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A New Hexahedral Solid Element for 3D FEM Simulation of Sheet Metal Forming

**Dissertation
Presented in Partial Fulfillment of the Requirement for the
Degree Doctor of Philosophy in the Graduate School of
The Ohio State University**

By

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ABSTRACT

A new 8-node hexahedral mixed element was developed to facilitate the analysis of large strain, nonlinear, sheet form operations with material having complex elastic-plastic constitutive behavior. A new approach was followed to maintain the desired simplicity of formulation while suppressing locking and hourglassing. Analysis of the deformation field showed that the hourglassing modes were mutually orthogonal and thus contributions to strain could be identified individually. With this result in mind, shear and volumetric locking were avoided by ignoring, respectively, the shear and dilatational strain terms attributable to hourglass modes, thus neglecting the work of deformation associated with them. By this simple and direct method, the new element has the simplest formulation among all mixed hexahedral elements that found in the literature. The new element preserves the 8 integration points of the linear hexahedral element and there is no enhanced strain term needed.

As with all solid elements developed for coarse mesh applications with large problems and nonlinear behavior, the work of deformation of an element is not evaluated precisely. More further, the only way to assess the trade-off between large element performance and

fine-mesh convergence (a characteristic only of pure displacement elements, which are generally impractical for large nonlinear problems) is through numerical testing and comparison.

The new element has been tested using a variety of special problems as well as two large-scale forming processes. Five other elements were also tested and all the results compared. The overall performance of the new element is the best among the 6 tested elements, it does not suffer locking and implementation of constitutive equations is simple and rapid.

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CHAPTER 1 INTRODUCTION

1.1 Introduction to Finite Element Method

Finite Element Method (FEM) is basically a numerical method that utilizes digital computers to solve partial differential equations where classic analytical solution, if exists, is too complicated or impossible to obtain. The results of FEM are rarely exact. However, the error can be estimated and decreased systematically until the desired accuracy is reached. The procedure of FEM analysis includes two steps:

1. The targeted continuum is divided into a finite number of small parts, called elements. Each part has simple geometry and therefore is very easy to analyze. The behavior of each of these elements is defined by the governing equation of the concerned problem with a finite number of parameters. This process is called discretization.
2. The approximation of the solution is obtained by solving a set of algebraic equations, which are the assembly of equations from each element.

FEM originated from methods of stress analysis used in the aircraft industry in the early 1960s. The technique has developed rapidly because of a large number of devoted researchers and the fast evolution of the digital computer. Its use is no longer limited to the field of stress analysis. Its application has been extended into many other fields, such as heat transfer, fluid mechanics and electromagnetic analysis. In fact, FEM can be found in almost all the engineering disciplines because of its effectiveness and general applicability.

1.2 The application of FEM in sheet metal forming

Sheet metal forming is a process where metal sheet is pressed by tools and into dies to form a desired shape. It has extensive application in industry. Sheet forming processes are characterized by large displacement, large rotation of the blank relative to the tools, and generally small strain of the blank. The final shape of the blank is often irregular and complex. The traditional trial-and-error design method in sheet metal forming is expensive and very time consuming. As competition in the forming industry intensifies, this method has become a costly bottle-neck of the manufacturing process.

Industrial practice shows that numerical simulations of forming processes by FEM can not only save a lot of time and cost in process design, but also help to introduce new forming technologies and new materials. By using FEM, the factors of material properties, tooling geometry, process parameters, friction and others can be taken into

account as input data. The overall geometry evolution of the blank, the distribution of strain and stress can be obtained as results. Based on these simulation results, modifications can be implemented for an improved design.

1.3 Background and motivation of the study

Shell elements are often used in FEM simulation of sheet metal forming processes. Shell theory takes advantage of the fact that the dimension of thickness is much smaller than the dimensions of the other two directions. It is based on the plane stress assumption in which the stress normal to the blank surface is neglected because of its small magnitude compared to the in-plane stresses. Two shell theories that are widely used; Kirchhoff-Love and Mindlin-Reissner, differ in their kinematics:

1. Kirchhoff-Love shell theory: A plane that is straight and normal to the midsurface before loading is assumed to remain straight and normal to the midsurface after loading. Thus the transverse shear strain is assumed to be zero. This theory is often called thin shell theory and it requires C^1 continuity [29]. The corresponding beam theory is called Bernoulli-Euler beam theory [47].
2. Mindlin-Reissner shell theory: A plane that is straight and normal to the midsurface before loading is assumed to remain straight but not necessarily normal to the midsurface after loading. A constant transverse shear deformation is allowed. This theory is often called thick shell theory and it requires C^0 continuity[29]. The corresponding beam theory is called Timoshenko beam theory [47].

Shell elements have good performance in FEM simulation of sheet metal forming if the plane stress assumption and either of the shell assumptions are well satisfied. But in some cases, shell elements give very poor results. For example, in hydro forming where stress normal to the sheet surface is the driving force of deformation and cannot be neglected, the plane stress assumption is no longer satisfied. Another example, in draw-bend using a small roller where the ratio of roller radius over sheet thickness is less than 5, shear strain through thickness is not constant. In these cases, neither shell theory can be applied with good accuracy.

Under these conditions, 3D solid element analysis has to be used. Solid elements have many advantages. They allow the accurate evaluation of the contact forces; provide accurate description of the evolution of contact and thickness during the forming process; and allow two-sided contact which is suitable for the simulation of the ironing process.

Many solid elements have been developed in the past twenty years, a complete survey can be found in [52, 53]. These elements have good performance but they also have some severe shortcomings. For example, the solid element developed by Belytschko includes material properties, such as Poisson's ratio, in its element formulation [13]. The element called JET3D, developed by Kaiping Li and Cescotto [1], uses the enhanced strain method. Its strain components include extra terms which make it difficult and awkward to add more constitutive laws. The element called HEXDS, developed by W.K.Liu, uses extremely complicated formulations [7]. Under these circumstances, a solid element with

simple formulation and that can easily work with more constitutive relations is needed for the FEM simulation of sheet metal forming processes.

1.4 Outline of the dissertation

This dissertation presents a new 8-node hexahedral element, which is specially formulated for finite element analysis of thin-walled structure. The dissertation is divided into 6 chapters.

Chapter 1 gives an introduction and the background and motivation of this work.

Chapter 2 is a literature review. Basic FEM procedures used in sheet metal forming analysis are presented. Key equations are derived in detail and locking phenomena are discussed. Two typical solid elements, namely JET3D and HEXDS, are discussed. Implementation of constitutive relation and contact treatment are also briefly presented.

In Chapter 3, development of a new 8-node hexahedral element is presented. Special effort is spent on explaining the new approach to avoid the locking phenomena. The formulation for infinitesimal analysis is discussed first. Then, the idea is extended to nonlinear finite deformation analysis.

Chapter 4 is numerical verification and discussion of the new element. The simulation results are compared with analytical solutions. The performance of the new element is compared with other elements.

In chapter 5, an improved Whitley's model for deep drawing of an axisymmetric cup is introduced. The new model gives much better prediction on Limited Drawing Ratio (LDR) than the original model by involving some simple modifications.

The conclusions about the new element are drawn in chapter 6. Some possible future work is also mentioned.

CHAPTER 2 LITERATURE REVIEW

In this chapter, basic principles and equations used in nonlinear FEM analysis are described. Constitutive models and contact treatment used in SHEET-3 are briefly discussed. The formulation of two newly published hexahedral elements are also discussed at the end of this chapter.

2.1 Presentation of the problem of solid mechanics

Considering a continuous body in the rectangular Cartesian coordinates (x, y, z) that is under static equilibrium under its body force and prescribed boundary conditions, the governing equations can be summarized as follows:

Equations of equilibrium:

$$\sigma_{ij,j} + b_i = 0 \quad (2.1.1)$$

Strain and displacement relations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.1.2)$$

Stress-strain relations

$$\sigma_{ij} = D_{ijkl} \epsilon_{kl} \quad (2.1.3)$$

Mechanical (force) boundary conditions

$$T_i = \bar{t}_i \text{ on } S_t \quad (2.1.4)$$

$$\text{where } T_i = \sigma_{ij} n_j \quad (2.1.5)$$

Geometrical (displacement) boundary conditions

$$U_i = \bar{u}_i \text{ on } S_u \quad (2.1.6)$$

2.2 Variational principles

2.2.1 Principle of virtual work

Principle of virtual work is also often called virtual displacement principle. This principle states that the equilibrium of a body requires that for any compatible infinitesimal virtual displacements imposed on the body in its state of equilibrium, the total internal work is equal to the total external work. By compatible infinitesimal virtual displacements, we mean that the virtual displacements are zero at the part of the surface of the body where the displacements are prescribed and also the virtual displacements won't change the equilibrium state and the boundary conditions of the body. The principle can be expressed as [28]:

$$\int \delta \epsilon^T \sigma dV = \int \delta \mathbf{u}^T \mathbf{b} dV + \int_{\mathcal{S}^f} \delta \mathbf{u}^T \mathbf{f} dS \quad (2.2.1)$$

with two conditions

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (\text{Displacement-Strain relation}) \quad (2.2.2)$$

$$\mathbf{u}^{S^u} = \mathbf{u}_p \quad (\text{Prescribed displacement}) \quad (2.2.3)$$

where, $\mathbf{b}$ is the body force. $\mathcal{S}^f$ the part of the surface where surface traction $\mathbf{f}$ is prescribed. $\mathcal{S}^u$ is part of the surface where displacements are prescribed by $\mathbf{u}_p$.

The left hand side of the equation is the internal work done by internal stress σ on the virtual strain $\delta \epsilon$, which is caused the virtual displacement $\delta \mathbf{u}$. The right hand side of the equation is the external work. It has two components. The first term is the work done by body force $\mathbf{b}$ on the virtual displacement $\delta \mathbf{u}$. the second term is the work done by surface traction $\mathbf{f}$ on the virtual displacement $\delta \mathbf{u}$ [27].

The principle of virtual work is precisely another mathematical expression of the equilibrium conditions and it is suitable for finite element analysis because it is in weak form (integral form). The derivation of the virtual displacement principle starting from the equilibrium conditions can be found in reference [28, 30]. The principle of virtual

work can also be derived from the stationary condition of total potential energy Π under the two conditions, Eq.(2.2.2) and Eq.(2.2.3).

$$\begin{aligned}\Pi(\mathbf{u}) &= \frac{1}{2} \int_V \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon} dV - \int_V \mathbf{u}^T \mathbf{b} dV - \int_{S'} \mathbf{u}^T \mathbf{f}^{S'} dS \\ &= \text{stationary}\end{aligned}\quad (2.2.4)$$

where, $\mathbf{D}$ is the stress-strain relation.

2.2.2 Hu-Washizu variational principle

The Hu-Washizu principle is an extension of the virtual displacement principle. Imposing the two conditions, Eq.(2.2.2) and Eq.(2.2.3), into Eq.(2.2.4) by Lagrange multipliers, a very general and powerful variational principle is obtained:

$$\begin{aligned}\Pi^* &= \Pi - \int_V \boldsymbol{\lambda}_\varepsilon^T (\boldsymbol{\varepsilon} - \boldsymbol{\partial} \mathbf{u}) dV - \int_{S'} \boldsymbol{\lambda}_u^T (\mathbf{u}^{S'} - \mathbf{u}_p) dS \\ &= \text{stationary}\end{aligned}\quad (2.2.5)$$

where $\boldsymbol{\lambda}_\varepsilon$ and $\boldsymbol{\lambda}_u$ are Lagrange multipliers and S'' is the surface on which displacements are prescribed. By invoking $\delta \Pi^* = 0$, the Lagrange multipliers $\boldsymbol{\lambda}_\varepsilon$ and $\boldsymbol{\lambda}_u$ are identified, respectively, as the stress $\boldsymbol{\sigma}$ and the tractions over S'' , $\mathbf{f}^{S''}$. So Eq.(2.2.5) can be rewritten as

$$\Pi_{HW} = \Pi - \int_V \boldsymbol{\sigma}^T (\boldsymbol{\varepsilon} - \partial_{\boldsymbol{\varepsilon}} \mathbf{u}) dV - \int_{\mathcal{S}} f^{S''} (\mathbf{u}^{S''} - \mathbf{u}_p) dS \quad (2.2.6)$$

This function is referred to as the Hu-Washizu functional. In the above functional, the independent variables in this functional are the displacements $\mathbf{u}$, strains $\boldsymbol{\varepsilon}$, stresses $\boldsymbol{\sigma}$, and the surface $f^{S''}$ with no subsidiary conditions [27, 30].

From Hu-Washizu variational principle, the so-called assumed strain method can be systematically formulated. Simo and Hughes proposed to impose an orthogonality condition on the stress field $\boldsymbol{\sigma}$ [17],

$$\boldsymbol{\sigma}^T (\boldsymbol{\varepsilon} - \partial_{\boldsymbol{\varepsilon}} \mathbf{u}) = 0 \quad (2.2.7)$$

A simplified form of Hu-Washizu principle can be obtained

$$\Pi^*_{HW} = \Pi - \int_{\mathcal{S}} f^{S''} (\mathbf{u}^{S''} - \mathbf{u}_p) dS \quad (2.2.8)$$

2.3 Basic formulation of nonlinear finite element analysis

The updated Lagrangian method is used in this research. The finite element formulation adopts an approach that was initially proposed by McMeeking and Rice [42]. Jaumann's stress rate [39] is used to ensure objectivity. The derivation of equations uses the method

proposed by Crisfield [26] and Belytschko [47]. In this section, key finite element equations for large-strain analysis are developed. Focus is placed on internal force vector and the tangent stiffness matrix.

Internal Virtual Work

$$W_i = \int_0 \mathbf{S} : \delta \mathbf{E}_v dV_0 = \int \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon}_v dV = \int_0 \boldsymbol{\tau} : \delta \boldsymbol{\epsilon}_v dV_0 \quad (2.3.1)$$

Internal Virtual Power

$$\dot{W}_i = \frac{dW_i}{dt} = \frac{1}{dt} \int_0 \mathbf{S} : \delta \dot{\mathbf{E}}_v dV_0 = \int_0 \mathbf{S} : \dot{\mathbf{E}}_v dV_0 = \int \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}_v dV = \int_0 \boldsymbol{\tau} : \dot{\boldsymbol{\epsilon}}_v dV_0 \quad (2.3.2)$$

$\mathbf{S}$: 2nd Piola-Kirchhoff stress tensor

$\mathbf{E}$: Green-Lagrange strain tensor

$\boldsymbol{\sigma}$: Cauchy stress tensor (true stress)

$\boldsymbol{\tau}$: Kirchhoff stress tensor

Throughout this section, the increments (δ 's) and rates ($\dot{}$'s) are loosely switched, as it is convenient to do so.

$$\delta \dot{W}_i = \delta(\mathbf{q}_i^T \dot{\mathbf{P}}_v) = \int_0 \boldsymbol{\tau} : \delta \dot{\boldsymbol{\epsilon}}_v dV_0 + \int_0 \dot{\boldsymbol{\tau}} : \boldsymbol{\epsilon}_v dV_0 = I_1 + I_2 \quad (2.3.3)$$

$$\delta \dot{\boldsymbol{\epsilon}}_v = \frac{1}{2} \delta \left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}_v^T}{\partial \mathbf{x}} \right) = \frac{1}{2} \delta (\mathbf{L}_v + \mathbf{L}_v^T) \quad (2.3.4)$$

$$\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}} = \mathbf{L}_v = \frac{\partial \mathbf{u}_v}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}_v}{\partial \mathbf{X}} \mathbf{F}^{-1} \quad (2.3.5)$$

$\mathbf{X}$ is the original configuration and $\mathbf{x}$ is the current configuration. Because $\mathbf{X}$ is fixed,

$$\delta\left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) = \mathbf{0} = \delta\left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) \mathbf{F} + \left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) \delta \mathbf{F} = \delta\left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) \mathbf{F} + \left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) \dot{\mathbf{F}} \quad (2.3.6)$$

$$\delta\left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) = \delta \mathbf{L}_v = -\left(\frac{\partial \mathbf{u}_v}{\partial \mathbf{x}}\right) \dot{\mathbf{F}} \mathbf{F}^{-1} = -\mathbf{L}_v \mathbf{L} \quad (\dot{\mathbf{F}} = \mathbf{L} \mathbf{F}) \quad (2.3.7)$$

$$\delta \dot{\boldsymbol{\varepsilon}}_v = -\frac{1}{2} (\mathbf{L}_v \mathbf{L} + \mathbf{L}^T \mathbf{L}_v^T) \quad (2.3.8)$$

$$I_1 = \int_0 \boldsymbol{\tau} : \delta \dot{\boldsymbol{\varepsilon}}_v dV_0 = - \int_0 \boldsymbol{\tau} : \mathbf{L}_v \mathbf{L} dV_0 = - \int_0 \boldsymbol{\tau} : \frac{\partial \mathbf{u}_v}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} dV \quad (\boldsymbol{\tau} \text{ is symmetric}) \quad (2.3.9)$$

$$I_2 = \int_0 \dot{\boldsymbol{\tau}} : \dot{\boldsymbol{\varepsilon}}_v dV_0 \quad (2.3.10)$$

Using Jaumann's rate of Kirchhoff stress

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}_J + \boldsymbol{\Omega} \boldsymbol{\tau} + \boldsymbol{\tau} \dot{\boldsymbol{\Omega}}^T = \mathbf{C}_{JK} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\Omega} \boldsymbol{\tau} + \boldsymbol{\tau} \dot{\boldsymbol{\Omega}}^T \quad (2.3.11)$$

$\mathbf{C}_{JK}$ is the tangential constitutive modular matrix related Jaumann's rate of Kirchhoff stress.

$$\begin{aligned} I_2 &= \int_0 (\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + (\dot{\mathbf{\Omega}}\boldsymbol{\tau} + \boldsymbol{\tau}\dot{\mathbf{\Omega}}^T) : \dot{\mathbf{e}}_v dV_0 \\ &= \int_0 (\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + 2\dot{\mathbf{\Omega}}\boldsymbol{\tau} : \dot{\mathbf{e}}_v dV_0 \end{aligned} \quad (2.3.12)$$

because $\dot{\mathbf{\Omega}} = \mathbf{L} - \dot{\mathbf{e}}$

$$I_2 = \int_0 [(\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + 2\mathbf{L}\boldsymbol{\tau} : \dot{\mathbf{e}}_v - 2\dot{\mathbf{e}}\boldsymbol{\tau} : \dot{\mathbf{e}}_v] dV_0 \quad (2.3.13)$$

because $\mathbf{AB} : \mathbf{C}^T = \mathbf{CA} : \mathbf{B}^T = \mathbf{BC} : \mathbf{A}^T$

$$\begin{aligned} I_2 &= \int_0 [(\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + 2\dot{\mathbf{e}}_v\mathbf{L} : \boldsymbol{\tau} - 2\dot{\mathbf{e}}_v\dot{\mathbf{e}} : \boldsymbol{\tau}] dV_0 \\ &= \int_0 [(\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + \mathbf{L}_v\mathbf{L} : \boldsymbol{\tau} + \mathbf{L}_v^T\mathbf{L} : \boldsymbol{\tau} - 2\dot{\mathbf{e}}_v\dot{\mathbf{e}} : \boldsymbol{\tau}] dV_0 \end{aligned} \quad (2.3.14)$$

$$\begin{aligned} \delta\dot{W}_i &= I_1 + I_2 = \int_0 [(\mathbf{C}_{JK} : \dot{\mathbf{e}}) : \dot{\mathbf{e}}_v + \mathbf{L}_v^T\mathbf{L} : \boldsymbol{\tau} - 2\dot{\mathbf{e}}_v\dot{\mathbf{e}} : \boldsymbol{\tau}] dV_0 \\ &\equiv \dot{\mathbf{P}}_v^T [\mathbf{K}_{I1} + \mathbf{K}_{I\sigma_1} - \mathbf{K}_{I\sigma_2}] \dot{\mathbf{P}} \end{aligned} \quad (2.3.15)$$

“ $\equiv$ ” means approximation after finite element discretization.

Using Jaumann's rate of Cauchy stress

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_J + \dot{\mathbf{\Omega}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\mathbf{\Omega}}^T \quad (2.3.16)$$

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \Rightarrow \dot{\boldsymbol{\tau}} = J\dot{\boldsymbol{\sigma}} + \dot{J}\boldsymbol{\sigma} = J(\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}tr(\dot{\mathbf{e}})) \quad \text{because } \dot{J} = Jtr(\dot{\mathbf{e}}) \quad (2.3.17)$$

$$\dot{\boldsymbol{\tau}} = \mathbf{C}_{JK} : \dot{\mathbf{e}} + \dot{\mathbf{\Omega}}\boldsymbol{\tau} + \boldsymbol{\tau}\dot{\mathbf{\Omega}}^T = J(\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}tr(\dot{\mathbf{e}})) = J\mathbf{C}_{JC} : \dot{\mathbf{e}} + \dot{\mathbf{\Omega}}\boldsymbol{\tau} + \boldsymbol{\tau}\dot{\mathbf{\Omega}}^T + \boldsymbol{\tau}tr(\dot{\mathbf{e}}) \quad (2.3.18)$$

$$\text{so that } \mathbf{C}_{JK} : \dot{\mathbf{E}} = J(\mathbf{C}_{JC} : \dot{\mathbf{E}} + \boldsymbol{\sigma} \text{tr}(\dot{\mathbf{E}})) \quad (2.3.19)$$

$\mathbf{C}_{JC}$ is the tangential constitutive modular matrix related to Jaumann's rate of Cauchy stress.

$$\begin{aligned} \delta \dot{W}_i &= \int [(\mathbf{C}_{JC} : \dot{\mathbf{E}}) : \dot{\mathbf{E}}_v + \mathbf{L}_v^T \mathbf{L} : \boldsymbol{\sigma} - 2\dot{\mathbf{E}}_v \dot{\mathbf{E}} : \boldsymbol{\sigma} + \text{tr}(\dot{\mathbf{E}}) \boldsymbol{\sigma} : \dot{\mathbf{E}}_v] dV \\ &\equiv \dot{\mathbf{P}}_v^T [\mathbf{K}_{i1} + \mathbf{K}_{i\sigma_1} - \mathbf{K}_{i\sigma_2} + \mathbf{K}_{i\sigma_3}] \dot{\mathbf{P}} \end{aligned} \quad (2.3.20)$$

$$\begin{aligned} \mathbf{K}_{i1} &= \int \mathbf{B}^T \mathbf{C}_{JC} \mathbf{B} dV \\ \mathbf{K}_{i\sigma_1} &= \int \mathbf{G}^T \hat{\boldsymbol{\sigma}} \mathbf{G} dV \\ \mathbf{K}_{i\sigma_2} &= \int \mathbf{B}^T \tilde{\boldsymbol{\sigma}} \mathbf{B} dV \\ \mathbf{K}_{i\sigma_3} &= \int \mathbf{B}^T \tilde{\boldsymbol{\sigma}} \mathbf{B} dV \end{aligned} \quad (2.3.21)$$

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \bar{\boldsymbol{\sigma}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{\sigma}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{\sigma}} \end{bmatrix} \quad \text{where } \bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (2.3.22)$$

$$\mathbf{B} = [\mathbf{B}^1 \quad \mathbf{B}^2 \quad \mathbf{B}^3 \quad \mathbf{B}^4 \quad \mathbf{B}^5 \quad \mathbf{B}^6 \quad \mathbf{B}^7 \quad \mathbf{B}^8] \quad (2.3.23)$$

where

$$\mathbf{B}^\alpha = \begin{bmatrix} \frac{\partial N_\alpha}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_\alpha}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_\alpha}{\partial z} \\ \frac{\partial N_\alpha}{\partial y} & \frac{\partial N_\alpha}{\partial x} & 0 \\ \frac{\partial N_\alpha}{\partial z} & 0 & \frac{\partial N_\alpha}{\partial x} \\ 0 & \frac{\partial N_\alpha}{\partial z} & \frac{\partial N_\alpha}{\partial y} \end{bmatrix} \quad \text{where } \alpha = 1 \dots 8 \quad (2.3.24)$$

$$\tilde{\sigma} = \begin{bmatrix} 2\sigma_{11} & 0 & 0 & \sigma_{12} & \sigma_{13} & 0 \\ 0 & 2\sigma_{22} & 0 & \sigma_{21} & 0 & \sigma_{23} \\ 0 & 0 & 2\sigma_{33} & 0 & \sigma_{31} & \sigma_{32} \\ \sigma_{21} & \sigma_{12} & 0 & \frac{\sigma_{11} + \sigma_{22}}{2} & \frac{\sigma_{23}}{2} & \frac{\sigma_{13}}{2} \\ \sigma_{31} & 0 & \sigma_{13} & \frac{\sigma_{32}}{2} & \frac{\sigma_{11} + \sigma_{33}}{2} & \frac{\sigma_{12}}{2} \\ 0 & \sigma_{32} & \sigma_{23} & \frac{\sigma_{31}}{2} & \frac{\sigma_{21}}{2} & \frac{\sigma_{22} + \sigma_{33}}{2} \end{bmatrix} \quad (2.3.25)$$

$$\bar{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{11} & \sigma_{11} & 0 & 0 & 0 \\ \sigma_{22} & \sigma_{22} & \sigma_{22} & 0 & 0 & 0 \\ \sigma_{33} & \sigma_{33} & \sigma_{33} & 0 & 0 & 0 \\ \sigma_{12} & \sigma_{12} & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{13} & \sigma_{13} & \sigma_{13} & 0 & 0 & 0 \\ \sigma_{23} & \sigma_{23} & \sigma_{23} & 0 & 0 & 0 \end{bmatrix} \quad (2.3.26)$$

$$\mathbf{G} = [\mathbf{G}^1 \quad \mathbf{G}^2 \quad \mathbf{G}^3 \quad \mathbf{G}^4 \quad \mathbf{G}^5 \quad \mathbf{G}^6 \quad \mathbf{G}^7 \quad \mathbf{G}^8] \quad (2.3.27)$$

$$\mathbf{G}^\alpha = \begin{bmatrix} \frac{\partial N_\alpha}{\partial x} & 0 & 0 \\ \frac{\partial N_\alpha}{\partial y} & 0 & 0 \\ \frac{\partial N_\alpha}{\partial z} & 0 & 0 \\ 0 & \frac{\partial N_\alpha}{\partial x} & 0 \\ 0 & \frac{\partial N_\alpha}{\partial y} & 0 \\ 0 & \frac{\partial N_\alpha}{\partial z} & 0 \\ 0 & 0 & \frac{\partial N_\alpha}{\partial x} \\ 0 & 0 & \frac{\partial N_\alpha}{\partial y} \\ 0 & 0 & \frac{\partial N_\alpha}{\partial z} \end{bmatrix} \text{ where } \alpha = 1 \dots 8 \quad (2.3.28)$$

2.4 Constitutive models

2.4.1 Introduction

The elastic-plastic constitutive relation is introduced in stress calculation in an incremental manner because of the nature of the plastic flow rules. The procedure is based on the elastic-plastic splitting methodology in which the incremental strain is additively decomposed into an elastic part and a plastic part,

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \quad (2.4.1)$$

and the coupling between these two parts is neglected for simplicity [3, 18]. The procedure is divided into two steps. The first is the elastic prediction. In this step, the stresses are calculated from total strain increment by the elastic constitutive relation and there is no plastic response. The stresses obtained this way are called elastic trial stresses. If the elastic trial stresses are found lying within the yield surface, they are the true stresses that are sought. The behavior of the material at this point is elastic or the material is unloaded from the yield surface. However, if the elastic trial stresses lie outside the yield surface, the second step is activated. The second step is the plastic correction. In this step, the elastic trial stresses are returned on to an updated yield surface. The described procedure is schematically shown in Fig.(2.1) and can be written as

Fig. (2.1) Geometric interpretation of the generalized stress return algorithm

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbf{C}^e(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p) = \boldsymbol{\sigma}^{Tr} - \mathbf{C}^e \Delta\boldsymbol{\varepsilon}^p \quad (2.4.2)$$

$$\Delta\boldsymbol{\varepsilon}^p = \Delta\lambda [(1-\eta)\mathbf{a}_n + \eta\mathbf{a}_{n+1}] \quad (2.4.3)$$

$$f_n = f(\boldsymbol{\sigma}_n, K_n) = f_{n+1} = f(\boldsymbol{\sigma}_{n+1}, K_{n+1}) = 0; \quad f^{Tr} = f(\boldsymbol{\sigma}^{Tr}, K^{Tr}) > 0; \quad (2.4.4)$$

where, $\mathbf{C}^e$ is the elastic response matrix. f is the yield function and K is a scalar parameter representing the magnitude of the yield surface. Here, only isotropic hardening is considered. $\boldsymbol{\sigma}^{Tr} = \boldsymbol{\sigma}_n + \mathbf{C}^e \Delta\boldsymbol{\varepsilon}$ is the elastic trial stress. $\Delta\lambda = \int \dot{\lambda} dt$ is the increment of the plastic strain-rate multiplier which is defined by

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \quad (2.4.5)$$

$\boldsymbol{\sigma}_{n+1}$ and $\boldsymbol{\sigma}_n$ are the equilibrium stresses at the end of step n and step $n+1$, respectively.

$\mathbf{a}_n$ and $\mathbf{a}_{n+1}$ are flow vectors defined by

$$\mathbf{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \quad (2.4.6)$$

for step n and step $n+1$, respectively.

For $\eta = 0$, the above equations become the explicit, the so-called forward-Euler algorithm, which is numerically conditionally stable.

For $0 < \eta < 1$, we obtain the generalized trapezoidal rule.

For $\eta = 1$, the above equations become the backward-Euler algorithm which is total implicit. Eqs.(2.4.2) and (2.4.3) become,

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{Tr} - \Delta\lambda \mathbf{C}^e \mathbf{a}_{n+1} \quad (2.4.7)$$

$$\text{with } \Delta\boldsymbol{\epsilon}^p = \Delta\lambda \mathbf{a}_{n+1} \quad (2.4.8)$$

The backward-Euler algorithm is numerically unconditionally stable [31]. It has been widely used in the elastoplastic analysis because of its accuracy and good convergence. Two algorithms based on it will be discussed in this chapter. One is the *tangent cutting plane* algorithm which is used in JET3D and the other is the *radial return* algorithm which is used in HEXDS.

2.4.2 The tangent cutting plane algorithm

The tangent cutting plane algorithm was proposed by Simo and Ortiz [18]. In this algorithm, the elastic stress predictor $\boldsymbol{\sigma}^{Tr}$ is relaxed to the updated yield surface in successive iterates. At each iterate, the $(i+1)th$ updated stresses of step $n+1$ $\boldsymbol{\sigma}_{n+1}^{(i+1)}$ are

$$\Delta \boldsymbol{\sigma}^{(i)} = - (\Delta \lambda \mathbf{C}^e \mathbf{a})^{(i)} \quad (2.4.10)$$

and increment of K is given by

$$\Delta K^{(i)} = (\Delta \lambda B)^{(i)} \quad (2.4.11)$$

where B is a function of stresses defined below,

$$\Delta \epsilon_{eq} = B(\boldsymbol{\sigma}) \Delta \lambda \quad (2.4.12)$$

$\Delta \epsilon_{eq}$ is the increment of equivalent strain. $B(\boldsymbol{\sigma}) = 1$ for many functions.

Substituting Eqs.(2.4.10) and (2.4.11) into Eq.(2.4.9), the expression of $\Delta \lambda^{(i)}$ can be obtained,

$$\Delta \lambda^{(i)} = \frac{f^{(i)}}{\left(\mathbf{a}^{(i)} \right)^T \mathbf{C}^e \mathbf{a}^{(i)} - \left(\frac{\partial f}{\partial K} \right)^{(i)} B^{(i)}} \quad (2.4.13)$$

Substitution of the value of $\Delta \lambda^{(i)}$ into Eqs.(2.4.10) and (2.4.11) results the updated stresses and K at the $(i+1)th$ iteration,

$$\sigma^{(i+1)} = \sigma^{(i)} + \Delta\sigma^{(i)} \quad (2.4.14)$$

$$\text{and } K^{(i+1)} = K^{(i)} + \Delta K^{(i)} \quad (2.4.15)$$

The above procedure iterates successively until the plastic consistency is satisfied within a prescribed tolerance, that is,

$$f(\sigma^{(i+1)}, K^{(i+1)}) \leq \text{Tolerance} \quad (2.4.16)$$

From the above description we can see that this algorithm is very simple and this is indeed the main purpose it is proposed. The convergence of this algorithm toward the final value of the state variables is obtained at a quadratic rate [18, 25].

2.4.3 The radial return algorithm

For the general 3D case, the von Mises yield function is

$$\begin{aligned} f &= \sigma_e - \sigma_0 = \sqrt{3}J_2^{1/2} - \sigma_0 \\ &= \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} - \sigma_0 \\ &= \sqrt{3} \left[\frac{1}{2} (s_x^2 + s_y^2 + s_z^2) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right]^{1/2} - \sigma_0 \\ &= \sqrt{\frac{3}{2}} (s^T L s)^{1/2} - \sigma_0 \end{aligned}$$

$$(2.4.17)$$

where, σ_e is the equivalent stress and σ_0 is the yield stress. $J_2 = 1/2 s_{ij}s_{ij}$. s is the deviatoric stress and L is defined by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (2.4.18)$$

Differentiating Eq.(2.4.17) gives,

$$\begin{aligned} a^T &= \frac{\partial f^T}{\partial \sigma} = \frac{1}{2\sigma_e} \{ (2\sigma_x - \sigma_y - \sigma_z), (2\sigma_y - \sigma_x - \sigma_z), (2\sigma_z - \sigma_y - \sigma_x), 6\tau_{xy}, 6\tau_{yz}, 6\tau_{zx} \} \\ &= \frac{3}{2\sigma_e} \{ s_x, s_y, s_z, 2\tau_{xy}, 2\tau_{yz}, 2\tau_{zx} \} = \frac{3}{2\sigma_e} (Ls)^T = \frac{\partial f^T}{\partial s} \end{aligned} \quad (2.4.19)$$

Using the above equation, we can obtain

$$C^e a = \frac{\sqrt{3}\mu}{\sqrt{J_2}} s = \frac{3\mu}{\sigma_e} s \quad (2.4.20)$$

where μ is shear modulus and

$$\mathbf{a}^T \mathbf{C}^e \mathbf{a} = 3\mu \quad (2.4.21)$$

Substitute Eq.(2.4.21) into Eq.(2.4.13) gives

$$\Delta\lambda_n = \frac{f_n}{3\mu - \left(\frac{\partial f}{\partial K}\right)_n B_n} \quad (2.4.22)$$

If stresses are splitted into deviatoric and volumetric components, Eq.(2.4.7) can be rewritten as,

$$\sigma_{n+1} = (\sigma_m)^{Tr} \mathbf{I} + \mathbf{s}^{Tr} - \Delta\lambda \mathbf{C}^e \mathbf{a}^{Tr} \quad (2.4.23)$$

where $\mathbf{I}$ is the unit matrix. Substituting Eq.(2.4.22) into Eq.(2.4.23) we obtain the updated stresses. Since $\mathbf{s}^{Tr}$ is orthogonal to $\mathbf{I}$, we can get

$$(\sigma_m)_{n+1} = (\sigma_m)^{Tr} \quad (2.4.24)$$

From above two equations, we can also get

$$\mathbf{s}_{n+1} = \boldsymbol{\sigma}^{Tr} = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_e^{Tr}}\right) \mathbf{s}^{Tr} \quad (2.4.25)$$

From this equation, we see that the final deviatoric stresses has the same direction as s^{Tr} and is also proportional to s^{Tr} . That is why this algorithm is called radial return, as shown in Fig.(2.3). From the formulations presented above, we can find that the radial return algorithm is a special form of backward-Euler algorithm. For this algorithm, iterations are not required for linear hardening. For nonlinear hardening, the return is still radial but a scalar Newton-Raphson iteration is needed to calculate $\frac{d}{dK}$. However, this algorithm is still computationally efficient.

Fig. (2.3) Geometric interpretation of the radial return algorithm

2.4.4 Backward-Euler return algorithm with Hill's 1948 anisotropic yield criterion

2.4.4.1 Hill's 1948 Anisotropic Yield Criterion

Here Hill's 1948 anisotropic yield criterion is written in square root form:

$$f = \sigma_e - \sigma_0 = \left(\frac{1}{2} \boldsymbol{\sigma}^T \mathbf{P} \boldsymbol{\sigma} \right)^{1/2} - \sigma_0 = 0 \quad (2.4.26)$$

where

$$\mathbf{P} = \begin{bmatrix} \alpha_{12} + \alpha_{31} & -\alpha_{12} & -\alpha_{31} & 0 & 0 & 0 \\ -\alpha_{12} & \alpha_{12} + \alpha_{23} & -\alpha_{23} & 0 & 0 & 0 \\ -\alpha_{31} & -\alpha_{23} & \alpha_{31} + \alpha_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\alpha_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 6\alpha_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 6\alpha_{66} \end{bmatrix} \quad (2.4.27)$$

Coefficients α_{ij} can be determined by experiments [3].

2.4.4.2 Backward-Euler Return Algorithm [26]

Step 1 Elastic predictor:

$$\boldsymbol{\sigma}^{Trial} = \boldsymbol{\sigma}_A + \mathbf{C}^e \Delta \boldsymbol{\varepsilon}$$

where, $\mathbf{C}^e$ is the elastic response matrix (Hooke's law).

Step 2 Check for yielding:

$$f(\boldsymbol{\sigma}^{Trial}, K_A) \leq 0?$$

where, K_A is the hardening parameter, such as effective plastic strain.

Yes, pure elastic deformation.

Set $\sigma_B = \sigma^{Trial}$; $K_B = K_A$; exit

No, yield happens, elastic-plastic deformation.

Set $i = 0$; $\sigma_0 = \sigma^{Trial}$; $K_0 = K_A$; $\Delta\lambda_0 = 0$

Step 3 Plastic corrector: (Newton-Raphson linearlization)

$$f_{i+1} = f_i + \dot{\sigma}_e^i - \dot{\sigma}_0^i = 0$$

$$\dot{\sigma}_e^i = - \frac{H \dot{\lambda}^i}{4\sigma_e^{i^2} - 2H\Delta\lambda_2^i}$$

$$\dot{\sigma}_0^i = A_i' \dot{\lambda}^i$$

$$\dot{\lambda}^{i+1} = \frac{f_i}{\frac{H}{4\sigma_e^{i^2} - 2H\Delta\lambda_2^i} + A_i'}$$

where, A_i' is the slope of the hardening curve

$$H = \sigma^{iT} \mathbf{P} \mathbf{B}_i^{-1} \mathbf{C}^T \mathbf{P} \sigma^i$$

$$\mathbf{B}_i = \mathbf{I} + \Delta\lambda_2^i \mathbf{C}^T \mathbf{P}$$

$$\Delta\lambda_2^i = \frac{\Delta\lambda^i}{2\sigma_e^i}$$

$$\sigma^{i+1} = \mathbf{B}_i^{-1} \sigma^i$$

$$\Delta\lambda^{i+1} = \Delta\lambda^i + \dot{\lambda}^{i+1}$$

$$K_{i+1} = K_i + \dot{\lambda}^{i+1}$$

Step 4 Convergence check:

$$f(\boldsymbol{\sigma}^{i+1}, K_{i+1}) \leq \text{Tolerance} ?$$

Yes, set $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}^{i+1}; K_B = K_{i+1}$; exit

No, set $i = i + 1$; goto step 3.

2.4.4.3 The Consistent Tangent Modulus

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{Trial} - \Delta\lambda \mathbf{C}^e \mathbf{a} \quad (2.4.28)$$

$$\text{where, } \mathbf{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{2\sigma_e} \mathbf{P} \boldsymbol{\sigma} \quad (2.4.29)$$

Linearize the above equation, we have

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^e \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \mathbf{C}^e \mathbf{a} - \Delta\lambda \mathbf{C}^e \frac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}} \quad (2.4.30)$$

$$\dot{\boldsymbol{\sigma}} = (\mathbf{I} + \Delta\lambda \mathbf{C}^e \frac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}})^{-1} \mathbf{C}^e (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \mathbf{a}) = \mathbf{R}(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \mathbf{a}) \quad (2.4.31)$$

Apply the consistency condition:

$$\dot{f} = \mathbf{a}^T \dot{\boldsymbol{\sigma}} - A' \dot{\lambda} = \mathbf{a}^T \mathbf{R} \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \mathbf{a}^T \mathbf{R} \mathbf{a} - A' \dot{\lambda} = 0 \quad (2.4.32)$$

From above, we can obtain the consistent tangent modular matrix $\mathbf{C}_{ep}$:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}_{ep} \dot{\boldsymbol{\varepsilon}} = [\mathbf{R} - \frac{\mathbf{R} \mathbf{a} \mathbf{a}^T \mathbf{R}}{\mathbf{a}^T \mathbf{R} \mathbf{a} + A'}] \dot{\boldsymbol{\varepsilon}} \quad (2.4.33)$$

2.5 Constraint – Contact treatment

The N-CFS (Normal Consistent Full Set) algorithm is an implicit finite element analysis method adopted in SHEET-3. CFS was originally developed by Saran and Wagoner [21]. After that, Y.Keum, D.Zhou and Wagoner[20, 22, 23] developed N-CFS by improving CFS. This algorithm is based on the mesh normal and using a consistent full set of equations, including both force equilibrium relations and the contact condition in the nonlinear set. This algorithm exhibits remarkable stability and strong convergence.

2.5.1 Contact and friction conditions

Contact nodes are classified as being in die contact or punch contact. The external contact nodal force due to frictional contact can be divided into two parts, the normal part and the tangential part,

$$\mathbf{F}_E = \mathbf{F}_n + \mathbf{F}_t \quad (2.5.1)$$

The normal part, along the mesh normal vector $\mathbf{n}$, is the reaction force pushing against the node to prevent its penetration into the tool. The tangential part depends on friction between tool surface and node. So, the tangential force always acts along the opposite direction of relative nodal displacement. Using the Coulomb friction law and normal contact force, $\mathbf{P}$, which is an additional primary variable, Eq.(2.5.1) can be rewritten

$$\mathbf{F}_E = (\mathbf{n} - \mu \mathbf{t}) \mathbf{P} \quad (2.5.2)$$

where $\mathbf{n}$ and $\mathbf{t}$ are the normal and tangential incremental displacement unit vectors, respectively. $\mathbf{t}$ is defined to perpendicular to $\mathbf{n}$ and the tangential increment displacement $\Delta \mathbf{u}_t$,

$$\mathbf{t} = \frac{[\Delta \mathbf{u} \times \mathbf{n}] \times \mathbf{n}}{\|[\Delta \mathbf{u} \times \mathbf{n}] \times \mathbf{n}\|} \quad (2.5.3)$$

For Coulomb's friction law, ever-slipping friction condition by a special function with a continuous first derivative was introduced by Saran and Wagoner [21].

$$\Phi(|\Delta \mathbf{u}_t|) = \tanh\left(\frac{\beta |\Delta \mathbf{u}_t|}{\delta}\right) \quad (2.5.4)$$

Thus the external contact, can be written as follows:

$$\mathbf{F}_E = (\mathbf{n} - \mu \tanh\left(\frac{\beta |\Delta \mathbf{u}_t|}{\delta}\right) \mathbf{t}) \mathbf{P} \quad (2.5.5)$$

At the k th node, we can write the equilibrium equation to emphasize the independent variables:

$$\mathbf{F}_I^k(\Delta \mathbf{u}) = \mathbf{F}_E^k(\Delta \mathbf{u}, \mathbf{P}) \quad (2.5.6)$$

where $\mathbf{F}_i^k$ is the internal force of the k th node.

The N-CFS algorithm requires that the contact nodes lie precisely on the tool surface,

$$\mathbf{D}_n = \pm |\mathbf{X}_{tool} - \mathbf{X}_{node}| = 0 \quad (2.5.7)$$

where $\mathbf{D}_n$ is a contact error defined as a gap measured between the node and tool surface along a specified direction represented $\mathbf{X}_{node} - \mathbf{X}_{tool} \cdot \mathbf{D}_n$ is a signed value such that when $\mathbf{D}_n > 0$, a node is outside tool surface and $\mathbf{D}_n < 0$ represents a penetrating node. In N-CFS algorithm, $\mathbf{X}_{node} - \mathbf{X}_{tool}$ is chosen parallel to the mesh normal of the considered node, as shown in Fig.(2.4).

Fig.(2.4) Definition of $\mathbf{D}_n$:

2.5.2 The N-CFS algorithm

In the N-CFS algorithm, a full set of governing equations at each time step includes both the force equilibrium, Eq.(2.5.6), and the geometric conditions, Eq.(2.5.7), for all nodes,

$$\mathbf{F}_I(\Delta \mathbf{u}) - \mathbf{F}_E(\Delta \mathbf{u}, \mathbf{P}) = 0 \quad (2.5.8)$$

$$\mathbf{D}_n(\Delta \mathbf{u}) = 0 \quad (2.5.9)$$

For non-contact nodes, both $\mathbf{P}$ and $\mathbf{D}_n$ are zero. $\mathbf{P}$ is the nodal contact force which is an additional unknown variable and $\mathbf{D}_n$ is the nodal contact error defined in Eq.(2.5.7). Since these equations are nonlinear, we have to solve them by iterative manner such as Newton-Raphson method. After Taylor expansion with truncation of higher order terms, Eq.(2.5.8) and Eq.(2.5.9) can be written as below [32]:

$$\mathbf{F}_I^{i-1} + \left. \frac{\partial \mathbf{F}_I}{\partial \Delta \mathbf{u}} \right|^{i-1} \delta \Delta \mathbf{u} - \left(\mathbf{F}_E^{i-1} + \left. \frac{\partial \mathbf{F}_E}{\partial \Delta \mathbf{u}} \right|^{i-1} \delta \Delta \mathbf{u} + \left. \frac{\partial \mathbf{F}_E}{\partial \mathbf{P}} \right|^{i-1} \delta \mathbf{P} \right) = 0 \quad (2.5.10)$$

$$\mathbf{D}_n^{i-1} + \left. \frac{\partial \mathbf{D}_n}{\partial \Delta \mathbf{u}} \right|_{i-1} \delta \Delta \mathbf{u} = 0 \quad (2.5.11)$$

Rearranging Eq.(2.5.10) and Eq.(2.5.11)

$$\begin{bmatrix} \frac{\partial F_l}{\partial \Delta u} - \frac{\partial F_E}{\partial \Delta u} & -\frac{\partial F_E}{\partial P} \\ \frac{\partial D_n}{\partial \Delta u} & 0 \end{bmatrix}^{i-1} \begin{bmatrix} \delta \Delta u \\ \delta P \end{bmatrix}^i = \begin{bmatrix} F_E \\ -D_n \end{bmatrix}^{i-1} - \begin{bmatrix} F_l \\ 0 \end{bmatrix}^{i-1} \quad (2.5.12)$$

Finally we can write the linearized system of equations in matrix form for one node as follow:

$$\begin{bmatrix} K_l - K_E & N \\ C_n^T & 0 \end{bmatrix}^{i-1} \begin{bmatrix} \delta \Delta u \\ \delta P \end{bmatrix}^i = \begin{bmatrix} F_E \\ -D_n \end{bmatrix}^{i-1} - \begin{bmatrix} F_l \\ 0 \end{bmatrix}^{i-1} \quad (2.5.13)$$

where,

$$K_l = \frac{\partial F_l}{\partial \Delta u}, K_E = \frac{\partial F_E}{\partial \Delta u}, N = -\frac{\partial F_E}{\partial P}, C_n = \frac{\partial D_n}{\partial \Delta u} \quad (2.5.14)$$

The major difference of N-CFS algorithm with the Lagrange multiplier method and the penalty method comes from the starting point of deriving the equations. This results in, as can be seen in Eq.(2.5.13), the different stiffness matrix and force vector because of the special frictional contact treatment. All of them have some advantages and disadvantages. N-CFS and Lagrange multiplier method satisfy constraint equation accurately at the end of equilibrium iterations, but additional computational effort should be required of unknown variables P or λ s. On the other hand, penalty method needs no additional d.o.f. for the contact treatment but it gives only approximate solution

depending on the choice of penalty parameter. Numerical tests have shown that the N-CFS algorithm has strong stability to overcome the difficulties caused by the changing of contact and friction conditions during sheet metal forming operations. It can also improve the efficiency of the computation.

2.5.3 Comments on The N-CFS algorithm used in SHEET-3

Contact treatment is very critical for current nonlinear FEM because a lot of the computation time is spent on contact treatment for complicated geometry and contact is the most important source of instability.

Using N-CFS algorithm involves a lot of effort to calculate the mesh normal and contact search along the normal, as shown Eqn.(2.5.12). The current implementation of N-CFS algorithm in SHEET-3 is not optimal because it does not take the advantage of features of elements. For example, if a shell element is used, the mesh normal of a node can be easily obtained because the rotation of the node is a d.o.f. of the element, just like displacement. Even for solid elements, simpler and more efficient methods of mesh normal calculation can be developed without little effort. Current implementation of normal calculation is based on triangle membrane element. If a quadrilateral element is used, such as JET shell in SHEET-3, the normal of a node is calculated by splitting the quadrilateral into 4 triangles and averaging the weighted values.

The contact search and check procedures should be consistent. Both procedures should be performed strictly along mesh normal direction. In the current version of SHEET-3, the contact search is along mesh normal direction, while the contact check is along the Z axis or punch motion direction. Along mesh normal direction the contact gap or penetration maybe very small, but if measured along punch move direction the gap or penetration can be much larger. This inconsistency can consume a lot of computation time. Also, in the future, adaptivity can be introduced into N-CFS algorithm to deal with complex geometry and to improve stability and efficiency.

2.6 Previous Works on Solid Elements

2.6.1 Locking phenomena

2.6.1.1 Shear Locking

Shear locking is a numerical phenomenon. In the finite element analysis of a thin structure which is subjected to bending-dominant deformation, if coarse mesh and low order element, such as the bilinear quadrilateral element, are used, excessive shear strain energy is obtained at the integration points. For example, a thin long beam deforms by an end bending moment M , as shown in Fig.(2.5).

Fig.(2.5) A beam subjected to end bending

In Fig.(2.6a), the solid lines show the deformed shape and the dotted lines show the original shape. The deformed shape is exaggerated for clarity. Distance d is the actual end tip displacement of the beam after deformation. In finite element analysis, the beam is divided into elements as shown in Fig.(2.6b) in dotted lines. In this case only 4 quadrilateral elements are used. The deformed shape is shown by solid lines. The end tip displacement d' can be calculated from standard finite element procedure. Comparing d and d' , we find $d' \ll d$. This phenomenon is called shear locking because the finite element mesh looks appear to be locked and thus its displacement is much smaller than it should be.

Fig.(2.6) Shear locking phenomenon

The cause of shear locking is that excessive shear strain energy is sampled at the integration points which are used to evaluate the element stiffness matrix. The problem of shear locking can be solved by using very fine mesh. But this will significantly increase the computation time, an impractical approach for large nonlinear problems such as sheet metal forming.

2.6.1.2 Volumetric locking

Volumetric locking is a numerical problem that often happens in finite element analysis.

As Poisson's ratio ν approaches 0.5, a material becomes elastically incompressible.

Values of ν near 0.5 occur in rubber-like materials and in materials that flow, such as fluids and plastic solids. The bulk modulus κ of an isotropic material is

$$\kappa = \frac{E}{3(1 - 2\nu)} \quad (2.6.1)$$

where E is the Young's modulus.

As $\nu \rightarrow 0.5$, $\kappa \rightarrow \infty$ and the volumetric strain $\epsilon_v \rightarrow 0$, and finally the mesh locks. Besides locking, there are other numerical troubles. ϵ_v is determined from the derivatives of displacement, which itself can't be predicted accurately because of the locking. So locking causes error in ϵ_v . Since ϵ_v is very small, any error in the predicted ϵ_v will appear

as a large error in the stresses, and this in turn also affects the displacement prediction because the external loads are balanced by the internal stresses in finite element analysis. Volumetric locking can be solved by selective reduced integration which means that the volumetric part of the strain is calculated by reduced integration, which will be explained in next section, and the deviatoric part of the strain is normally integrated [11].

2.6.2 Reduced integration and hourglass

Reduced integration means that the Gauss-Legendre integration rule [37] with reduced order is used to evaluate the element stiffness matrix. For example, in two dimensions, normal Gauss-Legendre rule requires 4 integration points for quadrilateral element, while in three dimensions, it requires 8 integration points for hexahedral element. With the reduced integration, only one integration point is used for both cases. Reduced integration provides tremendous benefits in non-linear finite element analysis because the amount of calculation used for evaluation element stiffness matrix is reduced substantially. Studies suggest that the rate of convergence of element using reduced integration is comparable to that of the normally integrated element.

The major drawback of the reduced integration is hourglass or often called mesh instability. Take a quadrilateral element subjected to pure bending as an example, as shown in Fig.(2.6.3). The dotted lines show the original shape of the element and the solid lines show the deformed shape. The stress in the element changes linearly as shown with arrows. With the reduced integration, only one integration point, which is at the

center of the element, is used to sample the element. As shown in Fig.(2.7), the stress at the center of the element is zero and thus no strain energy is obtained by the reduced integration. Once this spurious zero-energy mode happens it will spread quickly in the mesh and destroy the whole calculation as show in Fig.(2.8).

Fig.(2.8) shows an example problem. The problem is a simply supported beam subjected to a uniform pressure at the top. Because of symmetry, only half of the structure is used for analysis. Fig.(2.8) also shows the deformed configuration at different stages without any hourglass control. It can be seen that the hourglassing destroys the whole mesh.

Fig.(2.7) Hourglass phenomenon

Fig.(2.8) Hourglass phenomenon [47]

There are two ways to control hourglassing. One method is to add a correction matrix to the element stiffness matrix obtained by reduced integration as suggested by Koh and Kikuchi [2]. The correction matrix is constructed by first identifying the kinematic modes that cause hourglass and then using special treatment to stabilize them. The JET3D element in next section uses this technique to control hourglass and the second term of the right hand side of Eq.(2.6.22) is the correction matrix. The second method is to expand the stresses into Taylor series about the element centroid and then keep only the first and second derivative terms as suggested by Schulz [10]. The HEXDS element uses a similar method to control hourglass, as discussed in section (2.6.4). The difference is that it expands strains into Taylor series instead of stresses. The shortcoming of both elements will be discussed in section (2.6.5).

2.6.3 JET3D element

JET3D is a mixed 8-node hexahedral element developed by K.P.Li and S.Cescotto [1, 3] based on a quadrilateral element called JET2D proposed by P.Jetteur and S.Cescotto [5]. Both elements use reduced integration to overcome locking problems and hourglass control technique to prevent hourglass. In this section, the formulation of JET3D will be briefly discussed.

In JET3D, the symmetric part of the displacement gradient is divided into constant part and non-constant part, Eq.(2.6.2). The constant part represents the displacement gradient at the center of the element. The evaluation of the non-constant part of the displacement gradient is the origin of the locking and hourglass problems.

$$\nabla_s \mathbf{u} = \nabla_s \mathbf{u}^0 + \nabla_s \mathbf{u}^h \quad (2.6.2)$$

For constant part,

$$\nabla_s \mathbf{u}^0 = \bar{\mathbf{B}} \mathbf{U} \quad (2.6.3)$$

and for non-constant part,

$$\nabla_s \mathbf{u}^h = \mathbf{B}^h \mathbf{U} = \mathbf{h} \mathbf{\Gamma} \mathbf{U} = \mathbf{h} \boldsymbol{\varepsilon}^* \quad (2.6.4)$$

where $\boldsymbol{\varepsilon}^*$ is called the generalized hourglass strains and $\mathbf{h}$ is the interpolation function of $\boldsymbol{\varepsilon}^*$ in the element. $\mathbf{\Gamma}$ is the stabilization matrix composed of γ_k ($k = 1, 2, 3, 4$), Eqn.(3.2.6).

$\bar{\mathbf{B}}$ and $\mathbf{B}^h$ are the strain-displacement matrix for the constant part and non-constant part of $\nabla_s \mathbf{u}$, respectively. $\mathbf{U}$ is the displacement vector. Thus, Eq. (2.6.2) can be rewritten as,

$$\nabla_s \mathbf{u} = (\bar{\mathbf{B}} + \mathbf{B}^h) \mathbf{U} \quad (2.6.5)$$

To overcome locking and hourglass problems, the $\mathbf{h}$ matrix in $\nabla_s \mathbf{u}^h$ can be modified as the following,

$$\mathbf{h}' = \begin{bmatrix} \frac{2}{3}h_{1,1} & -\frac{1}{3}h_{1,2} & -\frac{1}{3}h_{1,3} & \cdots & \frac{2}{3}h_{4,1} & -\frac{1}{3}h_{4,2} & -\frac{1}{3}h_{4,3} \\ -\frac{1}{3}h_{1,1} & \frac{2}{3}h_{1,2} & -\frac{1}{3}h_{1,3} & \cdots & -\frac{1}{3}h_{4,1} & \frac{2}{3}h_{4,2} & -\frac{1}{3}h_{4,3} \\ -\frac{1}{3}h_{1,1} & -\frac{1}{3}h_{1,2} & \frac{2}{3}h_{1,3} & \cdots & -\frac{1}{3}h_{4,1} & -\frac{1}{3}h_{4,2} & \frac{2}{3}h_{4,3} \\ 0 & \beta_{23}h_{1,3} & \beta_{23}h_{1,2} & \cdots & 0 & \beta_{23}h_{4,3} & \beta_{23}h_{4,2} \\ \beta_{13}h_{1,3} & 0 & \beta_{23}h_{1,1} & \cdots & \beta_{13}h_{4,3} & 0 & \beta_{13}h_{4,1} \\ \beta_{12}h_{1,2} & \beta_{12}h_{1,1} & 0 & \cdots & \beta_{12}h_{4,2} & \beta_{12}h_{4,1} & 0 \end{bmatrix} \quad (2.6.6)$$

The $\frac{2}{3}$ and $-\frac{1}{3}$ are used to prevent volumetric locking by eliminating the dilatational part of $\nabla_s \mathbf{u}^h$. The β s are the parameters used to control shear locking. Their values need to be determined to reduce or eliminate the shear associate with bending. The authors of JET3D suggested several ways to calculate them which are related to the geometry and aspect ratio of the element [1, 3]. The method used in this study is as shown in Eq. (2.6.7)

$$\beta_{12} = \frac{\min\langle x_{mean}, y_{mean} \rangle}{\max\langle x_{mean}, y_{mean} \rangle}; \beta_{13} = \frac{\min\langle x_{mean}, z_{mean} \rangle}{\max\langle x_{mean}, z_{mean} \rangle}; \beta_{23} = \frac{\min\langle y_{mean}, z_{mean} \rangle}{\max\langle y_{mean}, z_{mean} \rangle} \quad (2.6.7)$$

with

$$x_{mean} = 1/4(|x_{17}| + |x_{28}| + |x_{35}| + |x_{46}|)$$

$$y_{mean} = 1/4(|y_{17}| + |y_{28}| + |y_{35}| + |y_{46}|)$$

$$z_{mean} = 1/4(|z_{17}| + |z_{28}| + |z_{35}| + |z_{46}|)$$

where $|x_{ij}| = |x_i - x_j|$; $|y_{ij}| = |y_i - y_j|$; $|z_{ij}| = |z_i - z_j|$ and i, j are the node numbers.

By the similar procedure, the strain field can also be divided into two parts: the constant part ε^0 and the non-constant part ε^h , respectively,

$$\varepsilon = \varepsilon^0 + \varepsilon^h \quad (2.6.8)$$

Assume the two parts of strain field equal to the two parts of symmetric part of the displacement gradient, respectively

$$\varepsilon^0 = \nabla_s u^0 = \overline{BU} \quad (2.6.9)$$

$$\varepsilon^h = h' \varepsilon \quad (2.6.10)$$

Using the above two equations, the discretized strain field can be written as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \mathbf{h}' \boldsymbol{\varepsilon}^* \quad (2.6.11)$$

By such a choice of the strain field, the locking problem can be avoided.

Similarly, the stress field can be divided into two parts: the constant part $\boldsymbol{\sigma}^0$ and the non-constant part $\boldsymbol{\sigma}^h$.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^h \quad (2.6.12)$$

and

$$\boldsymbol{\sigma}^0 = \mathbf{C}^\varepsilon \boldsymbol{\varepsilon}^0 \quad (2.6.13)$$

$$\boldsymbol{\sigma}^h = \mathbf{C}^\varepsilon \mathbf{h}' \boldsymbol{\varepsilon}^* = \mathbf{h}_\sigma \boldsymbol{\sigma}^* \quad (2.6.14)$$

where $\boldsymbol{\sigma}^*$ is called the generalized hourglass stresses and $\mathbf{h}_\sigma$ is the interpolation function of $\boldsymbol{\sigma}^*$ in the element. $\mathbf{C}^\varepsilon$ is linear elastic constitutive matrix. Finally, we get the discretized stress field as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \mathbf{h}_\sigma \boldsymbol{\sigma}^* \quad (2.6.15)$$

$$\boldsymbol{\sigma}^* = 2G \boldsymbol{\varepsilon}^* \quad (2.6.16)$$

$$\boldsymbol{\varepsilon}^0 = \bar{\mathbf{B}} \mathbf{U} \quad (2.6.17)$$

$$\boldsymbol{\varepsilon}^* = \boldsymbol{\Gamma} \mathbf{U} \quad (2.6.18)$$

where G is the shear modulus.

Since displacement field, strain field and stress field have all be discretized, this leads to a mixed formulation of these three field. The relations among the three fields can be established by applying Hu-Washizu variational principle.

Finally, the system equilibrium equation can be obtained as below,

$$f_{int} - f_{ext} = 0 \quad (2.6.19)$$

with

$$f_{int} = V \bar{\mathbf{B}}^T \boldsymbol{\sigma} + \boldsymbol{\Gamma}^T \mathbf{H} \boldsymbol{\sigma} \quad (2.6.20)$$

where

$$\mathbf{H} = \int_{\Omega} \mathbf{h}'^T \mathbf{h}_{\sigma} d\Omega \quad (2.6.21)$$

During the derivation of the above equations, an approximation was made to simplify the calculation of the volume integrals. If the shape of the 8-node hexahedral element is almost parallelepiped or rectangular, the Jacobian matrix $\mathbf{J}$ and its determinant J are constant in the element and can be approximated by the values at the center of the element. By using the above approximations, the formulations can be made much simpler and a lot of time can be saved in calculation of the volume integral. But the accuracy of the computation will be reduced and convergence will slow down if a highly irregular mesh is used or the mesh is severely distorted during the computation because of the nodal displacement since in such cases the shape of the element is no longer parallelepiped or rectangular.

By linearization of Eq.(2.6.19) and taking into account of Eq.(2.6.20), we have,

$$\mathbf{K}_{small} \dot{\mathbf{U}} = \dot{\mathbf{f}}_{ext} \quad (2.6.22)$$

with the element tangent stiffness matrix $\mathbf{K}_{small}$

$$\mathbf{K}_{small} = V \bar{\mathbf{B}} \mathbf{C}^e \bar{\mathbf{B}} + 2G \mathbf{\Gamma}^T \mathbf{H} \mathbf{\Gamma} \quad (2.6.23)$$

Here if the reduced integration scheme is used without introducing an hourglass control technique, the result will be,

$$\mathbf{f}_{int}^{red} = V \bar{\mathbf{B}} \boldsymbol{\sigma}^0 \text{ and } \mathbf{K}_{small}^{red} = V \bar{\mathbf{B}}^T \mathbf{C}^e \bar{\mathbf{B}} \quad (2.6.24)$$

Comparing Eqs.(2.6.20) and (2.6.23) with Eq.(2.6.24), we can see that it is the second term in Eqs.(2.6.20) and (2.6.23) that stabilizes the hourglass modes.

2.6.4 HEXD element

The HEXDS element proposed by Liu is a four-point-quadrature 8-node hexahedral finite element [7]. The formulation of HEXDS inherits much from the series 3D solid elements which were developed by Belytschko and co-workers and used in the commercial finite

element code DYNA-3D [12, 13, 14, 15]. Similar to the JET3D element described in the previous section, this element is also based on the classical trilinear 3D element, section (3.1). In this element, a different hourglass control technique proposed by Liu is used [9]. In this approach, a stabilization vector which is used to control the mesh instability associated with RI is obtained by taking the partial derivatives of the generalized strain vector with respect to the natural coordinates. By this method, there are no unknown or arbitrary stabilization parameters needed, as they are with JET3D approach. The strain vector is approximated by the sum of a constant term and other terms involving strain derivatives. Schulz established a similar hourglass control procedure by expanding the stresses in a Taylor series about the element centroid [10]. It is found that by retention of the first and second derivative terms and omitting other terms in the series the hourglass modes were suppressed. Volumetric locking is avoided by setting the dilatation part of the normal strain components to be constant which is same as in JET3D. 4 integration points are used in HEXDS, which can improve the solution when the material response is nonlinear because of the more accurate stress interpolation.

For HEXDS, the strain-displacement submatrices at the center of the element are expressed as,

$$B_{\alpha}(\theta) = \begin{bmatrix} N_{\alpha,x}(\theta) \\ N_{\alpha,y}(\theta) \\ N_{\alpha,z}(\theta) \end{bmatrix} \quad \alpha = 1, \dots, 8 \quad (2.6.24)$$

The strain rate, $\dot{\epsilon}$, expanded in a Taylor series about the element center up to bilinear terms, can be written as [8, 9],

$$\dot{\epsilon}(r, s, t) = \dot{\epsilon}(\mathbf{0}) + \dot{\epsilon}_{,r}(\mathbf{0})r + \dot{\epsilon}_{,s}(\mathbf{0})s + \dot{\epsilon}_{,t}(\mathbf{0})t + \dot{\epsilon}_{,rs}(\mathbf{0})rs + \dot{\epsilon}_{,st}(\mathbf{0})st + \dot{\epsilon}_{,rt}(\mathbf{0})rt \quad (2.6.25)$$

or

$$\dot{\epsilon} = \bar{\mathbf{B}}_{\alpha}(r, s, t)\dot{\mathbf{U}}_{\alpha} \quad \text{Summation on } \alpha \text{ and } \alpha = 1, \dots, 8 \quad (2.6.26)$$

where

$$\bar{\mathbf{B}}_{\alpha}(r, s, t) = \mathbf{B}_{\alpha}(\mathbf{0}) + \mathbf{B}_{\alpha,r}(\mathbf{0})r + \mathbf{B}_{\alpha,s}(\mathbf{0})s + \mathbf{B}_{\alpha,t}(\mathbf{0})t + \mathbf{B}_{\alpha,rs}(\mathbf{0})rs + \mathbf{B}_{\alpha,st}(\mathbf{0})st + \mathbf{B}_{\alpha,rt}(\mathbf{0})rt \quad (2.6.27)$$

$\dot{\mathbf{U}}_{\alpha}$ is the nodal speed vector at node α .

As seen from Eq.(2.6.25), the first term at the right hand side represents the strain rates obtained by RI. This term is evaluated at the centroid of the element. The other terms are linear or bilinear strain rate terms.

The strain operator $\bar{\mathbf{B}}_{\alpha}(r, s, t)$ can be decomposed into two parts, the dilatational part,

$\bar{\mathbf{B}}_{\alpha}^{dil}(r, s, t)$, and the deviatoric part, $\bar{\mathbf{B}}_{\alpha}^{dev}(r, s, t)$, both of which can be expanded into a

Taylor series about the element center as in Eq.(2.6.27),

$$\begin{aligned} \bar{\mathbf{B}}_{\alpha}^{dil}(r, s, t) = & \mathbf{B}_{\alpha}^{dil}(\mathbf{0}) + \mathbf{B}_{\alpha,r}^{dil}(\mathbf{0})r + \mathbf{B}_{\alpha,s}^{dil}(\mathbf{0})s + \mathbf{B}_{\alpha,t}^{dil}(\mathbf{0})t \\ & + \mathbf{B}_{\alpha,rs}^{dil}(\mathbf{0})rs + \mathbf{B}_{\alpha,st}^{dil}(\mathbf{0})st + \mathbf{B}_{\alpha,rt}^{dil}(\mathbf{0})rt \end{aligned} \quad (2.6.28)$$

and

$$\begin{aligned}\bar{\mathbf{B}}_{\alpha}^{dev}(r, s, t) = & \mathbf{B}_{\alpha}^{dev}(\mathbf{0}) + \mathbf{B}_{\alpha,r}^{dev}(\mathbf{0})r + \mathbf{B}_{\alpha,s}^{dev}(\mathbf{0})s + \mathbf{B}_{\alpha,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{\alpha,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{\alpha,st}^{dev}(\mathbf{0})st + \mathbf{B}_{\alpha,rt}^{dev}(\mathbf{0})rt\end{aligned}\quad (2.6.29)$$

To avoid volumetric locking, the dilatational part of the strain operators is evaluated only at one integration point, the centroid, which is same as RI,

$$\bar{\mathbf{B}}_{\alpha}^{dil}(r, s, t) = \bar{\mathbf{B}}_{\alpha}^{dil}(\mathbf{0}) = \mathbf{B}_{\alpha}^{dil}(\mathbf{0}) \quad (2.6.30)$$

To remove shear locking, the deviatoric strain submatrices can be written as followings,

$$\begin{aligned}\bar{\mathbf{B}}_{xx}^{dev}(r, s, t) = & \mathbf{B}_{xx}^{dev}(\mathbf{0}) + \mathbf{B}_{xx,r}^{dev}(\mathbf{0})r + \mathbf{B}_{xx,s}^{dev}(\mathbf{0})s + \mathbf{B}_{xx,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{xx,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{xx,st}^{dev}(\mathbf{0})st + \mathbf{B}_{xx,rt}^{dev}(\mathbf{0})rt\end{aligned}\quad (2.6.31a)$$

$$\begin{aligned}\bar{\mathbf{B}}_{yy}^{dev}(r, s, t) = & \mathbf{B}_{yy}^{dev}(\mathbf{0}) + \mathbf{B}_{yy,r}^{dev}(\mathbf{0})r + \mathbf{B}_{yy,s}^{dev}(\mathbf{0})s + \mathbf{B}_{yy,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{yy,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{yy,st}^{dev}(\mathbf{0})st + \mathbf{B}_{yy,rt}^{dev}(\mathbf{0})rt\end{aligned}\quad (2.6.31b)$$

$$\begin{aligned}\bar{\mathbf{B}}_{zz}^{dev}(r, s, t) = & \mathbf{B}_{zz}^{dev}(\mathbf{0}) + \mathbf{B}_{zz,r}^{dev}(\mathbf{0})r + \mathbf{B}_{zz,s}^{dev}(\mathbf{0})s + \mathbf{B}_{zz,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{zz,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{zz,st}^{dev}(\mathbf{0})st + \mathbf{B}_{zz,rt}^{dev}(\mathbf{0})rt\end{aligned}\quad (2.6.31c)$$

$$\bar{\mathbf{B}}_{xy}^{dev}(r, s, t) = \mathbf{B}_{xy}^{dev}(\mathbf{0}) + \mathbf{B}_{xy,t}^{dev}(\mathbf{0})t \quad (2.6.31d)$$

$$\overline{\mathbf{B}}_{yz}^{dev}(r, s, t) = \mathbf{B}_{yz}^{dev}(\mathbf{0}) + \mathbf{B}_{yz,r}^{dev}(\mathbf{0})r \quad (2.6.31e)$$

$$\overline{\mathbf{B}}_{xz}^{dev}(r, s, t) = \mathbf{B}_{xz}^{dev}(\mathbf{0}) + \mathbf{B}_{xz,s}^{dev}(\mathbf{0})s \quad (2.6.31f)$$

Here, only one linear term is left for shear strain components and the other terms that cause shear locking are omitted. The normal strain components retain all linear and bilinear terms given by Eq.(2.6.29).

Summation of Eq.(3.1.30) and Eq.(3.1.31) yields the strain submatrices which are free of shear locking and volumetric locking:

$$\begin{aligned} \overline{\mathbf{B}}_{xx}(r, s, t) = & \mathbf{B}_{xx}(\mathbf{0}) + \mathbf{B}_{xx,r}^{dev}(\mathbf{0})r + \mathbf{B}_{xx,s}^{dev}(\mathbf{0})s + \mathbf{B}_{xx,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{xx,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{xx,st}^{dev}(\mathbf{0})st + \mathbf{B}_{xx,rt}^{dev}(\mathbf{0})rt \end{aligned} \quad (2.6.32a)$$

$$\begin{aligned} \overline{\mathbf{B}}_{yy}(r, s, t) = & \mathbf{B}_{yy}(\mathbf{0}) + \mathbf{B}_{yy,r}^{dev}(\mathbf{0})r + \mathbf{B}_{yy,s}^{dev}(\mathbf{0})s + \mathbf{B}_{yy,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{yy,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{yy,st}^{dev}(\mathbf{0})st + \mathbf{B}_{yy,rt}^{dev}(\mathbf{0})rt \end{aligned} \quad (2.6.32b)$$

$$\begin{aligned} \overline{\mathbf{B}}_{zz}(r, s, t) = & \mathbf{B}_{zz}(\mathbf{0}) + \mathbf{B}_{zz,r}^{dev}(\mathbf{0})r + \mathbf{B}_{zz,s}^{dev}(\mathbf{0})s + \mathbf{B}_{zz,t}^{dev}(\mathbf{0})t \\ & + \mathbf{B}_{zz,rs}^{dev}(\mathbf{0})rs + \mathbf{B}_{zz,st}^{dev}(\mathbf{0})st + \mathbf{B}_{zz,rt}^{dev}(\mathbf{0})rt \end{aligned} \quad (2.6.32c)$$

$$\overline{\mathbf{B}}_{xy}(r, s, t) = \mathbf{B}_{xy}(\mathbf{0}) + \mathbf{B}_{xy,t}^{dev}(\mathbf{0})t \quad (2.6.32d)$$

$$\bar{\mathbf{B}}_{yz}(r, s, t) = \mathbf{B}_{yz}(\mathbf{0}) + \mathbf{B}_{yz,r}^{dev}(\mathbf{0})r \quad (2.6.32e)$$

$$\bar{\mathbf{B}}_{xz}(r, s, t) = \mathbf{B}_{xz}(\mathbf{0}) + \mathbf{B}_{xz,s}^{dev}(\mathbf{0})s \quad (2.6.32f)$$

where $\bar{\mathbf{B}}_{xx}, \bar{\mathbf{B}}_{yy}, \bar{\mathbf{B}}_{zz}, \bar{\mathbf{B}}_{xy}, \bar{\mathbf{B}}_{yz}$ and $\bar{\mathbf{B}}_{xz}$ are the strain submatrices corresponding to strain rates components $\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{zz}, \dot{\epsilon}_{xy}, \dot{\epsilon}_{yz}$ and $\dot{\epsilon}_{xz}$, respectively.

$\mathbf{B}_{xx}(\mathbf{0}), \mathbf{B}_{yy}(\mathbf{0}), \mathbf{B}_{zz}(\mathbf{0}), \mathbf{B}_{xy}(\mathbf{0}), \mathbf{B}_{yz}(\mathbf{0})$ and $\mathbf{B}_{xz}(\mathbf{0})$ are strain submatrices corresponding to RI and consisting of both dilatational part and deviatoric part.

In order to evaluate the element internal forces more accurately when the elements are skewed, the strain-displacement submatrices defined in Eq.(2.6.24) are replaced by uniform gradient matrices, $\tilde{\mathbf{B}}_{\alpha}$, defined by Flanagan[15, 16],

$$\tilde{\mathbf{B}}_{\alpha} = \frac{1}{V_e} \int_{\Omega_e} \mathbf{B}_{\alpha}(r, s, t) dV \quad (2.6.33)$$

where V_e is the element volume. Thus, Eq.(2.6.24) can be modified as,

$$\tilde{\mathbf{B}}_{\alpha}(\mathbf{0}) = \begin{bmatrix} \tilde{\mathbf{N}}_{\alpha,x}(\mathbf{0}) \\ \tilde{\mathbf{N}}_{\alpha,y}(\mathbf{0}) \\ \tilde{\mathbf{N}}_{\alpha,z}(\mathbf{0}) \end{bmatrix} \quad (2.6.34)$$

In order to trace the plastic front in the mesh during loading and unloading more accurately for elastic-plastic large deformation problems, a four point integration scheme is used [8]. The element internal force vector is evaluated at the following integration points and the weight of each integration point is 2.

$$\text{Point1: } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\text{Point2: } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\text{Point3: } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

$$\text{Point4: } \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

2.6.5 Comments on the two elements

From section 2.6.3 and section 2.6.4, we can see these two elements are similar. Both elements represent the latest development in the analysis of thin wall structure by the continuum approach. According to the numerical examples reported, including the comparison of simulation results with the some theoretical solutions [33], their performances are satisfactory and both of them are free of locking problems. The hourglass control procedure used in JET3D starts by identifying and separating the hourglass modes and then uses special treatment to stabilize them. The hourglass control technique adopted by HEXDS is from the Taylor expansion of the strain field. The reason that the reduced integration scheme causes problem is that it uses only the first constant term of the Taylor series. By recovering certain linear terms in the series, the hourglass mode can be stabilized. These two ways should be theoretically equivalent. HEXDS has

no stabilization parameters. All the formulations of HEXDS can be expressed in an explicit form and no numerical approximation is assumed. Even though stabilization parameters are used in JET3D, they make JET3D suitable for more general applications. For example, JET3D can also be used for bulk forming analysis. The choice of integration points in HEXDS seems to lack theoretical basis and their influence is not reported. Comparison of the performance of these two elements, especially for nonlinear analysis, is one goal of this research. Their efficiency in simulating sheet metal forming processes and tolerance to irregular meshes are also of great interest.

One common shortcoming of both elements is that their formulations are complicated. HEXDS involves more than a hundred equations. JET3D needs specially developed constitutive relations so that it can take into account both constant part and hourglass parts of strains and stresses. This is very disadvantageous in terms of programming, debugging and further development, particularly when non-standard constitutive equations are required to be implemented. It is this shortcoming, rather than the ultimate performance in terms of locking and instability, that motivates the development of a simple element. It remains to be shown whether such an alternate approach retains the advantages of these elements over pure displacement “brick” elements.

CHAPTER 3 WW3D – A NEW HEXAHEDRAL MIXED ELEMENT FOR LARGE ELASTOPLASTIC DEFORMATIONS

“To eliminate shear locking, the portion of the shear velocity strain due to the hourglass mode must be eliminated”.

Ted Belytschko [47]

3.1 Classical deformation field

Consider an 8-node brick element in a normalized coordinate system (r, s, t) as shown in Fig.(3.1) with side length equal to 2. Its shape functions are,

$$N_{\alpha} = \frac{1}{8}(1 + r_{\alpha}r)(1 + s_{\alpha}s)(1 + t_{\alpha}t) \quad \alpha = 1, \dots, 8 \quad (3.1.1)$$

where α is the node number as shown in Fig.(3.1).

Fig.(3.1) An 8-node brick element (a) Physical coordinates. (b) Normalized coordinates

The linear isoparametric interpolation for the positions and displacement in an element gives [1, 7],

$$x_i = \sum_{\alpha=1}^8 N_{\alpha} X_{i\alpha} = (S^T + l_1^T r + l_2^T s + l_3^T t + m_1^T st + m_2^T rt + m_3^T rs + m_4^T rst) X_i \quad i = 1, 2, 3 \quad (3.1.2)$$

$$\begin{aligned} u_i &= \sum_{\alpha=1}^8 N_{\alpha} U_{i\alpha} = (S^T + l_1^T r + l_2^T s + l_3^T t + m_1^T st + m_2^T rt + m_3^T rs + m_4^T rst) U_i \quad i = 1, 2, 3 \\ &= U_i^0 + U_i^{l_1} r + U_i^{l_2} s + U_i^{l_3} t + U_i^{m_1} st + U_i^{m_2} rt + U_i^{m_3} rs + U_i^{m_4} rst \end{aligned} \quad (3.1.3)$$

with

$$\begin{aligned} S^T &= \frac{1}{8} \{1, 1, 1, 1, 1, 1, 1, 1\}; & l_1^T &= \frac{1}{8} \{-1, 1, 1, -1, -1, 1, 1, -1\}; \\ l_2^T &= \frac{1}{8} \{-1, -1, 1, 1, -1, -1, 1, 1\}; & l_3^T &= \frac{1}{8} \{-1, -1, -1, -1, 1, 1, 1, 1\}; \\ m_1^T &= \frac{1}{8} \{1, 1, -1, -1, -1, -1, 1, 1\}; & m_2^T &= \frac{1}{8} \{1, -1, -1, 1, -1, 1, 1, -1\}; \\ m_3^T &= \frac{1}{8} \{1, -1, 1, -1, 1, -1, 1, -1\}; & m_4^T &= \frac{1}{8} \{-1, 1, -1, 1, 1, -1, -1, 1\}; \end{aligned}$$

$$X_i^T = \{X_{i1}, X_{i2}, X_{i3}, X_{i4}, X_{i5}, X_{i6}, X_{i7}, X_{i8}\}; \text{ nodal coordinates}$$

$$U_i^T = \{U_{i1}, U_{i2}, U_{i3}, U_{i4}, U_{i5}, U_{i6}, U_{i7}, U_{i8}\};$$

$$U_i^0 = S^T U_i; \quad U_i^{l_1} = l_1^T U_i; \quad U_i^{l_2} = l_2^T U_i; \quad U_i^{l_3} = l_3^T U_i;$$

$$U_i^{m_1} = m_1^T U_i; \quad U_i^{m_2} = m_2^T U_i; \quad U_i^{m_3} = m_3^T U_i; \quad U_i^{m_4} = m_4^T U_i; \quad (3.1.4)$$

The physical meaning of the various terms in Eq. (3.1.4) are given below:

The U_i^0 terms represent the displacements at the center of the element which account for 3 rigid body translations; the $U_i^{l,k}$ ($k = 1, 2, 3$) represent the constant terms in the displacement gradients. $U_i^{l,k}$ defines 9 displacement modes which include 3 uniform normal strain modes and 6 uniform shear strain modes. $U_i^{m,k}$ ($k = 1, 2, 3, 4$) represent 12 non-constant terms in the displacement gradients which include 6 pure bending modes, 3 torsion modes and 3 warping. Fig.(3.2) graphically shows the $U_i^{m,k}$ in x direction

Fig.(3.2) The deformation modes corresponding to $U_i^{m,k}$ in x direction

As shown in Eq.(3.1.3), the displacement field of a 8-node brick element can be written in terms of the linear combination of the 8 vectors. These vectors include one s vector which represents rigid body motion, 3 l vectors which represent constant deformation modes and 4 m vectors which represent hourglass deformation modes. It is easy to verify the following orthogonal conditions between these 8 base vectors.

$$s \cdot l_i = 0 \quad (3.1.5)$$

$$s \cdot m_j = 0 \quad (3.1.6)$$

$$l_i \cdot l_k = \frac{1}{64} \delta_{ik} \quad (3.1.7)$$

$$m_j \cdot m_n = \frac{1}{64} \delta_{jn} \quad (3.1.8)$$

$$l_i \cdot m_j = 0 \quad (3.1.9)$$

where $i, k = 1, 3$ and $j, n = 1, 4$

3.2 Orthogonality conditions

After defining the displacement field in the above section, the deformation gradient can be easily obtained using conventional methods. Using Eq.(3.1.2) the normalized coordinates (r, s, t) can be expressed as,

$$\begin{Bmatrix} r \\ s \\ t \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} x_1 - (\mathbf{S}^T + stm_1^T + rtm_2^T + rsm_3^T + rstm_4^T)X_1 \\ x_2 - (\mathbf{S}^T + stm_1^T + rtm_2^T + rsm_3^T + rstm_4^T)X_2 \\ x_3 - (\mathbf{S}^T + stm_1^T + rtm_2^T + rsm_3^T + rstm_4^T)X_3 \end{Bmatrix}, \quad (3.2.1)$$

where,

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} l_1^T X_1 & l_2^T X_1 & l_3^T X_1 \\ l_1^T X_2 & l_2^T X_2 & l_3^T X_2 \\ l_1^T X_3 & l_2^T X_3 & l_3^T X_3 \end{bmatrix}^{-1} \quad (3.2.2)$$

Substituting Eq.(3.2.1) into Eq.(3.1.3), obtains,

$$\begin{aligned}
u_i &= (\mathbf{G}_0^T + x_1 \mathbf{b}_1^T + x_2 \mathbf{b}_2^T + x_3 \mathbf{b}_3^T + st \gamma_1^T + rt \gamma_2^T + rs \gamma_3^T + rst \gamma_4^T) \mathbf{U}_i \\
&= (\mathbf{G}_0^T + x_1 \mathbf{b}_1^T + x_2 \mathbf{b}_2^T + x_3 \mathbf{b}_3^T + h_1 \gamma_1^T + h_2 \gamma_2^T + h_3 \gamma_3^T + h_4 \gamma_4^T) \mathbf{U}_i \\
&= (\mathbf{G}_0^T + x_j \mathbf{b}_j^T + h_k \gamma_k^T) \mathbf{U}_i
\end{aligned} \tag{3.2.3}$$

where,

$$j = 1, 2, 3 \text{ and } k = 1, 2, 3, 4$$

$$\mathbf{b}_j = \mathbf{C}_p \mathbf{I}_p \quad p = 1, 2, 3 \tag{3.2.4}$$

$$\mathbf{G}_0 = \mathbf{S} - (\mathbf{S}^T \mathbf{X}_q) \mathbf{b}_q \quad q = 1, 2, 3 \tag{3.2.5}$$

$$\gamma_k = \mathbf{m}_k - (\mathbf{m}_k^T \mathbf{X}_j) \mathbf{b}_j \tag{3.2.6}$$

$$h_1 = st; \quad h_2 = rt; \quad h_3 = rs; \quad h_4 = rst; \tag{3.2.7}$$

Using the above equations, it is easy to find the following consistency conditions and orthogonality conditions [2],

$$\mathbf{b}_i^T \mathbf{X}_j = \delta_{ij} \quad \text{where } \delta_{ij} \text{ is the Kronecker delta} \tag{3.2.8}$$

$$\mathbf{b}_i^T \mathbf{S} = 0 \tag{3.2.9}$$

$$\mathbf{b}_i^T \mathbf{m}_j = 0 \tag{3.2.10}$$

$$\mathbf{S}^T \mathbf{m}_j = 0 \tag{3.2.11}$$

Taking derivatives of Eq.(3.2.3), the deformation gradient can be obtained,

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = (\mathbf{b}_j^T + h_{k,j} \gamma_k^T) \mathbf{U}_i = \varepsilon_{ij} + \omega_{ij} = u_{i,j}^0 + u_{i,j}^h \quad k = 1, 2, 3, 4 \quad (3.2.12)$$

with

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}); & \omega_{ij} &= \frac{1}{2}(u_{i,j} - u_{j,i}) \\ u_{i,j}^0 &= \mathbf{b}_j^T \mathbf{U}_i; & u_{i,j}^h &= h_{k,j} \gamma_k^T \mathbf{U}_i \end{aligned} \quad (3.2.13)$$

where ε_{ij} is the symmetric part of the displacement gradient which represents the deformation field and ω_{ij} is the skew part of the displacement gradient which represents the rigid body rotation; $u_{i,j}^0$ is the constant part and $u_{i,j}^h$ is the non-constant part of the displacement gradient.

Using the orthogonality conditions Eq.(3.2.9~3.2.11), some interesting observations can be found:

- For linear displacement modes, their displacement gradients are always constant.

Consider a displacement field with complete linear base

$$\mathbf{U}_i = a_i \mathbf{S} + a_{ij} \mathbf{X}_j \quad (3.2.14)$$

Substitute the above expression into Eq.(3.2.13) and using the orthogonality conditions Eq.(3.2.8~3.2.9)

$$u_{i,j}^0 = \mathbf{b}_j^T (a_i \mathbf{S} + a_{ik} \mathbf{X}_k) = a_{ik} \delta_{jk} = a_{ij} \quad (3.2.15)$$

$$u_{i,j}^h = h_{k,j} \gamma_k^T (a_i \mathbf{S} + a_{in} \mathbf{X}_n) = 0 \quad (3.2.16)$$

As shown in Eq. (3.2.14), the first part of it represents the rigid body translation. This displacement field can represent the rigid body rotation if $a_{ij} = -a_{ji}$ and $a_{11} = a_{22} = a_{33} = 0$. It can also represent constant strain field if $a_{ij} \neq -a_{ji}$ and $a_{ii} \neq 0$.

- $\mathbf{m}_j$ are the hourglass displacement modes and their constant parts of the displacement gradient are always zero.

Consider the following hourglass nodal displacement fields

$$U_i = d_{ij} \mathbf{m}_j \quad (3.2.17)$$

Substitute it into Eq.(3.2.13) and using the orthogonality conditions Eq.(3.2.8~3.2.9)

$$u_{i,j}^0 = \mathbf{b}_j^T d_{ik} \mathbf{m}_k = 0 \quad (3.2.18)$$

$$u_{i,j}^h = h_{k,j} \gamma_k^T d_{in} \mathbf{m}_n = h_{k,j} d_{in} \mathbf{m}_k^T \mathbf{m}_n = h_{k,j} d_{in} \delta_{kn} = d_{in} h_{n,j} \quad (3.2.19)$$

These two types of displacement modes are orthogonal to each other, which means any one of them has no influence on any other. It also can be seen in Eq. (3.2.19) that there is no coupling among any of the hourglass modes.

If an element has a nodal displacement of the hourglass modes, using reduced integration scheme (RI), which means its displacement gradient is evaluated only at its centroid, it will suffer the mesh instability problem. That is the hourglass phenomena induced by RI [6].

3.3 Locking control techniques for infinitesimal analysis

Since the modes (m_1, m_2, m_3, m_4) which cause shear locking are identified, the portion of shear strain due to each of these hourglass modes can be directly eliminated at each integration point. By this way, shear locking can be avoided as mentioned by Belytschko [47].

To avoid volumetric locking, the dilatational part of the strain is constant in the element. It is only evaluated at the center of the element.

A new hexahedral element, called WW3D, is based on this approach.

New Way to Avoid Shear Locking

For example, rewrite one term in Eq.(3.2.13), such as $u_{1,2}$, in the expanded form,

$$u_{1,2} = (\mathbf{b}_2^T + h_{1,2}\gamma_1^T + h_{2,2}\gamma_2^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T) \mathbf{U}_1 \quad (3.3.1)$$

As shown in Eq.(3.3.1), $u_{1,2}$ includes 5 deformation modes as shown in Fig.(3.3).

Fig.(3.3) 5 deformation modes associated with $u_{1,2}$

(a) is the uniform shear mode and the shear strain it contributes is physical. (b) is the torsion mode. Shear strain caused by this mode varies linearly in the element. In pure torsion, only shear strain exists. So the part of shear strain caused by torsion is also physical. (c) and (d) are bending modes. (e) is warp mode. The shear strain caused by these 3 modes is the origin of the shear locking and should be eliminated to prevent locking. We can modify Eq.(3.3.1) so that the part of shear strain that causes locking is excluded, Eq.(3.3.2).

$$u_{1,2}^{new} = (\mathbf{b}_2^T + h_{1,2}\boldsymbol{\gamma}_1^T) \mathbf{U}_1 \quad (3.3.2)$$

New Way to Avoid Volumetric Locking

As discussed before, in order to avoid volumetric locking, the dilatational part of the strain must be constant in the element. Its value equals the dilatation at the center of the element. This can be achieved by modifying the normal component in Eq.(3.3.1). Take $u_{1,1}$ as an example, its expression may be rewritten in the expanded form, Eq.(3.3.3).

$$u_{1,1} = [\mathbf{b}_1^T + h_{1,1}\boldsymbol{\gamma}_1^T + h_{2,1}\boldsymbol{\gamma}_2^T + h_{3,1}\boldsymbol{\gamma}_3^T + h_{4,1}\boldsymbol{\gamma}_4^T] \mathbf{U}_1 \quad (3.3.3)$$

As shown in above equation, $u_{1,1}$ includes 5 dilatational parts caused by 5 deformation modes respectively, Fig.(3.4).

Fig.(3.4) 5 deformation modes associated with $u_{1,1}$

(a) is tension mode and the part of dilatational strain it contributes is uniform throughout the element. (b) is torsion mode. In torsion, there is no dilatation at all. So for $u_{1,1}$, we don't need to calculate this part. (c) and (d) are bending modes. The dilatation caused by these two modes varies linearly in the element. (e) is warp mode. The dilatation caused by this mode varies bilinearly in the element. To prevent volumetric locking, the dilatational strain caused by (c), (d) and (e) should be eliminated. So we can change Eq.(3.3.3) into Eq.(3.3.4).

$$u_{1,1}^{new} = [\mathbf{b}_1^T + \frac{2}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)]U_1 - \frac{1}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)U_2 - \frac{1}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)U_3 \quad (3.3.4)$$

Similarly, we can modify the other terms of the displacement gradient in Eq.(3.1.16). The modified displacement gradient for infinitesimal analysis, Eq.(3.3.5a ~ i), is free of the locking problems. In Eq.(3.3.5), the superscript 'new' is omitted for simplicity.

$$u_{1,1} = [\mathbf{b}_1^T + \frac{2}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)]U_1 - \frac{1}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)U_2 - \frac{1}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)U_3 \quad (3.3.5a)$$

$$u_{2,2} = [\mathbf{b}_2^T + \frac{2}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)]U_2 - \frac{1}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)U_1 - \frac{1}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)U_3 \quad (3.3.5b)$$

$$u_{3,3} = [\mathbf{b}_3^T + \frac{2}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)]U_3 - \frac{1}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)U_1 - \frac{1}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)U_2 \quad (3.3.5c)$$

$$u_{1,2} = (\mathbf{b}_2^T + h_{1,2}\gamma_1^T)U_1 \quad (3.3.5d)$$

$$u_{2,1} = (\mathbf{b}_1^T + h_{2,1}\gamma_2^T)U_2 \quad (3.3.5e)$$

$$u_{1,3} = (\mathbf{b}_3^T + h_{1,3}\gamma_1^T)U_1 \quad (3.3.5f)$$

$$u_{3,1} = (\mathbf{b}_1^T + h_{3,1}\gamma_3^T)U_3 \quad (3.3.5g)$$

$$u_{3,2} = (\mathbf{b}_2^T + h_{3,2}\gamma_3^T)U_3 \quad (3.3.5h)$$

$$u_{2,3} = (\mathbf{b}_3^T + h_{2,3}\gamma_2^T)U_2 \quad (3.3.5i)$$

3.4 Locking control techniques for nonlinear analysis

To extend the proposed locking control technique to the nonlinear finite deformation, the displacement and the strain are replaced by velocity and strain rate, respectively. The

modified velocity gradient for finite deformation analysis can be written as Eq.(3.4.1a ~

i)

$$D_{11} = \dot{u}_{1,1} = [\mathbf{b}_1^T + \frac{2}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)]\dot{\mathbf{U}}_1 - \frac{1}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)\dot{\mathbf{U}}_2 - \frac{1}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)\dot{\mathbf{U}}_3 \quad (3.4.1a)$$

$$D_{22} = \dot{u}_{2,2} = [\mathbf{b}_2^T + \frac{2}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)]\dot{\mathbf{U}}_2 - \frac{1}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)\dot{\mathbf{U}}_1 - \frac{1}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)\dot{\mathbf{U}}_3 \quad (3.4.1b)$$

$$D_{33} = \dot{u}_{3,3} = [\mathbf{b}_3^T + \frac{2}{3}(h_{1,3}\gamma_1^T + h_{2,3}\gamma_2^T + h_{4,3}\gamma_4^T)]\dot{\mathbf{U}}_3 - \frac{1}{3}(h_{2,1}\gamma_2^T + h_{3,1}\gamma_3^T + h_{4,1}\gamma_4^T)\dot{\mathbf{U}}_1 - \frac{1}{3}(h_{1,2}\gamma_1^T + h_{3,2}\gamma_3^T + h_{4,2}\gamma_4^T)\dot{\mathbf{U}}_2 \quad (3.4.1c)$$

$$D_{21} = \dot{u}_{2,1} = (\mathbf{b}_1^T + h_{2,1}\gamma_2^T)\dot{\mathbf{U}}_2 \quad (3.4.1d)$$

$$D_{12} = \dot{u}_{1,2} = (\mathbf{b}_2^T + h_{1,2}\gamma_1^T)\dot{\mathbf{U}}_1 \quad (3.4.1e)$$

$$D_{31} = \dot{u}_{3,1} = (\mathbf{b}_1^T + h_{3,1}\gamma_3^T)\dot{\mathbf{U}}_3 \quad (3.4.1f)$$

$$D_{13} = \dot{u}_{1,3} = (\mathbf{b}_3^T + h_{1,3}\gamma_1^T)\dot{\mathbf{U}}_1 \quad (3.4.1g)$$

$$D_{32} = \dot{u}_{3,2} = (\mathbf{b}_2^T + h_{3,2}\gamma_3^T)\dot{\mathbf{U}}_3 \quad (3.4.1h)$$

$$D_{23} = \dot{u}_{2,3} = (\mathbf{b}_3^T + h_{2,3}\gamma_2^T)\dot{\mathbf{U}}_2 \quad (3.4.1i)$$

3.5 The definition of the incremental corotational coordinate system in WW3D

A local coordinate system is embedded in each element and rotates with the element. All concerned variables are evaluated with respect to this local reference system with rigid body motion eliminated. There are several ways to define this local corotational reference system. For WW3D, the local reference system is defined in the same approach used in JET3D[3]. By using such a local coordinate system, the local axis Z_l is assumed to be coincident with the Z-axis of the material anisotropy and the angles between the local reference axes and material anisotropic axes fixed during the whole deformation process. The local corotational reference system can be determined by Eq.(3.5.1~4) and the graphic representation is shown in Fig.(3.5).

$$\mathbf{e}_{z_l} = \frac{\mathbf{e}_r \times \mathbf{e}_s}{\|\mathbf{e}_r \times \mathbf{e}_s\|} \quad (3.5.1)$$

$$\mathbf{e}_w = \frac{\mathbf{e}_{z_l} \times \mathbf{e}_r}{\|\mathbf{e}_{z_l} \times \mathbf{e}_r\|} \quad (3.5.2)$$

$$\mathbf{e}_{y_l} = \frac{\mathbf{e}_w + \mathbf{e}_s}{\|\mathbf{e}_w + \mathbf{e}_s\|} \quad (3.5.3)$$

$$\mathbf{e}_{x_l} = \mathbf{e}_{y_l} \times \mathbf{e}_{z_l} \quad (3.5.4)$$

Fig.(3.5) The setup of local corotational coordinate system

Point O is the center of the element, Points A and B are the centers of their faces, respectively.

3.6 Features of the new element

- The new element has 8 integration points for each element, which can provide good stress interpolation;
- There are no arbitrary or undermined parameters needed to prevent locking in the new element;
- Jaumann stress rate is used to ensure objectivity;
- Spin is constant within element;
- Updated Lagrange method is used and mid-point integration is used for strain increment evaluation;
- The new element can work with any existing constitutive relation without special modification.

CHAPTER 4 NUMERICAL EXAMPLES AND DISCUSSION

In this chapter, the performance of the new mixed hexahedral element developed in last chapter is tested through small-scale problems and two real processes. The small-scale problems include a patch test, cantilever beam bending, square plate bending, circular plate bending, pure bending, pinched cylinder and Scordelis-Lo roof. These examples are widely used to test the performance of new elements developed for the analysis of thin-walled structures. The real processes include deep drawing of a cylindrical cup and large curvature thick plate bending. The simulation results by the new element are compared against the analytical solutions as well as other hexahedral elements including JET3D [1, 3], ABAQUS C3D8, C3D8R, C3D8I and C3D8H [41]. The convergence and the consistency of the new element are also investigated and compared with these elements. The properties of all the hexahedral elements used in this study are summarized in Table (4.1).

Element name	Number of d.o.f. per element	Number of integration points	Feature
WW3D	24	8	New locking control method
JET3D	24	1	Reduced integration with hourglass control
C3D8R	24	1	Reduced integration with hourglass control
C3D8	24	8	Standard displacement element
C3D8I	37	8	Incompatible modes
C3D8H	24	8	Stress interpolation with constant pressure

Table (4.1) Properties of all the hexahedral elements used in this study

JET3D is an 8-node hexahedral element with 1 integration point and hourglass control, as discussed in section (2.6.3). Detailed information about this element can be found ref.[1, 3, 4]. The shear locking control parameters used in JET3D, Eq.(2.6.6), are determined by the automatic method suggested by the authors, which is related to the aspect ratio of the element.

ABAQUS C3D8 is a standard 8-node linear element with 8 integration points [41].

ABAQUS C3D8R is an 8-node hexahedral element with 1 integration point and hourglass control [41]. It uses the artificial stiffness method of Flanagan and Belytschko [12].

ABAQUS C3D8I is an 8-node hexahedral element with incompatible modes. It has 13 additional variables relating to the incompatible modes [41]. Some details can be found in [55]. For this type of element, additional quadratic models are added in order to overcome shear locking. This method was originally proposed by Wilson [46].

ABAQUS C3D8H is an 8-node linear hexahedral hybrid element with constant pressure [41]. Some details can be found in [55]. Hybrid finite element was first proposed by Pian [44] and a historical view about it can be found in [45]. This type of element uses the assumed stress field and is based on complementary energy principle.

4.1 Patch test

Name of the test: Patch test

Introduction: In patch test, distorted elements are used because rectangular element may give correct result while irregular element may fail. All boundary displacements in a patch test are prescribed. In order to pass patch test, the calculated strains and stresses should be in high order accuracy with the analytical solution or true solution.

Purpose: Patch test checks whether an element can exactly reproduce constant strains (linear displacement field) and thus give correct constant stresses throughout the patch. It gives the sufficient condition for convergence. Patch test can also be used for debugging programming. If an element fails the patch test, it either cannot correctly reproduce linear field exactly or the implementation of the element is not correct.

Reference: Zienkiewicz [28], Belytschko [47]

Implementation: Large elastic deformation, plane stress and plane strain.

Mesh refinement: No

Elements tested: WW3D

Material properties: Elastic, Young's modulus $E = 1.0$ and Poisson's ratio $\nu = 0.25$.

The first loading case is stretching along Y direction. The original dimension in Y is 3.0.

The patch is stretched in Y to 6.0. For plane stress case, assume $\sigma_x = 0$, by Hook's law the

true strains are $\epsilon_Y = \ln(2.0) = 0.693114718$, $\epsilon_x = \epsilon_z = -\nu \ln(2.0) = 0.17328680$. The

stresses are $\sigma_x = \sigma_z = 0$ and $\sigma_Y = E \ln(2.0) = 0.693114718$. For plane strain case, assume

$\epsilon_x = 0$. The true strains are $\epsilon_Y = \ln(2.0) = 0.693114718$, $\epsilon_z = -0.23104906$. The stresses

are $\sigma_x = 0.18483943$, $\sigma_Y = 0.739356993$ and $\sigma_z = 0$.

The second loading case is stretching along Z direction. The original dimension in Z is

3.0. The patch is stretched in Z to 6.0. For plane stress case, assume $\sigma_x = 0$, by Hook's

law the true strains are $\epsilon_z = \ln(2.0) = 0.693114718$, $\epsilon_x = \epsilon_Y = -\nu \ln(2.0) = 0.17328680$.

The stresses are $\sigma_x = \sigma_Y = 0$ and $\sigma_z = E \ln(2.0) = 0.693114718$. For plane strain case,

assume $\epsilon_x = 0$. The true strains are $\epsilon_z = \ln(2.0) = 0.693114718$, $\epsilon_Y = -0.23104906$. The

stresses are $\sigma_x = 0.18483943$, $\sigma_z = 0.739356993$ and $\sigma_Y = 0$.

The third loading case is simple shear. The displacement is 3.0. For plane stress case, the theoretical results are $\sigma_{zz} = -\sigma_{yy} = 0.183879078$ and $\tau_{yz} = 0.336588394$ [53].

Discussion and conclusion: As shown in table (4.2 ~ 4), the results of WW3D are very accurate compared to the theoretical solution. The error for the first 2 loading cases is less than 10^{-5} and the error of the third loading case is less than 2×10^{-4} . So WW3D passed the patch test for constant strain under the plane stress and the plane strain conditions.

Therefore, the membrane behavior of WW3D is satisfactory.

Fig. (4.1) Patch test

Fig. (4.2) Patch test results – Y stretch

Fig. (4.3) Patch test results – Z stretch

Fig. (4.4) Patch test results – Simple shear

Plane Stress		
	Stress	Normalized Result
Theoretical Result	0.69314718	1.000000000
Y Stretch	0.69314788	1.00000101
Z Stretch	0.69314788	1.00000101

Table (4.2) Patch test – Plane stress stretch

Plane Strain				
	Stress	Normalized Result	Stress	Normalized Result
Theoretical Result	0.73935699	1.00000000	0.18483925	1.00000000
Y Stretch	0.73935773	1.00000100	0.18483943	1.00000098
Z Stretch	0.73935773	1.00000100	0.18483943	1.00000098

Table (4.3) Patch test – Plane strain stretch

Simple Shear				
	Normal Stress	Normalized Result	Shear Stress	Normalized Result
Theoretical Result	0.18387908	1.00000000	0.33658839	1.00000000
Simulation Result	0.18384304	0.99980402	0.33664449	1.00016667

Table (4.4) Patch test – Simple Shear

4.2 Cantilever beam bending

Name of the test: Cantilever beam bending

Introduction: An elastic straight cantilever is subjected to an end shear force P as shown in Fig. (4.5). Bending a straight cantilever beam is a very popular numerical test for beam, plate and solid elements because it is simple and it involves many deformation modes, such as constant strain modes and linearly varying strain modes.

Purpose: To evaluate the bending performance of the WW3D element with comparison to other elements. If WW3D suffers shear locking, the tip vertical displacement at point A will be much less than the analytical value.

Reference: Belytschko [12], Timoshenko [33], Macneal [43].

Implementation: Small elastic deformation, plane stress and plane strain. Maximum principle strain is less than or equal to 3.0×10^{-3} .

Mesh refinement: No

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I and C3D8H.

Material properties: Elastic, Young's modulus $E = 3 \times 10^7$ and Poisson's ratio $\nu = 0.25$ or 0.4999.

Fig. (4.5) Cantilever beam bending

Since this problem is antisymmetric [1,7], only half of the beam is modeled. The analytical solution for infinitesimal deformation is given by Timoshenko and Goodier [33] as

$$\begin{aligned}
 u_x(x, y) &= \frac{-Py}{6\bar{E}I} \left[(6L - 3x)x + (2 + \bar{\nu})(y^2 - \frac{1}{4}D^2) \right] \\
 u_y(x, y) &= \frac{P}{6\bar{E}I} \left[3\bar{\nu}y^2(L - x) + \frac{1}{4}(4 + 5\bar{\nu})D^2x + (3L - x)x^2 \right]
 \end{aligned}
 \tag{4.2.1}$$

where $I = \frac{1}{12}D^3$;

For plane stress: $\bar{E} = E$ and $\bar{\nu} = \nu$

For plane strain: $\bar{E} = E/(1 - \nu^2)$ and $\bar{\nu} = \nu/(1 - \nu^2)$

A $4 \times 1 \times 1$ rectangular mesh with aspect ratio of 2, as shown in Fig. 4.5 (a), is used to assess the coarse mesh accuracy of the element. A $4 \times 1 \times 1$ irregular mesh characterized by angle θ , as shown in Fig.4.5 (b), is also used. In all cases, $P = 20000$, the height of the

half beam shown is 6, the thickness of the beam is 1. The following four cases are considered: 1. plane strain, Poisson ratio $\nu = 0.25$; 2. plane strain, $\nu = 0.4999$; 3. plane stress, $\nu = 0.25$; 4. plane stress, $\nu = 0.4999$. The analytical results for the end vertical deflection at point A for these four cases are: 0.3341667, 0.2740325, 0.3533333, and 0.3586653 respectively. The results, including calculated tip deflections and normalized results, are shown in Table (4.5 ~ 10).

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$
Analytical Solution	0.334167	0.274033	0.355333	0.358665
WW3D	0.308207	0.30917	0.330355	0.397644
JET3D	0.35522	0.294585	0.385536	0.397042
C3D8R	0.406415	0.339623	0.42924	0.434536
C3D8	0.260184	0.245965	0.273278	0.293134
C3D8I	0.316477	0.26185	0.334966	0.336755
C3D8H	0.260156	0.245968	0.273241	0.293139

Table (4.5) Tip deflection of regular mesh for cantilever beam bending

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$	SDNE
WW3D	0.922	1.128	0.93	1.109	0.099
JET3D	1.063	1.075	1.085	1.107	0.084
C3D8R	1.216	1.239	1.208	1.212	0.219
C3D8	0.779	0.898	0.769	0.817	0.191
C3D8I	0.947	0.956	0.943	0.939	0.054
C3D8H	0.779	0.898	0.769	0.817	0.191

Table (4.6) Normalized tip deflection of regular mesh for cantilever beam bending

1: SNDE stands for Standard Deviation of Normalized Error.

Normalized Error (NE) = Normalized result – 1.0

$$SDNE = \sqrt{\frac{\sum (NE)^2}{N}}$$

and N is the number of data.

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$
Analytical Solution	0.334167	0.274033	0.355333	0.358665
WW3D	0.29716	0.299849	0.317561	0.384237
JET3D	0.34319	0.30034	0.372389	0.384489
C3D8R	0.394804	0.330242	0.417007	0.422397
C3D8	0.253182	0.240035	0.266233	0.286331
C3D8I	0.307061	0.255738	0.324466	0.326106
C3D8H	0.253158	0.240037	0.266202	0.286336

Table (4.7) Tip deflection of irregular mesh, $\theta = 9.462^\circ$, for cantilever beam bending

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$	SDNE
WW3D	0.889	1.094	0.894	1.071	0.097
JET3D	1.027	1.096	1.048	1.072	0.066
C3D8R	1.181	1.205	1.174	1.178	0.185
C3D8	0.758	0.876	0.749	0.798	0.211
C3D8I	0.919	0.933	0.913	0.909	0.082
C3D8H	0.758	0.876	0.749	0.798	0.211

Table (4.8) Normalized tip deflection of irregular mesh, $\theta = 9.462^\circ$, for cantilever beam bending

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$
Analytical Solution	0.334167	0.274033	0.355333	0.358665
WW3D	0.271031	0.280606	0.287657	0.357431
JET3D	0.319798	0.283624	0.34716	0.361534
C3D8R	0.382247	0.320344	0.403718	0.409373
C3D8	0.237652	0.22711	0.250196	0.270634
C3D8I	0.287308	0.241817	0.302478	0.302482
C3D8H	0.237636	0.227112	0.250176	0.270636

Table (4.9) Tip deflection of irregular mesh, $\theta = 18.435^\circ$, for cantilever beam bending

Element	Plane strain $\nu = 0.25$	Plane strain $\nu = 0.4999$	Plane stress $\nu = 0.25$	Plane stress $\nu = 0.4999$	SDNE
WW3D	0.811	1.024	0.81	0.997	0.135
JET3D	0.957	1.035	0.977	1.008	0.030
C3D8R	1.144	1.169	1.136	1.141	0.148
C3D8	0.711	0.829	0.704	0.755	0.255
C3D8I	0.86	0.882	0.851	0.843	0.142
C3D8H	0.711	0.829	0.704	0.755	0.255

Table (4.10) Normalized tip deflection of irregular mesh, $\theta = 18.435^\circ$, for cantilever beam bending

Fig. (4.6) The effect of volumetric locking control on WW3D

Discussion and conclusion: For this beam bending problem, as Poisson's ratio approaching 0.5, the error of WW3D will increase quickly without Volumetric Locking Control (VLC), as shown in Fig. (4.6). For this reason, there is a switch inside WW3D. Once the Poisson's ratio is larger than a certain value, for example, in this case the value is 0.4, VLC will be automatically turned on. This value is problem dependent.

As shown in Table (4.7 ~ 10), we see WW3D gives reasonable results even for distorted meshes when compared to analytical solution. For Poisson's ratio $\nu = 0.4999$, WW3D does not have the volumetric locking problem, for both plane strain and plane stress cases. When compared to JET3D, the accuracy of these two elements is almost the same, as shown in their SDNE values. An interesting point noticed is that for both elements

distorted mesh gives better results. This is contradictory to what we used to think.

Theoretically the more regular the mesh, the better result. For $\nu = 0.25$, regular mesh gives the better results than distorted meshes and the tip deflection is smaller than analytical solution. With volumetric locking control, the results, in most times for both elements, are larger than the analytical solution. One possible explanation is that the distortion part of the internal strain energy has to become larger to compensate the ignorance of volumetric part of the internal strain energy. Because the internal energy must be equal to the external energy caused by the tip shear force along the tip deflection. ABAQUS C3D8R is too soft because the results it gives are at least 20% larger than the analytical results for all 4 cases when regular mesh is used. Also, the higher degree of mesh distortion, the better results C3D8R gives. C3D8I has the best performance among the 6 elements tested for the regular mesh. The results from C3D8 and C3D8H are almost same and their performance is poor, as expected.

4.3 Convergence test for cantilever beam bending

Name of the test: Convergence test for cantilever beam bending

Introduction: Cantilever beam bending problem same as the previous test.

Purpose: To evaluate the convergence of WW3D element with comparison to other elements.

Reference: Timoshenko [33], Macneal [43].

Implementation: Small elastic deformation, plane stress. Maximum principle strain is less than or equal to 3.0×10^{-3} .

Mesh refinement: Uniform mesh refinement with aspect ratio = 1. 3 mesh examples are shown in Fig. (4.7 ~ 9).

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, B21 and C3D20.

Material properties: Elastic, Young's modulus $E = 3 \times 10^7$ and Poisson's ratio $\nu = 0.25$.

h is the length of element side.

Fig.(4.7) 8×1 mesh, $l/h = 1$, convergence test for cantilever beam bending

Fig.(4.8) 32×4 mesh, $1/h = 4$, convergence test for cantilever beam bending

Fig.(4.9) 64×8 mesh, $1/h = 8$, convergence test for cantilever beam bending

Fig.(4.10) Convergence test for WW3D with comparison to ABAQUS elements and JET3D

Discussion and conclusion: As shown in Fig.(4.10), with the reduction of the element size, the accuracy of WW3D increases. So, we can conclude that WW3D is consistent. It gives good result even for coarse meshes. For coarse meshes, such as $1/h$ is less than 2, the results of ABAQUS C3D8 are much less than the analytical solution. This is not unexpected, because it suffers the shear locking. On the other hand, the results of ABAQUS C3D8R are much larger than the analytical solution. C3D8R uses reduced integration with hourglass control. Obviously, it is too soft when coarse mesh is used. We can also see from this figure that as the mesh becomes finer, the results of all the 8-node linear elements become closer to the analytical solution. However, the result of the 20 node quadratic element C3D20 becomes more inaccurate as the mesh is refined. With coarse mesh, the result of C3D20 is better than all the linear elements. The explanation to

this phenomenon is that as the number of d.o.f. increases, the quadratic element becomes too flexible. In dynamic analysis, when element sides become too flexible, they produce too much “noise” in the solution. This is why the high order elements are seldom used in the dynamic analysis [47]. ABAQUS B21 gives best performance in this test. B21 is a linear beam element and it has additional rotation d.o.f..

4.4 Cantilever beam bending with geometric nonlinearity

Name of the test: Cantilever beam bending with geometric nonlinearity

Introduction: Cantilever beam bending problem same as the previous test, section (4. 2), with geometric nonlinearity introduced by large tip displacement.

Purpose: To compare WW3D with other elements under the conditions of large displacement and large strain.

Reference: No

Implementation: Large elastic deformation, plane stress. Maximum principle strain is less than or equal to 0.2.

Mesh refinement: No

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, B21 and C3D20.

Material properties: Elastic, Young's modulus $E = 1 \times 10^6$ and Poisson's ratio $\nu = 0.25$.

Fig.(4.11) Mesh used for cantilever beam bending with geometric nonlinearity

Fig.(4.12) Comparison of WW3D to ABAQUS elements and JET3D

Discussion and conclusion: The mesh used in this test is shown in Fig. (4.11). For this test, the results of C3D20 can be treated as “true solution” because C3d20 is a quadratic element and it does not has locking problem. As shown in Fig. (4.12), the result by WW3D is very close the results of all the other solid elements. As expected, the result of beam element is very close to those of solid elements for small tip displacement and force. With the introduction of geometric nonlinearity, the tip displacement of solid elements is much smaller than that of the beam element.

4.5 Square plate bending

Name of the test: Square plate bending

Introduction: A thin square plane is subjected a concentrated load at its center as shown in Fig. (4.13). The dimension of the plate is 10×10×0.2. Two types of boundary condition are considered: simple supported edges and clamped edges. For the simple supported edge, only the vertical displacement of the edge nodes is fixed. For the clamped edge, all d.o.f. of the edge nodes are fixed. Because of the symmetry, only a quarter of the plate is simulated. The concentrated force $F = 100$. The analytical solution of the deflection at the center of the plate was given by Timoshenko [48] as

$$w_{\max} = 0.0116FL^2 / D \quad \text{for simple supported edge} \quad (4.5.1)$$

$$w_{\max} = 0.0056FL^2 / D \text{ for clamped edge} \quad (4.5.2)$$

$$\text{with } D = (Et^3)/(12(1-\nu^2))$$

Purpose: To study the bending performance and locking control of WW3D in 3D with regular meshes.

Reference: Koh[2], Timoshenko[48].

Implementation: Small elastic deformation. Maximum principle strain is less than or equal to 2.0×10^{-4} .

Mesh refinement: Up to 4 elements were used in thickness and up to 20 elements were used in each of the 2 in-plane directions.

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, and C3D8H.

Material properties: Elastic, Young's modulus $E = 3 \times 10^7$ and Poisson's ratio $\nu = 0.3$.

Fig.(4.13) Square plate bending

Discussion and conclusion: The calculated center displacements of WW3D, with the comparison to results by JET3D and ABAQUS elements, for various mesh sizes are shown in Table (4.11) for clamped edge and Table (4.12 ~ 17) show the normalized results for each element. Table (4.18) summarizes the results for simple supported edge and Table (4.19 ~ 24) show the normalized results. The first two numbers about the mesh indicate the number of elements used in each of the two in-plane directions, X and Y. The third number is the number of elements through thickness, Z. As shown in Table (4.12) and Table (4.19), WW3D gives very accurate results for this problem under both boundary conditions and shows very good consistency as mesh refines. This is a bending dominant problem. Only one layer of elements can't give very accurate results for WW3D element. However, for two or more layers, the element can have very good

results. Also shown on these two tables, the results by WW3D consistently improve as mesh refines in both through thickness direction and section.

As to JET3D and ABAQUS C3D8R, their results are not as good as WW3D, as shown in Table (4.13), Table (4.20), Table (4.14) and Table (4.21). Both elements do not show consistency as the mesh is refined. If only one layer of elements is used, the results by the elements are very poor. Only when very fine meshes are used, can both elements give reasonable results. For JET3D, as the mesh is refined in thickness direction, it shows consistency. However, it fails to show consistency as the mesh is refined in section view. As for ABAQUS C3D8R, it does not show consistency in either section or thickness directions.

As expected, C3D8 gives very poor results because of shear locking, as show in Table(4.15) and Table(4.22). If only one layer of elements is used, for fine meshes such as 16×16 and 20×20 , the results by C3D8 can even exceed the analytical solution. The reason for this unexpected phenomenon is the underestimation of bending since only one layer of elements is used. It is also interesting to see that C3D8 does not show consistency as mesh refines in thickness direction. However, the results by C3D8 do improve as more and more elements are used in section. C3D8H gives identical results as C3D8, as show in Table (4.17) and Table (4.24).C3D8I also gives very good results when compared to analytical solutions, as shown in Table (4.16) and Table (4.23). It gives best results in 26 out of 40 times. WW3D gives best results in 11 out of 40 times. JET3D wins only 3 times.

Mesh	Center Displacement					
Section x Thickness	WW3D	JET3D	C3D8R	C3D8	C3D8I	C3D8H
4x4x1	0.0020464	0.0033758	0.0018336	0.0002532	0.0020941	0.0002532
8x8x1	0.0020997	0.0035000	0.022523	0.0008689	0.0025276	0.0008689
12x12x1	0.0021132	0.0035372	0.048634	0.0016249	0.0025729	0.0016249
16x16x1	0.0021204	0.0035390	0.0564060	0.0023463	0.0025867	0.0023463
20x20x1	0.0021259	0.0035460	0.0589990	0.0029572	0.0025943	0.0029572
4x4x2	0.0023087	0.0026241	0.0003640	0.0002447	0.0020472	0.0002447
8x8x2	0.0023979	0.0027295	0.0021408	0.0007685	0.0024953	0.0007685
12x12x2	0.0024221	0.0027550	0.0030730	0.0013026	0.0025497	0.0013026
16x16x2	0.0024345	0.0027679	0.0033461	0.0017277	0.0025684	0.0017277
20x20x2	0.0024429	0.0027770	0.0034499	0.0020380	0.0025790	0.0020380
4x4x3	0.0023675	0.0025235	0.0001780	0.0002432	0.0020403	0.0002432
8x8x3	0.0024656	0.0026278	0.0013721	0.0007522	0.0024917	0.0007522
12x12x3	0.0024925	0.0026536	0.0023750	0.0012573	0.0025479	0.0012573
16x16x3	0.0025062	0.0026667	0.0027426	0.0016490	0.0025676	0.0016490
20x20x3	0.0025155	0.0026761	0.0028812	0.0019298	0.0025790	0.0019298
4x4x4	0.0023890	0.0024903	0.0001064	0.0002427	0.0020380	0.0002427
8x8x4	0.0024905	0.0025947	0.0009714	0.0007474	0.0024905	0.0007474
12x12x4	0.0025184	0.0026212	0.0020028	0.0001242	0.0025474	0.0001242
16x16x4	0.0025326	0.0026344	0.0024848	0.0016233	0.0025675	0.0016233
20x20x4	0.0025422	0.0026439	0.0026770	0.0018949	0.0025792	0.0018949

Table (4.11) Square plate bending with clamped edge

Center displacement by analytical solution = 0.0025458

WW3D	Section Refinement					SDNE
Thickness Refinement	4x4	8x8	12x12	16x16	20x20	
1	0.804	0.825	0.830	0.833	0.835	0.175
2	0.907	0.942	0.951	0.956	0.960	0.060
3	0.930	0.968	0.979	0.984	0.988	0.037
4	0.938	0.978	0.989	0.995	0.999	0.030

Table (4.12) Normalized results of WW3D for square plate bending

JET3D	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	1.326	1.375	1.389	1.390	1.393	0.375
2	1.031	1.072	1.082	1.087	1.091	0.076
3	0.991	1.032	1.042	1.047	1.051	0.039
4	0.978	1.019	1.030	1.035	1.039	0.030

Table (4.13) Normalized results of JET3D for square plate bending

C3D8R	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.720	8.847	19.104	22.156	23.175	16.302
2	0.143	0.841	1.207	1.314	1.355	0.453
3	0.070	0.539	0.933	1.077	1.132	0.470
4	0.042	0.382	0.787	0.976	1.052	0.519

Table (4.14) Normalized results of C3D8R for square plate bending

C3D8	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.099	0.341	0.638	0.922	1.162	0.531
2	0.096	0.302	0.512	0.679	0.801	0.581
3	0.096	0.295	0.494	0.648	0.758	0.592
4	0.095	0.294	0.049	0.638	0.744	0.696

Table (4.15) Normalized results of C3D8 for square plate bending

C3D8I	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.823	0.993	1.011	1.016	1.019	0.080
2	0.804	0.980	1.002	1.009	1.013	0.088
3	0.801	0.979	1.001	1.009	1.013	0.090
4	0.801	0.978	1.001	1.009	1.013	0.090

Table (4.16) Normalized results of C3D8I for square plate bending

C3D8H	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.099	0.341	0.638	0.922	1.162	0.531
2	0.096	0.302	0.512	0.679	0.801	0.581
3	0.096	0.295	0.494	0.648	0.758	0.592
4	0.095	0.294	0.049	0.638	0.744	0.696

Table (4.17) Normalized results of C3D8 for square plate bending

Mesh	Center Displacement					
Section x Thickness	WW3D	JT3D	C3D8R	C3D8	C3D8I	C3D8H
4x4x1	0.0043329	0.0082294	0.0018693	0.0007203	0.0048115	0.0007203
8x8x1	0.0043488	0.0082331	0.025851	0.0023788	0.0052819	0.0023788
12x12x1	0.0043599	0.0082459	0.07172	0.0042358	0.0053297	0.0042358
16x16x1	0.0043690	0.0082520	0.0939280	0.0058492	0.0053475	0.0058492
20x20x1	0.0043768	0.0082569	0.1032800	0.0071118	0.0053591	0.0071118
4x4x2	0.0050022	0.0058755	0.0004265	0.0006845	0.0047883	0.0006845
8x8x2	0.0050307	0.0058727	0.0033488	0.0020199	0.0052711	0.0020199
12x12x2	0.0050464	0.0058823	0.0058172	0.0032146	0.0053215	0.0032146
16x16x2	0.0050581	0.0058888	0.0066876	0.0040662	0.0053406	0.0040662
20x20x2	0.0050677	0.0058946	0.0069929	0.0046406	0.0053531	0.0046406
4x4x3	0.0051526	0.0055907	0.0002067	0.0006783	0.0047857	0.0006783
8x8x3	0.0051853	0.0055872	0.0019455	0.0019656	0.0052726	0.0019656
12x12x3	0.0052028	0.0055971	0.0042331	0.0030793	0.0053244	0.0030793
16x16x3	0.0052156	0.0056037	0.0053398	0.0038525	0.0053444	0.0038525
20x20x3	0.0052261	0.0056099	0.0057750	0.0043650	0.0053574	0.0043650
4x4x4	0.0052076	0.0054983	0.0001246	0.0006761	0.0047846	0.0006761
8x8x4	0.0052420	0.0054950	0.0012841	0.0019474	0.0052734	0.0019474
12x12x4	0.0052602	0.0055054	0.0033314	0.0030347	0.0053257	0.0030347
16x16x4	0.0052735	0.0055122	0.0046775	0.0037832	0.0053460	0.0037832
20x20x4	0.0052843	0.0055185	0.0052838	0.0042765	0.0053594	0.0042765

Table (4.18) Square plate bending with simple supported edge

Center displacement by analytical solution = 0.0052708

WW3D	Section Refinement					SDNE
Thickness Refinement	4x4	8x8	12x12	16x16	20x20	
1	0.822	0.825	0.827	0.829	0.830	0.173
2	0.949	0.954	0.957	0.960	0.961	0.044
3	0.978	0.984	0.987	0.990	0.992	0.015
4	0.988	0.995	0.998	1.001	1.003	0.006

Table (4.19) Normalized results of WW3D

JET3D	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	1.561	1.562	1.564	1.566	1.567	0.564
2	1.115	1.114	1.116	1.117	1.118	0.116
3	1.061	1.060	1.062	1.063	1.064	0.062
4	1.043	1.043	1.045	1.046	1.047	0.045

Table (4.20) Normalized results of JET3D

C3D8R	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.355	4.905	13.607	17.820	19.595	12.675
2	0.081	0.635	1.104	1.269	1.327	0.483
3	0.039	0.369	0.803	1.013	1.096	0.523
4	0.024	0.244	0.632	0.887	1.002	0.579

Table (4.21) Normalized results of C3D8R

C3D8	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.137	0.451	0.804	1.110	1.349	0.494
2	0.130	0.383	0.610	0.771	0.880	0.521
3	0.129	0.373	0.584	0.731	0.828	0.534
4	0.128	0.369	0.576	0.718	0.811	0.539

Table (4.22) Normalized results of C3D8

C3D8I	Section Refinement					SDNE
Thickness Refinement	4×4	8×8	12×12	16×16	20×20	
1	0.913	1.002	1.011	1.015	1.017	0.041
2	0.908	1.000	1.010	1.013	1.016	0.042
3	0.908	1.000	1.010	1.014	1.016	0.043
4	0.908	1.000	1.010	1.014	1.017	0.043

Table (4.23) Normalized results of C3D8I

C3D8H	Section Refinement					SDNE
Thickness Refinement	4x4	8x8	12x12	16x16	20x20	
1	0.137	0.451	0.804	1.110	1.349	0.494
2	0.130	0.383	0.610	0.771	0.880	0.521
3	0.129	0.373	0.584	0.731	0.828	0.534
4	0.128	0.369	0.576	0.718	0.811	0.539

Table (4.24) Normalized results of C3D8H

4.6 Circular plate bending

Name of the test: Circular plate bending

Introduction: A thin circular plate is subjected to a concentrated load at its center, as shown in Fig. (4.14). The radius of the plate $R = 100.0$ and thickness is $t = 1.0$. Because of the symmetry, only a quarter of the plate is modeled. The concentrated force $F = 10$. Both simple supported edge and clamped edge are considered. The analytical solution of the displacement at the center of the plate under the two boundary conditions is given as [7]

$$w_{\max} = 3(3 + \nu)(1 - \nu)FR^2 / 4\pi Et^3 \quad \text{for simple supported edge} \quad (4.6.1)$$

$$w_{\max} = 3(1 - \nu^2)FR^2 / 4\pi Et^3 \quad \text{for clamped edge} \quad (4.6.2)$$

Purpose: To study the bending performance and locking control of WW3D in 3D with irregular meshes.

Reference: Liu [7]

Implementation: Small elastic deformation. Maximum principle strain is less than or equal to 2.0×10^{-6} .

Mesh refinement: up to 8 elements were used in thickness and up to 75 elements were used in-plane.

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, and C3D8H.

Material properties: Elastic, Young's modulus $E = 1 \times 10^7$ and Poisson's ratio $\nu = 0.3$.

Fig.(4.14) Circular plate bending

Fig.(4.15) shows a mesh example used in the simulation. For this case, 48 elements were used in the plane. The computed center displacements were shown in Table (4.25) for clamped edge and Table (4.26 ~ 31) shows the normalized results for each element. Table (4.32) shows the calculated displacements for simple supported edge and Table (4.33 ~ 38) shows the normalized results for all elements. The first number about the mesh indicates the number of elements used in-plane and the second number is the number of elements through thickness.

Fig.(4.15) Mesh used for Circular plate bending

Discussion and conclusion: The results of this problem show exactly the same scenario of results of the square plate bending, except for C3D8I. One thing that should be pointed out and is different from the previous test is that for this problem element shape is irregular because of the geometry of the plate, as shown in Fig.(4.15). For square plate

bending, the element shape is always rectangular. As shown in Table (4.30) and Table (4.37), C3D8I gives very poor results, even for very fine meshes. For coarse meshes, the simulations were simply corrupted. The other meshes, it did not give any results.

Mesh	Center Displacement					
Section x Thickness	WW3D	JT3D	C3D8R	C3D8	C3D8I	C3D8H
12x1	0.0015967	0.0025586	0.0224300	0.0000128	Corrupted	0.0000128
12x2	0.0017740	0.0021153	0.0018002	0.0000127	0.0000000	0.0000127
12x4	0.0018250	0.0020291	0.0014582	0.0000127	0.0006211	0.0000127
12x8	0.0018383	0.0020088	0.0013902	0.0000127	0.0006207	0.0000127
27x1	0.0017140	0.0026502	0.0792100	0.0000277	Corrupted	0.0000277
27x2	0.0019368	0.0021979	0.0021856	0.0000274	0.0000000	0.0000274
27x4	0.0020142	0.0021093	0.0017341	0.0000274	0.0014643	0.0000274
27x8	0.0020322	0.0020884	0.0016428	0.0000274	0.0014633	0.0000274
48x1	0.0017440	0.0026804	0.1053970	0.0000479	Corrupted	0.0000479
48x2	0.0019913	0.0022306	0.0023180	0.0000472	0.0000000	0.0000472
48x4	0.0020652	0.0021423	0.0018381	0.0000471	0.0000000	0.0000471
48x8	0.0020846	0.0021215	0.0017404	0.0000470	0.0000000	0.0000470
75x1	0.0017546	0.0026958	0.1065650	0.0000750	0.0000000	0.0000750
75x2	0.0020098	0.0022473	0.0023775	0.0000740	0.0000000	0.0000740
75x4	0.0020868	0.0021594	0.0018728	0.0000731	0.0000000	0.0000731
75x8	0.0021070	0.0021386	0.0017694	0.0000731	0.0000000	0.0000731

Table (4.25) Circular plate bending with clamped edge

Analytical solution = 0.0021725.

WW3D	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.804	0.825	0.830	0.833	0.177
2	0.907	0.942	0.951	0.956	0.064
4	0.930	0.968	0.979	0.984	0.041
8	0.938	0.978	0.989	0.995	0.033

Table (4.26) Normalized results of WW3D

JET3D	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	1.178	1.220	1.234	1.241	0.219
2	0.974	1.012	1.027	1.034	0.026
4	0.934	0.971	0.986	0.994	0.037
8	0.925	0.961	0.977	0.984	0.045

Table (4.27) Normalized results of JET3D

C3D8R	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	10.325	36.460	48.514	49.052	38.441
2	0.829	1.006	1.067	1.094	0.103
4	0.671	0.798	0.846	0.862	0.219
8	0.640	0.756	0.801	0.814	0.256

Table (4.28) Normalized results of C3D8R

C3D8	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.006	0.013	0.022	0.035	0.981
2	0.006	0.013	0.022	0.043	0.979
4	0.006	0.013	0.022	0.034	0.982
8	0.006	0.013	0.022	0.034	0.982

Table (4.29) Normalized results of C3D8

C3D8I	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	@	0.000	0.286	0.286	N/A
2	@	0.000	0.674	0.674	N/A
4	@	0.000	0.000	0.000	N/A
8	0.000	0.000	0.000	0.000	1.000

Table (4.30) Normalized results of C3D8I

C3D8H	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.006	0.013	0.022	0.035	0.981
2	0.006	0.013	0.022	0.043	0.979
4	0.006	0.013	0.022	0.034	0.982
8	0.006	0.013	0.022	0.034	0.982

Table (4.31) Normalized results of C3D8H

Mesh	Center Displacement					
Section x Thickness	WW3D	JT3D	C3D8R	C3D8	C3D8I	C3D8H
12x1	0.0039157	0.0096019	0.2294170	0.0000422	Corrupted	0.0000422
12x2	0.0041553	0.0058554	0.0067338	0.0000420	0.0000000	0.0000420
12x4	0.0049208	0.0053776	0.0053991	0.0000420	0.0021983	0.0000420
12x8	0.0049855	0.0052723	0.0051440	0.0000420	0.0021979	0.0000420
27x1	0.0040780	0.0098592	0.2277110	0.0000949	0.0000000	0.0000949
27x2	0.0049553	0.0060126	0.0067701	0.0000940	0.0000000	0.0000940
27x4	0.0051496	0.0055243	0.0054219	0.0000938	0.0039303	0.0000938
27x8	0.0052191	0.0054163	0.0051632	0.0000938	0.0039300	0.0000938
48x1	0.0041594	0.0099692	0.2271430	0.0001679	Corrupted	0.0001679
48x2	0.0050013	0.0060720	0.0067848	0.0001651	0.0000000	0.0001651
48x4	0.0052708	0.0055799	0.0054228	0.0001644	0.0000000	0.0001644
48x8	0.0053432	0.0054712	0.0051595	0.0001643	0.0000000	0.0001643
75x1	0.0041886	0.0100183	0.2268680	0.0002604	0.0000000	0.0002604
75x2	0.0050384	0.0060992	0.0068034	0.0002539	0.0000000	0.0002539
75x4	0.0053138	0.0056057	0.0054232	0.0002522	0.0000000	0.0002522
75x8	0.0053871	0.0054967	0.0051544	0.0002519	0.0000000	0.0002519

Table (4.32) Circular plate bending with simple supported edge

Center displacement by analytical solution = 0.0055147

WW3D	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.710	0.739	0.754	0.760	0.260
2	0.753	0.899	0.907	0.914	0.148
4	0.892	0.934	0.956	0.964	0.069
8	0.904	0.946	0.969	0.977	0.058

Table (4.33) Normalized results of WW3D

JET3D	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	1.741	1.788	1.808	1.817	0.789
2	1.062	1.090	1.101	1.106	0.091
4	0.975	1.002	1.012	1.017	0.016
8	0.956	0.982	0.992	0.997	0.024

Table (4.34) Normalized results of JET3D

C3D8R	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	41.601	41.292	41.189	41.139	40.305
2	1.221	1.228	1.230	1.234	0.228
4	0.979	0.983	0.983	0.983	0.018
8	0.933	0.936	0.936	0.935	0.065

Table (4.35) Normalized results of C3D8R

C3D8	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.008	0.017	0.030	0.047	0.974
2	0.008	0.017	0.030	0.046	0.975
4	0.008	0.017	0.030	0.046	0.975
8	0.008	0.017	0.030	0.046	0.975

Table (4.36) Normalized results of C3D8

C3D8I	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	@	0.000	0.399	0.399	N/A
2	0.000	0.000	0.713	0.713	0.736
4	@	0.000	0.000	0.000	N/A
8	0.000	0.000	0.000	0.000	1.000

Table (4.37) Normalized results of C3D8I

C3D8H	Section Refinement				SDNE
Thickness Refinement	12	27	48	75	
1	0.008	0.017	0.030	0.047	0.974
2	0.008	0.017	0.030	0.046	0.975
4	0.008	0.017	0.030	0.046	0.975
8	0.008	0.017	0.030	0.046	0.975

Table (4.38) Normalized results of C3D8H

4.7 Pinched cylinder

Name of the test: Pinched cylinder

Introduction: A thin-walled cylinder of length l and of radius r is loaded by two normal and equal point loads applied centrally at the opposite sides of the cylindrical surface.

Both ends of the cylinder are free edges, Fig.(4.16). Because of the symmetry, only one octant of the cylinder is modeled. The radial displacement at the loading point is

computed and compared to the analytical solution [2, 49]. Analytical displacement = 0.1139

Purpose: To study the bending performance and locking control of WW3D for thin and curved structure.

Reference: Kikuchi [2], Ansys online manual [49].

Implementation: Small elastic strain. Maximum principle strain is less than or equal to 9.0×10^{-4} .

Mesh refinement: Up to 4 elements were used in thickness and up to 20 elements were used in each of the other 2 directions.

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, and C3D8H.

Material properties: Elastic, Young's modulus $E = 1.05 \times 10^6$ and Poisson's ratio $\nu = 0.3125$.

Fig. (4.16) Pinched cylinder [49]

$l = 10.35$; $r = 4.953$; $t = 0.094$; $F = 100$

The simulation results of WW3D, with the comparison to results by JET3D, ABAQUS C3D8, C3D8R, C3D8I and C3D8H, for various mesh sizes are shown in Table (4.39).

The normalized results for all the elements with mesh refinement are shown in Table (4.40 ~ 45). The first number about the mesh is the number of elements in the periphery. The second number indicates the number of elements used along the symmetric axis. The third number indicates the number of elements in radial direction R or through thickness.

Discussion and conclusion: It can be seen from Table (4.40 ~ 45), this test repeats what is shown in square plate bending and circular plate bending. In this test, WW3D gives

very accurate result for this curved structure and it does not suffer from membrane locking as shell element does for curved structure.

Mesh	Displacement					
Section x Thickness	WW3D	JT3D	C3D8R	C3D8	C3D8I	C3D8H
5x5x1	0.0502	0.0945	0.4719	0.0015	corrupted	0.0015
8x8x1	0.0885	0.1389	2.6888	0.0036	0.0000	0.0036
10x10x1	0.0930	0.1447	3.9092	0.0054	0.0000	0.0054
16x16x1	0.0958	0.1501	7.0889	0.0129	0.0000	0.0129
20x20x1	0.0963	0.1514	8.0310	0.0193	0.0000	0.0193
32x32x1	0.0969	0.1529	9.0358	0.0436	0.1212	0.0436
5x5x2	0.0548	0.0834	0.1009	0.0015	corrupted	0.0015
8x8x2	0.1026	0.1175	0.1320	0.0036	0.0000	0.0036
10x10x2	0.1085	0.1215	0.1405	0.0053	0.0000	0.0053
16x16x2	0.1124	0.1251	0.1523	0.0124	0.0000	0.0124
20x20x2	0.1130	0.1259	0.1559	0.0182	0.0000	0.0182
32x32x2	0.1138	0.1268	0.1605	0.0380	0.1211	0.0380
5x5x3	0.0558	0.0818	0.0885	0.0015	0.0618	0.0015
8x8x3	0.1059	0.1146	0.1127	0.0035	0.1102	0.0035
10x10x3	0.1121	0.1185	0.1192	0.0053	0.1159	0.0053
16x16x3	0.1162	0.1218	0.1280	0.0123	0.0000	0.0123
20x20x3	0.1169	0.1226	0.1308	0.0180	0.0000	0.0180
5x5x4	0.0561	0.0813	0.0848	0.0015	0.0618	0.0015
8x8x4	0.1070	0.1136	0.1072	0.0035	0.1101	0.0035
10x10x4	0.1134	0.1174	0.1130	0.0053	0.1158	0.0053
16x16x4	0.1175	0.1208	0.1209	0.0123	0.0000	0.0123
20x20x4	0.1183	0.1225	0.1234	0.0179	0.0000	0.0179

Table (4.39) Results of pinched cylinder

WW3D	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	0.441	0.777	0.816	0.841	0.845	0.850	0.279
2	0.481	0.901	0.953	0.987	0.992	0.999	0.217
3	0.489	0.930	0.984	1.020	1.026		0.231
4	0.492	0.939	0.996	1.032	1.039		0.230

Table (4.40) Normalized results of WW3D

JET3D	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	0.829	1.219	1.270	1.318	1.329	1.342	0.282
2	0.732	1.032	1.067	1.098	1.105	1.113	0.136
3	0.718	1.006	1.040	1.069	1.076		0.135
4	0.713	0.997	1.031	1.061	1.076		0.136

Table (4.41) Normalized results of JET3D

C3D8R	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	4.143	23.607	34.321	62.238	70.509	79.331	52.199
2	0.886	1.159	1.234	1.337	1.369	1.409	0.292
3	0.777	0.989	1.047	1.124	1.148		0.134
4	0.745	0.941	0.992	1.061	1.083		0.126

Table (4.42) Normalized results of C3D8R

C3D8	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	0.013	0.032	0.048	0.113	0.170	0.383	0.883
2	0.013	0.031	0.047	0.109	0.160	0.333	0.891
3	0.013	0.031	0.047	0.108	0.158		0.930
4	0.013	0.031	0.047	0.108	0.157		0.930

Table (4.43) Normalized results of C3D8

C3D8H	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	0.013	0.032	0.048	0.113	0.170	0.383	0.883
2	0.013	0.031	0.047	0.109	0.160	0.333	0.891
3	0.013	0.031	0.047	0.108	0.158		0.930
4	0.013	0.031	0.047	0.108	0.157		0.930

Table (4.44) Normalized results of C3D8H

C3D8I	Section Refinement						SDNE
Thickness Refinement	5x5	8x8	10x10	16x16	20x20	32x32	
1	@	0.000	0.000	0.000	0.000	1.064	N/A
2	@	0.000	0.000	0.000	0.000	1.063	N/A
3	0.542	0.967	1.017	0.000	0.000		0.665
4	0.542	0.967	1.017	0.000	0.000		0.665

Table (4.45) Normalized results of C3D8I

4.8 Scordelis-Lo Roof

Name of the test: Scordelis-Lo roof

Introduction: This test is a reinforced concrete single span cylindrical roof subjected to its own weight. Both ends of the roof are covered with rigid diaphragms that allow only axial displacement. Because of the symmetry, only one quarter of the roof is modeled, as shown in Fig.(4.17). The test result most frequently displayed is the vertical displacement of point A, which is the midpoint of the free edge. The theoretical vertical displacement of point A is 0.3086. But numerical tests show that the results of FEM simulations

converges to a smaller value of 0.3024[43]. So 0.3024 is used for comparison of finite element solution.

Purpose: To study the bending performance and locking control of WW3D for thin and curved structure.

Reference: Macneal [43]

Implementation: Small elastic strain. Maximum principle strain is less than or equal to 5.0×10^{-4} .

Mesh refinement: Up to 4 elements were used in thickness and up to 64 elements were used in each of the other 2 directions.

Elements tested: WW3D, JET3D, ABAQUS C3D8R, C3D8, C3D8I, and C3D8H.

Material properties: Elastic, Young's modulus $E = 4.32 \times 10^8$ and Poisson's ratio $\nu = 0.0$.

Fig. (4.17) Scordelis-Lo roof

$L = 50.0$; $R = 25.0$; $h = 0.25$; Load = 90.0/unit area

The simulation results are shown in Table (4.46). The normalized results with mesh refinement for each element are shown in Table (4.47 ~ 52). The first number about the mesh is the number of elements in the periphery. The second number indicates the number of elements used along the longitudinal direction. The third number indicates the number of elements through thickness direction R .

In this test, none of the 6 solid elements give good results though WW3D and JET3D perform better than the two ABAQUS elements. In fact, mesh refinement does not show any influence on WW3D, JET3D and C3D8I. In two cases, C3D8R cannot finish the computation. C3D20, which is a quadratic element in ABAQUS and it does not have

locking problem, was used for this problem. For 16×16×2 mesh, the result is 0.2514 or normalized 0.831. For 32×32×4 mesh, the result of C3D20 is 0.2523 or normalized 0.834. For such a fine mesh with high order solid element, the result is still not satisfactory. The possible explanation is that the analytical solution for this problem is no longer valid because it was derived based on shell theory.

Mesh	Center Displacement					
Section × Thickness	WW3D	JET3D	C3D8R	C3D8	C3D8I	C3D8H
8×8×1	0.2562	0.2753	0.0409	0.0371	0.2466	0.0371
10×10×1	0.256	0.2771	0.0578	0.0463	0.2523	0.0463
12×12×1	0.2559	0.2778	0.0861	0.0563	0.2542	0.0563
16×16×1	0.2558	0.2783	0.1773	0.0777	0.2553	0.0777
20×20×1	0.2557	0.2785	0.2864	0.0998	0.2556	0.0998
32×32×1	0.2558	0.2786	0.4862(0.9816)*	0.1585	0.2558	0.1585
64×64×1	0.2561	0.2786	0.4535(0.5545)*	0.2350	0.2561	0.2350
8×8×2	0.2562	0.2598	0.0395	0.0467	0.2467	0.0467
10×10×2	0.256	0.2609	0.0449	0.0582	0.2523	0.0582
12×12×2	0.2559	0.2613	0.0544	0.0704	0.2542	0.0704
16×16×2	0.2558	0.2615	0.0892	0.0963	0.2554	0.0963
20×20×2	0.2557	0.2616	0.1421	0.1225	0.2556	0.1225
32×32×2	0.2558	0.2616	0.2313	0.1529	0.2558	0.1529
8×8×4	0.2562	0.2566	0.0289	0.0371	0.2467	0.0371
10×10×4	0.256	0.2574	0.0299	0.0462	0.2523	0.0462
12×12×4	0.2559	0.2577	0.0319	0.0560	0.2543	0.0560
16×16×4	0.2558	0.2579	0.0400	0.0767	0.2554	0.0767
20×20×4	0.2558	0.2579	0.0556	0.0977	0.2556	0.0977
32×32×4	0.2558	0.2579	0.1394	0.1516	0.2558	0.1516

Table (4.46) Results of Scordelis-Lo roof

^: ABAQUS simulation stopped at the time increment indicated in the parenthesis. Total time increment is 1.0.

WW3D	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.847	0.847	0.846	0.846	0.846	0.846	0.847	0.154
2	0.847	0.847	0.846	0.846	0.846	0.846		0.154
4	0.847	0.847	0.846	0.846	0.846	0.846		0.154

Table (4.47) Normalized results of WW3D

JET3D	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.910	0.916	0.919	0.920	0.921	0.921	0.921	0.082
2	0.859	0.863	0.864	0.865	0.865	0.865		0.137
4	0.849	0.851	0.852	0.853	0.853	0.853		0.148

Table (4.48) Normalized results of JET3D

C3D8R	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.135	0.191	0.285	0.586	0.947	1.608	1.500	0.622
2	0.131	0.148	0.180	0.295	0.470	0.765		0.706
4	0.095	0.099	0.105	0.132	0.184	0.461		0.831

Table (4.49) Normalized results of C3D8R

C3D8	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.123	0.153	0.186	0.257	0.330	0.524	0.777	0.700
2	0.154	0.192	0.233	0.318	0.405	0.506		0.709
4	0.123	0.153	0.185	0.254	0.323	0.501		0.754

Table (4.50) Normalized results of C3D8

C3D8I	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.816	0.834	0.841	0.844	0.845	0.846	0.847	0.161
2	0.816	0.834	0.841	0.844	0.845	0.846		0.163
4	0.816	0.834	0.841	0.845	0.845	0.846		0.163

Table (4.51) Normalized results of C3D8I

C3D8H	Displacement							SDNE
Thickness Refinement	8×8	10×10	12×12	16×16	20×20	32×32	64×64	
1	0.123	0.153	0.186	0.257	0.330	0.524	0.777	0.700
2	0.154	0.192	0.233	0.318	0.405	0.506		0.709
4	0.123	0.153	0.185	0.254	0.323	0.501		0.754

Table (4.52) Normalized results of C3D8H

4.9 Pure bending

Name of the test: Pure bending

Introduction: A beam is subjected to a force couple at each of its ends. Because of the symmetry, only a half of the beam is simulated, as shown in Fig.(4.18). The material is linear elastic-linear hardening Von-Mises plastic. The ratio of elastic modulus E to plastic modulus E_p is 50. For pure bending, the only deformation component is the the strain along the longitudinal direction and it changes linearly in through-thickness direction. For linear elastic-linear hardening plastic material in pure bending, the stress distribution in

thickness direction is shown in Fig.(4.19) and σ_a can be calculated by Eq.(4.9.1) [50].

Because the stress distribution is known for this test, it can be used to check strain calculation and thus stress obtained from the Radial Return algorithm for Von-Mises plasticity.

$$\sigma_a = Y + \mu Y \left(\frac{1}{\gamma} - 1 \right) \quad \text{Eq.(4.9.1)}$$

where $\gamma = c/(h/2)$, $\mu = E_p/E$ and Y is the yield stress. c can obtained by Eq.(4.9.2)

$$M = \frac{1}{3} Y b \left(\frac{h}{2} \right)^2 \left\{ 3 - \gamma^2 + \frac{\mu}{\gamma} [2 - 3\gamma + \gamma^3] \right\} \quad \text{Eq.(4.9.2)}$$

where b is the width of the beam and M is the moment of the stress.

Purpose: To check the stress calculation of WW3D and the implementation of radial-return algorithm.

Reference: Yu and Zhang [50].

Implementation: Small elastic-plastic deformation, plane stress. Maximum principle strain is less than or equal to 5.0×10^{-4} .

Mesh refinement: No

Elements tested: WW3D

Material properties: Elastic-plastic, Young's modulus $E = 500$, Poisson's ratio $\nu = 0.25$, plastic modulus $E_p = 10.0$ and yield stress $= 0.2$. The hardening law is $\bar{\sigma} = E_p \bar{\epsilon}$.

Fig. (4.18) Pure bending

$h = 2.0$, $b = 5.0$, $l = 50.0$, and $F = 0.5$,

Fig.(4.19) Stress distribution of pure bending

Discussion and conclusion: Simulation results, as well as the analytical solution, are shown in Fig.(4.20). 4, 8, 12 and 16 layers elements through thickness were used. 16 elements were used along longitudinal direction, which is the Y-axis in this test. Because this is a plane stress problem, only one element in width direction was used. As shown in this figure, as the number of elements through thickness increases, the stress distribution from the simulation is getting closer to the analytical stress distribution. The stress distribution is also uniform along the longitudinal direction. The other stress components

are virtually zero except near the end where concentrated forces are applied as shown in Fig.(4.21 ~ 26).

Fig.(4.20) Comparison of pure bending results to theoretical solution

Fig.(4.21) σ_{xx} Distribution

V1

Fig. (4.22) τ_{xy} Distribution

V1

Fig. (4.23) τ_{xz} Distribution

97

Fig. (4.26) σ_{zz} Distribution

4.10 Deep drawing – Benchmark problem of NUMISHEET'02 [51]

Name of the test: Deep drawing

Introduction: This is a benchmark problem of NUMISHEET'02. Because of symmetry, only a quarter of the blank is simulated. Blank holder force is uniformly distributed on the nodes that are under blank holder.

Purpose: To apply WW3D to a real sheet forming process.

Reference: NUMISHEET'02 official web site [51].

Implementation: Elastic-plastic deformation. Maximum strain is less than or equal to 0.15.

Mesh refinement: No

Elements tested: WW3D, JET3D, and JET shell.

Material properties: Elastic-plastic,

DDQ (M2) manufactured and supplied by POSCO(Korea)

Young's modulus $E = 2.2137 \times 10^5$ Mpa, Poisson's ratio $\nu = 0.3$

Hardening law: $\bar{\sigma} = 547.763(\bar{\epsilon} + 0.008755)^{0.26921}$

Anisotropic parameters: $r_0 = 2.160$, $r_{45} = 1.611$, $r_{90} = 2.665$.

Fig.(4.27) Tools setup for Cylindrical cup drawing

$R_1 = 50.0$ mm, $R_2 = 51.25$ mm, $R_3 = 9.5$ mm, $R_4 = 7.0$ mm, R_0 : Radius of initial blank

Thickness = 1.0mm, Punch stroke = 40 mm, Friction coefficient = 0.1459

Blank holding force = 70,000N

Discussion and conclusion: As shown in Fig.(4.32) and Fig.(4.33), WW3D gives reasonable results for the punch reaction force and thickness strain distribution. JET3D overestimated the punch force because it does not completely eliminate shear locking. The shear locking control parameters used in JET3D are related to the reciprocal of the aspect ratio of the element. Because the aspect ratio is not large enough, so the shear locking is only partly eliminated, as discussed in section (2.6.3). The thickness strain distribution given by WW3D is closer to the thickness strain distribution given by JET Shell than JET3D does as shown in Fig.(4.10.7).

Fig.(4.28) Mesh used for JET shell simulation – 729 elements

Fig. (4.29) Deformed shape of shell simulation

Fig.(4.30) Mesh used for WW3D simulation – 396 elements

Fig.(4.31) Deformed shape of WW3D simulation

Fig.(4.32) Punch force vs. Travel

Fig.(4.33) Thickness strain distribution

4.11 Large curvature bending of a beam over a roller by a top uniform pressure

Name of the test: Large curvature bending of a beam over a roller by a top uniform pressure

Introduction: The test setup is as shown in Fig. (4.34). The dimension of the beam is 10.0×50.0×10.0. The radius of the roller is 10.0. The pressure is 0.1/unit area. If the ratio of bending curvature over thickness is less than 5, shell elements give inaccurate result because the shear strain through thickness is no longer constant. The vertical

displacement of the midpoint of the beam was recorded and compared to the result from a very fine mesh (100×20) by ABAQUS C3D8 element.

Purpose: The purpose of this test is to evaluate the performance of WW3D under the condition that the ratio of bending curvature over thickness is less than 5.

Reference: Original test, designed for this work.

Implementation: Large elastic-plastic deformation, plane stress. Maximum principle strain is less than or equal to 0.3.

Mesh refinement: Uniform mesh refinement and refinement in one direction were used.

Elements tested: WW3D, ABAQUS C3D8R, C3D8, C3D8I, and C3D8H.

Material properties: Elastic-plastic, Young's modulus $E = 500$, Poisson's ratio $\nu = 0.3$ and plastic linear hardening law is $\sigma = 10.0(\epsilon + 0.2)$.

Fig.(4.34) The simulation setup of the bending beam over a roller

Fig.(4.35) The deformed shape of the bending beam over a roller

Discussion and conclusion: As shown in Table (4.53) and Table (4.54), though WW3D gives reasonable results, about 7% error, there is still a difference between the results of WW3D and those of ABAQUS elements. The cause of the difference is that different contact search and treatment algorithms are used for SHEET-3 and ABAQUS. SHEET-3 uses N-CFS algorithm. The algorithm used in ABAQUS is not very clear. The beam-bending test is very sensitive to the contact condition. A small difference in contact can result in a large difference in the displacement. This is why the midpoint of the beam, instead of the tip displacement, is used for measuring the displacement.

MESH	ELEMENT TYPES					
Y×Z	WW3D	JET3D	C3D8	C3D8R	C3D8I	C3D8H
2×1	-17.527	-0.713	-11.704	-18.711	-15.248	-11.706
3×1	-21.506	-0.800	-18.949	-21.32	-19.756	-18.951
4×1	-20.913	-0.790	-21.096	-24.161	-20.413	-21.088
5×1	-21.405	-0.809	-22.343	-21.645	-21.208	-22.346
2×2	-14.848	-0.608	-10.05	-15.419	-14.435	-10.05
3×2	-18.462	-0.682	-15.342	-20.291	-18.594	-15.349
4×2	-19.264	-0.673	-18.478	-20.708	-19.363	-18.475
5×2	-18.482	-0.692	-18.085	-21.641	-19.235	-18.085
2×3	-14.667	-0.596	-9.813	-15.1	-14.19	-9.813
3×3	-18.021	-0.668	-14.806	-19.485	-18.417	-14.808
4×3	-18.989	-0.659	-17.942	-19.747	-19.44	-17.942
5×3	-18.079	-0.677	-17.246	-20.577	-18.863	-17.25
2×4	-14.537	-0.593	-9.759	-14.802	-14.404	-9.759
3×4	-17.821	-0.664	-14.67	-19.006	-18.531	-14.67
4×4	-18.833	-0.655	-17.866	-19.158	-19.464	-17.863
5×4	-17.83	-0.673	-17.086	-19.729	-19.044	-17.084
5×1	-21.405	-0.809	-22.343	-21.645	-21.208	-22.346
10×2	-19.586	-0.697	-21.174	-21.749	-20.679	-21.176
25×5	-19.123	-0.689	-20.401	-20.495	-20.395	-20.401
50×10	-18.978	-0.694	-20.371	-20.39	-20.355	-20.381
100×20	-18.976	-0.703	-20.356	-20.363	N/A	-20.356

Table(4.53) Displacement of the midpoint of the beam

MESH	ELEMENT TYPES					
Y×Z	WW3D	JET3D	C3D8	C3D8R	C3D8I	C3D8H
2×1	0.861	0.035	0.575	0.919	0.749	0.575
3×1	1.056	0.039	0.931	1.047	0.971	0.931
4×1	1.027	0.039	1.036	1.187	1.003	1.036
5×1	1.052	0.040	1.098	1.063	1.042	1.098
2×2	0.729	0.030	0.494	0.757	0.709	0.494
3×2	0.907	0.034	0.754	0.997	0.913	0.754
4×2	0.946	0.033	0.908	1.017	0.951	0.908
5×2	0.908	0.034	0.888	1.063	0.945	0.888
2×3	0.721	0.029	0.482	0.742	0.697	0.482
3×3	0.885	0.033	0.727	0.957	0.905	0.727
4×3	0.933	0.032	0.881	0.97	0.955	0.881
5×3	0.888	0.033	0.847	1.011	0.927	0.847
2×4	0.714	0.029	0.479	0.727	0.708	0.479
3×4	0.875	0.033	0.721	0.934	0.91	0.721
4×4	0.925	0.032	0.878	0.941	0.956	0.878
5×4	0.876	0.033	0.839	0.969	0.936	0.839
5×1	1.052	0.040	1.098	1.063	1.042	1.098
10×2	0.962	0.034	1.04	1.068	1.016	1.04
25×5	0.939	0.034	1.002	1.007	1.002	1.002
50×10	0.932	0.034	1.001	1.002	1	1.001
100×20	0.932	0.035	1	1	N/A	1

Table (4.54) Normalized displacement of the midpoint of the beam

The last parts of both Table (4.53) and Table (4.54) are the results of uniform mesh refine and aspect ratio is kept as 1.0.

4.12 Large curvature bending of a beam by a tip shear force

Name of the test: Large curvature bending of a beam by a tip shear force

Introduction: The test setup is as shown in Fig. (4.36). The dimension of the beam is $10.0 \times 50.0 \times 10.0$. The tip shear force is 20.0.

Purpose: The purpose of this test is to evaluate the performance of WW3D under the condition that the ratio of bending curvature over thickness is less than 5 without the interference of contact. The vertical displacement of the midpoint of the beam was recorded and compared to the result from a very fine mesh (100×20) by ABAQUS C3D8 element.

Reference: Original test, designed for this work.

Implementation: Large elastic-plastic deformation, plane stress. Maximum principle strain is less than or equal to 1.0.

Mesh refinement: Uniform mesh refinement and refinement in one direction were used.

Elements tested: WW3D, ABAQUS C3D8R, C3D8, and C3D8H.

Material properties: Elastic-plastic, Young's modulus $E = 500$, Poisson's ratio $\nu = 0.3$ and plastic linear hardening law is $\sigma = 10.0(\epsilon + 0.2)$.

Fig.(4.36) The simulation setup of the large curvature bending by a tip shear force

Fig. (4.37) The deformed shape of the large curvature bending by a tip shear force

MESH	ELEMENT TYPES				
	WW3D	JET3D	C3D8	C3D8R	C3D8H
1×1	-26.985	-0.932	-4.381	-17.103	-4.381
2×1	-14.985	-0.692	-9.471	-12.165	-9.47
3×1	-18.091	-0.792	-14.549	-15.039	-14.549
4×1	-19.638	-0.761	-17.009	-15.67	-17.074
5×1	-20.578	-0.786	-18.843	-19.618	-18.843
1×2	-22.001	-0.774	-4.106	-16.159	-4.106
2×2	-12.386	-0.595	-8.082	-11.597	-8.082
3×2	-15.294	-0.663	-12.164	-14.64	-12.163
4×2	-16.16	-0.637	-13.911	-15.455	-13.911
5×2	-16.819	-0.658	-15.231	-16.145	-15.23
1×3	-21.496	-0.783	-4.089	-16.1	-4.089
2×3	-12.062	-0.595	-7.877	-11.517	-7.877
3×3	-14.847	-0.670	-11.767	-14.474	-11.767
4×3	-15.625	-0.642	-13.396	-15.272	-13.396
5×3	-16.279	-0.665	-14.652	-15.97	-14.651
1×4	-21.195	-0.789	-4.073	-15.946	-4.073
2×4	-11.76	-0.602	-7.813	-11.353	-7.813
3×4	-14.531	-0.675	-11.658	-14.26	-11.658
4×4	-15.269	-0.644	-13.266	-14.984	-13.491
5×4	-15.878	-0.669	-14.492	-15.642	-14.491
1×5	-21.157	-0.777	-4.069	-16.032	-4.069
2×5	-11.754	-0.606	-7.7836	-11.372	-7.7836
3×5	-14.507	-0.677	-11.605	-14.28	-11.612
4×5	-15.259	-0.645	-13.193	-15.06	-13.194
5×5	-15.855	-0.671	-14.422	-15.683	-14.422
5×1	-20.578	-0.786	-18.843	-19.618	-18.843
10×2	-17.803	-0.653	-17.19	-17.031	-17.19
25×5	-17.131	-0.672	-16.923	-17.014	-16.923
50×10	-17.23	-0.678	-17.036	-17.075	-17.036
100×20			-17.366		

Table (4.55) Displacement of the midpoint of the beam

MESH	ELEMENT TYPES				
	WW3D	JET3D	C3D8	C3D8R	C3D8H
1×1	1.554	0.054	0.252	0.985	0.252
2×1	0.863	0.040	0.545	0.701	0.545
3×1	1.042	0.046	0.838	0.866	0.838
4×1	1.131	0.044	0.979	0.902	0.983
5×1	1.185	0.045	1.085	1.13	1.085
1×2	1.267	0.045	0.236	0.931	0.236
2×2	0.713	0.034	0.465	0.668	0.465
3×2	0.881	0.038	0.7	0.843	0.7
4×2	0.931	0.037	0.801	0.89	0.801
5×2	0.969	0.038	0.877	0.93	0.877
1×3	1.238	0.045	0.235	0.927	0.235
2×3	0.695	0.034	0.454	0.663	0.454
3×3	0.855	0.039	0.678	0.833	0.678
4×3	0.9	0.037	0.771	0.879	0.771
5×3	0.937	0.038	0.844	0.92	0.844
1×4	1.221	0.045	0.235	0.918	0.235
2×4	0.677	0.035	0.45	0.654	0.45
3×4	0.837	0.039	0.671	0.821	0.671
4×4	0.879	0.037	0.764	0.863	0.777
5×4	0.914	0.039	0.835	0.901	0.834
1×5	1.218	0.045	0.234	0.923	0.234
2×5	0.677	0.035	0.448	0.655	0.448
3×5	0.835	0.039	0.668	0.822	0.669
4×5	0.879	0.037	0.76	0.867	0.76
5×5	0.913	0.039	0.831	0.903	0.83
5×1	1.185	0.045	1.085	1.13	1.085
10×2	1.025	0.038	0.99	0.981	0.99
25×5	0.986	0.039	0.975	0.98	0.975
50×10	0.992	0.039	0.981	0.983	0.981
100×20			1		

Table (4.56) Normalized displacement of the midpoint of the beam

Discussion and conclusion: The deformed shape of the beam is shown in Fig.(4.37).

After deformation, the curvature at the fixed end after deformation is about 30.0, which is only 3 times of the beam thickness. The last parts of both Table (4.55) and Table (4.56) are the results of uniform mesh refine and aspect ratio is kept as 1.0. As shown in Table (4.55) and Table (4.56), without interference of contact, WW3D gives better results than the other 3 ABAQUS elements. In fact, in 22 out of 28 cases, WW3D has the best results.

4.13 Summary

12 numerical examples have been tested with WW3D, as well as with other 8-node hexahedral elements. The membrane behavior of WW3D is satisfactory because it passes patch test successfully. The stress calculation and the Radial-Return algorithm implemented in WW3D are correct and accurate, as shown in pure bending tests. The comparison of all 8-node hexahedral elements is shown in Table (4.57~615), respectively. Table (4.62) summarizes all the comparison. From Table (4.62), it can be shown that the performance of WW3D is the best among all the tested elements. JET3D is the second best. C3D8I gives very accurate results only when regular mesh is used. It performs poorly or even fails if the mesh is initially distorted. If only one layer of elements used through thickness for bending dominant problems, only C3D8I gives good results provided regular mesh is used. C3D8R does not show any bending stiffness under such a circumstance. WW3D has better performance than JET3D if only one layer of elements is used. WW3D gives good results if 2 or more layers of elements are used. For JET3D, at least 2 layers or sometime 3 layers of elements are needed in order for good

results. WW3D has better consistency than JET3D does as mesh refines. JET3D tends to over estimate the displacement while WW3D underestimates the results in most cases as expected for most FEM solution. Overall, C3D8, C3D8R and C3D8H are not good for the tests conducted in this work. In deep drawing, WW3D gives closer results to the results by JET shell than JET3D does. For large curvature bending tests, WW3D gives accurate results as discussed in section (4.11~12). The convergence of WW3D is also shown through all these numerical examples to be good and consistent.

Cantilever Beam Bending	Best	2nd Best	3rd Best	SDNE
WW3D	3	1	5	0.110
JET3D	4	7	1	0.060
C3D8I	5	3	4	0.093
C3D8R	0	1	1	0.276
C3D8/H	0	0	1	0.219

Table (4.57) Comparison on cantilever beam bending

Square Plate Bending	Best	2nd Best	3rd Best	SDNE
WW3D	11	26	3	0.093
JET3D	4	3	28	0.242
C3D8I	25	10	5	0.065
C3D8R	0	0	3	7.258
C3D8/H	0	3	1	0.563

Table (4.58) Comparison on square plate bending

Circular Plate Bending	Best	2nd Best	3rd Best	SDNE
WW3D	8	15	9	0.148
JET3D	23	9	0	0.256
C3D8I	0	0	0	N/A
C3D8R	1	8	23	19.687
C3D8/H	0	0	0	0.978

Table (4.59) Comparison on circular plate bending

Pinched Cylinder	Best	2nd Best	3rd Best	SDNE
WW3D	13	4	5	0.241
JET3D	6	13	3	0.187
C3D8I	0	0	0	N/A
C3D8R	3	5	10	27.261
C3D8/H	0	0	4	0.907

Table (4.60) Comparison on pinched cylinder

Scordelis-Lo Roof	Best	2nd Best	3rd Best	SDNE
WW3D	0	18	1	0.154
JET3D	18	1	0	0.124
C3D8I	0	0	18	0.162
C3D8R	1	0	0	0.719
C3D8/H	0	0	0	0.720

Table (4.61) Comparison on Scordelis-Lo roof

Summary	Best	2nd Best	3rd Best	SDNE
WW3D	35	64	23	0.149
JET3D	55	33	32	0.174
C3D8I	30	13	27	N/A
C3D8R	5	14	37	11.040
C3D8/H	0	3	6	0.677

Table (4.62) Summary of the all comparison

CHAPTER 5 A MODIFIED WHITLEY'S MODEL FOR PREDICTING LIMITED DRAWING RATIO

The original Whitley's model is based on following assumptions according to Hosford and Caddell [56]:

1. Only the deformation of material in the flange is considered. Work used for bend and unbend and against friction is not concluded;
2. The material does not work harden;
3. The deformation of flange material is assumed plane strain and plane stress;
4. Normal anisotropy is assumed;
5. Hill's anisotropic plasticity theory is used.

From Whitley's model, a conclusion can be drawn that Limited Drawing Ratio (LDR) for axisymmetric deep drawing is

$$\ln(LDR) = \frac{\sigma_w}{\sigma_f} \quad (5.1)$$

σ_w is flow stress in the wall and σ_f is the flow stress in the flange. Usually, they are

assumed to be equal. Using Whitley's model, $LDR = e = 2.72$. But in engineering

practice, the observed LDR is much less than e . So a coefficient η is to added to Eq.(5.1)

to compensate friction, binder pressure, sheet thickness, and die radius and Eq.(5.1) becomes Eq.(5.2). The typical value of for η is from 0.74 to 0.79.

$$\ln(LDR) = \eta \frac{\sigma_w}{\sigma_f} \quad (5.2)$$

A modified model was proposed in this study to include bending under tension effect and an equivalent friction force. For bending-under-tension test, Fox proposed Eq.(5.3) for the incremental force balance with both free and fixed rollers [57].

$$\mu = \frac{1}{\theta} \left(\frac{r + 0.5t}{r} \right) \ln \left(\frac{F_1 - F_b}{F_2} \right) \quad (5.3)$$

with

μ : Friction coefficient;

F_1 : Pulling force, fixed roller;

F_2 : Back force, fixed roller;

F_1^* : Pulling force, free roller;

F_2^* : Back force, free roller;

F_b : Force due to bending = $F_1^* - F_2^*$;

θ : Bending angle in radian;

r : Roller radius;

t : Sheet thickness.

Swift [58] derived the following equation by applying the principle of work to the complete process of bending and straightening.

$$\frac{F_b}{tw} = \frac{\sigma_y t}{2r} \quad (5.4)$$

where

w : Sheet width;

σ_y : Material flow stress.

Combining Eq.(5.3) and Eq.(5.4), it can obtain:

$$F_1 = F_2 e^{\frac{\mu \theta r}{r+0.5t}} + \frac{\sigma_y t^2 w}{2r}; w = 2\pi r_l \quad (5.5)$$

Back force F_2 has two parts: equivalent friction force on the flange because of blank holder pressure and force used to deform sheet.

$$F_2 = 2f\mu + 2\pi r_l \sigma_f \ln\left(\frac{d_0}{d_1}\right) \quad (5.6)$$

$$F_1 = 2\pi r_l t \sigma_w \quad (5.7)$$

Substituting Eq.(5.6) and Eq.(5.7) into Eq.(5.5) and after rearrangement, a new expression of Limited Drawing Ratio (LDR) can be obtained:

$$\ln(LDR) = \left(\frac{\sigma_w}{\sigma_f} - \frac{t}{2r} \right) e^{\frac{-\mu \theta r}{r+0.5t}} - \frac{f\mu}{\pi r_l t \sigma_f} \quad (5.8)$$

with

f : Blank holder pressure;

r_l : Die opening radius;

σ_f : Flow stress at the flange;

σ_w : Flow stress at the wall.

$$\frac{\sigma_w}{\sigma_f} = \sqrt{\frac{R+1}{2}} \text{ for Hill's anisotropic plasticity theory [56].}$$

For example, assuming:

Planar strain ratio, $R = 1.2$

Friction coefficient, $\mu = 0.15$

Blank holder pressure, $f = 20000\text{N}$

Die opening, $r_l = 50\text{mm}$

Sheet thickness, $t = 1.0\text{mm}$

Die radius, $r = 5.0\text{mm}$

Material flow stress, $\sigma_y = 300\text{Mpa}$

Eq.(5.8) gives $\ln(LDR) = 0.789$. If $R = 1.0$, Eq.(5.8) gives $\ln(LDR) = 0.745$.

By including bending under tension effect and an equivalent friction force, the proposed modified Whitley's model gives much better prediction of LDR than the original model without introducing any arbitrary coefficient.

CHAPTER 6 CONCLUSIONS AND FUTURE WORK

6.1 Summary and conclusions

A new 8-node solid hexahedral element, WW3D, was developed to combine the advantages of complex 3D solid elements with the simplicity and coding efficiency needed for introduction of arbitrary material constitutive equations. The new element eliminates shear locking by ignoring the shear strain terms involved in hourglass modes. The element avoids volumetric locking by eliminating the dilatational strain caused by hourglass modes. The performance of the new element was evaluated through 12 numerical examples. The results of the new element were compared with analytical solutions, where are available, and are compared with results obtained with other 8-node hexahedral elements. Based on the numerical examples tested in Chapter 4, the following conclusions can be drawn:

- WW3D does not suffer from locking problems, neither shear locking nor volumetric locking.

- **WW3D has the simplest formulation of 8-node assumed strain linear hexahedral elements found in the literature. It uses 8 integration points, thus avoiding special techniques to correct for reduced integration.**
- **WW3D gives accurate results and very good consistency. Its performance and consistency is as good as or better than JET3D or ABAQUS 8-node hexahedral elements. For bending dominant problems, 2 layers of elements through thickness give results similar to those obtained with 3 layers of JET3D element [1, 3].**
- **WW3D can be readily implemented with arbitrary constitutive relations because there are no enhanced strain terms involved. It can also be used for bulk forming simulation simply by reintroducing the omitted strain terms into the formulation.**
- **No predetermined or arbitrary parameters are needed for WW3D so the results are independent of arbitrary numerical choices.**
- **The practical use of WW3D for the simulation of large deformation processes has been demonstrated.**
- **Compared to other available hexahedral elements (JET3D, ABAQUS C3D8R, C3D8, C3D8I and C3D8H), WW3D gives the best overall performance**

among the six elements, as discussed in each test. JET3D is the next.

ABAQUS C3D8R holds the third place. ABAQUS C3D8 and C3D8H give identical results and they are the fourth. ABAQUS C3D8I is very unstable and unpredictable. It is the worst among all the tested 8-node hexahedral elements.

- The proposed modification to Whitley's model gives better prediction of LDR than the original model without introducing any arbitrary coefficient by including bending under tension effect and an equivalent friction force.

6.2 Future work

- Improve contact treatment. Contact treatment has become the impediment in the FEM simulation of sheet metal forming process. Accurately predicting and checking the contact status of every node is necessary condition of convergence. Good contact treatment algorithm can greatly enhance the capability of SHEET-3.
- Adding more constitutive laws. Currently, there are only Von Mises and Hill'48 plastic models in the new element. Since there are no extra strain terms in the kinetics of the new element, adding new material laws can be easy and accomplished readily.

- Using the new element to predict springback and compare it with shell elements, particular for small R/t values. At the current stage, the prediction of springback is done by using shell elements. Taking the advantage of the simple formulation and 8 integration points, using the new element in springback prediction is of great interest.

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