ON A DECOMPOSITION THEOREM FOR MEASURES
IN EUCLIDEAN n -SPACE

DISSERTATION

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INTRODUCTION

It is the purpose of this dissertation to extend a decomposition theorem of Mickle [1] (square brackets refer to the bibliography at the end of this dissertation) for $(n-1)$ -dimensional measures in Euclidean n -space R^n to k -dimensional measures in R^n for $0 < k < n$, k an integer. In this dissertation we define a measure $\mathcal{M}_n^k$ on the family $\mathcal{B}$ of Borel sets of R^n that satisfies the following conditions:

(a) If $B \in \mathcal{B}$ and $\mathcal{M}_n^k(B) < +\infty$, then $B = B_1 \cup B_2$, where B_1 is countably k -rectifiable (see 4.1) and $\mathcal{M}_n^k(B_2) = 0$.

(b) If $B \in \mathcal{B}$, then

$$(1) \quad F_n^k(B) \leq \mathcal{M}_n^k(B) \leq H_n^k(B),$$

where F_n^k is the Favard k -dimensional measure in R^n (see (7.3)) and H_n^k is the Hausdorff k -measure in R^n (see [2] 2.18). Furthermore (see 7.3),

$$(2) \quad F_n^k(B) = \mathcal{M}_n^k(B)$$

whenever $\mathcal{M}_n^k(B) < +\infty$ and

$$(3) \quad F_n^k(B) = \mathcal{M}_n^k(B) = H_n^k(B),$$

whenever B is countably k -rectifiable. Whether (2) holds for every $B \in \mathcal{B}$ is an open question.

(c) $\mathcal{M}_n^k$ is the smallest measure on $\mathcal{B}$ which satisfies a weak projection inequality in the following sense: For almost every R^k (in a sense given in 6.4), the Lebesgue k -dimensional measure of the projection of a Borel set B into R^k is less than or equal to $\mathcal{M}_n^k(B)$.

(d) If m is a positive integer such that $k < m < n$, and B is a Borel set in R^m , then (see 7.4)

$$\mathcal{H}_m^k(B) = \mathcal{H}_n^k(B) .$$

While our results are stated in terms of measures on Borel sets, it will be convenient to work with Borel regular Carathéodory outer measures (see 1.11). The procedures and proofs will follow closely those of Mickle [1] and Federer [2]. However it should be noted that in the work of Mickle [1] the geometrical arguments needed do not necessitate the use of the group of orthogonol transformations of R^n onto R^n that are used in this dissertation.

I. PRELIMINARY CONSIDERATIONS

1.1 Let R^n denote Euclidean n -space, $\mathcal{B}$ the Borel sets of R^n , G_n the group of orthogonal $n \times n$ matrices with real entries, and σ_n the Haar measure on G_n . Likewise let G_k denote the group of orthogonal $k \times k$ matrices, and σ_k the Haar measure in G_k . For g , an element of G_n , let g_i be the i th row vector of g . That is,

$$g_i = (g_{i1}, g_{i2}, \dots, g_{in}),$$

where g_{ij} is the entry of g in the i th row and j th column.

Let I^k be that $n \times n$ matrix formed from the identity of G_n by setting the last $n-k$ diagonal elements equal to 0. We shall use I^k both for the above matrix and for the mapping effected by the matrix, that is, for the projection of R^n onto the space spanned by the first k basis vectors of R^n . Thus, although I^k considered as a matrix has no inverse, we shall use $(I^k)^{-1}$ for the inverse of the projection mapping. For convenience, we shall set $R^k = I^k(R^n)$. Also let $I_{n-k} = I - I^k$, and $R_{n-k} = I_{n-k}(R^n)$. If y is an element of R^n , and g is an element of G_n , then the length of $I^k g(y)$ and $I_{n-k} g(y)$ are given by

$$|I^k g(y)| = \sqrt{\sum_{i=1}^k (y \cdot g_i)^2}$$

$$|I_{n-k} g(y)| = \sqrt{\sum_{i=k+1}^n (y \cdot g_i)^2}$$

where " $\cdot$ " denotes inner product.

1.2 Let S be an element of G_{k+1} . Then S is a $k+1$ by $k+1$ orthogonal matrix. We form an $n \times n$ orthogonal matrix S' from S as follows:

$$S'_i = (S_{i1}, S_{i2}, \dots, S_{i, k+1}, 0, \dots, 0) \text{ for } 1 \leq i \leq k+1,$$

$$S'_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in}) \text{ for } k+2 \leq i \leq n,$$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}.$$

S' may be considered as a continuous function of S .

1.3 Lemma. For S in G_{k+1} and y in R^n

$$S' I^{k+1}(y) = I^{k+1} S'(y)$$

Proof. Let

$$u = (u_1, \dots, u_n) = S' I^{k+1}(y)$$

$$v = (v_1, \dots, v_n) = I^{k+1} S'(y)$$

$$I^{k+1}(y) = (y_1, \dots, y_{k+1}, 0 \dots 0).$$

$$u_i = \sum_{p=1}^n S'_{ip} (I^{k+1}(y))_p = \sum_{p=1}^{k+1} S'_{ip} y_p$$

$$v_i = \begin{cases} \sum_{p=1}^n S'_{ip} y_p & \text{if } 1 \leq i \leq k+1 \\ 0 & \text{if } k+2 \leq i \leq n \end{cases}$$

But if $1 \leq i \leq k+1$ and $k+2 \leq p \leq n$, then

$$S'_{ip} = 0;$$

$$\text{so } v_1 = \sum_{p=1}^{k+1} S'_{ip} y_p = u_1 \text{ if } 1 \leq i \leq k+1.$$

Also if $k+2 \leq i \leq n$ and $1 \leq p \leq k+1$, then

$$S'_{ip} = 0;$$

$$\text{so } u_i = \sum_{p=1}^{k+1} S'_{ip} y_p = 0 = v_i \text{ if } k+2 \leq i \leq n.$$

Hence $u = v$.

1.4 Lemma. For S in G_{k+1} and y in R^n

$$|I^{k+1} S'(y)| = |I^{k+1}(y)|$$

Proof. S' is in G_n and hence S' preserves length. Therefore by 1.3,

$$|I^{k+1} S'(y)| = |S' I^{k+1}(y)| = |I^{k+1}(y)|$$

1.5 Lemma. For S in G_{k+1} and $\alpha > 0$ let

$$\mathcal{U}(S, \alpha) = \{x \mid x \in R^n, |I^k S'(x)| < \alpha [S'(x)]_{k+1}\}$$

$$\mathcal{V}(S, \alpha) = \{x \mid x \in R^n, |I^{k+1}(x)| < (\alpha^2 + 1)^{1/2} [S'(x)]_{k+1}\}$$

Then $\mathcal{U}(S, \alpha) = \mathcal{V}(S, \alpha)$

Proof. First note that for x in either set

$$[S'(x)]_{k+1} = (S'_{k+1} \cdot x) > 0.$$

Now $x \in \mathcal{U}(S, \alpha)$

if and only if

$$|I^k S'(x)|^2 = \sum_{i=1}^k (S'_i \cdot x)^2 < \alpha^2 [(S'(x))_{k+1}]^2$$

if and only if

$$|I^{k+1} S'(x)|^2 = \sum_{i=1}^{k+1} (S'_i \cdot x)^2 < (\alpha^2 + 1) [(S'(x))_{k+1}]^2.$$

By 1.4 then

$$x \in \mathcal{U}(S, \alpha)$$

if and only if

$$|I^{k+1}(x)|^2 < (\alpha^2 + 1) [(S'(x))_{k+1}]^2$$

if and only if

$$x \in \mathcal{V}(S, \alpha)$$

1.6 For x a real number, $x > 0$, let

$$y = \beta(x) = \left\{ 2 \left[1 - \frac{1}{\sqrt{x^2 + 1}} \right] \right\}^{1/2}.$$

Then $\lim_{x \rightarrow 0} \beta(x) = 0$

and

$$x = \beta^{-1}(y) = \frac{y\sqrt{4-y^2}}{2-y^2}, \quad \lim_{y \rightarrow 0} \beta^{-1}(y) = 0,$$

and

$$\lim_{x \rightarrow 0} \frac{\beta(x)}{x} = 1$$

$$1.7 \text{ Let } C^{k+1} = \{y \mid y \in \mathbb{R}^{k+1}, |y| = 1\}.$$

For y in C^{k+1} , $0 < \eta < \infty$, let

$$C^o(y, \eta) = \{z \mid z \in C^{k+1}, |z-y| < \eta\},$$

$$C(y, \eta) = \{z \mid z \in C^{k+1}, |z-y| \leq \eta\}.$$

$$D = \{x \mid x \in \mathbb{R}^n, |I^{k+1}(x)| > 0\}.$$

D is open in $\mathbb{R}^n$.

Let f be the mapping with domain D and range C^{k+1} given by

$$f(x) = \frac{I^{k+1}(x)}{|I^{k+1}(x)|}.$$

f is continuous onto C^{k+1} .

1.8 Lemma. For S in G_{k+1} , $\alpha > 0$,

$$f^{-1} C^0[S_{k+1}, \beta(\alpha)] = V(S, \alpha)$$

Proof.

$$f^{-1} C^0[S_{k+1}, \beta(\alpha)] \subset D$$

Also, for x in $V(S, \alpha)$, $[S'(x)]_{k+1} > 0$

which implies that $|I^{k+1} S'(x)| > 0$

which by 1.4 implies that $|I^{k+1}(x)| > 0$.

Hence for any x under consideration, $|I^{k+1}(x)| > 0$.

$$\text{Now } [S'(x)]_{k+1} = [I^{k+1} S'(x)]_{k+1}$$

$$\text{which by 1.3 } = [S' I^{k+1}(x)]_{k+1} =$$

$$S'_{k+1} \cdot [I^{k+1}(x)] = S_{k+1} \cdot [I^{k+1}(x)] = (S_{k+1} \cdot f(x)) (|I^{k+1}(x)|).$$

$$\text{Now } |S_{k+1}| = |f(x)| = 1$$

$$\text{So } |f(x) - S_{k+1}|^2 = 2[1 - S_{k+1} \cdot f(x)].$$

Thus x is an element of $V(S, \alpha)$ if and only if

$$|I^{k+1}(x)| < (\alpha^2 + 1)^{1/2} [S'(x)]_{k+1}$$

which is true if and only if

$$1 < (\alpha^2 + 1)^{1/2} (S_{k+1} \cdot f(x))$$

which is true if and only if

$$(S_{k+1} \cdot f(x)) > (\alpha^2 + 1)^{-1/2}$$

which is true if and only if

$$2 - (S_{k+1} \cdot f(x)) < 1 - (\alpha^2 + 1)^{-1/2} = \frac{[\beta(\alpha)]^2}{2}$$

which is true if and only if

$$|f(x) - S_{k+1}|^2 < [\beta(\alpha)]^2$$

1.9 Lemma. If $f(x) = S_{k+1}$, then $I^k S'(x) = 0$.

Proof. If $f(x) = S_{k+1}$ then for all $\alpha > 0$, $f(x)$ is an element of $C^0(S_{k+1}, \beta(\alpha))$, which by 1.5 and 1.8 means that x is an element of $\mathcal{U}(S, \alpha)$.

Thus for all $\alpha > 0$

$$|I^k S'(x)| < \alpha [S'(x)]_{k+1}.$$

Hence $|I^k S'(x)| = 0$.

1.10 Lemma. Let $\varphi: X \rightarrow Y$ be a mapping from a set X into a set Y . Let E be a subset of X . Then

$$\varphi^{-1} \mathcal{C} \varphi(E) \subset \mathcal{C}(E),$$

where

$\mathcal{C}(E)$ is the complement of E .

Proof. $x \in \varphi^{-1} \mathcal{C} \varphi(E)$ implies $\varphi(x) \in \mathcal{C} \varphi(E)$

which implies $\varphi(x) \notin \varphi(E)$ which implies $x \notin E$

and hence $x \in \mathcal{C}(E)$

1.11 By a Carathéodory Outer Measure (abbreviated C.O.M.) on a metric space X , with distance ρ , we shall mean a non-negative set

function $\wedge$ defined for all subsets of X , such that

- 1) $\wedge(\emptyset) = 0$ ($\emptyset$ is the empty set),
- 2) $E_1 \subset E_2 \subset X$ implies $\wedge(E_1) \leq \wedge(E_2)$,
- 3) $E = \bigcup_n E_n$ implies $\wedge(E) \leq \sum_n \wedge(E_n)$,
- 4) $\rho(E_1, E_2) > 0$ implies $\wedge(E_1) + \wedge(E_2) = \wedge(E_1 \cup E_2)$.

An outer measure is a non-negative set function defined on the subsets of X which satisfies only the first three above conditions.

By a $\wedge$ measurable subset of X we shall mean a set E such that for all subsets Q of X ,

$$\wedge(Q) = \wedge(Q \cap E) + \wedge(Q \cap E^c).$$

By a Borel regular C.O.M. we shall mean a C.O.M. such that for any subset E of X there is a Borel set B containing E for which $\wedge(E) = \wedge(B)$.

Lemma. Let A be a closed set in R^n , j a positive integer,
 $\wedge$ an outer measure in R^n . For Y a subset of C^{k+1} , let

$$\psi(Y) = \text{lub}_{\alpha r \leq 1/j} \frac{\wedge[A \cap K(\bar{0}, r) \cap f^{-1}(Y)]}{r^k} \quad \text{where}$$

$\bar{0}$ is the origin of R^n , and $K(x, r)$ is the open sphere of center x and radius r in R^n .

Then $\psi(Y)$ is an outer measure in C^{k+1} .

Proof. Since $f^{-1}(\emptyset) = \emptyset$, for all r greater than 0,

$$\wedge[A \cap K(\bar{0}, r) \cap f^{-1}(\emptyset)] / r^k = 0$$

and hence $\psi(\emptyset) = 0$.

For $Y_1 \subset Y_2$, $f^{-1}(Y_1) \subset f^{-1}(Y_2)$, and hence for all $r > 0$

$$\frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}(Y_1)]}{r^k} \leq \frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}(Y_2)]}{r^k}$$

and therefore $\psi(Y_1) \leq \psi(Y_2)$.

If $Y = \bigcup_1 Y_1$, then $f^{-1}(Y) = \bigcup_1 f^{-1}(Y_1)$ and hence for all $r > 0$

$$\frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}(Y)]}{r^k} \leq \sum_{i=1}^{\infty} \frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}(Y_i)]}{r^k}$$

which implies that

$$\psi(Y) \leq \limsup_{0 < r \leq 1/j} \sum_i \frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}(Y_i)]}{r^k} \leq \sum_i \psi(Y_i).$$

1.12 Lemma. For A closed $\subset \mathbb{R}^n$, j a positive integer, D as in 1.7, let

$$Z = f(A \cap D \cap K(\bar{0}, 1/j)).$$

Then Z is an analytic subset of C^{k+1} , and

$$\psi(C^{k+1} - Z) = 0.$$

Proof. Since $A \cap D \cap K(\bar{0}, 1/j)$ is a Borel set of $\mathbb{R}^n$, and f is continuous, Z is analytic. Also,

$$C^{k+1} - Z = C^{k+1} \cap C f[A \cap D \cap K(\bar{0}, 1/j)].$$

Hence $\psi(C^{k+1} - Z) =$

$$\limsup_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}\{C^{k+1} \cap C f[A \cap D \cap K(\bar{0}, 1/j)]\}]}{r^k}$$

which by 1.10 is less than or equal to

$$\limsup_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{0}, r) \cap D \cap C[A \cap D \cap K(\bar{0}, 1/j)]]}{r^k} = 0.$$

$$1.13 \text{ Lemma Let } p(y, \gamma) = \frac{\psi[C(y, \gamma)]}{H^k[C(y, \gamma)]}$$

where H^k is Hausdorff k measure. Let

$$H = \{y \mid y \in C^{k+1}, \limsup_{\gamma \rightarrow 0} p(y, \gamma) = +\infty\},$$

$$K = \{y \mid y \in C^{k+1}, \limsup_{\gamma \rightarrow 0} p(y, \gamma) > 0\}.$$

Then $H^k(K - (H \cup Z)) = 0$.

Proof. By 1.12 $\psi(C^{k+1} - Z) = 0$. So $\psi[C(y, \gamma) - Z] = 0$.

Hence

$$\psi[C(y, \gamma) \cap Z] \leq \psi[C(y, \gamma)] \leq \psi[C(y, \gamma) \cap Z] + \psi[C(y, \gamma) - Z].$$

$$\text{Therefore } p(y, \gamma) = \frac{\psi[C(y, \gamma) \cap Z]}{H^k[C(y, \gamma)]}.$$

Now, by [3] and [4],

$$\limsup_{\gamma \rightarrow 0} \frac{\psi[C(y, \gamma) \cap Z]}{H^k[C(y, \gamma)]} = 0 \text{ or } +\infty$$

H^k almost everywhere on $C^{k+1} - Z$. But on $K - H$

$$0 < \limsup_{\gamma \rightarrow 0} p(y, \gamma) < +\infty.$$

$$\text{So } H^k[K - (H \cup Z)] = 0.$$

1.14 The following are immediate consequence of 1.6.

$$\lim_{\gamma \rightarrow 0} \frac{H^k[C(y, \frac{\beta(\gamma)}{2})]}{\alpha(k) \gamma^k} = 2^{-k} > 0$$

$$\lim_{\gamma \rightarrow 0} \frac{H^k[C(y, \beta(\gamma))]}{\alpha(k) \gamma^k} = 1,$$

where $\alpha(k)$ is the Lebesgue k measure of the set

$$\{x \mid x \in \mathbb{R}^k, |x| \leq 1\}.$$

1.15 Lemma. Let

$$H^*(\Lambda, A) = \{S \mid S \in G_{k+1},$$

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap \mathcal{U}(S, \gamma)]}{\alpha(k) r^k \gamma^k} = +\infty\}.$$

Then S_{k+1} in H implies S is in $H^*(\Lambda, A)$.

Proof. S_{k+1} in H implies

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap f^{-1}C(S_{k+1}, \gamma)]}{r^k H^k[C(S_{k+1}, \gamma)]} = +\infty.$$

Hence by 1.6

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap f^{-1}C(S_{k+1}, \beta(\gamma))]}{r^k H^k[C(S_{k+1}, \beta(\gamma))]} = +\infty.$$

Hence since

$$f^{-1}[C(S_{k+1}, \beta(\gamma))] \subset f^{-1}[C^0(S_{k+1}, 2\beta(\gamma))],$$

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap f^{-1}[C^0(S_{k+1}, 2\beta(\gamma))]]}{r^k H^k[C(S_{k+1}, \beta(\gamma))]} = +\infty$$

implies

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\gamma))]]}{r^k H^k[C(S_{k+1}, \frac{\beta(\gamma)}{2})]} = +\infty.$$

Hence by 1.14, $+\infty =$

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\gamma))]]}{\alpha(k) r^k \gamma^k},$$

which by 1.5 and 1.8 equals

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma_j} \text{l.u.b.} \frac{\Lambda[\Lambda \cap K(\bar{0}, r) \cap \mathcal{U}(S, \gamma)]}{\alpha(k) r^k \gamma^k}.$$

Hence S is in $H^*(\Lambda, A)$.

1.16 Lemma. Let

$$K^*(\Lambda, A) = \left\{ S \mid S \in G_{k+1}, \limsup_{\gamma \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{0}, r) \cap \mathcal{U}(S, \gamma)]}{\alpha(k) r^k \gamma^k} > 0 \right\}.$$

Then S in $K^*(\Lambda, A)$ implies S_{k+1} is in K .

Proof. S in $K^*(\Lambda, A)$ implies

$$0 < \limsup_{\gamma \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{0}, r) \cap \mathcal{U}(S, \gamma)]}{\alpha(k) r^k \gamma^k},$$

which by 1.5 and 1.8 is less than or equal to

$$\limsup_{\gamma \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}[C(S_{k+1}, \beta(\gamma))]]}{\alpha(k) r^k \gamma^k},$$

which by 1.14 is less than or equal to

$$\limsup_{\gamma \rightarrow 0} \text{l.u.b.}_{0 < r \leq 1/j} \left\{ \left[\frac{\Lambda[A \cap K(\bar{0}, r) \cap f^{-1}[C(S_{k+1}, \beta(\gamma))]]}{\alpha(k) r^k \gamma^k} \right] \times \left[\frac{\alpha(k) \gamma^k}{h^k[C(S_{k+1}, \beta(\gamma))]} \right] \right\},$$

which by 1.6 equals $\limsup_{\gamma \rightarrow 0} p(S_{k+1}, \gamma)$. Hence S_{k+1} is in K .

1.17 Lemma. Let

$$E(S) = \left\{ x \mid x \in \mathbb{R}^n, \quad I^k S'(x) = \bar{0} \right\}.$$

Let

$$L^*(A) = \left\{ S \mid S \in G_{k+1}, (A - \bar{0}) \cap K(\bar{0}, 1/j) \cap E(S) \neq \emptyset \right\}.$$

Then S_{k+1} in Z (See 1.12) implies that S is in $L^*(A)$.

Proof. S_{k+1} in Z implies that there is an x in

$$A \cap D \cap K(\bar{0}, 1/j)$$

such that $f(x) = S_{k+1}$. Hence by 1.9 $I^k S'(x) = \bar{0}$, which implies that x is in $E(S)$. Now if x is in $A \cap D$, then x is not the origin. So x is in

$$(A - \bar{0}) \cap K(\bar{0}, 1/j) \cap E(S)$$

which is therefore not empty. Hence S is in $L^*(A)$.

1.18 Theorem.

$$\bigcap_{k+1} [K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))] = 0.$$

Proof. By 1.16, if S is in $K^*(\Lambda, A)$, then S_{k+1} is in K . By 1.15, if S is not in $H^*(\Lambda, A)$, then S_{k+1} is not in H . By 1.17, if S is not in $L^*(A)$, then S_{k+1} is not in Z .

Hence

$$K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))$$

is contained in the set of S in G_{k+1} such that S_{k+1} is in $K - (H \cup Z)$.

Hence the theorem follows from the fact that

$$H^k[K - (H \cup Z)] = 0 \quad (\text{see 1.13}).$$

II. DENSITIES

2.1 For a fixed $\gamma > 0$, a an element of R^n , A a subset of R^n , j a fixed positive integer, g an element of G_n , and $\wedge$ a C.O.M., let

$$P(g, \gamma, a) = \{x \mid x \in R^n, |I^k_g(x-a)| < \gamma \mid I_{n-k}_g(x-a)\}.$$

Let $M(\wedge, A, g, \gamma, r, a) =$

$$\wedge [A \cap K(a, r) \cap P(g, \gamma, a)] / \alpha(k) \gamma^k r^k.$$

Let $M_j(\wedge, A, g, \gamma, a) = \lim_{r \rightarrow 0} \sup_{r \leq 1/j} M(\wedge, A, g, \gamma, r, a).$

Let

$$U(g, a) = \{x \mid x \in R^n, I^k_g(x-a) = 0\}.$$

Let

$$W_j(\wedge, A) = \{(x, g) \mid (x, g) \in A \times G_n, \lim_{\gamma \rightarrow 0} \sup M_j(\wedge, A, g, \gamma, x) = +\infty\}.$$

Let

$$X_j(\wedge, A) = \{(x, g) \mid (x, g) \in A \times G_n, \lim_{\gamma \rightarrow 0} \sup M_j(\wedge, A, g, \gamma, x) = 0\}.$$

Let

$$V_j(A) = \{(x, g) \mid (x, g) \in A \times G_n, (A-x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\}.$$

2.2 Lemma. If A is closed in R^n , then $V_j(A)$ is a Borel set in the cartesian product space $A \times G_n$.

Proof. For p and q integers, p less than j , and $C(x, r)$ the closed sphere for radius r and center x in R^n , the set $S_{j,p,q}$ of elements of $A \times G_n$ for which

$$A \cap [C(x, 1/j - 1/p) - K(x, (p-j)/(jp+q))] \cap U(g, x) \neq \emptyset$$

is a closed set in $A \times G_n$, and

$$V_j = \bigcup_{p=j+1}^{\infty} \bigcup_{q=1}^{\infty} S_{j,p,q}$$

2.3 Lemma. For A a Borel set of R^n , Λ a C.O.M., $W_j(\Lambda, A)$ and $X_j(\Lambda, A)$ are Borel sets in $A \times G_n$.

Proof. It suffices to show that

$$\Lambda[A \cap K(x, r) \cap P(g, \gamma, x)]$$

is a lower semi-continuous function of

$$(g, \gamma, r, x).$$

For

$$(g_0, \gamma_0, r_0, x_0),$$

let λ be any number less than

$$\Lambda[A \cap K(x_0, r_0) \cap P(g_0, \gamma_0, x_0)].$$

There exists F , a closed set of R^n , such that

$$F \subset K(x_0, r_0) \cap P(g_0, \gamma_0, x_0), \text{ and}$$

$$\Lambda(A \cap F) > \lambda, \text{ and for}$$

$$(g, \gamma, r, x)$$

sufficiently close to

$$(g_0, \gamma_0, r_0, x_0),$$

$$F \subset K(x, r) \cap P(g, \gamma, x),$$

and

$$\Lambda[A \cap K(x, r) \cap P(g, \gamma, x)] \geq \Lambda(A \cap F) > \lambda.$$

2.4 For A a closed subset of R^n , and x in A , the sets

$$W_j^*(\Lambda, A, x) = \{g \mid g \in G_n, \limsup_{j \rightarrow 0} M_j(\Lambda, A, g, j, x) = +\infty\},$$

$$X_j^*(\Lambda, A, x) = \{g \mid g \in G_n, \limsup_{j \rightarrow 0} M_j(\Lambda, A, g, j, x) = 0\},$$

$$V_j^*(A, x) = \{g \mid g \in G_n, (\Lambda - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\}$$

are Borel sets in G_n .

2.5 For Λ a closed subset of R^n , g in G_n , the sets

$$\bar{W}_j(\Lambda, A, g) = \{x \mid x \in R^n, \limsup_{j \rightarrow 0} M_j(\Lambda, A, g, j, x) = +\infty\},$$

$$\bar{X}_j(\Lambda, A, g) = \{x \mid x \in R^n, \limsup_{j \rightarrow 0} M_j(\Lambda, A, g, j, x) = 0\},$$

$$\bar{V}_j(A, g) = \{x \mid x \in R^n, (\Lambda - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\}.$$

$$\bar{W}(\Lambda, A, g) = \bigcap_j \bar{W}_j(\Lambda, A, g), \bar{X}(\Lambda, A, g) = \bigcup_j \bar{X}_j(\Lambda, A, g), \text{ and}$$

$$\bar{V}(A, g) = \bigcap_j \bar{V}_j(A, g) \text{ are Borel sets in } R^n. \text{ Also note that}$$

$\bar{V}(A, g)$ is the set of all points x of R^n such that x is an accumulation point of $\Lambda \cap U(g, x)$.

2.6 For g in G_n , $j > 0$, and a in R^n , let

$$Q_1^+(g, j) = \{x \mid x \in R^n, |I^k g(x-a)| < (n-k)^{\frac{1}{j}} j [g(x-a)]_1 \text{ for } k+1 \leq i \leq n.\}$$

Likewise let

$$Q_1^-(g, j) = \{x \mid x \in R^n, |I^k g(x-a)| < -(n-k)^{\frac{1}{j}} j [g(x-a)]_1 \text{ for } k+1 \leq i \leq n.\}$$

Now let

$$F_1^+ = \left\{g \mid g \in G_n, \limsup_{j \rightarrow 0} \sup_{0 < r \leq 1/j} \frac{\Lambda[\Lambda \cap K(a, r) \cap Q_1^+(g, j)]}{\alpha(k) r^k j^k} > 0\right\}.$$

and let

$$F_1^- = \left\{ g \mid g \in G_n, \limsup_{\gamma \rightarrow 0} \frac{\bigwedge [A \cap K(a, r) \cap Q_1^-(g, \gamma)]}{\alpha(k) r^k \gamma^k} > 0 \right\}.$$

F_1^+ and F_1^- are Borel sets in G . The proof of this is essentially the same as that given in 2.3.

$$2.7 \text{ Lemma. } P(g, \gamma, a) \subset \bigcup_{i=k+1}^n [Q_1^+(g, \gamma) \cup Q_1^-(g, \gamma)].$$

Proof. If not, then there exists x in R^n such that

$$|I^k g(x-a)| < \gamma |I_{n-k} g(x-a)|$$

and for all $i, k+1 \leq i \leq n$,

$$|I^k g(x-a)| \geq (n-k)^{\frac{1}{2}} \gamma | [g(x-a)]_1 |.$$

Hence

$$(n-k) |I^k g(x-a)|^2 \geq (n-k) \gamma^2 \sum_{i=k+1}^n [g(x-a)]_i^2 = (n-k) \gamma^2 |I_{n-k} g(x-a)|^2.$$

This is a contradiction.

2.8 Lemma. Let $I(p, q)$ be the identity matrix with the p^{th} row replaced by the q^{th} row, and the q^{th} row replaced by the negative of the p^{th} row. Then, for $k+1 \leq p, q \leq n$,

$$I(p, q) \cdot [F_p^+ - (V_j^*(\lambda, a) \cup W_j^*(\lambda, \lambda, a))] = F_q^- - (V_j^*(\lambda, a) \cup W_j^*(\lambda, \lambda, a)).$$

where "." denotes cosetting with respect to $I(p, q)$.

Proof. For g in G_n , and $i \neq p, q$,

$$[I(p, q)g]_1 = g_1; \quad [I(p, q)g]_p = g_q; \quad [I(p, q)g]_q = -g_p.$$

Thus, since $k+1 \leq p, q \leq n$, we have

$$|I^k(I(p, q)g(x-a))| = |I^k g(x-a)|, \text{ and}$$

$$|I_{n-k}(I(p,q)g)(x-a)| = |I_{n-k}g(x-a)|.$$

Thus $Q_p^+(g, \gamma) = Q_q^-(I(p,q)g, \gamma)$, and hence

$$F_q^- = I(p,q) \cdot F_p^+.$$

Also $P(g, \gamma, a) = P(I(p,q)g, \gamma, a)$, and hence

$$W_j^*(\wedge, \Lambda, a) = I(p,q) \cdot W_j^*(\wedge, \Lambda, a).$$

Also $U(g, a) = U(I(p,q)g, a)$, and hence

$$V_j^*(\Lambda, a) = I(p,q) \cdot V_j^*(\Lambda, a).$$

Hence $I(p,q) \cdot [F_p^+ - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a))] = F_q^- - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a))$.

2.9 Lemma. For $k+1 \leq p, q \leq n$,

$$\sigma_n [F_p^+ - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a))] = \sigma_n [F_q^- - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a))].$$

Proof. By 2.8

$$F_q^- - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a)) = I(p,q) \cdot [F_p^+ - (V_j^*(\Lambda, a) \cup W_j^*(\wedge, \Lambda, a))],$$

and the lemma follows since σ_n is the Haar measure in G_n .

III. FURTHER DENSITY CONSIDERATIONS

3.1 Let A be a closed subset of R^n , g an element of G_n , and a an element of A .

Let

$$t(x) = g(x-a), \quad \text{for } x \text{ an element of } R^n.$$

Then t is a distance preserving homeomorphism of R^n onto R^n such that $t(a) = \bar{0}$, $t(K(a,r)) = K(\bar{0},r)$, and $t(A)$ is a closed subset of R^n .

3.2 Lemma. For s in G_{k+1} , $\gamma > 0$,

$$Q_{k+1}^+(S'g, \gamma) = t^{-1}[\mathcal{U}(S, (n-k)^{\frac{1}{2}}\gamma)].$$

Proof. $x \in Q_{k+1}^+(S'g, \gamma)$ if and only if

$$|I^k S'g(x-a)| < (n-k)^{\frac{1}{2}}\gamma [S'g(x-a)]_{k+1}$$

which is true if and only if

$$|I^k S'g(x-a)| < (n-k)^{\frac{1}{2}}\gamma [S'(t(x))]_{k+1},$$

which is true if and only if $t(x)$ is in $\mathcal{U}(S, (n-k)^{\frac{1}{2}}\gamma)$.

3.3 Lemma. For g in G_n , a in R^n , S in G_{k+1} , $\gamma > 0$,

$$t^{-1}(\mathcal{U}(S, \gamma)) \subset P(S'g, \gamma, a).$$

Proof. If x is in $t^{-1}(\mathcal{U}(S, \gamma))$, then $t(x)$ is in $\mathcal{U}(S, \gamma)$,

and therefore

$$|I^k S'g(x-a)| < \gamma [S'g(x-a)]_{k+1},$$

and hence

$$|I^k S'g(x-a)|^2 < \gamma^2 [S'g(x-a)]_{k+1}^2 \leq \gamma^2 |I_{n-k} S'g(x-a)|^2,$$

and therefore x is in $P(S'g, \gamma, a)$.

3.4 Lemma. For g in G_n , a in R^n , S in G_{k+1} ,

$$t^{-1}(E(S)) = U(S'g, a).$$

Proof.

x is in $t^{-1}(E(S))$ if and only if $t(x)$ is in $E(S)$, which is true if and only if

$$|I^k S'g(x-a)| = 0,$$

which is true if and only if x is in $U(S'g, a)$.

3.5 Lemma. For A a closed subset of R^n , a an element of A , g in G_n , S in G_{k+1} ; S is in $L^*(t, A)$ if and only if $S'g$ is in $V_j^*(A, a)$.

Proof. By 3.4

$$(A-a) \cap K(a, 1/j) \cap U(S'g, a) = t^{-1}[(t(A) - \bar{0}) \cap K(\bar{0}, 1/j) \cap E(S)].$$

So S is in $L^*(t, A)$ if and only if

$$(t(A) - \bar{0}) \cap K(\bar{0}, 1/j) \cap E(S) \neq \emptyset,$$

which is true if and only if

$$t[(A-a) \cap K(a, 1/j) \cap U(S'g, a)] \neq \emptyset,$$

which is true if and only if

$$(A-a) \cap K(a, 1/j) \cap U(S'g, a) \neq \emptyset,$$

which is true if and only if $S'g$ is in $V_j^*(A, a)$.

3.6 For $\wedge$ an outer measure, and E a subset of R^n , let

$$\wedge^*(E) = \wedge[t^{-1}(E)].$$

Then $\wedge^*$ is an outer measure in R^n .

3.7 Lemma. For Λ an outer measure, A a closed subset of R^n , g in G_n , S in G_{k+1} ; if S is in $H^*(\Lambda^*, t(A))$, then $S'g$ is in $W_j^*(\Lambda, A, a)$.

Proof. If S is in $H^*(\Lambda^*, t(A))$, then

$$+\infty = \limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma} 1.u.b. \frac{\Lambda^*[t(A) \cap K(\bar{0}, r) \cap \mathcal{U}(S, \gamma)]}{\alpha(k) r^k \gamma^k},$$

which by 3.3 is less than or equal to

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma} 1.u.b. \frac{\Lambda[A \cap K(a, r) \cap P(S'g, \gamma, a)]}{\alpha(k) r^k \gamma^k}.$$

Hence $S'g$ is in $W_j^*(\Lambda, A, a)$.

3.8 Lemma. If $S'g$ is in F_{k+1}^+ , then S is in $K^*(\Lambda^*, t(A))$.

Proof. If $S'g$ is in F_{k+1}^+ , then

$$0 < \limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma} 1.u.b. \frac{\Lambda[A \cap K(a, r) \cap Q_{k+1}^+(S'g, \gamma)]}{\alpha(k) r^k \gamma^k},$$

which by 3.2 equals

$$\limsup_{\gamma \rightarrow 0} \sup_{0 < r \leq \gamma} 1.u.b. \frac{\Lambda^*[t(A) \cap K(\bar{0}, r) \cap \mathcal{U}(S, (n-k)^{\frac{1}{2}} \gamma)]}{\alpha(k) r^k \gamma^k},$$

and hence S is in $K^*(\Lambda^*, t(A))$.

3.9 Theorem. For any g in G_n , and a in A , a closed subset of R^n , and Λ an outer measure,

$$\overline{\sigma}_{k+1} \{S \mid S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} = 0.$$

Proof. By 3.5, 3.7, and 3.8,

$$\begin{aligned} \{S \mid S'g \in [F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} &\subset \\ &K^*[\Lambda^*, t(A)] - \{H^*[\Lambda^*, t(A)] \cup L^*[t(A)]\}. \end{aligned}$$

Since $\bigwedge^*$ is an outer measure, and $t(A)$ is closed in R^n , we may apply 1.18.

3.10 Theorem. For $a \in A$, a closed subset of R^n , and $\bigwedge$ a C.O.M.

$$\sigma_n \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] = 0.$$

Proof. Let $\alpha : (G_{k+1} \times G_n) \rightarrow G_n$ be a mapping defined by

$\alpha(S, g) = S'g$. α is a continuous mapping. Hence the set

$$\left\{ (S, g) \mid (S, g) \in G_{k+1} \times G_n, S'g \in \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] \right\} \\ = \alpha^{-1} \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right]$$

is a Borel set in $G_{k+1} \times G_n$.

Let $c(S, g)$ equal 1 if $S'g$ is in $F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a))$,

and 0 otherwise. Then, since σ_n is a Haar measure in G_n ,

for any S in G_{k+1} ,

$$\sigma_n \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] = \\ \sigma_n \left\{ g \mid S'g \in \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] \right\} = \int_{G_n} c(S, g) d\sigma_n.$$

So by 3.9

$$\sigma_n \left[F_{k+1}^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] = \\ \int_{G_{k+1}} \int_{G_n} c(S, g) d\sigma_n d\sigma_{k+1} = \int_{G_n} \int_{G_{k+1}} c(S, g) d\sigma_{k+1} d\sigma_n = 0$$

3.11 Lemma. For $k+1 \leq p, q \leq n$, a, A , and $\bigwedge$ as in 3.10,

$$\sigma_n \left[F_p^+ - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] = \sigma_n \left[F_q^- - (W_j^*(\bigwedge, A, a) \cup V_j^*(A, a)) \right] = 0.$$

Proof. By 2.8 and 3.10,

$$\begin{aligned} \sigma_n[F_p^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a))] &= \sigma_n[F_q^- - (W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a))] = \\ \sigma_n[F_{k+1}^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a))] &= 0 \end{aligned}$$

3.12 Lemma. For a , A , and Λ as in 3.10

$$G_n - X_j^*(\Lambda, A, a) \subset \bigcup_{i=k+1}^n (F_i^+ \cup F_i^-).$$

Proof. If not, then there exists a g in G_n , such that

$$\lim_{\gamma \rightarrow 0} \sup M_j(\Lambda, A, g, \gamma, a) > 0,$$

and for all i , $k+1 \leq i \leq n$,

$$\begin{aligned} 0 &= \lim_{\gamma \rightarrow 0} \sup_{0 < r \leq 1/2} \text{l.u.b.} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \gamma)]}{\alpha(k) r^k \gamma^k} \\ &= \lim_{\gamma \rightarrow 0} \sup_{0 < r \leq 1/2} \text{l.u.b.} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \gamma)]}{\alpha(k) r^k \gamma^k}. \end{aligned}$$

Hence by 2.7,

$$\begin{aligned} 0 &< \lim_{\gamma \rightarrow 0} \sup_{0 < r \leq 1/2} \text{l.u.b.} \frac{\Lambda[A \cap K(a, r) \cap P(g, \gamma, a)]}{\alpha(k) r^k \gamma^k} \leq \\ &\sum_{i=k+1}^n \left[\lim_{\gamma \rightarrow 0} \sup_{0 < r \leq 1/2} \text{l.u.b.} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \gamma)]}{\alpha(k) r^k \gamma^k} + \right. \\ &\quad \left. \lim_{\gamma \rightarrow 0} \sup_{0 < r \leq 1/2} \text{l.u.b.} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \gamma)]}{\alpha(k) r^k \gamma^k} \right] = 0 \end{aligned}$$

This is a contradiction.

3.13 Lemma. For a , A and Λ as in 3.10

$$\sigma_n[G_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a))] = 0.$$

Proof. By 3.12

$$Q_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a)) \subset$$

$$\bigcup_{i=k+1}^n \left\{ \left[F_1^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a)) \right] \cup \left[F_1^- - (W_j^*(\Lambda, A, a) \cup V_j^*(\Lambda, a)) \right] \right\},$$

and the result follows from 3.11.

3.14 Lemma. For Λ , a closed subset of R^n , j an integer,

Λ a C.O.M.,

$$\Lambda \left[\Lambda - (\bar{W}_j(\Lambda, A, g) \cup \bar{X}_j(\Lambda, A, g) \cup \bar{V}_j(\Lambda, g)) \right] = 0$$

for σ_n almost every g in Q_n .

Proof. By 3.13 for any a in Λ ,

$$\sigma_n \left[Q_n - (W_j^*(\Lambda, A, a) \cup X_j^*(\Lambda, A, g) \cup V_j^*(\Lambda, a)) \right] = 0.$$

Apply Fubini's theorem to the characteristic function of

$$(\Lambda \times Q_n) - (W_j(\Lambda, A) \cup X_j(\Lambda, A) \cup V_j(\Lambda)) .$$

3.15 Theorem. For Λ a closed subset of R^n , Λ a C.O.M.,

$$\Lambda \left[\Lambda - (\bar{W}(\Lambda, A, g) \cup \bar{X}(\Lambda, A, g) \cup \bar{V}(\Lambda, g)) \right] = 0,$$

for σ_n almost every g in Q_n .

Proof. By 3.14, for every integer j , there is a subset

Z_j of Q_n , such that $\sigma_n(Z_j) = 0$, and for g in $Q_n - Z_j$

$$\Lambda \left[\Lambda - (\bar{W}_j(\Lambda, A, g) \cup \bar{X}_j(\Lambda, A, g) \cup \bar{V}_j(\Lambda, g)) \right] = 0.$$

Let $Z = \bigcup_{j=1}^{\infty} Z_j$. Then $\sigma_n(Z) = 0$, and for g in $Q_n - Z$, and

for all integers j ,

$$\Lambda \left[\Lambda - (\bar{W}_j(\Lambda, A, g) \cup \bar{X}_j(\Lambda, A, g) \cup \bar{V}_j(\Lambda, g)) \right] = 0, \text{ and}$$

thus, by 2.5,

$$\Lambda[A - (\bar{W}_j(\Lambda, A, g) \cup \bar{X}(\Lambda, A, g) \cup \bar{V}_j(\Lambda, g))] = 0.$$

Also, $\bar{W}_j(\Lambda, A, g)$ and $\bar{V}_j(\Lambda, g)$ are monotone decreasing sequences of sets. So

$$\begin{aligned} \Lambda - (\bar{W}(\Lambda, A, g) \cup \bar{X}(\Lambda, A, g) \cup \bar{V}(\Lambda, g)) &= \\ [A - \bar{X}(\Lambda, A, g)] \cap [\bigcup_j \bar{W}_j(\Lambda, A, g)] \cap [\bigcup_j \bar{V}_j(\Lambda, g)] &\subset \\ [A - \bar{X}(\Lambda, A, g)] \cap \bigcup_j [\bar{W}_j(\Lambda, A, g) \cap \bar{V}_j(\Lambda, g)] &= \\ \bigcup_j [A - [\bar{X}(\Lambda, A, g) \cup \bar{W}_j(\Lambda, A, g) \cup \bar{V}_j(\Lambda, g)]] &, \end{aligned}$$

and the result follows.

IV. $(\wedge, k)$ UNRECTIFIABLE SETS

4.1 By a Lipschitz mapping T from a metric space X , with metric ρ , to a metric space Y , with metric δ , we shall mean a mapping such that there exists a constant M such that for all x, y in X , $\delta [T(x), T(y)] \leq M \rho(x, y)$.

A subset of Euclidean n -space will be said to be k -rectifiable if it is the image under a Lipschitz mapping of a bounded subset of Euclidean k -space.

If $\wedge$ is an outer measure on Euclidean n -space, a subset E of n -space will be said to be $(\wedge, k)$ unrectifiable if every k -rectifiable subset of E is of $\wedge$ measure 0.

Lemma. Let $\wedge$ be an outer measure in R^n . Let A be a subset of R^n such that $d(A)$, the diameter of A , is less than $1/j$, for a fixed integer j . Let g be an element of G_n . $0 < \delta < 1$, $0 < \gamma < +\infty$, and assume that

$$(1) \quad M_j(\wedge, A, g, \gamma, x) < \gamma \text{ for } x \text{ in } A \text{ and } 0 < \gamma < \delta.$$

Let (2) $B =$

$$\{x \mid x \in A, A \cap P(g, \gamma, x) \cap K(x, r) \neq \emptyset \text{ for } r > 0, 0 < \gamma < 1\}.$$

For x in R^n , let

$$x' = I_{n-k}g(x); \text{ and}$$

let

$$x'' = I^k g(x). \text{ Also}$$

let

$$K^n(x^n, r) = I^k_g(K(x, r)).$$

Then for a in B, we have

$$(3) \quad \Lambda[B \cap (I^k_g)^{-1}(K^n(a^n, r))] < 2^{k+1} 10^{2k} \gamma \alpha(k) r^k,$$

for $0 < r < \delta/12j$.

Proof. Since $d(A) < 1/j$, (1) implies that

$$(4) \quad M(\Lambda, A, g, \gamma, r, x) < \gamma \text{ for } x \text{ in } A, 0 < \gamma < \delta,$$

$r > 0$.

Now for x, y in R^n

$$|y^n - x^n| = |I^k_g(y) - I^k_g(x)| = |I^k_g(y-x)|,$$

$$|y' - x'| = |I_{n-k}^g(y) - I_{n-k}^g(x)| = |I_{n-k}^g(y-x)|.$$

With this remark, the proof proceeds exactly as in ([1], 5.2).

For a in B , $0 < r < \delta/12j$, let $\gamma = 12rj$, $\epsilon = \gamma/12 = rj$.

Set $E = B \cap (I^k_g)^{-1}(K^n(a^n, r))$.

For x in E , set $h(x) = \text{l.u.b. } |y' - x'| \text{ for } y \text{ in}$

$$A \cap P(g, \epsilon, x) \cap (I^k_g)^{-1}(K^n(a^n, r)).$$

Then $0 < h(x) \leq 1/j$.

For x in E , let $\bar{x}$ in $A \cap P(g, \epsilon, x) \cap (I^k_g)^{-1}(K^n(a^n, r))$ be

such that

$$(5) \quad 12|\bar{x}' - x'| > 11h(x)$$

Since $\bar{x}$ is in $P(g, \epsilon, x)$

$$(6) \quad |\bar{x}^n - x^n| < \epsilon |\bar{x}' - x'|.$$

For x in E , set

(7) $Q(x)$ equal the set of y in E such that

$$|y^n - x^n| < 5\epsilon h(x).$$

The remainder of the proof is divided into five parts.

Part 1. For x in E , $Q(x) \subset P(g, \eta, x) \cup P(g, \eta, \bar{x})$.

Proof. If not, then there is an x in E , such that

$$|y^n - x^n| < 5\epsilon h(x), \quad |y^n - x^n| \geq \eta |y' - x'|, \quad |y^n - \bar{x}^n| \geq \eta |y' - \bar{x}'|.$$

$$\text{Then } 0 < 11\epsilon h(x) < 12\epsilon |x' - \bar{x}'| = \eta |x' - \bar{x}'| \leq$$

$$\eta |x' - y'| + \eta |y' - x'| \leq |y^n - \bar{x}^n| + |y^n - x^n| \leq$$

$$|x^n - \bar{x}^n| + 2|y^n - x^n| < \epsilon |x' - \bar{x}'| + 10\epsilon h(x) \leq 11\epsilon h(x).$$

This is a contradiction.

Part 2. For x in E , $K^n(x^n, \epsilon h(x)) \subset K^n(a^n, 2r)$.

Proof. If $|y^n - x^n| < \epsilon h(x)$, then, since $|x^n - a^n| < r$,

$$|y^n - a^n| < |y^n - x^n| + |x^n - a^n| < \epsilon h(x) + r < 2r.$$

Part 3. If y is in $Q(x)$, then $|y' - x'| < 5h(x)$.

Proof. Assume that there is a y in $Q(x)$ such that $|y' - x'| \geq 5h(x)$.

Then $|y^n - x^n| < 5\epsilon h(x) \leq \epsilon |y' - x'|$. Thus y is in

$$A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K(a^n, r)), \text{ and hence}$$

$$1/j \geq h(x) \geq |y' - x'| \geq 5h(x) > 0. \text{ This is a contradiction.}$$

Part 4. If x is in E , then $\bigwedge (Q(x)) \leq 2(84)^k \gamma_\alpha(k) [\epsilon h(x)]^k$.

Proof. By Part 3, if y is in $Q(x)$, then $|y - x| \leq |y' - x'| + |y^n - x^n|$

$$< 5h(x) + 5\epsilon h(x) = 5(\epsilon + 1)h(x) < 6h(x). \text{ Also } |y - \bar{x}| \leq$$

$$|y - x| + |x - \bar{x}| \leq 5(\epsilon + 1)h(x) + |x' - \bar{x}'| + |x^n - \bar{x}^n| <$$

$$5(\epsilon + 1)h(x) + (\epsilon + 1)|x' - \bar{x}'| < 6(\epsilon + 1)h(x) < (13/12) \cdot 6h(x) <$$

$$7h(x). \text{ Hence } Q(x) \subset K(x, 7h(x)) \cap K(\bar{x}, 7h(x)).$$

Hence by Part 1,

$$Q(x) \subset [A \cap P(g, \eta, x) \cap K(x, 7h(x))] \cup [A \cap P(g, \eta, \bar{x}) \cap K(\bar{x}, 7h(x))].$$

Thus by (4) and since $\eta = 12\epsilon$,

$$\Lambda[Q(x)] \leq 2 \gamma \alpha(k) (7h(x) \gamma)^k = 2(84)^k \gamma \alpha(k) (\epsilon h(x))^k.$$

Part 5. $\Lambda(E) \leq 2^{k+1} 10^{2k} \alpha(k) \gamma r^k.$

Proof. Since, for x in E , $I^k_g(E) \subset \bigcup K^n(x'', \epsilon h(x))$,

by a covering theorem of Morse, [5], there are $x_1, x_2, \dots$

in E such that $I^k_g(E) \subset \bigcup_{i=1}^{\infty} K^n(x_i'', \epsilon h(x_i))$, and for $p \neq q$

$$K^n(x_p'', \epsilon h(x_p)) \cap K^n(x_q'', \epsilon h(x_q)) = \emptyset.$$

Now for x in E , there is an x_1 such that $|x'' - x_1''| < \epsilon h(x)$,

and hence x is in $Q(x_1)$. Thus

$$E \subset \bigcup_{i=1}^{\infty} Q(x_i).$$

From Parts 2 and 4,

$$\Lambda(E) \leq \sum_{i=1}^{\infty} \Lambda(Q(x_i)) \leq 2(84)^k \alpha(k) \gamma \sum_{i=1}^{\infty} (\epsilon h(x_i))^k \leq$$

$$2(84)^k (2)^k \gamma \alpha(k) r^k \leq 2^{k+1} 10^{2k} \gamma \alpha(k) r^k.$$

4.2 Theorem. Let Λ be a C.O.M. in R^n , g in G_n , and let A be a (Λ, k) unrectifiable subset of R^n . Then (See 2.5)

$$\Lambda[\bar{X}(\Lambda, A, g)] = 0$$

Proof. It suffices to show that for j an integer $\Lambda[\bar{X}_j(\Lambda, A, g)] = 0$.

Also, since R^n is separable, we may assume $d(\bar{X}_j(\Lambda, A, g)) < 1/j$.

Now $\lim_{j \rightarrow \infty} \sup M_j(\Lambda, \bar{X}_j, g, \gamma, x) = 0$ for x in $\bar{X}_j$. Assume $\Lambda[\bar{X}_j(\Lambda, A, g)] > 0$.

Hence by 2.5, there is a set $\bar{A}_j$ contained in $\bar{X}_j$, and a

sequence of positive numbers $\delta_1, \delta_2, \dots$, such that

$$\Lambda(\bar{A}_j) > 0, \quad \bar{M}_j(\Lambda, \bar{A}_j, \varepsilon, \eta, x) < 1/1, \text{ for } 0 < \eta < \delta_1$$

and x in $\bar{A}_j$. Let B_j be the set of x in $\bar{A}_j$, such that for

$$r > 0, \quad 0 < \eta < 1,$$

$$\bar{A}_j \cap P(g, \eta, x) \cap K(x, r) \neq \emptyset.$$

Since $\bar{A}_j$ is (Λ, k) unrectifiable, by Federer [2] 4.3,

$$\Lambda(\bar{A}_j - B_j) = 0, \quad \Lambda[B_j] = \Lambda[\bar{A}_j] > 0.$$

Thus there is a point a in B_j such that

$$(1) \limsup_{r \rightarrow 0} \Lambda[B \cap (I_g^k)^{-1}(K^u(a, r))] / \alpha(k)r^k > \lambda > 0,$$

for some λ . Choose i such that $1/i < \lambda / 2^{k+1} 10^{2k}$.

By 4.1,

$$(2) \Lambda[B \cap (I_g^k)^{-1}(K^u(a, r))] < \alpha(k)r^k \lambda$$

for $0 < r < \delta_1/12j$.

Since (1) contradicts (2), it follows that

$$\Lambda(\bar{X}_j(\Lambda, A, g)) = 0.$$

V. LEBESGUE MEASURE OF PROJECTED SETS

5.1 Lemma. If Λ is a C.O.M. in R^n , g in G_n , and A is a subset of R^n , with $\Lambda(A) < +\infty$, then

$$L_k [I^k_g(\bar{W}(\Lambda, A, g))] = 0,$$

where L_k denotes Lebesgue measure in R^k .

Proof. For E in R^k , let

$$\Lambda_g(E) = \Lambda[A \cap (I^k_g)^{-1}(E)].$$

Then Λ_g is finite-valued C.O.M. in R^k , and for x'' in R^k

$$(1) \limsup_{t \rightarrow \infty} \Lambda_g(K''(x'', t)) / \alpha(k) t^k < +\infty,$$

for L_k almost every x'' in R^k . Now for x in $\bar{W}(\Lambda, A, g)$, and

$0 < \eta < 1$, $0 < r < +\infty$, the set $P(g, \eta, x) \cap K(x, r)$ is

contained in the set of y in R^n such that y'' is an element of $K''(x'', \eta r)$.

Thus there is a sequence

$$(\eta_1, r_1) \rightarrow (0, 0) \text{ for } i \rightarrow +\infty, \text{ such that}$$

$$+\infty = \lim_{i \rightarrow \infty} M(\Lambda, A, g, \eta_1, r_1, x) = \lim_{i \rightarrow \infty} \Lambda_g(K''(x'', \eta_1 r_1)) / \alpha(k) \eta_1^k r_1^k,$$

and therefore the result follows from (1).

5.2 Lemma. Let Λ be a C.O.M. in R^n , g in G_n , and A a Borel subset of R^n such that

$$\begin{aligned} \Lambda(A) &< +\infty \text{ and} \\ \Lambda(A) &\geq \int_{R^k} N(x'', I^k_{g,A}) dL_k, \end{aligned}$$

where $N(x'', I^k_{g,A})$ is the number of points (possibly ∞)

in $A \cap (I^k_g)^{-1}(x^*)$. Then

$$L_k(I^k_g(\bar{V}(A,g))) = 0$$

Proof. $N(x^*, I^k_g, A)$ is finite for L_k almost every x^* in R^k .

But x in $\bar{V}(A,g)$ implies $N(x^*, I^k_g, A)$ is infinite. Hence the result follows.

VI. THE μ_n^k MEASURE

6.1 Let $\mathcal{J}$ denote the family of subsets Z of G_n , such that $\sigma_n(Z) = 0$. For B a Borel subset of R^n , and Z in $\mathcal{J}$, let

$$\lambda_Z^*(B) = \text{l.u.b. } L_k[I^k g(B)], \text{ for } g \text{ in } G_{n-Z}.$$

Let $\lambda(B) = \text{g.l.b. } \lambda_Z^*(B)$, for Z in $\mathcal{J}$.

For E a subset of R^n , and $\epsilon > 0$, let

$\mu_\epsilon(E) = \text{g.l.b. } \sum \lambda(B_i)$, the sum being taken over a countable covering of E by Borel sets B_i such that $d(B_i) < \epsilon$, and the greatest lower bound being taken over the family of all such coverings.

For E a subset of R^n , let

$$\mu_n^k(E) = \lim_{\epsilon \rightarrow 0} \mu_\epsilon(E).$$

μ_n^k is a Borel regular C.O.M. in R^n .

6.2 Lemma. Let B_i be Borel sets for $i = 0, 1, 2, \dots$, and
assume $B_0 \subset \bigcup_1 B_i$, and Z is in $\mathcal{J}$. Then

$$\lambda_Z^*(B_0) \leq \sum_i \lambda_Z^*(B_i).$$

Proof. $\text{l.u.b. } L_k[I^k g(B_0)] \leq \text{l.u.b. } L_k[\bigcup_i [I^k g(B_i)]] \leq$
 $\text{l.u.b. } \sum_i L_k[I^k g(B_i)] \leq \sum_i \text{l.u.b. } L_k[I^k g(B_i)],$ all least
 upper bounds being taken over g in G_{n-Z} .

6.3 Lemma. For B a Borel set in R^n , there is a set Z in $\mathcal{Z}$ such that $\lambda(B) = \lambda^*_Z(B)$.

Proof. For any integer i , there is a set Z_i in $\mathcal{Z}$ such that

$$\lambda^*_{Z_i}(B) < \lambda(B) + 1/i. \text{ Let } Z = \bigcup_i Z_i. \text{ } Z \text{ is in } \mathcal{Z}, \text{ and}$$

$$\lambda(B) \leq \lambda^*_Z(B) < \lambda^*_{Z_i}(B) < \lambda(B) + 1/i. \text{ Hence for all } i,$$

$$\lambda(B) \leq \lambda^*_Z(B) < \lambda(B) + 1/i. \text{ Therefore } \lambda(B) = \lambda^*_Z(B).$$

6.4 Lemma. For $E \subset R^n$, there is a set Z in $\mathcal{Z}$ such that for g in G_{n-Z} , $\mu_n^k(E) \geq L_k I^k_g(E)$.

Proof. Since μ_n^k is Borel regular, it suffices to work with Borel sets. Let B be a Borel set, $\epsilon > 0$, and i an integer.

There is a countable covering of B by Borel sets B_j , $d(B_j) < \epsilon$, such that $\mu_n^k(B) + 1/i > \sum_{j=1}^{\infty} \lambda(B_j)$. By 6.3

for each j there is a Z_j in $\mathcal{Z}$ such that

$$\lambda(B_j) = \lambda^*_{Z_j}(B_j).$$

Set $Z^1 = \bigcup_{j=1}^{\infty} Z_j$. Then Z^1 is in $\mathcal{Z}$ and, (by 6.2)

$$\mu_n^k(B) + 1/i > \sum_{j=1}^{\infty} \lambda^*_{Z^1}(B_j) \geq \lambda^*_{Z^1}(B).$$

Now set $Z = \bigcup_{i=1}^{\infty} Z^i$. Then Z is in $\mathcal{Z}$, and

$$\mu_n^k(B) + 1/i > \lambda^*_Z(B) \geq L_k I^k_g(B), \text{ for } g \text{ in } G_{n-Z},$$

and for all i . Hence $\mu_n^k(B) \geq L_k I^k_g(B)$ for g in G_{n-Z} .

6.5 Lemma. Let Λ be a Borel regular C.O.M. in R^n . Assume that for any subset E of R^n , there is a set Z in $\mathcal{Z}$ such that for g in G_{n-Z} , $\Lambda(E) \geq L_k I_k^g(E)$. Then $\Lambda(E) \geq \mu_n^k(E)$.

Proof. Again it suffices to work only with Borel sets B . For $\epsilon > 0$, B may be written as a countable union of Borel sets B_i , where $d(B_i) < \epsilon$, and for $i \neq j$, $B_i \cap B_j = \emptyset$.

Further for all integers i , there is a set Z_i in $\mathcal{Z}$ such that for g in G_{n-Z_i} , $\Lambda(B_i) \geq L_k I_k^g(B_i)$. Set $Z = \bigcup_{i=1}^{\infty} Z_i$.

Then Z is in $\mathcal{Z}$, and

$$\Lambda(B_i) \geq \text{l.u.b.} [L_k I_k^g(B_i)] \geq \lambda(B_i) \text{ for } g \text{ in } G_{n-Z}.$$

$$\text{Hence } \Lambda(B) \geq \sum_{i=1}^{\infty} \lambda(B_i) \geq \mu_{\epsilon}(B).$$

The result follows since ϵ was arbitrary.

6.6 Lemma. Let Λ be a Borel regular C.O.M. in R^n .

Let Γ be a Borel regular C.O.M. in R^k such that for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z_E in $\mathcal{Z}$ such that for g in $G - Z_E$, $\Lambda(E) \geq \Gamma I_k^g(E)$. Then for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z in $\mathcal{Z}$ such that for all subsets $E^* \subset E$, and g in G_{n-Z} ,

$$\Lambda(E^*) \geq \Gamma [I_k^g(E^*)].$$

Proof. Again it suffices to consider only Borel sets B in R^n , and Borel subsets $B^* \subset B$. First, let $\mathcal{O}$ be the family of sets C such that C is the union of a finite number of spheres of rational center and rational radius in R^n . $\mathcal{O}$ is a countable family. If F is a compact subset of R^n , then there exists

$$C_1 \supset C_2 \supset \dots, C_i \in \mathcal{O}, F = \bigcap_{i=1}^{\infty} C_i.$$

We may further assume that B is bounded. Let

$$\mathcal{O}_B = \{E \mid E = B \cap C, C \in \mathcal{O}\}.$$

$\mathcal{O}_B$ is countable. For each E in $\mathcal{O}_B$, there is a Z_E in $\mathcal{J}$ such that, for g in $G_n - Z_E$, $\bigwedge (E) \geq \bigwedge [I^k_g(E)]$. Set $Z = \bigcup Z_E$,

for E in $\mathcal{O}_B$. Z is in $\mathcal{J}$, and for E in $\mathcal{O}_B$ and g in $G_n - Z$,

$$\bigwedge (E) \geq \bigwedge [I^k_g(E)].$$

Now take F closed with respect to B . Then there is a compact set $F^* \subset R^n$, such that $F = F^* \cap B$. Hence there is a sequence of sets

$$C_1 \supset C_2 \supset \dots \text{ such that } F^* = \bigcap_{i=1}^{\infty} C_i. \text{ Set } E_1 = C_1 \cap B.$$

$$F = \bigcap_{i=1}^{\infty} E_i, \text{ and } E_1 \supset E_2 \supset \dots. \text{ Also for all integers } i,$$

and for g in $G_n - Z$,

$$\bigwedge [I^k_g(F)] \leq \bigwedge [I^k_g(E_1)] \leq \bigwedge (E_1).$$

Since $\lim_{i \rightarrow \infty} \bigwedge (E_i) = \bigwedge (F)$, we have, for all g in $G_n - Z$,

$$\bigwedge [I^k_g(F)] \leq \bigwedge (F).$$

Now let O be open with respect to B . Then there is a bounded open set O^* in R^n , such that $O = O^* \cap B$, and a sequence of compact sets $F^*_1 \subset F^*_2 \subset \dots$, such that $O^* = \bigcup_{i=1}^{\infty} F^*_i$.

Set $F_1 = F^*_1 \cap B$. F_1 is closed relative to B , and $O = \bigcup_{i=1}^{\infty} F_i$.

For all integers i , and g in $G_n - Z$,

$$\bigwedge [I^k_g(F_1)] \leq \bigwedge (F_1) \leq \bigwedge (O).$$

Now $I^k_g(F_i) \subset I^k_g(F_{i+1})$, and $I^k_g(F_i)$ is Γ measurable, and

$$I^k_g(0) = \bigcup_{i=1}^{\infty} I^k_g(F_i). \text{ So we have that}$$

$$\Gamma[I^k_g(F_i)] \rightarrow \Gamma[I^k_g(0)] \text{ as } i \rightarrow +\infty, \text{ and thus}$$

$$\Gamma[I^k_g(0)] \leq \Lambda(0).$$

Now take B^* a Borel subset of B . Fix $\epsilon > 0$. Then there is a set 0 open relative to B , such that $0 \supset B^*$, and

$$\Lambda(0) < \Lambda(B^*) + \epsilon.$$

Then for g in G_{n-2} .

$$\Gamma[I^k_g(B^*)] \leq \Gamma[I^k_g(0)] \leq \Lambda(0) < \Lambda(B^*) + \epsilon.$$

Thus $\Gamma[I^k_g(B^*)] \leq \Lambda(B^*)$.

there is a set Z in $\mathcal{Z}$, such that for g in G_{n-2} that $\mu_n^k(E) < +\infty$,
there is a set Z in $\mathcal{Z}$, such that for g in G_{n-2} ,

$$\mu_n^k(E) \geq \int_{R^k} N(x^*, I^k_g, E) dL_k.$$

Proof. Again it suffices to work with Borel sets $B \subset R^n$.

By 6.6 there is a set Z in $\mathcal{Z}$ such that for all Borel subsets $B^* \subset B$, and g in G_{n-2} , $\mu_n^k(B^*) \geq L_k I^k_g(B^*)$.

For i an integer, R^n may be written as the union of a countable family of disjoint Borel sets B_{ij} , where $d(B_{ij}) < 1/i$.

For x^* in R^k , let $f_{ij}(x^*)$ be the characteristic function of $I^k_g(B \cap B_{ij})$, and let

$$f_i(x^*) = \sum_{j=1}^{\infty} f_{ij}(x^*). \text{ Then}$$

$$\int_{R^k} f_1(x'') dL_k = \sum_{j=1}^{\infty} \int_{R^k} f_{1j}(x'') dL_k = \sum_{j=1}^{\infty} L_k I_g^k(B \cap B_{1j}) \leq \\ \sum_{j=1}^{\infty} \mu_n^k(B \cap B_{1j}) = \mu_n^k(B).$$

Since $f_1(x'') \rightarrow N(x'', I_g^k, B)$ as $i \rightarrow +\infty$,

the result follows from the lemma of Fatou.

6.8 Lemma. Let E be a k -rectifiable set in R^n . Then

$$H^k(E) = \mu_n^k(E)$$

Proof. Again it suffices to work with Borel sets B . Let Q denote the unit cube in R^k , that is for x in R^n , and x in Q , and $x = (x_1, x_2, \dots, x_n)$, we have

$$0 \leq x_i \leq 1, \text{ for } 1 \leq i \leq k, \text{ and}$$

$$x_i = 0, \text{ for } k+1 \leq i \leq n.$$

Now B may be taken as the image under a Lipschitz mapping of a subset of Q . Also it follows immediately from 6.5 that

$$(*) \quad H^k(B) \geq \mu_n^k(B).$$

By [6], this Lipschitz mapping may be extended to all of Q , with the same Lipschitz constant. Hence, in view of (*) without loss of generality, we may assume that

$$B = T(Q), \quad T \text{ a Lipschitz mapping.}$$

Now let u be any point of Q , and g any element of G_n . Let

$J(u)$ be the Jacobian of T at u ;

$J_g^*(u)$ be the Jacobian of gT at u ;

$J_g(u)$ be the Jacobian of $I_g^k gT$ at u .

Now by ([7], 4.2), $J(u) = J_g^*(u)$. Also all of the above Jacobians exist L_k almost everywhere on Q . Hence there is a Borel subset E of Q such that

(1) T is univalent on E ,

(2) All of the Jacobians exist and have positive, finite absolute values everywhere on E ,

$$(3) \mu_n^k(B - T(E)) = H^k(B - T(E)) = 0.$$

Now let u be any point in E . By (2) there is a k -plane $\mathcal{T}$ which is spanned by the k column vectors of the differential matrix of T at u . Select g in G_n such that $g(\mathcal{T}) = R^k$. Then for this g ,

$$J(u) = J_g^*(u) = J_g(u)$$

Thus for $\epsilon > 0$ there can be selected a sequence $g_1, g_2, \dots$ in G_n such that the sets

$$(4) E_1^* = \{u \mid u \in E, 0 \leq |J(u) - J_{g_1}(u)| < \epsilon\}$$

cover E .

Let

$$E_1 = E_1^*, \quad E_1 = E_1^* - \bigcup_{j=1}^{i-1} E_j^*.$$

Now, from 6.7, if necessary, a second sequence $g'_1, g'_2, \dots$ can be selected from G_n , such that for u in E_1 ,

$$(5) 0 \leq |J(u) - J_{g'_1}(u)| < \epsilon,$$

and

$$(6) \mu_n^k(T(E_1)) \geq \int_{R^k} N(x^n, I^k_{g'_1}, T(E_1)) dL_k.$$

Hence, from ([7], 4.5), we have

$$\begin{aligned}
 H^k(B) &= \int_E J(u) \, dL_k = \sum_{i=1}^{\infty} \int_{E_i} J(u) \, dL_k \leq \\
 &\sum_{i=1}^{\infty} \int_{E_i} J_{g'_i}(u) \, dL_k + \epsilon = \\
 \sum_{i=1}^{\infty} \int_{R^k} N(x^n, I^k g'_i, T(E_i)) \, dL_k + \epsilon &\leq \\
 \sum_{i=1}^{\infty} \mu_n^k(T(E_i)) + \epsilon &= \mu_n^k(B) + \epsilon.
 \end{aligned}$$

Hence

$$H^k(B) \leq \mu_n^k(B).$$

From this result and (*), it follows that

$$H^k(B) = \mu_n^k(B).$$

VII. THE DECOMPOSITION THEOREM

7.1 Lemma. Let A be a closed (μ_n^k, k) unrectifiable set
in R^n , such that $\mu_n^k(A) < +\infty$. Then $\mu_n^k(A) = 0$.

Proof. For any g in G_n ,

$$A = \left[A - (\bar{W}(\mu_n^k, A, g) \cup \bar{X}(\mu_n^k, A, g) \cup \bar{V}(A, g)) \right] \cup \\ \left[\bar{W}(\mu_n^k, A, g) \cup \bar{X}(\mu_n^k, A, g) \cup \bar{V}(A, g) \right].$$

By 3.15

$$\mu_n^k \left[A - (\bar{W}(\mu_n^k, A, g) \cup \bar{X}(\mu_n^k, A, g) \cup \bar{V}(A, g)) \right] = 0,$$

for σ_n almost every g in G_n .

$$\text{By 6.4, } L_k \left[I_k^g (A - (\bar{W}(\mu_n^k, A, g) \cup \bar{X}(\mu_n^k, A, g) \cup \bar{V}(A, g))) \right] = 0$$

for σ_n almost every g in G_n .

Likewise by 4.2 and 6.4, we have that

$$L_k \left[I_k^g (\bar{X}(\mu_n^k, A, g)) \right] = 0, \text{ for } \sigma_n \text{ almost every } g \text{ in } G_n.$$

Likewise by 5.1,

$$L_k \left[I_k^g (\bar{W}(\mu_n^k, A, g)) \right] = 0, \text{ for every } g \text{ in } G_n.$$

Likewise by 5.2 and 6.7, we have that

$$L_k \left[I_k^g (\bar{V}(A, g)) \right] = 0, \text{ for } \sigma_n \text{ almost every } g \text{ in } G_n.$$

Hence, for σ_n almost every g in G_n ,

$$L_k [I^k_g(A)] = 0,$$

and therefore

$$\mu_n^k(A) = 0.$$

7.2 Theorem. Let A be a Borel set in R^n such that $\mu_n^k(A) < +\infty$. Then $A = A_1 \cup A_2$, where A_1, A_2 are Borel sets, $A_1 \cap A_2 = \emptyset$, A_1 is countably k -rectifiable, and $\mu_n^k(A_2) = 0$.

Proof. Let

$$\gamma = \text{l.u.b. } \mu_n^k(A'),$$

for A' a k -rectifiable subset of A . For each positive integer i , there is a k -rectifiable set $B_i \subset A$ such that

$$\mu_n^k(B_i) > \gamma - 1/i.$$

Then there is a Lipschitz transformation from a bounded set E_i in R^k onto B_i . By [6] this transformation can be extended to the whole of R^k with the same Lipschitz constant. If F_i is a bounded closed set in R^k containing E_i then the intersection of A and the image of F_i under this extended transformation is a Borel k -rectifiable subset of A containing B_i . Hence we may assume that B_i is a Borel set for each i .

$$\text{Set } A_1 = \bigcup_{i=1}^{\infty} B_i.$$

Then A_1 is a Borel and countably k -rectifiable set.

The set $A_2 = A - A_1$ is Borel and (μ_n^k, k) unrectifiable,

$A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. Since A_2 is the union of a countable

number of closed $(\mathcal{M}_n^k, k)$ unrectifiable sets and a set of $\mathcal{M}_n^k$ measure 0, it follows from 7.1 that $\mathcal{M}_n^k(A_2) = 0$.

7.3 Theorem. If A is a Borel subset of R^n and $\mathcal{M}_n^k(A) < +\infty$, then $\mathcal{M}_n^k(A) = F_n^k(A)$, where F_n^k is the integralgeometric Favard k -measure of A , in R^n . (See [2], 2.18, 5.11)

Proof. By 7.2

$A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, A_1, A_2 Borel, A_1 is countably k -rectifiable, and $\mathcal{M}_n^k(A_2) = 0$. Thus

$$F_n^k(A_2) = 0.$$

Also by 6.8 and ([2], 5.14),

$$\mathcal{M}_n^k(A_1) = H^k(A_1) = F_n^k(A_1).$$

7.4. Theorem. Let m, k and n be positive integers, $k < n < m$, and let A be a Borel subset of R^n , which may be thought of as a subspace of R^m . Then $\mathcal{M}_n^k(A) = \mathcal{M}_m^k(A)$.

Proof. Under the above assumptions Federer has shown ([8], 7) that $F_m^k(A) = F_n^k(A)$.

To begin, we assume $\mathcal{M}_n^k(A) < +\infty$,

Then, by 6.8 and 7.2, $A = A_1 \cup A_2$, A_1 Borel for $i = 1, 2$, $A_1 \cap A_2 = \emptyset$.

Also A_1 is a countably k -rectifiable subset of both R^n and R^m . So

$$\mu_n^k(A_1) = H^k(A_1) = \mu_m^k(A_1), \text{ and } \mu_n^k(A_2) = F_n^k(A_2) = 0.$$

$$\text{Hence } F_m^k(A_2) = 0 = \mu_m^k(A_2). \text{ Therefore } \mu_n^k(A) = \mu_m^k(A).$$

$$\text{We now wish to show that if } \mu_n^k(A) = +\infty, \text{ then } \mu_m^k(A) = +\infty.$$

$$\text{To show this it suffices to show that if } \mu_m^k(A) < +\infty, \text{ then}$$

$$\mu_m^k(A) = \mu_n^k(A). \text{ But this follows from exactly the same type of}$$

argument as used above.

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