) Perhaps someone has suggested the use of rectangles whose sum is always slightly less than that number which we would be willing to call the area under the curve while someone else has suggested the use of rectangles whose sum is always slightly more. If this has not already been done, how could these two methods be used simultaneously to suggest a number which we could call area? Outline the suggested method and illustrate with figures.
- (h) Carry out the process suggested in (g) for the area under the curve $y = x^2$ from $x = 1$ to $x = 3$ if you have time. Suggest some notation to distinguish between "greater than" area and "less than" area.
- (i) Do you see any advantages of the method in (h)? What?

EXERCISES.

1. What is the area under the curve $y = 8x^2$ between $x = 0$ and $x = 1$ and the x -axis as approximated by one of the methods used above?
2. How is the procedure of approximating an area like approximating a rate of change?

I. B. Areas. 1.
(continued)

EXERCISES (continued).

3. Try using graph paper to estimate the "area" under the curve $y = x^2$ between $x = 1$ and $x = 3$. How does your result agree with those obtained above from numerical methods?

This space contains some relations which
will be found useful later.

$$1. \quad S_{\infty} = \frac{a}{1 - r}$$

$$2. \quad 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$3. \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$4. \quad 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

I. B. Areas. 2.
More work on areas. Review.

EXERCISES.

1. Find the area under the curve $y = 2x^2$ between $x = 1$ and $x = 2$ by some method which leads to a sequence of approximate areas. Show work in a systematic fashion. (Ans. $4 \frac{2}{3}$)

2. Review. Short answers.
 - (a) The speed of a body moving under acceleration (nonuniform motion) we define to be _____

 - (b) The slope of a curve we define to be _____

 - (c) In (a) there are really two sequences: i) _____
and ii) _____
and what we are really saying is that as _____
we ASSUME THAT _____
_____ which we define as speed.
 - (d) Likewise, in (b), there are really two sequences: i) _____
and ii) _____
and what we are really saying is that as _____
we ASSUME THAT _____
_____ which we define as slope.
 - (e) the function of h which gives average speed and the function of h which gives average slope may be called "_____
_____".
 - (f) Hence so far we have made several important definitions and assumptions. Name or indicate and state them here.

3. As you see it WHAT REMAINS AS THE FUNDAMENTAL QUESTION WHICH WE MUST ANSWER?

4. In our process which we have developed does it make any difference how $h \rightarrow 0$? _____

I. B. Work. 3.

How can one compute work when the force is changing?

To consider another type of problem for which the calculus provides a method we ask how much work is done when a certain (changing) force is exerted through a given distance.

First we must review some elementary physics.

- (a) What is the definition of work?

What work is done when

- 8 lbs. are lifted 3 feet? _____
- 10 lbs. are lifted 2 1/2 feet? _____
- 6 lbs. are lifted through a foot? _____
- a lbs. are lifted through b feet? _____
- a force of 10 lbs. is exerted through 3 feet? _____
- a force of 10 lbs. is exerted through 0.1 feet? _____

- (b) Now for the spring. The law of the spring is $F = k s$ where F is force (lbs.) and s is the distance the spring is stretched (feet or inches or some other length unit). k is the "constant" of the spring in force per unit length. If $k = 10$ lbs. per inch stretch then the law becomes

$$F = 10 s.$$

What does this mean? Give (write) three examples here.

i)

ii)

iii)

- (c) Let us now consider a spring whose law is $F = 32x$.

- (d) What force is being required to hold (or keep pulled) the spring when the stretch x is $x = 2$?

What force is required when $x = 2 \frac{1}{2}$?

What force is required when $x = 3$?

What would be an approximation to the work required to pull the spring from $x = 2$ to $x = 3$ ft.?

I. B. Work. 3.
(continued)

- (e) What can be done to get a better approximation to the work done in computing the work done to move the spring from $x = 2$ to $x = 3$? Show work. Call this A_2 (approximation).
- (f) How can you obtain a still better approximation, A ? Show work.
- (g) Can you eventually set up a sequence of approximations to the work done in stretching the spring from $x = 2$ to $x = 3$? Do it and suggest an answer to this problem.
- (h) Summarize the process which you have developed as follows:
- i) We take smaller and smaller _____ and compute the _____ for each.
 - ii) We then _____ these "little works."
 - iii) Does it make any difference how we divide the distance? _____; the only restriction which we make is that _____.
 - iv) What fundamental assumption do we then make? _____.
 - v) How can you DEFINE work done on a spring whose force is always changing? DO THIS WELL. (*)

- (i) How is making computations to present a number as work done like finding "area" under a curve?
- You might like to consider $y = 3x^2$ (curve) and $f = 3x^2$ (spring). If you wish, suppose you divide the distance into two or three intervals, note similarities, note geometrical interpretations, to show the analogy between the two.

*We now have four fundamental definitions: speed, slope, area, work. See if you can state these well.

I. B. Work. 3.
(continued)

EXERCISES.

1. The "equation" of a spring is $F = 72x$. What number would you present as the "work" required to pull the spring from $x = 1$ to $x = 4$? Make several approximations until you feel you have reason to indicate an answer.
2. Two balloons have been charged alike with static electricity. Suppose the force of repulsion between them is given by

$$F = \frac{100}{x^2} \text{ dynes.}$$

How much work will be required to push them from 3 cms. apart to only 2 cm. apart? Show approximations and answer which you would present.

3. Let us consider F again in No. 2. What is the $R_x F$? How fast is this repulsive force changing with respect to x when x (distance) is 3 cms.? 2 cms.? When the distance is 3 cms. and a change of 0.1 cm. is made in the distance this will require how much force approximately?
4. An oriental hat shaped something like the figure at the right may be considered as formed by rotating the curve $y = x^2$ around the x -axis and taking the portion between (say) $x = 0$ and $x = 8$. What suggestions can you make for finding the volume of such a hat? Show a few approximations.

Optional: continue until you can present an answer with some confidence.

5. In all our area, work, speed, slope work we reduced the "x-interval" according to some law: as by halves, fourths, etc. rather than just taking the intervals smaller at random. The former we may call "ordered subdivision" and the latter "random subdivision." What is the advantage of the former?
6. Review: A box is to have a volume of 800 cu. ft. and the length is three times the width. What must the width be for minimum surface?

I. B. Work. 3.
(continued)

EXERCISES (continued).

7. Review: Find $R_x y$.

$$(a) \quad y = 3x^2 + 4x - \frac{1}{x}$$

$$(c) \quad 7 = (x + 2)^3$$

$$(e) \quad y = \frac{3}{x + 4}$$

$$(b) \quad y = \frac{8}{x} + 7x^2$$

$$(d) \quad y = \frac{x + 4}{x}$$

$$(f) \quad y = 3x^2 - 8x + \frac{1}{x} + 10$$

I. B. Areas. 4.

Can we develop an "approximating function"
for "areas?"

When we were attempting to find speed from a given distance-time law we saw the advantage of using the "h-method" which gave us an "approximating function" for the average speed. For example, when

$$s = 16 t^2$$

was the "law" we obtained

$$\frac{s_f - s_b}{t_f - t_b} = 32 t_b + 16h \quad (\text{a function of } h)$$

as the approximating function? What were several practical values in having such a function? _____

We note also that in the estimation of "areas" under curves and the "work done" when force is variable there is much numerical computation. Can we arrive at some kind of approximating function for these too?

- (a) Let us again consider the curve $y = x^2$ and attempt to approximate the "area" under the curve between $x = 0$ and $x = 5$. Let us divide the interval $(0,5)$ into n equal parts. What is the width of each subinterval?

What is the area of the "first" rectangle? _____; the second? _____; the third? _____; . . . the n th? _____.*

- (b) Set up an expression which gives the approximate area under the curve. What conclusion can you draw (or do you feel you can draw) as n becomes very large? It may be necessary to work on this expression until it is in more compact form before you feel that you can draw a conclusion. Use the reverse side of this sheet or other paper.

*What we are really doing here is that we are "stopping" our successive process of dividing the "area" in first 2 rectangles, 4 rectangles, 8 rectangles, etc. and picking out the step in which we used n rectangles. This case we then choose to set up and discuss.

I. B. Areas. 4.
(continued)

- (c) From your "answer" suggested in (b) and the approximating function of n which you developed what would be the expression for error (E) for any n ?
- (d) In your own words tell what you have done to arrive at the number which you present as the "area" under the curve.
- (e) What skills do you suggest it will be necessary to have or to develop to become adept at the use of this method? What evidence do you find from your text that the author either agrees or disagrees with you? (pp. 132-136)

EXERCISES.

1. Text. p. 138. Nos. 1,2,3,4,5,6.
2. What hypothesis does No. 6 suggest?
3. What elements of method are common to the use of approximating functions in determining a rate of change and in determining an area under a curve?
4. Review. Given $y = x^3 - 3x^2 - 9x + 10$ what is the rate of change of y with respect to x ?
5. In No. 4
 - (a) What is the rate of change of y with respect to x when $x = 2$?
 - (b) For what value of x is the rate of change equal to zero?
 - (c) For what value of x does the curve have a maximum?
Show the argument.
6. Write the same function (as in No. 4) in three other ways and then answer No. 5(a) in these three corresponding ways.
7. What do these words or expressions mean?
 - (a) an infinite sequence
 - (b) getting "bounds" on area
 - (c) an approximating function
 - (d) work
 - (e) fundamental relation involving actual change, rate of change, and error

I. C. The Fundamental Problems Recapitulated

We have now examined the two fundamental problems which form the basis of the calculus: the "general rate (slope) problem" and the "general area problem." Let us summarize what we have done by means of these

EXERCISES.

1. Use the two parallel columns as shown and discuss items suggested.

<u>GENERAL RATE PROBLEM</u>	<u>GENERAL AREA PROBLEM</u>
-----------------------------	-----------------------------

- (a) a figure

(b) questions which originate the problem	<p>i)
ii)
iii)</p>
(c) kinds of practical questions which originate the problem	
(d) mechanics to answer the general question; i.e. procedure you go through. Write carefully.	

I. C. The Fundamental Problems Recapitulated
(continued)

	<u>GENERAL RATE PROBLEM</u>	<u>GENERAL AREA PROBLEM</u>
(e) kind of approximating function you get (write a sample)		
(f) descriptive statement on what you wish to happen to "h" "n"		
(g) what essentially the "answer" is		
(h) what elements do the two solutions have in common	$\longleftrightarrow$	
(i) basic ideas which require definition	$\longleftrightarrow$	
(j) basic assumptions which require investigation	$\longleftrightarrow$	
(k) introduces us to what is called the the	_____ and _____ _____	_____ and _____ _____
and yet we shall find that even they are related.		

I. C. The Fundamental Problems Recapitulated
(continued)

EXERCISES. These help us to review and to note similarities in the fundamental problems. We learn by review.

1. A piece of metal in the form of a rectangular solid has a length which is always five more than the height. The width is twice the height. What is the rate of change of the volume with respect to the height? When the height is 5 an error of +1% in height will produce what approximate error in the volume?
2. If in No. 1 the height expands from 10.00 to 10.05 and the other sides expand correspondingly what will be the approximate volume change? What will be the approximate new volume?
3. If in No. 1 the height shrinks from 10.00 inches to 9.95 inches what will be the approximate volume change?
4. Find the area under the curve from $x = 0$ to $x = 2$. Suppose the equation of the curve is $y = 4x^2$. Let the x -axis be the other bound on the area.
5. A piece of cardboard is 12 inches by 10 inches. Squares are cut from the corners and the sides are folded up to make a box with no top. How high must the box be for maximum volume?
6. Given the function whose equation is $y = x^3 - 3x^2 + 8x$, what is the slope of its graph when $x = -2$? when $x = 0$? Give two possible interpretations of your answer depending upon what the $y = f(x)$ represents.
7. In No. 6 at what value of x is the slope equal to 5?
8. Is this (in No. 7) a point of maximum, a point of minimum, or neither? What is your argument?
9. Suggest (draw) a curve which would have a slope of zero at, let us say, $x = 2$ which is neither a point of maximum or a point of minimum. What is the name of such a point?
10. Find the speed of a body which moves according to this distance-time law:

$$s = \frac{t^2 - t}{t + 1}.$$

I. C. The Fundamental Problems Recapitulated
(continued)

EXERCISES (continued).

11. Draw a graph of $V = x^3$ (roughly) and illustrate the difference between the volume change as x changes from x_1 to x_2 as computed from the volume formula and as estimated by the rate of change method. What algebraic expression gives this difference?
12. A box is to have a volume of 800 cu. ft. The length is to be twice the width. The bottom material will cost 50 cents per sq. ft. and the side material costs only 20 cents per sq. ft. What must be the width for the cost to be a minimum? (no top)
13. What do you suggest as a definition or meaning to say that the sequence $\{a_n\} = a_1, a_2, a_3, \dots$ approaches a limit A?

II. A. Systematic Look at the Calculus

In the last section (I.C.) we made a resume of the two fundamental problems of the calculus; we noted that there were elements in common to the solutions of the two problems; each contained chosen sequences which were arranged to approach a limit, each employed an approximating function of some kind which appeared to approach a limit, and in each case this limit was assumed to be the answer to the problem.

Some fundamental questions which we raised were: i) _____, ii) _____, iii) _____. Do you have more? _____

Apparently the first word which we must tackle is that of "limit." What are some questions which we must answer about this?

i) _____, ii) _____, iii) _____

- (a) We shall assume that our sequences are made up of real numbers. (*)
- (b) We shall assume the definition of a sequence as has been stated before. For infinite sequences there must be a rule of formation, or, an expression for the nth term.
- (c) After class discussion or other sources explain what we shall mean when we say that a sequence $\{a_n\} \rightarrow A$.

- (d) Hence in technical language we shall say that

$$\lim_{n \rightarrow \infty} \{a_n\} = A \Rightarrow$$

- (e) Suggest meanings of the words

- i) converge
- ii) convergence
- iii) diverge.

*If it were desirable to base our logical system on natural numbers then it would be necessary for us to develop the real number system before going further. A more meagre beginning would be a set of postulates known as the Peano Postulates (four of these) from which the natural numbers can be developed (or what we call natural numbers) and then the reals finally. Such developments as these can be found in advanced texts.

II. A Systematic Look at the Calculus (continued)

We shall try to develop some ideas by means of the

EXERCISES.

1. Consider the sequence $1, 1/2, 1/3, 1/4, \dots 1/n, \dots$
 - (a) Write the sequence for several more terms.
 - (b) As n becomes infinite what is the limit of this sequence?
 - (c) How many terms must you use so that all the rest will be within 0.25 of the limit?
 - (d) How many terms must you use so that all the rest will be within 0.10 of the limit?
 - (e) How many terms must you use so that all the rest (i.e. from there on) will be within 0.00001 of the limit?
 - (f) Explain then, how you can argue that this sequence has a limit, as $n \rightarrow \infty$, by the definition of limit.
2. Consider the sequence $1, 1/2, 2/3, 3/4, \dots \frac{n-1}{n} \dots$
 - (a) Write several more terms.
 - (b) What do you suggest is the limit as $n \rightarrow \infty$?
 - (c) Make several statements as (c), (d), (e) above and then see if you can argue that the sequence has a limit and what it is.
3. Here are some sequences
 - (a) $3, 2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \dots, 2 + \frac{1}{2^{n-1}}, \dots$
 - (b) $2, 2\frac{1}{2}, 2\frac{3}{4}, \dots, 2 + \frac{2^{n-1}-1}{2^{n-1}}, \dots$
 - (c) $2, 2\frac{1}{2}, 2, 2\frac{3}{4}, 2, 2\frac{7}{8}, \dots \begin{cases} 2 \text{ for } n \text{ odd} \\ 2 + \frac{2^{\frac{n}{2}}-1}{2^{\frac{n}{2}}} \text{ for } n \text{ even} \end{cases}, \dots$
 - (d) $1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-1}}, \dots$
 - (e) $c, c, c, \dots, c, \dots$

Which of these, if any, would you suggest converge to a limit and, if so, what limit do you suggest?

II. A Systematic Look at the Calculus (continued)

EXERCISES (continued)

4. What is the "limit" of the sequence in No. 3(b)? _____
 What is the "limit" of the sequence in No. 3(d)? _____
 Suppose we set up a new sequence obtained from these two by adding the corresponding terms. Do this. Write the new sequence so you have about six or seven terms. What limit do you suggest for the "sum sequence?" _____
5. Try subtracting the terms of the sequence in (d) No. 3. from the corresponding ones in (b) of No. 3. Do you get a new sequence? Does it seem to converge? What limit do you suggest?
6. Let us take sequences No. 3 (b) and No. 3 (a) and see what happens if one forms a new sequence by multiplication.
7. What other way might be suggested to attempt to set up a new sequence? What precautions need we take? Do it.
8. What hypotheses on the formation of new sequences from "old ones" which are convergent can be made on the basis of Nos. 4,5,6,7 above? Make careful statements.
9. Try to deduce some of these hypotheses from the definition of limit. Then the results can be called theorems.
10. To study the limits of sequences for which the n th term is known (as $n \rightarrow \infty$) we need a somewhat simple technique which will be shown. With it find the limits of these sequences whose n th terms are as follows (as $n \rightarrow \infty$):

(a) $a_n = 3 + \frac{1}{n}$	(b) $a_n = \frac{3n + 1}{n}$
(c) $a_n = \frac{n^2 + 3n + 2}{n^2}$	(d) $a_n = \frac{n^2 + 3n + 2}{n^3}$
(e) $a_n = \frac{n^2 + 3n + 2}{n^2 + 2}$	(f) $a_n = \frac{n^2 + 3n + 2}{n}$
11. Text. p. 29. No. 4 (a), (c), (d), (e), (f).
12. A curve has the equation $y = x^2 - 6x + 8$. At what values of x does it cross the x -axis? At what value of x is the slope zero? Is this a point of maximum or minimum?
13. How is a sequence a function?

II. A Systematic Look at the Calculus. 2. Limits of Functions

In the speed problem and in the slope problems (and, therefore, in the _____ problem) we had two sequences. What are these?

i) _____

ii) _____

The study in the last section in which we defined limit and gave a type of test to see if a sequence approaches a limit enables us to discuss "how much" of the sequences above? _____

(That is, what limitations are placed upon our "sequences" at present?)

In the speed and in the slope problem we are hoping that as the time or independent variable interval sequence approaches zero the approximating function which gives either the _____ or the _____ will _____. Likewise, we are hoping in the area and in the work problem that as the _____ then the _____ which gives an approximate area or an approximate work for any n will _____.

In other words, we are led to what question, as suggested by the title of this page? (State well) _____

What are some questions which we might have to answer?

(a) _____

(b) _____

(c) _____

With the aid of your class discussion or other sources suggest a definition or meaning to say that

$$\lim_{x \rightarrow a} f(x) = K.$$

Figures will help. _____

When you think you have a good definition which you understand write it in your notebook. Definitions should be numbered. Many use D_1 ; D_2 ; etc.

II. A Systematic Look at the Calculus. 2. (continued)

More of this we shall study in the

EXERCISES.

1. Review: definitions (write carefully) of
 - (a) A is the limit of sequence $\{a_n\}$ as $n \rightarrow \infty$.
 - (b) speed (non-uniform motion)
 - (c) slope of a curve
 - (d) rate of change of y with respect to x where y is a function of x
 - (e) area under a curve
 - (f) work done when the force is variable
2. Review: Here are two sequences $\{a_n\} \rightarrow 8$ and $\{b_n\} \rightarrow 2$ as $n \rightarrow \infty$.
 - (a) what is the limit of the sequence $\{a_n + b_n\}$ as $n \rightarrow \infty$?
 - (b) what is the limit of the sequence $\{a_n \cdot b_n\}$ as $n \rightarrow \infty$?
 - (c) what is the limit of the sequence $\{a_n - b_n\}$ as $n \rightarrow \infty$?
 - (d) what is the limit of the sequence $\left\{ \frac{a_n}{b_n} \right\}$ as $n \rightarrow \infty$?
3. Let us consider now the function $f(x) = x^2 + x$. Suppose x itself is approaching 2 through the sequence $1, 1 \frac{1}{2}, 1 \frac{3}{4}, 1 \frac{7}{8}, \dots, 1 + (n-1)/2^{n-1}, \dots$.
 What do you suggest is the limit of the function $f(x)$ as $x \rightarrow 2$?
 Make calculations and exhibit results to substantiate your hypothesis.
4. If $\lim_{x \rightarrow 1} f(x) = 4$ and $\lim_{x \rightarrow 1} g(x) = 3$, what do you suggest about $\lim_{x \rightarrow 1} [f(x) + g(x)]$?
5. Make several more hypotheses on limits of functions. Here are more hypotheses for your notes.
6. Review. See if you can deduce the theorem suggested in No. 2(a) above. Make restrictions so that you have a special case if you wish. Be sure to state restrictions.
7. Text. p. 36. Nos. 1,2,3,4,5.
8. Of all the numbers what number plus its reciprocal ($1/2$ is the reciprocal of 2) gives a minimum value? Show your work.

II. A Systematic Look at the Calculus. 3.
Continuous Functions.

We begin this topic by consideration of some

EXERCISES.

1. What are the limits of these functions $f(x)$ as x approaches the value shown?

(a) $\lim_{x \rightarrow 2} (x^2 + x + 2)$

(b) $\lim_{x \rightarrow 2} \left(\frac{x^2 + 3x}{x + 1} \right)$

(c) $\lim_{x \rightarrow 2} \left(\frac{x^2 + 1}{x - 1} \right)$

(d) $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x + 1} \right)$

(e) $\lim_{x \rightarrow 3} \left(\frac{x^2 - x - 2}{x - 2} \right)$

(f) $\lim_{x \rightarrow \infty} (3x^2 + x + 2)$

(g) $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 2}{2x^2} \right)$

(h) $\lim_{x \rightarrow 0} \left(\frac{4x + 3}{x + 7} \right)$

2. What are the limits of these functions $f(x)$ as x approaches the value shown?

(a) $\lim_{x \rightarrow 1} \left(\frac{3}{x - 1} \right)$

(b) $\lim_{x \rightarrow 2} \left[\frac{x^2 - 4}{(x - 2)(x - 2)} \right]$

(c) $\lim_{x \rightarrow 0} \left(\frac{1}{x} \right)$

(d) $\lim_{x \rightarrow \infty} \left(\frac{3x + 1}{4} \right)$

(e) $\lim_{x \rightarrow 7} \left(\frac{3}{x - 7} \right)$

(f) $\lim_{x \rightarrow 1} \left(\frac{16 - x}{x - 1} \right)$

3. What important difference do you note between how one can find the limit of the functions as x approaches the given number in Problem 1 and 2?

II. A Systematic Look at the Calculus. 3.
(continued)

We are now ready to discuss the matter of functions which are said to be "continuous." There may appear to be two aspects to the idea of continuous functions but really these merge into one.

- (a) Just from the meaning of the word (as an ordinary word) what would you suggest about the graph of a function which you were told is called "continuous?" _____

- (b) Which of these graphs would have been functions which are "continuous?" (in the range of independent variable indicated)

(i)

(ii)

(iii)

(iv)

(v)

(vi)

(vii)

(viii)

(ix)

- (c) From the work on the previous page and observations above what general algebraic idea would describe the meaning of a "continuous" function? _____

- (d) Hence a "man-on-the-street" definition of a continuous function might be _____

II. A Systematic Look at the Calculus. 3.
(continued)

- (e) Our technical definition is three-fold. It is: A function $f(x)$ is continuous at $x = a$ if (i) _____
(ii) _____ (iii) _____
Perhaps this should go into your notebook?

EXERCISES.

1. Text. p. 40. Nos. 1,2.
2. Our definition of continuity of a function $f(x)$ at $x = a$ is a definition of continuity at a point. What, then, would you suggest as a definition of a function being continuous throughout an interval?
3. What does it mean to say that a function is discontinuous at $x = a$?
4. Draw some ways to illustrate how a function might be discontinuous at $x = a$.
5. We speak of a function having an infinite discontinuity at the point $x = a$ or, perhaps, only a finite discontinuity. What do you suggest is the difference in meaning? Illustrate by figures.
6. Consider the function $f(x)$ defined as follows:

$$f(x) = \frac{3}{x} + \frac{4}{x-2} \quad \text{for } x \neq 0, x \neq 2,$$

$$\begin{aligned} \text{but } f(x) &= 0 & \text{for } x &= 0 \\ f(x) &= 4 & \text{for } x &= 2. \end{aligned}$$

- (a) As $x \rightarrow 0$ what happens to this function?
 - (b) As $x \rightarrow 2$ what happens to this function?
 - (c) At what values of x do you say that this function is continuous?
 - (d) At what values of x do you say that this function is discontinuous?
 - (e) How can you evaluate the limit of this function as $x \rightarrow 3$?
 - (f) Construct a rough graph of this function to help confirm intuitively what you have said.
7. Here are two functions, say,
- $$\begin{aligned} f(x) &= x^2 - 2x & \text{defined for all } x \\ g(x) &= x^3 - \frac{10}{x} & \text{suppose defined for all } x \neq 0 \end{aligned}$$
- Both these functions are continuous at $x = 2$. What hypotheses would you be willing to accept about the continuity of the sum of these functions? the difference? the product? the quotient? State carefully.

II. A Systematic Look at the Calculus
(continued)

EXERCISES (continued).

8. Here is a sample approximating function for slope: $\frac{3xh + h^2}{h}$
Discuss the continuity of this function for $h = 0$.
9. If a function is continuous between $x = a$ and $x = b$ what are some properties which you suggest this function has (or its graph)?

II. A Systematic Look at the Calculus. 4.
Review (T)

EXERCISES.

- State carefully definitions for (a) speed at the end of a certain time t , (b) slope of a curve at a value of x , (c) rate of change of $f(x)$ with respect to x at a certain value of x , (d) area under a curve and between $x = a$ and $x = b$ and the x -axis, (e) work done when force is variable, (f) to say that $\{a_n\} \rightarrow A$ as $n \rightarrow \infty$, (g) to say that $f(x) \rightarrow k$ as $x \rightarrow a$; (h) abstract statement of (d) and (e).
- Draw a figure to show relations between exact change of $f(x)$ as x changes from x to $x + h$ and the change approximated by the rate of change idea. Show also the error in the approximation. This may be on volumes, distances, or any thing you wish.
- The volume of a box is to be 80 cu. ft. The length of the box is twice the width. The box has a top. The bottom material is to cost 50 cents per sq. ft., the side material 40 cents per square foot, whereas the top costs 100 cents per square foot.
 - What is an expression for the cost (c) of the box?
 - What is the rate of change of the cost with respect to the width?
 - If such a box has a width of 2 ft. and an error of $+.03$ is made in measuring the width what will be the approximate change in the cost?
 - For what value of the width does the cost have a minimum?
- Here is a curve $y = 4x^2$. By dividing into n intervals the length between $x = 0$ and $x = 2$ find the area under the curve and between these abscissae. See figure below. Use formulae on p. 31.
- Find these limits as requested.
 - $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x + 2}{x^3 + 6x} \right)$
 - $\lim_{n \rightarrow \infty} \left(\frac{n^2 + 3n + 7}{n^2 + 8} \right)$
 - $\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x - 2} \right)$
- State four theorems on limits of sequences; four theorem on limits of functions.
- Be able to use the "h-process" to compute R_{xy} when y is such as

$$y = \frac{x^2 + 2}{x}.$$

Mathematics 263

Name..... Date.....

I. Problems.

1. A body is moving such that its time-distance formula is given by

$$s = 8t^2 + 4.$$

By numerical methods arrive at a number which you would present as the speed at the end of the second second ($t = 2$). Arrange your work neatly. There are some formulae on the last page. Use

$$S_{\infty} = \frac{a}{1-r} \quad \text{to summarize if you can.}$$

2. Here is the equation of a curve: $y = 2x^2$. By numerical methods arrive at a number which you would present as the slope of the curve at $x = 3$. Arrange work neatly. Use formula to summarize if you can.

3. In No. 1 had we asked for the slope of the curve $s = 8t^2 + 4$ would the numerical work have been
 (a) different? (b) similar (c) exactly the same? _____

Mathematics 263

Name..... Date.....

II. Problems.

1. Here is the equation of a curve $y = 3x^2$. By numerical methods arrive at a number which you would present as the "area" (intuitive) under the curve and between $x = 2$ and $x = 4$. We would suggest that you use trapezoids. Arrange work in an orderly fashion.

2. The "equation" of a spring is $F = 3x$. By numerical methods arrive at a number which you would present as the work done in stretching the spring from $x = 1$ to $x = 3$. Arrange work well.

3. How are the two computations above the same?
4. In No. 1 had the equation $y = 4x^2$ been $F = 4x^2$ for a spring how would you interpret the answer to the question asked?

Mathematics 263

Name..... Date.....

III. Problems. Discussion.

1. In Nos. 1, 2 on page f52pnw you used a sequence of intervals of the independent variable (x or t).
 - (a) You were planning on this sequence to approach _____
 - (b) Does it make any difference how this interval sequence approaches _____? Answer: _____
 - (c) Illustrate what you mean in (b). Give some examples.
 - i) _____
 - ii) _____
 - (d) As this interval sequence was approaching its limit you hoped that the approximating function (for average speed or for slopes of secants) _____.
2. How can you summarize your work on Nos. 1 and 2 on page f52pnw-2?
 As the number of _____
 then _____
 (If the "blanks" bother you write it in your own words.)

S O M E F O R M U L A E

Work = force x distance

Area of trapezoid = $\frac{h(b_1 + b_2)}{2}$ $S_{\infty} = \frac{a}{1 - r}$

Mathematics 263

Name..... Date.....

I. Discussion.

1. State carefully worded definitions for the following:

- (a) infinite sequence $\{a_n\}$.
- (b) speed at the end of a certain time $t = t_0$.
- (c) slope of a curve at any point $x = a$.
- (d) area under a curve between the limits $x = a$ and $x = b$.
- (e) rate of change of a function $y = f(x)$ with respect to x at $x = x_0$.
- (f) work done in stretching spring, say, from $x = a$ to $x = b$ when the force is variable.
- (g) to say that a sequence $\{a_n\} \rightarrow A$ as a limit as $n \rightarrow \infty$.
- (h) to say that $f(x) \rightarrow f(a)$ as $x \rightarrow a$.
- (i) try to make an abstract statement of (d) and (f).

Mathematics 263

Name.....

Date.....

II. Problems.

1. A box is to have a volume of 120 cubic feet. The length of the box is three times the width. The box has a top. The bottom material is to cost 120 cents, the side material 80 cents, and the top is to cost 20 cents per square foot. For what value of the width will the cost be a minimum?

2. The cost of a manufactured article with width x is given by $c = 10x^2 - 3x + 2$.
 - (a) What is the rate of change of cost with respect to x ?
 - (b) When the width is $x = 2$ a certain machine went wrong and made articles with widths 2.04 inches. What was the approximate increase in the cost?

3. Here is a curve $y = x^2$. By dividing the length from $x = 0$ to $x = 4$ into n intervals and using rectangles find the area under the curve and between these abscissae. You will get an approximating function in terms of n .

4. Here is a function which tells the area of a circle: $A = \pi r^2$. The curve is a parabola. Draw a figure to show what is meant by exact change in area as the radius changes from r_1 to r_2 . Show what is meant by approximate change if the rate of change $R_r A$ is used. Show error. Do well.

Mathematics 263

Name..... Date.....

III. Problems. Discussion.

1. By the use of the h-process (show "h" work) compute the $R_x y$ if

$$y = \frac{3x^2 - 2}{x} .$$

2. Find the limits as requested.

$$(a) \lim_{x \rightarrow 2} \left(\frac{x^2 + 4}{x} \right)$$

$$(b) \lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x - 3} \right)$$

$$(c) \lim_{n \rightarrow \infty} \left(\frac{n + 3}{n + 1} \right)$$

$$(d) \lim_{n \rightarrow \infty} \left(\frac{n^2 + 4n + 2}{3n^2 + 2n} \right)$$

$$(e) \lim_{x \rightarrow 0} \left(\frac{3}{x} \right)$$

$$(f) \lim_{x \rightarrow 3} \left(\frac{x^2 - x - 6}{x - 3} \right)$$

3. State four theorems on sequences which are convergent. Suppose $\{a_n\} \rightarrow A$ and $\{b_n\} \rightarrow B$.

- (a)
(b)
(c)
(d)

II. A Systematic Look at the Calculus. 5.
(continued)

GENERAL REVIEW EXERCISES.

1. Given $y = \frac{t^2 - 3}{t + 4}$ compute $R_t y$ by the h-method.

2. Find the following limits:

(a) $\lim_{x \rightarrow \infty} \left(\frac{3x-2}{8x+7} \right)$

(b) $\lim_{x \rightarrow \infty} \left(\frac{5x^2+6x+8}{3x^2-3x+2} \right)$

(c) $\lim_{x \rightarrow \infty} \left(\frac{2x^4}{x^2+1} \right)$

(d) $\lim_{x \rightarrow 2} \left(\frac{x-2}{x^2-4} \right)$

(e) $\lim_{n \rightarrow \infty} \left(\frac{n^2+3n-6}{n^2-7n} \right)$

(f) $\lim_{x \rightarrow -6} \left(\frac{x^2-x-42}{x+6} \right)$

3. A farmer wishes to enclose a field so it will have an area of 600 sq. rods. One side of the field will be bounded by a hedge fence. The "length side" of the field will cost \$4 a rod for fencing and the two "width" sides will cost \$10 a rod. What dimensions do you suggest for the cost to be a minimum?

4. Explain, define, or suggest, as requested:

(a) $\lim_{n \rightarrow \infty} \{a_n\} = A$

(b) $\lim_{x \rightarrow a} f(x) = M$

(c) $f(x)$ is continuous at the point $x = a$

(d) work

(e) how estimations of "work" and "area" under a curve are similar

(f) what is implied when there is a slope of 10 and the x-change is 3?

II. A Systematic Look at the Calculus. 5.
(continued)

GENERAL REVIEW EXERCISES (continued).

- (g) the basic concepts of the calculus
 - (h) a way to obtain "bounds" for the "area" under a curve
 - (i) four theorems on combinations of convergent sequences and their limits
 - (j) four theorems on combinations of functions which have limits as $x \rightarrow a$ and their limits
5. Text. p. 96. Nos. 1,2,3,5.
6. Text. p. 56. Nos. 5,4,3.
7. Text. p. 51. Nos. 26,27,28,29.

[Recall from Analytic geometry that if you have a point (x_0, y_0) and a slope m one can write the equation of the line through the point with the given slope as

$$y - y_0 = m (x - x_0)] .$$

8. Given $s = t^3 - t^2 + 4t$ as a distance-time formula do you recall that we finally worked out an expression for speed for all times t and obtained $v = 3t^2 - 2t + 4$ and hence did not have to work with specific numbers for the time anymore?
9. N O W see if you can work out an expression which will give the area under $y = x^2$ from $x = 0$ to ANY x , say $x = a$. Divide the interval from $x = 0$ to $x = a$ into n intervals. Show work well.

II. A Systematic Look at the Calculus. 5.
(continued)

GENERAL REVIEW EXERCISES (continued).

10. These require only short answers. Write them on this sheet. We learn well by understanding "little things"; these coalesce into bigger ideas:

- (a) If, for any $\epsilon > 0$, we can find an N such that for all terms beyond the N th one we have $a_n > A - \epsilon$ then we can say that the _____
- (b) If (as above) $a_n + b_n > A + B - \epsilon$ then we can say that _____
- (c) If (as above) $a_n \cdot b_n > A \cdot B - \epsilon$ then we can say that _____
- (d) If, for any $\frac{\epsilon}{2} > 0$, we can find an N such that for all terms beyond the N th one, we have $a_n > A - \frac{\epsilon}{2}$ and, for any $\frac{\epsilon}{2} > 0$, we can find an M such that for all terms beyond the M th one, we have $b_n > B - \frac{\epsilon}{2}$, then (adding) we can say that if $M > N$ we have found an _____ such that for all terms beyond the _____th one, we have $a_n + b_n > A + B - \epsilon$
- (e) Why did we use N in one case and M in the other? _____
Which one will we have to use for the sum sequence? _____; why? _____
- (f) If, for some $\epsilon > 0$, we can find an R such that for all terms beyond the R th one, we have $a_n/b_n > A/B - \epsilon$, then we can say that the sequence _____
What restrictions must we place on the b_n sequence and on its limit for this to be true? _____

11. The "rate of change" problem leads to the _____; the "area" problem leads to the _____.
12. If $y = x^4 - x^2$ then R_{xy} is _____; at $x = 2$ R_{xy} is _____; this means that at $x = 2$ the y is _____.
13. Would you expect that three convergent sequences could be added and you would get a convergent sequence? _____
14. We now have hypotheses and theorems on limits of sequences, limits of functions, continuous functions, concerning sums, differences, products, quotients. No. 9 on the previous page might well be a "theorem" on "areas."

III. The Derivative. A. 1.

What is another name for the h-process?

We have observed that our "h-process" is one which can be applied to a variety of functions at a certain point x_1 in the range of the independent variable x and that it is a process which gives the slope of the curve at the point x_1 or the rate of change $R_x y$ of $y = f(x)$ at x_1 . Or, to repeat, we note that the h-process is one in which as the numbers $x_1 + h$ converge to x_1 then the numbers

$$\frac{f(x_1 + h) - f(x)}{h}$$

converge to the rate of change or the slope of the curve at x_1 .

The result of this process, which we have indicated by the symbol $R_x f$ is called by mathematicians the DERIVATIVE of $f(x)$ with respect to x . This process is called DIFFERENTIATION of $f(x)$ with respect to x . The verb is DIFFERENTIATE. Symbols used are $D_x f$ or $D_x y$.*

Further comments and studies on the derivative we shall help you consider by means of these

EXERCISES.

1. We have discovered that to find the derivatives of some functions it is not necessary to undergo the laborious process of replacing x by $x_1 + h$ and eventually obtain $R_x y$ or $D_x y$ (new symbol) but that we can write $D_x y$ at sight. Those which we cannot do at sight must be done (at present) by our "h-process." Part of our study will be to develop methods to differentiate quickly more complicated expressions.

To place the differentiation of functions at sight on a firm logical foundation try to deduce from the definition of the derivative that if $y = ax^n$ then $D_x y = nax^{n-1}$. This can be called a theorem on derivatives.

2. How is a derivative a limit?

Recalling theorems on the "algebra of limits" what statements would you be willing to make or suggest for investigation on derivatives? Mention several and examine if you wish. What you have deduced can be properly called theorems. Text (pp. 47-50) may be used to see if the author agrees with your theorems.

*Note Text p. 46 for other notations.

III. The Derivative. A. 1.
(continued)

EXERCISES (continued).

3. Text. p. 50. Nos. 1 to 25.
4. Text. p. 51. Nos. 26, 27.
5. Text. p. 51. Nos. 33, 34.
6. Text. p. 51. Nos. 37, 38.
7. Draw figures or graphs to show some functions which are not continuous at a certain point.
8. Two kinds of "discontinuities" are _____ and _____.
Label above.
9. Suggest the nature of a curve (by drawing) which might be continuous but which might not have unique slope or a unique derivative at a point.

In such cases the function is continuous but the derivative does not exist.

10. If a function has a first derivative at a certain point is it continuous there? _____. Why? _____
11. Review: What does it mean to say that
 - (a) $\lim_{x \rightarrow a} f(x) = f(a)$?
 - (b) $f(x)$ is continuous at $x = a$ (three aspects)
 - (c) $\{a_n\}$ is a convergent sequence
12. Review relation between x_f , x_b , y_f , y_b , $R_x y$, ϵ . Figure also.
13. See sheet 54 . No. 9. What would have been the area under the curve $y = x^2$ from $x = 0$ to $x = b$? What would you suggest (therefore) for the area between $x = a$ and $x = b$?

III. A. The Derivative. 2.

What are some other methods to make differentiation easier?

- (a) Some functions are expressed as PRODUCTS of other functions; e.g., $y = (x^2 - 3)(x^3 + 4x + 7)$.

How would you suggest that one could find the derivative $D_x y$? Do not do it; merely write the suggestion.

- (b) We seek a method by which we can do this more directly. Consider $f(x)$ as a product and write $f(x) = u(x) \cdot v(x)$, and let us see if we can deduce an expression for the derivative of $u \cdot v$. Consider the error-rate of change-actual change expression for $u(x) \cdot v(x)$. Develop this on the reverse side or on other paper. This would be a theorem for your notes.

- (c) Use this theorem on the example in (a) above.

- (d) What can you deduce about the derivative of $f(x) = \frac{u(x)}{v(x)}$ with the restriction that _____? Again we suggest the use of the actual change-approximate change-error relation. Show this development. Here is another theorem for your notes.

EXERCISES.

1. Text. p. 54. Nos. 1 to 10. These are applications of the theorems which you have deduced.
2. Text. p. 54. No. 11 (first part), No. 12.
3. Text. p. 96. Nos. 1,2,3,4.
4. Recapitulation: Tell how to differentiate at sight:
 - (a) ax^n
 - (b) ax^{-n}
 - (c) a product of functions
 - (d) a quotient of functions (denominator not zero for any point in the interval).
5. Review: ϵ - meaning or definition of $a_n \rightarrow A$
 ϵ, δ - meaning or definition that $\lim_{x \rightarrow a} f(x) = k$
 $f(x)$ continuous at $x = a$
6. Review: If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$ then
 $\lim_{n \rightarrow \infty} (a_n + b_n) = \underline{\hspace{2cm}}.$

III. A. The Derivative. 2.
(continued)

EXERCISES (continued).

$$\text{If } \lim_{x \rightarrow a} f(x) = k \text{ and } \lim_{x \rightarrow a} g(x) = 1$$

$$\text{then } \lim_{x \rightarrow a} [f(x) + g(x)] = \underline{\hspace{10cm}}$$

$$\text{Hence why is } D_x f(x) + D_x g(x) = D_x f(x) + g(x) \text{ ?}$$

III. A. The Derivative. 3.

How can we differentiate $y = f(x)$ if x depends upon some other variable?

The money you can spend on your college training perhaps depends upon how much you can borrow and this depends upon the backing your father can give. So, we can say that m is a function of b , which, in turn, is a function of f , or

$$m = F(b) \quad \text{and} \quad b = G(f),$$

whence, in the "last analysis," m is a function of _____.

Here m is called a function of a function or a compound function.

- (a) You write another example and show the meaning of a function of a function. Use words and symbols.
- (b) If $m = F(b) = b^2$ (for example) and $b = G(f) = 3f$ what rates of change or what derivatives would you suggest among these variables? Write in symbolic language and see if you can compute them.
- (c) Given $m = b^3$ but $b = 2f$ compute D_fm .
 Could this have been computed through the use of D_bm and D_fb ? How?
- (d) Given $m = b^2$ but $b = f^3$ compute D_fm . Could this have been computed through the use of D_bm and D_fb ? How?
- (e) Given $m = b^3$ but $b = 5f$ compute D_fm . Could this have been computed through the use of D_bm and D_fb ? How?
- (f) On the basis of examples studied above can you state a hypothesis as to relations among D_fm , D_fb , and D_bm ? If so, what?
- (g) Illustrate the use of your hypothesis with $y = u^2$, $u = 3x^2 - 5$ to compute D_xy .
- (h) Try to test your rule by deduction. We suggest the use of the actual change - approximate change by the rate of change or derivative-error relation.

This rule is called the CHAIN RULE. Do this on the reverse side or on other paper.

- (i) Consider the expression $y = (x^2 + 10)^4$. This is not a power of x but this expression is a power of a function of x . Suggest how you might differentiate this function quickly by the use of the chain rule.

III. A. The Derivative. 3.
(continued)

EXERCISES.

1. In mathematical symbolism review here in parallel columns theorems on the "algebra" of

<u>limits of sequences</u>	<u>limits of functions</u>	<u>continuity of functions</u>	<u>derivatives of functions.</u>
(given)	(given)	(given)	(given)
$\lim_{n \rightarrow \infty} \{a_n\} = A$	$\lim_{x \rightarrow a} f(x) = K$	$f(x)$ continuous at $x = a$	$D_x f$
$\lim_{n \rightarrow \infty} \{b_n\} = B$	$\lim_{x \rightarrow a} g(x) = L$	$g(x)$ continuous at $x = a$	$D_x g$
(then)	(then)	(then)	(then)
(a) $\lim_{n \rightarrow \infty} \{a_n + b_n\} = A + B$	;	_____;	_____;
(b) $\lim_{n \rightarrow \infty} \{a_n - b_n\} = A - B$	;	_____;	_____;
(c) $\lim_{n \rightarrow \infty} \{a_n b_n\} =$ _____	;	_____;	_____;
(d) $\lim_{n \rightarrow \infty} \left\{ \frac{a_n}{b_n} \right\} =$ _____	;	_____;	_____;

(Suggest any limitations which should be mentioned.)

III. A. The Derivative. 3.
(continued)

EXERCISES (continued).

2. Express theorems on differentiation by completing the following:

(a) $D_x(x^n) =$

(d) $D_x(f \cdot g) =$

(b) $D_x(ax^n) =$

(e) $D_x \left(\frac{f}{g} \right) =$

(c) $D_x(f + g) =$

(f) $D_x(K) =$ (K = a constant)

(g) $D_x(f - g) =$

(h) $D_t y$ where $y = f(x)$ but $x = g(t)$

(i) $y = (ax^3 + bx)^4$ then $D_x y =$

(j) $y = a f(x) + b g(x)$ then $D_x y =$

3. Text. p. 58. Nos. 1,2,3,4,5,6,7,8,9,10.

4. Short answers. How can you see if a point on a curve is a

(a) point of maximum?

(b) point of minimum?

(c) point of continuity of the function?

(d) point at which the function is increasing?

(e) point at which the function is decreasing?

(f) point at which the curve is "stationary"?

5. We recall that IF the curve is $y = x^2$ then the area under the curve from $x = 0$ to $x = a$ is $a^3/3$. The area under the curve from $x = 0$ to $x = b$ is _____.

Hence find the area under the curve $y = x^2$ from $x = 1$ to $x = 4$;

from $x = 2$ to $x = 5$.

III. A. The Derivative. 4.

- A. What is the derivative of an inverse function?
 B. What is the derivative $D_x y$ when both x and y are expressed in terms of a parameter?

- A. (a) First, what is an inverse function? Suppose we have given

$$y = 3x + 8. \quad (1)$$

Here y is expressed as a function of _____. If we solve for x we get

$$x = \frac{y - 8}{3}. \quad (2)$$

Here x is expressed as a function of _____.

(1) is said to be the INVERSE of (2).

- (b) One can TEST to confirm that (1) and (2) are inverse functions by replacing x in (1) by _____.
 That is,

$$y = 3x + 8 \text{ becomes } y = 3\left(\frac{y-8}{3}\right) + 8 \text{ or } y = y.$$

- (c) Consider $y = 3x + 8$ again.

The INVERSE function is $x = \frac{y - 8}{3}$ (2)
 where now y is the _____ variable and x is the _____ variable. If we replace the independent variable y by the (customary) x and the dependent variable x by the (customary) y (2) becomes

$$y = \frac{x - 8}{3} \quad (3)$$

This shows that a FUNCTION IS INDEPENDENT of notation used.
 (2) and (3) are exactly the same functions and both are inverses of (1).

EXERCISES.

1. Given these functions you set up inverse functions:

(a) $y = 6x$

(b) $y = 3x + 2$

(c) $y = 4x - 2$

(d) $y = \frac{x + 4}{4}$

(e) $y = \frac{x - 5}{2}$

III. A. The Derivative. 4.
(continued)

EXERCISES (continued).

2. Check in each case that the function which you have found really is an inverse to the given function.
3. Given $y = f(x)$ we can write the INVERSE as $x = g(y)$. What must be true notationally about these two functions?
4. From the functions which are given in No. 1 and the inverses you have written in No. 2 compute and place here

$D_x y$ (from the function) $D_y x$ (from the inverse)

(a)

(b)

(c)

(d)

(e)

5. What do you note in No. 4 on relations between $D_x y$ and $D_y x$?
6. Hence if $y = f(x)$ and $x = g(y)$ are inverse functions what hypothesis do you propose as to relations between derivatives? State carefully.
7. See if you can establish this hypothesis deductively by the use of the chain rule.
8. Review: Find $D_x y$ if (a) $y = (x^2 - 3x + 4)^3$ (b) $y = (3x^2 - 10)^4$
9. Text. p. 62. Nos. 1,2,3,4,5.
10. Rapid review.

(a) $D(f + g) =$

(f) $D_x(1/x^n) =$

(b) $D(f/g) =$

(g) $D_x(bx^{r/s}) =$

(c) $D(f \cdot g) =$

(h) $D_x(af + bg) =$

(d) $x = f(t)$ but $t = g(u)$
then $D_u x =$

where f and g are functions
of x

(e) $D_{xy} \cdot D_y x =$

III. A. The Derivative. 4.
(continued)

B. (a) Given
$$\begin{cases} y = t^2 + 3t \\ x = 3t. \end{cases}$$

Both x and y are expressed in terms of a third variable t called a PARAMETER and x and y are said to be PARAMETRIC FUNCTIONS of t .

- (b) Our functions (above) x and y are expressed in terms of t BUT WE SEEK TO COMPUTE $D_x y$.

Think of two ways by which you might do this and compare results.

(1)

(2)

- (c) See if you can write a formal argument (using previous theorems) of a method which you suggest above.

EXERCISES.

1. Review. Deduce the product rule (i.e. rule for differentiating a product).
2. Review. Deduce the quotient rule.
3. Review. Deduce the chain rule.
4. If $y = 3t^2 + 6t$ but $t = 4x^2 + 3x$ what is $D_x y$?
5. If $y = 3t^2 + 6t$ and $x = 4t^2 + 3t$ what is $D_x y$?
6. If $y = 4t^2$ and $x = t^3$ what is $D_x y$?

III. A. The Derivative. 5.

How does one differentiate functions expressed implicitly?

(a) We review. Find $D_x y$.

i) $y = (x^2 - 4)^2$

ii) $y = (x^3 - 3x + 2)^{\frac{3}{2}}$

iii) $y = (x^2 + 10x)^3$

iv) $y = \left(\frac{1}{x^2}\right)^3$

(b) An implicit function is one which implies a relation but it does not show directly (i.e. without some manipulation) how the values of one variable are obtained from those of the other. An explicit function is one which tells how to compute the value of one variable when that of the other is known.

Which of these express y implicitly as a function of x and which of these express y explicitly as a function of x ?

i) $y = 3x^2 - 6x$

ii) $y + 3x^2 = 9$

iii) $3y - x^2 = 10$

iv) $3y^2 + 6x = x^2 + 3$

v) $y^2 = 6x$

We should make a special note here. When y is expressed in terms of x (explicitly) we have certain criteria to see if y is a function of x : there must be pairing, a law, and, for calculus, there must be continuity and existence of derivatives. When the relation is implicit, however, the question arises: when does $F(x,y) = 0$ express y as a function of x ? Criteria which must be satisfied are discussed in advanced courses in mathematics.

(c) Suggest a method to find $D_x y$ for each of the above. Do it. Use the reverse side of this sheet if necessary.

III. A. The Derivative. 5.
(continued)

- (d) If $x^2 + 10y^2 = 16$ then find $D_x y$.
 (e) What is the slope of the ellipse $3x^2 + 4y^2 = 10$?

EXERCISES.

1. Review. Find $D_x y$ in each of these cases.

(a) $y = (x^2 + 2)(x^3 - 3x)$ (b) $y = \frac{x^2 + 1}{x^2}$
 (c) $y = (x^3 + 3x)^2$ (d) $y = \left(\frac{x+3}{x+2}\right)^2$
 (e) $y = u^2 + 3u$ but $u = 3x^2$ (f) $\begin{cases} y = 3t^2 \\ x = 2t^3 + 3 \end{cases}$

2. Text. p. 65. Nos. 1,2,3,4,5,6. Implicit functions.
 3. Text. p. 96. Nos. 6,7,8.

4. Review: If $\lim_{n \rightarrow \infty} \{a_n\} = A$ and $\lim_{n \rightarrow \infty} \{b_n\} = B$
 then $\lim_{n \rightarrow \infty} \{a_n + b_n\} = A + B$. Deduce this.

5. Review: If $y = f(x)$ and $y = g(x)$ both possess first derivatives for all values of x over the same range then

$$D_x f + D_x g = \underline{\hspace{2cm}}.$$

6. How are the results in Nos. 4 and 5 related?
 7. An investigation: What is the area under the curve $y = x^3$ between the x -axis and $x = 0$ and $x = a$? Divide the length $x = 0$ to $x = a$ into n intervals and derive an expression for area.
 8. Review: Show that if $y = ax^n$ then $D_x y = nax^{n-1}$.
 9. Given the expression y^3 where y is a function of x explain why

$$D_x y^3 = 3y^2 D_x y.$$

III. A. The Derivative. 6.

Is there an operation "inverse" to differentiation?

(a) What is the inverse of

- i) addition? ii) multiplication?
- iii) taking square root? iv) cubing a number?
- v) taking the logarithm of a number?

(b) What, in general, does an inverse operation do?

(c) Let us suppose that $y = x^3 + x^2$. What is $D_x y$?

What do you suggest that the inverse of differentiation should do to $D_x y$?

(d) If $D_x y = 2x + 7$ what could y be?
 what else? what else?
 what else? what else?

How are these answers alike?
 How are these answers different?

(e) How could you summarize the result for y in (d)?

(f) Hence find the general expression for y if

- i) $D_x y = 3x^2 + 7x + 10$
- ii) $D_x y = 4x^3 + 6x^2 - 3x + 8$
- iii) $D_x y = x^3 - x^2 + 7x + 8$
- iv) $D_x y = ax^2 + bx + c$
- v) $D_x y = 3x^3 + 2x^2 + 3x - 80$
- vi) $D_x y = 5$
- vii) $D_x y = 5x + 10$

(g) If $D_x y = 0$ then $y =$

(h) If $D_x y = ax^n$ then $y =$

(i) If $D_x y = x^{1/2}$ then $y =$

III. A. The Derivative. 6.
(continued)

- (j) If $D_x y = x^{-1/2}$ then $y =$
- (k) If $D_x y = x^{-4}$ then $y =$
- (l) The process which we have used to find y could be called
-

EXERCISES.

1. Consult the Text, p. 105, Section 3. How do the names given in the book compare with what you suggested in (l) just above? What are other names used?
2. Illustrate from (h) what is meant by a primitive function.
3. Text, p. 109. Find the primitive functions as requested in Nos. 1 to 25.
4. The slope of a curve is given by $D_x y = 3x^2$.
What is y ?
What is the equation of that curve which goes through the point (1,5)?
5. Text, p. 51. Nos. 37, 38, 39, 40.
6. Text, p. 66. Nos. 1 to 10.
7. Rapid review.
 - (a) What can you infer about $f(x)$ at $x = a$ if it has a derivative there?
 - (b) What does it mean to say that $f(t)$ and $g(t)$ are parametric functions?
 - (c) What does it mean to say that $y = f(x)$ and $x = g(y)$ are inverse functions?
 - (d) How are the graphs of $f(x)$ and $g(y)$ in (c) related?
 - (e) How are $D_x y$ and $D_y x$ related in (c)?
8. Review: If $y = x^2$ what do we find to be the area above the x -axis and under the curve between $x = a$ and $x = b$? If the curve is $y = x^3$?

III. A. The Derivative. 7.

GENERAL REVIEW EXERCISES.

1. Differentiate by definition.

(a) $y = \frac{t+1}{t-1}$

(b) $y = \frac{t^2 - 6t}{t+4}$

(c) $y = (t+1)(t^2 - 1)$

(d) $y = (3t^2 + 1)^2$

2. Check your work in No. 1 by the use of rules for differentiation.

3. What are several interpretations which can be given to the derivative?

4. We have learned rules for differentiation to apply to

(a) $y = f(u)$ but $u = g(x)$

(b) $y = f(t)$, $x = g(t)$

(c) y and x are implicitly related

(d) $y = f(x)$ and $x = g(y)$ are inverse relations.

Using symbols, tell the methods which we have developed in each case to compute $D_x y$.

5. Is there any
- one
- of these rules which becomes the basis or a help in developing others? If so, which?

- 6.
- Text
- . p. 65. Nos. 5, 6.

- 7.
- Text
- . p. 67. Nos. 11, 13, 15, 17, 19, 20.

8. We recall that we developed a method for finding the rate of change (slope) of
- y
- with respect to
- x
- at an arbitrary point
- x_1
- by either departing
- forward
- from
- x_1
- (i.e. by the use of
- $x_1 + h$
-) or by departing
- backward
- from
- x_1
- (i.e. by the use of
- $x_1 - h$
-).

Is there any other way which you would suggest that we approach the problem of estimating slope at the point x_1 ? If so, outline your method and compute the slope of the curve $y = x^2$ at $x = 2$ if you wish.

- 9.
- Text
- . p. 97. Nos. 9, 10, 11, 12.

10. The slope of a curve at any point
- x
- is given by
- $D_x y = 3x^2 - 6x$
- . What is the equation of that curve of the pencil which goes through (2,6)? (-1,6)?

III. A. The Derivative. 7.
(continued)

GENERAL REVIEW EXERCISES (continued).

11. See what you can find on Fermat's contribution to the calculus. Illustrate his method of working a maximum-minimum problem. When did Fermat live? What are some important events which took place in his lifetime?
12. State carefully worded definitions for
 - (a) speed of a body under nonuniform motion
 - (b) slope of a curve
 - (c) rate of change of y with respect to x
 - (d) would the definition in (c) be the same as the definition of the DERIVATIVE of y w.r.t. x ? How or why?
 - (e) area under a curve between $x = a$ and $x = b$ and also bounded by the x -axis
 - (f) work done to move a body from $x = a$ to $x = b$ if the force is variable
 - (g) an abstraction of definitions in (e) and (f)
 - (h) saying that $\lim_{x \rightarrow a} f(x) = K$
13. Complete: A derivative is really a _____ of a _____ as $h \rightarrow 0$. Hence theorems on derivatives come directly from theorems on _____ or _____.
14. Deduce either the "power rule," the "product rule," the "quotient rule," or the "chain rule" from the definitions of derivatives which are more suitable for the deduction.

Mathematics 263
Helps for Test

I. You should be able to

- (a) define the derivative of $y = f(x)$ as the result of a several step process (the h-process).
- (b) deduce either the product rule or the quotient rule from the fundamental relation on error, etc.
- (c) deduce the chain rule
- (d) explain test to see if two functions $y = f(x)$ and $x = g(y)$ are inverse
- (e) deduce rule for finding $D_x y$ when x and y are expressed in terms of a parameter t .

II. You should be able to compute $D_x y$ of

- (a) $y = \frac{x+1}{x}$ by definition (h-process)
- (b) $y = x^3 + 3x + 10$
- (c) $y = \frac{x^2 + 6}{x + 4}$
- (d) $y = (x^2 + 3x + 5)^3$
- (e) $y^2 + 7y + 10 = x$
- (f) $3x^2y + 4y^3 + 3xy^2 = 0$
- (g) $y = 3t^2$; $x = 4t^2 + 3t$
- (h) $y = 6x^2 + 3x + 4$ but $x = 7u^2 + 2$
- (i) $y = (x^2 + 3)(4x^3 + 10)^2$.

III. Inverse differentiation as if $D_x y = 3x^2 + 5x + 10$ then $y = \underline{\hspace{1cm}}$?

IV. Practical problems:

- (a) If the slope of a curve is $D_x y = 3x - 3$ find the equation of that curve of the pencil which goes through the point $(1, -1)$.
- (b) maximum, minimum problems of types on p. 96. Text, Nos. 1 to 10.

V. Argue that if $\lim_{n \rightarrow \infty} \{a_n\} + \lim_{n \rightarrow \infty} \{b_n\} = \lim_{n \rightarrow \infty} \{a_n + b_n\} = A + B$
then we would expect $D_x f + D_x g$ to be equal to $D_x(f + g)$.

VI. Fundamental definitions: slope, speed, the derivative, area under a curve, work when force is variable.

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Name.....

Date.....

I. Short Answers.

1. If $y = 3t^2 + 4t$ then $D_t y = \underline{(1)}$ (1) _____
2. If $y = (3x^2)(6x)$ then $D_x y = \underline{(2)}$ (2) _____
and $\underline{(3)}$. Answer to show product rule. (3) _____
4. If $y = \frac{x+1}{3x}$ then $D_x y = \underline{(4)}$ (4) _____
and $\underline{(5)}$. (5) _____
6. If $y = (x^2 + 3x + 10)^3$ then $D_x y = \underline{(6)}$. (6) _____
(Do not work out.)
7. If $3y^2 + 6xy + 10 = 0$ then the expression (7) _____
involving $D_x y$ is $\underline{(7)}$; solving for $D_x y$ (8) _____
gives $\underline{(8)}$.
9. If $y = f(x)$ and $x = g(y)$ are inverse (9) _____
functions then what relation holds (10) _____
between them? $\underline{(9)}$
10. If $y^2 + 6y + 10 = x$ then $D_x y$ is $\underline{(10)}$; (11) _____
 $\underline{(11)}$. (12) _____
12. If $y = 3t^2$ and $x = 4t$ then $D_x y = \underline{(12)}$. (13) _____
13. If $y = 6x^2 + 4x$ but $x = 3u^2$ then $D_u y =$ (14) _____
 $\underline{(13)}$; $\underline{(14)}$.
15. If $y = f(x)$ but $x = g(t)$ then $D_t y$ can be (15) _____
computed by $\underline{(15)}$. (16) _____
16. This is called the $\underline{(16)}$ rule. (17) _____
17. If $D_x y = 3x$ then $y = \underline{(17)}$. (18) _____
18. If $D_x y = x^2 + 4x + 10$ then $y = \underline{(18)}$. (19) _____
19. If $y = f(t)$ and $x = g(t)$ (i.e. y and x (20) _____
expressed parametrically as functions of (21) _____
 t) then $D_x y = \underline{(19)}$.
20. Given $y = f(x)$ then four important steps (22) _____
in computing $D_x y$ are (in order) $\underline{(20, 21, 22, 23)}$. (23) _____

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Short Answers.

24. What are two interpretations of $D_x y$? (24) _____
 ____ (24,25) ____ (25) _____
26. If a function has a maximum value at $x = a$ (26) _____
 we know that the slope ____ (26) there.

II. Discussions.

1. Using the fundamental relations as

$$\frac{y_f - y_b}{x_f - x_b} = D_x y + \epsilon \quad \text{or} \quad \frac{y_f - y_b}{x_f - x_b} - \epsilon = D_x y$$

- (a) DEDUCE the rule for computing $D_x y$ when $y = f(u)$ but $u = g(x)$.
- (b) DEDUCE the rule for computing $D_t y$ in terms of $D_t x$ and $D_t y$ when $y = f(t)$ and $x \cong g(t)$.
- (c) Show (deduce) either the rule for the differentiation of a product or the rule for the differentiation of a quotient of two functions.

2. Is $x = \frac{y + 10}{4}$ a function which is inverse to $y = 4x - 10$?

Answer yes or no and then show test.

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III. Problems.

1. Compute $D_x y$ in each case. Express as a function of the handiest variable in each case but if the work becomes too involved do not finish it.

(a) $y = u^2 - 4u + 10, \quad u = \frac{x+1}{x^2}$

(b) $y = \left(\frac{x+4}{x} \right)^3$

(c) $y = \sqrt{1+t^2}, \quad x = \sqrt{1+t^2}$

2. Now go back to No. 1(b) and find the slope of the curve whose equation is given there at the point (1,-2).

3. If you have time compute $D_x y$ by the h-process. $y = \frac{x+3}{x}$

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IV. Problems.

1. If the slope of a curve is always 5 less than three times the x for any x then find the equation of that curve which goes through the point $(2,6)$.

2. A farmer has 120 rods of fence and he wishes to fence a field as shown. What dimensions will make the area a maximum?

3. Find two numbers whose sum is 20 such that the square of the one times the other gives a product which is the maximum.
4. A piece of cardboard is 20 inches square. By cutting out equal squares from the corners and bending up the sides one can make a box. How far in must one cut for the volume to be a maximum?

III. B. Higher Derivatives. 1.

A. Can one compute the derivative of a derivative?

B. If so, what interpretation does it have? To what uses can the second derivative be put?

A. (a) Given $y = 3x^4 + 2x^3 + 4x^2 + 3x + 10$ Compute $D_x y =$

Can this be differentiated? _____ If so, do it.

=

What symbol do you suggest? _____

Continue with more differentiations until you get zero.

=

=

=

(b) Here are other symbols for first and second derivatives; you complete to introduce third and fourth derivatives.

$D_x y$	,	$D_x^2 y$	,	_____	_____
y'		y''		_____	_____
$\frac{dy}{dx}$		$\frac{d^2 y}{dx^2}$		_____	_____
f'		f''		_____	_____
$\dot{y}$		$\ddot{y}$		_____	_____

(Use
care
in
read-
ing)

This will help you to know what is meant when you read other books on calculus or when you read texts and books which use the calculus.

B. (a) A student once said that " $D^2 y$ would tell about D_y the same that $D_x y$ tells about y ." What do you think of this statement?

III. B. Higher Derivatives. 1.
(continued)

- B. (b) What does $D_x y$ tell about y relative to changes in x ?

Hence what can you say about y relative to x -changes when $D_x y$ is positive? (Illustrate also.)

when $D_x y$ is negative? (Illustrate also.)

when $D_x y = 0$ (Illustrate also.)

- (c) In a similar manner when $D_x^2 y$ is positive at $x = a$ what can you say about the slope $D_x y$ at $x = a$?

when $D_x^2 y$ is negative at $x = a$?

when $D_x^2 y$ is zero at $x = a$?

Draw figures to illustrate curves which have these properties as x moves through $x = a$ from $x < a$ to $x > a$.

- (d) Consider these figures.

At $x = a$

i) we have a point of

ii) $D_x y =$

iii) what slope does as x goes through a

iv) $D_x^2 y$ is

_____	_____	_____	_____
_____	_____	_____	_____
_____	_____	_____	_____
_____	_____	_____	_____

- (e) Hence suggest or hypothesize a quick test to see if at $x = a$ there is a point

i) of maximum?

ii) of minimum?

iii) (might be) of inflection?

- (f) Apply your tests which you hypothesize to examine the stationary points of this function to see if they are points of maximum, minimum, or inflection: $y = x^3 - 5x^2 + 7x - 3$.

III. B. Higher Derivatives. 1.
(continued)

EXERCISES.

1. Text. p. 70. Nos. 1 to 12.
2. Text. p. 74. Nos. 1,2,3,4,5.
3. Text. p. 97. Nos. 13, 14, 15, 16. Use the second derivative test for maximum and minimum.
4. Review.
 - (a) Define the derivative of $y = f(x)$ with respect to x .
 - (b) Is there just one definition or could there be several definitions? Explain.
 - (c) Define the second derivative of $y = f(x)$.
 - (d) Draw a figure of a curve which is continuous at a certain point but which does not have a first derivative there. What comparable statement would you be willing to make on the continuity of the first derivative and the existence of the second derivative? What could you say about the second derivative and the third derivative?
 - (e) If $s = f(t)$ represents distance traversed by a moving body then $D_t s$ can be interpreted as _____; $D_t^2 s$ can be interpreted as _____.
 - (f) The antiderivative of acceleration a would be an expression giving _____; the antiderivative of the antiderivative of a would give _____.
5. A student once said that "The second derivative tells the same about the first derivative as the first derivative tells about the original function." Comment on this.
6. Given $y = f(x)$; if $D_x y$ exists at $x = a$ then the function is continuous at $x = a$. What would insure the continuity of $D_x y$ at $x = a$? What can you say about the converse?
7. If the derivative $D_x y$ exists at $x = a$ then $f(x)$ is continuous at $x = a$. What is the converse of this statement? the inverse? the contrapositive? What do you say about the truth of these statements if the original statement is true?
8. The vanishing of the second derivative at $x = a$ is a necessary condition for the existence of a point of inflection at $x = a$ but not a sufficient condition. What does this mean?
9. Does the parabola $y = ax^2 + bx + c$ have a point of inflection? Why or why not? What statement can you make about the possibility of existence of a point of inflection on any conic? Support your statement.

III. B. Higher Derivatives. 1.
(continued)

EXERCISES (continued).

10. $D_{xy} > 0$, $D_{xy} < 0$, $D_{xy} = 0$ at $x = a$ tells what about the function $y = f(x)$?
 $D_{xy}^2 > 0$, $D_{xy}^2 < 0$, $D_{xy}^2 = 0$ at $x = a$ tells what about the first derived function?

III. B. Higher Derivatives. 2.

How can derivatives help in the study of rectilinear motion?

- (a) Below is a list of terms and ideas which are important in any discussion of motion. Suggest meanings of these words and compare with those of the author of our text (pp. 74-79):

- i) rectilinear motion
- ii) accelerated motion
- iii) motion which has positive acceleration
- iv) motion which has negative acceleration
- v) motion which has an acceleration of 0
- vi) uniform motion
- vii) what we shall mean by positive velocity
- viii) what we shall mean by negative velocity
- ix) negative acceleration
- x) how is a body behaving when the velocity is positive but the acceleration is negative?
- xi) how is a body behaving when the velocity is negative but the acceleration is positive?
- xii) write another variation of x) and xi) and explain.
- xiii) write another variation of x) and xi) and explain.

- (b) Recall from previous work in science and mathematics how a vertically falling body behaves near the earth. What is the expression for distance, velocity, acceleration?

- (c) A body moves along a straight line in such a way that its distance from a fixed point at a time t seconds is given by

$$s = t^3 - 6t^2 + 9t + 10.$$

What expression gives the velocity of the body?

III. B. Higher Derivatives. 2.
(continued)

For what value or values of t is the velocity zero?

For what range of times t is the body moving in a positive direction?
In a negative direction?

What expression gives the acceleration?

For what value of t is the acceleration increasing?

Is there a time t for which the rate of change of the velocity is zero?

Try to sketch a figure of this motion (i.e. where is the body at $t = 0$? at the other values of t mentioned above?). Describe the motion in these time and distance intervals.

- (d) If $a = D_t^2 s = 32$ feet per second per second
then $v = D_t s =$
and $s =$

Since the acceleration is downward and we call it $+32 \text{ ft./sec.}^2$ how must we interpret a velocity which is positive and a distance which is positive?

If we wish to interpret a distance which is positive as upward then the downward acceleration to the earth (32 ft./sec.^2) must have what sign?

If s be interpreted as distance upward then the above expressions become

$$a =$$

$$v =$$

$$s =$$

III. B. Higher Derivatives. 2.
(continued)

- (e) A ball is thrown upward from a building which is 150 feet high and the ball strikes the ground 9 seconds later.

- i) What is the distance s (measured from the ground) at the end of 9 seconds?
- ii) What is the initial distance?
- iii) How can you find the initial velocity?

Do it.

- iv) At what time t does the distance s reach a maximum? Show work.
- v) How high does the ball rise?
- vi) At what speed does the ball strike the ground?
- vii) Use the reverse side of this sheet or a piece of graph paper to construct a rough graph or curve which shows the relation between the distance and the time through which the ball is moving.

EXERCISES.

1. Text. p. 79. Nos. 1,2,3,4,5,6.
2. A body is dropped from the top of a building which is 200 feet high. When will it reach the ground and with what velocity?
3. A ball is thrown upward from a building 100 feet high and the ball hits the ground 7 seconds later. What was the initial velocity? at what time did it reach maximum height? what is this height? with what velocity does the ball strike the ground?
4. Text. p. 97. Nos. 17, 18, 19.

III. B. Higher Derivatives. 2.
(continued)

EXERCISES (continued).

5. Review simply by a series of figures a method for finding the area under a curve $y = f(x)$ between $x = a$ and $x = b$.

6. Short answers.

- (a) four-step method (definition) for computing the derivative of $y = f(x)$.
- (b) given $y = x^2 - 6x$. For what value of x does the curve have a stationary point (point of zero slope)? Is this point one of maximum or minimum? Use second derivative test.
- (c) given $y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 10$. For what values of x does this function have maximum and minimum values? Use second derivative test.
- (d) review: if $y = f(t)$ and $x = g(t)$ show that $D_x y = D_t y / D_t x$. Upon what previous theorems does this development depend?
- (e) given a spring whose law is $F = 16x$ how much work is required to stretch it from $x = 2$ to $x = 4$? Make three approximations.
- (f) The answer in No. (e) above is the same as the area under what curve and between what limits?
- (g) We have found that the area between $x = a$ and $x = b$ under the curve $y = x^2$ and above the x -axis is $b^3/3 - a^3/3$ ($b > a$). What have we found for the area if the curve is $y = x^3$? What would you hypothesize for the area if the curve is $y = x^4$? $y = x^5$? $y = x^n$?
- (h) Hence under your hypothesis what is the area under $y = x^4$ between $x = 1$ and $x = 3$?

III. B. Higher Derivatives. 3.
Review

REVIEW EXERCISES.

1. Given that $\{a_n\} \rightarrow A$ and $\{b_n\} \rightarrow B$
 - (a) deduce that $\{a_n + b_n\} \rightarrow A + B$
 - (b) deduce that $\{a_n b_n\} \rightarrow A \cdot B$
2. Hence write an argument for if $\lim_{x \rightarrow a} f(x) = f(a)$ and if $\lim_{x \rightarrow a} g(x) = g(a)$ then $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = f(a) \cdot g(a)$.
3. What does it mean to say that a function $f(x)$ is continuous at the point $x = a$? Why do calculus students like continuous functions?
4. Illustrate two kinds of discontinuity at a point for a function $f(x)$. Why is each bad?
5. Review (state) the tests to use the second derivative to distinguish between a point of maximum and a point of minimum.
6. From the hypothesis on areas which we have developed what would be the area between the curve $y = x^2$ and the x-axis and the lines $x = 1$ and $x = 2$?
7. What would be the area under the curve x^3 with the same bounds as above?
8. What would you suggest as the area under the curve $y = x^3 + x^2$ with the same bounds as above?
9. Carefully plot the curve in No. 8 above. Show what area we are getting by your method.
10. How would it change the area in No. 6 to consider the curve $y = 2x^2$? to consider $y = 5x^2$? What argument can you suggest without reworking all your proofs?
11. Hence can you complete this table? (see following page)

III. B. Higher Derivatives. 3.
(continued)

REVIEW EXERCISES (continued).

curve	limits	area between limits, curve, and x-axis
$y = kx$	$x = a \quad x = b$	
$y = kx^2$	$x = a \quad x = b$	
$y = kx^3$	$x = a \quad x = b$	
$y = kx^4$	$x = a \quad x = b$	

III. B. Higher Derivatives. 4.

How can we compute higher derivatives of parametric functions?

- (a) Let us consider $y = t^2 + 4t$ and $x = 3t^3$.

We hope to compute D_{xy}^2 . Suggest how this can be done directly. Do it. Use the reverse side of this or the previous sheet if necessary.

- (b) What is D_{xy} ?

What are the three functions of t which we now have? How are they related?

- (c) How would you suggest that we compute $D_t(D_{xy})$? Do it.

- (d) How would you suggest that we compute $D_x(D_{xy})$ or D_{xy}^2 ? Do it.

- (e) How does the result in (d) compare with that of (a)? Hence what statement can you make on your new method?

- (f) Compute $D_x(D_{xy}^2)$ or D_{xy}^3 .

- (g) If $y = f(t)$ and $x = g(t)$ then D_{xy} is a function of t . Let us call this $\phi(t)$. With the help of experience you have had above develop a general formula for D_{xy}^2 .

III. B. Higher Derivatives. 4.
(continued)

- (h) How does your result compare with that of Cooley on p. 80?
- (i) Would it be easier to remember this formula for $D_x^2 y$ or to simply carry out processes to yield the second derivative when necessary?

EXERCISES.

1. Text. p. 81. Nos. 1 to 8.
2. Text. p. 97. Nos. 20, 21, 22, 23, 24.
3. Given $y = f(t)$, $x = g(t)$ develop an expression for $D_x^3 y$.
4. Review. Text. p. 63. Nos. 8,9.
5. Given $y = f(f)$ but $u = g(x)$ explain what is meant by the chain rule.
6. If $y = f(x)$ and $x = g(y)$ are inverse functions show how the inverse rule comes from the chain rule.
7. If $y = f(t)$ and $x = g(t)$ are both functions of the parameter t explain how the rule for forming $D_x y$ (or the parametric rule) comes from the chain rule and the inverse rule.
8. Construct a diagram to show relations among these three rules.
9. If $s = f(t)$ is a distance-time law, what expression gives the velocity? the acceleration?
10. A coal mine works x men per shift. The output per shift is given by

$$\text{Output} = \frac{x^2}{25} \left(3 - \frac{x}{12} \right)$$
 tons of coal. What is the size of the shift for maximum output?
11. Deduce the chain rule.
12. Deduce the product rule.

III. B. Higher Derivatives. 4.
(continued)

EXERCISES (continued).

13. The acceleration of a moving body is always 10 feet per second per second.

Find an expression for the velocity. Find an expression for the distance traversed. Now if at $t = 2$ the distance is 10 feet and if at $t = 4$ the distance is 20 feet what is the equation or the formula for the distance in terms of time? (The latter conditions help you to evaluate the constants.)

14. The acceleration of a moving body is always $3t - 4$. Write an expression for the velocity; for the distance in terms of t . Now suppose when $t = 5$ sec. the velocity is 8 feet per second; when $t = 5$ sec. the distance is 50 feet per second. What is the value of each of the constants and hence what is the actual equation or expression for distance?

15. Write the first line only to begin to compute $D_x y$.

(a) $3y^4 + x^2y^3 + 10y = 8$

(b) $(y^2 + 6)^3 + 3x = 10$

(c) $8y^4 + (x^2 + 3y)^2 + (y + 6)^6 = y^3$

16. Given $y = \frac{2x^3}{3} + \frac{5x^2}{2} - 6x^2 + 10$.

For what value of x does y have minimum value? maximum value?
Use second derivative test.

III. B. Higher Derivatives. 5.

How can one compute higher derivatives of implicit functions?

(a) If we have $2x^2 + y^2 = 4$

what is $D_x y$?

(b) What is $D_x^2 y$?

(c) What is $D_x^3 y$?

EXERCISES.

1. Text. p. 82. Nos. 1, 2, 3, 4, 5, 6.
2. Text. p. 90. Nos. 3(a, c, e), 5, 6, 7.
3. Text. p. 98. No. 25.
4. Review: Given $y = x^2 + 5x + 10$ find the area between the lines $x = 1$ and $x = 3$ and under the curve and above the x -axis. Have we proved our process as a theorem or is this a hypothesis?
5. Define area under a curve $y = f(x)$ between $x = a$ and $x = b$.
6. How much of the above (no. 4) is hypothesis and how much is on solid footing as having been deduced from the definition and limit theorems?
7. Find the rate of change of y with respect to x .

III. C. The Inverse Rate Problem Continued. 1.
How can inverse differentiation be used?

- (a) In III. A. 7. we were introduced to the inverse of differentiation. We carried on the process rather intuitively. Let us consider this idea more formally. We recall these theorems on differentiation:
- i) $D_x c = 0$
 - ii) $D_x(u + v) = D_x u + D_x v$
 - iii) $D_x(au) = a D_x u$ where a is a constant
- D_x , of course, is a symbol for an operation--that of differentiation.
- (b) D_x^{-1} is a symbol for the inverse operation, or, for the operation of inverse differentiation. Some suggest a symbol made from the D_x with the -1 on top, as

$$D_x^{-1}.$$

What statements do you think the theorems above might suggest for investigation on the subject of inverse differentiation?

- i)
 - ii)
 - iii)
- (c) How can the above suggested theorems be justified? Do it.
- (d) If $D_x y = ax^n$ what do you suggest y equals? What is the justification for your suggestion?
- (e) Hence if $D_x y = x^3 + 2x^2$ what do you suggest for y ? How can you justify your answer in two ways?

EXERCISES.

1. Text. p. 109. Nos. 26, 28, 30, 32, 37.
2. Text. p. 116. Nos. 1,2,3,4,5.
3. Quick review: List three "meanings" or "interpretations" of $D_x y$; meanings of finite discontinuity, infinite discontinuity, to say that a function is continuous at $x = a$, four theorems of sequences, these give rise to what theorems on limits of functions, how is $D_x y$ a limit of a function?

III. C. Inverse Rate Problem Continued. 2.

Given the slope how can one find the equation of the curve?

How can one find the length of a curved segment?

- (a) We have had some experience in answering the first question asked in the title. Consider this as a review: A curve is such that at every point its slope is always four more than the square of the abscissa at that point. What is the equation of that curve which satisfies this relation and which goes through the point (3,25)?

- (b) Here is a curve with segment $S \approx s_f - s_b$ as shown in the figure. The figure suggests that we can think of the segment S as being created by two changes.

What changes would you suggest?

What would be a good notation for these changes?

If you suggest $x_f - x_b$ and $y_f - y_b$ what relation would you write which holds approximately for $s_f - s_b$, $x_f - x_b$, and $y_f - y_b$?

- (c) In (b) suppose the curve C is described by the parametric relations

$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (t \text{ real})$$

What expression involving derivatives can you develop from the relation expressed in (b) and the above?

- (d) How could you use the relation suggested in (c) to compute the length of a curved line segment? (Merely describe.)

III. C. Inverse Rate Problem Continued. 2.
(continued)

- (e) Let us consider now a curve given by

$$C: \begin{cases} x = 3t^2 \\ y = 4t^2 \end{cases} \quad (t \text{ real})$$

Using the method you have suggested in (d) compute the length of the curve segment from $t = 0$ to $t = 3$.

- (f) Suppose that we study this curve further. What kind of graph does this curve in (e) have? Construct the graph.

- (g) What other method could you use to find the length of the segment from $t = 0$ to $t = 3$?
Do it. How does the result compare with that of the work in (e)?

- (h) Given a curve C:

$$\begin{cases} x = \frac{t^2}{2} - t + 10 \\ y = \frac{4t^2}{3} + 7 \end{cases}$$

Find the function S (curve length) in terms of t and then compute the curve length from $t = 2$ to $t = 5$.

III. C. Inverse Rate Problem Continued. 2.
(continued)

EXERCISES.

1. Text. p. 127. Nos. 1 to 12.

2. Find the coefficients of the cubic

$$y = ax^3 + bx^2 + cx + d$$

so that it has a maximum at $(-1, 10)$ and a point of inflection at $(1, -6)$.

3. Given C: $x = \frac{t^2}{2} - 2t + 8$, $y = \frac{4\sqrt{2}}{3} t^{3/2} + 10$.

Find the function S (curve length) in terms of t and then compute the curve length which is generated as t varies from $t = 0$ to $t = 4$?

4. From your experience in No. 3 and the problems in (e) and (h) what can you say about the choice of curves at present?
5. Text. p. 98. No. 26. This is an interesting application of calculus to economics. It would be very good to try this problem independently of the discussion in the text; use it as a last resort.
6. If $y = f(x) = x^3 + 3x^2 + 2x + 10$ how can this be expressed in terms of a parameter t ? (Suggestion: let $x = t = g(t)$. Then what does y become?)
7. Using the result of No. 6 how does $D_x y$ computed directly from $y = f(x)$ compare with the value of $D_x y$ when using the method for parametric functions?
8. Are you interested in a mathematical exploration? In No. 2 above there are four coefficients to be determined which seems that, for a cubic of that kind, four "conditions" are necessary to completely determine the cubic. These four conditions could be furnished by four points--each point "absorbs" one condition. How many conditions are absorbed by telling where a maximum is? where a point of inflection is? This whole problem related to lines, general conics, general cubics, points, points of maximum, points of minimum, points of inflection, given tangents, etc. becomes a very fascinating and absorbing one.
9. The "demand law" for tea has been found to be $x = 100 - 2p$ where x is the demand in lbs. and p is the price (let us say, cents). The total revenue R is xp . For what price is the total revenue a maximum?

III. C. Inverse Rate Problem Continued. 2.
(continued)

EXERCISES (continued).

10. Review. We have found that work under a force-law $F = 12x^2$ can be geometrically considered the same as _____.
Hence what is the work required to pull a spring from $x = 1$ out to $x = 3$?

III. C. Inverse Rate Problem Continued. 3.
Can one develop a general formula for the anti-
differentiation of ax^n ?

- (a) We recall from previous work that $D_x x^n = \underline{\hspace{2cm}}$ and that $D_x u^n = \underline{\hspace{2cm}}$ where u is a function of x .
- (b) If we replace n by $n + 1$ in the general formula for $D_x u^n$ in (a) what do we get?
- (c) Operate on both sides D_x^{-1} and see if you can suggest a method or a formula for finding $D_x^{-1}(u^n D_x u)$ where u is a function of x . Are any limitations necessary?
- (d) Consider $(2x^4 - 5) \cdot 8x^3$. If $(2x^4 - 5)$ is called $u(x)$ then what is $8x^3$? How will the result in (c) help you to compute

$$D_x^{-1} [(2x^4 - 5)^3 \cdot 8x^3] ?$$

Do it.

- (e) Consider $(3x^5 + 8)^4 \cdot 15x^4$. If we call $3x^5 + 8 = u(x)$ then $15x^4$ can be called $\underline{\hspace{2cm}}$.

Use the result from (c) to compute

$$D_x^{-1} [(3x^5 + 8)^4 \cdot 15x^4].$$

III. C. Inverse Rate Problem Continued. 3.
(continued)

(f) Consider $(3x^5 + 8)^4 \cdot 10x^4$.

Using the method of the preceding exercises what can we call the $(3x^5 + 8)$? _____. Is $10x^4$ the derived expression? _____.; if not, can we make changes so there will be a derived expression? How? Suggest a solution to

$$D_x^{-1} [(3x^5 + 8) \cdot 10x^4]$$

EXERCISES.

1. Text. p. 114. Nos. 1 to 13.
2. How can the result in (d) be verified in two ways? Do it. What possibility is there of expecting some discrepancies in results from different methods?
3. In fields of economics, physics, etc. it sometimes happens that equations of curves are not easily obtainable. Yet the investigator may be interested in a first derivative curve. Of course, such a derived curve can be only an approximate one. Study this curve and sketch as well as you can a first derivative curve. Points on the curve have been labeled to help you gather data. E.g., the slope at B is zero; hence at the abscissa of B the ordinate is zero.

4. Text. p. 98. Nos. 27, 28.
5. Review. Definition of area under a curve; definition of work done when force is variable; a definition which can be abstracted from these two. All we need is a name for it!

Mathematics 263
Helps for Test

I. You should be able to

- (a) derive the expression for the derivative of the quotient u/v where u and v are functions of x .
- (b) define (merely) the derivative of the product uv .
- (c) explain what $D_x y$ tells about a curve; i.e.,
what $D_x y$ positive means about the curve as x increases.
what $D_x y$ negative means about the curve as x increases.
what $D_x y = 0$ means about the curve as x increases.
Illustrate these with figures.
- (d) explain what $D_x^2 y$ tells about $D_x y$ as x increases, i.e.,
what $D_x^2 y$ positive tells about $D_x y$ as x increases.
what $D_x^2 y$ negative tells about $D_x y$ as x increases.
what $D_x^2 y = 0$ tells about $D_x y$ as x increases.
Illustrate these with figures.
- (e) state and use tests for maximum, minimum, and possible points of inflection.
- (f) explain what we mean by "idealization" of natural phenomena, conjectures about the universe, checking our conjectures.
- (g) explain "calculus relations" between s , v , a .
- (h) explain meaning of antiderivative or inverse derivative.
- (i) explain if $\bar{D}^{-1}(f + g) = \bar{D}^{-1}f + \bar{D}^{-1}g$? if $\bar{D}^{-1}(f - g) = \bar{D}^{-1}f - \bar{D}^{-1}g$?

II. You should be able to compute or work

- (a) higher derivatives of parametric functions. E.g. if $y = 4t^2$, $x = 3t^2 + 4t$ what is $D_x^2 y$?
- (b) higher derivatives of implicit functions. E. g. given $3x^2y^3 + 4y = 0$ compute $D_x^2 y$.
- (c) lengths of curves. E.g. if $y = 4t^2$ and $x = 3t^2$ what is the length of curve from $t = 0$ to $t = 5$?
- (d) equations going through a certain point. E.g. if the slope of a curve is always four more than the abscissa what is the equation of that curve which goes through $(-1, 1)$?
- (e) maximum, minimum problems of the type p. 97 (text). Nos. 9, 11, 12, 13, 14, 15, 16.
- (f) derivative, distance, velocity, acceleration problems as on sheets 76, 77, 78 and p. 116 of the Text.
- (g) antiderivative problems as on page 91, 92 (sheets) or Text p. 114.
- (h) rate problems as on p. 65 Text.
- (i) fundamental definitions: convergent sequence, limit of a function, slope of a curve at a point, speed, derivative, area, work.

Mathematics 263

Name..... Date.....

I. Short Answers.

1. If D_y is positive as x increases through a point $x = a$ then what can we say about the curve in the vicinity of $x = a$? (1); if D_y is negative then what can be said? (2); if D_y is zero then what can be said? (3)
(4)
(5)
(6)
4. $D_x(uv)$ is defined to be (4).
5. $D_x(u/v)$ is defined to be (5).
6. If D_{xy}^2 is negative as x increases through $x = a$ then what can be said about D_{xy} ? (6). Can anything be said about the y ? (7). (If so, what?)
(8)
(9)
8. As x increases through $x = a$ what is D_{xy} doing as indicated by the figure? (10) What can you say about D_{xy}^2 as x goes through $x = a$ in a positive direction? (9)
(12)
10. State derivative or "calculus" relations between s (distance), v (velocity), and a (acceleration). (10), (11)
(14)
12. $D^{-1}v$ (v meaning velocity) gives (12); $D^{-1}a$ gives (13). (This is a continuation of ideas in Nos. 10 and 11.)
(15)
(16)
14. A test for a function to have a maximum at the abscissa $x = a$ is (14) and (15).
(18)
16. To say that Y is an antiderivative of y with respect to x means that (16).
(19)
17. If $D_x y = 4x^2 - 3x - 10$ then $y =$ (17).
18. A test for a function to have a minimum at the point whose abscissa is $x = a$ is (18) and (19).

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II. Discussion. Problems.

1. Deduce the rule for computing $D_x(u/v)$ in terms of $D_x u$ and $D_x v$.

2. Given:
$$\begin{cases} y = 3t^2 + 2 \\ x = t^3 + 3 \end{cases}$$

compute $D_x y$ and $D_x^2 y$.

3. If $3xy + 4y^2 = 0$ what is $D_x^2 y$?

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III. Problems (continued).

4. Given the curve $\begin{cases} x = 4t^2 \\ y = 7\frac{1}{2}t^2 \end{cases}$

expressed parametrically in terms of t . What is the length of the curve from $t = 0$ to $t = 3$?

5. The slope of a certain curve at any point x is three less than the abscissa at that point. What is the equation of that curve which goes through the point $(4,0)$?

6. A rectangular box is to have a square base and a volume of 40 cubic feet. There is no top. If the bottom material costs 10 cents per square foot and the side material costs five cents per square foot what dimensions will give the least cost?
SHOW TESTS.

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Name..... Date.....

IV. Problems (continued).

7. (a) Given $s = t^2 - t - 6$ for what range of values of t is the distance positive?
negative?
- (b) For what range of values of t is the velocity positive?
negative?
- (c) Is the speed always increasing or always decreasing or does it sometimes do one and then the other?

8. Compute these antiderivatives.

(a) $D_x^{-1} [(x^2 + 3)^3 \cdot 2x]$

(b) $D_x^{-1} [(4x^2 + 10x)^2 \cdot (4x + 5)]$

(c) $D_x^{-1} [x \sqrt{x^2 + 3}]$

IV. The General Area Problem Treated

A. The Definite Integral. 1.

The "time is ripe" for us to consider what mathematicians call the definite integral--we have many of the fundamental ideas at hand now. Throughout our preceding experiences(*) we have been developing and using ideas and techniques which turn out to be parts of the so-called integral calculus.

During our experiences referred to below(*) we have developed nearly a "panorama of methods" for computing the area under a curve from estimation from purely arithmetic methods to the suggested result of taking the limit of a sequence of sums of areas of rectangles--assuming that this limit exists--and we have gone even further to abstract the last statement to the limit of a sequence of _____.

To recapitulate on what we have done let us review and then go forward by the use of these

EXERCISES:

1. We have defined area under a curve $y = f(x)$ between the lines $x = a$ and $x = b$ and above the x -axis to be _____

Suggest three ways by which we can set up rectangles.

(a)

(b)

(c)

*Note especially:

1. pp. 29-42. Computations from which we "presented" "area" answers.
2. p. 54. No. 9. Area under $y = x^2$ from $x = 0$ to $x = a$.
3. p. 57. No. 13. Area under $y = x^2$ from $x = 0$ to $x = b$.
4. p. 62. No. 5. Area under $y = x^2$ from $x = a$ to $x = b$.
5. p. 67. No. 7. Area under $y = x^3$ from $x = 0$ to $x = a$.
6. p. 69. No. 8. Area under $y = x^3$ from $x = a$ to $x = b$.
7. p. 71. No. 12 (e,f,g). Review of definitions.
8. p. 79. No. 6(g,h). Hypothesis on area under $y = x^4$.
9. pp. 80, 81. Nos. 6,7,8,9,10,11. Is the area process distributive with respect to addition?
10. p. 85. Nos. 4,5. Area under $y = a$ polynomial.
11. p. 92. No. 5. Definitions. Name for abstracted process?

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

Draw to illustrate the three cases which you mentioned in (a) (b) (c) above.

Can one have "area" below the x-axis too? _____

2. We have defined work done under a variable force $F = f(x)$ between $x = a$ and $x = b$ to be _____

Suggest some various ways in which we could have set up these "products" which "sum" to give "work estimates." Illustrate.

(a)

(b)

(c)

3. We have noted that from our statements of area defined and work defined we can abstract the definition of SOMETHING (for which we lack a name). This

SOMETHING = _____

4. How can such an abstraction be written symbolically? Do well on this. We stop at some nth case. We shall show you how this is written. _____

5. Can you think of any other physical or economic or mathematical quantities to which the ideas in No. 3 might apply? What? _____

6. Returning to No. 1 we have found also that when the curve has an equation of $y = x^2$ then the area between $x = a$ and $x = b$ and the x-axis is given by _____;
for the area under the curve $y = x^3$ we found that $A =$ _____;
for the area under the curve $y = x^n$ we hypothesized that
 $A =$ _____ between $x = a$ and $x = b$.

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

7. We then raised the question as to the area under $y = x^3 + x^2$ and between $x = a$, $x = b$, and the x -axis. This was (perhaps) first hypothesized and then (later) deduced to be

$$A = \underline{\hspace{2cm}}.$$

8. Outline the deduction of the result in No. 7 above.

9. The idea in No. 3 and the expression written as an abstraction in No. 4 is defined to be the DEFINITE INTEGRAL of $f(x)$ with respect to x from $x = a$ to $x = b$.

That is,

$$\text{the } \lim_{m \rightarrow \infty} \sum_{i=1}^m f(x_i) h_i \text{ where } nh = b-a$$

is called the definite integral of $f(x)$ w.r.t. x from $x = a$ to $x = b$. This is symbolized by a new character:

Hence

$$\int_a^b f(x) dx \quad \text{MEANS} \quad \lim_{m \rightarrow \infty} \sum_{i=1}^m f(x_i) h_i$$

or, the value of the integral means that the limit has been taken as $n \rightarrow \infty$ and the value of this limit is the value of the integral. We are assuming, of course, that there is a limit.*

10. Hence, from No. 6

$$A \equiv \lim_{m \rightarrow \infty} \sum_{i=1}^m f(x_i^2) h_i = \frac{b^3}{3} - \frac{a^3}{3}$$

means that

$$\int_a^b x^2 dx \equiv \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

No. 6 tells us therefore that

$$\int_a^b x^3 dx \equiv \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

No. 7 tells us therefore that

$$\int_a^b (x^3 + x^2) dx \equiv \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

*At present we are assuming that all the h_i are equal; hence the i could be deleted. Also $f(x)$ must be at least piecewise continuous; or, one might make the requirement that $f(x)$ be bounded in the interval $[a, b]$. This will be discussed.

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

11. There are several symbols for the integral of $f(x)$ w.r.t. x between $x = a$ and $x = b$. Some are

$$\int_a^b f \quad \int_a^b f(x) \quad \int_a^b f(x) dx \quad I_a^b f \quad I_a^b f(x) \quad I_a^b f(x)$$

The $\int_a^b f(x) dx$ is commonly used though the use of $I_a^b f$ has some advantages. $I_a^b f$ is somewhat analogous to $D_x f$ and it can be looked upon as an operator in much the same manner as $D_x f$ is sometimes considered. That is, $I(f)$ may be thought of the operation which gives the integral of $f(x)$ just as D is thought of the operator which gives the derivative of $f(x)$.

12. Some terms relative to the integral of a function are these. (Suggest what you think they mean.)

(a) to integrate $f(x)$ w.r.t. x between $x = a$ and $x = b$.

(b) limits of integration

(c) integrand

13. In No. 1 were suggested several different ways of forming the "rectangles" or the "products"--area can drop out of the picture now, since we are considering abstract products. Do you think the method of doing this makes any difference? _____
Why? _____

What limitations do we seem to be placing on the h_i ? _____

What assumption do we seem to be making about the limit of this sequence of sums of products? _____

This assumption we shall not prove.

14. Review. Prove (deduce) by the use of n -intervals again that

$$\int_0^b x^2 dx = I_0^b x^2 = b^3/3.$$

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

15. At present what theorems can you propose on integrals $\int_a^b f(x)dx$?

(a)

(b)

(c)

16. With your introductory thoughts in mind see how they compare with these itemized steps in defining

$$\int_a^b f(x) dx \quad \text{or} \quad I_a^b(f).$$

1. For each natural number n choose a subdivision of $[a, b]$ with division points $\{x_j^{(n)}\}$

$$a = x_0^{(n)} < x_1^{(n)} < \dots < x_j^{(n)} < \dots < x_n^{(n)} = b$$

Put
$$\delta^{(n)} = \max (x_j^{(n)} - x_{j-1}^{(n)}) \quad (1 \leq j \leq n)$$

and require

$$\lim_{n \rightarrow \infty} \delta^{(n)} = 0$$

2. For each natural number n choose in each of the j th subintervals

$$[x_{j-1}^{(n)}, x_j^{(n)}] \quad \text{a point} \quad \xi_j^{(n)},$$

$$x_{j-1}^{(n)} \leq \xi_j \leq x_j^{(n)} \quad j = 1, 2, \dots, n.$$

3. For each natural number form the sum S_n of the products of the value of f at the chosen point $\xi_j^{(n)}$ by the length of the interval $[x_{j-1}^{(n)}, x_j^{(n)}]$

$$S_n = \sum_{j=1}^n f(\xi_j^{(n)}) \cdot (x_j^{(n)} - x_{j-1}^{(n)})$$

4. If S_n approaches a limit as n tends to ∞ no matter how steps 1) and 2) are carried out, then it is shown by advanced methods of analysis* that the value of $\lim S_n$ does not depend upon how one carries out steps 1) and 2). The function f is said to have a definite integral $I(f)$ or I_f over the interval $[a, b]$ whose value is $\lim S_n$.

*The interested student might consult Richard Courant's Differential and Integral Calculus, pp. 131-35 for a readable proof. Chapter Two introduces the integral, pp. 76-88.

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

5. If the function f is continuous in $[a, b]$ then it is shown in advanced mathematics that this function f always has a definite integral over $[a, b]$.

Most of the elementary functions f which one meets in experience are defined and are continuous over some closed interval $a \leq x \leq b$. According to 5) the function f has a definite integral $I(f)$ over $[a, b]$ whose value may, according to 4), be obtained by carrying out steps 1) and 2) in any fashion. In our past experience, we have often carried out steps 1) and 2) as follows:

- 1'. For each natural number n divide $[a, b]$ into n intervals of equal length. The division points are

$$x_j^{(n)} = a + j \left(\frac{b-a}{n} \right)$$

and the intervals are

$$(x_{j-1}^{(n)}, x_j^{(n)}) = \left[a + (j-1) \left(\frac{b-a}{n} \right), a + j \left(\frac{b-a}{n} \right) \right]$$

with equal lengths

$$x_j^{(n)} - x_{j-1}^{(n)} = \frac{b-a}{n}$$

so that

$$\delta^{(n)} = \frac{b-a}{n} \quad \text{and clearly} \quad \lim_{n \rightarrow \infty} \delta^{(n)} = 0$$

- 2'. For each natural number n we choose the point $s_j^{(n)}$

in the interval $(x_{j-1}^{(n)}, x_j^{(n)})$

in one of the following manners; either

i) $s_j^{(n)}$ is always the left end point $x_{j-1}^{(n)}$;

$$s_j^{(n)} = x_{j-1}^{(n)} = a + (j-1) \left(\frac{b-a}{n} \right) \quad j = 1, 2, \dots, n$$

or

ii) $s_j^{(n)}$ is always the right end point $x_j^{(n)}$;

$$s_j^{(n)} = x_j^{(n)} = a + j \left(\frac{b-a}{n} \right) \quad j = 1, 2, \dots, n$$

or

iii) $s_j^{(n)}$ is always the midpoint of $(x_{j-1}^{(n)}, x_j^{(n)})$:

$$s_j^{(n)} = \frac{x_{j-1}^{(n)} + x_j^{(n)}}{2} = a + \left(\frac{2j-1}{2} \right) \cdot \left(\frac{b-a}{n} \right) \quad j = 1, 2, \dots, n$$

Observe that we have not and will not prove in this course that any function (except possibly $I_a^b(k)$ or

$$\int_a^b k dx)$$

IV. A. The Definite Integral. 1.
(continued)

EXERCISES (continued).

has a definite integral although we shall find the value of many definite integrals, for we do not show that the sums S_n have a limit no matter how steps 1) and 2) are carried out.

17. Use the results which you have found below and theorems which you suggested in No. 15

$$\int_a^b k dx = k(b-a)$$

$$\int_a^b kx^2 dx = k \left(\frac{b^3}{3} - \frac{a^3}{3} \right)$$

$$\int_a^b kx dx = k \left(\frac{b^2}{2} - \frac{a^2}{2} \right)$$

$$\int_a^b kx^3 dx = k \left(\frac{b^4}{4} - \frac{a^4}{4} \right)$$

to compute these integrals:

(a) $\int_0^2 x^2 dx$	(b) $\int_0^3 2x^2 dx$	(c) $\int_2^3 x^2 dx$
(d) $\int_{-1}^2 x^3 dx$	(e) $\int_2^0 4x^3 dx$	(f) $\int_{-1}^2 3x^2 dx$
(g) $\int_0^2 (x^2 + x + 3) dx$	(h) $\int_{-1}^2 (2x^2 + 3x - 10) dx$	

18. What would you suggest is the value of each of these integrals?

(a) $\int_0^t x^2 dx$	(b) $\int_u^{2u} x^3 dx$	(c) $\int_x^0 t^2 dt$
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19. Hence can integrals be used to define functions as well as derivatives can? _____ Each of the above integrals is a function of what variable? (a) _____ (b) _____ (c) _____
20. Is there any way which you can conceive that you can evaluate an integral "at sight?" Could you pass directly from the integrand to the value of the integral with the aid of something which you have done before? _____
What? _____

Some answers to above: (a) $2 \frac{2}{3}$, (b) 18, (c) $6 \frac{1}{3}$, (d) $3 \frac{3}{4}$, (e) -16, (f) 9, (g) $10 \frac{2}{3}$, (h) $1 \frac{1}{2}$.

IV. A. The Definite Integral. 2.
Fundamentals, Theorems, and Applications

EXERCISES.

1. Review.

- (a) definition of $\int_a^b f(x) dx$.
- (b) if one takes the left-hand endpoint of each interval to compute the height of the successive rectangles does one always get a "lower sum?" Draw figures to illustrate. If one takes the right-hand endpoint of each interval does one always get an "upper sum?" Illustrate. Draw a figure to show the use of midpoints of intervals to obtain heights of successive rectangles.
- (c) what is the first thing which one does to prepare to set up expressions for areas--"draw rectangles" or "take the intervals?"
- (d) what can be shown in higher analysis about methods of taking intervals and "heights?"
- (e) what assumption are we making on the definition of the integral in any particular case?
- (f) given a curve how did we arrive at an expression for "area" under a curve from $x = a$ to $x = b$?

2. Hence from No. 1(e) above what theorems on integration might you suggest (some we have already suggested and deduced)? The accompanying figure may help. Use integral symbols.

3. Using theorems which you suggest and your suggestion as to evaluation of integrals at sight find the values of each of these integrals. We shall show you another method of writing results.

(a) $\int_0^2 (x^2 + x) dx$ (b) $I_1^3 (x^3 + 2x - 5)$ (c) $I_{-1}^0 (-x^3)$

IV. A. The Definite Integral. 2.
(continued)

EXERCISES (continued).

4. Text. p. 153. Nos. 1,2,3,4,5,6,7,8. Use your discovery and theorems.

5. Find the areas bounded by these curves and the ordinates given or simply by the curves and the x-axis.

6. From the figures above (No. 5) what interpretation do you suggest when the "answer" to an area problem is negative?

7. Review. Find the antiderivatives. That is, what is

(a) $D^{-1} (2x \sqrt{x^2 + 4})$ (b) $D^{-1} [(x^2 + 4) \cdot 2x]$ (c) $D^{-1} [(x^2 - 6) \cdot x]$
 (d) $D^{-1} x \sqrt{x^2 - 7}$ (e) $D^{-1} 3x \sqrt{x^2 - 2}$ (f) $D^{-1} [4x^2 (x^3 - 4)^2]$

8. Review. If $x = t^2 + 3$ and $y = t^3$ compute D_{xy}^2 .

IV. A. The Definite Integral. 2.
(continued)

1. Review.

- (a) what is the "quick way" which we have found to evaluate an integral such as $\int_a^b f(x)$? State the various steps which seem to give the answer.
- (b) in what cases has the above hypothesis been verified?
- (c) what yet remains to be done on this problem discussed in (a) and (b) above? _____
- (d) the integral $\int_a^b f(x) dx$ is DEFINED TO BE _____ but we have discovered that it can be evaluated quickly merely by _____
- (e) how can integrals define functions?
- (f) in evaluating an integral BY DEFINITION what assumption do we make about the ordinates chosen?
- (g) if the sign on an area comes out negative what does this indicate about the area?

2. We showed BY DEFINITION THAT $\int_a^b kx^2 dx = k \left(\frac{b^3}{3} - \frac{a^3}{3} \right)$.

Using this result see if you can show that these two theorems written in class hold: that

(a) $\int_a^b x^2 dx = \int_a^c x^2 dx + \int_c^b x^2 dx$ ($a < c < b$)

(b) $\int_a^b x^2 dx = - \int_b^a x^2 dx$

3. What do you suggest as the result of subtracting these integrals?

$$\int_a^c f(x) - \int_a^b f(x) = \quad (a < b < c)$$

See if it holds using the definition in No. 2. I.e. try this for $f(x) = x^2$.

4. How high must the rectangle be to have the same area as that under the curve?

IV. A. The Definite Integral. 2.
(continued)

5. How high must this rectangle be to have the same area as that under the curve? How can you find the x -value of the curve whose ordinate is the same as this height?

6. In Nos. 4 and 5 we had continuous curves (functions) $y = f(x)$ and we had an interval from $x = a$ to $x = b$. What general statement can you make on what we did and found? Write this mathematically if you can.
7. Find the x so that its ordinate can serve as the height of a rectangle whose base is the length of interval given such that the area of the rectangle will be equal to the area under the curve within the limits given.
- (a) $y = x^2 - 2x + 3$ from $a = 2$, $b = 5$ (or $x = 2$, $x = 5$)
- (b) $y = \frac{1}{x^2}$ from $a = 1$ to $b = 2$.
8. Text. p. 170. Nos. 1,3,5,6,7,8,13.
9. Review. Given $s = t^2 - 3t - 40$.
- (a) for what values of t will the distance be positive? negative?
- (b) for what value of t will the speed be positive? negative?
- (c) what is the sign of the acceleration? what does this mean about the motion?
10. Review. Given $y = (x - 1)^2 (x + 1)^3$. Note this is a product. Examine for points of maximum or minimum and possible points of inflection.

IV. B. The Mean-Value Theorem
Some Applications of Integrals

1. Review.

- (a) A rectangular box with a top is to have a square base and a capacity of 64 cubic feet. The material for the bottom costs 30 cents per square foot, the material for the sides costs 15 cents per square foot, and the material for the top costs 50 cents per square foot. Find the dimensions so the cost will be the least. (Answer $x = 3\sqrt{24}$)
- (b) The velocity of a body given in terms of t is $v = 3t^2 - 6t + 10$. What is the distance traversed between $t = 1$ and $t = 2$ seconds?
- (c) A particle moves along the x -axis starting from the point $(-4, 0)$ with an initial velocity of 10 units per second to the left. It accelerates in such a way that after t seconds ($t \geq 0$) the acceleration is $12t - 8$. What is the expression for v ? for s ? What is its position after 3 seconds? What is the speed at the end of 3 seconds? Which way is it going at the end of this time?

(d) Complete:

i) $\int_a^b f + \int_b^c f =$

vi) $D(f \cdot g) =$

ii) $I_a^c f - I_a^b =$

vii) $D\left(\frac{f}{g}\right) =$

iii) $I_a^b kf =$

viii) $D_x x^n =$

iv) $I_a^b f \stackrel{?}{=} I_b^a f$

ix) $D_x x =$

v) $D(f + g) =$

x) $\text{Lim } (a_n + b_n) =$

(e) Given $\begin{cases} x = \frac{3}{2} t^2 \\ y = 2t^2 \end{cases}$

find the length of the curve from $t = 0$ to $t = 2$ secs.

2. Consider the curves $y = x^2$ and $y = x^2 + 4$. What is the area between these curves, the ordinate $x = 1$, and the ordinate at $x = 4$?

3. Text. p. 170. Nos. 16, 17, 18, 21, 22. Do work carefully.

IV. B. The Mean-Value Theorem
Some Applications of Integrals.
(continued)

4. Nos. 6 and 7 on p. 103 tell us what about the integral

$$\int_a^b f(x) \, dx \quad \text{and} \quad (b-a)?$$

5. What would your result tell or what relation would be suggested for these integrals?

(a) $\int_c^d f(x) \, dx =$

(b) $\int_a^{a+h} f(x) \, dx =$

(c) $\int_a^{2a} f(x) \, dx =$

(d) $\int_u^{u+2h} f(x) \, dx =$

Will the $f(x)$'s have to be continuous? _____

6. Above we have been making use of (and have been formulating) what is called the MEAN VALUE THEOREM.* State it here, or, if you cannot, consult your Text pp. 155, 156.
7. Find the value of x which "satisfies" the Mean Value Theorem if $f(x)$ is x^2 and x goes from $x = 1$ to $x = 5$.
8. The force required to push a certain body is always $f = x^3$. How much work will be required to push it from $x = 0$ to $x = 3$? Think how you would do this by summing small bits of work, then taking the limit of this sequence of sums--but NOW we have a shorter method! It still needs logical substantiation.
9. Text. p. 182. Study example.
10. Text. p. 185. Nos. 1, 3.

*We could call this a First Mean Value Theorem or the Mean Value Theorem for Integrals.

Some Answers: p. 104, No. 1 (a) 24, (b) 8 units, (c) expression for speed is $v = 6t^2 - 8t - 10$, distance $s = 2t^3 - 4t^2 - 10t - 4$, position = -16 units, speed 20 u/s to right. No. 2. 12 sq. units.
P. 105, No. 7. $x = 3.2$, No. 8. 20 1/4 units of work.

IV. B. The Fundamental Theorem
Some Applications of Integrals

1. Review.

- (a) Draw a figure (and state) to show the fundamental rate of change-approximation-error relation.
- (b) Make suitable changes in it to show a definition of the derivative.
- (c) State another form of the definition of the derivative (h-process).
- (d) By the use of (b) show that $D(uv) = uD_x v + v D_x u$.
- (e) Complete: $D_x(f + g) =$; $\int (f + g) =$; $\lim_{n \rightarrow \infty} (a_n + b_n) =$
Is there one of these which is the "basis" for the other two? _____; which? _____
- (f) a derivative is the limit of a sequence of _____
- (g) an integral is the limit of a sequence of _____
- (h) hence at the "basis" of each discipline (derivatives or integrals) we use theorems on _____
- (i) state the Mean Value Theorem. Use $f(x)$ continuous between $x = a$ and $x = b$.
- (j) an integral with constant limits defines a _____; but if one of the limits is a variable then the integral defines a _____

2. Given the function defined by $\int_a^t f(x) dx$ where $f(x)$ is continuous we then get a function of _____.

What is the derivative of this function? See if you can develop it or work it out by the definition of the derivative--i.e. by the use of the four-step "h" process. In your work perhaps the Mean Value Theorem will help.

3. Hence what does the result of your work tell you about the derivative of a function which is defined by an integral?

IV. B. The Fundamental Theorem
Some Applications of Integrals
(continued)

4. Hence what is the value of each of these?

(a) $D_t \int_a^t f(x) dx$ (b) $D_t \int_b^t g(u) du$ (c) $D_x \int_b^x h(v) dv$
(d) $D_t \int_t^a f(x) dx$ (e) $D_t \int_a^{t^2} f(x) dx$ (f) $D_t \int_a^w f(x) dx$

(Assume that the functions are continuous on the necessary closed intervals.)

5. Hence what is the value of each of these?

(a) $D_t \int_a^t (x^2 + 3x) dx$ (b) $D_u \int_a^u (x^3 + 5x^2) dx$ (c) $D_t I_a^t (x^2 + 10x + 5) dx$

6. The result which you may have stated in No. 3 and the idea which you have used in Nos. 4 and 5 is called (by us) (by Cooley) The FUNDAMENTAL THEOREM OF THE CALCULUS. It is stated differently by some authors.

7. Find the areas. Set up definition, use definite integrals to evaluate, integrate by the use of our discovery. (The validity of which as a device must still be substantiated logically.)

Ans.

- (a) $5 \frac{1}{3}$
(b) $5 \frac{1}{3}$
(c) $\frac{1}{3}$
(d) $\frac{14\sqrt{3}}{9}$

8. Set up volumes by definition, use integral to evaluate, integrate by the use of our discovery. Perhaps you may need to formulate a definition on volume first.

9. Text. p. 177. Nos. 1,2,3,5,7,9,10.

IV. B. The Fundamental Theorem
Some Applications of Integrals
(continued)

1. Review.

(a) Examine for a point of maximum, a point of minimum, and/or a point of inflection: $y = x^3 - 8x^2 + 20x - 16$.

(b) Find the dimensions of the largest rectangle which can be inscribed in a circle of radius 8 inches.

(c) The current I from a battery is given by $I = \frac{E}{R + r}$

where R is the external resistance in the circuit and r is the internal resistance of the cell. The electrical energy used in this circuit is $E = I^2 R$. The r is a constant but we shall make the external resistance R a variable. For what value of the external resistance R is the energy E a maximum?

(d) Text. p. 98. No. 28.

(e) State the Mean Value Theorem for $\int_a^b f(x)dx$ (function continuous.

(f) A fundamental idea of the base of the calculus is that of _____.

(g) Practice in the use of symbolism and operators:

$$\text{i) } D^{-1} Df = \quad \text{ii) } D^{-1} D Df = \quad \text{iii) } D^{-1} D^3 F =$$

(Do not forget constants of integration.)

2. What is the derivative of the function $\int_a^u f(x) dx$ w.r.t. u ?

Show this by definition of the derivative.

3. The result in No. 2 is the conclusion of what is called the Fundamental Theorem of the Calculus. See if you can formulate the "given part" and state the entire theorem here.

4. Hence

$$(a) D_t I_a^t f(x) dx = \quad (b) D_u I_a^u g(x) dx = \quad (c) D_u I_a^u (x^3 - 6x^2 + 10) dx =$$

5. Using practice and ideas gained in No. 1(g) and the Fundamental Theorem can you confirm our hypothesis that one can evaluate integrals by the use of antiderivatives?

$$\text{Use (a) } D_t I_a^t f(x) dx = \quad (b) D_t I_a^t (x^3 + x^2) dx =$$

IV. B. The Fundamental Theorem
Some Applications of Integrals
(continued)

6. In No. 5 we see the culmination of our logical support of our hypothesis that we can _____
7. Hence we can with assurance of its validity now _____
8. How are the processes of evaluating an integral by definition and evaluating an integral by the use of antidifferentiation conceptually different? _____

Are the results in general equal? _____

9. Text. p. 162. Nos. 1,3,5,6,7,8,9,12,13,14.
10. Text. P. 178. Nos. 11, 12, 13, 14.
11. Text. p. 185. No. 4.
12. Review. A particle starts from the origin with initial velocity 5 units per second directed toward the right. Its acceleration is given by $a_x = 2t$ and $a_y = 6t^2$.
- (a) What are the parametric equations of v_x and v_y ?
- (b) What are the parametric equations of s_x and s_y (s_x means distance in x direction)?
13. From (a) above and from this figure what would you suggest for the velocity along the path through which the particle is traveling? (Recall vectors in physics).

Answers. No. 1 (a). 14.7 min., (b) $x = \sqrt{32}$, (c) $R = r$.
 No. 12. $v_x = t^2 + 5$, $v_y = 2t^3$, $s_x = t^3/3 + 5t$, $s_y = \frac{t^4}{2}$
 No. 13. $\sqrt{(2t^3)^2 + (t^2 + 5)^2}$

IV. C. Some Applications of Integrals

1. Review.

- (a) To say that the sequence $\{a_n\}$ approaches A as a limit as $n \rightarrow \infty$ means what (ϵ language)?
- (b) To say that the sequence $\{b_n\}$ approaches B as a limit as $n \rightarrow \infty$ means what?
- (c) Hence show that $\lim_{n \rightarrow \infty} \{a_n + b_n\} = A + B$.
- (d) Hence show that $\lim_{n \rightarrow \infty} \{a_n \cdot b_n\} = A \cdot B$.
- (e) The above sequences are called _____ sequences; those which do not have this property are said to be _____
- (f) From the "error expression" and other means deduce the chain rule.
- (g) Define $\int_a^b f(x)dx$.
- (h) What limitations do we need to place on the $f(x)$ in (g) above?

2. Review. Find

(a) $D_x (x^2 - 3x)$	(b) $D_x (x^2 - 3x)^4$
(c) $D_x^{-1} (x^2 + 6x)^2$	(d) $D_x^{-1} [2x(x^2 + 6x)^2]$
(e) $D_x^{-1} \frac{2x}{\sqrt{x^2 - 5}}$	(f) $D_u (x^3 + 3x + 4)^2$
(g) $D_t (x^2 - 6x)$	(h) $D_u (t^3 + 3t)^3$

3. Review. Trace the logical development of our substantiation of the hypothesis that one can evaluate an integral merely by the process of antidifferentiation. This now becomes a theorem.

4. Text. p. 185. Nos. 5,6,7.

5. Text. p. 180. Nos. 1,2,3,4.

6. Review. A square is four inches on a side. Find the rate of change of the area w.r.t. to edge by four methods.

IV. C. Some Applications of Integrals
(continued)

1. Review. Construct

- (a) Construct a graph of this function $f(x) = \begin{cases} f(x) = x & \text{for } x > 3 \\ f(x) = 6 & \text{for } x = 3 \\ f(x) = 1 & \text{for } x < 3 \end{cases}$

For what value or values of x is this function continuous?

For what value or values of x is this function discontinuous?

- (b) Consider this function (above) from $x = 0$ to $x = 8$.
Is there an "area?" _____

What statement can you make on evaluating $\int_a^b f(x)dx$ by definition (with respect to continuity throughout the interval)?

What statement can you make on evaluating $\int_a^b f(x)dx$ by the use of the Mean Value Theorem-Antiderivative Method (with respect to continuity throughout the interval)?

- (c) For what ranges of x are these functions defined and continuous?

$$\text{i) } f(x) = \frac{x}{x-3} \quad \text{ii) } f(x) = \frac{1}{x} \quad \text{iii) } f(x) = \frac{x^2-9}{x-3} = 6 \text{ at } x = 3$$

- (d) Hence what processes can be carried on in these ranges?

2. Text. p. 180. Nos. 7, 10, 5, 6.

3. The pressure of a liquid on a surface is equal to the weight of the cubic units of liquid above it. Also in a liquid the pressure is the same in all directions.

Hence

- (a) If water weighs 60 lbs. per cu. ft. what is the pressure on a surface 1 foot below the top of the water?
- (b) What is the pressure on a surface 4 feet below the surface of the water?
- (c) What is the pressure on the surface lying horizontally under the water shown in (c)?

IV. C. Some Applications of Integrals
(continued)

- (d) What is the force on a plate lying horizontally 4 feet below the surface of the water if the area of the plate is 6 square feet?
- (e) What is the force on a horizontal plate which is 5' by 2' and is 3' below the surface of water?

IV. C. Some Applications of Integrals
(continued)

1. Review.

- (a) Explain carefully the meanings or definitions of i) the derivative of $f(x)$ w.r.t. x at $x = a$
 ii) the integral of $f(x)$ w.r.t. x from $x = a$ to $x = b$
 iii) a quick way to evaluate this integral if $f(x)$ is continuous
 iv) to say that $f(x)$ is continuous at $x = a$
 v) to say that $f(x)$ is continuous through an interval $[a, b]$
- (b) Some properties of continuous functions.
- (c) If the derivative of a function exists at $x = a$ what does this imply about the function at $x = a$?
- (d) List in order steps required to evaluate an integral by definition.
- (e) List in order steps required to evaluate an integral by the use of the antiderivative method.
- (f) The function $f(x) = \frac{3}{1-x}$ is not continuous at _____. Why?
- (g) We like continuous functions because _____

- (h) Would you say that continuous or non-continuous functions are more descriptive of natural phenomena? Support your contention.

2. Review.

- (a) Given $f(x)$ on the interval $[a, b]$. If $\int_a^b f(x)dx$ exists then what does the Mean Value Theorem say?
- (b) Did we prove the Mean Value Theorem or merely make its content plausible?
- (c) What does the Fundamental Theorem of the Calculus say about the integral in (a)?
Did we deduce this?
- (d) After deducing the Fundamental Theorem how did we show that one needed only to take the antiderivative of the integrand, make some substitutions, subtract, to evaluate the integral?

IV. C. Some Applications of Integrals
(continued)

3. Review. The volume of a cylinder is to be 180 cu. in. What must be the radius for the total surface to be a minimum?
4. What is the approximate force on this plate which is sitting vertically in water?
5. Text. p. 190. Nos. 1,2,3,4.

Mathematics 263
Helps for Test

For this test

I. You should be able to

carefully state tests for a function to have a maximum or a minimum

define a definite integral $\int_a^b f(x)dx$

explain meaning of the "Mean Value" Theorem

state the Fundamental Theorem of the Calculus

deduce the Fundamental Theorem of the Calculus

explain (or deduce) how some of the theorems on integrals come to be--or why they are true

explain

assumptions we make on integrals as to the existence of a limit

assumptions we make (but can be shown to be true) on methods of

subdivision of the integral and on ways of forming products

show that the use of the antiderivative in evaluating integrals is valid

II. You should be able to

evaluate integrals by methods of antidifferentiation (as Text 162)

work problems on finding areas (as sheets 101, 104, 107) (Text 170-71)

work simple problems on finding volumes (as sheet 107) (Text 177)

work harder problems on finding volumes (as Text 178)

work simple problems on work when force is variable

Mathematics 263
An Attempt at Relations.

APPENDIX D

CROSS-TESTS

TEST MADE FOR TRADITIONAL SECTION A

TEST MADE FOR EXPERIMENTAL SECTION ONE

TEST MADE FOR TRADITIONAL SECTION B

Mathematics 263

Test Made for Traditional Section A

Part I

Name.....

Show all work. Simplify all answers.

1,2. Define $\int_a^b f(x) dx$.

3. Given $\int_1^2 f(x) dx = 10$, $\int_2^4 f(x) dx = 14$, $\int_1^3 g(x) dx = 5$, and
 $\int_3^4 g(x) dx = 6$, find $\int_1^4 (f(x) - g(x)) dx$ by using the theorems
on integrals.

4,5. Use the definition in No. 1 to evaluate $\int_0^1 x^2 dx$.

Show every step of your work clearly, showing plainly that your evaluation fits the definition. You may need one or more of the following.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

- 6,7. The Mean Value Theorem for Integrals states that if $f(x)$ is continuous for $a \leq x \leq b$, then

$$\int_a^b f(x) dx = f(t) (b-a), \text{ where } a \leq t \leq b. \text{ For } a = 0, b = 1,$$

$f(x) = x^2$, find t . Complete the figure below to illustrate the geometric meaning of the theorem for the given a , b , and $f(x)$. Be sure to mark t on the figure.

8,9. Given $f(u) = \int_2^u (6x^2 - 2x)dx$, find an explicit formula for $f(u)$.

10,11. Evaluate $\int_0^4 \frac{x \, dx}{x^2 - 7}$.

Mathematics 263

Test Made for Traditional Section A

Part II

Name.....

Show all work. Simplify all answers.

1. a) Sketch the graph of
- $y = f(x)$
- , where

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } 0 \leq x \leq 1 \\ 2 & \text{for } x > 1 \end{cases}$$

- b) For what value or values of
- x
- is the function in No. 1 discontinuous?

2. Find $\lim_{x \rightarrow 1} \left(\frac{x^3 - 1}{x - 1} \right)$.

3. Define "derivative of $f(x)$ with respect to x for $x = x_1$."
- 4,5. Use the above definition to find $D_x(x^2)$. Show every step of your work clearly, showing plainly that your method fits the definition.

6. Define "f(x) is continuous for $x = a$."

7,8. Give one value of x for which each of the following functions is discontinuous and state how your definition is violated for that value of x .

a) $f(x) = x$ for $x \neq 1$.

b) $f(x) = \frac{x^2 - 4}{x - 2}$.

c) $f(x) = x$ for $x \neq 1$, $f(1) = 0$.

d) $f(x) = \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$.

9. A man proposes to fence in a rectangular field to contain 30,000 square yards. The fence on three sides of the field costs 30 cents per yard, but the fourth side is along a road and costs \$1.50 per yard. Find the dimensions which will make the cost of the whole fence least.

10. If $y = \frac{x}{\sqrt{x^2 - 1}}$, find $D_x y$.

11. Find $D^{-1} [5x(x^2 - 3)^{1/3}]$.

Mathematics 263

Test Made for Experimental Section One

Part I

Name..... Date.....

1. By $\sum_{i=1}^n f(x_i)h_i$ is meant (1) (1)_____
2. The definite integral $\int_a^b f(x)dx$ is defined to be (2) (2)_____
3. From Nos. 1 and 2 what relation exists between h and n ? (assuming all the h_i to be equal) (3) (3)_____
4. To evaluate the integral in No. 2 by definition requires what minimum continuity conditions on $f(x)$? (4) (4)_____
5. Under what continuity conditions can No. 1 not be evaluated by definition? (5)_____
6. The Fundamental Theorem of the Calculus states that (6) (6)_____
7. This means (for example) that (you make up an example) (7) (7)_____
8. Does antidifferentiation mean the same as integration? (8) ; (8)_____
- if so, how? (9) . If not, place (9)_____
- a row of x's in this blank. (10)_____
10. The antiderivative of $(x^2 - 2x)^3$ (11)_____
- $(x-1)$ is (10) (12)_____
11. The Mean Value Theorem states that (13)_____
- given $\int_a^b f(x)dx$ [f(x) continuous] (14)_____
- then (11) (15)_____
12. Given $y = f(x)$. A test for a relative maximum at $x = a$ is to see if (12) (16)_____
- and (13) . To test for a relative minimum one sees if (14) and (15) (17)_____
16. If at $x = a$, D_{xy}^2 is negative, what does (21)_____
- this tell about D_{xy} as x increases through (16)
- a ? (16) What, if anything, does this tell about y ? (17)

Mathematics 263

Name..... Date.....

Part I (continued)

18. Suggest some ways of forming the "heights" in the rectangles which we use to enable us to present a number which we call the "area" under a curve. __ (18, 19, 20, 21) __

II. Problems. Discussion.

1. Deduce the Fundamental Theorem.

Given: $\int_a^b f(x) dx$ $[f(x) \text{ continuous}]$

You are going to show that: _____

Argument:

2. Show how the result from No. 1 leads you to an assurance of validity of evaluating an integral by the method of antidifferentiation.

3. To evaluate an integral by antidifferentiation requires what minimum condition of continuity? _____
Why? _____

4. Find the area indicated. $y = x^3 + x^2 - 6x$

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III. Problems

1. Find the volume made by rotating the area enclosed by $y = x^2 - 6x$ and the x-axis around the x-axis. Show work.

2. Find the volume made by rotating the area within the curves $y = x^2$ and $x = y^2$ around the x-axis. (Area shown.)

3. The "force-distance law" of a certain spring is

$$F = x - 5 \quad (\text{pounds})$$

What is the work required to stretch the spring from 5 inches to 8 inches?

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Test Made for Experimental Section One

Part II
Final Examination I

Name..... Date.....

I. Short Answers.

1. If $y = 4x^3$ then $D_x y = \underline{\hspace{1cm}}(1)\underline{\hspace{1cm}}$
If $x = 2$ this means that $\underline{\hspace{1cm}}(2)\underline{\hspace{1cm}}$ or that
 $\underline{\hspace{1cm}}(3)\underline{\hspace{1cm}}$
3. If $y = 4x^3$ but $x = 1/t^2$ then $D_x y$ equals
 $\underline{\hspace{1cm}}(4)\underline{\hspace{1cm}}$
5. If x, y are given by the relations
 $x = 6t^2, y = \sqrt{t}$, then $D_x y = \underline{\hspace{1cm}}(5)\underline{\hspace{1cm}}$
6. If $y = (x^2 - 3x)^3$ then $D_x y = \underline{\hspace{1cm}}(6)\underline{\hspace{1cm}}$
7. If $y = x^3 + 2x^2$ then $D_x y = \underline{\hspace{1cm}}(7)\underline{\hspace{1cm}}$
8. It is possible to differentiate the
expression termwise because
 $D_x(f + g) = \underline{\hspace{1cm}}(8)\underline{\hspace{1cm}}$ where f and g
are functions of x .
9. The relation in No. 8 comes from what
basic theorem on limits? $\underline{\hspace{1cm}}(9)\underline{\hspace{1cm}}$
10. Suggest a theorem in our logical
structure of the calculus "intermediate"
between that in No. 9 and in No. 8.
 $\underline{\hspace{1cm}}(10)\underline{\hspace{1cm}}$
11. If $f(x) = \frac{x+3}{x}$ then $D_x f = \underline{\hspace{1cm}}(11)\underline{\hspace{1cm}}$
12. The area of a rectangle is $A = 2x^2 + 5x$.
At $x = 5$ a measurement of x is made but
it is 0.02 in. in error. What is the
resulting approximate error in area?
 $\underline{\hspace{1cm}}(12)\underline{\hspace{1cm}}$
13. $\int [f(x) + g(x)] = \underline{\hspace{1cm}}(13)\underline{\hspace{1cm}}$
14. What logical foundation is there for the
statement in No. 13? $\underline{\hspace{1cm}}(14)\underline{\hspace{1cm}}$

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II. Short Answers.

15. $D_x y \equiv \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. What (15) _____
 limitation is placed upon the h ? (15) _____
16. To say that $\lim_{n \rightarrow \infty} \{a_n\} = A$ means that (16) _____
 (17) _____
17. To say that the $\lim_{n \rightarrow \infty} \{b_n\} = B$ is to say (17) _____
 that (17) _____ (18) _____
18. To work with the sequences in No. 16 and (18) _____
 No. 17 simultaneously (i.e., both at the (19) _____
 same time) what care should we take?
 (18) _____ (20) _____
19. If $x = 3y^2 + 6y$ then $D_x y =$ (19) _____ (21) _____
20. Given $y = f(x)$ and the derivatives (20) _____
 $D_x y$, $D_x^2 y$, and $D_x^3 y$. At $x = a$ which
 of the above must equal zero if there
 is a maximum at $x = a$? (20) _____ (22) _____
 there is a minimum at $x = a$? (21) _____
 a point of inflection (horizontal) at
 $x = a$? (22) _____ (23) _____
23. A function $f(x)$ is continuous at $x = a$. (23) _____
 Does this mean or insure that a deriva-
 tive exists at $x = a$? (23) _____ (24) _____
24. A function $f(x)$ has a derivative at (24) _____
 $x = a$. Does this mean or insure that the
 function is continuous at $x = a$? (24) _____ (25) _____
25. $D_x(uv)$ is defined to be (25) _____ (26) _____
26. If $D_x^2 y$ is negative at $x = a$ what can be (26) _____
 said about $D_x y$ as x increases through
 $x = a$? (26) _____ What can be said about
 y ? (27) _____ (27) _____

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III. Short Answers.

28. If v means velocity then $D_t^{-1}v$ can be (28) _____
 physically interpreted to mean (28) _____;
 $D_t^{-2}v$ would have what interpretation, (29) _____
 if any? (29) _____ (30) _____
30. Given $y = f(x)$. Four important steps (31) _____
 in computing $D_x y$ are (30, 31, 32, 33) _____
34. Can these steps be applied to differentiat- (32) _____
 ing a function defined by an integral? (33) _____
 (34) _____
35. If $D_x y = x^2 - 3x + 10$ then $y =$ (35) _____ (34) _____
36. The $D_x f(x)$ and $\int f(x)dx$ are connected or (35) _____
 are related by the (36) _____ through (37) _____
 and (38) _____ (36) _____
39. A farmer wishes to know what shape of (37) _____
 silo will hold the most. Is this a (38) _____
 problem of the differential or the (39) _____
 integral calculus? (39) _____
40. $D_x y = 3x - 5$ could be an equation sug- (40) _____
 gested by what problem? (40) _____
41. As x increases through (41) _____
 $x = a$ what is $D_x y$ doing (42) _____
 as indicated by the
 figure? (41) _____ What
 can you say about $D_x^2 y$ as
 x increases through $x = a$?
 (42) _____

DISCUSSION.

1. Given $A = 3r^2$ as the area of a circle explain how you could find the rate of change of the area with respect to the radius when the radius is 4 inches, if you did not know any calculus.

Start the process but you need not finish it unless you have time.
 Do enough to indicate that you understand.

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IV. Discussion.

2. Fundamentally what would you suggest
- (a) as the concept most basic to the calculus?
 - (b) definition or concept of a derivative?
 - (c) definition or concept of the integral?
3. Deduce either the (a) quotient rule, (b) the product rule, or the (c) chain rule for differentiation of functions.
- Given for (a) $f(x)$, $g(x)$
- (b) $f(x)$, $g(x)$
 - (c) $y = f(x)$ but $x = g(u)$.
4. Explain or describe the logical thread which connects the process of differentiation with that of integration. You might start with fundamental concepts of each and develop or mention properties or facets of each until they meet on "some common ground." You may wish to use a chart with descriptive phrases. Use reverse side if there is not room below.

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Test Made for Traditional Section B

Name.....Date.....

I. Problems.

1. Evaluate the integral

$$\int_0^5 2(x^2 - 3)^2 dx.$$

2. Evaluate the integral

$$\int_2^{\sqrt{8}} \frac{5x dx}{8 - x^2}.$$

3. Find the value of
- x
- which satisfies the mean-value theorem for
- $f(x) = 8x - x^2$
- , when
- $a = 2$
- and
- $b = 6$
- .

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Name..... Date.....

II. Problems (continued).

4. Find the area bounded by the parabola $y^2 = 4x$, the y-axis, and the lines $y = 4$ and $y = 6$.

5. Find the volume generated by revolving the area bounded by $x^2 = y$, the line $x = y - 2$, and the line $x = 0$, about the x-axis.

6. Evaluate the integral $\int_1^2 (1 - 2x)^3 dx$.

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Name..... Date.....

III. Problems.

7. Find the volume generated by rotating the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ about the x-axis.

APPENDIX E

EVALUATION QUESTIONNAIRE

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An Attempt at Evaluation

Name (not necessary if you do not wish to write it)

1. Did you find the use of sheets helpful to learn calculus? _____
2. Did you like the more detailed style of the first sheets or do you prefer the style of the latter sheets? _____
Why? _____
3. We suggested that we have a different approach to the teaching of the calculus. We would appreciate your sincere answers to these questions. The answers need not be favorable simply because you are writing them on this sheet.
 - (a) Do you think that this approach really is different? (i.e. different from the traditional method of teaching mathematics) _____
How or why? _____
 - (b) Do you think you were given any more of an opportunity for self-growth by this method? _____ If so, could you name an instance? _____
 - (c) Do you think that understanding was encouraged? _____
How? _____
 - (d) Do you feel that what you know you know because of what you were told? or did you see some of it grow?
 - (e) Do you feel that you understand logical connections in the elementary calculus? _____ If so, name some.
 - (f) Do you think that we stressed theory too much? _____; too little? _____.
 - (g) Do you feel that we ever went "too fast" for you? _____; not fast enough? _____
 - (h) Did we expect too much work from you? _____
 - (i) What would you suggest as the main factor which prevented our experience-discovery approach from being more successful?

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- (j) Did you spend about two hours a day in study for this course?
 _____; more? _____; less? _____
- (k) Were there days or classes for which you made no preparation?
 _____; about how many? _____;
 why? _____
- (l) What good points do you have to say about this course and our approach?
- (m) What criticisms (adverse) do you have to make?
- (n) Have you had sources of help other than "your own," class discussion, the book? _____; What? _____
- (o) Are there any techniques of teaching used in this course which you would recommend be dropped? If so, what?
- (p) What feature of our work did you find most helpful: class reviews, work on sheets, discussion of sheets in classes, quizzes, studying from the book, my working problems on the board, reviews on the daily sheets? Perhaps you wish to name several, perhaps none, perhaps others.
- _____
- (q) Do you think that the numerical work at the beginning of the course helped you to understand the nature of the calculus?

4. Had you known the course would be conducted in this manner would you have enrolled in it? _____
5. Any further comment you have we would be glad for on the reverse side of this sheet.
6. Why are you studying calculus? _____
 Do you intend to take calculus next quarter? _____
 What is your major? _____
 Do you feel that this course opened an "undreamed-of field" for you?
7. One must never forget that emotion or "feeling" is a very real part of one's living. We would like to know how you feel about calculus.
- (a) In this approach (or during this quarter) did you feel any great sense of satisfaction in "understanding something?" If so, mention what, to you, was the occasion of a deep feeling of satisfaction.

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- (b) Did you experience any "thrill of discovery" large or small?
If so, mention when, or in what topic.
 - (c) Did you at any time feel that you began to see how mathematics grows? If so, mention an instance, if you can.
 - (d) Did you sense a "logical thread" running through our treatment?
What do you feel all calculus is based upon?
 - (e) Do you feel any "joy" or "deep satisfaction" in seeing ideas arranged in an orderly (logical) fashion?
 - (f) Did you at any time sense the immense power of the calculus?
When was this particularly evident in your thinking?
 - (g) Did you enjoy this study of calculus?
8. Do you think you gained more from this approach in any way whatever than you would have under traditional methods: lecture, study the book, test, lecture, study the book, test, etc.? In what way (if so)?
9. Do you feel that you made any contributions to the building of our course through any suggestion which you made--large or small?
10. Did you see any of your ideas grow either in or out of class?

BIBLIOGRAPHY

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I.	REFERENCES QUOTED OR MENTIONED IN THIS STUDY	353
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I. REFERENCES QUOTED OR MENTIONED IN THIS STUDY

- Bell, E. T. Men of Mathematics. New York: Simon and Schuster, 1937.
- Burkhart, Sarah Maybelle. "A Study of Concept Learning in Differential Calculus," Dissertation Abstracts, XVI (No. 11), 2102. Ann Arbor: University Microfilms, 1956.
- Cooley, Hollis R. First Course in Calculus. New York: John Wiley and Sons, Inc., 1954.
- Corey, Stephen M. Action Research to Improve School Practices. New York: Bureau of Publications, Teachers College, Columbia University, 1953.
- _____. "Measuring Attitudes in the Class Room," Elementary School Journal, XLIII (April, 1943), 457-461.
- Courant, Richard. Differential and Integral Calculus. Vol. I (translated by E. J. McShane). New York: Interscience Publishers, Inc., 1937.
- Durell, Fletcher. "Applications of the New Education to the Differential and Integral Calculus." American Mathematical Monthly, I (January, 1894), 15-19.
- Evans, Paul L. Mathematics for Technical Training: Calculus. New York: Ginn and Company, 1942.
- Fagerstrom, William Henry. Mathematical Facts and Processes Prerequisite to the Study of the Calculus. Contributions to Education, No. 572. New York: Bureau of Publications, Teachers College, Columbia University, 1933.
- Fawcett, Harold P. The Nature of Proof. Thirteenth Yearbook. Washington, D.C.: National Council of Teachers of Mathematics, 1938.
- _____. Lectures. Education 840. The Ohio State University. Summer, 1953.
- Filano, Albert E. "The Study of Effects of Various Teaching Methods on Achievement in College Algebra and Analytic Geometry with Special Inquiry into Student Participation," Pennsylvania State Review of Educational Research, VI, No. 2 (May, 1955), 13-16.
- Fine, Henry B. Calculus. New York: MacMillan Company, 1929.

- Granville, William A. and Smith, Percy F. Elements of the Differential and Integral Calculus. New York: Ginn and Company, 1911.
- Hooper, Alfred. Makers of Mathematics. New York: Random House, 1948.
- Hutchinson, C. A. (editor of mathematics education section) American Mathematics Monthly, XLVI (May, 1939), 280.
- Kells, Lyman M. Calculus. New York: Prentice-Hall, Inc., 1944.
- Kowalewski, Gerhard. Grundzüge der Differential- und Integralrechnung. Berlin: B. G. Teubner, 1928.
- Landau, Edmund G. H. Differential and Integral Calculus (translated from the German by Melvin Hausner and Martin Davis). New York: Chelsea Publishing Company, 1951.
- Maxwell, E. A. An Analytical Calculus for School and University. Volume I. Cambridge: University Press, 1954.
- Meland, Bernard E. Higher Education and the Human Spirit. Chicago: The University of Chicago Press, 1953.
- Middlemiss, Ross R. Differential and Integral Calculus. New York and London: McGraw-Hill Book Company, 1946.
- Moore, E. H. "On the Foundations of Mathematics," Bulletin of the American Mathematical Society, IX (May, 1903), 402-424.
- Murnaghan, F. D. Differential and Integral Calculus. Brooklyn: Remsen Press, 1947.
- _____. "The Teaching of College Mathematics," The American Mathematical Monthly, XIII (October, 1946), 419-425.
- National Council of Teachers of Mathematics. The Teaching of Mathematics in the Secondary School. Eighth Yearbook. Washington, D.C.: The Council, 1933.
- _____. The Learning of Mathematics; Its Theory and Practice. Twenty-First Yearbook. Washington, D.C.; the Council, 1953.
- _____. Emerging Practices in Mathematics Education. Twenty-Second Yearbook. Washington, D.C.: the Council, 1954.
- New Century Encyclopedia of Names. New York: Appleton-Century-Crofts, 1954.
- Pierce, C. S. "How to Make Our Ideas Clear," Popular Science Monthly, XII (January, 1878), 286-302.

- Perry, John. Calculus for Engineers. London: Edward Arnold, 1897.
- Polya, G. Induction and Analogy in Mathematics. Princeton: Princeton University Press, 1954.
- _____. How to Solve It. Princeton: Princeton University Press, 1946.
- Ransom, William R. The Calculus (According to a New Plan). Medford 55, Mass.: Tufts College Bookstore, 1949.
- Saxelby, F. M. A Course in Practical Mathematics. London: Longmans, Green, and Company, 1907.
- Schaaf, Oscar. "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," Unpublished Doctor's dissertation, The Ohio State University, 1954.
- Schultze, Arthur. The Teaching of Mathematics in Secondary Schools. New York: MacMillan, 1912.
- Sherwood, G. E. E. and Taylor, Agnus E. Calculus. New York: Prentice-Hall, Inc., 1946.
- Sobel, Max A. "Concept Learning in Mathematics," The Mathematics Teacher, XLIX (October, 1956), 425-430.
- Thorndike, R. L. "How Children Learn the Principles and Techniques of Problem-Solving," Learning and Instruction, Part I. National Society for the Study of Education, Forty-Ninth Yearbook, Chicago: the Society, 1950, 192-216.
- U. S. Department of Health, Education, and Welfare. Guidelines for Science and Mathematics. Washington, D.C.: Office of Education (May, 1958).
- Walker, Helen M. "Methods of Research," Review of Educational Research XXVI (June, 1956), 323-343.
- Woods, Frederick S. and Bailey, Frederick H. Elementary Calculus. New York: Ginn and Company, 1928.
- Young, J. W. Fundamental Concepts of Algebra and Geometry. New York: MacMillan Company, 1911.

II. Other Helpful References

- Allen, R. G. D. Mathematical Analysis for Economists. New York: The MacMillan Company, 1939.
- Bacon, Harold Maile. Differential and Integral Calculus. New York and London: McGraw-Hill Book Company, Inc., 1942.
- Butler, Charles H. and Wren, F. Lynwood. The Teaching of Secondary Mathematics. New York: McGraw-Hill Book Company, Inc., 1941.
- Breslich, E. R. The Technique of Teaching Secondary School Mathematics. Chicago: University of Chicago Press, 1930.
- Christofferson, H. C. Geometry Professionalized for Teachers. Menasha, Wisconsin: George Banta Publishing Company, 1933.
- Dragee, R. C. "Teaching the Calculus," National Mathematics Magazine, XIX (January, 1945), 186-193.
- Goodstein, R. L. "Teaching the Calculus," The Mathematical Gazette, XXV (October, 1941), 208-217.
- Hobson, E. W. The Theory of Functions of a Real Variable and the Theory of Fourier's Series. Cambridge: University Press, 1927.
- Menger, Karl. Calculus. A Modern Approach. Chicago: Illinois Institute of Technology Bookstore, 1953.
- Perry, John. "Address to the Engineering Section of the British Association," Science (New Series), XVI, No. 411 (November 14, 1902), 761-782.
- Slobin, H. L. and Solt, M. R. A First Course in Calculus. New York: Farrar and Rinehart, 1935.
- Wells, Volney H. Elementary Calculus. New York: D. Van Nostrand Co., Inc., 1941.
- Young, J. W. A. The Teaching of Mathematics. New York: Longmans, Green, and Company, 1914.

AUTOBIOGRAPHY

I, Kenneth Burdette Cummins, was born in New Washington, Ohio, July 27, 1911. I received my secondary school education in the public schools of New Washington. My undergraduate work, with majors in chemistry and mathematics, was taken at Ohio Wesleyan University, which granted me the Bachelor of Arts Degree in 1933. At Bowling Green State University I continued the study of mathematics and received the Master of Arts degree in 1939. During the summer and fall quarters of 1940 I held an assistantship in the Department of Mathematics at The Ohio State University and continued with more study in mathematics. Beginning again in the summer of 1953 with work in mathematics and mathematics education, I have been completing requirements for the degree Doctor of Philosophy. For four quarters during the year 1955-56 it was my good fortune to be the recipient of an E. I. duPont de Nemours and Company scholarship.

My teaching experience has been in the field of science and mathematics at the Sulphur Springs and New Washington High Schools, in Crawford County, Ohio. During the past year I served as an assistant professor of mathematics at Kent State University, Kent, Ohio.