SECOND MOMENT OF THE CENTRAL VALUES OF THE SYMMETRIC SQUARE L-FUNCTIONS

DISSERTATION

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By

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ABSTRACT

In this thesis we obtain the sharp upper bound for the moments of L -functions of conductor $K^{2+\epsilon}$ in short interval at the center of the critical strip. This includes the case of the second moment of symmetric square L -function associated to holomorphic cusp forms of weight k , over the interval $K \leq k \leq K + K^{\frac{1}{3}}$, and the first moment of Rankin-Selberg L -function of symmetric square lift of holomorphic cusp forms of weight k and a fixed Maass cusp form over the interval $K \leq k \leq K + K^{\frac{1}{3}}$.

To my mother

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TABLE OF CONTENTS

Abstract	ii
Dedication	ii
Acknowledgments	iv
Vita	v

CHAPTER		PAGE
1	Introduction	1
	1.1 Definitions and Notations	1
	1.1.1 Holomorphic cusp forms	1
	1.1.2 Maass forms and Eisenstein series	4
	1.1.3 Hecke Theory	6
	1.1.4 Automorphic L -functions	7
	1.2 Main Results	10
	1.2.1 Local large sieve inequalities for $GL(2)$ cusp forms	10
	1.2.2 Mean value of symmetric square L -functions	13
	1.2.3 First moment of Rankin-Selberg L -functions.	13
2	Large sieve inequalities	15
	2.1 Holomorphic cusp forms of integral weight k	15
	2.1.1 First manipulations	16
	2.1.2 The diagonal contribution	17
	2.1.3 The off-diagonal contribution	17
	2.1.4 Proof of Lemma 2.1.2	19
	2.2 Holomorphic cusp forms of half integral weight	21
	2.2.1 First manipulation	21
	2.2.2 The diagonal contribution	22
	2.2.3 The off-diagonal contribution	22
	2.3 Maass cusp forms	34
	2.3.1 First Manipulation	35
	2.3.2 The diagonal contribution	36

	2.3.3 The off-diagonal contribution	36
3	The Second moment of the central values of the symmetric square L -functions	40
	3.1 Shortening the length of summation	41
	3.2 Further manipulation	45
	3.3 The off-diagonal contribution	47
	3.3.1 Restricting the sum over c	47
	3.3.2 Applying the stationary phase lemma	49
	3.3.3 Applying the Poisson summation formula	51
	3.3.4 Application of the two dimensional Van der Corput lemma	53
4	The first moment of the central values of the Rankin-Selberg L -functions	56
	4.1 First manipulation	58
	4.2 Further reduction to the sum over c	61
	4.3 Applying the Stationary Phase Lemma and the Hecke relation	64
	4.4 Evaluation of the Sums of the Kloosterman sum	68
	4.5 Applying Voronoi formula	72
	4.6 Reduction of the range of sum over n	73
	4.7 Bounding the Oscillatory integral	77
	4.8 Estimating the sum of twisted exponential sums	80
	4.9 Proof of Lemma 4.4.1	85
	4.10 Concluding the proof of the main theorem	88
	Bibliography	92

CHAPTER 1

INTRODUCTION

*Men wanted for hazardous journey. Low wages,
bitter cold, long hours of complete darkness.
Safe return doubtful. Honour and recognition
in event of success*

—Ernest Shackleton

1.1 Definitions and Notations

In this section we will introduce basic concepts and definitions in the theory of automorphic forms including results that we will use throughout this thesis.

1.1.1 Holomorphic cusp forms

Let $\mathbb{H}$ be the upper half plane, $SL_2(\mathbb{Z})$ be the modular group, i.e. group of 2×2 matrices with integral entries and determinant 1. For each integer q , one can define the Hecke congruence subgroup of level q , denoted $\Gamma_0(q)$, by

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(q) \right\}$$

and the stabilizing group of a group Γ by

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

The modular group (hence the Hecke congruence subgroups) acts on the upper half plane as follows: for each $z \in \mathbb{H}$, $g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$g \cdot z := \frac{az + b}{cz + d}.$$

The cusps of $\Gamma_0(q)$ are rational numbers and ∞ . Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be a function on the upper half plane. We define for each k

$$(f|_k g)(z) := j(g, z)^{-k} f(g \cdot z)$$

$$j(g, z) := cz + d \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A holomorphic modular form of weight k , level q , and trivial nebentypus (which we assume from now on and will not mention again in this thesis) is a holomorphic function f on $\mathbb{H}$, such that

$$f|_k \gamma = f \text{ for all } \gamma \in \Gamma_0(q)$$

and is holomorphic at all cusps of $\Gamma_0(q)$.

By holomorphic at a cusp, we mean the following: Let $\mathfrak{a}$ be a cusp of $\Gamma_0(q)$, then there exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $\gamma_{\mathfrak{a}}$ is the generator of $\Gamma_0(q)_{\mathfrak{a}}$. Let $f_{\mathfrak{a}} := f|_k \sigma_{\mathfrak{a}}$. We have the Laurent series expansion

$$f_{\mathfrak{a}}(z) = \sum_{n \geq n_{\mathfrak{a}}} \hat{f}_{\mathfrak{a}}(n) e(nz)$$

(where $e(z) = \exp(2\pi iz)$) for some integer $n_{\mathfrak{a}}$, and $\hat{f}_{\mathfrak{a}}(n_{\mathfrak{a}}) \neq 0$. We say that f is holomorphic at $\mathfrak{a}$ if it is meromorphic and $n_{\mathfrak{a}} \geq 0$. If moreover, we have $n_{\mathfrak{a}} > 0$, we say that f vanishes at $\mathfrak{a}$. If f vanishes at all cusps, f is said to be a cusp form.

In the case when k is half integral, we have to be slightly more careful, as it involves taking fractional root of a complex numbers. In this case, everything is the same as before, except we define

$$j(\gamma, z) := \theta(\gamma \cdot z) / \theta(z) \text{ for } \gamma \in \Gamma_0(4)$$

$$\text{where } \theta(z) := \sum_{-\infty}^{\infty} e(n^2 z).$$

The set of holomorphic cusp forms of weight k (integral or half-integral) and level q (in the case of half-integral weight forms, we require that $4|q$) forms a finite dimensional vector space, which is denoted by $S_k(\Gamma_0(q))$. $S_k(\Gamma_0(q))$ has an inner product called the Petersson inner product defined as follows: For $f, g \in S_k(\Gamma_0(q))$.

$$\langle f, g \rangle := \frac{1}{\text{vol}(X_0(q))} \int_{X_0(q)} y^k f(z) \overline{g(z)} \frac{dx dy}{y^2}$$

where $X_0(q) = \Gamma_0(q) \backslash \mathbb{H}$.

For each $f \in S_k(\Gamma_0(q))$, f has a Fourier expansion $f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e(nz)$. Define $\rho_f(n)$ as

$$\rho_f(n) = \sqrt{\frac{(k-1)!}{(4\pi)^{k-1}}} a_f(n).$$

Take any orthonormal basis $B_{k,q}$ of $S_k(\Gamma_0(q))$, we have the following Petersson trace formulae:

Theorem 1.1.1. *For integer $k > 1$. For any positive integers m, n ,*

$$\frac{1}{\text{vol}(X_0(q))} \sum_{f \in B_{k,q}} \rho_f(m) \overline{\rho_f(n)} = (k-1) \delta_{mn}$$

$$+ 2\pi i^{-k} (k-1) \sum_{c \equiv 0(q)} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right)$$

where

$$S(m, n; c) = \sum_{\substack{d(c) \\ (d,c)=1}} e \left(\frac{dm + \bar{d}n}{c} \right)$$

is the Kloosterman sum, $J_{k-1}(x)$ is the J -Bessel function.

Theorem 1.1.2. For half integral $k > 0$. For any m, n ,

$$\frac{1}{\text{vol}(X_0(q))} \sum_{f \in B_{k,q}} \rho_f(m) \overline{\rho_f(n)} = (k-1) \delta_{mn} \\ + 2\pi i^{-k} \sum_{c \equiv 0(M)} \frac{K_k(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right)$$

where $K_k(m, n; c)$ is the generalized Kloosterman sum,

$$K_k(m, n; c) = \sum_{d(c)} \epsilon^{-2k} \left(\frac{c}{d} \right) e \left(\frac{m\bar{d} + nd}{c} \right)$$

with $d\bar{d} \equiv 1(c)$, $\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1(4) \\ i & \text{if } d \equiv -1(4) \end{cases}$, and $\left(\frac{c}{d} \right)$ is the extended Kronecker symbol (see P.388 of [8]).

1.1.2 Maass forms and Eisenstein series

Let $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ be the hyperbolic Laplace operator. A Maass form of level q is a function $f \in L^2(\Gamma_0(q) \backslash \mathbb{H})$ such that

$$\Delta f = s(1-s)f \text{ for some } s \in \mathbb{C}$$

and

$$f|_\gamma = f \text{ for all } \gamma \in \Gamma_0(q).$$

Similar to the holomorphic, one can define cuspidal Maass forms (see P.6). Each Maass forms f has a Fourier expansion (at the cusp ∞)

$$f(z) = ay^s + by^{1-s} + \sqrt{y} \sum_{n \neq 0} a_f(n) K_{s-\frac{1}{2}}(2\pi|n|y) e(nx)$$

where $\Delta f = s(1-s)f$ and $K_s(y)$ is the K -Bessel function. Similar to holomorphic cusp forms, one can define an inner product as follows: For two Maass forms f, g of level q ,

$$\langle f, g \rangle := \frac{1}{\text{vol}(X_0(q))} \int_{X_0(q)} f(z) \overline{g(z)} \frac{dx dy}{y^2}.$$

For each cusp $\mathfrak{a}$ of $\Gamma_0(q)$, we can define the Eisenstein series

$$E_{\mathfrak{a},q}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} (\text{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z)^s$$

where $\Gamma = \Gamma_0(q)$, $\Gamma_{\infty} = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \infty = \infty\}$, and $\Gamma_{\mathfrak{a}} = \sigma_{\mathfrak{a}} \Gamma_{\infty} \sigma_{\mathfrak{a}}^{-1}$. At each cusp $\mathfrak{b}$, it has the Fourier expansion

$$E_{\mathfrak{a},q}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a},\mathfrak{b}} y^s + \phi_{\mathfrak{a},\mathfrak{b},q} y^{1-s} + \sqrt{y} \sum_{n \neq 0} \phi_{\mathfrak{a},\mathfrak{b},q}(n, s) K_{s-\frac{1}{2}}(2\pi|n|y) e(nx).$$

The analogue of the Petersson trace formula in this case is the Kuznetsov trace formula. Let (u_j) be an orthonormal basis of Maass forms for $\Gamma_0(q)$ with $q \geq 1$ and let $\lambda_j = s_j(1-s_j)$ with $s_j = \frac{1}{2} + it_j$ be the corresponding eigenvalues of the Laplace operator. Let

$$u_j = ay^{s_j} + by^{1-s_j} + \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{it_j}(2\pi|n|y) e(nx)$$

be the Fourier expansion of u_j at the cusp ∞ . Let $\mathfrak{a}$ be a cusp of $\Gamma_0(q)$, and $\tau_{\mathfrak{a},q}(n, r) = \phi_{\mathfrak{a},\infty,q}\left(n, \frac{1}{2} + ir\right)$. Then for a function $h : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\begin{cases} h(t) = h(-t), \\ \text{is holomorphic in } |\text{Im}(t)| \leq \frac{1}{2} + \delta, \\ \text{and } h(t) \ll (|t| + 1)^{-2-\delta} \end{cases}$$

for some $\delta > 0$. We have

Theorem 1.1.3. (*Kuznetsov trace formula*) for any $m, n > 0$,

$$\begin{aligned} & \sum_{j=1}^{\infty} \overline{\rho_j(m)} \rho_j(n) \frac{h(t_j)}{\cosh \pi t_j} + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \overline{\tau_{\mathfrak{a},q}(m, r)} \tau_{\mathfrak{a},q}(n, r) \frac{h(r) dr}{\cosh \pi r} \\ &= \delta_{mn} g_0 + \sum_{c \equiv 0(q)} \frac{S(m, n; c)}{c} g\left(\frac{4\pi\sqrt{mn}}{c}\right) \end{aligned}$$

where

$$g_0 = \frac{1}{\pi^2} \int_{-\infty}^{\infty} rh(r) \tanh \pi t \, dr$$

and

$$g(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir}(x) \frac{rh(r) \, dr}{\cosh \pi r}.$$

1.1.3 Hecke Theory

Fix integers $k \geq 1$, $q \geq 1$, $n \geq 1$. One defines

$$(T_n f)(z) := \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right).$$

It is well-known that for each $(n, q) = 1$, T_n sends holomorphic cusp forms of weight k and level q to holomorphic cusp forms of weight k and level q .

In the Maass form case, the Hecke operator is defined differently and will still be denoted by T_n ,

$$(T_n f)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right).$$

It also sends Maass cusp forms of level q to Maass cusp forms of level q . Of crucial importance is the following well-known Theorem.

Theorem 1.1.4. *There is an orthonormal basis of the space of cusp forms (holomorphic or Maass) of level q which consists of eigenfunctions of all the Hecke operators T_n for $(n, q) = 1$.*

A (L^2 -normalized) holomorphic cusp form or Maass cusp form of level q who is the common eigenfunction of all T_n with $(n, q) = 1$ is called a Hecke eigenform and for a Hecke eigenform f such that $T_n f = \lambda_n f$. We have in the holomorphic case

$$\lambda_n a_f(1) = a_f(n) \text{ if } (n, q) = 1$$

where $f(z) = \sum_{n \geq 1} a_f(n) n^{(k-1)/2} e(nz)$, in this case

$$|a_f(1)|^2 = \frac{(4\pi)^{k-1}}{(k-1)!} \frac{2\pi^2}{qL(1, \text{sym}^2 f)} \text{vol}(X_0(q));$$

and in the Maass form case

$$\lambda_n \nu(1) = \nu(n) \text{ if } (n, q) = 1$$

where

$$\nu(n) = \sqrt{\frac{|n|q}{\cosh \pi t_j}} \rho(n)$$

and

$$f(z) = \sqrt{y} \sum_{n \neq 0} \rho(n) K_{it_j}(2\pi |n|y) e(nx)$$

with $t_j \in \mathbb{R}$ or $\frac{1}{2} < s_j < 1$.

By Theorem 2.3.1, for each integral k, q , we can find an orthonormal basis consisting of Hecke eigenforms (called a Hecke basis). Similarly in the Maass form case.

Take any q' and d with $dq' | q$. If f is a holomorphic cusp form of weight k and level q' then $f_d(z) := f(dz)$ is a holomorphic cusp form weight k and level q . A primitive holomorphic cusp form (or holomorphic new cusp form) is a holomorphic cusp form f such that f can not be written as g_d for any cusp form g of lower level.

1.1.4 Automorphic L -functions

Let f be a primitive holomorphic cusp forms (i.e. not induced from a lower level) of weight k and level q . Let

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

be its Fourier expansion at the cusp ∞ , with $\lambda_f(n)$ being its n -th Hecke eigenvalue.

Then

$$L(f, s) = \sum_{n \geq 1} \lambda_f(n) n^{-s} = \prod_p (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1}$$

is an L -function of degree 2 with conductor q and gamma factor given by

$$\gamma(f, s) = 2^{\frac{3-k}{2}} \sqrt{\pi} (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right).$$

Then the analytic conductors are

$$q(f, s) \leq q(|s| + |k| + 3)^2,$$

$$q(f) \asymp qk^2.$$

Similarly, let ϕ be a primitive Maass cusp form of level q which is an eigenfunction of the Laplace operator with eigenvalue $\lambda = \frac{1}{4} + r^2$, where $r \in \mathbb{R}$ or $ir \in \left[0, \frac{1}{2}\right]$. Writing its Fourier expansion at ∞ in the form

$$\phi(z) = \sqrt{y} \sum_{n \neq 0} \rho(n) K_{ir}(\pi|n|y) e(nx)$$

we associate with ϕ the L -function

$$L(\phi, s) = \sum_{n \geq 1} \rho(n) n^{-s} = \prod_p (1 - \rho(p) p^{-s} + p^{-2s})^{-1}$$

of conductor q , with gamma factor

$$\gamma(\phi, s) = \pi^{-s} \Gamma\left(\frac{s + \delta + ir}{2}\right) \Gamma\left(\frac{s + \delta - ir}{2}\right)$$

where $\delta = 0$ if ϕ is even (i.e. $\rho(n) = \rho(-n)$) and $\delta = 1$ otherwise. Then the analytic conductors are

$$q(\phi, s) \leq q(|s| + |r| + 3)^2$$

$$q(\phi) = q(|r| + 3)^2 \asymp \lambda q.$$

The next L -function we are going to introduce is the symmetric square L -function associated to holomorphic cusp forms. It is proved in 1976 by S. Gelbart and H. Jacquet that the symmetric square lift of a $GL(2)$ automorphic form is a $GL(3)$ automorphic form, this is one of the first validation of the functoriality conjecture of

R. Langlands. The symmetric square L -function is the L -function associates to this $GL(3)$ automorphic form. The symmetric square L -function associated to holomorphic cusp forms is defined as follows: Let k be an even positive number, f be a holomorphic Hecke cusp form of weight k , level 1 and of L^2 -norm 1. The symmetric square L -function associated to f is defined to be

$$L(s, \text{sym}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

where $\lambda_f(m)$ is the m -th Hecke eigenvalue. $L(s, \text{sym}^2 f)$ analytically extends to an entire function on $\mathbb{C}$ and satisfies the functional equation

$$L_{\infty}(s)L(s, \text{sym}^2 f) = L_{\infty}(1-s)L(1-s, \text{sym}^2 f)$$

where

$$\begin{aligned} L_{\infty}(s) &= \pi^{-\frac{3}{2}s} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right) \\ &= \pi^{-\frac{3}{2}s + \frac{1}{2}} 2^{2-s-k} \Gamma\left(\frac{s+1}{2}\right) \Gamma(s+k-1). \end{aligned}$$

Let

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e(nz)$$

be the Fourier expansion of f . Then we have the following relation between $\lambda_f(n)$ (Hecke eigenvalue) and $a_f(m)$ (Fourier coefficient),

$$a_f(n) = a_f(1) \lambda_f(n)$$

with

$$|a_f(1)|^2 = \frac{(4\pi)^{k-1}}{(k-1)!} \frac{2\pi^2}{L(1, \text{sym}^2 f)}.$$

For an even level 1 Hecke-Maass form ϕ with n -th Hecke eigenvalue $\lambda_f(n)$. Then one can define the following Rankin-Selberg L -function:

$$L(s, \phi \otimes \text{sym}^2 f) = \sum_{m, n \geq 1} \frac{\lambda_{\phi}(m) a_F(m, n)}{(mn^2)^s}$$

where

$$a_F(m, n) = \sum_{d|(m, n)} \lambda_F\left(\frac{m}{d}, 1\right) \lambda_F\left(\frac{n}{d}, 1\right) \mu(d)$$

and

$$\lambda_F(r, 1) = \sum_{s^2 t = r} \lambda_f(t^2).$$

Let

$$\begin{aligned} \Lambda_{\phi, f}(s) &= \pi^{-3s} \Gamma\left(\frac{s+k-1+it_\phi}{2}\right) \Gamma\left(\frac{s+k-1-it_\phi}{2}\right) \\ &\times \Gamma\left(\frac{s+k+it_\phi}{2}\right) \Gamma\left(\frac{s+k-it_\phi}{2}\right) \Gamma\left(\frac{s+1+it_\phi}{2}\right) \\ &\times \Gamma\left(\frac{s+1-it_\phi}{2}\right) L(s, \phi \otimes \text{sym}^2 f) \end{aligned}$$

where $\frac{1}{4} + it_\phi^2$ is the Laplace eigenvalue of ϕ . Then $\Lambda_{\phi, f}(s)$ admits analytic continuation to $\mathbb{C}$ as an entire function and satisfies the functional equation

$$\Lambda_{\phi, f}(s) = \Lambda_{\phi, f}(1-s).$$

1.2 Main Results

1.2.1 Local large sieve inequalities for $GL(2)$ cusp forms

The classical theory of large sieve has its genesis in the work of Linnik on the least quadratic non-residue. It was subsequently refined and extended by Bombieri, Gallagher, Renyi, Roth and many others. It is probably best known for its role in partial results toward the Goldbach conjecture (works by Rényi, Bombieri, Vinogradov, and Chen who has the best result to this date). An instance of the large sieve inequality for character sum is the following.

Theorem 1.2.1. *For any complex numbers $\{a_n\}$, N a large positive integer.*

$$\sum_{q \leq Q} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} \left| \sum_{n=N}^{2N} a_n \chi(n) \right|^2 \ll (N + Q^2) \sum_{n=N}^{2N} |a_n|^2.$$

Note that the cardinality of the primitive characters modulo q , $q \leq Q$, is of size $\sim Q^2$ and N is the cardinality of the set $\{a_n\}_{n=N}^{2N}$. It is of crucial importance in application that the two cardinalities are separated, i.e. $Q^2 + N$ instead of Q^2 and N intertwining multiplicatively. This feature is indeed what we desire for the large sieve inequality in general.

Holomorphic and Maass cusp forms, being the $GL(2)$ analogues of Dirichlet character, have their corresponding large sieve inequality. For holomorphic cusp forms, it was previously studied by Jutila and Motohashi, and for Maass forms, the most notable contributors are Deshoullier and Iwaniec, Jutila, Luo, Motohashi:

Theorem 1.2.2. (*Lemma 8 in [15], [2],[14],[21] and [27]*) For any $1 \leq G \leq K$, $\{a_n\}$ complex numbers,

$$\begin{aligned} & \sum_{\substack{K \leq k \leq K+G \\ 2|k}} \sum_{f \in H_k} \left| \sum_{n=N}^{2N} a_n \lambda_f(n) \right|^2 \\ & + \sum_{K \leq t \leq K+G} \sum_{f \in \mathfrak{H}_t} \left| \sum_{n=N}^{2N} a_n \lambda_f(n) \right|^2 \\ & \ll_{\epsilon} (KN)^{\epsilon} (KG + N) \sum_{n=N}^{2N} |a_n|^2 \end{aligned}$$

where H_k is a basis for the space of holomorphic cusp forms of level 1 and weight k consisting of common eigenfunctions for all Hecke operators T_n and $\lambda_f(n)$ is the n -th Hecke eigenvalue of f ; and $\mathfrak{H}_t$ is a basis for the space of Maass forms of level 1 consisting of common eigenfunctions for all Hecke operators T_n and that $f \in \mathfrak{H}_t$ if f has Laplace eigenvalue $\frac{1}{4} + t^2$.

These settle the $GL(2)$ large sieve inequalities in the cases of family in integral weight for holomorphic cusp forms and family in the Laplacian eigenvalue for the Maass form.

In this thesis, the above Theorem is generalized in two respects: The first is that we take level into consideration and the second is that we prove a large sieve type inequality for half-integral weight holomorphic cusp forms.

For integral weight holomorphic cusp forms and Maass forms, we prove that

Theorem 1.2.3. (see [17]) *Let M be square-free number, $\{a_n\}$ complex numbers,*

$$\begin{aligned} & \sum_{\substack{K \leq k \leq K+G \\ 2|k}} \sum_{f \in H_{k,M}^*} \left| \sum_{n=N}^{2N} a_n \lambda_f(n) \right|^2 \\ & + \sum_{K \leq t \leq K+G} \sum_{f \in \mathfrak{H}_{t,M}^*} \left| \sum_{n=N}^{2N} a_n \lambda_f(n) \right|^2 \\ & \ll_{\epsilon} (MNK)^{\epsilon} (KGM + N) \sum_{n=N}^{2N} |a_n|^2 \end{aligned}$$

where $H_{k,M}^*$ is a Hecke basis for the space of new forms of weight k and level M , $\mathfrak{H}_{t,M}^*$ is a Hecke basis for the space of new Maass forms with Laplace eigenvalue $\frac{1}{4} + t^2$ and level M and $\lambda_f(n)$ is the n -th Hecke eigenvalue of f .

In the half-integral weight case, we have

Theorem 1.2.4. (see [17]) *For any natural number M divisible by 4, $\{a_n\}$ complex numbers,*

$$\sum_{K \leq k \leq K+G} \sum_{f \in B_{k,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2 \ll_{\epsilon} (MNK)^{\epsilon} (KGM + N) \sum_{n=N}^{2N} |a_n|^2.$$

Here $B_{k,M}$ is an orthonormal basis (with respect to the Petersson inner product defined above) of the space of holomorphic cusp forms of weight k with level M and $\rho_f(n)$ as in Theorem 2.4.

1.2.2 Mean value of symmetric square L -functions

Some of the simplest examples of higher rank L -functions are the symmetric power L -functions. The first detailed study of the symmetric square L -functions associated to holomorphic cusp forms was done in 1973 by G. Shimura (see [29]). In this thesis, we investigate the second moment of symmetric square L -functions. Namely, we study the sum

$$\sum_{K \leq k \leq K+G} \sum_{f \in H_k} L^2 \left(\frac{1}{2}, \text{sym}^2 f \right).$$

Ideally one wishes to show

$$\sum_{f \in H_k} L^2 \left(\frac{1}{2}, \text{sym}^2 f \right) \ll_{\epsilon, t} K^{1+\epsilon}$$

seems out of reach we manage to prove

Theorem 1.2.5. *For $K^{\frac{1}{3}} \leq G \leq K$,*

$$\sum_{K \leq k \leq K+G} \sum_{f \in H_k} L^2 \left(\frac{1}{2}, \text{sym}^2 f \right) \ll_{t, \epsilon} K^{1+\epsilon} G.$$

This improves on the previous result of W. Luo ([23]) where G can be taken to be taken to be $K^{1/2} \leq G \leq K$.

1.2.3 First moment of Rankin-Selberg L -functions.

There has been a lot of work on the mean value of Rankin-Selberg L -function. For example, Li established in [20] the sharp bound for the mean value of $GL(3) \times GL(2)$ Rankin-Selberg L -functions when the $GL(2)$ forms vary in a range of spectral parameter. In my dissertation I study the mean value of $GL(3) \times GL(2)$ Rankin-Selberg L -functions for a fixed $GL(2)$ cusp form and the $GL(3)$ forms coming from a family of self-dual forms. More precisely, we prove that

Theorem 1.2.6. *Let ϕ a fixed Hecke-Maass form, then*

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \frac{L\left(\frac{1}{2}, \phi \otimes \text{sym}^2 f\right)}{L(1, \text{sym}^2 f)} \ll_{\epsilon} GK^{1+\epsilon}$$

whenever $K \geq G \geq K^{\frac{1}{3}}$.

Luo and Sarnak proved the case when $G = K$ in [24] as a corollary of their work on Quantum variance of Hecke cusp forms. Theorem 1.2.1 can be viewed as an extension of my result on mean value of the second moment of symmetric square L -functions. Indeed, we can view Theorem 1.2.1 as the case when $\phi(z)$ is replaced by the Eisenstein series $E\left(z, \frac{1}{2} + it\right)$.

CHAPTER 2

LARGE SIEVE INEQUALITIES

In this chapter, we prove Theorem 1.2.3 and Theorem 1.2.4. One common ingredient in all the proofs is the embedding method of H. Iwaniec, which we now describe. Take any prime p and for any function f in $B_{k,M}$ (respectively Maass cusp form of level M and for the half integral case $4|M$), it is automatically a holomorphic cusp form of level pM (respectively Maass cusp form of level pM). Hence we can embed B_k into $B_{k,pM}$ (respectively B_M into $B_{k,pM}$). We then apply the Petersson trace formula (respectively Kuznetsov trace formula), which gives both diagonal and non-diagonal terms. The management of the diagonal term is straight-forward while the non-diagonal term involves the Bessel functions which traditionally are the source of troubles. The advantage of the embedding method is by choosing p well, we can completely avoid the troublesome transitional range of the Bessel functions.

2.1 Holomorphic cusp forms of integral weight k

In this section we prove the large sieve inequality for holomorphic cusp forms of integral weight.

2.1.1 First manipulations

Notations as in section 1.2.1. For any small $\epsilon > 0$. Assume $K^{1-\delta} \geq G \geq (KMN)^\delta$ for some $\delta > 0$, otherwise the result is well-known (see section 1.2.1). We assume $M < K^D$ for some $D > 0$ otherwise the result is trivial. We also assume $N \ll K^E$ for some $E > 0$ otherwise the result follows easily from direct estimation using Deligne's bound. We will now choose the test function that will smooth our sum. Let $g(x) \in C_0^\infty(0, \infty)$ such that $\text{supp}(g(x)) \subset \left[\frac{1}{2}, \frac{5}{2}\right]$, $g^{(j)} \ll_j 1$ for all $j \geq 0$, and $g(x) = 1$ for $x \in [1, 2]$. To prove the large sieve inequality, it suffices to bound, for sufficiently large K , the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{\substack{k \geq 1 \\ 2|k}} g\left(\frac{k-K}{G}\right) \sum_{f \in B_{k,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2.$$

Choose prime p in the interval

$$\left[(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\}, 2(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\} \right].$$

Such prime exists by the Bertrand postulate.

For any $f \in H_{k,M}$, f is holomorphic cusp forms of weight k and level pM . On the other hand,

$$\begin{aligned} & \frac{1}{\text{vol}(X_0(pM))} \int_{X_0(pM)} f(z) \overline{f(z)} y^k \frac{dx dy}{y^2} \\ &= \frac{1}{\text{vol}(X_0(M)) [\Gamma_0(pM) : \Gamma_0(M)]} \int_{X_0(pM)} f(z) \overline{f(z)} y^k \frac{dx dy}{y^2} \\ &= \frac{1}{\text{vol}(X_0(M))} \int_{X_0(M)} f(z) \overline{f(z)} y^k \frac{dx dy}{y^2} = 1. \end{aligned}$$

Also if f is an Hecke eigenform for all T_q with $(q, M) = 1$, then f is an eigenform for all T_q with $(q, pM) = 1$. Hence we can embed $H_{k,M}$ into $H_{k,pM}$.

By positivity, $S_M \leq S_{pM}$. Applying the Petersson trace formula to S_{pM} , $S_{pM} =$

$$\text{vol}(X_0(pM)) \sum_{k \text{ even}} g\left(\frac{k-K}{G}\right) (k-1) \sum_{n=N}^{2N} |a_n|^2 \quad (2.1.1)$$

$$+ \sum_{N \leq n_1, n_2 \leq 2N} a_{n_1} \overline{a_{n_2}} \text{vol}(X_0(pM)) \sum_{c \equiv 0(pM)} \frac{S(n_1, n_2; c)}{c} 2\pi \times \quad (2.1.2)$$

$$\sum_{k \text{ even}} i^k g\left(\frac{k-K}{G}\right) (k-1) J_{k-1}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right).$$

We call (2.1.1) the diagonal term and (2.1.2) the off-diagonal term.

2.1.2 The diagonal contribution

The diagonal term contributes at most $\sum_{m=N}^{2N} |a_m|^2$ times

$$KG(pM)^{1+\epsilon} \ll (KM)^\epsilon pKGM \ll (KNM)^{2\epsilon} \max\{KGM, N\}.$$

2.1.3 The off-diagonal contribution

Since $J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}$ (see (5.10.2) in [19]) and $\frac{4\pi\sqrt{n_1 n_2}}{c} \leq \frac{8\pi N}{c}$, so the contribution from those c with $c > \frac{24\pi N}{K}$ is exponentially small. Thus it suffices to consider those c less than or equal $\frac{24\pi N}{K}$. The sum of weighted Bessel function has the following integral representation.

Lemma 2.1.1. *For $h \in C_0^\infty(0, \infty)$, we have*

$$\sum_{\substack{k \geq 1 \\ 2|k}} 2\pi(-1)^{\frac{k}{2}} J_{k-1}(x) h(k-1) = -2\pi \int_{-\infty}^{\infty} \hat{h}(t) \sin(x \cos(2\pi t)) dt$$

where $\hat{h}(t) = \int_{-\infty}^{\infty} h(y) e(ty) dy$ is the Fourier transform of $h(t)$.

Proof. See Lemma 4.1 of [24]. □

Let $h(y) = yg\left(\frac{y-(K-1)}{G}\right)$, then

$$\begin{aligned}
\hat{h}(t) &= \int_{-\infty}^{\infty} yg\left(\frac{y-(K-1)}{G}\right) e(yt) dy \\
&= \int_{-\infty}^{\infty} [y-(K-1)]g\left(\frac{y-(K-1)}{G}\right) e(yt) dy \\
&\quad + (K-1) \int_{-\infty}^{\infty} g\left(\frac{y-(K-1)}{G}\right) e(yt) dy \\
&= e((K-1)t) \int_{-\infty}^{\infty} xg\left(\frac{x}{G}\right) e(xt) dx \\
&\quad + (K-1)e((K-1)t) \int_{-\infty}^{\infty} g\left(\frac{x}{G}\right) e(xt) dx \\
&= e((K-1)t)\hat{h}_1(t) + (K-1)e((K-1)t)\hat{h}_2(t)
\end{aligned}$$

where $h_1(x) = xg\left(\frac{x}{G}\right)$ and $h_2(x) = g\left(\frac{x}{G}\right)$.

Hence,

$$\begin{aligned}
&\sum_{\substack{k \geq 1 \\ 2|k}} (-1)^{\frac{k}{2}} J_{k-1}(x) h(k-1) \\
&= - \sum_{j=1}^2 (K-1)^{j-1} \int_{-\infty}^{\infty} \hat{h}_j(t) e((K-1)t) \sin(x \cos(2\pi t)) dt.
\end{aligned}$$

We will show in the next subsection that assuming $\frac{x}{KG} \ll (KMN)^{-\epsilon}$ and $G \leq K^{1-\delta}$ for some $\epsilon, \delta > 0$ then for $j = 1, 2$

Lemma 2.1.2.

$$\int_{-\infty}^{\infty} \hat{h}_j(t) e((K-1)t) \sin(x \cos(2\pi t)) dt \ll_B (KMN)^{-B}$$

for all $B > 0$.

Assuming Lemma 2.1.2 and takes $x = \frac{2\sqrt{n_1 n_2}}{c}$, $N \leq n_1, n_2 \leq 2N$, the off-diagonal contribution

$$\begin{aligned} & \ll \sum_{N \leq n_1, n_2 \leq 2N} |a_{n_1} a_{n_2}| (pM)^{1+\epsilon} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{|S(n_1, n_2; c)|}{c} (KMN)^{-B} \\ & \ll (pM)^{1+\epsilon} N^{2+\epsilon} (KMN)^{-B} \sum_{n=N}^{2N} |a_n|^2 \end{aligned}$$

which is much less the diagonal contribution upon taking, say, $B = 2015$. This established the large sieve inequality of holomorphic cusp forms of integral weight.

2.1.4 Proof of Lemma 2.1.2

We first prove the following basic Lemma in Fourier analysis.

Lemma 2.1.3. *Let $h \in C_c^\infty(0, \infty)$ be such that $\text{supp } h \subset [K, K+G]$ and $h^{(j)} \ll_j G^{-j+1}$ for all $j > 0$. Then*

$$\hat{h}(t) \ll |t|^{-m} G^{-m+1} K$$

for all $m > 0$.

Proof. For each $m \in \mathbb{N}$, integrating by parts m times

$$\hat{h}(t) = \left(-\frac{1}{2\pi i t} \right)^m \int_{-\infty}^{\infty} h^{(m)}(x) e^{-2\pi i x t} dx.$$

Hence,

$$|\hat{h}(t)| \leq \frac{1}{(2\pi|t|)^m} \int_{-\infty}^{\infty} |h^{(m)}(x)| dx \ll |t|^{-m} G^{-m+1} K.$$

□

As a corollary,

$$\int_{|t| \geq \frac{K^{\frac{5}{6}}}{G}} \hat{h}_j(t) e((K-1)t) \sin(x \cos(2\pi t)) dt$$

is negligible. With a little bit of algebra, we can consider instead the integrals

$$\int_{-\frac{K^{\frac{\epsilon}{2}}}{G}}^{\frac{K^{\frac{\epsilon}{2}}}{G}} \hat{h}_j(t) e((K-1)t \pm x \cos(2\pi t)) dt.$$

Let $\omega_{\pm}(t) = 2\pi(K-1)t \pm x \cos(2\pi t)$. We claim that for $|t| \leq \frac{K^{\epsilon/2}}{G}$, $x \leq (MNK)^{-\epsilon}KG$, we have

$$|\omega'_{\pm}(t)| > \frac{K}{2}.$$

To see this, note that

$$\omega'_{\pm}(t) = 2\pi(K-1) \mp x \sin(2\pi t) = 2\pi(K-1) \mp (2\pi)xt \frac{\sin(2\pi t)}{2\pi t}.$$

Since $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$ and $G \gg K^{\delta}$, we have for $|t| \leq \frac{K^{\epsilon/2}}{G}$ that $\left| \frac{\sin(2\pi t)}{2\pi t} \right| \leq \frac{3}{2}$.

From which we have

$$\left| (2\pi)xt \frac{\sin(2\pi t)}{2\pi t} \right| \leq (2\pi)\frac{3}{2}|xt| \leq (2\pi)\frac{3}{2}(KNM)^{-\epsilon}KG \frac{K^{\epsilon/2}}{G} < \frac{K}{2}.$$

Hence our claim.

Note that

$$\begin{aligned} \hat{h}_1^{(\nu)}(t) &= (2\pi i)^{\nu} \int_{-\infty}^{\infty} x^{\nu+1} g\left(\frac{x}{G}\right) e(xt) dx \\ &= (2\pi i G)^{\nu} G \int_{-\infty}^{\infty} \left(\frac{x}{G}\right)^{\nu+1} g\left(\frac{x}{G}\right) e(xt) dx \ll G^{\nu+2}. \end{aligned}$$

Similarly, $\hat{h}_2^{(\nu)}(t) \ll G^{\nu+1}$.

We now apply integrating by parts to the oscillatory integral $\int_{-\frac{K^{\frac{\epsilon}{2}}}{G}}^{\frac{K^{\frac{\epsilon}{2}}}{G}} \hat{h}_j(t) e^{\omega_{\pm}(t)} dt$

with $|\omega'_{\pm}(t)| > \frac{K}{2}$, $\hat{h}_j^{(\nu)}(t) \ll G^{\nu+3-j}$, and $\omega_{\pm}^{(k)}(t) \ll \frac{x}{KG}$ for $k > 1$ to conclude the proof of Lemma 2.1.2. $\square$

2.2 Holomorphic cusp forms of half integral weight

In this section we prove the large sieve inequality for holomorphic cusp forms of half integral weight.

2.2.1 First manipulation

Notations as in section 1.2.1. For any small $\epsilon > 0$. Assume without loss of generality that $(KMN)^\delta \leq G \leq K^{1-\delta}$ for some $1 > \delta > 0$. Also assume $N \leq K^D$ for some $D > 0$ otherwise the result is trivial using Cauchy-Schwarz inequality and mean value estimate of the Fourier coefficient of half integral weight holomorphic cusp forms (see P.70 of [9]). Let $g(x) \in C_0^\infty(0, \infty)$ such that $\text{supp}(g) \subset \left[\frac{1}{2}, \frac{5}{2}\right]$, $g^{(j)} \ll 1$ for all $j \geq 1$, and $g(x) = 1$ for $x \in [1, 2]$. To prove the large sieve inequality, it suffices to bound, for sufficiently large K , the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in B_{k,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2.$$

Choose a prime p such that

$$p \in \left[(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\}, 2(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\} \right].$$

For the same reason as was presented in 2.1.1

$$S_M \leq \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in B_{k,pM}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2.$$

By Petersson formula,

$$\begin{aligned}
& \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in B_{k,pM}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2 \\
&= \text{vol}(X_0(pM)) \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{N \leq n_1, n_2 \leq 2N} a_{n_1} \overline{a_{n_2}} \{(k-1)\delta_{n_1 n_2} \\
&+ 2\pi i^{-k}(k-1) \sum_{c \equiv 0(pM)} \frac{K_k(n_1, n_2; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right)\}
\end{aligned}$$

where we define for each complex number $z \neq 0$ and real number ν ,

$$z^\nu = |z|^\nu \exp(i\nu \arg z) \text{ with } -\pi < \arg z \leq \pi.$$

2.2.2 The diagonal contribution

The diagonal term

$$\text{vol}(X_0(pM)) \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} (k-1) g\left(\frac{k-K}{G}\right) \sum_{m=N}^{2N} |a_m|^2$$

contributes at most $\sum_{m=N}^{2N} |a_m|^2$ times

$$KG(pM)^{1+\epsilon} \ll (KM)^\epsilon pKGM \ll (KNM)^{2\epsilon} \max\{KGM, N\}.$$

2.2.3 The off-diagonal contribution

Since $J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}$ (see (5.10.2) in [19]) and $\frac{4\pi\sqrt{n_1 n_2}}{c} \leq \frac{8\pi N}{c}$, so the contribution from those c with $c > \frac{24\pi N}{K}$ is exponentially small. Thus it suffices to consider those c less than or equal to $\frac{24\pi N}{K}$, i.e. we can consider instead

$$\text{vol}(X_0(pM)) \sum_{\substack{k \geq 1 \\ k - \frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{N \leq n_1, n_2 \leq 2N} a_{n_1} \overline{a_{n_2}} \times \quad (2.2.1)$$

$$\left\{ 2\pi i^{-k}(k-1) \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{K_k(n_1, n_2; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{n_1 n_2}}{c} \right) \right\}.$$

Note that we have the following integral representation of the J -Bessel function (see P.231 of [30]): For $l \in \mathbb{N} \cup \{0\}$,

$$J_{l+\frac{1}{2}}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e \left(\left(l + \frac{1}{2} \right) t \right) e^{-ix \sin 2\pi t} dt + \frac{(-1)^l}{\pi} \int_0^\infty e^{-(l+\frac{1}{2})t-x \sinh t} dt.$$

Hence, (2.2.1)

$$\begin{aligned} &= 2\text{vol}(X_0(pM)) \sum_{\alpha=1}^2 \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \sum_{N \leq n_1, n_2 \leq 2N} \sum_{\substack{l \text{ even} \\ l \geq 2}} \frac{K_{l+\frac{1}{2}}(n_1, n_2; c)}{c} a_{n_1} \overline{a_{n_2}} \times \\ & \quad i^{-(l+\frac{1}{2})} \left(l - \frac{1}{2} \right) g \left(\frac{l + \frac{1}{2} - K}{G} \right) I_{l,\alpha} \left(\frac{4\pi \sqrt{n_1 n_2}}{c} \right) \end{aligned}$$

where

$$\begin{aligned} I_{l,1}(x) &= \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} e \left(\left(l - \frac{1}{2} \right) t \right) e^{-ix \sin 2\pi t} dt \text{ and} \\ I_{l,2}(x) &= \int_0^\infty e^{-(l-\frac{1}{2})t-x \sinh t} dt. \end{aligned}$$

Opening the twisted Kloosterman sum (see 1.1.1 for the equation of $K_k(n_1, n_2; c)$) and by our definition of z^ν , (2.2.1) =

$$\begin{aligned} &\sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{2\text{vol}(X_0(pM))}{c} \sum_{N \leq n_1, n_2 \leq 2N} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{\bar{d}} \right) \\ &\times e \left(\frac{n_1 d + n_2 \bar{d}}{c} \right) a_{n_1} \overline{a_{n_2}} \sum_{\substack{l \text{ even} \\ l \geq 2}} \left(l - \frac{1}{2} \right) i^{-m(2l+1)-(l+\frac{1}{2})} \times \\ &g \left(\frac{l + \frac{1}{2} - K}{G} \right) I_{l,1} \left(\frac{4\pi \sqrt{n_1 n_2}}{c} \right) + \end{aligned} \tag{2.2.2}$$

$$\begin{aligned}
& \sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{2\text{vol}(X_0(pM))}{c} \sum_{N \leq n_1, n_2 \leq 2N} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{d}\right) \\
& \times e\left(\frac{n_1 d + n_2 \bar{d}}{c}\right) a_{n_1} \overline{a_{n_2}} \sum_{\substack{l \text{ even} \\ l \geq 2}} \left(l - \frac{1}{2}\right) i^{-m(2l+1) - (l + \frac{1}{2})} \\
& \times g\left(\frac{l + \frac{1}{2} - K}{G}\right) I_{l,2}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right).
\end{aligned} \tag{2.2.3}$$

We will first deal with (2.2.3), first of all $I_{l,2}(x)$ can be evaluated explicitly with an error term. Let $\omega(t) = -\left(l - \frac{1}{2}\right)t - x \sinh t$, integrating by parts, we have

$$I_{l,2}(x) = \int_0^\infty e^{\omega(t)} dt = \frac{-1}{l - \frac{1}{2} + x} + \int_0^\infty e^{\omega(t)} \frac{\omega''(t)}{[\omega'(t)]^2} dt.$$

Applying integrating by parts repeatedly, we can show that $I_{l,2}(x)$ is a sum of

$$\frac{x^a}{\left(l - \frac{1}{2} + x\right)^{b+1+a}}$$

for some non-negative natural numbers a, b , with an error term $O(x^{-A})$ for all $A > 0$.

The contribution of the error term $O(x^{-A})$ is

$$\begin{aligned}
& \ll pMKG \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} c^A \left(\sum_{n=N}^{2N} \frac{|a_n|}{n^{A/2}}\right)^2 \ll \frac{pMKG(pM)^A}{N^{A-1}} \sum_{t=1}^{\frac{24\pi N}{KpM}} t^A \sum_{n=N}^{2N} |a_n|^2 \\
& \ll \frac{N^2}{K^{A-1}} \sum_{n=N}^{2N} |a_n|^2.
\end{aligned}$$

Since we assume a priori that $N \ll K^D$ for some $D > 0$, by taking $A = B + 1$, we have the term above $\ll N \sum |a_n|^2$. Hence to show (2.2.3) $\ll (MKN)^\epsilon (KGM + N) \sum_{n=N}^{2N} |a_n|^2$, it suffices to show

Lemma 2.2.1.

$$\text{vol}(X_0(pM)) \sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{1}{c} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{d}\right) \sum_{j=\pm 1} i^{-1+\frac{j}{2}} \tag{2.2.4}$$

$$\begin{aligned}
& \sum_{\substack{l \text{ even} \\ l \geq 2}} \left(l - \frac{1}{2}\right) i^{l(j-2)} g\left(\frac{l + \frac{1}{2} - K}{G}\right) \\
& \times \sum_{N \leq n_1, n_2 \leq 2N} e\left(\frac{n_1 \bar{d} + n_2 d}{c}\right) \frac{a_{n_1} \overline{a_{n_2}} \left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right)^a}{\left(l - \frac{1}{2} + \frac{4\pi\sqrt{n_1 n_2}}{c}\right)^{b+1+a}} \\
& \ll (MKN)^\epsilon (KGM + N) \sum_{n=N}^{2N} |a_n|^2
\end{aligned}$$

for all non-negative natural number a, b .

To prove Lemma 2.2.1 we need the following hybrid large sieve inequality.

Theorem 2.2.2. *For any complex numbers $\{a_n\}$, $M < n \leq M + N$, let $x_1 \cdots, x_R$ be real numbers which are distinct mod 1. Let $\delta = \min_{\substack{r,s \\ r \neq s}} \|x_r - x_s\|$, where if $R \geq 2$, $\|x\| := \min_{k \in \mathbb{Z}} |x - k|$; $\delta := \infty$, if $R = 1$. Then*

$$\sum_{r=1}^R \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 \leq (\pi N + \delta^{-1}) \sum_{n=N+1}^{M+N} |a_n|^2.$$

Proof. See Theorem 2.1 in [28] □

Corollary 2.2.3. *Notations as before. Let Q, s be natural numbers. Then*

$$\sum_{\substack{q \leq Q \\ q \equiv 0(s)}} \sum_{\substack{(l,q)=1 \\ 0 < l < q}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{nl}{q}\right) \right|^2 \ll \left(N + \frac{Q^2}{s}\right) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Proof. For $(l, q) = 1$, $0 < l < q$, $q \leq Q$, $q \equiv 0(s)$, $q = sv$, $q' = sv'$, let $x_{l,q} = \frac{l}{q}$. Then for each $q \neq q'$,

$$\|x_{l,q} - x_{l',q'}\| = \left\| \frac{l}{q} - \frac{l'}{q'} \right\| = \left\| \frac{l}{sv} - \frac{l'}{sv'} \right\|$$

for some $v, v' \in \mathbb{N}$. Hence, (since $sv', sv \leq Q$)

$$\|x_{l,q} - x_{l',q'}\| \geq \frac{1}{svv'} \geq \frac{1}{\frac{Q^2}{s}}.$$

Hence taking $\delta = \frac{s}{Q^2}$ in Theorem 2.2.2,

$$\sum_{\substack{q \leq Q \\ q \equiv 0(s)}} \sum_{\substack{(l,q)=1 \\ 0 < l < q}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{nl}{q}\right) \right|^2 \ll \left(N + \frac{Q^2}{s}\right) \sum_{n=M+1}^{M+N} |a_n|^2.$$

□

Proposition 2.2.4. *Notations as before. For $a_n \in \mathbb{C}$, let*

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then, for $T \geq 1$,

$$\sum_{\substack{q \leq Q \\ q \equiv 0(s)}} \sum_{\substack{(l,q)=1 \\ 0 < l < q}} \int_{-T}^T \left| \sum_{n=1}^{\infty} a_n e\left(\frac{nl}{q}\right) n^{-it} \right|^2 dt \ll \sum_{n=1}^{\infty} \left(\frac{TQ^2}{s} + n \right) |a_n|^2.$$

Proof. See the proof of Theorem 5.1 in [28].

□

We define, for any complex sequence $\{b_n\}_{N \leq n \leq 2N}$,

$$B(\{b_n\}, c, N, t) = \sum_{m=0,1} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left| \sum_{n=N}^{2N} b_n e\left(\frac{nd}{c}\right) n^{-it} \right|^2.$$

We need to show for all non-negative a, b that

Lemma 2.2.5.

$$\begin{aligned} & \text{vol}(X_0(pM)) \sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{1}{c} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{d}\right) \times \\ & \sum_{N \leq n_1, n_2 \leq 2N} e\left(\frac{n_1 \bar{d} + n_2 d}{c}\right) a_{n_1} \bar{a}_{n_2} \frac{\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right)^a}{\left(l - \frac{1}{2} + \frac{4\pi\sqrt{n_1 n_2}}{c}\right)^{b+1+a}} \\ & \ll \left(\frac{N}{K^{b+2}} + \frac{pM}{K^b}\right) \sum |a_n|^2 \end{aligned} \tag{2.2.5}$$

Proof. We follow the argument on P.256 of [2]. The above sum remains unchanged if a smooth weight $h\left(\frac{\sqrt{n_1 n_2}}{N}\right)$ is attached to each term provided that

$$h(x) = \begin{cases} 1 & \text{if } x \in (1, 2] \\ 0 & \text{if } x \notin (\frac{1}{2}, 3]. \end{cases}$$

In what follows we demand $h(x)$ to be of C^∞ class. Then

$$\frac{h(x)}{\left(l - \frac{1}{2} + \frac{4\pi}{c}xN\right)^{b+1+a}} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} H(s)x^{-s}ds$$

with

$$H(s) = \int_0^\infty h(x) \frac{1}{\left(l - \frac{1}{2} + \frac{4\pi}{c}xN\right)^{b+1+a}} x^{it} dx.$$

Hence, (2.2.5) =

$$\begin{aligned} & \sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{1}{2\pi i c} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{d}\right) \sum_{N \leq n_1, n_2 \leq 2N} e\left(\frac{n_1 \bar{d} + n_2 d}{c}\right) \\ & \times \text{vol}(X_0(pM)) a_{n_1} \bar{a}_{n_2} \left(\frac{4\pi \sqrt{n_1 n_2}}{c}\right)^a \int_{1-i\infty}^{1+i\infty} H(s) \left(\frac{\sqrt{n_1 n_2}}{N}\right)^{-s} ds \\ & = \frac{\text{vol}(X_0(pM))}{2\pi i} \sum_{m=0,1} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{(4\pi)^a}{c^{a+1}} \sum_{\substack{d(c) \\ d \equiv (-1)^m(4)}} \left(\frac{c}{d}\right) \int_{1-i\infty}^{1+i\infty} H(s) \times \\ & \left(\sum_{n_1=N}^{2N} a_{n_1} n_1^{\frac{a}{2}} \left(\frac{n_1}{N}\right)^{-\frac{s}{2}} e\left(\frac{n_1 \bar{d}}{c}\right) \right) \\ & \times \left(\sum_{n_2=N}^{2N} \bar{a}_{n_2} n_2^{\frac{a}{2}} \left(\frac{n_2}{N}\right)^{-\frac{s}{2}} e\left(\frac{n_2 d}{c}\right) \right) ds \end{aligned}$$

$$\begin{aligned}
&\ll \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-\infty}^{\infty} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) |H(1+2it)| dt \\
&\leq \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-1}^1 B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) |H(1+2it)| dt \\
&+ \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \sum_{\nu \geq 0} \int_{J_\nu} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) |H(1+2it)| dt
\end{aligned}$$

where $J_\nu = [-2^{\nu+1}, -2^\nu] \cup [2^\nu, 2^{\nu+1}]$. Integrating by parts,

$$H(1+it) \ll \begin{cases} (1+|t|)^{-\frac{1}{2}} \frac{1}{(l - \frac{1}{2} + \frac{4\pi N}{c})^{b+1+a}} \\ t^{-2} \frac{1}{(l - \frac{1}{2} + \frac{4\pi N}{c})^{b+1+a}} \text{ for } |t| > \frac{16\pi N}{c}. \end{cases}$$

Then, (2.2.5)

$$\begin{aligned}
&\ll \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-1}^1 B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) \left(\frac{c}{N}\right)^{b+1+a} dt + \sum_{v=0}^{\infty} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} \\
&\times \frac{pMN}{c^{a+1}} \int_{J_\nu} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) \min \left\{ \frac{1}{\sqrt{1+2^v}}, 2^{-2v} \right\} \left(\frac{c}{N}\right)^{b+1+a} dt \\
&= \frac{pM}{N^{b+a}} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} c^b \int_{-1}^1 B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) dt + \frac{pM}{N^{b+a}} \sum_{v=0}^{\infty} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} c^b \\
&\times B(2^\nu, c) \min \left\{ \frac{1}{\sqrt{1+2^v}}, 2^{-2v} \right\}
\end{aligned}$$

where

$$B(T, c) = \int_{[T, 2T] \cup [-2T, -T]} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) dt.$$

By the Abel summation formula,

$$\begin{aligned}
& \frac{pM}{N^{b+a}} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} c^b B(T, c) = \frac{(pM)^{b+1}}{N^{a+b}} \sum_{u=1}^{\frac{24\pi N}{KpM}} u^b B(T, pMu) \\
& \ll \frac{(pM)^{n+1}}{N^{a+b}} \left(\sum_{u=1}^{\frac{24\pi N}{KpM}} B(T, pMu) \right) \left(\frac{24\pi N}{KpM} \right)^b \\
& + b \int_1^{\frac{24\pi N}{KpM}} x^{b-1} \left(\sum_{u=1}^x B(T, pMu) \right) dx \frac{(pM)^{b+1}}{N^{a+b}} \\
& \ll \frac{pM}{N^a K^b} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} B(T, c) + \frac{(pM)^{b+1}}{N^{a+b}} \int_1^{\frac{24\pi N}{KpM}} x^{b-1} \left(\sum_{\substack{c \equiv 0(pM) \\ c \leq xpM}} B(T, c) \right) dx
\end{aligned}$$

By Proposition 2.2.4,

$$\begin{aligned}
& \sum_{\substack{c \equiv 0(pM) \\ c \leq xpM}} B(T, c) \ll \sum_{n=N}^{2N} (Tx^2pM + n) |a_n|^2 n^{a-1} \\
& \ll (Tx^2pMN^{a-1} + N^a) \sum_{n=N}^{2N} |a_n|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{pM}{N^{b+a}} \sum_{\substack{c \equiv 0(pM) \\ c \leq \frac{24\pi N}{K}}} c^b B(T, c) \ll \frac{pM}{N^a K^b} \left(T \frac{N^2}{K^2 pM} N^{a-1} + N^a \right) \sum_{n=N}^{2N} |a_n|^2 \\
& + \frac{(pM)^{b+1}}{N^{a+b}} \int_1^{\frac{24\pi N}{KpM}} x^{b-1} (Tx^2pMN^{a-1} + N^a) dx \sum_{n=N}^{2N} |a_n|^2 \\
& \ll \left(\frac{TN}{K^{b+2}} + \frac{pM}{K^b} \right) \sum_{n=N}^{2N} |a_n|^2 \\
& + \frac{(pM)^{b+1}}{N^{a+b}} \left[TpMN^{a-1} \left(\frac{N}{KpM} \right)^{b+2} + N^a \left(\frac{N}{KpM} \right)^b \right] \sum_{n=N}^{2N} |a_n|^2 \\
& \ll \left[\frac{TN}{K^{b+2}} + \frac{pM}{K^b} \right] \sum_{n=N}^{2N} |a_n|^2.
\end{aligned}$$

Hence, (2.2.5)

$$\ll \left[\frac{N}{K^{b+2}} + \frac{pM}{K^b} \right] \sum_{n=N}^{2N} |a_n|^2.$$

□

Proof. (of Lemma 2.2.5)

By Lemma 2.2.5, (2.2.4)

$$\begin{aligned} &\ll \sum_{k \geq 1} g \left(\frac{2k - K + \frac{1}{2}}{G} \right) \left(2k - \frac{1}{2} \right) \left(\frac{N}{K^2} + pM \right) \sum_{m=N}^{2N} |a_m|^2 \\ &\ll KG \left(\frac{N}{K^2} + pM \right) \sum_{m=N}^{2N} |a_m|^2 \ll (MKN)^\epsilon (KGM + N) \sum_{m=N}^{2N} |a_m|^2. \end{aligned}$$

Thus concludes the proof of Lemma 2.2.1

□

We now develop the appropriate Neumann theory for (2.2.2). We start with

Lemma 2.2.6. *For $h \in C_0^\infty(0, \infty)$, $\alpha \in \mathbb{R}$,*

$$\begin{aligned} &\sum_l h(l) e(\alpha l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e \left(\left(l + \frac{1}{2} \right) t \right) e^{-ix \sin 2\pi t} dt \\ &= \int_{-\infty}^{\infty} \hat{h}(t) e \left(\frac{\lfloor \alpha - t \rfloor + \alpha - t}{2} \right) e^{-ix \sin 2\pi(\alpha - t)} dt \end{aligned}$$

Proof. Let $g_x(t) = e \left(\frac{t}{2} \right) e^{-ix \sin 2\pi t}$ and

$$f(t) = \begin{cases} g_x(t) & t \in \left(-\frac{1}{2}, \frac{1}{2} \right] \\ 0 & \text{otherwise} \end{cases}$$

then, $\int_{-\frac{1}{2}}^{\frac{1}{2}} e(lt) g_x(t) dt = \int_{-\infty}^{\infty} e(lt) f(t) dt = \hat{f}(l)$, the Fourier transform of f at l . By Poisson summation formula, denoting the Fourier transform by $\mathfrak{F}$,

$$\sum_l h(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e \left(\left(l + \frac{1}{2} \right) t \right) e^{-ix \sin 2\pi t} dt$$

$$\begin{aligned}
&= \sum_l h(l) \hat{f}(l) = \sum_l \mathfrak{F}(\hat{h}(-\cdot) * f)(l) \\
&= \sum_l (\hat{h}(-\cdot) * f)(l) = \sum_l \int_{-\infty}^{\infty} \hat{h}(-t) f(l-t) dt \\
&= \int_{-\infty}^{\infty} \hat{h}(t) \sum_l f(l+t) dt.
\end{aligned}$$

We now have to show

$$\sum_l f(l+t) = e^{\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t}}.$$

For each $t \in \mathbb{R}$, if $t = 2n + y$ for some $y \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{Z}$, then

$$\begin{aligned}
\sum_l f(l+t) &= f(y) = g_x(y) = e^{\left(\frac{y}{2}\right) e^{-ix \sin 2\pi y}} \\
&= e^{\left(\frac{2n+y}{2}\right) e^{-ix \sin 2\pi(2n+y)}} = e^{\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t}},
\end{aligned}$$

if $t = 2n + 1 + y$ for some $y \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{Z}$, then

$$\begin{aligned}
\sum_l f(l+t) &= f(y) = g_x(y) = e^{\left(\frac{y}{2}\right) e^{-ix \sin 2\pi y}} \\
&= -e^{\left(\frac{2n+1+y}{2}\right) e^{-ix \sin 2\pi(2n+1+y)}} = e^{\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t}}.
\end{aligned}$$

Hence $\sum_l f(l+t) = e^{\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t}}$.

Applying the above to $h_\alpha(t) = h(t)e(\alpha t)$, then

$$\begin{aligned}
&\sum_l h(l)e(\alpha l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t}} dt \\
&= \sum_l h_\alpha(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t}} dt \\
&= \int_{-\infty}^{\infty} \widehat{h_\alpha}(t) e^{\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t}} dt.
\end{aligned}$$

Notice that

$$\begin{aligned}\widehat{h_\alpha}(t) &= \int_{-\infty}^{\infty} h(y)e(\alpha y)e(-yt)dy \\ &= \int_{-\infty}^{\infty} h(y)e((\alpha - t)y)dy = \hat{h}(\alpha - t).\end{aligned}$$

Hence,

$$\begin{aligned}& \sum_l h(l)e(\alpha l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t} dt \\ &= \int_{-\infty}^{\infty} \hat{h}(\alpha - t) e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix \sin 2\pi t} dt \\ &= \int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{\lfloor \alpha - t \rfloor + (\alpha - t)}{2}\right) e^{-ix \sin 2\pi(\alpha - t)} dt.\end{aligned}$$

□

In particular, take $\alpha = \frac{1}{4}$

$$\begin{aligned}& \sum_l h(l) e\left(\frac{l}{4}\right) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t} dt \\ &= \sum_{l \text{ even}} (-1)^{\frac{l}{2}} h(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t} dt \\ &+ i \sum_{l \text{ odd}} (-1)^{\frac{l-1}{2}} h(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t} dt.\end{aligned}$$

Hence

$$\begin{aligned}& \sum_{l \text{ odd}} (-1)^{\frac{l-1}{2}} h(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right)t\right) e^{-ix \sin 2\pi t} dt \\ &= \text{Im} \left(\int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix \cos 2\pi t} dt \right).\end{aligned}$$

To get rid of the annoying floor function $\lfloor \frac{1}{4} - t \rfloor$, note that $\hat{h}$ is essentially supported in $\left[-\frac{1}{G}, \frac{1}{G}\right]$ whenever h is supported in $[K, K+G]$ and its j -derivative is majorized by G^{-j+1} , i.e. $h(t) = tg\left(\frac{t - (K-1)}{G}\right)$ satisfies the conditions in Lemma 2.1.3. Thus,

Proposition 2.2.7. Notation as above. For all $A > 0$,

$$\begin{aligned} & \int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt \\ &= \int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{1}{8} - \frac{t}{2}\right)} e^{-ix \cos 2\pi t} dt + O(K^{-A}). \end{aligned}$$

Proof. Fix $0 < \delta < 1$ such that $\frac{G^\delta}{G} < \frac{1}{4}$,

$$\begin{aligned} & \int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt \\ &= \int_{|t| \leq \frac{G^\delta}{G}} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt \\ &+ \int_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt. \end{aligned}$$

By Lemma 2.1.3,

$$\begin{aligned} & \int_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt \\ &\ll KG^{-m+1} \left(\frac{G^\delta}{G}\right)^{-m+1} = KG^{\delta(-m+1)}. \end{aligned}$$

Taking $m = \left\lfloor 3\frac{A+1}{\delta} \right\rfloor + 2$,

$$\int_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right)} e^{-ix \cos 2\pi t} dt \ll K^{-A}.$$

On the other hand when $|t| \leq \frac{G^\delta}{G} \leq \frac{1}{4}$, $\left\lfloor \frac{1}{4} - t \right\rfloor = 0$. Hence,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2} \right)} e^{-ix \cos 2\pi t} dt \\
&= \int_{|t| \leq \frac{G^\delta}{G}} \hat{h}(t) e^{\left(\frac{1}{8} - \frac{t}{2} \right)} e^{-ix \cos 2\pi t} dt + O(K^{-A}) \\
&= \int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{1}{8} - \frac{t}{2} \right)} e^{-ix \cos 2\pi t} dt + O(K^{-A}) \\
&\text{by using } \int_{|t| > G^{\delta-1}} \hat{h}(t) e^{\left(\frac{1}{8} - \frac{t}{2} \right)} e^{-ix \cos 2\pi t} dt = O(K^{-A}). \quad \square
\end{aligned}$$

From this we conclude that

$$\begin{aligned}
& \sum_{l \text{ even}} (-1)^{\frac{l}{2}} h(l-1) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\left(\left(l + \frac{1}{2} \right) t \right)} e^{-ix \sin 2\pi t} dt \\
&= \text{Im} \left(\int_{-\infty}^{\infty} \hat{h}(t) e^{\left(\frac{1}{8} - \frac{t}{2} \right)} e^{-ix \cos 2\pi t} dt \right) + O(K^{-A}).
\end{aligned}$$

As before,

$$\hat{h}(t) = e((K-1)t) \hat{h}_1(t) + (K-1) e((K-1)t) \hat{h}_2(t)$$

where $h_1(x) = xg\left(\frac{x}{G}\right)$ and $h_2(x) = g\left(\frac{x}{G}\right)$. Hence

$$\begin{aligned}
& \sum_{l \text{ even}} (-1)^{\frac{l}{2}} h(l-1) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\left(\left(l + \frac{1}{2} \right) t \right)} e^{-ix \sin 2\pi t} dt \\
&= \text{Im} \left(\int_{-\infty}^{\infty} \hat{h}_1(t) e^{\left(\frac{1}{8} + \left(K - \frac{3}{2} \right) t - \frac{x}{2\pi} \cos 2\pi t \right)} dt \right. \\
&\quad \left. + (K-1) \int_{-\infty}^{\infty} \hat{h}_2(t) e^{\left(\frac{1}{8} + \left(K - \frac{3}{2} \right) t - \frac{x}{2\pi} \cos 2\pi t \right)} dt \right).
\end{aligned}$$

The rest of the proof is then completely analogous to that of Theorem 1.2.3.

2.3 Maass cusp forms

In this section we will prove the large sieve inequality for the Maass cusp forms.

2.3.1 First Manipulation

For any small $\epsilon > 0$. Assume without loss of generality that $(KMN)^\delta \leq G \leq K^{1-\delta}$ for some $\delta > 0$. Also assume $N \leq K^D$ for some $D > 0$ otherwise the result is trivial using Cauchy-Schwarz inequality and mean value estimate of the Fourier coefficient of Maass cusp forms.

Let $h(t) = e^{-\left(\frac{t-K}{G}\right)^2} + e^{-\left(\frac{t+K}{G}\right)^2}$. To prove Theorem 1.2.3, it suffices to bound, for sufficiently large K , the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{f \in H_M} \frac{1}{\cosh \pi t_f} h(t_f) \left| \sum_{n=N}^{2N} a_n \rho_{t_f}(n) \right|^2.$$

Choose a prime p such that

$$p \in \left[(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\}, 2(KMN)^\epsilon \max \left\{ \frac{N}{MKG}, 1 \right\} \right].$$

Under the assumption that $K^{1-\delta} \geq G$ for some $\delta > 0$, if $t \in i\mathbb{R}$ then

$$h(t) \ll e^{-\frac{K^2}{G^2}} \leq e^{-K^{2\delta}}.$$

Thus the contribution from the contribution from the exceptional eigenvalues is negligible.

We embed H_M into H_{pM} and since the contribution from the exceptional eigenvalues is small, we can consider instead

$$\begin{aligned} & \sum_{f \in H_{pM}} \frac{1}{\cosh \pi t_f} h(t_f) \left| \sum_{n=N}^{2N} a_n \rho_{t_f}(n) \right|^2 \\ & + \sum_{\mathfrak{a} \in \mathfrak{P}_{pM}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \sum_{n=N}^{2N} a_n \phi_{\mathfrak{a}\infty}^{\Gamma_0(pM)} \left(n, \frac{1}{2} + ir \right) \right|^2 \frac{h(r) dr}{\cosh \pi r}. \end{aligned}$$

Applying Kuznetsov formula (see P.409 of [12])

$$\begin{aligned}
& \sum_{f \in H_{pM}} \frac{1}{\cosh \pi t_f} h(t_f) \left| \sum_{n=N}^{2N} a_n \rho_{t_f}(n) \right|^2 \\
& + \sum_{\mathfrak{a} \in \mathfrak{P}_{pM}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \sum_{n=N}^{2N} a_n \phi_{\mathfrak{a}\infty}^{\Gamma_0(pM)} \left(n, \frac{1}{2} + ir \right) \right|^2 \frac{h(r) dr}{\cosh \pi r} \\
& = \sum_{N \leq n_1, n_2 \leq 2N} a_{n_1} \overline{a_{n_2}} \text{vol}(X_0(pM)) \times \\
& \quad \left\{ \frac{\delta_{n_1 n_2}}{\pi^2} \int_{-\infty}^{\infty} r h(r) \tanh \pi r dr + \frac{2i}{\pi} \right. \\
& \quad \left. \sum_{c \equiv 0(pM)} \frac{S(n_1, n_2; c)}{c} \int_{-\infty}^{\infty} J_{2ir} \left(\frac{4\pi \sqrt{n_1 n_2}}{c} \right) r h(r) \frac{dr}{\cosh \pi r} \right\}.
\end{aligned}$$

2.3.2 The diagonal contribution

The diagonal terms contribute at most $\sum_{n=N}^{2N} |a_n|^2$ times

$$KG(pM)^{1+\epsilon} \ll (KM)^\epsilon pKGM \ll (KNM)^{2\epsilon} \max\{KGM, N\}.$$

Thus it remains to treat the off-diagonal term.

2.3.3 The off-diagonal contribution

Let $C = NK^{\frac{1}{4}\epsilon-1}$, for $c > C$, the integral

$$\frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir} \left(\frac{4\pi \sqrt{n_1 n_2}}{c} \right) \frac{r h(r)}{\cosh \pi r} dr$$

can be made as small as we wish (see P. 631 of [21]). Hence, it suffices to consider those c less than $NK^{\frac{1}{4}\epsilon-1}$. We have the following asymptotic expansion for $J_{2ir}(x)$ (P.627 in [21])

$$J_{2ir}(x) = \frac{e^{i\omega_x(2r) + \pi r - i\frac{\pi}{4}}}{\pi \sqrt{2}} \left(\sum_{m=0}^{B-1} t_m (4r^2 + x^2)^{-\frac{m}{2} - \frac{1}{4}} + O\left((4r^2 + x^2)^{-\frac{B}{2}}\right) \right)$$

where $\omega_x(r) = \sqrt{r^2 + x^2} - r \log \left(\frac{r}{x} + \sqrt{\left(\frac{r}{x}\right)^2 + 1} \right)$ and $\sum_{n=0}^{\infty} |t_n| S^{-n} < \infty$ for some $S > 0$. We assume $K \geq S$ from now on.

Using the above asymptotic expansion we will show for $K^{1-\frac{\epsilon}{4}} \ll x \ll \frac{N}{pM}$,

$$\int_{-\infty}^{\infty} J_{2ir}(x) rh(r) \frac{dr}{\cosh \pi r} \ll (MKN)^{-10}.$$

Indeed,

$$\begin{aligned} & \int_{-\infty}^{\infty} J_{2ir}(x) rh(r) \frac{dr}{\cosh \pi r} \\ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}} \sum_{m=0}^{B-1} t_m \int_{-\infty}^{\infty} rh(r) \frac{e^{\pi r}}{\cosh \pi r} (4r^2 + x^2)^{-\frac{1}{4}-\frac{m}{2}} e^{i\omega_x(2r)} dr \\ &+ O\left(\int_{-\infty}^{\infty} rh(r) \frac{e^{\pi r}}{\cosh \pi r} (4r^2 + x^2)^{-\frac{B}{2}} dr\right). \end{aligned}$$

Taking $B = \left\lfloor \frac{\log(MNK)}{\log K} 100 \right\rfloor$ and bounding trivially

$$\int_{-\infty}^{\infty} rh(r) \frac{e^{\pi r}}{\cosh \pi r} (4r^2 + x^2)^{-\frac{B}{2}} dr \ll KK^{-B}G \ll (MKN)^{-10}.$$

For each $0 \leq m \leq B-1$, let

$$h_{m,x}(r) = rh(r) \frac{e^{\pi r}}{\cosh \pi r} (4r^2 + x^2)^{-\frac{1}{4}-\frac{m}{2}}.$$

Then by Leibniz's rule,

$$\begin{aligned} h_{m,x}^{(l)}(r) &= \sum_{j=0}^l \binom{l}{j} \left[\frac{d^{l-j}}{dr^{l-j}} \left(rh(r) \frac{e^{\pi r}}{\cosh \pi r} \right) \right] \\ &\times \left(\frac{d^j}{dr^j} (4r^2 + x^2)^{-\frac{1}{4}-\frac{m}{2}} \right). \end{aligned}$$

And by induction, for $K \leq r \leq K+G$,

$$\frac{d^j}{dr^j} (4r^2 + x^2)^{-\frac{1}{4}-\frac{m}{2}} \ll m^j K^{-\frac{1}{2}-m-j}$$

and

$$\begin{aligned} & \frac{d^{l-j}}{dr^{l-j}} \left(rh(r) \frac{e^{\pi r}}{\cosh \pi r} \right) \\ &= 2 \frac{d^{l-j}}{dr^{l-j}} (rh(r)) - 2 \frac{d^{l-j}}{dr^{l-j}} \left(rh(r) \frac{e^{-\pi r}}{\cosh \pi r} \right) \\ &\ll G^{-(l-j)} K. \end{aligned}$$

Hence,

$$\begin{aligned} h_{m,x}^{(l)}(r) &\ll \sum_{j=0}^l \binom{l}{j} m^j G^{-l+j} K^{\frac{1}{2}-m-j} = G^{-l} K^{\frac{1}{2}-m} \sum_{j=0}^l \binom{l}{j} \left(\frac{Gm}{K}\right)^j \\ &= G^{-l} K^{\frac{1}{2}-m} \left(\frac{Gm}{K} + 1\right)^l \leq G^{-l} K^{\frac{1}{2}-m} B^l. \end{aligned}$$

We can now estimate $\int_{-\infty}^{\infty} h_{m,x}(r) e^{i\omega_x(2r)} dr$.

Lemma 2.3.1.

$$\int_{-\infty}^{\infty} h_{m,x}(r) e^{i\omega_x(2r)} dr \ll (KMN)^{-10}.$$

Proof. We first have to estimate the size of $\omega'_x(2r)$ in the range $K \leq r \leq K+G$ and

$$K^{1-\frac{\epsilon}{4}} \ll x := x_{n_1, n_2} = \frac{4\pi\sqrt{n_1 n_2}}{c} \ll \frac{N}{pM}.$$

Let $u = \frac{2r}{x}$, then $\frac{pKM}{N} \ll u \ll K^{\frac{\epsilon}{4}}$ and

$$\omega'_x(2r) = -\log(u + \sqrt{u^2 + 1}) = -u \frac{\log(u + \sqrt{u^2 + 1})}{u} < 0.$$

We will find a lower bound of $f(u) := \frac{\log(u + \sqrt{u^2 + 1})}{u}$ in the given range.

The derivative

$$f'(u) = \frac{u - \sqrt{u^2 + 1} \log(\sqrt{u^2 + 1} + u)}{u^2 \sqrt{u^2 + 1}} := \frac{h(u)}{u^2 \sqrt{u^2 + 1}}$$

has exactly one zero u_0 in $(0, \infty)$ and $f'(u) > 0$ for all $0 < u < u_0$; $f'(u) < 0$ for all $u_0 < u$. To see this notice that $h'(u) = \frac{u \log(\sqrt{u^2 + 1} + u)}{\sqrt{u^2 + 1}} < 0$ for all $u > 0$ (hence $h(u)$ is decreasing in $(0, \infty)$), $\lim_{u \rightarrow 0^+} h(u) > 0$ and $\lim_{u \rightarrow \infty} h(u) = -\infty$. Thus for $\frac{pMK}{N} \ll u \ll K^{\frac{\epsilon}{4}}$,

$$f(u) \gg \min\left\{\lim_{u \rightarrow 0^+} f(u), f(K^{\frac{\epsilon}{4}})\right\} \geq K^{-\frac{\epsilon}{4}}.$$

Hence,

$$\omega'_x(2r)^{-1} \ll \frac{x}{r} K^{\frac{\epsilon}{4}}$$

and by induction,

$$\omega_x^{(s)}(2r) \ll r^{-s+1}.$$

Recall also that $h_{m,x}^{(l)}(r) \ll G^{-l} K^{\frac{1}{2}-m} B^l$, we are done after applying integration by parts repeatedly to the oscillatory integral $\int_{-\infty}^{\infty} h_{m,x}(r) e^{i\omega_x(2r)} dr$ $\square$

The proof of the Theorem is completed after applying the above Lemma and Weil's bound of the Kloosterman sum to the off-diagonal term.

CHAPTER 3

THE SECOND MOMENT OF THE CENTRAL VALUES OF THE SYMMETRIC SQUARE L -FUNCTIONS

In this chapter, we prove Theorem 1.2.5. Since we assume the level 1 throughout, we write H_k instead of $H_{k,1}$. The proof is an adaptation of the proof of the main Theorem in [24]. In their proof they have to consider shifted convolution sum of the Fourier coefficients of Hecke forms. In particular, in the unshifted case, they have to consider a sum of the form

$$\sum_{K \leq k \leq K+G} \sum_{f \in H_k} \sum_{r_1, r_2 \sim K} \frac{a_f(r_1^2) a_f(r_2^2)}{\sqrt{r_1 r_2}}.$$

The fact that they can restrict the sum to those r_1, r_2 of size K (i.e. $r_1, r_2 \sim K$) is due to the presence of a compactly supported function coming from the incomplete Poincare series. In our case such feature does not exist, hence we have to consider

$$\sum_{K \leq k \leq K+G} \sum_{f \in H_k} \sum_{r_1, r_2 \ll K^{1+\epsilon}} \frac{a_f(r_1^2) a_f(r_2^2)}{\sqrt{r_1 r_2}}.$$

To overcome the difficulty thus presented, we apply a large sieve type inequality (see Lemma 3.1.4) so that we can consider instead

$$\sum_{K \leq k \leq K+G} \sum_{f \in H_k} \sum_{K^{\frac{2}{3}+\epsilon} \ll r_1, r_2 \ll K^{1+\epsilon}} \frac{a_f(r_1^2) a_f(r_2^2)}{\sqrt{r_1 r_2}}.$$

which turns out to be good enough for the adoption of the proof of Luo and Sarnak and the result obtained here is in agreement with theirs, i.e. we prove that the theorem is true when we take $G \geq K^{\frac{1}{3}}$.

In the first section we shorten the range of summation of $L\left(\frac{1}{2}, \text{sym}^2 f\right)$ using an approximate functional equation and a large sieve type inequality for holomorphic cusp forms. In the second and third section we follow P.887-889 of [24] to arrive at our main result.

3.1 Shortening the length of summation

We start with the approximate functional equation.

Lemma 3.1.1. *For $f \in H_k$,*

$$L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n \ll k^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) + O_\epsilon\left(k^{-\frac{1}{2}+\epsilon}\right)$$

where

$$\tilde{V}_k(n) = \frac{1}{2\pi i} \int_{\text{Re } y=A} G(y) \frac{\zeta(1+2y)}{y} \left(\frac{k}{n}\right)^y dy$$

for any $A > 0$ with

$$G(y) = \frac{\pi^{-\frac{3}{2}y} 2^{-y} \Gamma\left(\frac{y}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

decays exponentially on vertical lines.

Proof. By Lemma 2.2 of [16],

$$L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n=1}^{\infty} \frac{a_f(n^2)}{\sqrt{n}} V_k(n)$$

where

$$V_k(x) = \frac{1}{2\pi i} \int_{\text{Re } y=A} \frac{L_\infty\left(\frac{1}{2} + y\right)}{L_\infty\left(\frac{1}{2}\right)} \zeta(1+2y) x^{-y} \frac{dy}{y}$$

$$V_k(n) \ll_A \left(\frac{k}{n}\right)^A \text{ for any } A > 0.$$

By Corollary 2.3 of [5],

$$L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n \leq k^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} V_k(n) + O_{\epsilon, A}(k^{-A})$$

for all $A > 0$.

By (2.7) in [16],

$$\begin{aligned} L\left(\frac{1}{2}, \text{sym}^2 f\right) &= 2 \sum_{n \leq k^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \\ &+ O\left(\sum_{n \leq k^{1+\epsilon}} \frac{|a_f(n^2)|}{\sqrt{n}} k^{-1} \left(\frac{k}{n}\right)^\epsilon\right) + O_{\epsilon, A}(k^{-A}) \end{aligned}$$

for all $A > 0$. Using Deligne's bound,

$$L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n \leq k^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) + O_\epsilon(k^{-\frac{1}{2}+2\epsilon}).$$

□

Since K is large, for each $K \leq k \leq 3K$ we have

$$L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n \leq K^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) + O_\epsilon(K^{-\frac{1}{2}+2\epsilon}).$$

The weight function $\tilde{V}_t$ and its derivatives behave nicely with respect to t . More precisely,

Lemma 3.1.2.

$$\frac{d^a}{dt^a} \tilde{V}_t(n) \ll \frac{1}{t^a} \left(\frac{t}{n}\right)^A$$

for all $a \geq 0$, $A > 0$.

Proof. For any $a \geq 0$, $A > 0$, $\frac{d^a}{dt^a} \tilde{V}_t(n) =$

$$\begin{aligned} &\frac{1}{2\pi i} \frac{1}{t^a} \int_{(A)} G(y) \zeta(1+2y) G(y) (y-1) \cdots (y-a+1) \left(\frac{t}{n}\right)^y dy \\ &\ll \frac{1}{t^a} \left(\frac{t}{n}\right)^A \end{aligned}$$

□

We can now restrict the sum over n to those n less than $K^{1+\epsilon}$.

Lemma 3.1.3.

$$\begin{aligned} & \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)} \frac{L^2\left(\frac{1}{2}, \text{sym}^2 f\right)}{k-1} \\ & \ll \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \left| \sum_{n \leq K^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2 \times \\ & \quad \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} + O(G). \end{aligned}$$

Proof. By Lemma 2.1,

$$\begin{aligned} & \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)} \frac{L^2\left(\frac{1}{2}, \text{sym}^2 f\right)}{k-1} \\ & = \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \\ & \quad \left| \sum_{n \leq K^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) + O_\epsilon\left(k^{-\frac{1}{2}+\epsilon}\right) \right|^2. \end{aligned}$$

Since $|a+b|^2 \ll |a|^2 + |b|^2$, the error term contributes

$$O\left(\sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{K^{-1+2\epsilon}}{L(1, \text{sym}^2 f)(k-1)}\right).$$

Using the following bound (see P. 6 of [1]),

$$k^{-\epsilon} \ll L(1, \text{sym}^2 f)$$

The error term contributes $O(G)$. □

The following Large Sieve type inequality will be crucial later, making the adoption of the proof in [24] possible.

Lemma 3.1.4. *Notations as before. Let $\theta(t, x)$ be a smooth function such that*

$$\partial_2 \theta(t, x) \ll x^{-1} t^\epsilon \text{ for all } \epsilon > 0$$

$$\theta(t, x) \ll x^\epsilon \text{ for all } \epsilon > 0$$

and let $\{b_n\}_{n=N}^{2N}$ be a set of complex numbers. Then for $1 \leq G \leq K$

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n=N}^{2N} a_f(n) \theta(k, n) b_n \right|^2 \ll (KN)^\epsilon (KG + N) \sum_{n=N}^{2N} |b_n|^2.$$

Proof. Let $A_f(x) = \sum_{n=N}^x a_f(n) b_n$. Then by the Abel summation formula,

$$\sum_{n=N}^{2N} a_f(n) b_n \theta(k, n) = A_f(2N) \theta(k, 2N) - \int_N^{2N} A_f(x) \partial_2 \theta(k, x) dx.$$

Hence by Cauchy-Schwartz inequality,

$$\begin{aligned} \left| \sum_{n=N}^{2N} a_f(n) b_n \theta(k, n) \right|^2 &\ll |A_f(2N)|^2 N^\epsilon + \int_N^{2N} |A_f(x)|^2 dx \int_N^{2N} |\partial_2 \theta(k, x)|^2 dx \\ &\ll |A_f(2N)|^2 N^\epsilon + N^\epsilon \int_N^{2N} |A_f(x)|^2 dx \int_N^{2N} x^{-2} k^\epsilon dx. \end{aligned}$$

Using Lemma 8 of [15], we get

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n=N}^{2N} a_f(n) \theta(k, n) b_n \right|^2 \ll (KN)^\epsilon (KG + N) \sum_{n=N}^{2N} |b_n|^2.$$

□

Taking $\theta(t, x) = \tilde{V}_t(x)$, we get

$$\begin{aligned} b_n &= \begin{cases} n^{-\frac{1}{4}} & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases} \\ \sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n \ll K^{\frac{2}{3} + \epsilon}} \frac{a_f(n^2) \tilde{V}_k(n)}{\sqrt{n}} \right|^2 \\ = \sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n \ll K^{\frac{4}{3} + 2\epsilon}} a_f(n) \theta(k, n) b_n \right|^2. \end{aligned}$$

We then apply the large sieve inequality and sum dyadically, for each $N = 2^t$, $0 \leq t \leq \left(\frac{4}{3} + 2\epsilon\right) \log_2 K$,

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n=N}^{2N} a_f(n) \theta(k, n) b_n \right|^2 \ll K^\epsilon (KG + N) \sum_{n=N}^{2N} |b_n|^2.$$

Summing over all these N , we get

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n \ll K^{\frac{2}{3} + \epsilon}} \frac{a_f(n^2) \tilde{V}_k(n)}{\sqrt{n}} \right|^2 \ll K^\epsilon (KG + K^{\frac{4}{3} + 2\epsilon}) \ll K^{1+\epsilon} G.$$

Hence from now on, we can consider instead

$$\sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \left| \sum_{K^{\frac{2}{3} + \epsilon} \ll n \ll K^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2.$$

3.2 Further manipulation

We then apply Lemma 3.1.4 and sum up dyadically.

For each $N = 2^t$, $0 \leq t \leq \left(\frac{4}{3} + 2\epsilon\right) \log_2 K$,

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n=N}^{2N} a_f(n) \theta(k, n) b_n \right|^2 \ll K^\epsilon (KG + N) \sum_{n=N}^{2N} |b_n|^2.$$

Summing over all these N , we get

$$\sum_{\substack{K \leq k \leq K+G \\ k \text{ even}}} \sum_{f \in H_k} \left| \sum_{n \ll K^{\frac{2}{3} + \epsilon}} \frac{a_f(n^2) \tilde{V}_k(n)}{\sqrt{n}} \right|^2 \ll K^\epsilon (KG + K^{\frac{4}{3} + 2\epsilon}) \ll K^{1+\epsilon} G.$$

Hence from now on, we can consider instead

$$\begin{aligned} \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \\ \left| \sum_{K^{\frac{2}{3} + \epsilon} \ll n \ll K^{1+\epsilon}} \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2. \end{aligned} \tag{3.2.1}$$

We now perform a smooth dyadic argument. Let η be a C^∞ function such that

$$\eta(x) = \begin{cases} 1 & \text{if } x \in (1, 2] \\ 0 & \text{if } x \notin (\frac{1}{2}, 3); \end{cases}$$

$S(x) = \sum_{j=0}^{\infty} \eta\left(\frac{x}{2^j}\right)$; and let for each $0 \leq j$, $g_{2^j}(x) = \frac{\eta\left(\frac{x}{2^j}\right)}{S(x)}$. Then for any $x \geq 1$,

$\sum_{j=0}^{\infty} g_{2^j}(x) = 1$. Thus (3.2.1) =

$$\sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \quad (3.2.2)$$

$$\left| \sum_{j=0}^{\infty} \sum_{K^{\frac{2}{3}+\epsilon} \ll n \ll K^{1+\epsilon}} g_{2^j}(n) \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2$$

Since for $K^{\frac{2}{3}+\epsilon} \ll n \ll K^{1+\epsilon}$, $g_{2^j}(n) = 0$ whenever $j \geq (1+2\epsilon) \log_2 K$ or $j \leq \left(\frac{2}{3} - \epsilon\right) \log_2 K$. Let $v_1 = \left(\frac{2}{3} - \epsilon\right) \frac{\log K}{\log 2}$, $v_2 = (1+2\epsilon) \frac{\log K}{\log 2} - 1$, then (3.2.2)

$$\begin{aligned} &\leq \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \\ &\quad \sum_{t=v_1}^{v_2} \left| \sum_{n=1}^{\infty} g_{2^t}(n) \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2 \left(\sum_{t=v_1}^{v_2} 1 \right) \\ &\ll K^\epsilon \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \\ &\quad \sum_{t=v_1}^{v_2} \left| \sum_{n=1}^{\infty} g_{2^t}(n) \frac{a_f(n^2)}{\sqrt{n}} \tilde{V}_k(n) \right|^2. \end{aligned}$$

We will show for each $T = 2^t$, $v_1 \leq t \leq v_2$,

$$\begin{aligned} &K^\epsilon \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{f \in H_k} \frac{2\pi^2}{L(1, \text{sym}^2 f)(k-1)} \times \\ &\quad \left| \sum_{n \geq 1} \frac{g_T(n) a_f(n^2) \tilde{V}_k(n)}{\sqrt{n}} \right|^2 \ll \frac{T^2 K^\epsilon}{K G^2}. \end{aligned}$$

By Petersson formula, the above sum

$$= K^\epsilon \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{n_1, n_2} \frac{g_T(n_1)g_T(n_2)\tilde{V}_k(n_1)\tilde{V}_k(n_2)}{\sqrt{n_1 n_2}} \times \\ \left[\delta_{n_1, n_2} + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(n_1^2, n_2^2; c)}{c} J_{k-1}\left(\frac{4\pi n_1 n_2}{c}\right) \right].$$

By (5.102) in [19], $J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}$, so the contribution from those $c > \frac{24\pi n_1 n_2}{K}$ is exponentially small. Thus it suffices to consider those c less than or equal to $\frac{24\pi n_1 n_2}{K}$, i.e.

$$K^\epsilon \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{n_1, n_2} \frac{g_T(n_1)g_T(n_2)\tilde{V}_k(n_1)\tilde{V}_k(n_2)}{\sqrt{n_1 n_2}} \quad (3.2.3) \\ \times \left[\delta_{n_1, n_2} + 2\pi i^{-k} \sum_{c \leq \frac{24\pi n_1 n_2}{K}} \frac{S(n_1^2, n_2^2; c)}{c} J_{k-1}\left(\frac{4\pi n_1 n_2}{c}\right) \right].$$

Note that the diagonal contribution is $\ll K^{2\epsilon}G$.

3.3 The off-diagonal contribution

Thus, it remains to treat the off-diagonal term, that is K^ϵ times

$$\sum_{n_1, n_2} \frac{g_T(n_1)g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \leq \frac{24\pi n_1 n_2}{K}} \frac{S(n_1^2, n_2^2; c)}{c} \times \quad (3.3.1)$$

$$\sum_{k \text{ even}} i^{-k} h\left(\frac{k-K-1}{G}\right) \tilde{V}_k(n_1)\tilde{V}_k(n_2) J_{k-1}\left(\frac{4\pi n_1 n_2}{c}\right).$$

3.3.1 Restricting the sum over c

We now show that we can further restrict the sum over c to those $c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}$.

The proof is essentially the same as on P.880-882 of [24].

Let $f(t) := f_{n_1, n_2}(t) = h\left(\frac{t-K}{G}\right) \tilde{V}_t(n_1) \tilde{V}_t(n_2)$.

By Lemma 4.1 in [24],

$$\begin{aligned} & \sum_{k \text{ even}} i^{-k} h\left(\frac{k-K-1}{G}\right) \tilde{V}_t(n_1) \tilde{V}_t(n_2) J_{k-1}\left(\frac{4\pi n_1 n_2}{c}\right) \\ &= \int_{-\infty}^{\infty} \hat{f}(t) \sin\left(\frac{4\pi n_1 n_2}{c} \cos 2\pi t\right) dt. \end{aligned}$$

When $c \gg \frac{n_1 n_2 K^{\epsilon/2}}{KG}$, $\frac{4\pi n_1 n_2}{c} G^{-4} \ll K^{1-\epsilon/2} G^{-3} \ll K^{-\epsilon/2}$.

Using (4.4), (4.5), Lemma 4.2 in [24] and in view of the fact that $\int_{-\infty}^{\infty} |\hat{f}(t) t^r| dt \ll$

G^{-r} (since $\frac{d^r}{dt^r} h\left(\frac{t-K}{G}\right) \ll G^{-r}$ and see also Lemma 3), the sum

$$\begin{aligned} & \sum_{n_1, n_2} \frac{g_T(n_1) g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \gg \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{c} \times \\ & \int_{-\infty}^{\infty} \hat{f}(t) \sin\left(\frac{4\pi n_1 n_2}{c} \cos 2\pi t\right) dt \\ &= \sum_{n_1, n_2} \frac{g_T(n_1) g_T(n_2)}{\sqrt{n_1 n_2}} \left| \sum_{c \gg \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{c} \times \right. \\ & \left. \int_0^{\infty} f^{(p)}\left(\sqrt{\frac{8\pi n_1 n_2 y}{c}}\right) \sin\left(y + \frac{4\pi n_1 n_2}{c} - \frac{\pi}{4}\right) \frac{dy}{\sqrt{y}} \right| + O(K^{-A}) \end{aligned}$$

for all $A > 0$ upon taking $p > \lfloor \frac{2A+200}{\epsilon} \rfloor + 1$. Note that

$$f^{(p)}\left(\sqrt{\frac{8\pi n_1 n_2 y}{c}}\right) = h^{(p)}\left(\frac{\sqrt{\frac{8\pi n_1 n_2}{c}} y - K}{G}\right),$$

hence $\frac{1}{\sqrt{y}} \sim \sqrt{\frac{8\pi n_1 n_2}{c}} \frac{1}{K}$, and

$$\begin{aligned} & \frac{d^j}{dy^j} f^{(p)}\left(\sqrt{\frac{8\pi n_1 n_2 y}{c}}\right) \\ & \ll \max \left\{ \left(\frac{1}{G} \sqrt{\frac{8\pi n_1 n_2}{cy}}\right)^j, \dots, y^{-j+1} \left(\frac{1}{G} \sqrt{\frac{8\pi n_1 n_2}{cy}}\right) \right\}. \end{aligned}$$

If $\frac{n_1 n_2}{c} \ll \frac{KG}{K^{\frac{\epsilon}{2}}}$ then we get

$$\frac{1}{G} \sqrt{\frac{8\pi n_1 n_2}{cy}} \ll K^{-\frac{\epsilon}{2}}$$

and

$$y^{-1} \ll \frac{G}{K} K^{-\frac{\epsilon}{2}}.$$

Hence if $\frac{n_1 n_2}{c} \ll \frac{KG}{K^{\frac{\epsilon}{2}}}$, by applying partial integration sufficiently many times, we see that

$$\int_0^\infty f^{(p)} \left(\sqrt{\frac{8\pi n_1 n_2 y}{c}} \right) \sin \left(y + \frac{4\pi n_1 n_2}{c} - \frac{\pi}{4} \right) \frac{dy}{\sqrt{y}}$$

is negligible. Hence the total contribution is small. We can now concentrate on

$$\sum_{n_1, n_2} \frac{g_T(n_1) g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{c} \int_{-\infty}^\infty \hat{f}(t) \sin \left(\frac{4\pi n_1 n_2}{c} \cos 2\pi t \right) dt.$$

3.3.2 Applying the stationary phase lemma

As in P.882 of [24], we can consider instead

$$\begin{aligned} & \max_{p_1 + p_2 + p_3 \leq 14N} \left\{ \left| \sum_{n_1, n_2} \frac{g_T(n_1) g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{c} e \left(\frac{2n_1 n_2}{c} \right) \right. \right. \\ & \times \int_0^\infty \frac{\partial^{p_1} h}{\partial y^{p_1}} \left(\frac{y - K - 1}{G} \right) \frac{\partial^{p_2} \tilde{V}_y}{\partial y^{p_2}}(n_1) \frac{\partial^{p_3} \tilde{V}_y}{\partial y^{p_3}}(n_2) \times \\ & \left. \left. \int_{-\infty}^\infty e \left(\frac{2n_1 n_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3} \pi^4 t^4 \right) + ty \right) dt dy \right| \right\}. \end{aligned}$$

We now apply the principle of stationary phase to evaluate asymptotically the integral

$$\int_{-\infty}^\infty e \left(\frac{2n_1 n_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3} \pi^4 t^4 \right) + ty \right) dt$$

as in P.883-884 of [24].

Let $f(t) = \frac{4\pi n_1 n_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3}\pi^4 t^4 \right) + 2\pi t y$. Then

$$f'(t) = \frac{4\pi n_1 n_2}{c} \left(-4\pi^2 t + \frac{8}{3}\pi^4 t^3 \right) + 2\pi y$$

and

$$f''(t) = \frac{4\pi n_1 n_2}{c} (-4\pi^2 + 8\pi^4 t^2).$$

Since $-4\pi^2 t + \frac{8}{3}\pi^4 t^3$ has the roots $0, \sqrt{\frac{3}{2}}\pi^{-1}$ and $-\sqrt{\frac{3}{2}}\pi^{-1}$ and $\frac{yc}{4\pi n_1 n_2} \sim \frac{Kc}{n_1 n_2} \ll \frac{K^{\epsilon/2}}{G}$, the integral has stationary phases at $t_1, \sqrt{\frac{3}{2}}\pi^{-1} - t_2, -\sqrt{\frac{3}{2}}\pi^{-1} - t_3$ with $t_i = t_i \left(y, \frac{4\pi n_1 n_2}{c} \right) \sim \frac{yc}{4\pi n_1 n_2} \ll \frac{K^{\epsilon/2}}{G}$ for $i = 1, 2, 3$ by the mean value theorem.

Let β be any critical point as above. Then β satisfies $\beta_y = \frac{c}{8\pi^2(1 - 2\pi^2\beta^2)}$,

$$\beta_{n_i} = \frac{-yc}{8\pi^2 n_1 n_2 n_i (1 - 2\pi^2\beta^2)} \text{ and } |\beta_{n_i}| \sim \frac{cK}{n_1 n_2 n_i}. \quad (3.3.2)$$

We have the same asymptotic expression for our oscillatory integral as in P.884 of [24], where their x is our $\frac{4\pi n_1 n_2}{c}$. To see that the $O\left(\frac{c}{n_1 n_2}\right)$ term contributes $O(GK^\epsilon)$ one notices that Weil's bound gives $S(n_1^2, n_2^2; c) \ll_\epsilon (n_1^2, n_2^2, c)^{\frac{1}{2}c^{\frac{1}{2}+\epsilon}}$. Hence the contribution of the $O\left(\frac{c}{n_1 n_2}\right)$ term is

$$\begin{aligned} &\ll G \sum_{n_1, n_2} \frac{g_T(n_1)g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{(n_1^2, n_2^2, c)^{\frac{1}{2}c^{\frac{1}{2}+\epsilon}}}{n_1 n_2} \\ &\ll G \frac{1}{T^3} \sum_{T \leq n_2 < 2T} \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} \sum_{T \leq n_1 < 2T} (n_1^2, c)^{\frac{1}{2}c^{\frac{1}{2}+\epsilon}} \\ &\ll \frac{G}{T} \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{\frac{1}{2}+\epsilon} \ll G \frac{1}{T} \left(\frac{T^2 K^{\epsilon/2}}{KG} \right)^{\frac{3}{2}+\epsilon} \ll G \frac{K^\epsilon T^2}{K^{\frac{3}{2}} G^{\frac{3}{2}}}. \end{aligned}$$

As stated in P.884 of [24], we only have to consider

$$\max_{p_1+p_2+p_3=p \leq 14N} \left\{ \left| \sum_{n_1, n_2} \frac{g_T(n_1)g_T(n_2)}{\sqrt{n_1 n_2}} \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{c} \right| \right\} \quad (3.3.3)$$

$$\begin{aligned} & \times e\left(\frac{2n_1n_2}{c}\right) \int_0^\infty \frac{\partial^{p_1} h}{\partial y^{p_1}} \left(\frac{y-K-1}{G}\right) \frac{\partial^{p_2} \tilde{V}_y}{\partial y^{p_2}}(n_1) \frac{\partial^{p_3} \tilde{V}_y}{\partial y^{p_3}}(n_2) \sqrt{\frac{c}{n_1n_2}} \\ & \times \left| -4\pi^2 + 8\pi^4 t_1^4 \right|^{-\frac{1}{2}} e\left(\frac{2n_1n_2}{c} \left(-2\pi^2 t_1^2 + \frac{2\pi^4}{3} t_1^4\right) + t_1 y\right) dy \Bigg\}. \end{aligned}$$

$$\begin{aligned} \text{Let } F_c(x_1, x_2) &= \int_0^\infty h^{(p_1)}(y) \partial_1^{p_2} \tilde{V}(Gy + K + 1, x_1) \partial_1^{p_3} \tilde{V}(Gy + K + 1, x_2) \\ & \times \left| -4\pi^2 + 8\pi^4 t_1^4 \right|^{-\frac{1}{2}} \\ & \times e\left(\frac{2x_1x_2}{c} \left(-2\pi^2 t_1^2 + \frac{2\pi^4}{3} t_1^4\right) + Gt_1y + (K+1)t_1\right) \frac{dy}{x_1x_2} \end{aligned}$$

with the convention that $\tilde{V}_t(x) = \tilde{V}(t, x)$ and ∂_i means differentiating with respect to the i th coordinate. Note that we are suppressing the dependence in p_1, p_2 and p_3 as it won't fundamentally change the analysis.

A typical sum in (3.3.3) is (after a change of variable)

$$G \sum_{n_1, n_2} g_T(n_1) g_T(n_2) \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{\sqrt{c}} e\left(\frac{2n_1n_2}{c}\right) F_c(n_1, n_2). \quad (3.3.4)$$

3.3.3 Applying the Poisson summation formula

As in [24], we will separate the sum over n_1, n_2 into congruence classes modulo c and then apply the Poisson summation formula. To see that all but the $(0, 0)$ -Fourier transform are insignificant, we need the following lemma.

Lemma 3.3.1. *For $K^{\frac{2}{3}+\epsilon} \ll x_1, x_2 \ll K^{1+\epsilon}$, $\frac{x_1x_2}{c} \gg \frac{KG}{K^{\epsilon/2}}$,*

$$\frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} F_c(x_1, x_2) \ll K^{-\frac{\epsilon(m_1+m_2)}{2}}$$

for all $m_1, m_2 > 0$.

Remark 3.3.2. *The fact that we can assume $x_1, x_2 \gg K^{\frac{2}{3}+\epsilon}$ is crucial here. This is the main point of departure from [24] in which they have $x_1, x_2 \sim K$.*

Proof. (of Lemma 3.3.1) From

$$\tilde{V}(t, u) = \frac{1}{2\pi i} \int_{(A)} G(y) \frac{\zeta(1+2y)}{y} \left(\frac{t}{u}\right)^y dy,$$

it is easy to see that $\partial_1^a \partial_2^b \tilde{V}(t, u) \ll \frac{1}{t^a} \frac{1}{u^b} \left(\frac{t}{u}\right)^A$ for all $a > 0$, $b > 0$ and $A > 0$. We see from (3.3.2) that

$$\begin{aligned} & \frac{\partial^{m_1+m_2}}{\partial x_1^{m_1} \partial x_2^{m_2}} F_c(x_1, x_2) \ll \left(K \frac{\partial t_1}{\partial x_1}\right)^{m_1} \left(K \frac{\partial t_1}{\partial x_2}\right)^{m_2} \\ & + \left(\frac{x_1 x_2}{c} t_1 \frac{\partial t_1}{\partial x_1}\right)^{m_1} \left(\frac{x_1 x_2}{c} t_1 \frac{\partial t_1}{\partial x_2}\right)^{m_2} \\ & \ll \left(\frac{K^2 c}{x_1^2 x_2}\right)^{m_1} \left(\frac{K^2 c}{x_1 x_2^2}\right)^{m_2} \ll K^{-\frac{\epsilon(m_1+m_2)}{2}}. \end{aligned}$$

□

Then we can consider instead

$$G \sum_{n_1, n_2} g_T(n_1) g_T(n_2) \sum_{c \ll \frac{n_1 n_2 K^{\epsilon/2}}{KG}} \frac{S(n_1^2, n_2^2; c)}{\sqrt{c}} e\left(\frac{2n_1 n_2}{c}\right) F_c(n_1, n_2). \quad (3.3.5)$$

Since $\text{supp } g_T \subset (T/2, 3T)$, (3.3.5) can be replaced by

$$\begin{aligned} & G \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} \frac{1}{\sqrt{c}} \sum_{a, b(c)} S(a^2, b^2; c) e\left(\frac{2ab}{c}\right) \\ & \times \sum_{\substack{n_1 \equiv a(c) \\ n_2 \equiv b(c)}} F_c(n_1, n_2) g_T(n_1) g_T(n_2) \\ & = G \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{-\frac{5}{2}} \sum_{a, b(c)} \sum_{u, v(c)} S(a^2, b^2; c) e\left(\frac{2ab + ua + vb}{c}\right) \times \\ & \sum_{n_1, n_2} F_c(n_1, n_2) g_T(n_1) g_T(n_2) e\left(-\frac{u}{c} n_1\right) e\left(-\frac{v}{c} n_2\right). \end{aligned}$$

By the Poisson summation formula, (3.3.5) can be replaced by

$$G \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{-\frac{5}{2}} \sum_{a, b(c)} \sum_{u, v(c)} S(a^2, b^2; c) e\left(\frac{2ab + ua + vb}{c}\right) \times$$

$$\sum_{m_1, m_2} \iint_{\mathbb{R}^2} F_c(x_1, x_2) e\left(\left(-\frac{u}{c} + m_1\right) x_1\right) e\left(\left(-\frac{v}{c} + m_2\right) x_2\right) dx_1 dx_2.$$

3.3.4 Application of the two dimensional Van der Corput lemma

Picking $|u|, |v| \leq \frac{c}{2}$ and integrating by parts sufficiently many times (using Lemma 3.3.1), (3.3.5) can be replaced by

$$\begin{aligned} & G \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{-\frac{5}{2}} \sum_{a, b(c)} \sum_{u, v(c)} S(a^2, b^2; c) e\left(\frac{2ab + ua + vb}{c}\right) \\ & \times \iint_{R_T} F_c(x_1, x_2) e\left(\frac{-ux_1 - vx_2}{c}\right) dx_1 dx_2 \end{aligned} \quad (3.3.6)$$

where R_T is the rectangle in which $x_1 \sim T$ and $x_2 \sim T$.

Let

$$\phi_{u, v, c}(x_1, x_2, z) = \frac{2x_1 x_2}{c} \left(-2\pi^2 t_1^2 + \frac{2\pi^4}{3} t_1^4 \right) + Gz t_1 + (K + 1) t_1$$

where t_1 depends on both x_1, x_2 and z (see Section 3.3.2).

Then, (3.3.6)

$$\begin{aligned} & = G \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{-\frac{5}{2}} \sum_{a, b(c)} \sum_{u, v(c)} S(a^2, b^2; c) e\left(\frac{2ab + ua + vb}{c}\right) \times \\ & \int_0^\infty \iint_{R_T} h^{(p_1)}(z) \partial_1^{p_2} \tilde{V}(Gz + K - 1, x_1) \partial_1^{p_3} \tilde{V}(Gz + K - 1, x_2) \times \\ & |4\pi^2 - 8\pi^4 t_1^2|^{-\frac{1}{2}} e(\phi_{u, v, c}(x_1, x_2, z)) e\left(\frac{-ux_1 - vx_2}{c}\right) \frac{dx_1 dx_2}{x_1 x_2} dz \end{aligned}$$

Differentiating $\phi_{u, v, c}$ (which we write as ϕ for brevity),

$$\begin{aligned} \phi_{x_1} & = \frac{2x_2}{c} \left(-2\pi^2 t_1^2 + \frac{2\pi^4}{3} t_1^4 \right) + \frac{2x_1 x_2}{c} \left(-4\pi^2 t_1 \frac{\partial t_1}{\partial x_1} + \frac{8\pi^4}{3} t_1^3 \frac{\partial t_1}{\partial x_1} \right) \\ & + (Gz + K + 1) \frac{\partial t_1}{\partial x_1}. \end{aligned}$$

By (2.5)

$$\frac{\partial t_1}{\partial x_1} = \frac{c(Gz + K + 1)}{8\pi^2 x_1^2 x_2 (1 - 2\pi^2 t_1^2)}.$$

Hence,

$$\phi_{x_1} \sim \frac{-K^2 c}{16\pi^2 x_1^2 x_2} \text{ and } \phi_{x_2} \sim \frac{-K^2 c}{16\pi^2 x_1 x_2^2}.$$

Thus, the integral has no stationary phase unless

$$|u|, |v| \sim \frac{K^2 c^2}{T^3}.$$

More precisely, if u, v do not satisfy the above conditions, then

$$\phi_{x_1} - \frac{u}{c}, \phi_{x_2} - \frac{v}{c} \sim \frac{cK^2}{T^3}.$$

Integrating by parts A times. The oscillatory integral is bounded by

$$\left(\frac{T^3}{cK^2}\right)^A \left(\frac{1}{T}\right)^A \ll \frac{1}{c^A}.$$

Hence the contribution is negligible if $c \gg K^\epsilon$ by taking A large enough and the contribution from $c \ll K^\epsilon$ is also negligible by direct estimation.

Argue as in P.888 of [24],

$$\begin{aligned} \phi_{x_1 x_2}, \phi_{x_2 x_1} &\sim \frac{K^2 c}{16\pi^2 x_1^2 x_2^2} \sim \frac{K^2 c}{T^4}, \phi_{x_1 x_1} \sim \frac{K^2 c}{8\pi^2 x_1^3 x_2} \sim \frac{K^2 c}{T^4} \\ \phi_{x_2 x_2} &\sim \frac{K^2 c}{8\pi^2 x_1 x_2^3} \sim \frac{K^2 c}{T^4} \text{ and } \left| \frac{\partial(\phi_{x_1}, \phi_{x_2})}{\partial(x_1, x_2)} \right| \gg \frac{K^4 c^2}{T^8} \end{aligned}$$

Integrating by parts sufficiently many times, (3.3.6)

$$\begin{aligned} &= G \int_0^\infty \sum_{c \ll \frac{T^2 K^\epsilon/2}{KG}} c^{-\frac{5}{2}} \sum_{|u|, |v| \sim \frac{K^2 c^2}{T^3}} \sum_{a, b(c)} S(a^2, b^2; c) \times \\ &e \left(\frac{2ab + ua + vb}{c} \right) \iint_{R_T} h^{(p_1)}(z) \partial_1^{p_2} \tilde{V}(Gz + K - 1, x_1) \times \\ &\partial_1^{p_3} \tilde{V}(Gz + K - 1, x_2) |4\pi^2 + 8\pi^4 t_1^4|^{-\frac{1}{2}} \times \\ &e(\phi_{u, v, c}(x_1, x_2, z)) e \left(\frac{-ux_1 - vx_2}{c} \right) \frac{dx_1 dx_2}{x_1 x_2} dz + O(K^{-A}) \end{aligned}$$

for all $A > 0$.

Applying Lemma 4.7 in [24], the integral is

$$= O\left(\frac{1 + \log K + \log c}{K^2 c T^{-2}}\right).$$

Using (4.15) in [24],

$$\begin{aligned} & \sum_{a,b(c)} S(a^2, b^2; c) e\left(\frac{2ab + ua + vb}{c}\right) \\ &= \begin{cases} O\left((v, c_1) c_1^{3/2} c_2^{5/2+\epsilon}\right) & \text{if } (v, c_1) | u \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by writing $c = c_1 c_2$, $(2, c_1) = 1$ and $c_2 | 2^\infty$.

Putting all these together, (3.3.6)

$$\begin{aligned} & \ll GK^\epsilon \sum_{c \ll \frac{T^2 K^{\epsilon/2}}{KG}} c^{-\frac{5}{2}} \sum_{\substack{|u|, |v| \sim \frac{K^2 c^2}{T^3} \\ (v, c_1) | u}} (v, c_1) c_1^{\frac{3}{2}} c_2^{\frac{5}{2}+\epsilon} K^{-2} c^{-1} T^2 + O(GK^\epsilon) \\ & \ll GK^\epsilon \sum_{c_1 \ll \frac{T^2 K^{\frac{\epsilon}{2}}}{KG}} \frac{c_1^{-\frac{5}{2}-1+\frac{3}{2}} T^2}{K^2} \left(\frac{K^2 c_1^2}{T^3}\right)^2 + O(GK^\epsilon) \\ & \ll \frac{GK^{2+\epsilon}}{T^4} \sum_{c_1 \ll \frac{T^2 K^{\frac{\epsilon}{2}}}{KG}} c_1^2 + O(GK^\epsilon) \\ & \ll GK^\epsilon \frac{T^2}{KG^3} + O(GK^\epsilon) \end{aligned}$$

Summing over $T = 2^t$, $\left(\frac{2}{3} - \epsilon\right) \log_2 K \leq t \leq (1 + 2\epsilon) \log_2 K - 1$. The resulting sum is less than or equal to GK^ϵ whenever $G \geq K^{\frac{1}{3}}$ as is desired.

CHAPTER 4

THE FIRST MOMENT OF THE CENTRAL VALUES OF THE RANKIN-SELBERG L -FUNCTIONS

*It is not the critic who counts; not the man
who points out how the strong man stumbles,
or where the doer of deeds could have done
them better. The credit belongs to the man who
is actually in the arena, whose face is marred
by dust and sweat and blood; who strives
valiantly; who errs, who comes short again and
again, because there is no effort without error
and shortcoming; but who does actually strive
to do the deeds; who knows great enthusiasms,
the great devotions; who spends himself in a
worthy cause; who at the best knows in the end
the triumph of high achievement, and who at
the worst, if he fails, at least fails while daring
greatly, so that his place shall never be with
those cold and timid souls who neither know
victory nor defeat.*

—Theodore Roosevelt

In this chapter we will prove Theorem 1.2.6.

It was shown by Luo and Sarnak in [25] (see P.787-P.789 in their paper) that

$$\sum_{\substack{K \leq k \leq 2K \\ k \text{ even}}} \sum_{f \in H_k} \frac{L\left(\frac{1}{2}, \phi \otimes \text{sym}^2 f\right)}{L(1, \text{sym}^2 f)(k-1)} \ll_{\epsilon} K^{1+\epsilon}.$$

Our result improved on it in the sense that that family we average over is of size as small as $\sim K^{\frac{4}{3}}$ instead of $\sim K^2$.

In this paper we first follow the strategy of [24] with the help of a few lemmas in [25] to arrive at an expression of roughly the form

$$\begin{aligned} & \sum_c \frac{1}{c} \sum_{\substack{t_1 \sim T_1 \\ t_2 \sim T_2}} S(t_1^2, t_2^2; c) e\left(\frac{2t_1 t_2}{c}\right) \frac{\lambda_{\phi}(t_1)}{\sqrt{t_1 t_2}} \\ & \times \int_0^{\infty} g(y) \int_{-\infty}^{\infty} e\left(\frac{2t_1 t_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3}\pi^4 t^4\right) + ty\right) dt dy \end{aligned}$$

with test function g and $t_1 t_2^2 \ll K^{2+\frac{\epsilon}{4}}$. Then as in [25], we separate the arithmetic and analytic parts. Then comes the main departure from [24]. In [24], they dealt with the analytic part using the Poisson summation formula and estimated the sum of the Kloosterman sums (the arithmetic part) using evaluations of the Gauss sum. However in our situation, we can't use the Poisson summation formula, for the simple reason that $\lambda_{\phi}(t_1)$ is not a smooth function in t_1 and also that t_1 and t_2 might not be in the same range. Instead of the Poisson summation formula, we appeal to the Voronoi formula for the Maass cusp forms. The classical Voronoi formula is of roughly the same strength as the Poisson summation formula as is demonstrated in Section 4.5 of [12]. Our result can be seen as a demonstration of such principle in the $GL(2)$ case. On the other hand the arithmetic part is more delicate in our situation and require lengthy if not slightly tedious treatment.

In section 4.1 we apply the approximate functional equation and the Petersson trace formula; in section 4.2 we shorten the length of summation over c ; section 4.3, 4.5, 4.6, 4.7 and 4.9 are the core of the analytic components of the proof, involving the

stationary phase lemma, Van der Corput's Lemma and Voronoi's formula; section 4.4 and 4.8 are the core of the arithmetic components of the proof, consisting of estimating various twisted exponential sum; in section 4.10 we put everything together and finish the proof.

4.1 First manipulation

We start with the approximate functional equation

Proposition 4.1.1. (P.787 of [25])

$$L\left(\frac{1}{2}, \phi \otimes \text{sym}^2 f\right) = 2 \sum_{m,n \geq 1} \frac{\lambda_\phi(m) a_F(m, n)}{\sqrt{mn^2}} G_k(\pi^3 mn^2)$$

where

$$G_k(x) = \frac{1}{2\pi i} \int_{(2)} U_k(s) \left(\frac{4x}{(k-1)^2} \right)^{-s} \frac{ds}{s}$$

with

$$U_k(s) = (1 + T_k(s)) \frac{\Gamma\left((s + \frac{1}{2}) + 1 + it_\phi\right) \Gamma\left((s + \frac{1}{2}) + 1 - it_\phi\right)}{\Gamma\left(\frac{\frac{1}{2} + 1 + it_\phi}{2}\right) \Gamma\left(\frac{\frac{1}{2} + 1 - it_\phi}{2}\right)}$$

and

$$T_k(s) = \sum_{1 \leq r \leq 6} \frac{p_{r+1}(s)}{(k-1)^r} + O\left(\frac{(|s| + 1)^8}{(k-1)^7}\right);$$

$p_{r+1}(s)$ being polynomial of degree at most $r + 1$.

Using the Hecke relation and the Petersson formula as in P.787-788 of [25], we

have

$$\begin{aligned}
& \sum_{k \text{ even}} h\left(\frac{k-1-K}{G}\right) \sum_{f \in H_k} \frac{L\left(\frac{1}{2}, \phi \otimes \text{sym}^2 f\right)}{L(1, \text{sym}^2 f)(k-1)} \\
&= 2 \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2, t_1, t_2 \geq 1} \frac{\lambda_{\phi}(ds_1^2 t_1)}{\sqrt{s_1^2 t_1 s_2^4 t_2^2}} \\
&\times G_k(\pi^3 d^3 s_1^2 s_2^4 t_1 t_2) \left[\delta_{t_1 t_2} + 2\pi i^{-k} \sum_{c \geq 1} \frac{S(t_1^2, t_2^2; c)}{c} J_{k-1}\left(\frac{4\pi t_1 t_2}{c}\right) \right] \\
&= 2 \sum_{k \text{ even}} h\left(\frac{k-K-1}{G}\right) \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2, t \geq 1} \frac{\lambda_{\phi}(ds_1^2 t)}{s_1 t^{\frac{3}{2}} s_2^2} G_k(\pi^3 d^3 s_1^2 t^3 s_2^4) \\
&+ \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{k \text{ even}} \sum_{s_1, s_2, t_1, t_2 \geq 1} \frac{\lambda_{\phi}(ds_1^2 t_1) G_k(\pi^3 d^3 s_1^2 s_2^4 t_1 t_2)}{s_1 \sqrt{t_1 s_2^2 t_2}} \sum_{c \geq 1} \frac{S(t_1^2, t_2^2; c)}{c} \\
&\times 4\pi h\left(\frac{k-K-1}{G}\right) i^{-k} J_{k-1}\left(\frac{4\pi t_1 t_2}{c}\right).
\end{aligned}$$

Using the integral representation of $G_k(s)$, the diagonal term contributes (see P.788 of [25] for details)

$$2GL(1, \text{sym}^2 \phi) \int_0^{\infty} h(x) dx + O\left(GK^{-\frac{1}{2}+\epsilon}\right).$$

For the non-diagonal term, first notice that by the definition of $G_k(x)$ we can consider the t_1, t_2 -terms such that $t_1 t_2^2 \ll K^{2+\epsilon}$ in which case the contribution from error term below is negligible) and now writing

$$\begin{aligned}
& G_k(x) \\
&= H\left(\frac{4x}{(k-1)^2}\right) + \sum_{1 \leq r \leq 6} \frac{1}{(k-1)^r} H_r\left(\frac{4x}{(k-1)^2}\right) + O\left(\frac{1}{(k-1)^7}\right)
\end{aligned}$$

where H and H_r , $1 \leq r \leq 6$, are smooth and decay exponentially as $x \rightarrow \infty$ (see P.789 of [25]), the non-diagonal term is

$$\begin{aligned}
& \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{k \text{ even}} \sum_{s_1, s_2, t_1, t_2 \geq 1} \frac{\lambda_{\phi}(ds_1^2 t_1)}{s_1 \sqrt{t_1 s_2^2 t_2}} H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{(k-1)^2}\right) \sum_{c \geq 1} \frac{S(t_1^2, t_2^2; c)}{c} \\
&\times 4\pi h\left(\frac{k-K-1}{G}\right) i^{-k} J_{k-1}\left(\frac{4\pi t_1 t_2}{c}\right) + O(1)
\end{aligned}$$

Applying the bound $J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}$ (see (5.102) of [19]) we can reduce the range to $c \leq \frac{24\pi t_1 t_2}{K}$.

Let η be a C^∞ function such that

$$\eta(x) = \begin{cases} 1 & x \in (1, 2) \\ 0 & x \notin (\frac{1}{2}, 3); \end{cases}$$

$S(x) = \sum_{j \geq 0} \eta\left(\frac{x}{2^j}\right)$; and let for each $0 \leq j$, $g_{2^j}(x) = \frac{\eta(\frac{x}{2^j})}{S(x)}$. Applying this smooth partition of unity, the non-diagonal term is

$$\begin{aligned} &= 4\pi \sum_{T_1, T_2 | 2^\infty} \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2, t_1, t_2 \geq 1} \frac{\lambda_\phi(ds_1^2 t_1)}{s_1 \sqrt{t_1} s_2^2 t_2} g_{T_1}(t_1) g_{T_2}(t_2) \times \\ &\quad \sum_{k \text{ even}} H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{(k-1)^2}\right) \sum_{c \leq \frac{24\pi T_1 T_2}{K}} \frac{S(t_1^2, t_2^2; c)}{c} \\ &\times h\left(\frac{k-1-K}{G}\right) i^{-k} J_{k-1}\left(\frac{4\pi t_1 t_2}{c}\right) + O(1). \end{aligned}$$

Since $H(x)$ decays exponentially as $x \rightarrow \infty$, we can consider instead

(4.1.1)

$$\begin{aligned} &4\pi \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2 \geq 1} \frac{1}{s_1 s_2^2} \sum_{\substack{T_1, T_2 | 2^\infty \\ d^3 s_1^2 T_1 s_2^4 T_2^2 \ll K^{2+\frac{\epsilon}{4}}}} \sum_{t_1, t_2 \geq 1} \frac{\lambda_\phi(ds_1^2 t_1)}{\sqrt{t_1} t_2} g_{T_1}(t_1) g_{T_2}(t_2) \\ &\times \sum_{k \text{ even}} H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{(k-1)^2}\right) \sum_{c \leq \frac{24\pi T_1 T_2}{K}} \frac{S(t_1^2, t_2^2; c)}{c} \\ &\times h\left(\frac{k-1-K}{G}\right) i^{-k} J_{k-1}\left(\frac{4\pi t_1 t_2}{c}\right) + O(1). \end{aligned}$$

Using Lemma 4.2 in [24], (4.1.1) =

$$\begin{aligned} &4\pi \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2 \geq 1} \frac{1}{s_1 s_2^2} \sum_{\substack{T_1, T_2 | 2^\infty \\ d^3 s_1^2 T_1 s_2^4 T_2^2 \ll K^{2+\frac{\epsilon}{4}}}} \sum_{t_1, t_2 \geq 1} \frac{\lambda_\phi(ds_1^2 t_1)}{\sqrt{t_1} t_2} g_{T_1}(t_1) g_{T_2}(t_2) \\ &\times \sum_{c \leq \frac{24\pi T_1 T_2}{K}} \frac{S(t_1^2, t_2^2; c)}{c} \int_{-\infty}^{\infty} \hat{f}(t) \sin\left(\frac{4\pi t_1 t_2}{c} \cos 2\pi t\right) dt + O(1) \end{aligned}$$

where

$$f(t) = h\left(\frac{t-K}{G}\right) H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2}\right).$$

4.2 Further reduction to the sum over c

We assert that we can restrict the c -sum to those $c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}$. To this end, we consider the Taylor expansion of $\cos 2\pi t$, take the first two terms and treat the rest trivially, precisely, let $x = \frac{2\pi t_1 t_2}{c}$ then for each $N \in \mathbb{N}$,

$$\begin{aligned} \sin(x \cos 2\pi t) &= \sin(x(1 - 2\pi^2 t^2)) \sum_{0 \leq n, m \leq N-1} c_{n,m} (xt^4)^{2n} t^{2m} \\ &+ \cos(x(1 - 2\pi^2 t^2)) \sum_{\substack{1 \leq n \leq N \\ 0 \leq m \leq N-1}} d_{n,m} (xt^4)^{2n-1} t^{2m} \\ &+ O(\max\{(xt^4)^{2N}, (xt^4)^{4N}\}) + O(\max\{t^{2N}, t^{4N}\}). \end{aligned}$$

The largest error term gives

$$\int_{-\infty}^{\infty} |\hat{f}(t) x t^4| dt \ll |x| \int_{-\infty}^{\infty} |\widehat{f^{(4)}}(t)| dt$$

where $f(t) = h\left(\frac{t-K}{G}\right) H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2}\right)$.

Note that

$$\begin{aligned} \frac{d^j}{dt^j} h\left(\frac{t-K}{G}\right) &\ll_j G^{-j} \text{ and} \\ \frac{d^j}{dt^j} H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2}\right) &\ll_j \left(\frac{4\pi^3 d^3 s_1^2 s_2^4 t_1 t_2^2}{t^2}\right)^j t^{-j}. \end{aligned}$$

Since $\frac{24\pi T_1 T_2}{K} \geq c \geq 1$ and $\frac{4\pi^3 d^3 s_1^2 s_2^4 t_1 t_2^2}{t^2} \ll K^\epsilon$, we have when $t \sim K$

$$\frac{d^j}{dt^j} H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2}\right) \ll_{j,\epsilon} K^{-j} K^{\epsilon j}.$$

From which we conclude that

$$\int_{-\infty}^{\infty} \left| \hat{f}(t) x t^4 \right| dt \ll G^{-4} |x| \ll G^{-4} \frac{T_1 T_2}{c}.$$

Under the assumption $c \gg \frac{T_1 T_2}{K G} K^{\frac{\epsilon}{2}}$,

$$\int_{-\infty}^{\infty} \left| \hat{f}(t) x t^4 \right| dt \ll G^{-4} K^{-\frac{\epsilon}{2}} K G \leq K^{-\frac{\epsilon}{2}}.$$

whenever $K \geq G \geq K^{\frac{1}{3}}$.

Thus, under the assumption that $c \gg \frac{T_1 T_2}{K G} K^{\frac{\epsilon}{2}}$, a typical integral one has to consider is

$$\int_{-\infty}^{\infty} \hat{f}(t) \sin(x(1 - 2\pi^2 t^2)) (x t^4)^l dt, \quad \int_{-\infty}^{\infty} \hat{f}(t) \cos(x(1 - 2\pi^2 t^2)) (x t^4)^l dt$$

where x and f as before. Since the treatment of other terms are similar, we will only deal with

$$\int_{-\infty}^{\infty} \hat{f}(t) \sin(x(1 - 2\pi^2 t^2)) dt$$

here.

Our goal is then to show the above integral is negligible under our assumption.

Applying the Fresnel integral (see Lemma 4.2 of [25])

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{f}(t) \sin(x(1 - 2\pi^2 t^2)) dt &= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} f(\sqrt{2yx}) \sin\left(y + x - \frac{\pi}{4}\right) \frac{dy}{\sqrt{y}} \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} h\left(\frac{\sqrt{2yx} - K}{G}\right) H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{(\sqrt{2yx})^2}\right) \sin\left(y + x - \frac{\pi}{4}\right) \frac{dy}{\sqrt{y}} \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} h\left(\frac{\sqrt{2yx} - K}{G}\right) H\left(\frac{\pi^2 d^3 s_1^2 s_2^4 t_2 c}{y}\right) \frac{1}{\sqrt{y}} \sin\left(y + x - \frac{\pi}{4}\right) dy. \end{aligned}$$

Note that $K \leq x = \frac{2\pi t_1 t_2}{c} \ll KGK^{-\frac{\epsilon}{2}}$, $\sqrt{2yx} \sim K$ and $d^3 s_1^2 s_2^4 T_1 T_2^2 \ll K^{2+\frac{\epsilon}{4}}$.

For each $j > 0$,

$$\begin{aligned} & \frac{d^j}{dy^j} h\left(\frac{\sqrt{2xy} - K}{G}\right) \\ & \ll \max \left\{ \left(\frac{1}{G}\sqrt{\frac{x}{y}}\right)^j, \frac{1}{y} \left(\frac{1}{G}\sqrt{\frac{x}{y}}\right)^{j-1}, \dots, \left(\frac{1}{y}\right)^{j-1} \left(\frac{1}{G}\sqrt{\frac{x}{y}}\right) \right\}, \\ & \frac{d^j}{dy^j} H\left(\frac{\pi^3 d^3 s_1^2 s_2^4 t_2 c}{y}\right) \\ & \ll \max \left\{ \left(\frac{d^3 s_1^2 s_2^4 t_2 c}{y^2}\right)^j, \left(\frac{d^3 s_1^2 s_2^4 t_2 c}{y^2}\right)^{j-1} \frac{1}{y}, \dots, \frac{d^3 s_1^2 s_2^4 t_2 c}{y^2} \frac{1}{y^{j-1}} \right\} \end{aligned}$$

and

$$\frac{d^j}{dy^j} y^{-\frac{1}{2}} \ll y^{-\frac{1}{2}-j}.$$

Since $yx \sim K^2$, $y^{-1} \sim \frac{x}{K^2} \ll \frac{G}{K} K^{-\frac{\epsilon}{2}}$; $\frac{1}{G}\sqrt{\frac{x}{y}} \sim \frac{1}{G} \frac{x}{K} \ll K^{-\frac{\epsilon}{2}}$; and $\frac{d^3 s_1^2 s_2^4 T_2 c}{y^2} \sim \frac{d^3 s_1^2 s_2^4 T_2 c x^2}{K^4} \sim \frac{d^3 s_1^2 s_2^4 T_2 T_1^2 T_2^2}{K^4 c} \ll \frac{d^3 s_1^2 s_2^4 T_2^2 T_1}{K^4} KGK^{-\frac{\epsilon}{2}} \ll \frac{G}{K} K^{-\frac{\epsilon}{4}}$.

Thus, the j -th derivative of all the three functions are $\ll K^{-\frac{\epsilon j}{4}}$. From which we conclude that the integral

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty h\left(\frac{\sqrt{2yx} - K}{G}\right) H\left(\frac{\pi^2 d^3 s_1^2 s_2^4 t_2 c}{y}\right) \frac{1}{\sqrt{y}} \sin\left(y + x - \frac{\pi}{4}\right) dy$$

is negligible whenever $c \gg \frac{T_1 T_2}{KG} K^{\frac{\epsilon}{2}}$. This shows that we can restrict the sum over c to those less than $\frac{T_1 T_2}{KG} K^{\frac{\epsilon}{2}}$.

4.3 Applying the Stationary Phase Lemma and the Hecke relation

Following P.882 of [24]) we can consider instead (all the others being similar)

$$\begin{aligned}
& 4\pi \sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2 \geq 1} \frac{1}{s_1 s_2^2} \sum_{\substack{T_1, T_2 | 2^\infty \\ d^3 s_1^2 s_2^4 T_1 T_2^2 \ll K^{2+\frac{\epsilon}{4}}}} \sum_{\substack{c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}}} \frac{1}{c} \sum_{t_1, t_2} S(t_1^2, t_2^2; c) \\
& \times e\left(\frac{2t_1 t_2}{c}\right) \frac{\lambda_\phi(ds_1^2 t_1)}{\sqrt{t_1 t_2}} g_{T_1}(t_1) g_{T_2}(t_2) \int_0^\infty h\left(\frac{y - K - 1}{G}\right) \\
& \times H\left(\frac{4\pi^3 d^3 s_1^2 t_1 s_2^4 t_2^2}{y^2}\right) \int_{-\infty}^\infty e\left(\frac{2t_1 t_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3}\pi^4 t^4\right) + ty\right) dt dy
\end{aligned}$$

with the cost of an admissible error term.

Using the fact that $H(x)$ decays exponentially as $x \rightarrow \infty$, we can restricts the d -, s_1 -, s_2 -, T_1 -, T_2 - sums to

$$\sum_{d \ll K^{\frac{2}{3}+\epsilon}} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} \sum_{\substack{T_2 | 2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1 | 2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{\substack{s_1 \geq 1 \\ d^3 s_1^2 s_2^4 T_1 T_2^2 \ll K^{2+\frac{\epsilon}{4}}}}.$$

Let

$$\psi(x) = g_{T_1}(x) \int_0^\infty h\left(\frac{y - K - 1}{G}\right) H\left(\frac{4\pi^3 d^3 s_1^2 s_2^4 t_2^2 x}{y^2}\right) \int_{-\infty}^\infty e(\kappa(t)) dt dy$$

where $\kappa(t) = \frac{2xt_2}{c} \left(-2\pi^2 t^2 + \frac{2}{3}\pi^4 t^4\right) + ty$. Then as a consequence of the stationary phase lemma (the proof will be given in the Section 4.9), we have

Lemma 4.3.1.

$$\begin{aligned}
\psi(x) &= \frac{e\left(-\frac{1}{8}\right) g_{T_1}(x)}{\sqrt{2x}} \sqrt{\frac{c}{t_2}} \int_0^\infty h\left(\frac{y - K - 1}{G}\right) H\left(\frac{4\pi^3 d^3 s_1^2 s_2^4 t_2^2 x}{y^2}\right) \\
&\times \frac{e(\kappa(\gamma))}{\sqrt{|-4\pi^2 + 8\pi^4 \gamma^4|}} dy + O\left(G\left(\frac{c}{T_1 t_2}\right)^{\frac{3}{2}}\right)
\end{aligned}$$

where $\gamma = \frac{2yc}{xt_2} (1 + o(x^{-1}))$.

Indeed as we will see in section 10, γ is the root of $\kappa(t)$ that is closest to 0. By the Implicit Function Theorem, we get

$$\frac{\partial^n \gamma}{\partial x^n} = (-1)^n \frac{2n! y c}{x^{n+1} t_2} (1 + o(x^{-1})).$$

Let θ be such that $\lambda_\phi(n) \ll n^\theta$ then progress toward Ramanujan conjecture for the Maass form indicates that we can take $\theta = \frac{1}{4}$. The error term $O\left(G\left(\frac{c}{T_1 t_2}\right)^{\frac{3}{2}}\right)$ contributes

$$\begin{aligned} & G \sum_{d \ll K^{\frac{2}{3}+\epsilon}} \frac{1}{d^{\frac{3}{2}}} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} \frac{1}{s_2^2} \sum_{\substack{T_2 | 2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1 | 2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{s_1 \geq 1} \frac{1}{s_1} \\ & \times \sum_{c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}} c^{\frac{1}{2}} \sum_{\substack{t_1 \sim T_1 \\ t_2 \sim T_2}} \frac{|S(t_1^2, t_2^2; c)| |\lambda_\phi(ds_1^2 t_1)|}{T_1^2 T_2^{\frac{5}{2}}} \\ & \ll_\epsilon G \sum_{d \ll K^{\frac{2}{3}+\epsilon}} d^{-\frac{3}{2}} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} s_2^{-2} \sum_{\substack{T_2 | 2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1 | 2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{s_1 \geq 1} s_1^{-1} \\ & \times \sum_{c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}} c^{1+\epsilon} \frac{(ds_1^2 T_1)^\theta}{T_1 T_2^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned}
&\ll G \sum_{d \ll K^{\frac{2}{3}+\epsilon}} d^{-\frac{3}{2}+\theta} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} s_2^{-2} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{\substack{s_1 \geq 1 \\ d^3 s_1^2 s_2^4 T_1 T_2^4 \ll K^{2+\frac{\epsilon}{4}}}} \\
&\quad s_1^{-1+2\theta} \left(\frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{K G} \right)^{2+\epsilon} T_1^{\theta-1} T_2^{-\frac{3}{2}} \\
&\ll G \sum_{d \ll K^{\frac{2}{3}+\epsilon}} d^{-\frac{3}{2}+\theta} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} s_2^{-2} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_2 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \left(\frac{K^{2+\frac{\epsilon}{4}}}{d^3 s_2^4 T_1 T_2^4} \right)^\theta \\
&\quad \times \frac{T_1^{1+\theta+\epsilon} T_2^{\frac{1}{2}+\epsilon}}{K^2 G^2} \\
&\ll G \sum_{d \ll K^{\frac{2}{3}+\epsilon}} d^{-\frac{3}{2}-2\theta} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} s_2^{-2-4\theta} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \frac{K^{2\theta-2+\epsilon}}{G^2} \\
&\quad \times T_1^{1+\epsilon} T_2^{\frac{1}{2}-4\theta+\epsilon} \\
&\ll G \frac{K^{2\theta-2+\epsilon}}{G^2} K^{2+\epsilon} = G \frac{K^{2\theta+2\epsilon}}{G^2} \ll G \left(\frac{K^{\frac{1}{4}}}{G} \right)^2 K^{2\epsilon} \ll G K^{2\epsilon},
\end{aligned}$$

note that the Weil bound on the Kloosterman sum was deployed.

Applying Lemma 4.3.1, we get

(4.3.1)

$$\begin{aligned}
&\sum_{d \ll K^{\frac{2}{3}+\epsilon}} \frac{\mu(d)}{d} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} \frac{1}{s_2^2} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{\substack{s_1 \geq 1 \\ d^3 s_1^2 s_2^4 T_1 T_2^4 \gg K^{2+\frac{\epsilon}{4}}}} \\
&\quad \sum_{\substack{c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{K G} \\ t_2 \geq 1}} c^{-\frac{1}{2}} \sum_{t_2 \geq 1} \frac{g_{T_2}(t_2)}{t_2^{\frac{3}{2}}} \sum_{t_1 \geq 1} S(t_1^2, t_2^2; c) e\left(\frac{2t_1 t_2}{c}\right) \\
&\quad \times \frac{\lambda_\phi(ds_1^2 t_1)}{(ds_1^2)^{\frac{3}{2}}} \Psi(ds_1^2 t_1)
\end{aligned}$$

where

$$\begin{aligned}
\Psi(x) &= \frac{g_{T_1}\left(\frac{x}{ds_1^2}\right)}{x} \int_0^\infty h\left(\frac{y-K-1}{G}\right) H\left(\frac{4\pi^3 d^2 s_2^4 t_2^2 x}{y^2}\right) \\
&\quad \times \frac{e(\Phi(x, y))}{\sqrt{|-4\pi^2 + 8\pi^4 \gamma^2|}} dy
\end{aligned}$$

with

$$\Phi(x, y) = \frac{2xt_2}{cds_1^2} \left(-2\pi^2\gamma^2 + \frac{2}{3}\pi^4\gamma^4 \right) + \gamma y \text{ and } \gamma = \frac{ycds_1^2}{2xt_2}(1 + o(x^{-1})),$$

$$\text{i.e. } \Psi(x) = \psi\left(\frac{x}{ds_1^2}\right).$$

Breaking t_1 into congruence classes modulo c , (4.3.1)

$$\begin{aligned} &= \sum_{d \ll K^{\frac{2}{3}+\epsilon}} \frac{\mu(d)}{d} \sum_{s_2 \ll K^{1+\frac{\epsilon}{2}}} \frac{1}{s_2^2} \sum_{\substack{T_2 | 2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1 | 2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{\substack{s_1 \geq 1 \\ K^{2+\frac{\epsilon}{4}} \gg d^3 s_1^2 s_2^4 T_1 T_2^2}} \\ &\quad \frac{1}{(ds_1^2)^{\frac{3}{2}}} \sum_{\substack{c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}}} c^{-\frac{3}{2}} \sum_{t_2 \geq 1} \frac{g_{T_2}(t_2)}{t_2^{\frac{3}{2}}} \sum_{u(c)} \sum_{a(c)} S(a^2, t_2^2; c) \\ &\times e\left(\frac{2at_2 + ua}{c}\right) \sum_{n \geq 1} \lambda_\phi(ds_1^2 n) e\left(-\frac{un}{c}\right) \Psi(ds_1^2 n). \end{aligned}$$

Let $ds_1^2 = q_1^{r_1} \cdots q_m^{r_m}$ be the unique prime factorization. Then using the Hecke relation,

$$\lambda_\phi(r_1 r_2) = \sum_{d|(r_1, r_2)} \mu(d) \lambda_\phi\left(\frac{r_1}{d}\right) \lambda_\phi\left(\frac{r_2}{d}\right),$$

we have

$$\begin{aligned} &\sum_{n \geq 1} \lambda_\phi(q_1^{r_1} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\ &= \sum_{(q_1, n)=1} \lambda_\phi(q_1^{r_1} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\ &\quad + \sum_{(q_1, n)=q_1} \lambda_\phi(q_1^{r_1} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\ &= \lambda_\phi(q_1^{r_1}) \sum_{(q_1, n)=1} \lambda_\phi(q_2^{r_2} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\ &\quad + \lambda_\phi(q_1^{r_1}) \sum_{(q_1, n)=q_1} \lambda_\phi(q_2^{r_2} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\ &\quad - \lambda_\phi(q_1^{r_1-1}) \sum_{(q_1, n)=q_1} \lambda_\phi(q_2^{r_2} \cdots q_m^{r_m} \frac{n}{q_1}) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \end{aligned}$$

$$\begin{aligned}
&= \lambda_\phi(q_1^{r_1}) \sum_{n \geq 1} \lambda_\phi(q_2^{r_2} \cdots q_m^{r_m} n) e\left(-\frac{un}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} n) \\
&- \lambda_\phi(q_1^{r_1-1}) \sum_{n \geq 1} \lambda_\phi(q_2^{r_2} \cdots q_m^{r_m} n) e\left(-\frac{uq_1 n}{c}\right) \Psi(q_1^{r_1} \cdots q_m^{r_m} q_1 n).
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\sum_{n \geq 1} \lambda_\phi(ds_1^2 n) e\left(-\frac{un}{c}\right) \Psi(ds_1^2 n) = \lambda_\phi(ds_1^2) \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{un}{c}\right) \Psi(ds_1^2 n) \\
&- \sum_{j=1}^m \lambda_\phi(ds_1^2/q_j) \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{uq_j n}{c}\right) \Psi(ds_1^2 q_j n) \\
&+ (-1)^2 \sum_{1 \leq j_1 \neq j_2 \leq m} \lambda_\phi\left(\frac{ds_1^2}{q_{j_1} q_{j_2}}\right) \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{uq_{j_1} q_{j_2} n}{c}\right) \Psi(ds_1^2 q_{j_1} q_{j_2} n) \\
&+ \cdots + (-1)^m \lambda_\phi\left(\frac{ds_1^2}{\prod_{j=1}^m q_j}\right) \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{u \prod_{j=1}^m q_j n}{c}\right) \Psi(ds_1^2 \prod_{j=1}^m q_j n).
\end{aligned}$$

There are 2^m terms where $m = \omega(ds_1^2)$ is the number of distinct prime factor of ds_1 which is of size at most $\frac{\log ds_1^2}{\log \log ds_1^2}$.

4.4 Evaluation of the Sums of the Kloosterman sum

For each c , we can write $c = c_1 c_2$ where $(c_1, c_2) = 1$ and $c_2 | 2^\infty$, i.e. when c_2 is a power of 2. In this section we will evaluate

$$\sum_{a(c)} S(a^2, t_2^2; c) e\left(\frac{2at_2 + ua}{c}\right)$$

and determine the conditions for which the above sum is non-zero. In particular, we will show that the above sum is zero unless

$$((t_2 + \bar{4}u)u, c_1) = \prod_{i=1}^l p_i^{k_i-1}$$

and k_i odd for all $1 \leq i \leq l$, here $c_1 = p_1^{k_1} \cdots p_l^{k_l}$ is the prime factorization of c_1 . This is done via evaluation of the quadratic Gauss sum with prime power modulus.

By P.59, (4.12) of [7], let $\overline{c_1}, \overline{c_2}$ be such that $\overline{c_2}c_2 \equiv 1(c_1)$ and $\overline{c_1}c_1 \equiv 1(c_2)$. Then we have

$$\begin{aligned}
& \sum_{a(c)} S(a^2, t_2^2; c) e\left(\frac{2at_2 + ua}{c}\right) \\
&= \sum_{a(c)} S(a^2\overline{c_2}, t_2^2\overline{c_2}; c_1) S(a^2\overline{c_1}, t_2^2\overline{c_1}; c_2) e\left(\frac{2at_2 + ua}{c_1c_2}\right) \\
&= \sum_{\substack{a_1(c_1) \\ a_2(c_2)}} S((a_1c_2 + a_2c_1)^2\overline{c_2}, t_2^2\overline{c_2}; c_1) S((a_1c_2 + a_2c_1)^2\overline{c_1}, t_2^2\overline{c_1}; c_2) \\
&\times e\left(\frac{2(a_1c_2 + a_2c_1)t_2 + u(a_1c_2 + a_2c_1)}{c_1c_2}\right) \\
&= \sum_{\substack{a_1(c_1) \\ a_2(c_2)}} S(a_1^2c_2, t_2^2\overline{c_2}; c_1) S(a_2^2c_1, t_2^2\overline{c_1}; c_2) \\
&\times e\left(\frac{2a_1t_2 + ua_1}{c_1}\right) e\left(\frac{2a_2t_2 + ua_2}{c_2}\right) \\
&= \sum_{a_2(c_2)} \left[\sum_{a_1(c_1)} S(a_1^2c_2, t_2^2\overline{c_2}; c_1) e\left(\frac{2a_1t_2 + ua_1}{c_1}\right) \right] \\
&\times S(a_2^2c_1, t_2^2\overline{c_1}; c_2) e\left(\frac{2a_2t_2 + ua_2}{c_2}\right).
\end{aligned}$$

We now evaluate the sum over c_1

$$\begin{aligned}
& \sum_{a_1(c_1)} S(a_1^2c_2, t_2^2\overline{c_2}; c_1) e\left(\frac{2a_1t_2 + ua_1}{c_1}\right) \\
&= \sum_{a_1(c_1)} \sum_{\substack{d(c_1) \\ (d, c_1)=1}} e\left(\frac{a_1^2c_2}{c_1}d\right) e\left(\frac{t_2^2\overline{c_2}}{c_1}\overline{d}\right) e\left(\frac{2a_1(t_2 + \overline{2}u)}{c_1}\right) \\
&= \sum_{\substack{d(c_1) \\ (d, c_1)=1}} \sum_{a_1(c_1)} e\left(\frac{[a_1c_2 + (t_2 + \overline{2}u)\overline{d}]^2\overline{c_2}d}{c_1}\right) e\left(\frac{-(t_2 + \overline{2}u)^2\overline{dc_2} + t_2^2\overline{c_2}d}{c_1}\right) \\
&= \sum_{\substack{d(c_1) \\ (d, c_1)=1}} \epsilon_{c_1} \left(\frac{d\overline{c_2}}{c_1}\right)_L \sqrt{c_1} e\left(\frac{-(t_2 + \overline{4}u)u\overline{c_2}d}{c_1}\right) \\
&= \epsilon_{c_1} \sqrt{c_1} \sum_{\substack{d(c_1) \\ (d, c_1)=1}} \left(\frac{d}{c_1}\right)_L e\left(\frac{-(t_2 + \overline{4}u)u\overline{c_2}d}{c_1}\right)
\end{aligned}$$

since $\left(\frac{d}{c_1}\right)_L = \left(\frac{\bar{d}}{c_1}\right)_L$ where $(\cdot)_L$ is the Jacobi-Legendre symbol and

$$\epsilon_c = \begin{cases} 1 & \text{if } c \equiv (4) \\ i & \text{if } c \equiv -1(4). \end{cases}$$

Hence,

$$\begin{aligned} \sum_{a_1(c_1)} S(a_1^2 c_2, t_2^2 \bar{c}_2; c_1) e\left(\frac{2a_1 t_2 + u a_1}{c_1}\right) \\ = \epsilon_{c_1} \sqrt{c_1} G\left(-(t_2 + \bar{4}u) u \bar{c}_2^2; c_1\right) \end{aligned}$$

where

$$G(\beta, c_1) = \sum_{\substack{d(c_1) \\ (d, c_1)=1}} \left(\frac{d}{c_1}\right)_L e\left(\frac{\beta d}{c_1}\right)$$

is the quadratic Gauss sum.

Write $c_1 = p_1^{k_1} \cdots p_l^{k_l}$ where $p_1, \dots, p_l$ are distinct primes. Then by Lemma 3 in [8],

$$G(\beta, c_1) = \prod_i G\left(\beta \prod_{j \neq i} \overline{p_j^{k_j}}; p_i^{k_i}\right).$$

Lemma 4.4.1. *Let p be an odd prime number, $(n, p)=1$ and $q = p^k$. Then*

$$G(n, q) = \begin{cases} p^{k-\frac{1}{2}} \epsilon_p \left(\frac{n/p^{k-1}}{p}\right) & \text{if } k \geq 1 \text{ is odd and } (n, p^k) = p^{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By lemma 12.4 of [12], if $(n, p) = 1$ then

$$G(n, q) = \epsilon_p \sqrt{p} \left(\frac{n}{p}\right)_L \sum_{v^2 \equiv 0(p)} e\left(\frac{2v}{p}\right).$$

Thus if $k = 1$, when $(n, p) = 1$

$$G(n, q) = \epsilon_p \sqrt{p} \left(\frac{n}{p}\right)_L.$$

In the case when $(n, p) = p$ and $k = 1$, a direct calculation shows that $G(n, q) = 0$.

For k even,

$$\begin{aligned} G(n, p^k) &= \sum_{d(p^k)} \left(\frac{d}{p} \right)_L^k e \left(\frac{nd}{p^k} \right) \\ &= \sum_{d(p^k)} e \left(\frac{nd}{p^k} \right) = 0. \end{aligned}$$

For $k > 1$ and odd, for each $0 \leq d < p^k$, write $d = up + v$ where $0 \leq u < p^{k-1}$ and $0 \leq v < p$. Then

$$\begin{aligned} G(n, p^k) &= \sum_{\substack{u(p^{k-1}) \\ v(p)}} \left(\frac{v}{p} \right)_L e \left(\frac{n(up + v)}{p^k} \right) \\ &= \sum_{v(p)} \left(\frac{v}{p} \right)_L e \left(\frac{nv}{p^k} \right) \sum_{u(p^{k-1})} e \left(\frac{un}{p^{k-1}} \right). \end{aligned}$$

Hence if $(n, p^k) \neq p^{k-1}$, $G(n, q) = 0$. Otherwise,

$$\begin{aligned} G(n, q) &= p^{k-1} \sum_{v(p)} \left(\frac{v}{p} \right)_L e \left(\frac{n'v}{p} \right) \\ &= p^{k-\frac{1}{2}} \epsilon_p \left(\frac{n'}{p} \right)_L \end{aligned}$$

where $n' = n/(n, p^k)$. This concludes the proof of our lemma. $\square$

Thus, we conclude that for

$$\sum_{a_1(c_1)} S(a_1^2 c_2, t_2^2 \overline{c_2}; c_1) e \left(\frac{2a_1 t_2 + ua_1}{c_1} \right)$$

to be non-zero, we must have $-(t_2 + \overline{4}u)u, c_1) = \prod_{i=1}^l p_i^{k_i-1}$ (this includes the case $k_i = 1$), in which case

$$\begin{aligned} &\sum_{a_1(c_1)} S(a_1^2 c_2, t_2^2 \overline{c_2}; c_1) e \left(\frac{2a_1 t_2 + ua_1}{c_1} \right) \\ &= \epsilon_{c_1} \frac{c_1^{\frac{3}{2}}}{\sqrt{\prod_{i=1}^l p_i}} \prod_{i=1}^l \epsilon_{p_i} \prod_{i=1}^l \left(\frac{-(t_2 + \overline{4}u)u \overline{\prod_{j \neq i} p_j^{k_j}} / p_i^{k_i-1}}{p_i} \right)_L. \end{aligned}$$

4.5 Applying Voronoi formula

We now apply the following Voronoi formula

Lemma 4.5.1. *(Theorem 2 in [26]) Let $f \in C_0^\infty(\mathbb{R})$ be a smooth function with compact support in $(0, \infty)$, and for any integer $C \geq 1$ and $(a, C) = 1$, we have*

$$\begin{aligned} & \sum_{n \geq 1} \lambda_\phi(n) e\left(\frac{an}{C}\right) f(n) = \frac{\pi i}{C \sinh t_\phi} \\ & \times \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{\bar{a}n}{C}\right) \int_0^\infty f(x) \left(J_{2it_\phi} \left(\frac{4\pi\sqrt{nx}}{C} \right) - J_{-2it_\phi} \left(\frac{4\pi\sqrt{nx}}{C} \right) \right) dx \\ & + \frac{4 \cosh t_\phi}{C} \sum_{n \geq 1} \lambda_\phi(-n) e\left(\frac{\bar{a}n}{C}\right) \int_0^\infty f(x) K_{2it_\phi} \left(\frac{4\pi\sqrt{nx}}{C} \right) dx \end{aligned}$$

where

$$\Delta\phi = \left(\frac{1}{4} + t_\phi^2 \right) \phi$$

and

$$a\bar{a} \equiv 1(C).$$

From Section 4, for each $ds_1^2 = q_1^{r_1} \cdots q_m^{r_m}$, we know that we have to deal with the sum of the form

$$\sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{u\beta'n}{c'_1 c'_2}\right) \Psi(ds_1^2 \beta n)$$

where β is a product of distinct prime factors of ds_1^2 ,

$$\beta' = \frac{\beta}{(\beta, c)}, \quad c'_1 = \frac{c_1}{(\beta, c_1)} \text{ and } c'_2 = \frac{c_2}{(\beta, c_2)}.$$

We can't assume that $(u, c_2) = 1$. This presents a minor difficulty and to overcome it we have to break the sum $\sum_{\substack{u(c) \\ (u, c_1)=1}}$ into

$$\sum_{\substack{u(c) \\ (u, c_1)=1}} = \sum_{j=0}^{\infty} \sum_{\substack{u(c) \\ (u, c_1)=1 \\ (u, c'_2)=2^j}}$$

Fix $(u, c'_2) = 2^j$. We now apply the Voronoi formula to

$$\sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{u' \beta' n}{c_1'' c_{2,j}''}\right) \Psi(ds_1^2 \beta n)$$

where

$$c_{2,j}'' = \frac{c_2'}{(u, c_2')}, \quad u' = \frac{u}{(u, c_2')(u, c_1')} \quad \text{and} \quad c_1'' = \frac{c_1'}{(u, c_1')}$$

and obtain

$$\begin{aligned} & \sum_{n \geq 1} \lambda_\phi(n) e\left(-\frac{u' \beta' n}{c_1'' c_{2,j}''}\right) \Psi(ds_1^2 \beta n) = \frac{\pi i}{c_1'' c_{2,j}'' \sinh t_\phi} \\ & \times \sum_{j=1}^2 \sum_{n \geq 1} \lambda_\phi(n) e\left(\frac{\overline{u' \beta' n}}{c_1'' c_{2,j}''}\right) \int_0^\infty \Psi(ds_1^2 \beta x) J_{(-1)^j 2it_\phi} \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''}\right) dx \\ & + \frac{4 \cosh t_\phi}{c_1'' c_{2,j}''} \sum_{n \geq 1} \lambda_\phi(n) e\left(\frac{\overline{u' \beta' n}}{c_1'' c_{2,j}''}\right) \int_0^\infty \Psi(ds_1^2 \beta x) K_{2it_\phi} \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''}\right) dx \end{aligned}$$

First of all note that $K_{2it}(z) \ll \min\{1 + |\log z|, z^{-\frac{1}{2}} e^{-z}\}$ (see P.2416 of [22]) and since $n \in \mathbb{N}$, $ds_1^2 s_2^4 T_1 T_2^2 \ll K^{2+\frac{\epsilon}{4}}$, $\frac{ds_1^2 \beta x}{ds_1^2} \sim T_1$ and $c \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG}$,

$$\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} \gg \frac{\sqrt{T_1}}{\sqrt{ds_1^2 c}} \gg \frac{s_2^2 T_1 T_2}{c K^{1+\epsilon}} \gg \frac{G}{K^{\frac{3\epsilon}{2}}}.$$

Thus, the term involving $K_{2it_\phi} \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''}\right)$ is negligible. Hence we only have to deal with terms involving the J-Bessel function.

4.6 Reduction of the range of sum over n

The main goal of this section is to reduce the range of the sum over n . This is done using the asymptotic expansion of the J -Bessel function and then integrating by parts to show the contribution from the large n is negligible. The treatment for the J_{2it_ϕ} and J_{-2it_ϕ} are quite similar, we would thus only deal with the term involving J_{2it_ϕ}

here. We will use the following asymptotic expansion for $J_{2it}(z)$ (see [4], 7.13.1, (3), P.87), for any complex $\nu > 0$, $-\pi < \arg z < \pi$ and $B > 1$,

$$J_\nu(z) = \left(\frac{\pi z}{2}\right)^{-\frac{1}{2}} \left\{ \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{B-1} t_{m,\nu} z^{-2m} + (|z|^{-2B}) \right] \right. \\ \left. - \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{B-1} t'_{m,\nu} z^{-2m-1} + O(|z|^{-2B-1}) \right] \right\}$$

for some constants $t_{m,\nu}$, $t'_{m,\nu}$.

Since $\Psi(x)$ is supported in $\left[\frac{1}{2}ds_1^2T_1, 3ds_1^2T_1\right]$, $c \ll \frac{T_1T_2K^{\frac{\epsilon}{2}}}{KG}$ and $d^3s_1^2s_2^4T_1T_2^2 \ll K^{2+\frac{\epsilon}{4}}$,

$$\frac{4\pi\sqrt{nx}}{c'_1c''_{2,j}} \gg \frac{\sqrt{nT_1}}{c\sqrt{ds_1^2}} \gg \sqrt{n}GK^{-\frac{5\epsilon}{8}},$$

the asymptotic expansion for the J -Bessel function stated above can be applied to the integral

$$\int_0^\infty \Psi(ds_1^2\beta x) J_{2it_\phi}\left(\frac{4\pi\sqrt{nx}}{c'_1c''_{2,j}}\right) dx.$$

The error terms in the asymptotic expansion contributes

$$O\left(\int_0^\infty |\Psi(ds_1^2\beta x)| \left(\frac{4\pi\sqrt{nx}}{c'_1c''_{2,j}}\right)^{-A} dx\right)$$

where $A = 2B$ or $2B + 1$.

We will now show that the error term is admissible. To this end, we need to bound $\Psi(x)$. It is useful later to have a good bound on the derivatives of $\Psi(x)$. Thus we prove that for each $p \geq 0$,

Lemma 4.6.1.

$$\Psi^{(p)}(x) \ll_p \frac{G}{ds_1^2T_1} \left[\left(\frac{K^{\frac{\epsilon}{4}}}{T_1}\right)^p + \left(\frac{K^2c}{ds_1^2T_1^2T_2}\right)^p \right] \ll \frac{G}{ds_1^2T_1} \left(\frac{K^2c}{ds_1^2T_1^2T_2}\right)^p.$$

Proof. First recall that

$$\Psi(x) = \frac{g_{T_1}\left(\frac{x}{ds_1^2}\right)}{x} \int_0^\infty h\left(\frac{y-K-1}{G}\right) H\left(\frac{4\pi^3d^2s_2^4t_2^2x}{y^2}\right)$$

$$\frac{e(\Phi(x, y))}{|-4\pi^2 + 8\pi^4\gamma^2|} dy$$

with

$$\Phi(x, y) = \frac{2xt_2}{cds_1^2} \left(-2\pi^2\gamma^2 + \frac{2}{3}\pi^4\gamma^4 \right) + \gamma y$$

and

$$\gamma = \frac{ycds_1^2}{2xt_2} (1 + o(x^{-1})).$$

Since $g_{T_1}(z)$ is supported in $\left[\frac{1}{2}T_1, 3T_1\right]$ and h is supported in $\left[\frac{1}{2}, \frac{5}{2}\right]$, we conclude that $y \sim K$, $x \in \left[\frac{1}{2}ds_1^2T_1, 3ds_1^2T_1\right]$. Also we know a priori that $c \ll \frac{T_1T_2K^{\frac{\epsilon}{2}}}{KG}$. With these in mind,

$$\begin{aligned} \gamma &\ll \frac{Kcds_1^2}{T_1T_2ds_1^2} \ll \frac{K^{\frac{\epsilon}{2}}}{G} \\ \frac{\partial^p \gamma}{\partial x^p} &\ll \frac{Kcds_1^2}{x^p T_2} \ll \frac{Kcds_1^2}{(ds_1^2T_1)^p T_2}, \text{ for each } p \geq 0. \end{aligned}$$

Hence, for each $p > 0$,

$$\begin{aligned} \left(\frac{\partial \Phi}{\partial x}\right)^p &\ll \left(\frac{K^2cds_1^2}{(T_1ds_1^2)^2T_2}\right)^p = \left(\frac{K^2c}{T_1^2T_2ds_1^2}\right)^p \ll \left(\frac{K^{1+\frac{\epsilon}{2}}}{ds_1^2T_1G}\right)^p \\ \frac{\partial^p \Phi}{\partial x^p} &\ll \left(\frac{ds_1^2T_1T_2}{cds_1^2}\right) \left(\frac{Kcds_1^2}{T_1T_2ds_1^2}\right)^2 \cdot \frac{1}{(ds_1^2T_1)^p} + \frac{K^2cds_1^2}{(ds_1^2T_1)^p T_2} \\ &\ll \frac{K^2cds_1^2}{(ds_1^2T_1)^p G}. \end{aligned}$$

Since $d^3s_1^2s_2^4T_1T_2^2 \ll K^{2+\frac{\epsilon}{4}}$, we get

$$\frac{d^2s_2^4t_2^2}{y^2} \ll \frac{d^2s_2^4T_2^2}{K^2} \ll \frac{K^{2+\frac{\epsilon}{4}}}{K^2T_1} = \frac{K^{\frac{\epsilon}{4}}}{T_1}.$$

Putting all these together concludes our proof. $\square$

The error term contributes (by Lemma 4.6.1)

$$\begin{aligned} &O\left(G\left(\frac{c_1''c_{2,j}''}{\sqrt{nds_1^2T_1}}\right)^A\right) = O\left(G\left(\frac{T_1T_2}{\sqrt{nT_1}\sqrt{ds_1^2}}\frac{c}{T_1T_2}\right)^A\right) \\ &= O\left(G\left(\frac{\sqrt{T_1T_2}}{\sqrt{n}\sqrt{ds_1^2}}\frac{K^{\frac{\epsilon}{2}}}{KG}\right)^A\right) = O\left(G\left(\frac{K^{\epsilon}}{\sqrt{n}G}\right)^A\right) \end{aligned}$$

which is negligible when $A = 2B$ ($2B+1$ respectively) is large enough (say $B = 2015$).

We will only consider

$$\begin{aligned} & \int_0^\infty \Psi(ds_1^2 \beta x) \left(\frac{\sqrt{nx}}{c_1'' c_{2,j}''} \right)^{-\frac{1}{2}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi \pi}{2} - \frac{\pi}{4} \right) dx \\ &= \frac{\sqrt{c_1'' c_{2,j}''}}{n^{\frac{1}{4}}} \int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi \pi}{2} - \frac{\pi}{4} \right) dx \end{aligned}$$

since the treatment for the other terms are quite similar.

We now show that we can restrict the sum over n to those $n \ll \frac{K^{4+\frac{\epsilon}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2}$.

Indeed, integrating by parts p times,

$$\begin{aligned} & \frac{\sqrt{c_1'' c_{2,j}''}}{n^{\frac{1}{4}}} \int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi \pi}{2} - \frac{\pi}{4} \right) dx \\ & \ll \frac{\sqrt{c_1'' c_{2,j}''}}{n^{\frac{1}{4}}} \int_{x \sim ds_1^2 T_1} \left(\frac{K^2 c}{T_1^2 T_2} \right)^p x^{\frac{p}{2}} \left(\frac{c_1'' c_{2,j}''}{\sqrt{n}} \right)^p dx \\ & \ll \frac{\sqrt{c_1'' c_{2,j}''} ds_1^2 T_1}{n^{\frac{1}{4}}} \left(\frac{K^2 c c_1'' c_{2,j}'' \sqrt{ds_1^2 T_1}}{T_1^2 T_2 \sqrt{n}} \right)^p \\ & = \frac{\sqrt{c_1'' c_{2,j}''} ds_1^2 T_1}{n^{\frac{1}{4}}} \left(\frac{K^4 c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2 n} \right)^{\frac{p}{2}}. \end{aligned}$$

If $\frac{K^4 c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2 n} \ll K^{-\frac{\epsilon}{100}}$ then the integral is negligible by taking p large. Hence we can assume $\frac{K^4 c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2 n} \gg K^{-\frac{\epsilon}{100}}$. In this case, we get

$$n \ll \frac{K^{4+\frac{\epsilon}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2}.$$

Putting all these together, for each ds_1^2 and j we have to deal with

$$\sum_{\substack{u(c) \\ (u, c_2')=2^j}} \sum_{a(c)} S(a^2, t_2^2; c) e \left(\frac{2at_2 + ua}{c} \right) \quad (4.6.1)$$

$$\begin{aligned}
& \times \sum_{n \ll \frac{K^{4+\frac{\epsilon}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2}} \lambda_\phi(n) e \left(\frac{\overline{u' \beta' n}}{c_1'' c_{2,j}''} \right) \frac{1}{n^{\frac{1}{4}} \sqrt{c_1'' c_{2,j}''}} \\
& \times \int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi}{2} - \frac{\pi}{4} \right) dx.
\end{aligned}$$

By Section 4.5, (4.6.1)

$$\begin{aligned}
& = \frac{\epsilon_{c_1} \prod_{i=1}^l \epsilon_{p_i} c_1^{\frac{3}{2}}}{\sqrt{c_1'' c_{2,j}''} \sqrt{\prod_{i=1}^l p_i}} \prod_{i=1}^l \left(\frac{-\overline{\prod_{j \neq i} p_j^{k_j}}}{p_i} \right)_L \sum_{n \ll \frac{K^{4+\frac{\epsilon}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2}} \frac{\lambda_\phi(n)}{n^{\frac{1}{4}}} \\
& \times \sum_{a_2(c_2)} S(a_2^2 c_1, t_2^2 \overline{c_1}; c_2) e \left(\frac{2a_2 t_2}{c_2} \right) \\
& \times \sum_{\substack{(u,c) \\ ((t_2 + \overline{4}u)u, c_1) = \prod_{i=1}^l p_i^{k_i-1} \\ (u, c_2) = 2^j}} \left(\frac{(t_2 + \overline{4}u)u / (p_1^{k_1-1} \cdots p_l^{k_l-1})}{p_1 \cdots p_l} \right)_L e \left(\frac{ua_2}{c_2} \right) \\
& \times e \left(\frac{\overline{u' \beta' n}}{c_1'' c_{2,j}''} \right) \int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi \pi}{2} - \frac{\pi}{4} \right) dx.
\end{aligned}$$

4.7 Bounding the Oscillatory integral

In this section we will apply the Van der Corput Lemma (see Lemma 4.7.1 below) to get a better bound on the oscillatory integral

$$\int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} e \left(\pm \frac{2\sqrt{nx}}{c_1'' c_{2,j}''} \right) dx$$

(and hence on $\int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi}{2} - \frac{\pi}{4} \right) dx$). In particular, we will show that

$$\int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} e \left(\pm \frac{2\sqrt{nx}}{c_1'' c_{2,j}''} \right) dx$$

$$\ll G(ds_1^2)^{-\frac{3}{4}} \frac{T_1^{\frac{1}{4}} \sqrt{T_2}}{\sqrt{c} K \sqrt{\beta}}.$$

We will make heavy use of the following well-known Van der Corput lemma.

Lemma 4.7.1. *Suppose that $\phi : [a, b] \rightarrow \mathbb{R}$ is a phase obeying the bound $|\phi^{(k)}(x)| \geq \lambda$ for all $x \in [a, b]$ for some $k \geq 2$ and $\lambda > 0$, and $\psi : [a, b] \rightarrow \mathbb{C}$ is a function such that $\psi' \in L^1[a, b]$ then*

$$\int_a^b e^{i\phi(x)} \psi(x) dx \ll \lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right].$$

The same claim also holds for $k = 1$ provided that $\phi'(x)$ is monotone.

Recall that

$$\begin{aligned} & \int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} e\left(\pm \frac{2\sqrt{nx}}{c_1'' c_{2,j}''}\right) dx \\ &= \int_0^\infty h\left(\frac{y - K - 1}{G}\right) \int_{x \sim ds_1^2 T_1} H\left(\frac{4\pi^3 d^2 s_2^4 t_2^2 ds_1^2 \beta x}{y^2}\right) \frac{x^{-\frac{1}{4}}}{x \sqrt{|-4\pi^2 + 8\pi^4 \gamma^2|}} \\ &\times e\left(\Phi(ds_1^2 \beta x, y) \pm \frac{2\sqrt{nx}}{c_1'' c_{2,j}''}\right) dx dy \end{aligned}$$

where

$$\Phi(x, y) = \frac{2xt_2}{cds_1^2} \left(-2\pi^2 \gamma^2 + \frac{2}{3} \pi^4 \gamma^4 \right) + \gamma y$$

and

$$\gamma \sim \frac{ycds_1^2}{2xt_2}.$$

Then

$$\begin{aligned} & \frac{\partial \Phi}{\partial x} = \frac{2t_2}{cds_1^2} \left(-2\pi^2 \gamma^2 + \frac{2}{3} \pi^4 \gamma^4 \right) + \frac{2xt_2}{cds_1^2} \left(-4\pi^2 \gamma + \frac{8}{3} \pi^4 \gamma^3 \right) \frac{\partial \gamma}{\partial x} \\ &+ \frac{\partial \gamma}{\partial x} y \\ &\sim \frac{-4\pi^2 t_2}{cds_1^2} \left(\frac{Kcds_1^2}{2xt_2} \right)^2 + \frac{-8\pi^2 xt_2}{cds_1^2} \left(\frac{Kcds_1^2}{2xt_2} \right) \left(\frac{-Kcds_1^2}{2x^2 t_2} \right) \\ &- \frac{K^2 cds_1^2}{2x^2 t_2} \\ &= \left(\pi^2 - \frac{1}{2} \right) \frac{K^2 cds_1^2}{x^2 t_2}. \end{aligned}$$

In the case of $\Phi(ds_1^2\beta x, y) + \frac{2\sqrt{nx}}{c_1''c_{2,j}''}$, when $x \sim ds_1^2T_1$, by differentiating the function

$$\frac{\partial}{\partial x}\Phi(ds_1^2\beta x, y) + \frac{\partial}{\partial x}\left(\frac{2\sqrt{nx}}{c_1''c_{2,j}''}\right)$$

with respect to x we can see that it is monotonic and we get

$$\frac{\partial}{\partial x}\Phi(ds_1^2\beta x, y) + \frac{\partial}{\partial x}\left(\frac{2\sqrt{nx}}{c_1''c_{2,j}''}\right) \gg \frac{K^2c\beta}{T_1^2T_2}$$

from the above calculation.

Thus, applying the Van der Corput Lemma,

$$\begin{aligned} & \int_0^\infty \Psi(ds_1^2\beta x) x^{-\frac{1}{4}} e\left(\frac{2\sqrt{nx}}{c_1''c_{2,j}''}\right) dx \\ & \ll \frac{G}{ds_1^2T_1} (ds_1^2T_1)^{-\frac{1}{4}} \frac{T_1^2T_2}{K^2c\beta} = \frac{GT_1^{\frac{3}{4}}T_2}{cK^2\beta(ds_1^2)^{\frac{5}{4}}} \end{aligned}$$

which is superior to what is claimed in the beginning of the section.

In the case of $\Phi(ds_1^2\beta x, y) - \frac{2\sqrt{nx}}{c_1''c_{2,j}''}$,

$$\frac{\partial}{\partial x}\left(\Phi(ds_1^2\beta x, y) - \frac{2\sqrt{nx}}{c_1''c_{2,j}''}\right) = \left(\pi^2 - \frac{1}{2}\right) \frac{K^2cds_1^2\beta}{x^2t_2} - \frac{\sqrt{n}}{c_1''c_{2,j}''\sqrt{x}}(1 + o(x^{-1}))$$

which is bounded away from $\frac{K^2c\beta}{T_1^2T_2}$ unless

$$\frac{K^4c^2c_1''^2c_{2,j}''^2ds_1^2\beta^2}{T_1^3T_2^2} \left(\pi^2 - \frac{1}{2}\right)^2 \frac{1}{27} \leq n \leq \frac{K^4c^2c_1''^2c_{2,j}''^2ds_1^2\beta^2}{T_1^3T_2^2} \left(\pi^2 - \frac{1}{2}\right)^2 8$$

when $\frac{ds_1^2T_1}{2} \leq x \leq 3ds_1^2T_1$.

Thus, when n is not in the above range we proceed as before using Van der Corput's Lemma which will yield the same upper bound for our oscillatory integral as before. When n is in the above range, we again use the Van der Corput lemma. Note that

$$\frac{\partial^2}{\partial x^2}\left(\Phi(ds_1^2\beta x, y) - \frac{2\sqrt{nx}}{c_1''c_{2,j}''}\right)$$

$$= \left[-2 \left(\pi^2 - \frac{1}{2} \right) \frac{K^2 c (ds_1^2)^2 \beta}{x^3 T_2} + \frac{\sqrt{n}}{2x^{\frac{3}{2}} c_1'' c_{2,j}''} \right] (1 + o(x^{-1})).$$

which is bounded below by

$$\frac{K^2 c \beta}{ds_1^2 T_1^3 T_2}$$

in our range of n . Hence,

$$\begin{aligned} & \int_0^\infty \Psi(x) x^{-\frac{1}{4}} e \left(-\frac{2\sqrt{nx}}{c_1'' c_{2,j}''} \right) dx \ll \frac{G(ds_1^2 T_1)^{-\frac{1}{4}}}{ds_1^2 T_1} \sqrt{\frac{ds_1^2 T_1^3 T_2}{K^2 c \beta}} \\ &= G(ds_1^2)^{-\frac{3}{4}} \frac{T_1^{\frac{1}{4}} \sqrt{T_2}}{\sqrt{c} K \sqrt{\beta}}. \end{aligned}$$

4.8 Estimating the sum of twisted exponential sums

The main purpose of this section is to estimate the sum of exponential sums twisted by the Jacobi-Legendre symbol. From Section 4.4, we know that for the sum in consideration to be non-zero we must have

$$((t_2 + \bar{4}u)u, c_1) = p_1^{k_1-1} \dots p_l^{k_l-1} \quad (4.8.1)$$

with k_i odd for all i . Let $q_i = c_2 \prod_{j \neq i} p_j^{k_j}$.

Assume without loss of generality that $t_2 < c_1$ (by taking $t_2 \pmod{c_1}$ if necessary). For each $0 \leq u, t_2 < c_1$, we write

$$u = \sum_{i=1}^l x_i q_i \bar{q}_i + x c_1, \quad 0 \leq x_i < p_i^{k_i}, 0 \leq x < c_2$$

and

$$t_2 = \sum_{i=1}^l y_i q_i \bar{q}_i + y c_1, \quad 0 \leq y_i < p_i^{k_i}, 0 \leq y < c_2$$

such that $q_i \bar{q}_i \equiv 1(p_i^{k_i})$ and $c_1 \bar{c}_1 \equiv 1(c_2)$. Then (4.8.1) is true if and only

$$\left(\left(\sum_{i=1}^l (y_i + \bar{4}x_i) q_i \bar{q}_i + (x + y) c_1 \right) \left(\sum_{i=1}^l x_i q_i \bar{q}_i + x c_1 \right), p_j^{k_j} \right) = p_j^{k_j-1}$$

for all $1 \leq j \leq l$. This is equivalent to

$$((y_i + \bar{4}x_i)x_i, p_i^{k_i}) = p_i^{k_i-1}$$

for all $1 \leq i \leq l$, since $(q_i^2 \bar{q}_i^2, p_i^{k_i}) = 1$ for all i . We now state what (4.8.1) means to x_i .

Lemma 4.8.1. *For each $1 \leq i \leq l$, (4.8.1) is equivalent to*

(a) *if $(y_i, p_i^{k_i}) = 1$ then either*

$$x_i = p_i^{k_i-1}4 - 4y_i$$

given that $k_i > 1$ or

$$x_i = p_i^{k_i-1}b$$

for any $0 < b < p_i$;

(b) *if $(y_i, p_i^{k_i}) = p_i^t$ for some $0 < t < \frac{k_i-1}{2}$ then*

$$x_i = p_i^{k_i-1-t}b$$

for any $0 < b < p_i$;

(c) *if $(y_i, p_i^{k_i}) = p_i^{\frac{k_i-1}{2}}$, i.e. $y_i = p_i^{\frac{k_i-1}{2}}a$ for some $0 < a < p_i$ then*

$$x_i = p_i^{\frac{k_i-1}{2}}b$$

where $b \not\equiv -4a(p_i)$;

(d) *if $(y_i, p_i^{k_i}) = p_i^t$ with $t > \frac{k_i-1}{2}$ then*

$$x_i = p_i^{\frac{k_i-1}{2}}b$$

where $0 < b < p_i$.

Proof. Since the proofs are similar in nature, we will only prove (b) and (c). Note also that k_i is odd throughout the proofs below.

(b) We first show that $p_i | x_i$. Assuming the contrary then

$$((y_i + \bar{4}x_i)x_i, p_i^{k_i}) = 1$$

which is a contradiction. Hence $x_i = p_i^r b$ for some $0 < b < p_i$, then

$$(y_i + \bar{4}x_i)x_i = \left(p_i^{t-\min\{t,r\}} a + \bar{4}p_i^{r-\min\{t,r\}} b \right) p_i^{r+\min\{r,t\}} b.$$

Since $2t < k_i - 2$ and $p_i^2 \nmid p_i^{t-\min\{t,r\}} a + \bar{4}p_i^{r-\min\{t,r\}} b$, we must have $r \neq t$. Then either $t - \min\{r, t\} \neq 0$ or $r - \min\{r, t\} \neq 0$, in any case, $(p_i^{t-\min\{t,r\}} a + \bar{4}p_i^{r-\min\{t,r\}} b) = 1$. Thus, $r + \min\{r, t\} = k_i - 1$. We conclude in this case that $r = k_i - 1 - t$.

(c) Write $x_i = p_i^r b$ then

$$(y_i + \bar{4}x_i)x_i = \left(p_i^{\frac{k_i-1}{2}-\min\{r, \frac{k_i-1}{2}\}} a + \bar{4}p_i^{r-\min\{\frac{k_i-1}{2}, r\}} b \right) p_i^{r+\min\{r, \frac{k_i-1}{2}\}} b.$$

If $r < \frac{k_i-1}{2}$ then $r + \min\{r, \frac{k_i-1}{2}\} = 2r = k_i - 1$ which is impossible.

If $r > \frac{k_i-1}{2}$ then $r + \min\{r, \frac{k_i-1}{2}\} = r + \frac{k_i-1}{2} > k_i - 1$.

If $r = \frac{k_i-1}{2}$ and $b \not\equiv 4a(p_i)$. □

With Lemma 4.8.1, we can break $\sum_{u(c)} ((t_2 + \bar{4}u)u, c_1) = \prod_{i=1}^l p_i^{k_i-1}$ into a product of local sums.

For example if $(y_1, p_1^{k_1}) = 1$ and $(y_i, p_i^{k_i}) = p_i^t$ for some $t > \frac{k_i-1}{2}$ for all $i \geq 2$ then

$$\sum_{\substack{u(c) \\ ((t_2 + \bar{4}u)u, c_1) = \prod_{1 \leq i \leq l} p_i^{k_i-1} \\ (u, c_2) = 2^j}} = \sum_{\substack{x_1 = p_1^{k_1-1} 4 - 4y_1 \\ x_i = p_i^{\frac{k_i-1}{2}} b \\ 0 < b_i < p_i \\ \text{for all } i \geq 2 \\ (u, c_2) = 2^j}} + \sum_{\substack{x_i = p_i^{k_i-1} b_i \\ 0 < b_i < p_i \\ \text{for all } i \\ (u, c_2) = 2^j}}.$$

In each of these summands, $\text{val}_{p_i}(c_1'')$ is the same. Thus the integral

$$\int_0^\infty \Psi(ds_1^2 \beta x) x^{-\frac{1}{4}} \cos \left(\frac{4\pi \sqrt{nx}}{c_1'' c_{2,j}''} - \frac{it_\phi \pi}{2} - \frac{\pi}{4} \right) dx$$

can be pulled out of each of the sum.

Note that

$$\begin{aligned} & \left(\frac{(t_2 + \bar{4}u)u/(p_1^{k_1-1} \cdots p_l^{k_l-1})}{p_1 \cdots p_l} \right)_L e\left(\frac{ua_2}{c_2}\right) e\left(\frac{\overline{u'\beta'n}}{c_1''c_{2,j}''}\right) \\ &= \prod_{i=1}^l \left(\frac{[(y_i + \bar{4}x_i)x_i/p_i^{k_i-1}]\delta_i}{p_i} \right)_L e\left(\frac{xc_1\bar{c}_1a_2}{c_2}\right) e\left(\frac{\overline{\frac{u}{(u,c_1')}\beta'n}}{c_1''c_{2,j}''}\right) \end{aligned}$$

for some $(\delta_i, p_i) = 1$.

Write $c_1' = \frac{c_1}{(\beta, c_1)} = p_1^{k_1'} \cdots p_l^{k_l'}$. Since $(u, c_1') = \prod_i (u, p_i^{k_i'}) = \prod_i (x_i, p_i^{k_i'})$, we have

$$\begin{aligned} & \left(\frac{(t_2 + \bar{4}u)u/(p_1^{k_1-1} \cdots p_l^{k_l-1})}{p_1 \cdots p_l} \right)_L e\left(\frac{ua_2}{c_2}\right) e\left(\frac{\overline{u'\beta'n}}{c_1''c_{2,j}''}\right) \\ &= \prod_{i=1}^l \left(\frac{[(y_i + \bar{4}x_i)x_i/p_i^{k_i-1}]\delta_i}{p_i} \right)_L e\left(\frac{xa_2}{c_2}\right) \\ &\times e\left(\frac{\sum_i \frac{x_i}{(x_i, p_i^{k_i'})} \frac{q_i \bar{q}_i}{2^j \prod_{t \neq i} (x_t, p_t^{k_t'})} + \frac{x}{(x, 2^j)} c_1 \bar{c}_1 \beta' n}{c_1''c_{2,j}''}\right) \\ &= \prod_{i=1}^l \left\{ \left(\frac{[(y_i + \bar{4}x_i)x_i/p_i^{k_i-1}]\delta_i}{p_i} \right)_L e\left(\frac{\overline{\frac{x_i}{(x_i, p_i^{k_i'})} \delta_i' \beta' n}}{p_i^{k_i'}}\right) \right\} \\ &\times e\left(\frac{xa_2}{c_2}\right) e\left(\frac{\overline{\frac{x}{(x, 2^j)} \beta' \delta' n}}{c_{2,j}''}\right) \end{aligned}$$

for some $(\delta_i', p_i^{k_i'}) = 1$ and $(\delta', c_{2,j}'') = 1$.

We are going to estimate the local sum (the sum is over all x_i satisfying the conditions on P.82)

$$S_{i,n} := \sum \left(\frac{[(y_i + \bar{4}x_i)x_i/p_i^{k_i-1}]\delta_i}{p_i} \right)_L e\left(\frac{\overline{\frac{x_i}{(x_i, p_i^{k_i'})} \delta_i' \beta' n}}{p_i^{k_i'}}\right)$$

according to the different cases in Lemma 4.8.1.

Case 1: $(y_i, p_i^{k_i}) = 1$, $k_i > 1$ and $x_i = p_i^{k_i-1}4 - 4y_i$.

In this case, the sum is over a single element. Hence,

$$|S_{i,n}| \leq 1.$$

Case 2: $(y_i, p_i^{k_i}) = 1$ and $x_i = p_i^{k_i-1}b$, $0 < b < p_i$ (see Lemma 4.4.1).

$$S_{i,n} = \sum_{b=1}^{p_i-1} \left(\frac{\delta_i(y_i + \overline{4}p_i^{k_i-1}b)}{p_i} \right)_L e \left(\frac{\overline{b}\delta'_i\overline{\beta}'n}{p_i^{k'_i}} \right) \begin{cases} = \sqrt{p_i} \text{ if } (n, p_i^{k'_i}) = p_i^{k'_i-1} \\ = 0 \text{ if } k'_i = 1, (n, p_i) = p_i \\ \leq p_i \text{ if } k'_i > 1. \end{cases}$$

Case 3: $y_i = p_i^t a_i$ for some $0 < t < \frac{k_i-1}{2}$.

$$S_{i,n} = \sum_{b=1}^{p_i-1} \left(\frac{b(a_i + \overline{4}p_i^{k_i-1-2t}b)\delta_i}{p_i} \right)_L e \left(\frac{\overline{b}\delta'_i\overline{\beta}'n}{p_i^{k'_i}} \right).$$

Then,

$$|S_{i,n}| \leq \begin{cases} \sqrt{p_i} \text{ if } k'_i = 1 \\ p_i \text{ if } k'_i > 1. \end{cases}$$

Case 4: $y_i = p_i^{\frac{k_i-1}{2}} a_i$ for some $0 < a_i < p_i$.

$$S_{i,n} = \left(\frac{\delta'_i}{p_i} \right)_L \sum_{\substack{0 < b < p_i \\ b \neq -4a_i(p_i)}} \left(\frac{(a_i\overline{b} + \overline{4})\delta_i}{p_i} \right)_L e \left(\frac{\overline{b}\delta'_i\overline{\beta}'n}{p_i^{k'_i}} \right).$$

Then,

$$|S_{i,n}| \leq \begin{cases} \sqrt{p_i} \text{ if } k'_i = 1, (n, p_i) = 1 \\ 1 \text{ if } k'_i = 1, (n, p_i) = p_i \\ p_i \text{ if } k'_i > 1. \end{cases}$$

Case 5: $y_i = p_i^t a_i$ for some $t > \frac{k_i-1}{2}$ and $0 < a_i < p_i$.

$$S_{i,n} = \left(\frac{\delta'_i}{p_i} \right)_L \sum_{0 < b < p_i} e \left(\frac{\overline{b}\delta'_i\overline{\beta}'n}{p_i^{k'_i}} \right).$$

Hence,

$$|S_{i,n}| \leq \begin{cases} p_i & \text{if } k'_i = 1, (n, p_i) = p_i \\ 1 & \text{if } k'_i = 1, (n, p_i) = 1 \\ p_i & \text{if } k'_i > 1. \end{cases}$$

In case 2-5, when $k'_i > 1$, we have $(x_i, p_i^{k'_i}) \geq p_i$ and hence

$$(c''_1, p_i^{k_i}) \leq \frac{(c_1, p_i^{k_i})}{p_i}$$

and

$$c''_1 \leq c_1 \prod_{k'_i > 1} \frac{1}{p_i}.$$

4.9 Proof of Lemma 4.4.1

We need the following stationary phase lemma from [6] (Lemma 5.5.4 on P.114).

Lemma 4.9.1. *Let $f(x)$ be a real function, four times continuously differentiable for $\beta_1 \leq x \leq \beta_2$, and let $g(x)$ also be a real function, three times continuously differentiable for $\beta_1 \leq x \leq \beta_2$. Suppose that there are positive parameters M, N, S and U with*

$$M \geq \beta_2 - \beta_1, \quad N \geq \frac{M}{\sqrt{S}}$$

and positive constants C_r such that for $\beta_1 \leq x \leq \beta_2$,

$$|f^{(r)}(x)| \leq \frac{C_r S}{M^r}, \quad |g^{(s)}(x)| \leq \frac{C_s U}{N^s}$$

for $2 \leq r \leq 4$ and $0 \leq s \leq 3$, and

$$f''(x) \leq -\frac{S}{C_2 M^2}. \quad \left(\text{resp. } f''(x) \geq \frac{S}{C_2 M^2} \right)$$

Suppose also that $f'(x)$ changes sign from positive (resp. negative) to negative

(reps. positive) at a point γ with $\beta_1 < \gamma < \beta_2$. If S is sufficiently large in terms of the constant C_r , then we have

$$\begin{aligned}
& \int_{\beta_1}^{\beta_2} g(x)e(f(x))dx = \frac{g(\gamma)e(f(\gamma) - \frac{1}{8})}{\sqrt{|f''(\gamma)|}} \left(\text{resp. } \frac{g(\gamma)e(f(\gamma) + \frac{1}{8})}{\sqrt{|f''(\gamma)|}} \right) \\
& + \sum_{j=1}^2 \frac{(-1)^j g(\beta_j)e(f(\beta_j))}{2\pi i f'(\beta_j)} \left(\text{resp. } \sum_{j=1}^2 \frac{(-1)^{j-1} g(\beta_j)e(f(\beta_j))}{2\pi i f'(\beta_j)} \right) \\
& + O\left(\frac{M^4 U}{S^2} \left(1 + \frac{M}{N}\right)^2 \left(\sum_{j=1}^2 \frac{(-1)^{j-1}}{(\gamma - \beta_j)^3}\right)\right) \\
& + O\left(\frac{MU}{S^{\frac{3}{2}}} \left(1 + \frac{M}{N}\right)^2\right)
\end{aligned}$$

Proof. To prove the respective statement in parenthesis we just have to take complex conjugate and notice that $-f$ satisfies the conditions in Lemma 5.5.4 in [6]. $\square$

Lemma 4.9.2. For each $p \in \mathbb{N} \cup \{0\}$, let $\kappa(t) = z \left(-2\pi^2 t^2 + \frac{2}{3}\pi^4 t^4 \right) + ty$, where $y \sim K$, $z \gg \frac{KG}{K^{\frac{\epsilon}{2}}}$. Then (i) the roots of $\kappa'(t)$ are $\gamma_1 = -\sqrt{\frac{3}{2}}\frac{1}{\pi} - t_1$, $\gamma := \gamma_2 = t_2$, $\gamma_3 = \sqrt{\frac{3}{2}}\frac{1}{\pi} - t_3$ with $t_j := t_j(y, z) \sim \frac{y}{z} \ll \frac{K^{\frac{\epsilon}{2}}}{G}$; (ii) let β_j , $j = 1, \dots, 4$ be such that $\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3 < \gamma_3 < \beta_4$; we have

$$\begin{aligned}
\int_{-\infty}^{\infty} e(\kappa(t))dt &= \sum_{j=1}^3 \frac{e(\kappa(\gamma_j) + (-1)^{j-1}\frac{1}{8})}{\sqrt{|\kappa''(\gamma_j)|}} \\
&+ \sum_{j=1}^4 \frac{(-1)^{j-1} e(\kappa(\beta_j))}{\pi i \kappa'(\beta_j)} + O(z^{-\frac{3}{2}})
\end{aligned}$$

Proof. (i) A simple consequence of intermediate value Theorem (see P.883 of [25]).

(ii)

$$\begin{aligned}
\int_{-\infty}^{\infty} e(\kappa(t))dt &= \int_{\beta_4}^{\infty} e(\kappa(t))dt + \sum_{j=1}^3 \int_{\beta_j}^{\beta_{j+1}} e(\kappa(t))dt \\
&+ \int_{-\infty}^{\beta_1} e(\kappa(t))dt.
\end{aligned}$$

We get the main term by applying Lemma 4.6.1 to each $\int_{\beta_j}^{\beta_{j+1}} e(\kappa(t))dt$ with $g(t) = 1$, $S = z$, $N = 1$, $M = \beta_{j+1} - \beta_j$, $U = 1$, $C_r = (\beta_{j+1} - \beta_j)^r 100\pi^4$. We will now show that

$$\int_{-\infty}^{\beta_1} e(\kappa(t))dt = \frac{e(\kappa(\beta_1))}{2\pi i \kappa'(\beta_1)} + O(z^{-2})$$

and

$$\int_{\beta_4}^{\infty} e(\kappa(t))dt = -\frac{e(\kappa(\beta_4))}{2\pi i \kappa'(\beta_4)} + O(z^{-2}).$$

Integrating by parts twice

$$\begin{aligned} \int_{-\infty}^{\beta_1} e(\kappa(t))dt &= \frac{e(\kappa(\beta_1))}{2\pi i \kappa'(\beta_1)} + \frac{\kappa''(\beta_1)e(\kappa(\beta_1))}{2\pi i (\kappa'(\beta_1))^3} \\ &\quad - \frac{1}{(2\pi i)^2} \int_{-\infty}^{\beta_1} \frac{d}{dt} \left(\frac{\kappa''(t)}{(\kappa'(t))^3} \right) \cdot e(\kappa(t)) dt \\ &= \frac{e(\kappa(\beta_1))}{2\pi i \kappa'(\beta_1)} + O(z^{-2}) \end{aligned}$$

Similarly,

$$\int_{\beta_4}^{\infty} e(\kappa(t))dt = -\frac{e(\kappa(\beta_4))}{2\pi i \kappa'(\beta_4)} + O(z^{-2}).$$

□

We now turn to the proof of Lemma 4.4.1. First of all, for each j integrating by parts repeatedly shows that

$$\begin{aligned} &\frac{g_{T_1} \left(\frac{x}{ds_1^2} \right) e \left(z \left(-2\pi^2 \beta_j^2 + \frac{2}{3} \pi^4 \beta_j^4 \right) \right)}{\sqrt{x}} \int_0^{\infty} h \left(\frac{y - K - 1}{G} \right) \\ &\quad \times H \left(\frac{4\pi^3 d^2 s_2^2 t_2^2 x}{y^2} \right) \frac{e(\beta_j y)}{z \left(-4\pi^2 \beta_j + \frac{8}{3} \pi^4 \beta_j^3 \right) + y} dy \end{aligned}$$

contributes $O(G^{-A})$ for all $A > 0$. Hence the contribution from

$$\sum_{j=1}^4 \frac{(-1)^{j-1} e(\kappa(\beta_j))}{\pi i \kappa'(\beta_j)}$$

is negligible.

For $j \neq 2$, let

$$\chi(y) = z \left(-2\pi^2 \gamma_j^2 + \frac{2}{3} \pi^4 \gamma_j^4 \right) + \gamma_j y$$

then differentiating with respect to y

$$\chi'(y) = \left[z \left(-4\pi^2 \gamma_j + \frac{8}{3} \pi^4 \gamma_j^3 \right) + y \right] \frac{\partial \gamma_j}{\partial y} + \gamma_j.$$

Since $\kappa'(\gamma_j) = 0$, one gets $z \left(-4\pi^2 \gamma_j + \frac{8}{3} \pi^4 \gamma_j^3 \right) + y = 0$ hence $\chi'(y) = \gamma_j \asymp 1$.

Integrating by parts repeatedly show that the integral involving γ_j , $j \neq 2$ contributes $O(K^{-A})$ for all $A > 0$. On the other hand $O\left(\left(\frac{T_1 T_2}{c}\right)^{-\frac{3}{2}}\right)$ contributes

$$\begin{aligned} & O\left(\int_0^\infty h\left(\frac{y-K-1}{G}\right) \left| H\left(\frac{4\pi^3 d^3 s_1^2 s_2^4 t_2^2 T_1}{y^2}\right) \right| \left(\frac{c}{T_1 T_2}\right)^{\frac{3}{2}} dy\right) \\ &= O\left(G\left(\frac{c}{T_1 T_2}\right)^{\frac{3}{2}}\right). \end{aligned}$$

This concludes the proof of Lemma 4.4.1 by taking $z = \frac{2xt_2}{c}$.

4.10 Concluding the proof of the main theorem

For each ds_1^2 such that $d \ll K^{\frac{2}{3}+\frac{\epsilon}{2}}$, $s_2 \ll K^{1+\frac{\epsilon}{2}}$, $T_2 |2^\infty$, $T_2 \ll K^{1+\epsilon}$, $T_1 |2^\infty$, $T_1 \ll \frac{K^{2+\frac{\epsilon}{2}}}{T_2^2}$ and $d^3 s_1^2 s_2^4 T_1 T_2^2 \ll K^{2+\frac{\epsilon}{4}}$. (4.6.1) is at most a sum of $\omega(c_1)$ (recall that $\omega(n)$ is the number of distinct prime factor of n), each of which is

(4.10.1)

$$\begin{aligned}
& \left[\frac{c_1^{\frac{3}{2}} c_2^{\frac{5}{2} + \epsilon}}{\sqrt{c_1'' c_{2,j}''}} \sqrt{\prod_{i=1}^l p_i} \frac{G(ds_1^2)^{-\frac{3}{4}} T_1^{\frac{1}{4}} \sqrt{T_2}}{\sqrt{c} K \sqrt{\beta}} \sum_{\substack{n \ll \frac{K^{4+\frac{6}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2} \\ (n, (t_2, c_1')) = 1}} \right. \\
& + \sum_{i=1}^l \sum_{\substack{n \ll \frac{K^{4+\frac{6}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2} \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} + \sum_{\substack{i_1 \neq i_2 \\ 1 \leq i_1, i_2 \leq l}} \sum_{\substack{n \ll \frac{K^{4+\frac{6}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2} \\ p_{i_1} p_{i_2} | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } j \neq i_1, i_2}} \\
& \left. + \cdots + \sum_{\substack{n \ll \frac{K^{4+\frac{6}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2} \\ p_1 \cdots p_l | (n, (t_2, c_1'))}} \right] \frac{|\lambda_\phi(n)| \prod_i |S_{i,n}|}{n^{\frac{1}{4}}}
\end{aligned}$$

Using the bound $\sum_{n \sim A} |\lambda_\phi(n)|^2 \ll A$ (see P.246 (13.53) of [7]), and summing $\sum_{\substack{n \ll B \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} |\lambda_\phi(n)| n^{-\frac{1}{4}} \prod_i |S_{i,n}|$ dyadically,

$$\begin{aligned}
& \sum_{\substack{n \ll B \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} |\lambda_\phi(n)| n^{-\frac{1}{4}} \prod_i |S_{i,n}| = \sum_{\substack{T | 2^\infty \\ T \ll B}} \sum_{\substack{n \sim T \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} |\lambda_\phi(n)| n^{-\frac{1}{4}} \prod_i |S_{i,n}| \\
& \leq \prod_{\substack{k'_j > 1 \\ j \neq i}} p_j \prod_{\substack{k'_j = 1 \\ j \neq i}} \sqrt{p_j} \cdot p_i \sum_{\substack{T | 2^\infty \\ T \ll B}} \sum_{\substack{n \sim T \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} |\lambda_\phi(n)| n^{-\frac{1}{4}} \\
& \ll \prod_{\substack{k'_j > 1 \\ j \neq i}} p_j \prod_{\substack{k'_j = 1 \\ j \neq i}} \sqrt{p_j} \cdot p_i \sum_{\substack{T | 2^\infty \\ T \ll B}} \frac{|\lambda_\phi(p_i)|}{T^{\frac{1}{4}}} \sqrt{\sum_{\substack{n \sim T \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} |\lambda_\phi(n)|^2 \sum_{\substack{n \sim T \\ p_i | (n, (t_2, c_1')) \\ p_j \nmid (n, (t_2, c_1')) \\ \text{for all } i \neq j}} 1}
\end{aligned}$$

$$\begin{aligned}
&\ll \prod_{\substack{k'_j > 1 \\ j \neq i}} p_j \prod_{\substack{k'_j = 1 \\ j \neq i}} \sqrt{p_j} \cdot p_i \frac{B^{\frac{3}{4}} |\lambda_\phi(p_i)|}{p_i} \\
&\ll \prod_{\substack{k'_j > 1 \\ j \neq i}} p_j \prod_{\substack{k'_j = 1 \\ j \neq i}} \sqrt{p_j} \cdot \sqrt{p_i} B^{\frac{3}{4}}
\end{aligned}$$

using bound toward Ramanujan's conjecture for Hecke -Maass forms (we use $\lambda_\phi(p) \ll p^{\frac{1}{2}}$ here). It can be shown in a similar fashion that for any $1 \leq i_1, \dots, i_r \leq l$,

$$\sum_{\substack{n \ll B \\ p_{i_1} \cdots p_{i_r} | (n, (t_2, c'_1)) \\ p_j \nmid (n, (t_2, c'_1)) \\ \text{for all } j \neq i_1, \dots, i_r}} \frac{|\lambda_\phi(n)|}{n^{\frac{1}{4}}} \prod |S_{i,n}| \ll \prod_{k'_i > 1} p_i \prod_{k'_i = 1} \sqrt{p_i} B^{\frac{3}{4}}.$$

Hence using Weil's bound for the Kloosterman sum in dealing with the sum over c_2 and the fact that

$$c_1'' \leq c_1 \prod_{k'_i > 1} \frac{1}{p_i},$$

we have (4.10.1)

$$\begin{aligned}
&\ll \frac{2^{\omega(c_1)} c_1^{\frac{3}{2}} c_2^{\frac{5}{2} + \epsilon}}{\sqrt{c_1'' c_{2,j}''} \sqrt{\prod_{i=1}^l p_i}} \prod_{k'_i > 1} p_i \prod_{k'_i = 1} \sqrt{p_i} \frac{G(ds_1^2)^{-\frac{3}{4}} T_1^{\frac{1}{4}} \sqrt{T_2}}{\sqrt{c} K \sqrt{\beta}} \\
&\times \left(\frac{K^{4 + \frac{\epsilon}{100}} c^2 c_1''^2 c_{2,j}''^2 ds_1^2}{T_1^3 T_2} \right)^{\frac{3}{4}} \\
&\ll \frac{2^{\omega(c_1)} G K^{2 + \epsilon} c c_1^{\frac{3}{2}} c_2^{\frac{5}{2} + \epsilon} c_1 c_2}{T_1^2 \sqrt{T_2}}.
\end{aligned}$$

Recall that

$$\omega(c_1) \ll \frac{\log c_1}{\log \log c_1}.$$

Thus, fix any $\epsilon > 0$ for c_1 large enough (the case when c_1 small is trivial),

$$\omega(c_1) \ll c_1^\epsilon.$$

Hence, (4.3.1)

$$\begin{aligned}
& \ll K^\epsilon \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \sum_{\substack{c_1 c_2 \ll \frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG} \\ c_2|2^\infty \\ (c_1, c_2)=1}} (c_1 c_2)^{-\frac{3}{2}} \sum_{t_2 \sim T_2} \frac{1}{t_2^{\frac{3}{2}}} \\
& \times \frac{G K^{2+\epsilon} c_1^{\frac{7}{2}} c_2^{\frac{9}{2}+\epsilon}}{T_1^2 \sqrt{T_2}} \\
& \ll G K^{2+3\epsilon} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_2 \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \frac{1}{T_1^2 T_2} \left(\frac{T_1 T_2 K^{\frac{\epsilon}{2}}}{KG} \right)^3 \\
& \ll G K^{2+5\epsilon} \sum_{\substack{T_2|2^\infty \\ T_2 \ll K^{1+\epsilon}}} \sum_{\substack{T_1|2^\infty \\ T_1 \ll \frac{K^{2+\frac{\epsilon}{4}}}{T_2^2}}} \frac{T_1 T_2^2}{K^3 G^3} \ll \frac{G K^{5\epsilon} K}{G^3} \\
& \leq G K^{5\epsilon}
\end{aligned}$$

whenever $G \geq K^{\frac{1}{3}}$. This concludes the proof of our main Theorem.

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