THE RATIONALIZATION OF INDUSTRIAL
DECISION PROCESSES

DISSERTATION

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PREFACE

This study is of a sort which is rarely undertaken in the field of engineering. It deals with the very foundations of engineering thought and the philosophical bases for engineering practice. As such it is not the kind of effort which the practicing engineer finds the time or interest to undertake. Nor do any of its conclusions appear to require changes in the theory or technique of present day engineering work. Rather it is an attempt to construct an academic foundation upon which the scope and competence of engineering technique may be built and extended. It has led the writer into many diverse fields, such as Philosophy, Mathematical Statistics, and Economics, in which he can claim no special competence. The problems discussed are almost without exception interdisciplinary in nature, and it is hoped that the violence done in discussing them has not been unnecessarily severe.

Special gratitude is due Dr. Paul N. Lehoczky for the opportunities and atmosphere which he has provided in the Department of Industrial Engineering, and for his guidance and encouragement which brought the work to a conclusion. The study would never have been undertaken had it not been for the spark of curiosity com-

municated by Dr. Loring G. Mitten through his boundless enthusiasm in teaching and research. Dr. Robert J. Wherry and Dr. D. Ransom Whitney read the manuscript and suggested important revisions.

It is impossible to indicate the magnitude of the contribution of Miriam L. Morris who suffered all the agonies attendant to such an undertaking, who willingly sacrificed a normal family life, and who personally brought the text to a tolerable level of readability.

William T. Morris

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Chapter 1.

INTRODUCTION AND HISTORICAL NOTE

The Significance of Decision

Attempting to justify a course of research on the decision process is much like attempting to justify research on human activity in general. The problem of decision making is of interest to every individual and group. One can think of all human activity as being made up of decisions and actions. Explicit concern with the decision problem is antique, but the last decade has brought a great acceleration of activity in several of the sciences. The role of the engineer as adviser (at least) on decisions of some consequence motivates this examination of the decision process as it occurs in industrial situations.

Scope

Views of the decision process, which we will call "theories of decision making", may be divided immediately into normative theories and descriptive theories. Normative theories prescribe methods of decision making which are thought to be most likely to bring success to the decision maker: they give advice on

how decisions ought to be made. Descriptive theories report how people actually do make decisions. For example, Economists are much concerned with the prediction of economic decisions made by consumers. This study is concerned exclusively with normative theories of decision making. Attention is directed toward methods which will enable the engineer to improve the decisions he makes. Thus the final product of this effort will consist of rules for making decisions in various situations. The situations of interest here are those which are commonly encountered by the engineer, the manager, the military commander, and the like. The words, "engineering" and "industrial" will be used synonymously to denote these situations.

We do not reject theories of decision which are descriptive as being of no interest. Such theories provide information which enters the engineering decision problem as evidence. For example, predictions of consumers' behavior in deciding among goods may be made on the basis of market research and may be significant for the engineer as part of the information used in his decisions about, say, product design. Research on descriptive theories is primarily the province of the social scientist, however. As examples of two very

useful descriptive studies see the work of Katona (27) and that of Simon (56).

The objectives of this study will be fourfold.

1. The fundamental requirement is, of course, an exact and complete statement of what we will call the generalized engineering decision problem. As far as can be discovered, this has not been attempted in the engineering literature. This study makes no attempt to discuss a general normative theory of decision. It seeks rather to make some progress toward the understanding of the normative principles which are applicable to a restricted class of decision problems which are to be called engineering or industrial decision problems.
2. Any theory of decision involves a theory of value, and thus efforts must be made to understand the theory of value applicable to industrial decision situations. Our effort here will primarily be devoted to showing the relationship between recently developed value theories and the industrial situation. We shall be particularly concerned with aspects of value theory which might be used as engineering tools, rather than the axiomatic development of a theory.

3. No discussion of normative decision making can neglect the recent interest in theories of consistent decision, such as that of Savage (51) or Ramsey (47). We distinguish here between the logic of consistency and the logic of truth. For example, one might operate an accounting system on the assumption that two plus two equals five. As long as this assumption was not violated throughout the system we would say that it is logically consistent. Experience (experiment) would indicate that we will not long succeed with such a system. We would say, therefore, that it is not logically truthful. If one examines a text book on Engineering Economy the system presented there is logically consistent but not necessarily logically truthful. That is, a system of manipulating estimates and making decisions is presented, but there is no evidence that these estimates are the correct ones or that the decisions made according to the rules will succeed in application.

Our efforts will be directed toward pointing out the relation of theories of consistent decision in the face of uncertainty and theories of value (some of which possess a consistency interpretation) to

the engineering decision problem.

4. Our final contribution will be to study the theory of decision in the face of uncertainty about the future as applied to some situations which are of interest to the engineers. Probability theory is rich in normative implications, many of which have not been explicitly stated. We will also comment on such problems as the timing of decisions.

These statements will seem a vague and possibly confusing discussion of what is to follow, but unfortunately it is difficult to avoid this until the tools are at hand for a more precise explanation.

The Theory and Engineering Practice

We feel that it is important to indicate how this effort bears upon the practice of engineering. More directly, we hope to anticipate the question, "How does all this theory construction help the decision maker concerned with real problems?" Now there are two aspects of this and any other academic discussion of industrial problems which discourage the practical man. One is the "philosophical basis" with which such discussions begin, the other the mathematical formulae with which they end.

Of the philosophical basis, the practical man says, fairly, that it may be fine but it is irrelevant to the problems which he must solve every day. With this we tend to agree but justify our concern with the foundations of the industrial decision problem with the following statements:

1. The development of engineering tools cannot proceed on anything but a chance basis without the establishment of a philosophical foundation.
2. The chances of success for a tool are greater if its basic aspects have been carefully worked out, We think that time study may be a case in point here.
3. The examination of the philosophical basis itself may provide a surprising wealth of guidance for daily use. For example, engineers agree that they should act in a rational or reasonable way but are often at pains to indicate why a given decision was rational. We think the provision of a definition of rationality may be of considerable practical importance.

Of the mathematical formulae which result from academic treatments, the applied viewpoint says that it is impossible to know the quantities involved with anything

like the precision implied by the formulae, and that no formula can be rich enough so that it cannot be discarded in favor of "human judgment". To the first of these objections we reply:

1. The usefulness of mathematical models is in no sense restricted to occasions when numbers are available for substitution therein. Much use can be made of such models by simple understanding of the implications.
2. It becomes increasingly evident that the information available from accounting systems is not at all appropriate for use in engineering problems. (For the accountant is an historian.) There are almost certain to be changes in the kind of information collected by industry in the near future.

To the second objection a reply is far more difficult. We feel that in the field of decision making much of the lack of completeness in the theory is due to the inadequate methods of value measurement. This problem we deal with at length below. Beyond this perhaps the best reply is that of Laplace who says of this type of mathematical formulation, "it makes us appreciate with exactitude that which the exact minds feel by a sort of instinct without being able to give a reason for it

Even in the things which cannot be submitted to calculus it gives the surest hints that can guide our judgment. . . and teaches us to avoid the illusions that oftentimes confuse us." (35)

Historical Note

The proper historical frame of reference for this study can only be supplied by an examination of two fields which until recently have developed more or less independently; Decision Theory and Engineering Economy. The development of the field of Decision Theory has been reported in some detail by Edwards (11), Arrow (2), and Savage (51). It will be discussed here in outline form only. The development of the field of Engineering Economy has been discussed by Lesser (30), but with emphasis almost entirely on the position of interest rate calculations in the techniques of engineering decision making. The important point is that these two fields are to some extent faced with common problems, and a gradual assimilation of the techniques of the more highly developed Decision Theory into engineering practice is at least regarded as an attractive possibility, if not an actual movement under way.

The Development of Decision Theory

Much of the work which has gone on in the sciences could be broadly labelled as directed at the problem of decision making, from either a normative or descriptive viewpoint. There is however, a far more restricted field of study which is specifically labelled as being concerned with decision making, and it is this latter field which seems to hold promise for the enrichment of engineering techniques. Several lines of thought have contributed importantly to the development of Decision Theory. The development of classical probability theory was the result of the need for information upon which to base decisions in playing games of chance. Bayes provided an early and important normative rule by asserting that choices should be made in these situations so as to maximize the expected value of the money return. This was the beginning of the theory of decision making in the face of incompletely known consequences. Historically, however, there has been much more interest in describing how people actually do behave in making decisions, rather than how they ought to behave. This has been the approach of the Social Sciences. Economics has been explicitly concerned with the prediction of decision making behavior ever since the time of Bentham. Most classical theories of decision making behavior are founded on the assumption of "Economic Man".

The attributes of "Economic Man" include complete information as to what courses of action are open to him, and what the outcome of each course of action will be. He is rational in the sense that, for any two states of the world, he can always state that he prefers one of them or is indifferent between the two. These preference statements must be transitive; if he prefers A to B and also prefers B to C, then he also prefers A to C. A second aspect of the assumption of rationality is that "Economic Man" always makes decisions so as to maximize something. Two things are especially interesting about this set of assumptions. First, it implies full knowledge of the future, thus theories based on this foundation are known as theories of riskless choice. Secondly, although the Economists were interested essentially in a descriptive theory, the assumption of "Economic Man" can, with proper interpretation, provide the basis for a normative theory. It must be agreed that the kind of rationality ascribed to "Economic Man" would be eminently desirable for engineering decision making. It was also a part of the classical theory that what was to be maximized was the utility to the decision maker of bundles of commodities among which he chose. Until very recently this school has been concerned about whether utility could be measured on an ordinal scale or an

interval scale. This controversy revolved about the question of whether the classical analysis based on indifference curves was all that could be expected, or whether some stronger scale arising out of decisions made in the face of uncertainty could be used to measure utility.

Bayes was not concerned with utility in his investigation of choices under conditions of risk because he had assumed that the money values of outcomes were an appropriate expression of their utility. Daniel Bernoulli introduced the notion of maximizing the expected utility rather than the money value in order to explain the behavior of people who purchased insurance where the expected money return is negative. The most significant development in the area of risky decision making came in 1944 when von Neumann and Morgenstern showed that if the assumption of "Economic Man" could be modified to include a partial ordering of probability combinations of states, then utility was measurable on an interval scale having an arbitrary zero point and equal intervals. This made utility a continuous variable which corresponded to the expected value of the utilities of a probability combination of states.

With the work of von Neumann and Morgenstern and also

Wald began the theory of choice in the face of uncertainty. Uncertainty implies that one has no way of assigning probabilities to outcomes as is done in the theory of choice under conditions of risk. The theory of games deals with situations where the outcomes are dependent upon the action of an opponent or competitor. Wald's work deals with situations where a statistician "plays" against nature, and the problem is called a statistical game. Both problems lead to "strategies" based on the minimax principle, minimizing the maximum expected loss. This work does not describe how people actually do make decisions, but is normative in nature.

There is extensive literature on the interrelated problems of utility measurement and the interpretation to be made of the calculus of probability for decision making purposes. Also considerable work has been done exploring the principles of choice which have been proposed both in normative and in descriptive theories. Since these topics are discussed in some detail below, they are only noted here.

It is tempting to suppose that many of these ideas which have been developed entirely outside the field of engineering can be transplanted and directly applied to the engineering problem. There are some serious obsta-

cles to such a program, but also some real possibilities, both of which will be the subject of this study.

The Development of Engineering Economy

Following the general perspective of the field of Engineering Economy given by Lesser (30), the work of Wellington (64), Fish (16), and Grant (23), will be considered as representative of the origin of the field, its intermediate development, and its modern conception, respectively. Lesser has discussed the introduction and progress of the use of interest calculations upon which much of the formalism in Engineering Economy has been based. All of these writers have advocated the use of interest calculations in various ways. This concept is discussed at length in Chapter 3 below. It is more significant to examine the progress made by these writers in their views of value measurement, uncertainty or incomplete information, judgment and irreducible data, and the methods by which the information needed for decision making should be obtained. It is in these areas that the greatest possibilities for development seem to appear.

Value

Wellington recognizes clearly that the objectives of

the systems in which the engineer operates may be something other than dollar profits. (I) He sets the tone for future writers by indicating that any attempt to measure values of outcomes for other objectives would be futile. He recognizes implicitly however, the very important concept that value is relative to the state of the system, or that the marginal utility of any income is not necessarily constant. This surprisingly advanced view of the problem of value has not been supported to any extent by either Fish or Grant. Grant admits the possibility of existence of other objectives besides profit but relegates this to a discussion of the relation of engineering decisions to welfare economics. His implication seems to be clearly that the profit motive "should" be recognized as the only one.

I. "Neither does it follow that the deciding motive is direct pecuniary profit; for the line may be of great value to the inventors and the public and yet never pay such profit. In fact the railway system of the world taken as a whole, and especially that of the United States, has been only moderately profitable in any direct form: owing to the very large proportion of lines which have been built simply to increase the value of land, to afford local transportation facilities, to bring traffic to the main line, and for similar purposes. Yet the resulting gain to the community, from these indirect advantages alone, has been vast beyond computation " (64, p. 14).

None of these writers make any attempt to indicate how values should be measured beyond the use of exchange values and the use of interest rates to express the time value of money. Thinking about values seems to have been much influenced by the kind of analysis developed by Marston and Agg. (39) The concept of exchange value eliminates all aspects of a system of commodity which cannot be transferred from the seller to the buyer.

Uncertainty

Throughout the development of Engineering Economy the problem of uncertainty about the future and incomplete information has been recognized. (II) Various rules for dealing with the future have been implied but seldom explicitly stated. These rules might be reduced to such statements as "Select the most probable outcome and act

II. "To reach correct decisions as to what ought to be done requires that...we should foresee exactly the volume and sources of future traffic.... This we cannot do. To foresee such details perfectly is impossible. To foresee them in any degree we are obliged to do that most hazardous thing- to look forward to and "discount" the future; to make - and act upon - what is after all, nothing more than a guess at the probable course of future events." (64, p. 15)

In this respect it is important to note that Wellington later uses the word "discount" in situations where it is clear that he does not necessarily imply the reduction of future values.

as though it were certain", "Base the decision on the range of possible outcomes" (the exact principle of choice is usually not stated), "Base the decision on the expected values of the outcomes", or "Be conservative". None of these writers present clear, unequivocal or rational rules for dealing with the future.

Judgment and Irreducibles

Beginning with Wellington there has been a general recognition of the necessity for the submission of engineering analysis to the final arbiter of mature judgment. This may be interpreted as an implicit recognition of the inadequacy or incompleteness of engineering techniques. There has been little comment on the exact role which judgment should play, or how judgments could best be made. (III) Fish discussed the methods of combining judgments, but the subject progressed little beyond his preliminary analysis. See Chapter 6 below.

There has been a general recognition of the problems of irreducible data which arise in engineering decisions, upon which judgment must somehow operate. There is some confusion however, as to the exact nature of these irre-

III. Wellington notes however, that every attempt must be made to develop "some guide which will reduce unavoidable guesswork to its lowest terms". (64, p. 23)

ducibles. If one admits only profits as the objectives of a system, then any aspect of an outcome which implies profits but which cannot be expressed as an equivalent profit is an irreducible aspect. If one admits other objectives as well as profits for the system, then the range of irreducibles is considerably broadened. The problem of reducing aspects of outcomes to quantitative equivalents of profits and/or of the other objectives, as well as the problem of reducing such quantifications to a common basis of measurement arises. The first problem is the one which generally appears to be implied by Engineering Economy writers, but the second interpretation might also be implied. (IV, V, VI)

IV. "Physical value of a property is the value of that part of the property which has a physical existence so that it can be apprehended by the senses." (38, p. 12)
 "The intangible value of a property is the value of that part of the property which does not have physical existence." (38, p. 13)

V. "The intangible analysis will attempt to evaluate the worth-whileness in terms of those factors which are difficult to evaluate because the factors of human judgment enter largely into their valuation." (5, p. v.)
 "The economy analysis dealt with those factors pertaining to the project which could be described with monetary symbols. The intangible analysis has to do with a study of the project from the point of view of the factors that cannot be described with mathematical or monetary symbols." (5, p. 169)

VI. "A quality whose effect on choice can be numerically expressed we shall call a reducible datum: and we shall give the name irreducible datum to a quality whose effect on choice cannot be so expressed." (16, p.9)

Methods of Obtaining Information

It is difficult to understand how the development of more and more complex formal decision making techniques could have progressed since the time of Wellington with such a lack of attention devoted to the problems of obtaining information. Any decision making formalism is useless, of course, unless the information required to apply it to an actual problem can be obtained. Further, it must be possible not only to obtain the necessary information, but it must be economically obtainable. The literature has been completely free of any discussion of this problem, beyond the casual reference to the need for making judgments. In recent years a number of references have appeared to the fact that most industrial systems do not actually apply the standard techniques of Engineering Economy to their decision making problems. One of the most important reasons given for this is that the necessary information is not at hand, or could not economically be obtained. This problem now begins to appear as one of the more embarrassing problems facing engineering decision makers.

VI. (Cont'd.) "All the irreducible data considered in this chapter are of concern of course to the investor, and many of them are directly pertinent to questions which the engineer must answer." (16, p. 128)

Recent Work

Recently there has been a definite interest in enriching the tools applied to the problem of engineering decision making and in broadening the scope of the problems considered. This tendency seems to mark the beginning of an effort to incorporate the more advanced methods which might roughly be grouped under the heading of "Decision Theory" into the analysis of engineering decision problems. For examples of this effort see Mitten (43), Gitzendanner (21), Symonds (61), and Stanly (59). The present study represents an effort to contribute to this trend.

Chapter 2.

RATIONALITY FOR ENGINEERING DECISIONS

Basic Model of the Decision Process

It is convenient to adopt initially the basic model of the decision process which is rather commonly used in discussions of this sort. We define the decision process simply as the process of selecting between alternative courses of action. For the decision process to take place a situation must exist involving the presentation to the decision maker of more than one possible course of action. The selection must be between alternatives which are actually courses of action, or operations which lead to outcomes or events having some observable effect on the system within which the decision is made. We will consider the following abstract elements of the decision situation;

1. alternative courses of action
2. states of the world or states of the environment in which the actions are to be taken
3. outcomes which result from the adoption of particular courses of action in particular states of the world

In carrying out the process of decision the decision maker undertakes the following operations;

1. identification of the courses of action which are candidates for selection

2. description of the states of the environment which are relevant to the outcomes of the decision
3. prediction of what the outcomes will actually be for each action and each state of the world
4. evaluation of the outcomes in the value system appropriate for the system in which the decision maker operates
5. selection among the courses of action according to a principle of choice which looks in some sense toward the maximization of the value of the outcomes

We shall modify the basic model as the need arises, but for the present it will provide a basis for our discussions.

The following symbolism will be used;

a_i = a course of action ($i = 1, \dots, n$)

s_j = a state of the world ($j = 1, \dots, m$)

$a_i(s_j)$ = the outcome of the i th course of action undertaken when the j th state of the world obtains

u_{ij} = the value of $a_i(s_j)$ measured on the utility scale of the decision maker. This may also be written $u[a_i(s_j)]$.

We will symbolize a decision situation by a matrix of the following form;

	s_1	s_2	s_3
a_1	u_{11}	u_{23}	u_{13}
a_2	u_{21}	u_{22}	u_{23}
a_3	u_{31}	u_{32}	u_{33}
a_4	u_{41}	u_{42}	u_{43}

Rationality for Engineering Decisions

Our first task is to state as clearly as possible what we take the engineering decision problem to be. Our second, is then to indicate the characteristics which we feel a solution to the problem must have, so that if we should find such a solution it will be recognized as such. We do not argue that any of the things which we shall say are exclusively true of engineering decisions.

The engineering decision problem is, of course, of a structure identical with that we have described as our model of the basic decision problem. It has for its elements courses of action, outcomes, values, and states of the environment. We shall define the engineering decision problem then, not by the nature of its basic elements, but by the nature of the system within which it occurs. We think of the engineering decision problem as arising and being carried out in the course of operation of a productive system or task-oriented organization. It is the restrictions which the containing system places on the decision process which raise the problems of engineering decision making.

The most important kind of restriction which all of the systems of concern to us place on the decision problem

is that of a value system. A value system to us is a method of assigning values to the outcomes of courses of action which is particular to the system within which the action is taken. The values of an industrial system may be expressed by writing down what the Economist calls its utility function. A utility function is a function which assigns to every possible outcome of a decision situation a number which measures the value of that outcome to the system. In general the utility function of a system is applicable only to that particular system. Thus we cannot make decisions in general, but must evaluate the outcomes we predict on a separate scale for each system context. We think however, that there are some characteristics of the utility functions of productive organizations which are common enough to take as particular restrictions which arise in the engineering decision problem. The most significant of these general restrictions arising out of the utility functions which apply to the engineer, we think are the following:

1. Time: Time itself has a value on the utility scales of most productive organizations which results in a deadline of some sort before which the decision must be made. This will be made clear when we state the problem of the timing of engineering

decisions.

2. Information: The amount of information or evidence which may be amassed prior to the making of a decision is restricted because of time, the value of the resources which must be expended to obtain the information, and the capabilities of the system. By the capabilities of the system we mean the ability to know, realising that it is impossible for the system to obtain or provide some kinds of information.
3. Resources: The value of the resources which are consumed simply in carrying out the decision process itself is such that for many situations the process must be greatly abbreviated.
4. The Value of Information: The utility function is generally such that information has no intrinsic value of itself. It has value only if it improves the results of the decision making process.

We have in mind in this discussion productive systems which will ordinarily include a number of human decision makers. We shall ignore completely for the present the sociological aspects of this situation by considering a single impersonal decision maker who carries out in the abstract the decisions made within each pro-

ductive system. Thus when we use the term "the decision maker" we refer to a single individual or group of individuals as the case may be. We are interested really in the decision making function alone. We presume further that it is not necessary to distinguish between the information available to the decision maker and that available to the system as a whole, nor is it necessary to distinguish between the value system of the decision maker and that of the organization. In this of course, we neglect all the work that has been done in the theory of organization, which we consider a part of our concern but must regretfully omit.

Axiom of Long Run Success

We state now a fundamental axiom upon which some important parts of the discussion of engineering decisions will be based. This axiom may be called the "Axiom of Long Run Success". It may be stated very simply as follows: "The best decision rules are those which succeed in the long run." This axiom has a strong intuitive appeal for those in engineering systems, and may be justified in a number of ways. The reader will soon discover that this axiom makes it possible to avoid a number of important theoretical difficulties in the discussion to follow. It is hoped that the justifi-

cation for the axiom is sufficiently strong that it will not be regarded simply as a device for theoretical simplification. It must be clear however, that the a priori case in favor of this axiom may not be stronger than the appeal of other axioms, and it is our purpose here to explore the consequences of the Axiom of Long Run Success rather than to argue for its adoption as a universal basis for engineering decisions. We now state the axiom in a somewhat formal manner for clarification.

Consider a particular sequence of decision situations which may arise over time in the existence of an industrial system. The sequence may be finite, or if the system is one for which indefinite existence is contemplated, the sequence may be infinite. Each of these decision situations requires a decision rule by which the system may choose between the available alternatives. As the result of making such a choice the system is thought to pass from one state to another. More specifically, the outcome of any particular decision in the sequence is thought of as changing the system into a new state. The purpose of the Axiom of Long Run Success is to indicate what sort of sequence of decision rules should be used in dealing with such a particular sequence of decision situations. These decision situa-

tions are interdependent in the sense that the outcome of any decision situation depends upon past states of the system and thus upon past decisions.

The Axiom of Long Run Success states that one must apply a sequence of decision rules such that the value of the final state of the system is greater than the value of the final state of the system resulting from the application of any other sequence of decision rules. In other words, the axiom requires the use of the sequence of decision rules which maximize the value of the final state or the last state of the sequence.

The above statement is essentially a deterministic one. In the case where the value of the state resulting from the application of a particular decision rule to a particular decision situation is a random variable then the axiom requires the use of the sequence of decision rules which satisfies the following statement:

If the best set of decision rules is repeatedly applied to the sequence of decision situations in question, then the probability that the value of the final state is greater than the value of the final state for any other sequence of decision rules approaches one as the number of applications increases.

For the very special case in which the decision situations are independent and the values of outcomes are additive, then the axiom may be reduced to the following statements. We consider a particular sequence of n decision situations, the i th of which we denote by d_i . We consider also sequences of decision rules which might be applied to the sequence of situations, denoting a particular sequence of rules by N . Then in the deterministic decision problem the application of the sequence N (or a particular member of N) to the situation d_i results in an outcome of value $u[N(d_i)]$. The Axiom of Long Run Success states that one must use a sequence of decision rules N^* such that

$$\sum_i \{ u[N^*(d_i)] - u[N'(d_i)] \} > 0$$

where N' is any other sequence of decision rules.

In the probabilistic decision problem $u[N(d_i)]$ is a random variable. The axiom requires the selection of a sequence of decision rules N^* such that for repeated application of N^*

$$\Pr \left\{ \sum_i u[N^*(d_i)] \geq \sum_i u[N'(d_i)] \right\} \rightarrow 1$$

as the number of applications increases.

In another special case where infinite sequences of

decision situations are of interest we may consider a particular infinite sequence of situations and particular finite sequences of decision rules. The axiom might then be interpreted as requiring a sequence N^* satisfying the appropriate one of the two statements below:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \{ u[N^*(d_i)] - u[N'(d_i)] \} > 0$$

where the limit is assumed to exist;

or

$$\Pr \left\{ \sum_{i=1}^n u[N^*(d_i)] \geq \sum_{i=1}^n u[N'(d_i)] \right\} \rightarrow 1$$

as n approaches infinity.

Marschak also gives a rule of long run success which is similar to this one, but our use and interpretation of it is completely different. (36)

This axiom makes the objectivistic or relative frequency interpretation of probability theory directly applicable. It also provides a direct justification for the rule of maximization of the expected utility which is used exclusively in engineering decision making at present. This axiom is the one upon which all engineering application of probability theory rests.

Now it is still possible that it may be necessary to

eliminate from consideration under the axiom of long run success those decision situations which clearly involve uncertainty of the kind which is not expressible in relative frequency terms. If there are decisions which can be shown to be in some sense strictly one time decisions, possibly because of the magnitudes of the values of the outcomes, then a strong argument must be admitted for the inapplicability of our axiom. We feel however, that all such situations can be handled by the same rules one would apply to relative frequency situations by appealing to the concepts used by Ramsey in the development of his personalistic interpretation of probability. (47) Ramsey says, "The very idea of partial belief involves reference to a hypothetical or ideal frequency belief of degree m/n is the sort of belief which leads to action which would be best if repeated n times in m of which the proposition in question is true." If one accepts this view, then regardless of the number of times a particular decision was expected to occur, one would act as though the decision were to be made in a large number of hypothetical instances.

We discuss now the justifications of the axiom of long run success which arise out of the context within which

we are working. First, we think that the establishment of the set of decision rules which is to be used in engineering decision problems is equivalent to the establishment of policy. The very idea of policy involves the establishment of a set of rules for action in a number of instances. It seems reasonable to fix policies (as management must) so that the policy will be best for a number of instances of application. We say then that we are talking about policy formulation, rather than decision making on an "emergency" basis.

Finally we say that if one considers the class, say, of decisions which are to be made concerning the replacement of a given type of machine, the number of decisions will be thought small for any reasonable period of time in the life of a productive organization. When however, it is realized that the rules we propose are to be applicable to all the decisions which the organization makes in the course of its operation, then we very quickly bring a large number of decision situations under consideration, and the axiom again appears rational.

The Axiom of Long Run Success leads directly to the rule of selecting among prospects so as to maximize the expected utility. (See chapter 4 below.) This result

arises in various ways in all systems including those of Ramsey (47), von Neumann and Morgenstern (18), and Savage (51). It arises in different ways in each case depending upon the axioms which are selected. If it is true that the selection of axioms is entirely a matter of taste and that they can all be criticised as gratuitous, then we advance the Axiom of Long Run Success as simply an axiom which is appealing to the engineer.

Solutions to the Engineering Decision Problem

Having stated at least a tentative definition of the engineering decision problem and the Axiom of Long Run Success, we are in a position to state the characteristics of a solution to the engineering decision problem. We say simply that such a solution is a set of rules which guide behavior in such a way as to satisfy the Axiom of Long Run Success. In other words, we seek a set of rules for various engineering decision situations which will be optimal in the sense of this axiom. As we shall show, there are some problems in which no such set of rules may be obtained. We also take the Axiom of Long Run Success to be a definition of rationality for engineering decisions. Thus we adopt the basic premise that to be rational in engi-

neering decisions is to make choices which will satisfy the axiom. In cases where no such choice may be made, we say that rationality is not possible. Our problem is thus how to find the sets of decision rules which will in fact guide us to choices that will satisfy the axiom. We enter now on a discussion which we hope will lead us toward such sets of rules. This method we call rationalization.

Rationalization

The method by which we propose to examine engineering decisions we will call rationalization. It is related to some views of this term found in the literature but considerably different from others, so we shall take some pains to indicate exactly what we mean by rationalization in this context. Rationalization of the process of decision making we take to have three aspects; explicitness, logical consistency, and logical truth.

By explicitness we mean that the decision process be carried out with the conscious inclusion of the steps which we have indicated in our model of the general decision process. That is, we require that the decision maker be able to tell us explicitly what alternatives he considers, how he forecasts their outcomes,

how he assigns values, and the like. It is not necessary that the decision maker use the model exactly as we have outlined it, but only that he be able to provide us with sufficient information so that we may reconstruct the decision. The requirement for explicitness arises of course, out of the necessity for gaining an understanding of good and bad decision making performance. Without explicitness we have no method of criticising and modifying decision making behavior. We are primarily interested in avoiding decisions which are based on intuition alone, judgment alone, or some "crystal ball" process where the elements which lead to choice cannot be clearly stated. We do not deny that decisions arise in our context which must be based in part on judgment, but we hope that judgment alone is not the sole basis for business decisions as is sometimes professed. We shall later show that the requirement for the use of judgment arises in quite a different way from what is generally thought and that there are some rules which may be used not only to be consistent in the use of judgments which we have made, but also to make judgments themselves in certain better ways.

The second aspect of rationality which we require is that of logical consistency. By this we mean of course,

the kind of consistency which we find in mathematics, the kind, for example, manifested by the theory of probability. This requirement becomes important in the following way. When we have answered a particular question relative to the making of a decision, whether the question is answered by observation and inference, by judgment, or by deduction, we must then behave as though the answer to the question is as we have found it throughout the decision at hand and such others for which the answer may be relevant. This is simply to say that if we agree that the sum of two plus two is four, we do not somewhere in our reasoning say that four plus four is seven. Rationality thus requires that our decision making behavior in engineering situations be free of contradictions. Ramsey's comment on this point leads us into the requirement for logical truth (47) "... the most generally accepted parts of logic, namely formal logic, mathematics and the calculus of probabilities are all concerned simply to ensure that our beliefs are not self contradictory." In our context we may well read "decisions" for "beliefs". Ramsey says further, "We put before ourselves a standard of consistency and construct these elaborate rules to ensure its observance. But this is obviously not enough; we want our beliefs to be consistent

not merely with one another but also with the facts; nor is it even clear that consistency is always advantageous; it may well be better to be sometimes right than never right."

The aspect of logical truth we take to mean that our decisions, to be rational, must be in agreement as far as possible with our observations about our environment, or our inferences from these observations. This is the critical requirement, without which one could hardly hope to find a set of rules which would satisfy the axiom of long run success. We are concerned with how well a set of rules will perform when actually applied to the real environment, and we think it is essential that inferences about the real environment be a part of these rules wherever possible.

Probability and Values

The two most important kinds of data required for most engineering decisions are probability information and value or utility information. We draw now the very important distinction between subjective probability and objective probability and between subjective utility and objective utility. We say rather loosely that subjective utilities and subjective probabilities are data that arise as the result of judgments, or the

consultation of ones accumulated experience, and in general they may be called one's beliefs. Whereas we think of objective probabilities and utilities as arising as the result of some process of experimentation including observation and inference. Objective probabilities are usually thought of as limits of relative frequencies, and interpretations of the calculus of probability in these terms is called the relative frequency interpretation. Objective values are not so clearly differentiated, and we shall simply call them the value data determined on the basis of some more or less objective experimental process.

Both the calculus of probability and the calculus of values (as we shall try to present it) can be interpreted in either subjective or objective terms. From the viewpoint of the development of these calculi it makes no difference which interpretation is made in most cases, if indeed any interpretation at all must be made. The important point with respect to engineering work seems to be that if one inserts subjective data into either calculus one then produces a theory of decision which is consistent with the data. Thus we meet the first two requirements of rationality, explicitness and logical consistency. In order to achieve the

third requirement we must insert into the calculi objective data which are in the form of inferences based on controlled observations about the environment.

It is not appropriate to this study to enter the discussions on the subject of the foundations of probability theory or of the interpretation of the calculus of probabilities which is acceptable for engineering decisions. Nor is it clear that any finding as to the superiority of any particular interpretation of the calculus would in any way at all change the methods by which engineering decisions would be made. This is possibly true of research in the foundations of any field. Although rationality is here presented in a form which favors the frequency theorists, there can be no serious objection to an attempt to base a theory of engineering decision on a personalistic interpretation. Nor can there be any objection to claiming a relationship between the two interpretations, such as the idea that experience will produce personalistic probabilities which are close to objective probabilities. The complete presentation of the problem is given by Nagel. (44)

The Status of Engineering Economy

It would be rather easy in the light of recent literature on the decision process to be extremely critical of the Engineering Economy approach to a normative theory of engineering decision making. We think however, that within the bounds which it has set for itself, Engineering Economy has been adequate as a guide for decision making. Its basic philosophy has been to deal with decision situations in which the outcome which will result from each course of action takes a single known value. The great emphasis in Engineering Economy has been on questions of value. It has assumed a particular kind of a utility function in which the value of an outcome is expressed in terms of dollars, according to the rules of the accountant or some special rules set up by the discipline itself. As a result most of the effort expended on behalf of Engineering Economy has dealt in one way or another with compound interest calculations. We feel that thinking among Engineering Economists has advanced beyond the state which one gleans from the current Industrial Engineering literature. Thus, instead of undertaking an elaborate critical review of the subject, we shall simply state a few of the most important limitations which we hope will be in some way

affected by this study.

1. Engineering Economy does not deal with the observational aspects of the engineering decision problem in any way. The rules of Engineering Economy are explicit in some respects and are primarily concerned with the provision of logical consistency. We would therefore say that in our context they are only partially rationalized. They say nothing about observation of the environment or inferences from these observations. The Engineering Economist begins with some predictions which are deterministically expressed, but nothing is said about how these predictions are obtained.

2. Engineering Economy provides no means for including the so called "intangibles" or "irreducibles" in the decision process. This is one reason why writers on the subject emphasize the need for judgment. Any part of an outcome which was not reducible to a dollar value according to the rules provided was excluded from the formal part of the decision process. This makes the decision model useless whenever the irreducibles are of sufficient importance to change the outcome of the decision as it would be made on the basis of dollar valued out-

comes alone.

3. Engineering Economy neglects uncertainty on the part of the decision maker. We think the theory of probability is an admirable way of expressing uncertainty about the future which is a very real part of many decisions. We shall have much more to say about the nature of this uncertainty and the problem of choice in the face of it. Suffice it to say here that we think there is no longer any reason for avoiding the enrichment which the use of probability theory can bring to normative rules for engineering decisions. Although the interest rate is sometimes used as a device to express uncertainty, we think the theory of probability is far more appropriate and useful. As a matter of fact, in certain situations the use of the interest rate will be shown to be definitely misleading.

4. Our final criticism is that of the assumption of a rigid set of systems postulates within which the Engineering Economy is cast. This set of postulates assumes that maximizing dollar profits is equivalent to the maximization of utility for all systems to which it applies. It makes no provision for other utility functions, nor even for such phenomena as

the diminishing marginal utility of money. No postulates are admitted with respect to the restrictions of time, information completeness, and the consumption of resources in the decision process.

Chapter 3.

THE MEASUREMENT OF VALUES

Theories of Value for Engineering Decisions

One of the criticisms which has been levelled at Engineering Economy is that its value system is of too limited a range for the engineering decision problem. We therefore wish to discuss the possibilities for a more comprehensive view of values which will increase the scope of engineering decisions.

We think of the problem of value as arising in engineering decisions in the following way. We are of course, concerned with productive organizations and we think of them as having explicit goals or objectives. Thus any one outcome brings the system closer to its goal, removes it further from the goal, or preserves the status quo. In the comparison of outcomes we require a measure which tells us the relative effectiveness of outcomes for attainment of the goal. Such a measure we would call the value of the outcome. We have already defined utility functions as functions which attach such values to outcomes, thus we seek the utility function that applies to the particular system in which we happen to be making decisions.

Now the problem of determining the goals or objectives for the organization we will consider as a problem of choice involving entirely ethical judgments and will not treat it in the theory. We take these goals as given for any system within which our methods are to apply.

There are however, certain problems associated with these goals which are properly the concern of the engineering decision maker. For example, it is common to find that an organization appears not to have a single well defined goal, but rather a set of goals, some of which may conflict directly with others. Often we find an organization which tries to simultaneously maximize a set of interdependent variables. This problem is handled in theory by saying that there is some grand goal which subsumes the set of subgoals which conflict, and that all outcomes may then be evaluated by the utility function which is derived from this grand goal. In other words, the organization's objectives may be restated as the maximization of the attainment of the grand goal.

This problem of conflicting goals may be viewed as a problem of value measurement for goals, as compared to problems of value measurement for outcomes which we have

been discussing. One may either attempt to establish the relationship between goals or use the casuistic methods of value measurement which can be applied to goals, and thus resolve the problem of conflict. For an example of such an application see Churchman and Ackoff (9).

A second problem which one encounters in attempting to establish a statement of the goals of an organization is that very often these goals are dynamic in nature. They change with time as the organization and its environment react to each other. We can at present merely acknowledge this as one of the serious shortcomings of the theory of value for which we do not know a simple, immediate solution. We think that perhaps some light can be shed on this by injecting some sort of dynamics into value theory but we do not attempt here to even speculate on this.

The third difficulty with the goal concept is that very often the organization has not made its goals explicit. There may be general disagreement among members of the organization as to what the goals are, or there may be a general lack of knowledge. We shall assume that the goals of the organization with which we are concerned are possible of formulation with

sufficient stimulation and guidance of the personnel within the organization. Here again we think the casuistic methods will be of some assistance.

In any value problem encountered in engineering decision making situations, one may distinguish the following elements of the problem; there are some objectives or goals which the system seeks to achieve and some outcomes which are to be evaluated in terms of the extent of attainment of the goals which they represent. A utility function is a function which attaches to each outcome a number which represents its degree of attainment of the objectives in question. Alternatively the utility function may be thought of as a value measuring scale which can be applied to the outcomes. Now the basic value problem that arises in Economics is whether or not a utility function exists for the individual. This rests on two other questions; whether or not individual behavior can be characterized as being directed toward the attainment of some goals, and whether or not a method of measurement can be devised which will reveal the utility of various outcomes for these goals if they exist. Now we do not think that we will do extensive violence to the problem of values in engineering decisions if we assume away the first of these

economic problems. We think that one might safely assume that for any system of interest in an engineering theory a set of goals might be assumed to exist. We do not argue in any sense that these goals are in every case explicitly stated nor even explicitly known by any single individual within the system, nor do we insist that they are constant over long periods of time. The problem of whether any system can long exist without some degree of constancy of objectives is an interesting one which we cannot consider here. In any event we assume that the organization does possess some goals, and thus any given outcome may advance the system toward some of its goals, remove it further from others, or maintain the status quo. Thus the engineering problem is not one of whether or not a utility function exists, but how to determine it. In other words, our need is for a system of value measurement for use in engineering decisions. (Which we have defined as decisions occurring within a specific kind of system.)

Relation of Value Theory to Engineering Economy

The Industrial Engineer has rather commonly assumed that the dollar values of various states of the world could be used as an expression of their utility to the systems within which he works. Dollar values would presumably

be determined according to the rules of Accounting or Engineering Economy. Thus the maximization of profits or the minimization of losses would correspond to maximizing utility. This approach has several difficulties. Any state of the world which is not expressible in dollars according to one or the other sets of rules cannot be included in the decision model. It makes the decision model useless whenever these "irreducibles" are sufficient to change the outcome of the decision, as it would be made solely on the basis of dollar valued consequences.

One of the first problems in developing a theory of decision is thus to determine a measure of value which is appropriate to the system. Some real, practical difficulties arise here. One must determine the objectives of the system and then relate the states of the world to a utility measure which expresses their value for this objective. In real industrial systems for example, it is reasonable to expect that such a great variety of objectives exist that no general criterion will be applicable. Further, for any given system a number of objectives are usually found, some of which conflict with others. Thus the problem of relating these objectives in terms of some super-

objective arises. A further difficulty arises when these objectives change with time and a static utility function is not applicable.

von Neumann and Morgenstern have shown that if the decision maker can say for any two probability combinations of states of the world which he prefers or that he is indifferent, if these preferences are transitive, and if three other very weak assumptions hold, then a utility function exists. This function has the property that the utility of a state A is greater than that of a state B if, and only if, state A is preferred to state B. This conclusion of course is extremely important as a basis for any discussion of decisions in the face of incompletely known consequences, situations for which a normative theory is extremely important. However, this demonstration of the existence of a utility function is not much help to the Industrial Engineer who must actually find out what the function is over at least a part of its range if he is to apply the theory. The approach of the Economist to deriving utility functions appears to rest on inferences which are to be made from the actual behavior of the decision maker. To do this one must assume that the decision maker is behaving according to some definition of rationality. It appears that one immediately then

becomes involved in the question of subjective probability. Marschak (34) points out three cases of determination on the basis of manifest decisions.

1. If utilities are given and a person behaves so as to maximize the expected utility, then from his decision one may infer the probabilities (subjective) he attaches to the various states of the world. (Bayes Norm)
2. If the probabilities are given which a person attaches to various states of the world, then from his decisions one may infer the utilities for him of the states of the world if he is behaving so as to maximize the expected utility. (D. Bernoulli Norm)
3. It can also be shown that given the decisions of a person who behaves so as to maximize the expected utility, both his subjective probabilities and his utility function may be derived. (Ramsey Norm)

This approach to the determination of utilities which involves as well the idea of subjective probability can lead to several results. Given the information thus obtained, one could predict the behavior of the decision maker, that is, formulate a descriptive model. This is of interest to the Social Scientist. One might also construct a normative theory which would tell the decision maker how to behave, consistent with his

own utilities and subjective probabilities. This is however, a logical consistency, which in the case of interest to the Industrial Engineer does not assure that the decision maker will obtain good results with respect to the external world. The element of logical truth is missing. Thus a decision maker following such a theory would be consistent, but his probabilities and utilities might not be such that he could ever in fact move toward the attainment of his objective. He might simply have the wrong set of utilities and probabilities about the state of the real world. This assumes that neither is modified on the basis of experience. If a learning method of some kind is present, they may be improved.

There still remains the question as to whether the utilities which a decision maker holds for various states of the world are appropriate for the attainment of his objectives. Thus if ones objective is profit it is no easy task to set up a preference ordering of states which is somehow actually related to profits. So it would seem that for decisions where the objective is stated, some sort of investigation would be required to reveal the actual relationship between the two states of the world and the objective. Effectively, what is wanted is not only a normative theory which provides internal consistency given utilities

and probabilities, but a theory which will also improve the results of the decision maker's behavior in terms of some objective.

Even if one accepts, as is often done, the objective of profits as the only one admissible for a system, some important difficulties remain. Profits are not objectively given data in any sense. They must be measured, and the method of measurement which is appropriate to the system objectives must be selected. Thus even if we deal only with systems which seek profits, we must be explicit enough to define clearly just what profits are in a particular case. The system must manifest the relationship between present profits and future profits which are appropriate. No system can be held rigorously to the statement that its objective is profit alone, for profits are achieved only as the results of the incurrence of various kinds of costs, and almost all organizations will not incur certain kinds of costs. This condition manifests the existence of objectives other than profits alone. In almost all cases it is necessary to temper the concept of the maximization of profits. (27) Note that the attempt to measure utility for other objectives in terms of profits does not mean that profits alone

are the system's only objective.

Elements of the Engineering Value Problem

We present now a characterization of the general static value problem as it arises in engineering decisions. The discussion we now undertake depends heavily on progress in the theory of consumer or individual value which has been studied extensively by Economists for many years. Our problem is different from the problem of individual values in two respects. The engineering problem is in some sense easier, since we do not concern ourselves so much with the proof of the existence of utility functions and the assumptions required for such proofs. On the other hand, we take the engineering value problem to be more difficult in that any theory of engineering values is essentially useless unless it yields a method of value measurement which can actually be employed under the conditions which engineers commonly operate. Likewise, as we have repeatedly pointed out, we are not interested in developing a theory of value which will explain the behavior of people, but rather a normative theory which can be used as a guide for behavior.

The general engineering value problem arises in the

context of a system which displays a multitude of objectives. Some of these objectives will be conflicting in the sense that the simultaneous achievement of all of them is not possible. We can distinguish several important cases of expression of these objectives. They may be explicitly stated and known or implicitly stated and unknown. We assume clearly that the existence of a system implies the existence of objectives, although the objectives may not be explicitly clear to the decision makers in the system, either individually or considered as an impersonal whole. We distinguish, for purposes of measurement, objectives which can be ascribed to the system as a whole, such as the maximization of dollar profits, and objectives which must be ascribed to some individuals within the system, such as the maximization of "satisfaction" for the head of the system. To the present time engineering analysis has ignored this second type of objective. We feel that there are no a priori reasons why this is not the province of the engineer when viewed in his broadest role, and that given a method for including such objectives in the theory of engineering decisions the usefulness of the engineer can be greatly increased. There appears to be no reason other than lack of methodology which restrains the theory of normative decisions

from including this sort of objective.

Parenthetically it must be noted that there is by no means general acceptance of the idea that the engineer should or is able to assume the responsibility for value measurement in engineering decision problems. As with most specialized disciplines the engineer tends to draw a line beyond which he feels no obligation to go in discharging his duties. To the present time this line has not included the problem of value measurement in anything like the generality which they are viewed here. The question as to what, if any, meaningful limits may be placed on the engineer's activity remains unanswered. Likewise the question of what is to be done when the determinants of value for a system lie outside the system itself, as in the case of consumers or stockholders. The measurement of values which are dependent on the mental states of the persons within the system takes the engineer immediately into the realm of the Psychologist and the Economist, and the superior skill of these specialists in attacking the problem must be recognized. Further, the engineer must obtain value measurements often on the basis of judgments by laymen with respect to the value of highly technical outcomes which only the engineer himself can

understand. These and other vexing questions we simply pass over in order to study the possibility of increasing the scope and effectiveness of engineering decision making as an incentive to eventually answering some of these questions.

Analysis of State Values

It is clearly established, both in philosophic and economic discussions, that no object has value. Rather an object may be valued by virtue of its existence in a particular state of the world. Its value is not a function of its own condition but of the conditions of the world in which it exists. This is reflected in a simpler, more direct way in the economic discussions of marginal utility. The value of a unit of any resource depends on how much of the resource is already possessed. Money, for example, is usually thought to have a diminishing marginal utility for consumers, thus the value of a dollar depends on the assets of the consumer at the time he receives it. It seems clear that if values are to be discussed in connection with industrial systems that we must not only recognize the concept of variable marginal utility, but because of the complication of combining utilities in accordance with overall objectives, we must in fact talk

about the values of states of the system. Further, if we press this idea, it is clear that the value of any state of an industrial system is a function of the state of the world in which it occurs. Thus we must in fact talk about states of the world. We shall proceed to speak about states of the system with the understanding that the more general idea of states of the world is meant, or that we are in fact carrying on some sort of marginal analysis where those aspects of the world which are more or less indirectly related to the system are thought of as being held constant.

The sort of utility function appropriate to the measurement of state values was first suggested by Edgeworth in 1881. (49) It appeared in an article by Smith et. al. whose basic ideas are followed here. (58) Unfortunately the latter article is not a complete exposition of the subject, so that one must imply much of what is meant.

Let: $U(x)$ = the value of the state of the system x ,
where

x = the vector $(x_1, x_2, \dots, x_1, \dots, x_n)$

x_1 = measurement or description of an aspect
of the state

$$U(x) = \phi(x_1, x_2, \dots, x_n)$$

$$\frac{\partial U(x)}{\partial x_1} \quad \text{marginal utility of } x_1$$

If we think of an industrial system passing from state to state and fail to introduce uncertainty as classical Economics fails to do, then this analysis becomes trivial. With the introduction of uncertainty however, we think of the system moving from one state to the next according to a set of probabilities. There is then a relationship between the values of successive states which may be fruitfully examined. We define the value of a state as the expected value of the states to which the system may directly pass. Thus, let:

$$p(xy) = \begin{array}{l} \text{probability of passing from state } x \text{ directly} \\ \text{to state } y \end{array}$$

$$Q(x) = \sum_y p(xy)Q(y)$$

To illustrate the important relationships between the values of states, consider the following diagramatic representation of the future of a system. In this simple case the future consists of only an intermediate set of states (y), and a set of final states (z). The notion of final or "trapped" states is particularly important. These are states from which the system cannot move, or states which in some sense terminate the operating existence of the system. Just what turns out to be a final state depends on the viewpoint of the decision

maker. For example, if an entrepreneur terminates his connection with a particular system upon its sale to others, then sale is for him a final state, but if he still retains some connection with the system, or plans to in the future, then sale would not be a final state of the system. In industrial systems final states are often sale, bankruptcy, merger, liquidation, dissolution, etc. Consider the diagram below of the future of a system now in state x .

$$Q(y_1) = p(y_1 z_1)Q(z_1) + p(y_1 z_2)Q(z_2)$$

$$Q(y_2) = p(y_2 z_3)Q(z_3) + p(y_2 z_4)Q(z_4)$$

Thus the value of a state in the intermediate set depends on the values of the final set. Further:

$$Q(x) = p(xy_1)Q(y_1) + p(xy_2)Q(y_2)$$

$$Q(x) = p(xy_1)p(y_1 z_1)Q(z_1) + p(xy_1)p(y_1 z_2)Q(z_2) + \\ p(xy_2)p(y_2 z_3)Q(z_3) + p(xy_2)p(y_2 z_4)Q(z_4)$$

Thus the value of the state x also depends on the values of the set of final states. It may now be seen that the value of any state of the system depends on the values of the set of final states and the transition probabili-

ties. Thus if we could establish the values of the set of final states and if we knew the transition probabilities, the problem of value for the system would be solved.

Notice that the final states are defined so that the system eventually reaches the set of final states with probability one. Further, under the definition of the value of a state, the transition probabilities from the set of final states are all zero and thus their value should be all zero. Instead however, one may assign or measure the values of the final states, and by doing so, determine the values for all states of the system. In applied situations the probability of entering the set of final states in the foreseeable future is usually so small that the decision maker is unwilling to consider it. The number of intermediate states that must be examined before the probability reaches anywhere near the value one may be extremely large. Thus this serves to define one meaning of state value but does not usually provide a very useful method of measuring it.

The chief usefulness of these relations between the values of states arises in the following manner. If a set of states can be established such that the probability of the system entering the set is one, and if the values

of these states can be measured or assigned in some way, then the values of all previous states are determined. The set of states we now discuss need not be final states. Thus we might work with a set which the system will enter with probability one in the relatively near future. Under the definition of state value given above the assignment of values to this set of states implies knowledge of the values of the final states and the transition probabilities to them. Thus in principle we are no better off. However, we propose that in many situations one can obtain a satisfactory basis of assignment for values of a set of states in the near future, and thus use the derived values as a basis for decision.

As an example of this approach consider the following case. There is an intermediate set of states such that the system enters the set with the probability one. Let this set be the states y_i . It is known that the value of any state in the set is given by equation.

$$Q(y_i) = a + e_i \qquad a = \text{constant for all } i$$

This is a frequently occurring case in which a part of the value of any state y_i is made up of the expected outcome of a decision which is independent of the choice of y_i , and the remaining part of the value is independent of the outcome of this decision. Then one can,

without violating any of our definitions, assign values to the set of states y by considering only the e_1 .

The question now arises as to how far in the future we shall select the set of states y to work with, and what care we shall take to see that the values we assign to them are in agreement with the long chain of states leading to the final set and in agreement with the transition probabilities. It seems clear that one can almost never do the latter. It would be foolish however, to concentrate one's attention on some aspect of the immediate future to the exclusion of all else. Such behavior is called decision making on the basis of "expediency". To the extent which one attempts to consider the extended future, one acts on the basis of "principle". Attainable action is usually a compromise between expediency and principle. We shall examine later some methods by which this compromise can best be made. It is clear that principle is the desirable basis from the viewpoint of rationalization, and one turns to methods such as we have just described in order to bring the necessary expediency into as close an agreement as possible with the desired principle.

In many systems it is useful to distinguish between "terminal" value and "instrumental" value. Instrumen-

tal value is defined as value for the eventual achievement of some end, or utility for the attainment of terminal value. Terminal value is then defined simply as the compliment of instrumental value. These might correspond to a situation in which the objective is profits. Money removed from the business in the form of profits has terminal value, whereas a machine which has the potential of generating future profits has instrumental value. Consider the simple case in which the value of a state of the system is given by the sum of the terminal values and the instrumental values for the state

Let a_i = the terminal value of state i

b_i = the instrumental value of state i

Then $Q_i = a_i + b_i$

But by definition it is also true that

$$Q_i = \sum_j p_{ij} (a_j + b_j)$$

and by the definition of instrumental value

$$b_i = \sum_j p_{ij} (a_j + b_j)$$

Consider a set K of states such that the system goes from state j into the set K in one transition with probability one. Suppose the set K is a set of trapped states in

the true sense of the word, that is that the system cannot move out of the set K.

Then $Q_K = a_K$ for all K

$$b_j = \sum_K p_{jK} a_K$$

Thus

$$Q_i = a_i + \sum_j p_{ij} (a_j + \sum_K p_{jK} a_K)$$

This indicates that the value of any state may be given in terms of terminal values only.

The previous analysis of the relationship and meaning of state values can provide only a basis for understanding the structure of the value problem in engineering decision making. At best it can provide only a basis for translating values measured for one state to values of other states. We now turn to the problem of actually making value measurements for a particular state. In engineering problems one almost always turns attention to the measurement of the values of the states of the system to which it will pass directly as the result of the course of action selected from those available in the decision situation. These immediate states are called the consequences of the decision or of the course of action. Having indicated the nature of the relationship between state values, we now discuss the problem

of measuring the value of a given state. The first question we discuss is that of the kind of scale on which values must be measured for engineering decisions.

Scale Requirements for Engineering Decisions

The kinds of scales on which one must measure value for engineering decision making are generally different from those found in descriptive economics. The following kind of scales are useful in the engineering value problem. The definitions we give are similar to those of Coombs, Raiffa, and Thrall. (10)

1. An ordinal scale is merely a ranking. A partly ordered scale is defined by the relation "equal to or greater than".
2. An interval scale consists of a constant unit of measurement and an arbitrary zero point.
3. A ratio scale consists of an absolute zero point and a constant unit of measurement.

We have listed the scales in the order which might be taken from the most primitive to the most refined, and also from the most inexpensive and obtainable to the most expensive and unattainable in engineering situations. We now examine various engineering decision situations with respect to the scale refinements re-

quired for value measurement. The objective of course, is to determine how much freedom from the requirements of precision and accuracy one can be allowed a priori and still have the decision remain invariant with respect to the course of action selected. It should be reemphasised here that we are in no sense attempting to examine the general problem of value measurement, but only attempt as much measurement as is necessary to make an engineering decision.

1. In the ordinary engineering economic decision situation where only one state of nature is pertinent and there is no uncertainty, an ordinal scale of values for the outcomes is sufficient to determine the decision. This is the case of classical economics where ordinal value is sufficient to predict consumers' behavior when no uncertainty is present. Considerable attention has been devoted to the existence and derivation of utility functions which provide a cardinal measure of utility. It is well known in the theory of riskless choice that if $\phi(x)$ is a utility function which preserves the ordinal relation, then any monotonic transformation $\phi(x)$ is also a utility function.

2. The simplest kind of uncertainty in the value

sense that one may introduce is the existence of two relevant outcomes, which may be referred to as success or failure. Under these conditions, which we call a "binomial" decision problem, it is shown below that the only relation required is that the value of success is greater than the value of failure.

3. When more than two outcomes are possible within the decision matrix, then a scale of more refinement is required; the interval scale.

4. Note that in any decision problem the matrix may be reduced using the relation of dominance on the basis of an ordinal scaling only. Alternative A is said to dominate B when A is better than B, no matter what the state of the world.

It is interesting to note that the most severe scaling requirement that appears in engineering decisions is that of an interval scale. This would appear to obviate the attempts of accountants and others to measure value on a ratio scale, at least for decision making purposes. All types of decision situations which have been defined may be handled with an interval scale or a weaker scale.

The relation described in 3. above will now be demonstrated. When more than two outcomes are possible in

a decision situation an ordinal scale is not sufficient but an interval scale may be used. This is shown for the case of three possible outcomes, which may be called the trinomial decision problem.

Consider three possible states p , q , and r with utilities $u(p)$, $u(q)$, and $u(r)$. For any course of action a_i let the a priori probabilities of p , q , and r be P_i , Q_i , and R_i respectively. The decision maker, choosing so as to maximize the expected utility would then select a_i such that

$$P_i u(p) + Q_i u(q) + R_i u(r) = \max_i E u(a_i)$$

Let a_k be the course of action thus selected. Then for any other a_i ($i \neq k$)

$$\begin{aligned} P_k u(p) + Q_k u(q) + R_k u(r) &> P_i u(p) + Q_i u(q) + R_i u(r) \\ (P_k - P_i) u(p) + (Q_k - Q_i) u(q) + (R_k - R_i) u(r) &> 0 \end{aligned}$$

$$\begin{aligned} \text{But } R_k &= 1 - P_k - Q_k \\ R_i &= 1 - P_i - Q_i \end{aligned}$$

$$R_k - R_i = P_i - P_k + Q_i - Q_k$$

$$\text{Thus } (P_k - P_i)(u(p) - u(r)) > (Q_k - Q_i)(u(r) - u(q))$$

It is clear that whether or not this relation holds depends on the magnitude of the differences in utilities. Thus a ranking is not sufficient to preserve the relationship. However, addition of a constant to all utilities will not change the relationship, nor will multiplication by a constant, thus an interval scale is sufficient.

The theoretical structure of the engineering value problem may be sketched rather simply. We encounter systems which appear to have a variety of conflicting goals. With each of these goals must be associated a measuring scale such that the scale can be applied to any state of the system. The scale then furnishes a measure of the degree to which the measured state represents an achievement of the goal in question. We must have a method of combining the measurements on all such scales into a measurement on some grand value scale which indicates how the variety of goals combine to form a "super-goal". In terms of these combined measurements or overall measurements one may compare the values of states of the system.

The most difficult practical problem is the achievement of a workable method of value measurement. The scale requirements for measurement can be reduced for certain

types of decision situations. From the viewpoint of methodology we may consider two types of goals; psychological and physical. By psychological goals we mean such things as job satisfaction, accomplishment feelings, gratification of personal associations, and the like. By physical goals one would mean such things as profits, output, and the achievement of particular states of material wealth. We think of the value of a state as being a number subject to repeated observation, and, in theory at least, to controlled experimentation and verification. In the engineering problem we may distinguish two kinds of observations of importance in this problem. One may consult one's own accumulated experience or that of another, and produce a judgment about the value of a state of the system. Alternatively, one may make an observation which provides experimental evidence to answer the question of value. In most cases the actual process will be a combination of these two methods of measurement, but we shall discuss them separately. We shall refer to these two methods as judgment and observation respectively. Further we can distinguish between what will be called explicit measurement of the value of a state where one identifies sub-goals, evaluates the state in terms of these sub-goals, and then combines the resulting measurements in an evaluation of

the state with respect to the grand goal, and implicit measurement where one tries to observe or judge the grand goal value of a state directly.

The methods of explicit and implicit measurement can best be understood by exhibiting symbolically their respective structures. As has been already indicated, in the cases of interest to the engineer, that is decision in the face of uncertainty, it turns out that nothing stronger than preference statements is necessary to determine a useful measure of utility. Thus this analysis is concerned with obtaining preferences.

Consider; the objectives of the system: $\theta_1, \theta_2, \dots, \theta_m$
 outcomes or states of the system: $o_1, o_2, \dots$
 where an outcome is a vector

$$o_1 = (x_1, x_2, \dots, x_n)$$

Explicit measurement:

1. For each objective a scale is given such that for each outcome o_1 , the scale can be applied to the appropriate members of $(x_1, \dots, x_n)$, and the utility of that state for the objective in question is obtained.
2. The result of applying the scales for each objective is a vector which represents the "exploded utility"

of the outcome $(u(o_1), u(o_2), u(o_3) \dots u(o_m))$

3. The final step is the criterion function or ϕ function, into which the elements of the above vector are inserted as arguments, which produces a scalar utility measurement. This measurement may then be compared to similar measurements for other states and the preference relation between them obtained. Note that it is necessary to obtain a cardinal measurement in step 1 in order to determine the preference relation.

Implicit measurement:

The preference relation is determined directly by applying some method of preference determination directly to the states considered in toto. For example, one form of implicit measurement is that commonly used in economics where the consumer is asked to judge directly which of two states he prefers.

In order to illustrate some of the problems which arise let us consider the following simple illustration. We deal with a system which we assume has some objectives. As is usually the case, the decision makers in the system have not previously made the objectives explicit. We could ask them to make judgments about the values of

the system without either providing explicit descriptions of the states or insisting that the objectives and their relationship be made explicit. Alternatively, we could ask the decision maker in various ways to make the objectives explicit and their interrelation clear. If the objectives cannot be made explicit then value measurement can be carried on only by the decision maker. The engineer cannot inquire into his method of assigning values. The engineer can aid the decision maker in being logically consistent, but the full requirements for rationalization are not met. Suppose the objectives are made partially explicit, as is usually the case. The engineer may then work with those which are explicitly stated, but the final valuation is reserved to the decision maker.

Even if the objectives are made explicit, their interrelation presents an additional problem. Suppose the objectives of the system are stated to be first profits, second job satisfaction for all employees. Given any state of the system one must first find a way to measure the amount of profits and the amount of job satisfaction involved. One must obtain two measurements which are experimentally verifiable. Given these two measurements one must then find out how to combine them into a

measure of the overall value of the state of the system. If one could simply add the two measurements, then the entire theory of engineering values would be vastly simplified.

The problem of selecting among the various methodologies open to us for obtaining value measurement is one which seems to have led to considerable confusion in the literature. We shall investigate the methodology of measurement in some detail. We consider the two sources of value measurements to be those which we have already pointed out above, observation and judgment. Observation includes the process of collecting data specifically for a given value problem and making inductive inferences from the data. Judgement we think of as being the consultation of accumulated experience. Usually judgment is relied upon in the less important decision situations and observation in the more important ones. Both observations and judgments as we consider them must be subject to the criteria of logical and experimental verification. It is interesting to note that since a value determination is in general a statement about the indefinite future, it can never be completely corroborated. We do not deny that a judgment is certainly a sort of observation, and that there is a

middle ground where value determinations may actually be made as the result of some combination of judgment and observation (as we have defined them). For purposes of our analysis we propose to keep them distinct from each other. We have already indicated the possibility of considering the value problem implicitly and attempting to assign values directly to states or considering it explicitly and working with aspects of the state and objectives of the system. Here again there is a continuum of degrees of explicitness, but we distinguish two important kinds of explicitness, that of objectives and that of aspects of the state. From the viewpoint of the engineer's methodology it is useful to consider all states of the system as being made up of two kinds of aspects, the aspects which are mental states of the persons in the system, and the aspects which are purely physical states of the system. Thus we have objectives as well which are either physical states of the system or mental states of persons within the system. These we shall call physical and psychological objectives respectively. By considering various combinations of these conditions one can generate the possible value measurement problems which the engineer must face.

Methods of Obtaining Preferences

Consider states of a system which are composed of either psychological aspects or physical aspects, but not both. In evaluating these states the engineer may employ either observation or judgment, and be either implicit or explicit in his approach. This leads to twelve problems of preference determination. The elements of these problems are:

I. Observation	III. Implicit	V. Psychological
II. Judgment	IV. Explicit	VI. Physical

Case 1: (II, III, V) In this case the engineer makes a judgement about the preferences of the system among states which are psychological in nature. All aspects of a state are considered implicitly. This case, together with Case 2: II, III, VI

3: II, IV, V

4: II, IV, VI

are considered in some detail in Chapter 6 below. Special attention is devoted to the relation of effectiveness of explicit and implicit judgments, and to methods for improving the judgments which the engineers make. Cases 2 and 4 are related to the problems of Psychophysics, and the methods used there are suggestive. (See 62)

Case 5: (I, III, V) In this case the engineer observes the preference of the system among implicitly considered states which are psychological in nature. This is the case with which the economic literature deals almost exclusively, with the economist assuming the role of engineer. The problems here turn chiefly on the methods of observation to be used. In the economic problem some writers accept uncontrolled observations, some require controlled observations; some accept verbal expressions of preferences, and some require behavioral expression of preferences. The methods of observation used may be classified roughly into three types:

Casuistry: Preferences are here obtained by consulting historical cases in which they have been manifested by actual choice. This method has the advantage that the observations yield behavioral data which are often regarded as superior to (more valid than) verbal expressions of preference. Casuistry suffers from the disadvantage that the data are largely uncontrolled. It is seldom that a case history is rich enough so that assurance can be had as to the nature of conditions surrounding the choice.

False Casuistry: The method of false casuistry or simulation involves the creation of an artificial

decision situation from which an expression of preference is obtained, Simulation may consist only of a description of the states and a verbal expression of preference, or it may be carried to elaborate lengths in an attempt to obtain behavioral expression. In any event there are serious doubts as to the validity of such an artificial procedure.

Controlled Experimentation: The opportunities for carrying out controlled experiments, devoid of the artificiality of simulation, are few in the problem of engineering value measurement. While the data thus obtained would be clearly the most acceptable, it is doubtful if such a method would be possible or economic in most systems.

Case 6: (I, III, VI) In this case the engineer observes preferences for implicitly considered physical states. This case is almost never found in use because data which describe such states implicitly are not available.

Case 7: (I, IV, V) This case is analagous to the economic problem of Case 5 above, with the additional problem that a function must be obtained which combines the values of aspects of a state into value for the state

itself. This problem is treated in economic theory by the use of various functions of the form

$$u(x) = \phi(x_1, x_2, \dots, x_n)$$

such as we have used above. An artificially simple case which has been studied extensively assumes that the values of aspects are additive.

Case 8: (I, IV, VI) If the concept of observation is broadened to include the idea of prediction this case covers most ordinary engineering analyses of value. This is the case with which engineers are perhaps best equipped to deal.

Now we drop the requirement that states be either entirely physical in aspect or entirely psychological, and admit states which are partly physical in aspect and partly psychological. This case is not considered in any of the literature on value measurement. Many of the problems are those of a composition of the cases discussed above.

Case 9: (I, III, V & VI) and Case 10 (II, III, V & VI): It is doubtful whether these cases exist, for it is difficult to understand either an observation or a judgment which is not at least explicit to the extent that the physical and psychological aspects are considered

separately. One must then obtain a ϕ function which gives state value from the values of the two classes of aspects.

Case 11: (I, IV, V & VI) This is a composition of Cases 7 and 8 with the extension of the ϕ function to include both physical and psychological aspects.

Case 12: (II, IV, V & VI) This is a composition of Cases 3 and 4 again with the necessary extension of the ϕ function.

The engineer may, if he wishes, broaden his interests to the employment of any of these methods of preference determination. All have problems associated with them. Some are clearly more in agreement with rationalization than others. Our point is simply that whatever method of preference determination is used the result may be translated through indifference curve analysis into an ordering of states. If preferences among probability combinations of states can be established, by whatever method, then an interval scale may be obtained.

Indifference Curve Analysis

The standard technique for dealing with consumer values in economic theory is the method of indifference curves.

This is not really a method of measuring values but a way of describing the value system of a consumer. We now restate the tenets of this kind of analysis in terms of the values of the states of engineering systems. It is an analysis in which no uncertainty is admitted and simply expresses values on an ordinal scale.

The basic assumption of indifference curve analysis are that the decision maker, given any two states, can tell which he prefers or that he is indifferent between them. Again, under the more general interpretation of preference determination such as has been made above, this is taken to mean that a method is available for the determination of which of two states has the larger value for the system or that they are equal in value. One proceeds then to organize all the states among which the decision maker is indifferent into groups. The groups are then ranked with respect to preference. This simply represents a way of organizing the preference field of the decision maker. Under appropriate assumptions the groups of indifference states are plotted as functions of some aspects of the states, and one obtains the ordinary indifference map. The lines on the indifference map pass through states among which the system is indifferent. As one moves from one line to the next in

the appropriate direction the value increases or decreases.

Of special interest in the engineering decision problem is the use of indifference curve analysis to organize values for incomes differing with respect to their time of occurrence. (1) If x represents a state of the system at time t_1 , and y represents a state of the system at time t_2 , and if continuously varying x and y denote other states of the system at their respective time, then an indifference function $f(x, y) = a$ may be plotted. Here "a" is a parameter denoting the level of preference. The preference statements are of the form, "a state x_1 at t_1 and a state y_1 at t_2 is equivalent to a state x_2 at t_1 and a state y_2 at t_2 ." As an example, suppose that x and y denote states which need only be described as incomes of x and y dollars now and at some time in the future respectively. Then, according to the value relationship which is commonly ascribed to incomes differing with respect to time in engineering problems, the indifference lines would be continuous, non-intersecting, downward sloping, and increase in value as one moves upward and to the right from the origin.

Values for Uncertain Prospects

The utility theory of von Neumann and Morgenstern seems to have created the greatest interest among Economists, and it appears to have useful implications for deriving utility functions for engineering decision problems. We now paraphrase the axioms which they have shown are required for the existence of a utility function.

1. The decision maker (organization) offered any two possible outcomes can decide which is preferable or that they are equally preferable with respect to goal attainment.
2. This ordering is transitive; if A is preferred to B, and B is preferred to C, then A is preferred to C.
3. Any probability combination of equally desirable outcomes is just as desirable as either outcome by itself.
4. If A, B, and C are as in axiom 2, then there exists a probability combination of A and C which is just as desirable as B. (Axiom of Continuity)
5. If A and B are equally desirable, then $pA + (1-p)C$ and $pB + (1-p)C$ are equally desirable.

The proof that the utility function exists under these axioms is beyond the scope of this study and is given

in (51).

We first give an example of the use of the operations suggested by this axiom system for the measurement of utilities. We begin with two outcomes whose utilities are known. If they are not known, we must at least know the preference relation of the organization, which applies to them. In this latter case supposing A is preferred to B, we then may assign any two numbers such that the first is greater than the second to A and B respectively. We then call these numbers the utilities of A and B on an arbitrary scale. The scale is an interval scale having an arbitrary zero point and a constant unit of measurement. Now consider the problem of finding the utility of another outcome C on our measuring scale. One of the following statements is true of C:

1. $A \geq C \geq B$
2. $C > A$
3. $C < B$

In case 1 the utility of C is found by finding the probability combination of A and B which is equivalent to C as far as the organizational goal is concerned. That is, we find a p such that the organization is indifferent between the prospect of A with a probability p or B with probability 1-p, and C. The utility of C is then computed from the equation below, where all the terms

on the right hand side are known;

$$C = pA + (1-p)B$$

or in our symbolism

$$u(C) = pu(A) + (1-p) u (B)$$

In the second case we find a p which makes the organization indifferent between C with probability p or B with probability $1-p$, and A . Then the utility of C is given by

$$C = \frac{A - (1-p)B}{p}$$

Similarly in the third case;

$$C = \frac{B - (1-p)A}{p}$$

Now the major difficulty associated with the use of this scheme is the problem of whether or not preferences among probability combinations of states can actually be established for the system. It is certainly questionable whether all or any of the problems in preference determination which have been discussed above can be reinterpreted in terms, not of states alone, but of probability combinations of states. It is questionable for example, whether persons in the system whose mental states form part of the objectives of the system can

find any real meaning in questions about preferences among probability distributions. They may not be able to understand the notion of a probability combination of states to the extent of being able to form a meaningful judgment of their preferences. The question of practicality cannot be decided without recourse to experience in engineering situations.

There are several other methods of value measurement. Some, such as those of Churchman and Ackoff (9) and those of Suppes and Winet (60), rest on the assumption that values are additive and thus limit their usefulness. Others have been developed which are extremely complicated but avoid the necessity of preferences among probability combinations of states while yielding an interval scale for utility.

Discounting Future Incomes

A very important question which absorbs much of the literature of Engineering Economics is the question of how to evaluate incomes occurring at different times in the future. The use of the interest calculations to express the time value of money was introduced by Wellington (64) and is still advocated universally in Engineering Economics texts. The reasons for the use of interest cal-

culations to express the time utility of incomes are given by Grant (23):

1. Actual interest charges on borrowed funds
2. Opportunity cost of investment opportunities foregone
3. Compensation for risk
4. Compensation for deferred consumption
5. Contentment return policy

The first two reasons are clearly applicable for the computation of values if they in fact are real factors in the decision situation. The fifth is in reality a method for selecting among alternative courses of action. In the absence of any of the other reasons listed above and in view of the Axiom of Long Run Success there is little to recommend it as a method of choice. Reason 3 is acceptable, but the requirement of explicitness would indicate an analysis of risk in terms of the probabilities and consequences of the events involved. Discounting future incomes as a recognition of the need for deferred consumption is the subject of the analysis to follow. It is interesting to note that Grant justifies this reason on the basis of actual behavior which may or may not be rational in the sense of this study.

In purely economic discussions another reason is usually

given for the discounting of future incomes. This argument is based on the notion that information about the future is less certain than information about the present. Thus future incomes are less certain than present incomes, and one should attach more value to a prospective income in the near future than to the same prospective income in the distant future. This view is made most explicit by Fair (14) in the following discussion of profits.

Let $\sigma(t)$ = the standard deviation of the profit rate at time t . (Assumed increasing with t .)

Define $L = \frac{\sigma^2(0)}{\sigma^2(t)} = L(t)$ = the information level

Then the value of a stream of profit extending into the future is given by

$$P = \int_0^T L(t) \left[\frac{dP}{dt} \right] dt \quad \text{where } t \rightarrow \infty$$

Thus the only effect of time is to increase that standard deviation of the probability distribution of the profit rate. But suppose the utility function of a system is such that the value of profit is a non-decreasing function of the dollar profit. Then maximizing the expected value of the profit expressed in dollars is equivalent to maximizing the expected utility. As will

be shown below, the Axiom of Long Run Success leads to the conclusion that rational choice among probability distributions of value means choosing so as to maximize the expected utility. Thus the fact that a prospect of profit in the near future has a small standard deviation by comparison to a prospect in the distant future would not in any way affect the choice. The choice would be determined by the expected values only. The expression for the present value of a stream of profits extending into the future should be given by

$$P = \int_0^T \left[\frac{dE(P)}{dt} \right] dt$$

It may be conceded that discounting the future is perhaps an adequate description of behavior, but as a normative principle it appears to require at least some further analysis. Let us eliminate from the discussion immediately those decision situations in which an interest bearing investment is actually an alternative course of action. In these cases the techniques used in Engineering Economy to express the time value of money appear to be perfectly legitimate. In the absence of such an investment opportunity the question arises as to whether an income in the near future should be valued differently from the same income in the more distant

future. In the light of the discussion of state values above, it seems clear that the answer to this question depends on what the states of the system are at the two points in time under consideration. Suppose we are concerned with an income of a fixed amount or a "commodity bundle".

Let x_i = a possible state of the system

B = an income of fixed amount

$(x_i + B)$ = the event "an income of amount B when the system is in state x_i "

$Q(x_i + B)$ = the value of $(x_i + B)$

q_{i0} = the probability of the system being in state x_i at time t_0

q_{i1} = the probability of the system being in state x_i at time t_1

We are concerned with the relation between

$$\sum_i q_{i0} Q(x_i + B) \quad \text{and} \quad \sum_i q_{i1} Q(x_i + B)$$

Discounting B at t_1 implies that

$$\sum_i q_{i0} \{ Q(x_i + B) - Q(x_i) \} > \sum_i q_{i1} \{ Q(x_i + B) - Q(x_i) \}$$

Discounting can be shown to be rational in a system for which the condition of diminishing marginal utility of income prevails, provided the following relation holds:

$$\sum_i q_{i0} Q(x_i) < \sum_i q_{i1} Q(x_i)$$

If

$$Q(x_j) < Q(x_k)$$

Then

$$Q(x_j + B) - Q(x_j) > Q(x_k + B) - Q(x_k)$$

and

$$\sum_i q_{i0} \{ Q(x_j + B) - Q(x_i) \} > \sum_i q_{i1} \{ Q(x_i + B) - Q(x_i) \}$$

Which indicates that discounting applies under this particular set of conditions. Thus the rather surprising conclusion emerges that if the system is faced with an improving future it is rational to discount future incomes. In other words, the more optimistic the decision maker is about the future, the stronger his tendency to discount the future incomes.

Consider a system which may enter the set of states (m) where

$$m = \{ x_i \mid Q(x_i + B) = Q(x_i) \}$$

The set m might be considered a set of "disaster" states in which an income of amount B is of no value to the system.

Let

$$Q(x_1 + B) - Q(x_1) = \begin{cases} K & \text{for } x_1 \notin m \\ 0 & \text{for } x_1 \in m \end{cases}$$

Then

$$\sum_i q_{i0} Q(x_i + B) = \sum_{i \notin m} q_{i0} Q(x_i + B) + \sum_{i \in m} q_{i0} Q(x_i)$$

and

$$\begin{aligned} \sum_i q_{i0} \{ Q(x_i + B) - Q(x_i) \} &= \sum_{i \notin m} q_{i0} \{ Q(x_i + B) - Q(x_i) \} \\ &= \sum_{i \notin m} q_{i0} K \end{aligned}$$

Also

$$\begin{aligned} \sum_i q_{i1} \{ Q(x_i + B) - Q(x_i) \} &= \sum_{i \notin m} q_{i1} \{ Q(x_i + B) - Q(x_i) \} \\ &= \sum_{i \notin m} q_{i1} K \end{aligned}$$

Thus discounting will be rational if

$$\sum_{i \notin m} q_{i0} > \sum_{i \notin m} q_{i1} \quad \text{or} \quad \sum_{i \in m} q_{i0} < \sum_{i \in m} q_{i1}$$

The magnitude of $\sum_{i \in m} q_{i1}$ indicates in a sense the

degree of pessimism in the decision maker's outlook. It is interesting to note that as the tendency toward pessimism increases, the tendency to discount the future increases. These two situations indicate two sorts of utility functions which lead to the rationality of

discounting the future.

In the cases discussed so far, B was taken to be a known constant and x_1 to be a random variable. Thus $Q(x_1 + B)$ was a random variable depending on the distribution of x_1 . It is also interesting to consider the case where $x_1 = x$, a known constant, and B is a random variable such that $E(B)$ is constant over time and the variance of B is changing over time. In this case the value of $(x + B)$ is given by $\phi(x, B)$. In Chapter 4 certain kinds of

ϕ functions will be examined which lead to the necessity for considering the variance of B .

Chapter 4.

BASIC THEOREMS FOR PROBABILISTIC DECISIONS

When the outcome resulting from a course of action may be considered a random variable, many of the important theorems on random variables may be interpreted in terms of the model of a decision situation which has been established above. This interpretation yields various normative implications, the most important of which is the result that the Axiom of Long Run Success establishes the rationality of choosing so as to maximize the expected utility. To introduce this analysis the inequalities of Chebyshev and Kolmogorov are examined for their normative significance. These theorems are stated in the forms analagous to those given by Feller (15) where the proofs of their original forms may be found.

1. Chebyshev's Inequality: Consider a sequence of identical decision situations such that if we choose repeatedly an alternative a_1 , then the utility of the outcome is a random variable $u(a_1)$ with mean $E[u(a_1)]$ and variance

$$\sigma_{u(a_1)}^2 : \text{ for any } t > 0$$

$$\Pr \left\{ \left| u(a_1) - E[u(a_1)] \right| \geq t \right\} \leq \frac{\sigma_{u(a_1)}^2}{t^2}$$

In this form the theorem simply indicates in a rough way the kinds of deviations one may expect between the outcome of a course of action and the expected outcome. Since the theorem applies to random variables in general define:

S_n^* = the sum of the outcomes of n independent decisions in which the decision maker chooses so as to maximize the expected value of the outcome

S_n = the sum of the outcomes when any other choice is made

Suppose for S_n^* and S_n we know only the respective means m_n^* and m_n , and the variances σ_n^{*2} and σ_n^2 .

Consider $\Pr \{ S_n \geq S_n^* \}$

If the density functions of the outcomes were known one could choose directly the alternative which would minimize this probability. If, as we have stated above, only the means and variances are known, then the best thing which can be done is to make use of a limit theorem which allows a statement to be made which is independent of the form of the distribution. Thus, instead of actually minimizing the probability, the

available information allows the decision maker to minimize only the upper bound of the probability.

Then

$$\Pr \left\{ \left| S_n^* - m_n^* \right| \geq t_1 \right\} \leq \frac{\sigma_n^{*2}}{t_1^2}$$

$$\Pr \left\{ \left| S_n - m_n \right| \geq t_2 \right\} \leq \frac{\sigma_n^2}{t_2^2}$$

$$\Pr \left\{ \left[S_n - S_n^* \right] + \left[m_n^* - m_n \right] \geq t_3 \right\} \leq \frac{\sigma_n^{*2} + \sigma_n^2}{t_3^2}$$

(For $t_3 > 0$)

Let $t_3 = m_n^* - m_n > 0$

$$\Pr \left\{ S_n \geq S_n^* \right\} \leq \frac{\sigma_n^{*2} + \sigma_n^2}{(m_n^* - m_n)^2}$$

Minimizing the upper bound of this probability is equivalent to maximizing

$$\frac{(m_n^* - m_n)^2}{\sigma_n^{*2} + \sigma_n^2}$$

Thus choosing so as to maximize the expected utility minimizes the upper bound of the probability that any

other method of choice will be better in a sequence of n independent decisions. Notice that given two sequence decisions, each of which results in an expected value of m_n^* , the decision maker will prefer the one with the smaller variance.

Another application of Chebyshev's Inequality arises when the outcome of any alternative a_i in a decision situation is a random amount of money, and only the means, $\bar{x}_i$, and the variances σ_i^2 are known. Suppose the utility function for money is of the form

$$u(x_i) = x_i \text{ for } x_i \geq r, \quad u(x_i) = 0 \text{ for } x_i < r$$

In this situation the Principle of Safety First, which is suggested by Roy (48), can be shown to sometimes agree with the principle of maximizing the expected return. The Principle of Safety First suggests that the decision maker choose so as to minimize the probability that x_i falls below r . If, for example, the outcomes were normally distributed, then minimizing the upper bound of the probability that x_i falls below r is equivalent to minimizing the probability itself. Note that this use of a very general limit theorem yields only very weak statements about the upper bounds. If

$$\frac{x_i - r}{\sigma_i} > 1 \text{ then one can only say that the upper bound}$$

of the probability is less than or equal to one.

$$\Pr \left\{ \left| x_1 - \bar{x}_1 \right| \geq \bar{x}_1 - r \right\} \leq \frac{\sigma_1^2}{(\bar{x}_1 - r)^2}$$

$$\Pr \left\{ \bar{x}_1 - x_1 \geq \bar{x}_1 - r \right\} \leq \frac{\sigma_1^2}{(\bar{x}_1 - r)^2}$$

$$\Pr \left\{ x_1 \leq r \right\} \leq \frac{\sigma_1^2}{(\bar{x}_1 - r)^2}$$

Minimizing the upper bound of this probability is equivalent to maximizing $\frac{(\bar{x}_1 - r)^2}{\sigma_1^2}$ or $\frac{\bar{x}_1 - r}{\sigma_1}$

Thus the Principle of Safety First would lead the decision maker to choose so as to maximize the ratio of the mean to the standard deviation. In the case where the standard deviation is the same for all alternatives this is equivalent to maximizing the expected value of the money return. This in turn is equivalent for this type of utility function, to maximizing the expected utility.

Parenthetically, it might be indicated that an extremely important problem in economics is the problem of choice

among random amounts of money when the utility function of money assumes various forms. This is also a central problem in the utility theory for engineering decision making. An example of one sort of result is given by the analysis above. A more general and important result which is well known in Economics is the following: suppose that the utility function of money is a continuous function having at least a first and a second derivative. If the first derivative is positive and the second derivative is negative, the utility function is said to exhibit "decreasing and positive marginal utility of money". If the decision maker chooses so as to maximize the expected utility of money, then given two prospects having the same expected money return, he will prefer the one having the smaller variance. See Marschak (36), Friedman and Savage (17), and Marshall (37). The utility of an amount of money, x , may be expanded into a Taylor series.

$$u(x) = u(a) + (x - a) u'(a) + (x - a)^2 u''(a) + \dots$$

Then letting $a = E(x)$ and taking the expected values;

$$Eu(x) = u(E(x)) + E(x - E(x))^2 u''(E(x)) + \dots$$

$$= u(E(x)) + \sigma_x^2 u''(E(x)) + \dots$$

2. Kolmogorov's Inequality: Consider a sequence of independent decision situations having outcomes

$$u_1 [a_1(S_j)] , u_2 [a_1(S_j)] , \dots , u_n [a_1(S_j)]$$

and means $Eu_1, Eu_2, \dots, Eu_n$

and standard deviations $\sigma_{u_1}, \sigma_{u_2}, \dots, \sigma_{u_n}$.

Let S_K be the sum of the first K outcomes,

$$m_K = Eu_1 + Eu_2 + \dots + Eu_K$$

$$\sigma_{S_K}^2 = \sigma_{u_1}^2 + \sigma_{u_2}^2 + \dots + \sigma_{u_K}^2$$

Then for every $t > 0$ the probability of the simultaneous realization of the n inequalities

$$|S_K - m_K| < t \sigma_{S_n} \quad K = 1, \dots, n$$

is at least $1 - t^2$.

This theorem leads to results analagous to those given above but much stronger. For example, Kolmogorov's Inequality implies that choosing so as to maximize the expected utility maximizes the probability that this method of choice will be better than any other on each of a sequence of n decisions, rather than simply at the n th decision.

The most important result, however, follows from the weak law of large numbers.

3. Weak Law of Large Numbers: Consider a sequence of decision situations such that the outcome of choosing the i th alternative on the k th decision is a random variable $U_K [a_i(S_j)]$. These outcomes are independent but not necessarily identically distributed. They must be uniformly bounded, that is $|u_K [a_i(S_j)]| \leq A$ for all i, j, K . Then for any $\epsilon > 0$ as $n \rightarrow \infty$

$$\Pr \left\{ \left| \frac{S_n - m_n}{n} \right| > \epsilon \right\} \rightarrow 0$$

Suppose that in each decision situation the decision maker chooses so as to maximize the expected utility of the outcome. Let the sum of the first n outcomes resulting from the application of this method of choice be S_n^* , and let the expected value of this sum be m_n^* . For any other method of choice let the sum be S_n , and the expected value of the sum be m_n . The weak law of large numbers then leads to the conclusion that choosing in each decision situation so as to maximize the expected utility is rational in the sense that it agrees with the Axiom of Long Run Success. This may be shown as follows:

$$\Pr \left\{ \left| \frac{S_n - m_n}{n} \right| > \epsilon_1 \right\} \rightarrow 0$$

$$\Pr \left\{ \left| \frac{S_n^* - m_n^*}{n} \right| > e_2 \right\} \rightarrow 0$$

$$\Pr \left\{ -S_n^* + S_n + m_n^* - m_n > n e_3 \right\} \rightarrow 0 \quad (\text{for } e_3 > 0)$$

$$\Pr \left\{ S_n > S_n^* + n e + m_n - m_n^* \right\} \rightarrow 0$$

Since $m_n - m_n^* < 0$ for all n

$$\Pr \left\{ S_n > S_n^* + n e \right\} \rightarrow 0$$

and since $n e$ may be made arbitrarily small

$$\Pr \left\{ S_n > S_n^* \right\} \rightarrow 0$$

or

$$\Pr \left\{ S_n^* > S_n \right\} \rightarrow 1$$

which proves the assertion.

The weak law of large numbers thus indicates that for any particular (large) value of n , the probability that S_n is better than S_n^* is small. This does not mean, however, that maximizing the expected utility will yield better results for all large values of n , but only that instances in which it is not better occur at infrequent times. This is all that is required or implied

by the Axiom of Long Run Success. The strong law of large numbers permits a stronger statement.

4. Strong Law of Large Numbers: Consider a sequence of decision situations such that the outcome of choosing the i th alternative on the k th decision is a random variable $u_K [a_i(S_j)]$. These outcomes are independent but not necessarily identically distributed. The series $\sum \frac{\sigma_K^2}{K^2}$ must converge. (Kolmogorov Cri-

terion) Then for every pair $\epsilon > 0$, $\delta > 0$ there exists an N such that there is probability $1 - \delta$ or better that for every $r > 0$ all r inequalities

$$\frac{|S_n - m_n|}{n} < \epsilon \quad n = N, N + 1, \dots, N + r$$

will be satisfied.

In this form the law implies that for all values of $n > N$ it is almost certain that $\frac{|S_n - m_n|}{n}$ remains

small. By methods analagous to those used with the weak law of large numbers it can be shown that there is an overwhelming probability that S_n^* is better than S_n for $n > N$. This is a much more appealing statement than that permitted by the weak law or required by the Axiom

of Long Run Success. It would appear that there are many sequences of decision situations which would satisfy the requirements for the strong law to hold.

5. Central Limit Theorem: If the two conditions indicated for the previous theorem hold together with the condition that σ_u^2 exists, then

$$\sigma_u [a_1(S_j)]$$

$$\Pr \left\{ \alpha < \frac{S_n - nEu}{\sigma_u [a_1(S_j)]} < \beta \right\} = \phi(\beta) - \phi(\alpha)$$

for fixed α, β ($\alpha < \beta$) where $\phi(x) =$ normal distribution.

By substituting $S_n^* - S_n$ for S_n in the above expression and making corresponding changes in the mean and variance one may estimate the probabilities of variance deviations among the results of two decision methods for large n .

As an example of the implications of these theorems a special type of decision situation is discussed below.

Binomial Decision Problems

There is a very interesting class of decision problems

which may be called binomial decision problems because the statistics of Bernoulli trials applies directly to them. These are decisions in which there are only two possible outcomes, for example, success or failure.

Let $u [a_i(S_j)] = u(p)$ or $u(q)$ for all i, j where

$p \equiv$ the outcome, success

$q \equiv$ the outcome, failure

$P_j =$ probability of S_j

One would then select the a_i such that

$$Eu(a_i) = \sum_j P_j u [a_i(S_j)] = \max_i Eu(a_i)$$

$$\begin{aligned} \text{For any } a_i \text{ let } \sum P_j \text{ over all } S_j \ni u [a_i(S_j)] \\ = u(p) = P_i \end{aligned}$$

$$\text{and let } \sum P_j \text{ over all } S_j \ni u [a_i(S_j)] = u(q) = Q_i$$

$$Q_i = 1 - P_i$$

In these terms one would select a_k such that

$$Eu(a_k) = P_k u(p) + Q_k u(q) = \max_i Eu(a_i)$$

Let a_k be the action thus selected:

$$P_k u(p) + Q_k u(q) - P_i u(p) - Q_i u(q) \geq 0$$

$$\begin{aligned} P_k u(p) - P_i u(p) &\geq Q_i u(q) - Q_k u(q) \\ &\geq (1 - P_i) u(q) - (1 - P_k) u(q) \\ &\geq P_k u(q) - P_i u(q) \end{aligned}$$

But if $u(p) > u(q)$ this will be true for $P_k > P_i$. Thus we need not know the utility function in order to make the choice. We require only the relation $u(p) > u(q)$, ("The value of success is greater than that of failure.") Then one may select so as to maximize P_i , the probability of success.

We now state some of the well known theorems on Bernoulli trials and interpret them in terms of binomial decisions. Our statements of the theorems follow Feller (15).

Weak Law of Large Numbers

As the number of decisions, N , increases, the probability that the average number of successes deviates from P_i by more than any preassigned ϵ , tends to zero.

$$\Pr \left\{ \left| \frac{X}{N} - P_i \right| > \epsilon \right\} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

$$\text{or } \Pr \left\{ \left| x - NP_1 \right| < N \epsilon_1 \right\} \rightarrow 1 \text{ as } N \rightarrow \infty$$

$$\Pr \left\{ \left| xu(p) - NP_1 u(p) \right| < \epsilon_2 \right\} \rightarrow 1 \text{ as } N \rightarrow \infty$$

Thus it is seen that selecting so as to maximize P_1 agrees with the Axiom of Long Run Success.

$$\Pr \left\{ \left| (x_i u(p) - x_j u(p)) - (NP_1 u(p) - NP_j u(p)) \right| < \epsilon_3 \right\} \rightarrow 1$$

Strong Law of Large Numbers

For every $\epsilon > 0$, $\delta > 0$ $\exists$ $r \geq 0$ for all v the probability of the simultaneous realization of the v inequalities

$$\left| \frac{x_r + K}{r + K} - P_1 \right| < \epsilon \quad \text{for } K = 1, \dots, v$$

is less than $1 - \delta$.

We now consider sequences of N choices in which a_1 is selected:

For N decisions

$$Eu(a_1) = NP_1 u(p) + NQ_1 u(q)$$

$$= NP_1 u(p) + Nu(q) - NP_1 u(q)$$

$$= NP_1 [u(p) - u(q)] + Nu(q)$$

$$\overline{u(a_1)}^2 = \overline{xu(p) + (N-x)u(q)}^2 = \overline{x [u(p) - u(q)] + Nu(q)}^2$$

$$= [u(p) - u(q)]^2 \sum_x^2 = [u(p) - u(q)]^2 N P_1 Q_1$$

Now suppose a utility function of the form commonly ascribed to industrial organizations is assumed. Say

$$Ux(p) = xu(p) \quad x < a$$

$$Ux(p) = -y \quad x \geq a$$

$$Eu(a_1) = \sum_{x=0}^a f(x)(-y) + \sum_{x=a+1}^N xu(p) f(x)$$

$$= \sum_{x=0}^a \frac{N!}{x!(N-x)!} P_1^x Q_1^{N-x} (-y) +$$

$$\sum_{x=a+1}^N xu(p) \frac{N!}{x!(N-x)!} P_1^x Q_1^{N-x}$$

Minimizing the probability of $x \leq a$ agrees with maximizing P_1 , and $Eu(a_1)$.

Diversification

We are now in a position to give a highly simplified analysis of the well known principle of diversification which is widely advocated. It would appear, however, that its exact nature is little understood in financial circles and possibly even less understood in industrial and engineering circles. We restrict this analysis to opportunities which are of the form of Binomial Decisions

which we have just been discussing.

We consider a decision maker with an initial capital of say v dollars who has two alternatives open to him;

a_1 ; invest v dollars in an opportunity which has a probability P_1 of returning $2v$ dollars and a probability $1 - P_1$ of returning zero dollars

a_2 ; invest v/n dollars in n independent opportunities each of which has probabilities and returns as in a_1

Under the Axiom of Long Run Success we should select so as to maximize $Eu(a_i)$.

$$Eu(a_1) = P_1(2v) + (1 - P_1)0 = 2vP_1$$

$$Eu(a_2) = \frac{v}{n} nP_1 + 0(n - nP_1) = 2vP_1$$

This rather surprising result means that in the long run diversification is of no real benefit to the decision maker. In fact, if there is a charge associated with the undertaking of each opportunity which is independent of size of investment (such as a constant brokerage fee) the diversification is more costly in the long run.

Diversification is, however, often described as a

strictly conservative device. We shall show that this is correct in the sense that diversification greatly reduces the probability of a total loss of capital. Consider this probability for each of the alternatives;

$$\text{For } a_1: \Pr(\text{total loss}) = 1 - P_1$$

$$\text{For } a_2: \Pr(\text{total loss}) = \frac{n!}{x!(n-x)!} P_1^x (1 - P_1)^{n-x}$$

$$\text{for } x = 0$$

$$= (1 - P_1)^n$$

Thus in this sense diversification is preferable.

Chapter 5.

DECISION MAKING WITH INCOMPLETE INFORMATION

Classification of Decision Situations

A decision situation may be described in terms of the way the decision maker organizes and abstracts the situation. In every instance the decision maker chooses to consider certain alternatives, to regard certain elements of the decision as known deterministically, certain elements as known probabilistically, and possibly certain elements as completely unknown. We shall avoid for the moment the question of how best to arrive at a set of abstractions for a particular decision situation. We shall call the elements of any such abstraction the assumptions about the situation.

We proceed then to classify decision situations into the following types:

1. The alternative courses of action are assumed known but nothing else. For convenience we shall distinguish the case where two alternatives exist from the case where more than two exist.
2. The alternative courses of action are assumed known and the outcomes resulting from each course of action

are assumed known deterministically.

3. The alternatives are assumed known, the relevant states of the world known, and the outcome resulting from the selection of a course of action under a particular state of the world is assumed known deterministically. Which state of the world actually obtains is known neither probabilistically nor deterministically. This is a problem of decision in the face of risk.

4. The same as type 3, except that the outcome is known probabilistically for each state of the world. That is, for each course of action the conditional probability density function of the outcomes is known for each state of the world.

5. The same as type 3, but the probability distribution of the states of the world is known. This is a problem of choice in the face of uncertainty.

6. The same as type 4, but the probability distribution of the states is known.

7. The alternatives are assumed known and the probability density function of the outcomes resulting from each alternative is assumed known.

We do not consider the question of what it would mean to say that an alternative course of action was assumed

known deterministically, but instead we shall assume that each alternative is well defined for the decision maker. We shall in general assume that the set of alternatives which are considered in any situation are collectively exhaustive (but not necessarily mutually exclusive).

It is helpful now to present each type of decision situation in its matrix form. Probabilities are considered in the sense of the frequency interpretation, unless otherwise indicated.

Type 1

a_1	
a_2	

a_1	
a_2	
$\vdots$	
a_i	
$\vdots$	
a_{n-1}	
a_n	

Type 2

S = known state of the world

$$u_i = u [a_i(S)]$$

Type 3

$$u_{ij} = u [a_i(S_j)] = \text{A number}$$

Type 4

	s_1	s_2	s_3	
a_1	$f_{11}(u)$	$f_{12}(u)$	$f_{13}(u)$	
a_2	$f_{21}(u)$	$f_{22}(u)$		
a_3	$f_{31}(u)$			

$f_{ij}(u)$ = Probability that the value of the outcome of the i th action taken in the j th state of the world equals u .

Type 5

Type 3 plus $f(s_j)$ = Probability that the j th state of the world obtains.

Type 6

Type 4 plus $f(s_j)$ known.

Type 7

Now up to this point in our classification of decision situations we have neglected the relationship of outcomes and the value of outcomes. This was done for simplicity, but we may now add a description of the possibilities for this relationship.

1. Given a deterministically known outcome, the value of the outcome may be given deterministically.
2. Given a deterministically known outcome, the value of the outcome may be known probabilistically. Note that here we depart importantly from the usual notion of utility functions.
3. In the case where we know the probability distribution of the outcomes for a course of action and state of the world, we may, for a given outcome, know
 - a. the value deterministically
 - b. the conditional distribution of the value

In both cases the utility of a course of action and state of the world is a random variable.

It is interesting to note that in general utility is a random variable, where the randomness enters from one or more of the following sources:

- a. density function of states of the world
- b. density function of outcomes conditional on each state of the world
- c. density function of values conditional on each outcome

We shall indicate the rules which may be given for each of the seven types of decision situations which we have described above. We shall also show when these rules meet the criterion for solutions to the engineering decision problem, that is, they satisfy the requirements of the Axiom of Long Run Success. It is interesting to note that Type 2 decision situations are the type dealt with in Engineering Economy. It seems intuitively appealing to attempt to reduce all other types to type 2. We shall be particularly cognizant of the realities of engineering decisions in our discussion of rules, noting especially that almost always some elements of the decision are unknown or probabilistically unknown.

Now in any situation except type 2 there may exist an opportunity to enrich the set of abstractions which the decision maker holds about the situation at hand. One opportunity that may exist is that of making an observation. Note that observation may be made which is designed to yield information about a variety of elements in the decision situation. For example, we make observations relative to the possible states of the world, the density function of the states, the outcome of an alternative and a state, the density function of outcomes, the value of an outcome, the density function of values, and the like. All these sorts of observations are thought of as being possible of accomplishment prior to the selection of an alternative. A collection of observations relative to any one or any combination of these elements or density functions will be termed an observation program.

We may also find that another kind of opportunity for the improvement of the abstractions of a decision maker exists. This sort of opportunity may consist only of actually carrying out the decision, recording the results of the choice, and using this information as a basis for subsequent decisions. While this is also a kind of observation program we shall differentiate

it from those described above for reasons which arise from the system's restrictions which are important for our problems. We shall call this kind of an opportunity a learning program. We shall examine the problem of rules for carrying out learning programs.

It is also evident that our theory of normative decision making must provide rules for the selection among the alternatives once the above programs have been carried out. These rules are called principles of choice.

This discussion puts us in a somewhat better situation to indicate in detail what we expect from the sets of rules which will compose the theory. We say that the normative theory for engineering decisions must consist of the following parts;

1. a method of value measurement
2. rules for conducting observation programs
3. rules for conducting learning programs
4. principle of choice

These requirements we consider the minimum, and will attempt to provide rules about additional operations in the decision process.

In discussing the problems associated with decisions in

the face of incomplete information we trade on the similarities between the situation of the engineering decision maker and that of the Statistician. One cannot press this similarity with any success at all, for the Statistician deals with problems that are generally far richer in given information than those of the engineering decision maker.

The possibility of this similarity is also recognized by Arrow (2), who says: "The problem of statistics can be formulated roughly as follows: It is known that one out of a number of hypotheses about a given situation is true. The statistician has the choice of one of a number of different experiments (including drawing a sample of any size) the outcome of any one of which is a random variable depending upon which of the unknown hypotheses is true. On the basis of the outcome the statistician must take some action (accept or reject a hypotheses, estimate the mean of a distribution to be some particular value, accept or reject a lot of goods) the consequences of which depend on the action taken and the hypothesis which is actually true." Many characteristics of engineering decision problems are similar, including the information gathering aspects of the problems. It is to be emphasized however, that

statistical models can be only employed as suggestive of normative behavior for the engineering decision maker. They cannot be followed in any degree exactly, but are suggestive of rules that would be useful if the engineering problem were sufficiently rich in given information.

To summarize, we take the fundamental problem of engineering decision making to be the problem of normative decision making behavior in the face of incomplete information. We propose to deal with this problem within the framework of the classification of decisions given above. We shall deal with the problem in two steps; first the problem of choice in the face of these situations, and secondly, the problem of moving from one situation to another. The second problem may be characterized as the dynamics of moving from one situation to another by either observation or learning.

The simplest type of decision situation to deal with is of course, the type 2 situation. One simply selects the course of action which will yield the maximum utility. This is the situation in which, for the purposes of decision at least, the future is considered to be determined (deterministically known). This agrees with the deterministic form of the Axiom of Long Run

Success.

We now begin to destroy this ultimate state of knowledge by introducing randomness at various points in the decision maker's view of his situation. Initially we can introduce randomness in two ways. The type 7 situation introduces randomness into the utility associated with each outcome. This might arise either through randomness of the outcomes themselves, or through randomness in the utility of the outcomes which are deterministically known. Based upon the discussions of the Axiom of Long Run Success, and utility functions, the appropriate principle of choice would be to select the course of action which will maximize the expected utility.

Another source of randomness which might be introduced at this point is the view of the situation as consisting of a number of states of the world whose a priori probability distribution is known, where the states are relevant to the outcome in the sense defined above. In this case, type 5, one would also select so as to maximize the expected utility, and the same rule would apply to situations of type 6.

Consider next situations of type 3 and type 4 where the relevant states of the world have been identified, but

their probability distribution is not known. This type of situation has received much attention in the literature. The following principles of choice have been suggested for this type of situation:

1. Assume the probability of each outcome equal to one divided by the number of outcomes and select the alternative which maximizes the expected value of the utility.
2. Select the alternative which will minimize the maximum loss (or maximize the minimum gain).
3. Select the alternative which will maximize the maximum gain.
4. Choose a number A in the interval zero to one. For each alternative examine the outcomes for the one having the largest utility and the one having the smallest utility. Multiply the largest utility by A and the smallest by $1-A$, then select the course of action for which the sum of these products is maximum.
5. Construct a new matrix by subtracting the utility attainable with a given course of action from the ~~maximum~~ utility attainable under a particular state of the world. The new matrix is called a regret matrix and choice is made by the computation of an optimal

game strategy.

None of these principles of choice necessarily agrees with the Axiom of Long Run Success, thus none can be selected as a guide for rational choice. Each has appeal for certain systems and certain situations, but none has any compelling properties which would recommend it generally as more likely to bring long run success than others. The properties of these principles have been investigated by Milnor (41) and others.

It is important to note that none of these principles is consistent with the Axiom of Long Run Success in the situations described so far. However, in engineering decisions where the elements of conflict are present the decision becomes a game in the sense of von Neumann and Morgenstern. In this case the minimax principle can be taken consistent with the axiom, and the decision maker may obtain optimal strategies in the game theoretic sense. Another special situation in which a modified minimax principle is consistent with the axiom is given in the example on page 97 above.

Thus if the decision maker must make a choice in the face of the decision situations we have described, he can achieve rationality or agreement with the Axiom of

Long Run Success only for decisions of types 2, 5, 6, and 7. Our concept of rationality is not sufficiently rich to provide guides for choices in decisions of types 1, 3, and 4. In most cases the decision maker may improve the set of abstractions which he holds, and is not required to make a choice at any particular point on the path of improvement. We enter now on a discussion of methods of transition from one situation to another.

Analysis of Type 1 Situations

We begin with an analysis of what we have called type 1 decision situations, that is, situations in which the decision maker views the problem in terms of the alternatives alone. He has no basis for forming abstractions about any of the other elements of the decision. Shubik points out, "It is fairly obvious that there is rarely an economic situation in which the entrepreneur has no information at all". (55) We think that this is perhaps not the case and that, in fact, many important decision situations are in reality of this form or take this form at some point in their development.

We shall consider initially the case where only two alternatives appear. We distinguish the situation in

which observations are possible from that in which we must resort to what we have called a learning program, or an actual carrying out of the decision. Now it will be argued that both of these situations are observational situations and may be treated identically in the theory. We think that reasons will shortly emerge which arise from the kind of systems restrictions we are considering which will indicate the meaningfulness of this distinction. For the present let us merely try to make the distinction clear.

By an observation we mean either the recourse to historical data or to indirect measures which are obtained without actually carrying out the decision in the full context of the system and with full realization of the consequences. By a learning situation we mean the case where there is no indirect or historical way of obtaining information relative to the decision, but the decision maker must actually carry out the decision and then begin to "learn by experience".

Case I - Observations Possible

We are supposing that the decision maker can abstract nothing from the decision situation except that there is a choice between two known alternatives. We propose to

undertake some process which will transform this view of the problem into one of the other more manipulable types of decision situations which we have outlined above. Let us begin by supposing that the outcomes of these two courses of action are functions of the state of the world which obtains when the choice is made. The first question which then arises is what states of the world are relevant, that is, have some effect on the outcomes.

In the ordinary problem of statistics the decision maker must decide which of a number of known states of the world actually obtains. In the problem we have posed no way is available to construct the hypotheses among which to decide. In other words, we would like to list some states of the world along the top of columns of our decision matrix, but no way is available for doing so. We do not know explicitly which states of the world are relevant. We might in time imagine the eventual possibility of constructing some kind of description of all states of the world, but these would necessarily be incomplete, and we have no way of determining their relevancy to the decision problem at hand at the moment.

We digress briefly to indicate what is meant by a relevant state of the world, or really a description of a

relevant state for a particular decision problem.

As before, S_j is a description of a possible state. Let

$\{S_j\}$ be the set of descriptions for which the following conditions hold:

$$(1) \quad u[a_1(S_k)] = u[a_1(S_1)] \quad \text{for all } S_k, S_1 \text{ in } \{S_j\}$$

and

$$(2) \quad u[a_1(S_k)] \neq u[a_1(S_p)] \quad \text{for all } S_k \text{ in } \{S_j\} \\ \text{and all } S_p \text{ not in } \{S_j\}$$

Now if this is true for all a_1 then $\{S_j\}$ is a set of states which is in some way relevant to the decision.

Under the Axiom of Long Run Success the case where $u[a_1(S_j)]$ is a random variable can be handled by inserting expected values in these statements.

Suppose that all possible states of the world were organized into mutually exclusive and collectively exhaustive subsets $\{S_1\}, \{S_2\}, \dots, \{S_j\}, \dots, \{S_m\}$ the

elements of which satisfy conditions (1) and (2) above.

Let m be finite. We then abbreviate $\{S_j\}$ by S_j and call S_j a relevant state for the decision at hand.

Now in order to give the description of the state which S_j represents we need not and cannot describe all the

aspects of the state. We need only include the aspects of the set $\{ S_j \}$ which led us to reduce it to a single description S_j . That is, those aspects such that if A_i is a statement about an aspect A of the i -th state of the world;

$$A_i \equiv A_j \quad \text{for all } i, j \text{ in } \{ S_j \}$$

$$A_i \neq A_k \quad \begin{array}{l} \text{for all } i \text{ in } \{ S_j \} \\ \text{for all } k \text{ not in } \{ S_j \} \end{array}$$

Those aspects which meet these requirements we might call the distinguishing features of S_j . For these are the features which distinguish it from another description S_k . To summarize, S_j is a description which applies to those states of the world which are relevant for a decision. The description may be given by stating those aspects which distinguish these states from the other sets of relevant states.

Now we return to the problem stated originally. This cannot be interpreted directly as a problem in statistics since we do not know what the relevant states of the world are, about which we should make observations. If we assume that suitable methods can be found, this decision might be handled by transforming it into any one of the other types of decisions which we have

described. Little need be said here about the methods of making such inferences, for they are the subject of statistics. It should be pointed out however, that if the cost of observations is zero, then it will in general pay to make observations. There is nothing to be lost by collecting information and making inferences from it. As an illustration, consider a type 6 decision situation. If this had been viewed as a type 1 situation there would have been little choice (assuming no observation or learning) but to select one of the alternatives at random. The expected return would then be

$$\frac{1}{n} \sum_i \sum_j Eu [a_i(S_j)] f(S_j)$$

By transforming the problem to one of type 3 or type 4 one could eliminate all the alternatives which were dominated, and then selecting at random from the reduced set of alternatives would improve the expected return. If no dominance exists and if one chooses according to the minimax or some such principle, then it cannot be said that the gain in information would in general improve the expected return. Upon transforming the problem to one of type 5 or type 6 the alternative which will maximize the expected return may be selected, and the

payoff will be $\max_i \sum_j Eu [a_i(S_j)] f(S_j)$ which will

be clearly larger than or equal to that for the type 1 abstractions. Again if inferences may be made to reduce the problem to one of type 2 or type 7 the payoff may be further improved.

Aside from the possibility of making direct inferences on the basis of observations, there are some interesting special approaches for dealing with a problem of type 1. The type 1 problem is not sufficiently rich as it stands to present any opportunities for the formulation of normative rules. To make a problem, we must either begin to add information and transform it into one of the other types of situations as we have described above, or we must begin to make some assumptions which will enrich the problem. Now here again is a place where judgment can be brought into play, namely in the selection of which assumptions to make. As in all cases the selection of assumptions on the basis of judgment may be criticized as entirely gratuitous, and this criticism cannot be refuted without recourse to some actual observations about the applicability of the assumptions. It often occurs however, that by making the process of judgment explicit one may convince the critic of the applicability of a particular set of assumptions.

Without regard for these matters we examine the consequences of two kinds of assumptions which might be made. In the first case we assume that the a posteriori distribution of a sample may be obtained and the problem reduced to a test of a composite hypothesis against a composite alternative.

Consider a type 1 problem in which there are two alternatives. We cannot tell in advance whether it is rational to undertake any sort of an observation program at all, but suppose one is to be undertaken regardless of whether or not it is economic. We are interested in testing the hypothesis

$$H: Eu(a_1) > Eu(a_2)$$

against the alternative

$$A: Eu(a_1) \leq Eu(a_2)$$

We might observe pairs of outcomes, one for each alternative, and compute $d = U(a_1) - U(a_2)$

After taking a sample of say N pairs of observations, the decision maker would like to decide about the hypothesis. The losses suffered by the decision maker might be given by the following table:

$$\text{Let } D = Eu(a_1) - Eu(a_2)$$

	$D > 0$	$D \leq 0$
Accept H (choose a_1)	0	D
Accept A (choose a_2)	-D	0

The acceptance of the hypothesis is taken to be equivalent to choosing a_1 , and the rejection of the hypothesis, to choosing a_2 . Now suppose on the basis of the N pairs of observations, the decision maker could infer the probability distribution of D , say $p(D|N)$. Then he would choose a_1 (accept the hypothesis) if

$$\int_{-\infty}^0 p(D|N) \geq \int_0^{\infty} p(D|N)$$

otherwise he would choose a_2 . While it may well be impossible to make any inference about the probability distribution of D , the scheme given above at least indicates the general method by which a type 1 problem might be approached.

Another assumption of particular interest which is often made in engineering analyses of the type 1 decision is that of adding an alternative whose consequences are presumed to be known independent of the states of the world. Such an alternative is called a standard invest-

ment opportunity in Engineering Economy. In many decisions the added alternative may be the "maintenance of the status quo", or in cases where the added alternative is a purely fictitious one it may be thought of as a standard in the generally accepted sense of the word. In these cases one may compute the expected utility of the known alternative and then take observations to determine whether or not the expected utilities of the unknown alternatives are greater or less than this. This is the kind of statistical problem which is very commonly encountered. In cases where observation is relatively expensive the problem arises as to whether or not the second unknown alternative should be tested if the first has already been shown to be better than the known alternative. As far as is known, there is no convincing argument on this point. This question of method becomes itself a decision problem of the same type as the original problem.

Case II - Observations Not Possible

We turn now to the type 1 decision problem when it is either not possible or not economic to make observations of the type we have discussed above. Instead one is forced to actually carry out the decision situation and accumulate experience as a basis for future decisions.

This problem, in which we deal with sequences of decisions, the first of which is a strict type 1, and those following depend on the experience gain from carrying out the previous situations. This is what we have called a learning problem. It is clearly a type of observation problem but of a far more restricted sort than discussed above.

We begin with a descriptive analysis which will bring out some interesting problems of a general nature.

Consider the case of a decision problem where only two alternatives are presented. We can think of experience as observation and apply the methods of inserting an additional alternative with known outcome, and of testing the composite hypothesis about the mean utility, as we have done in the observation case. But some sort of a sequential process immediately suggests itself, so that at any stage in the process of our learning we may evaluate the evidence on hand and form some kind of rationale for handling the next experience.

To this end suppose we begin on the first decision in the sequence by selecting according to the outcome of a chance device which assigns probability one half to each possibility. If we symbolize the two alternatives

by a_1 and a_2 , we can suppose without loss of generality that we know after the first decision the value of an outcome for a_1 . What is ultimately desired is a determination of the relation of $Eu(a_1)$ and $Eu(a_2)$, that is $Eu(a_1) \left\{ \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \right\} Eu(a_2)$. After the first decision we know a value for $u(a_1)$. This does not really tell us very much about the density function of $u(a_1)$. It would be especially useful to know whether or not $u(a_1)$ has a point distribution. The question of how to use this information in making the next choice cannot be settled with any reasonable argument. It usually occurs that a judgment of some sort is made about the adequacy of the first outcome and then it is argued either that things turned out satisfactorily by choosing a_1 so why risk finding out about $u(a_2)$, or that it might be that $Eu(a_1) < Eu(a_2)$ and thus one should try a_2 on the second choice, etc.

One possible approach would be to carry out the decision N times selecting a_1 and N times selecting a_2 , and proceed as in the observation program outlined above. It is clear however, that such a learning process is carried on only at a cost, and it would be desirable to terminate such a process as early as possible and proceed on a firm and hopefully correct policy of choice.

The question of when to terminate the learning program and establish a fixed decision policy thus arises.

A special kind of sequential learning program might be used in this situation. After each step in the learning program two qualities are computed:

1. the expected return if the learning program was terminated and the best possible decision policy established on the basis of the evidence at hand
2. the expected return if the learning program was continued and if at each future step the best possible decision was made

The Axiom of Long Run Success then leads the decision maker to choose the alternative, continuing or terminating, which has the largest expected return. There are a number of difficulties associated with the application of such a program to a type 1 situation. The type 1 situation has nothing like the richness necessary to compute the expected returns required for the choice as to continuation or termination. Thus insufficient information prevents the decision maker from achieving rationality. Further, if one considers an infinite sequence of decision situations over which learning might be carried, there will usually be no

stopping point. In this respect it differs importantly from sequential observation programs to be discussed below. In an indefinite sequence of decisions it does not pay to terminate learning until the risk of making a wrong decision can be reduced no further. However, suppose we consider a sequence of N decisions where N is a finite number. How many of these should be devoted to the learning process?

Let N = number of decisions in the sequence under consideration

n = number of these decisions to be used for learning

a_1, a_2 = alternatives

Suppose the decision maker uses $n/2$ pairs a_1, a_2 for the learning process and then makes the best firm choice on the basis of the evidence at hand.

$q(n)$ = a posteriori probability of a_1

$q(0) = \frac{1}{2}$

Cost of program = $\frac{n}{2} Eu(a_1) + \frac{n}{2} Eu(a_2)$

Return = $(N-n) \{ q(n) Eu(a_1) + [1-q(n)] Eu(a_2) \}$

As an example suppose $q(n) = \frac{N+n}{2N}$

$$G = \text{total return} = \frac{n}{2} [Eu(a_1) + Eu(a_2)] + \\ (N-n) \left\{ \frac{N+n}{2N} Eu(a_1) + \left[1 - \frac{N+n}{2N} \right] Eu(a_2) \right\}$$

$$\frac{\partial G}{\partial n} = \frac{Eu(a_1) + Eu(a_2)}{2} - \frac{n}{N} Eu(a_1) - Eu(a_2) + \frac{n}{N} Eu(a_2) = 0$$

$$\frac{Eu(a_1) - Eu(a_2)}{2} + \frac{n}{N} (Eu(a_2) - Eu(a_1)) = 0$$

$$\frac{n}{N} = \frac{1}{2}$$

Thus if N is finite and if $q(n)$ can be known, there is an optimum finite number of decisions to devote to a learning program such as we have described above. If $q(n)$ is known in advance to be a function of n only, then the optimum n may be computed in advance as above. In this case a fixed n learning program is best. Generally $q(n)$ will not be known in advance. The problem then is one of designing a sequential learning program much like the design of sequential observation programs to be discussed below. If some value of n exists, say n^* , such that for all values of n greater than n^* , $q(n)$ remains constant, then there is nothing to be gained from extending the observation program beyond n^* . Note also that in the absence of this condition, as N be-

comes very large n does also, and the decision maker may never stop learning if he considers the indefinite future.

Again if one can adjudge certain assumptions acceptable for the problem at hand, the type 1 situation may be enriched to the point where interesting solutions can be used. If one can assume that the outcomes which would fill in the decision matrix if known are not random variables but deterministic quantities, and if one can assume that for one reason or another a minimax strategy is appropriate, then the method of fictitious play has some possibilities. There is an additional requirement that the result of any trial must not only make available to the decision maker the particular outcome which occurs, but also he must be able to observe the outcomes for the given state of nature if any of the other alternative courses of action had been chosen, and the outcomes for the course of action actually chosen if any of the other states of nature had prevailed. That is, he must observe not only the actual outcome but all the other outcomes in the row and column of the matrix which intersect at the actual outcome. The iterative method is described in (39).

Such an iterative method need not be restricted to the computation of minimax solutions. It can be adapted to the conduct of learning programs which will eventually yield the alternative for which the expected utility is maximum.

Suppose the outcome from taking a course of action a_1 , when a state of the world S_j prevails is a random variable u_{1j} . Suppose that the decision maker begins a learning program which yields not only the outcome from the particular course of action taken, but also information as to what the outcome would have been if any of the other available actions had been taken. Thus the decision maker obtains through learning not only $u(a_1)$ when action a_1 is taken, but also $u(a_k)$ for $k = 1, 2, \dots, n$. Suppose the outcomes are numbered $u_1(a_1), u_2(a_1), \dots, u(a_n), \dots$. Then after making say H decisions, no matter what alternative was chosen on this decision, the decision maker may compute

$$\sum_{h=1}^H u_h(a_1) \quad \text{for all } a_1.$$

On the basis of this evidence the best choice for the decision maker on the next decision would be to select the a_1 for which

$$\sum_{h=1}^H u_h(a_1) \text{ is maximum.}$$

This corresponds to the common sense notion of making the best possible decision at any stage in the development of the learning process on the basis of the evidence then at hand. The weak law of large numbers leads directly to the conclusion that this process converges in probability to the alternative which maximizes the quantity $Eu(a_1)$.

Transformation of Type 3 Decisions to Type 5

Several approaches suggest themselves for the transformation of type 3 decision problems into type 5 problems. One might for example, either by observation or by learning begin to develop estimates of the relative frequencies of the S_j . Note that if such estimates are not available, the decision maker may satisfy the logical consistency requirement of rationalization by using personalistic probabilities in making the decisions.

The statistical model for this transformation requires that the decision maker be able to compute the a posteriori probabilities of the S_j on the basis of a sample of observations.

Let $x =$ a set of N observations, $(x_1, x_2, \dots, x_N)$

$f_x(S_j)$ = the a posteriori probability of S_j
given the N observations

$p(x|S_j)$ = probability of the sample given S_j

Then the a posteriori probability of the state S_j may be computed from the relation;

$$f_x(S_j) = \frac{p(x|S_j)}{\sum_j p(x|S_j)}$$

The decision maker would then select the alternative a_i for which

$$\sum_j u[a_i(S_j)] f_x(S_j) \text{ is maximum.}$$

This may be called a fixed sample size observation program. It assumes essentially that observations are costless and that it is necessary or desirable to determine in advance the number of observations to be made.

A more adequate statistical model for this transformation is a sequential one which takes into account the cost of observations and the value of the information gained from them. Such a sequential scheme will be outlined below. The general theory is given in Blackwell and Girshick (3).

We consider truncated sequential observation programs, that is programs which can consist of at most say N observations because of system restrictions which place a limit on the amount of observation which may be done. The observation program may consist of $0, 1, \dots, N$ observations, and at any point where observation is terminated there must be a rule for making the best choice in the light of the accumulated evidence.

Let $x =$ a particular sequence of N observations

$$(x_1, x_2, \dots, x_N)$$

$c =$ cost of making each observation (assumed constant for simplicity)

$ck =$ cost of the first k observations

$$R_k(x) = \max_i \sum_j u [a_i(S_j)] f_k(S_j) - ck$$

where $f_k(S_j) =$ a posteriori probability of S_j
given the k observations

To construct a sequential observation program we make the following definitions:

$$a_N(x) = R_N(x)$$

$$a_{N-1}(x) = \max \{ R_{N-1}(x) ; E_{N-1}(a_N(x)) \}$$

$$a_{N-2}(x) = \max \left\{ R_{N-2}(x); E_{N-2}(a_{N-1}(x)) \right\}$$

$$\vdots$$

$$a_0(x) = \max \left\{ R_0(x); E(a_1(x)) \right\}$$

where $E_k(a_{k+1}(x))$ is the conditional expectation

of $a_{k+1}(x)$ given the k observations which have already

been taken, and

$$R_0(x) = \max_i \sum_j u[a_i(S_j)] f_0(S_j)$$

Now at this point a very important difficulty is encountered. The type 3 decision situation was specifically defined as being one in which the probability distribution of the states was unknown. In the above definition the term $f_0(S_j)$ is this unknown distribution, the a priori probabilities of the states. It will be clear shortly that it is necessary to have the a priori probabilities of the states in order to begin the sequential observation program. Without it the decision maker cannot determine whether any observations should be made at all. Two approaches have been recommended for the decision maker at this point. He may assume that all relevant

states of the world are equally likely and proceed with the decision problem. From the viewpoint of rationality this method has absolutely nothing to recommend it.

Alternatively the decision maker may use a personalistic probability for the a priori probability of the state. This approach has the advantage of being logically consistent in the sense of agreeing with the decision maker's beliefs, but these beliefs may have no particular relation to the requirement for logical truth. Thus a sequential observation program is not strictly applicable to the type 3 problem, but we continue its development both for its normative implications and for its applicability to later types of decision problems. Overlooking these difficulties, the construction of the sequential observation program proceeds as follows.

After k observations have been made, $R_k(x)$ is the expected return if the decision maker stops taking observations and makes the best possible choice on the basis of the evidence in hand. $E_k(a_{k+1}(x))$ is the expected return from continuing the observation program if the best decision is made at each future stage. The decision maker stops if $R_k(x) \geq E_k(a_{k+1}(x))$ and continues to the next stage of the observation program otherwise.

The program can be specified by indicating the "stopping regions" or sequences of observations after which the decision maker maximizes the expected return by stopping and making the best choice. All sequences of k observations which form a stopping region are defined by

$$X_k^* = \{ x \mid R_k(x) < \sigma_k(x) \text{ for } K < k \text{ and } R_k(x) = \sigma_k(x) \}$$

and the sequential program is defined by

$$\underline{X}^* = (X_0^*, X_1^*, \dots, X_N^*)$$

The expected return from the use of this sequential observation program is given by

$$E(R) = \sum_{k=0}^N \sum_{x=X_k^*} (\max_i \sum_j f_k(S_j) E u [a_i(S_j)]) \cdot \sum_j f_0(S_j) p(x|S_j)$$

where $p(x|S_j)$ probability of the sequence x when S_j prevails.

The sequential observation program $\underline{X}^*$ agrees with the Axiom of Long Run Success in the sense that for any a priori distribution of the states $f_0(S_j)$, the expected return for the use of $\underline{X}^*$ is greater than for any other sequential program. Since the fixed sample size program may be considered a special kind of sequential program,

then X^* has a greater expected return than any fixed sample size program also. Proof of these statements is given in Blackwell and Girshick(3).

Transformation of Type 4 Decisions to Type 6

Type 4 decision problems may be transformed into type 6 problems by the same methods which have been outlined above. The only difference in the analysis would arise if the decision maker uses a learning program in which it is necessary to infer the state of nature which prevails from the actual outcome of a course of action. In the type 3 problem the outcomes were determinate and thus the prevailing state of nature could be uniquely identified on the basis of an outcome. If the decision maker attempts to do the same thing with type 4 problems the outcomes are random variables and it is not in general possible to identify with certainty which state of nature prevailed. If density functions of the outcomes are available, however, the a posteriori probabilities of the states may be computed. Suppose the decision maker observes say N outcomes, where he knows which alternative led to each outcome. Let those outcomes be numbered $u^1_i, u^2_i, u^3_i, \dots, u^k_i, \dots, u^N_i$ where the values of i are not necessarily the same for

each u_i^k . The a posteriori distribution of the states is then given by

$$f_N(S_j) = \frac{\prod_{k=1}^N f_{ij}(u_i^k)}{\sum_j \prod_{k=1}^N f_{ij}(u_i^k)}$$

and the decision maker may choose on the basis of this information.

Transformation of Type 5 Decisions to Type 2

In a type 5 situation one would select the a_i such that $\sum_j u_{ij} f(S_j)$ is maximum over all i . If an observation

procedure were available which would permit the determination of the state of nature which prevailed without error then the long run return from such a method of observation would be

$$\sum_j f(S_j) \left[\max_i u_{ij} \right] .$$

This will exceed $\max_i \sum_j u_{ij} f(S_j)$ unless

$\max_i u_{ij} = u_{i',j}$ for all j , in which case all courses of action except $a_{i'}$ would be eliminated by the relation of dominance and no observations would be made since no decision would exist.

Thus one would be willing to invest in an observation program of this errorless type no more than

$$\sum_j f(S_j) \left[\max_i u_{ij} \right] - \max_i \sum_j u_{ij} f(S_j)$$

Since this number may be computed in advance we may distinguish those observation programs which will pay from those which will not.

The more general type of observation program is a sequential one, such as we have outlined above, which will determine the a posteriori distribution of the states of nature after the observations have been taken. In this case the sequential method described above must be used.

Type 5 is exactly suited to a sequential observation program, since by definition the a priori distributions of the states are given. It is similar in structure to the usual sequential statistical decision problem. (See (3)). The analysis of the transformation of type 6 to type 7 is similar.

General View of Decision Making with Incomplete Information

At this point it would be well to summarize the results of this discussion so far. In choosing among alternative courses of action in the face of incomplete information the decision maker may be rational in the sense of

agreeing with the Axiom of Long Run Success only in decisions of types 5, 6, 7, and 2. In types 1, 3, and 4 rationalization may be carried to the extent of being explicit and logically consistent, but no further.

The above discussion indicates further the methods of progression from type 1 problems where only the alternatives are known, to type 2 where the exact outcome for each action is known. Assuming the appropriate opportunities are permitted by the restrictions imposed by the system, the decision maker may develop his view of the decision to any level he wishes and then make the best possible choice which is available at that level. The question of interest is the extent to which this development should be carried, or "To what level should the decision maker develop his view of the problem before making the choice?"

As has been demonstrated, transformations starting with types 5 or 6 may be made by a process of sequential observation. Further, a sequential program may be constructed, the expected return from which is greater than from any other sort of transformation program or lack thereof. Thus the Axiom of Long Run Success requires that such a program be used. Considering transformations beginning with types 5 or 6, the exact stage

to which the transformation should be carried may be ascertained, providing the necessary information is at hand. The principle is very simple. At each stage the decision maker calculates the expected return from stopping and making the best choice, and the expected return from continuing the transformation, and acts accordingly.

Parenthetically, it is well to reemphasize here that there appear to be few if any engineering decision situations in which anything like the richness of understanding necessary to actually compute the quantities necessary to carry out an optimal transformation program exists. The normative implications of the statistical model are clear and they may provide a very useful guide to behavior, however roughly applied.

In transformations from type 1 up through type 5 or 6 it is impossible to tell under the Axiom of Long Run Success whether or not it will be rational to make the transformation. One can imagine a sort of continuum of abstractions from type 1 to type 2. The decision maker may move along this continuum by undertaking various kinds of observation and learning programs. The kinds of programs may change as the transformation

progresses. At any point along the progression the decision maker would like to know whether to continue or stop and make a choice. If stopping is indicated, he would also like to know how the choice should be made. Until types 5 or 6 are reached the process of rationalization has not been carried far enough to yield what we have called a solution to the engineering decision problem. Up until this point the decision maker may fulfill the requirements of explicitness and logical consistency in certain respects, but the lack of logical truth prevents agreement with the Axiom of Long Run Success.

Chapter 6.

JUDGMENT IN ENGINEERING DECISIONS

It is almost universally agreed in the literature of engineering decision making that any analysis of an engineering decision problem must be submitted to the final authority of the mature judgment of experienced persons. Evans (13) has examined many of the problems associated with judgments and has discussed the role of judgment in engineering decisions. There is, however, little guidance available on just how judgments should be made, how they may be improved, or how they can best be used in decision making. The theories of personalistic probability and the economic theory of consumer values can be interpreted as dealing with, in part, how to make explicit the judgments of probability and value which are implicit in our actions. These theories further indicate methods for assuring that our behavior does not contradict these judgments. Both of these concepts have been discussed above. At this point we wish to make a few remarks about judgments in general.

There is some disagreement in the field of engineering as to what role judgment should play in the process

of decision making. Some feel that it is of first importance while others feel that its use should be minimized. It seems clear that the method of rationalization by which we attempt to achieve agreement with the Axiom of Long Run Success requires that judgment be kept to a minimum in engineering decisions. We have required in our definition of rationalization two things which ordinary judgments often do not satisfy, explicitness and logical truth. It is certainly true that judgments of the ordinary kind are very often rendered when it is impossible for the judge to describe explicitly the evidence upon which his judgment is based. The lack of logical truth is of course not general, but can be amply illustrated by cases of mistakes in judgment on the part of anyone. We say therefore, that engineering decisions should, as far as possible, be made independent of judgments. This perhaps is the basic meaning of the rationalization of engineering decisions.

We now face the problem of saying what we mean by a judgment, a problem which we will have only limited success in solving. If we ask, for example, what the difference is between judgments based on data explicitly collected for the purpose at hand and judgments based on cumulated but not explicit experience, intuition,

guesses, or feelings, we cannot give a clear answer. There are certainly elements of judgment in every kind of decision situation that is of interest. We will try to make clear what we mean by judgments by the three following classes of behavior.

We will say that behavior is objective if the accepted methods of inference are used in evaluating the elements of a decision situation. That is, to be objective one must obtain data and draw inferences from it according to the rules given under the general heading of the design of experiments.

We say that behavior is subjective when we disregard these rules. That is, when we do not follow the guides of experimental inference. We are subjective when we evaluate elements of a decision on the basis of our accumulated experience or on the basis of data which we are unable to make explicit.

We distinguish a third class of behavior which involves both the use of subjective behavior and objective behavior in the decision process. We think that there is actually a continuum of behaviors ranging from completely subjective to as objective as it is possible to be. We can never be "completely objective". We will

take judgments as being the process of determining the elements of a decision through subjective behavior.

In terms of the things that the engineering decision maker is called upon to make judgments about, three fields are especially important;

1. value judgements, or judgements as to the degree to which an outcome moves the system toward attainment of its objectives,
2. probability judgments, or judgments as to the probability of an event. These may be interpreted as estimates of relative frequency or expressions of degree of belief, as the situation demands.
3. ethical judgments, where we restrict this term to judgments which lead to the selection of the overall goals of the organization. We have rather arbitrarily ruled these judgments out of our consideration of the engineering decision problem.

From the method of rationalization which has been adopted here as the fundamental approach to engineering decisions, the following conclusions may be drawn about judgments.

1. In most engineering decision problems the system restrictions will make it impossible to avoid the use of judgments. It may be economic to base decisions

at least in part upon judgments, or systems requirements may prevent the obtaining of information to which more logical truth may be ascribed.

2. The process of making judgments explicit and incorporating them in the decision process helps the decision maker to achieve rationality by meeting the requirements of explicitness and logical consistency. The engineer is, however, much less ready to ascribe to judgments the characteristic of logical truth.

3. In the absence of evidence to which greater weight is attached, the decision maker should clearly use his judgments in an explicit fashion in order to achieve at least consistency. It is implicit in our view of rationality that relative frequency probabilities are consistent with logical truth, whereas personalistic probabilities are not necessarily consistent, for example. If personalistic probabilities are the only evidence available they should clearly be employed.

Below are discussed two techniques which are well known in the literature, but have not previously been discussed with care. These techniques appear to have possibilities for increasing the effectiveness of judgments with respect to the characteristic of logical truth. Engineering judgments are directed at the production of data

which are in theory at least obtainable by other means. Thus we think of these judgments as being possible of validation.

We are not concerned with the characteristic of the persons who make judgments, or with the relationship between the experiences of these persons and the degree of logical truth manifested by their judgments. We restrict this discussion to engineering techniques which are generally applicable and which tend to lessen the tendency to depend on particular persons for judgments, as well as to improve the validity of their judgments.

Method of Reduction of Required Discrimination

We are concerned here with the judgment of numerical magnitudes such as values, probabilities, and the like. These kinds of judgments are usually called by engineers estimates or "guesstimates". What we shall say is based upon the assumption that the behavior of persons making these kinds of judgments is similar to the behavior of persons making judgments in a psychophysical laboratory experiment. In general this means that it is easier to discriminate between widely different magnitudes than those which are nearly equal, and that the result of independently repeated judgments of a magnitude may be

reasonably viewed as a random variable.

Our first method for improving the judgments entering engineering decisions consists simply of taking steps where possible to reduce the fineness of discrimination required of the judge, thus improving the probability of a correct judgment of the magnitude. More exactly, we hope to improve the results of the decisions based on the discrimination. The principle upon which this method is based is the following, "When the magnitude of a variable entering the decision process must be judged, find the value of the variable for which the choice shifts from one alternative to another. The judgment required is then only whether or not the magnitude of the variable is greater or less than this critical value."

Consider for example, a decision of the deterministic type where the alternatives, the states of the world, and the outcomes are known. Suppose that the $u [a_i(S_j)]$ are known for all $i \neq k$ and $u [a_k(S)]$ is unknown. If it is required that a judgment be made as to the value of the $u [a_k(S)]$ then the judgment, as a basis for action, can be reduced to a choice of one among the two following propositions;

$$1. \quad u [a_k(S)] \geq \max_{i \neq k} u [a_i(S)]$$

$$2. \quad u [a_k(S)] < \max_{i \neq k} u [a_i(S)]$$

The judgment required has thus been reduced from the selection of one of an infinite number of possible values for the outcome, to one of a choice between the two statements above. It would certainly seem as though the reduced choice is more likely to be made with confidence by the decision maker. Once one of the statements has been selected, then the course of action is clear. It should be noted that the only merit claimed for this method is the following: in decision situations where it appears impossible to avoid the admission of judgment, this reduced type of judgment is likely to be easier to make or easier to obtain from others.

A situation which is more taxing on the judgment of the decision maker is presented by the case where the $u [a_i(S)]$ are unknown for more than one i . The best approach here might be to apply the method of paired comparisons to the unknown alternatives to generate a ranking of the values of their outcomes. One would then return to the case discussed above and compare the highest ranked unknown value to the maximum of the

known values.

Consider the following decision situation:

	S_1	S_2
a_1	u_{11}	u_{12}
a_2	u_{21}	u_{22}

Suppose the decision maker wishes to judge the probabilities of the S_j , that is $f(S_j)$. Now if $Eu(a_1) =$

$Eu(a_2)$ the decision maker is indifferent toward the two alternatives.

$$Eu(a_1) = f(S_1) u_{11} + \{1 - f(S_1)\} u_{12}$$

$$Eu(a_2) = f(S_1) u_{21} + \{1 - f(S_1)\} u_{22}$$

The value of $f(S_1)$ at which indifference prevails is

$$\hat{f}(S_1) = \frac{u_{22} - u_{12}}{u_{11} - u_{12} - u_{21} + u_{22}}$$

The decision maker may then judge whether $f(S_j)$ is greater than or less than $\hat{f}(S_j)$. Having made this judgment and knowing the values of the u_{ij} the decision is determined.

If the decision situation includes two alternatives and three states of the world, we may write the following equations for indifference:

$$f(S_1)u_{11} + f(S_2)u_{12} + \{1-f(S_1)-f(S_2)\} u_{13} =$$

$$f(S_1)u_{21} + f(S_2)u_{22} + \{1-f(S_1)-f(S_2)\} u_{23}$$

which reduces to

$$f(S_1) \{u_{11}-u_{13}-u_{21}+u_{23}\} + f(S_2) \{u_{12}-u_{13}-u_{22}+u_{23}\} =$$

$$u_{23}-u_{13}$$

The value of $f(S_1)$ and $f(S_2)$ which satisfy this relation lie on a straight line which divides the unit square in two regions. In one of these regions a_1 is preferred and in the other a_2 is preferred. The decision is required only to judge in which region the pair of probabilities lies, and the decision is determined.

If we extend this type of analysis to a decision involving three alternatives and say two states of the world, we can write three equations expressing the indifference conditions:

$$f(S_1)u_{11} + \{1-f(S_1)\} u_{12} = f(S_1)u_{21} + \{1-f(S_1)\} u_{22}$$

$$f(S_1)u_{11} + \{1-f(S_1)\} u_{12} = f(S_1)u_{31} + \{1-f(S_1)\} u_{32}$$

$$f(S_1)u_{21} + \{1-f(S_1)\}u_{22} = f(S_1)u_{31} + \{1-f(S_1)\}u_{32}$$

These equations yield three values of $f(S_1)$:

$$f(\hat{S}_1) = \frac{u_{22} - u_{12}}{u_{11} - u_{12} - u_{21} + u_{22}}$$

$$f(\hat{\hat{S}}_1) = \frac{u_{32} - u_{12}}{u_{11} - u_{12} - u_{31} + u_{32}}$$

$$f(\hat{\hat{\hat{S}}}_1) = \frac{u_{32} - u_{22}}{u_{21} - u_{22} - u_{31} + u_{32}}$$

These three critical values of $f(S_1)$ divide the interval 0 to 1 into four sub-intervals. In each of these sub-intervals the preference relations are determined, thus the decision maker must judge only in which interval the probability lies.

In general for m states of the world the decision maker must deal with regions in $m-1$ space. If there are n alternative courses of action in the decision, there will be $\frac{n!}{(n-2)!2!}$ hyperplanes which divide $m-1$ space

into regions in which the decision is determined.

This analysis was suggested by the work of Blackwell and Girshick (3) and applications of the techniques to engineering problems have been made by Gitzendanner (21).

Important reductions of this type for sequential observation programs are given in (3).

Estimating for Functions of More than One Variable

The topic of the combination of errors in Engineering estimates was apparently first raised in the literature by Fish (16). Unfortunately his presentation was incomplete and his conclusions somewhat misleading. Fish states that whenever estimates must be combined as the arguments of any function whatsoever, the individual estimates should be made and then combined as dictated by the function. As is indicated below this method is best only in certain cases. Unfortunately Fish's conclusion has been reinforced more or less uncritically throughout the engineering economy literature since its appearance. He stated neither the assumptions upon which his analysis was based, nor the fact that his results were only approximations. His work was based on the treatment by Merriman (40).

The advice that is often given for estimating a quantity that is a function of several other variables is to estimate these other variables individually and then combine them by substitution in the function. Thus, for example, one is urged in estimating a total cost, to

estimate first the unit cost, then the quantity, and then perform the multiplication. We will now show explicitly the conditions under which this method is best, and the conditions under which it is definitely poor. We assume again that the results of independent judgments about a magnitude are random variables. We further assume that these variables may or may not have a mean equal to the true value of the variable, that is the judge may have a constant bias.

Consider $B = f(b_1, b_2, \dots, b_n)$

The problem is to estimate B by one of the two following methods:

1. estimate B directly
2. estimate each b_i and then compute B.

Let $\hat{B}$ = an estimate of B

$\hat{b}_i$ = an estimate of b_i

$\hat{b}_i$ are independent in the probability sense

We will select sets out of the following list of assumptions to define problems:

$$1. E(\hat{B}) = B$$

$$2. E(\hat{b}_i) = b_i$$

$$3. \sigma_{\hat{B}} = K$$

$$3a. \sigma_{\hat{B}} = 1B$$

$$4. \sigma_{\hat{b}_i} = k$$

$$4a. \sigma_{\hat{b}_i} = 1b_i$$

Case I

Assume 1, 2, 3, 4

$$B = b_1 \pm b_2 \pm \dots \pm b_n$$

Method 1: $\hat{\sigma}_B = K$

$$\begin{aligned} \text{Method 2: } \hat{\sigma}_B &= \sqrt{\hat{\sigma}_{b_1}^2 + \hat{\sigma}_{b_2}^2 + \dots + \hat{\sigma}_{b_n}^2} \\ &= k \sqrt{n} \end{aligned}$$

Thus method 2 is better only if $k \sqrt{n} < K$

If $k = K$ then method 1 is preferred.

Case II

Assume 1, 2, 3, 4

$$B = b_1 b_2$$

Method 1: $\hat{\sigma}_B = K$

$$\begin{aligned} \text{Method 2: } \hat{\sigma}_B &= \left\{ [E(b_1)]^2 \hat{\sigma}_{b_2}^2 + [E(b_2)]^2 \hat{\sigma}_{b_1}^2 + \right. \\ &\quad \left. \hat{\sigma}_{b_1}^2 \hat{\sigma}_{b_2}^2 \right\}^{\frac{1}{2}} \\ &= \left\{ b_1^2 \hat{\sigma}_{b_2}^2 + b_2^2 \hat{\sigma}_{b_1}^2 + \hat{\sigma}_{b_1}^2 \hat{\sigma}_{b_2}^2 \right\}^{\frac{1}{2}} \\ &= \left\{ b_1^2 k^2 + b_2^2 k^2 + k^4 \right\}^{\frac{1}{2}} \end{aligned}$$

Thus method 2 is better only if

$$\left\{ b_1^2 k^2 + b_2^2 k^2 + k^4 \right\}^{\frac{1}{2}} < K$$

If $K=k$ method 1 is preferred.

Case III

Assume 1, 2, 3a, 4a

$$B = \sum_i + b_i \quad \text{or} \quad \sum_i - b_i$$

Method 1:

$$\sigma_B = 1B \qquad \sigma_B^2 = 1^2 B^2$$

Method 2:

$$\sigma_B = 1\sqrt{\sum b_i^2} \qquad \sigma_B^2 = 1^2 \sum b_i^2$$

But $B^2 = [\sum b_i]^2$

$$[\sum b_i]^2 - \sum b_i^2 = \sum_i \sum_j b_i b_j \qquad i \neq j$$

Thus method 2 is always preferred.

If B is a sum of both positive and negative quantities then either method 1 or method 2 may be preferred, depending on whether the sum of the cross products is negative or positive.

Case IV

Assume 1, 2, 3a, 4a

$$B = b_1 b_2$$

Method 1: $\sigma_B = 1B$

$$\begin{aligned} \text{Method 2: } B &= 1 \left\{ b_1^2 b_2^2 + b_2^2 b_1^2 + 1^2 b_1^2 b_2^2 \right\}^{\frac{1}{2}} \\ &= 1 \left\{ (1^2 + 2)(b_1^2 b_2^2) \right\}^{\frac{1}{2}} \end{aligned}$$

$$1B - 1b_1 b_2 \left\{ 1^2 + 2 \right\}^{\frac{1}{2}} = 1 - \sqrt{1^2 + 2}$$

But $1 - \sqrt{1^2 + 2} < 0$ for all 1

Thus method 1 is always preferred.

This is a particularly important case because in the estimation of expected values one deals with functions of the form $\sum_i u(x_i)f(x_i)$. If the set of assumptions

of Case IV do hold then the terms $u(x_i)f(x_i)$ should be estimated as units, rather than by the estimation of utility and probability separately. The difficulty of course is that there is no intuitive interpretation which allows one to judge or estimate $u(x_i)f(x_i)$.

An approximate relationship for the general case may be derived as follows:

Let $B = f(b)$ where f is any function

B is a random variable $= \bar{B} + x$

b is a random variable $= \bar{b} + x$

$\bar{B} + x = f(\bar{b} + x)$ which may be expanded into

$$f(\bar{b}+x) = f(\bar{b}) + f^1(\bar{b})x + \frac{f^{11}(\bar{b})x^2}{2} + \frac{f^{111}(\bar{b})x^3}{3!} + \dots$$

Then since $\bar{B} = f(\bar{b})$;

$$x = f^1(\bar{b})x + \frac{f^{11}(\bar{b})x^2}{2} + \frac{f^{111}(\bar{b})x^3}{3!} + \dots$$

$$x^2 = (f^1(\bar{b}))^2 x^2 + \frac{(f^{11}(\bar{b}))^2 x^4}{4} + f^1(\bar{b})f^{11}(\bar{b})x^3 + \dots$$

$$E(x^2) = \sigma_B^2 = f^1(\bar{b})^2 \sigma_b^2 + E(x^4) \frac{(f^{11}(\bar{b}))^2}{4} + E(x^3)f^1(\bar{b})f^{11}(\bar{b})$$

But $E(x^3) = 0$, etc.

If $f(b)$ is approximately linear we may neglect terms containing second and higher derivatives.

$$\text{Thus: } \sigma_B^2 = \{f^1(\bar{b})\}^2 \sigma_b^2$$

If $B = f(b_1, b_2, \dots, b_n)$ by similar methods

$$\sigma_B^2 = \{f^1(\bar{b}_1)\}^2 \sigma_{b_1}^2 + \{f^1(\bar{b}_2)\}^2 \sigma_{b_2}^2 + \dots + \{f^1(\bar{b}_n)\}^2 \sigma_{b_n}^2$$

Chapter 7.

THE TIMING OF DECISIONS

One of the important and essentially unsolved problems in engineering decision making is the problem of the timing of decisions. We are concerned here not with the timing of action but with the more immediate and fundamental problem of the timing of the decision process itself.

The simplest timing problem is that of the order in which a series of decisions are made. It may well be that a criterion measure may be developed which is a function of the ordering of the decisions, and that possible orderings may be examined for those which maximize the criterion measure. Interesting problems arise here in connection with the technological restrictions placed on the orderings and in connection with the nature of the interdependence among decisions in the sequence. For example, one might find that once a given decision is made, some following decision situations are dissolved because a choice no longer exists, or else the choice on a decision may form a part of the information upon which later decisions are based.

One can also ask the question, "Why not make and remake a decision continuously until we have taken a course of action which dissolves the decision situation?" If maintaining the status quo by inaction is considered decision making, this is in fact what one does much of the time. However, one usually does this without an explicit examination of other courses of action. The real question is when one should undertake the expense of examining the alternatives.

The problems of the timing of action also have many aspects to be investigated. We may express these problems in terms of our model of the decision process by including in the matrix as many courses of action of the form "take action A at time T" as are necessary to express the freedom open to the decision maker. If it could be known how the payoff changes for different values of T, then possibly an optimal decision point could be chosen.

Consider three important modes of decision making which can occur, in the following simplified situation. We consider an interval in time within which one may make the decision and within which some other event may occur which changes the payoff matrix in some way. In this

situation one might:

- a. make a prediction about the event and then select an appropriate course of action well in advance based this prediction,
- b. make no prediction about the event at all, wait until after the event has actually taken place and then select a course of action based on the knowledge of the outcome of the event,
- c. base the decision partly upon a prediction about the outcome of an event and partly upon information obtained by being closer to the event or by taking time to make relevant observations.

We might call these modes of behavior pure programming behavior, pure control behavior, and combination programming and control behavior respectively. We proceed now to formalize and investigate the characteristics of these types of behavior. The first of these modes is essentially that upon which Engineering Economic analyses are based, the second is that which is formalized in the theory of control, but the third mode has not been explicitly studied.

We begin by discussing a simple timing problem to illustrate its structure. Consider an interval of time during which the state of the world may change from

the present state to one of a number of other states. Neither the new state nor the moment at which the change takes place are within the control of the decision. At some moment within this interval the decision maker must select one of a number of courses of action. The outcome from the action selected depends both on the new state of the world and on the relationship between the time of the decision maker's choice and the time of the change of state.

Let $S_j = \{S_1, S_2\}$ = states to which the world may change

$t_N = \{t_N^1, t_N^2\}$ = times at which the change may take place

$a_1 = \{a_1, a_2\}$ = alternatives open to the decision maker

$t_d = \{t_d^1, t_d^2, t_d^3\}$ = times at which the choice may be made

$Z(a_1, S_j, (t_N - t_d))$ = return from choice of a_1 at t_d
when nature changes to S_j at t_N

We may thus construct a decision matrix with the events

$S_j t_N^k = \{S_j x t_N^k\}$ on one side and the events

$a_1 t_d^1 = \{a_1 x t_d^1\}$ along the other. If the values of

$Z(a_i, S_j, (t_N - t_d))$ are known and the a priori probabilities of the $\{S_j x t_N^k\}$ the decision maker may decide in advance which $\{a_i x t_d^1\}$ will be best. If as time passes neither the decision maker nor nature takes any action, then the probabilities of the $\{S_j x t_N^k\}$ will change and the decision may be remade.

A more interesting problem in the timing of choice is one in which one of the reasons for postponing choice is to gather information or make observations which will improve the return to the decision maker. Consider the situation in which the return to the decision maker is based on the following considerations:

1. Observations may be made by the decision maker which will yield information as to what changes of state will take place and the timing of the changes. These observations are made at a cost to the decision maker, both in resources and in time. With respect to time, the longer the decision maker postpones his choice the more observations he can make, and the better will be his choice because of the additional knowledge.
2. Early decisions are of value to the system because they allow preparations to be made which increase the

effectiveness of the action taken.

3. If a change of state takes place before the decision maker has chosen, he has the advantage of knowing what the new state of the system is, but incurs a cost for being tardy with his choice.

Consider first the case in which the decision maker makes observations which yield information as to what the next state of the system will be. These observations themselves do not provide any information as to when the change of states will take place.

Define:

R = a time interval just sufficient to carry out a single observation; includes collecting the information, making inferences, and necessary choices

c_1 = cost of an observation, assumed to be constant

t_n = time at which change of state takes place
($0 \leq t_n \leq T$)

t_d = time at which the decision maker makes his choice
($0 \leq t_d \leq T$)

$Eu(a_i, S_j, t_n - t_d)$ = expected return to the decision maker from taking action a_i when nature changes to state S_j , the action being selected $t_n - t_d$ time units in advance of the change of state
(for $t_n - t_d > 0$)

$f(S_j)$ = a priori probability of S_j

$f_k(S_j)$ = a posteriori probability of S_j , given k observations

$$Eu(a_1, S_j, t_n - t_d) = Eu_1(a_1, S_j) \quad \text{for } t_n - t_d \leq 0$$

(x) = a particular sequence of N observations
 $(x_1, x_2, \dots, x_N)$

(k) = the first k observations in the sequence (x)

$p(k | S_j)$ = probability of the k observations given
that state S_j prevails

$\phi(t_n)$ = the a priori probability that the change of
state will take place at time t_n

It is assumed that

1. all of the above information can be known at the appropriate point in the observation program,
2. that the observations made by the decision maker are random variables whose probabilities depend on which state of nature prevails,
3. an observation only partially completed has neither cost nor value to the system
4. the observation program must be truncated at N observations. ($N \leq T$)

Suppose that the decision maker begins observations at time $t = 0$ and that by time $t = t$ he has completed k observations. The expected return from stopping the observation program at this point and selecting a_1 would be

$$\frac{\int_t^T Eu(a_i, S_j, t_n - t_d) \phi(t_n) dt_n}{\int_t^T \phi(t_n) dt_n}$$

if the system were to change to state S_j . After taking these k observations the decision maker would compute the probability of the states on the basis of the information yielded by these observations.

$$f_k(S_j) = \frac{p(k|S_j) f_0(S_j)}{\sum_j p(k|S_j) f(S_j)}$$

The cost of these k observations is kc_1 . Thus the expected return from stopping the observation at this point and choosing so as to maximize the expected utility would be

$$\max_i \frac{\sum_j f_k(S_j) \int_t^T Eu(a_i, S_j, t_k - t_d) \phi(t_n) dt_n}{\int_t^T \phi(t_n) dt_n} - kc_1$$

If instead of stopping and choosing at this point the decision maker goes on with the observation program, he may be able to make as many as $N - k$ additional observations. As soon as the change of state takes place, however, further observation is useless, since the new state is assumed to be known then. As soon as the change

takes place the return is given by the new utility function $Eu_L(a_1, S_j)$ which we have taken to be generally less than the corresponding return $Eu(a_1, S_j, t_n - t_d)$ for positive values of $t_n - t_d$. "There is a penalty associated with making late decisions". Thus in considering the prospect of continuing the observation program, the decision maker must reckon with the information to be gained from $k_1, k_2, \dots, N$ observations together with the anticipation that the change of state may take place at any time during the interval from time t to time T .

Consider a particular sequence of N observations, (x) . If the decision maker starts at $t = 0$ and makes all N of these observations, the expected return will be

$$R_N(x) = \max_i \frac{\sum_j \int_{NR}^T Eu(a_1, S_j, t_n - NR) f_N(S_j) \phi(t_n) dt_n}{\int_{NR}^T \phi(t_n) dt_n} - C_1 N$$

Now let $\alpha_N(x) = R_N(x)$

$E_{N-1}(\alpha_N(x))$ = conditional expectation of $\alpha_N(x)$ given a particular set of $N-1$ observations.

After taking $N-1$ observations the decision maker may stop and select a course of action or he may continue observation. Thus we define

$$\alpha_{N-1}(x) = \max \left[R_{N-1}(x); E_{N-1}(\alpha_N(x) \frac{\int_{NR}^T \phi(t_n) dt_n}{\int_{(N-1)R}^T \phi(t_n) dt_n} + \frac{\int_{(N-1)R}^{NR} \phi(t_n) dt_n}{\int_{(N-1)R}^T \phi(t_n) dt_n} \sum_j f_{N-1}(S_j) Eu_L(a_i^*, S_j) \right]$$

In this expression

$$\frac{\int_{NR}^T \phi(t_n) dt_n}{\int_{(N-1)R}^T \phi(t_n) dt_n} \quad \text{is the condi-}$$

tional probability that the change of state will take place after the Nth observation has been completed, and

$$\frac{\int_{(N-1)R}^{NR} \phi(t_n) dt_n}{\int_{(N-1)R}^T \phi(t_n) dt_n} \quad \text{is the probability that the}$$

change of state will take place during the time the Nth observation is being made.

Note that after the change of state occurs the new state is known and the best alternative for the new state may then be selected. We let a_i^* be the best alternative for a known state S_j . Then $Eu_L(a_i^*, S_j)$ is the expected

return from selecting a_i^* .

The return from stopping after a particular set of k observations is given by:

$$R_k(x) = \max_i \frac{\sum_j f_k(S_j) \int_{kR}^T Eu(a_i, S_j, t_n - kR) \phi(t_n) dt_n}{\int_{kR}^T \phi(t_n) dt_n} - C_1 k$$

In the same manner as $\alpha_N(x)$ and $\alpha_{N-1}(x)$, we define

$$\begin{aligned} \alpha_{N-2}(x) = \max \left[R_{N-2}(x); \frac{\int_{(N-2)R}^T \phi(t_n) dt_n}{\int_{(N-2)R}^T \phi(t_n) dt_n} E_{N-2}(\alpha_{N-1}(x)) \right. \\ \left. + \frac{\int_{(N-2)R}^{(N-1)R} \phi(t_n) dt_n}{\int_{(N-2)R}^T \phi(t_n) dt_n} \sum_j f_{N-2}(S_j) Eu_L(a_i^*, S_j) \right] \end{aligned}$$

and similarly for $\alpha_{N-3}(x)$, $\alpha_{N-4}(x)$, ..., $\alpha_1(x)$,

$$\begin{aligned} \alpha_0(x) = \alpha_0 = \max \left[R_0; \frac{\int_0^R \phi(t_n) dt_n}{1} E(\alpha_1(x)) \right. \\ \left. + \int_0^R \phi(t_n) dt_n \sum_j f(S_j) Eu_L(a_i^*, S_j) \right] \end{aligned}$$

where

$$R_0 = \max_i \sum_j f(S_j) \int_0^T Eu(a_i, S_j, t_n) \phi(t_n) dt_n$$

The sequential observation program is then defined in the following way: There is a set of sequences of observations for which the first j observations fulfill the following requirements:

$$R_i(x) < \alpha_i(x) \text{ for } i < j$$

$$R_j(x) = \alpha_j(x) \text{ for } j$$

For any sequence in this set the decision maker stops after the first j observations. A "stopping region" is thus the set

$$X_j^* = \{x \mid R_i(x) < \alpha_i(x) \text{ for } i < j, R_j(x) = \alpha_j(x)\}$$

The program is specified by giving all such stopping regions:

$$X^* = \{X_0^*, X_1^*, \dots, X_N^*\}$$

The expected return from the use of such a program is

$$\begin{aligned} Eu(X^*) = & \sum_{k=0}^N \sum_{x \in X_k^*} \max_i \sum_j f_k(S_j) \frac{\int_{kR}^T Eu(a_i, S_j, t_n - kR) \phi(t_n) dt_n}{\int_{kR}^T \phi(t_n) dt_n} \\ & \times \sum_j f(S_j) p(x | S_j) \end{aligned}$$

Note that every observation on S_j successfully completed also furnishes information on $\phi(t_n)$. This is expressed by conditional probabilities of the change of state which change as time passes.

Observations on t_n

Instead of making observations which yield information about S_j , the decision maker could make observations about t_n .

Let: C_2 = cost of an observation, assumed constant

$\phi_k(t_n)$ = a posteriori probability of t_n , given a particular set of k observations

As before:

$$R_N(x) = \max_i \sum_j f(S_j) \int_{NR}^T Eu(a_i, S_j, t_n - NR) \phi_N(t_n) dt_n - C_2 N$$

$$R_k(x) = \max_i \sum_j f(S_j) \int_{kR}^T Eu(a_i, S_j, t_n - kR) \phi_k(t_n) dt_n - C_2 k$$

Then:

$$\alpha_N(x) = R_N(x)$$

$$\alpha_{N-1}(x) = \max \left[R_{N-1}(x); \int_{NR}^T E_{N-1}(\alpha_N(x) \phi_{N-1}(t_n) dt_n + \int_{(N-1)R}^{NR} \phi_{N-1}(t_n) dt_n \sum_j f(S_j) Eu_L(a_i^*, S_j) \right]$$

$$\alpha_{N-2}(x) = \max \left[\int_{(N-1)R}^T \phi_{N-2}(t_n) dt_n E_{N-2}(\alpha_{N-1}(x)) + \right. \\ \left. \int_{(N-2)R}^{(N-1)R} \phi_{N-2}(t_n) dt_n \sum_j f(S_j) Eu_L(a_i^*, S_j) \right]$$

and so forth to

$$\alpha_0(x) = \alpha_0 = \max \left[R_0; \int_R^T \phi(t_n) dt_n E(\alpha_1(x)) + \right. \\ \left. \int_0^R \phi(t_n) dt_n \sum_j f(S_j) Eu_L(a_i^*, S_j) \right]$$

Note that $\sum_j f(S_j) Eu_L(a_i^*, S_j)$ is a constant which may be computed in advance.

The sequential observation program is defined by stopping regions

$$X_j^* = \{ x \mid R_1(x) < \alpha_1(x) \text{ for } i < j, R_j(x) = \alpha_j(x) \}$$

Which then together specify the program:

$$X^* = \{ X_0^*, X_1^*, \dots, X_N^* \}$$

If the decision maker can make no observations at all, then the best policy is to select an alternative immediately. (under the assumption that

$$Eu_L(a_i, S_j) \leq Eu(a_i, S_j, t_n - t_d)$$

$$(\text{for all } i, j, t_n - t_d > 0)$$

This can be shown by examining the expression

$$Eu(t_d) = \max_i \sum_j f(S_j) \int_{t_d}^T Eu(a_i, S_j, t_n - t_d) \phi(t_n) dt_n +$$

$$\sum_j f(S_j) Eu_L(a_i^*, S_j) \int_0^{t_d} \phi(t_n) dt_n$$

Under the above assumption, $Eu(t_d)$ will be maximized with respect to t_d , when $t_d = 0$.

Appendix

We investigate here some special decision making methods which have been widely advocated in the financial world and have interesting implications for engineering decision making. These problems represent a special sort of a transformation from a type 7 decision problem to a type 2. Our purpose in examining these problems is to illustrate that some rather surprising results are obtained when one applies the method of rationalization to decision making methods which have been none too carefully stated. For example, it is commonly held that the method of dollar cost averaging is the best method for maximizing the probability that an investment portfolio will be "in the black". If one actually wishes to maximize this probability without placing any other restrictions on the method, one obtains a nonsense result as we shall see below. We make some assumptions which immediately divorce our analysis from the actual problem of the financial world, but the idealized model is of some interest of itself.

We consider a commodity which the decision maker may purchase at various times. These opportunities for purchase are numbered $t_0, t_1, \dots, t_N$. The unit

price, x , of the commodity at any t_1 is a random variable whose density function is given by $f(x)$. The decision maker may either buy or sell any amount of the commodity at the unit price prevailing at the particular opportunity in question.

Problem I: It is desired to buy a total of K units of the commodity within N opportunities. The rule often advocated is to buy only when the price is less than or equal to some constant, say a . If $x \leq a$, then D dollars are spent. At the end of the sequence of opportunities, on t_N , enough units are bought (or sold) to meet the requirement of K units purchased. On any opportunity the probability that a purchase will be made is

$\int_0^a f(x)dx$. The expected number of units obtained on each opportunity is given by $\int_0^a \frac{D}{x} f(x)dx$. The expected cost of obtaining K units under this plan may be computed as follows:

$$E(c) = D(N-1) \int_0^a f(x)dx + \left[K - (N-1) \int_0^a \frac{D}{x} f(x)dx \right] E(x)$$

Differentiating and setting the result equal to zero,

$$\frac{dE(c)}{da} = D(N-1) f(a) + \left[-(N-1) \frac{D}{a} f(a) \right] E(x) = 0$$

$$E(x) = a$$

Thus a should be taken equal to $E(x)$ and this result is independent from the form of $f(x)$, the value of D , and the value of N .

As an example, consider the case where $f(x) = \frac{1}{u-x}$

$$E(x) = \frac{u+1}{2} = a$$

Then

$$\begin{aligned} E(c) &= KE(x) - \frac{(N-1)DE(x)}{2} \\ &= \left\{ \frac{u+1}{2} \right\} \left\{ K - \frac{(N-1)D}{2} \right\} \end{aligned}$$

Thus the expected cost decreases with increasing N and with increasing D . A negative cost indicates a profit made by over-buying and reselling.

Problem 2: We now relax the requirement that K units must be purchased within N opportunities. Instead, the decision maker may buy on the first N opportunities and then sell. He seeks to maximize the expected value of his capital gains. He uses the following rule in deciding whether or not to make a purchase; "Spend D dollars whenever the unit price is less than or equal to a constant, a ." Let $E(G)$ be the expected value of the capital gains.

$$E(G) = -ND \int_0^a f(x) dx + N \int_0^a \frac{D}{x} f(x) dx E(x)$$

$$\frac{\partial E(G)}{\partial a} = NDE(x) \frac{f(a)}{a} - ND f(a)$$

$$E(x) = a$$

Again for the case where $f(x) = \frac{1}{u - x}$

$$\begin{aligned} E(G) &= NDE(x) \left\{ \frac{1}{2u} + \frac{1}{2E(x)} \right\} - \frac{ND}{2} \\ &= \frac{NDE(x)}{2u} \end{aligned}$$

Problem 3: We study now the rule which is a highly simplified form of formula planning. At each opportunity

if $x \leq a$ buy α units

if $x \geq b$ sell β units

α, β may be functions of x

It is required that

$$\alpha \int_0^a f(x) dx = \beta \int_b^u f(x)$$

So that the system tends neither to a "long" nor a "short" position.

The expected value of the capital gains (per opportunity) is given by:

$$E(G) = \int_b^u \beta x f(x) dx - \int_l^a \alpha x f(x) dx$$

Taking the partial derivatives of this function and setting them equal to zero yields only the values of a and b which minimize the function. However, the following simple example illustrates the type of solution obtained:

$$\text{Let } f(x) = \frac{1}{u-x} \quad u = 5 \quad l = 1$$

Note that for simplicity in the rule for buying or selling we require that $a \leq b$

Then:

α	β	b	a	$E(G)$
10	10	5	1	0
10	10	4.5	1.5	3.1
10	10	4	2	7.5
10	10	3	3	10

Thus the maximum occurs at $a=b=3$.

This indicates the following interesting conclusions:

1. The ordinary schemes for formula planning where $a \leq b - c$ are not optimal in this case.
2. Either a purchase or a sale is made on every opportunity.

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I, William Thomas Morris, was born in Concord, New Hampshire, November 12, 1928. I received my secondary school education at Saint Paul's School in Concord, New Hampshire. My undergraduate study was done at the Massachusetts Institute of Technology where I received the degree Bachelor of Science in Business and Engineering Administration in 1950. From the Ohio State University I received the Master of Science degree in Industrial Engineering in 1953, the greater part of the work for this degree being done while on active duty with the Air Force. In 1954 I became first part time, and later full time Instructor in the Department of Industrial Engineering at Ohio State University. This position, together with that of Research Engineer, part time, I held while completing the requirements for the degree Doctor of Philosophy.