On the Use of Physical Basis Functions in a Sparse Expansion for Electromagnetic
Scattering Signatures

THESIS

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By

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Abstract

Radar images are created from measurements of the electromagnetic field scattered from an object or scene of interest. The scattered field defines the radar signature as a function of frequency and aspect angle. High resolution radar images and radar signatures are used for target recognition, tracking, and hardware-in-the-loop testing. High resolution radar images of electrically large targets may require a large amount of data to be measured, stored, and processed. A sparse representation of this data may allow the radar signature to be efficiently measured, stored, and rapidly reconstructed on demand.

Compressed sensing is applied to obtain the sparse representation without measuring the full data set. “Compressed sensing” has different interpretations, but in this thesis it refers to using non-adaptive, random samples of the measured signal, with no *a priori* knowledge of the signal. According to compressed sensing theory, this is possible if the radar signature can be expressed in terms of a sparse basis. If a signal $\mathbf{y}$ can be approximated by K non-zero coefficients in the sparse basis (“ K -sparse”), the coefficients may be obtained with random sampling of the signal at sub-Nyquist rates provided that K is much smaller than the total number of Nyquist samples. The random sampling is non-adaptive (i.e., future samples are independent of previous samples) and the number of

samples required is primarily related to the sparseness of the signal, and not the bandwidth nor the size of the dictionary from which the basis functions are selected.

The objective of this thesis is to investigate the effectiveness of physical basis functions, defined as point scatter functions with frequency-dependent amplitudes characteristic of physical scattering mechanisms, to provide an improved sparse basis in which to expand radar signatures. The goal is to represent a radar signature accurately with the fewest terms possible and with the fewest measurements. Use of physical basis functions also provides insight into the scattering mechanism that is the source of the scattering. If the scattering mechanism is not known *a priori* or if a combination of scattering mechanisms is present in a single scattering center, a combined physical basis function is shown to provide a much more efficient representation. The angular dependence, which is usually not as simple as the frequency dependence, is incorporated using a low-order polynomial defined over limited angular sectors.

The closed-form physical optics solution for the far-field electromagnetic backscatter from flat perfect electrically conducting plates is used to demonstrate the form of the physical basis functions and their efficacy as a sparse basis in which to expand the scattered signal. The coefficients of the physical basis functions are found using the Orthogonal Matching Pursuits algorithm to solve the l_0 -minimization problem. Convergence of the physical basis function reconstruction of the signal is verified in terms of the mean square error with respect to the full data set. Convergence is also demonstrated in terms of the number of measured samples in the compressed sensing

algorithm—an important consideration for practical applications that is often overlooked in theoretical works.

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Chapter 1: Introduction

Radar images are created from measurements of the electromagnetic (EM) field scattered from an object or scene of interest. The scattered field defines the radar signature as a function of frequency and aspect angle. High resolution radar images and radar signatures are used for target recognition, tracking, and hardware-in-the-loop testing. High resolution radar images of electrically large targets may require a large amount of data to be measured, stored, and processed [1]. A sparse representation of this data may allow the radar signature to be efficiently measured, stored, and rapidly reconstructed on demand.

Compressed sensing [2] has several interpretations and many applications, but in this thesis it refers to the reconstruction a radar signature using non-adaptive, random samples of the measured signal. If the radar signature can be expressed in terms of a sparse basis, it may be possible to reconstruct a signature without measuring the full data set, while minimizing the error in the reconstructed signal. According to compressed sensing theory, this is possible if the radar signature can be expressed in terms of a sparse basis. If a signal $\mathbf{y}$ can be approximated by K non-zero coefficients in the sparse basis (“ K -sparse”), the coefficients may be obtained with random sampling of the signal at sub-Nyquist rates provided that K is much smaller than the total number of Nyquist samples. A signal is K -sparse if it can be represented as the sum of K basis functions a_{mn} , i.e. the signal is the column vector $\mathbf{y} = (y_1, y_2, \dots, y_M)^T$,

$$y_m = \sum_{n=1}^K x_n a_{mn} \quad (1)$$

and x_n are the coefficients in the sparse basis. Most real signals are not truly K -sparse, but a signal is said to be compressible if the magnitudes of the coefficients x_n decay rapidly as n increases. The random sampling is non-adaptive and the number of samples required is related to the sparseness of the signal, not the bandwidth nor the size of the dictionary from which the basis functions are selected.

For electrically large objects, the scattered field may be expanded as the sum of individual scattering centers [3]. Potter describes the frequency dependence of the scattered EM field predicted by the Geometrical Theory of Diffraction (GTD) and defines a scattering model based on the sum of individual point scatter functions with frequency dependent amplitudes [4]. The closed-form physical optics (PO) solution to the scattered electromagnetic field also shows that the high-frequency scattered field can be expanded as the sum of individual scattering centers with frequency dependent amplitudes.

Several authors have developed parametric models of radar scattering based on the form of PO and GTD scattering. Gerry et al. present a parametric model of radar scattering as a functions of frequency and aspect angle in which the parameters are physically significant [5]. The parameter estimation in the image domain provides an estimate of the position, size, shape, and orientation of the scattering centers. Jackson, et al. derives parametric models for several canonical shapes for 3D bistatic synthetic aperture radar scattering [6]. Trintinalia, et al. parameterize backscatter data using a

modified version of the adaptive Gaussian representation (AGR) algorithm to adaptively extract the parameters of their aspect-dependent amplitude scattering center model [7].

O'Donnell shows that the representation of the radar signature in a simple point scatter basis allows compression of the data required to characterize a radar target [8].

The use of compressed sensing for radar signals to increase resolution or reduce sampling rate is described by Whiteloni and Ling in [9] and Herman and Strohmer in [10].

Burns reviews adaptive methods for expanding radar signals into physically-based basis functions, including Matching Pursuits and Basis Pursuits [11]. McClure and Carin use Matching Pursuits as developed by Mallat [12] on a wave-based dictionary of basis functions based on wavefronts, resonances, and chirps related to physical characteristics of the target. The results are used for target identification [13].

The objective of this thesis is to investigate the effectiveness of physical basis functions, defined as point scatter functions with frequency-dependent amplitudes characteristic of physical scattering mechanisms, to provide an improved sparse basis in which to expand radar signatures, as compared to a simple point scatter basis. Use of physical basis functions also provides insight into the scattering mechanism that is the source of the scattering. If the scattering mechanism is not known *a priori* or if a combination of scattering mechanisms is present in a single scattering center, a combined physical basis function is shown to provide a much more efficient representation. The angular dependence, which is usually not as simple as the frequency dependence, is incorporated using a low-order polynomial defined over limited angular sectors.

In this thesis, the physical optics solution for the far-field electromagnetic backscatter from flat perfect electrically conducting (PEC) plates is used to demonstrate the form of the physical basis functions. As an example, the scattering from flat plates is used to compare the effectiveness of physical basis functions as a sparse basis in which to expand the scattered signal. The coefficients of the PBFs are found using the Matching Pursuits and Orthogonal Matching Pursuits algorithms. Convergence of the PBF expansions is verified in terms of the mean square error (MSE). The goal is to represent a radar signature in a sparse basis with the fewest terms possible while maintaining a very small MSE. The approach taken is as follows:

1. Derive the physical optics approximation for the far-field plane wave scattering from a PEC object. Specifically, derive the closed-form, analytical solution for the backscatter from a rectangular plate and a circular plate.
2. Extract the physical basis functions from the physical optics solutions with coefficients that are a function of frequency and aspect angle of the incident plane wave. These physical basis functions are each associated with a physical scattering mechanism.
3. Use the Matching Pursuits (MP) and Orthogonal Matching Pursuits (OMP) algorithms to find the coefficients of the PBFs from simulated data to represent the scattered field in the sparse basis.
4. Compare results using PBFs and combinations of PBFs for simple objects and orientations for several frequency bandwidths and angular apertures.
5. Compare results using PBFs to results using polynomial basis functions.

6. Using compressed sensing theory and the sparse representation, reconstruct scattered fields with fewer than Nyquist-rate samples of the scattered field.

Chapter 2: Physical Optics Scattering

The electromagnetic field due to a surface current density can be written in terms of the magnetic vector potential. The magnetic vector potential due to a surface current $\vec{J}_s$ on the surface S is [14]

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \iint_S \vec{J}_s(\vec{r}') \frac{e^{-jkR}}{R} dS' \quad (2)$$

where $R = |\vec{r} - \vec{r}'|$, $k = \frac{2\pi}{\lambda}$, and λ is the wavelength of the incident plane wave.

Figure 1 Definition of coordinate system, source point, and observation point

Using the physical optics approximation, the surface currents are approximated by relating them directly to the incident electromagnetic fields

$$\vec{J}_s = 2\hat{n} \times \vec{H}_i \quad (3)$$

where $\vec{H}_i$ is the incident plane wave magnetic field and $\hat{n}$ is the unit vector normal to the surface. Physical optics is a good approximation in the far field when the target is electrically large and the incident angle of the incident field is not too far from broadside.

By the definition of vector potential, the scattered magnetic field is

$$\vec{H}_s(\vec{r}) = \frac{1}{\mu} \nabla \times \vec{A}(\vec{r}) \quad (4)$$

Substituting for $\vec{A}(\vec{r})$ in (4) with (2), the scattered magnetic field is

$$\vec{H}_s(\vec{r}) = \frac{1}{\mu} \nabla \times \frac{\mu}{4\pi} \iint_S \vec{J}_s(\vec{r}') \frac{e^{-jkR}}{R} dS' \quad (5)$$

or putting the curl inside the integral,

$$\vec{H}_s(\vec{r}) = \frac{1}{4\pi} \iint_S \nabla \times \left[\frac{e^{-jkR}}{R} \vec{J}_s(\vec{r}') \right] dS' \quad (6)$$

Using the identity $\nabla \times (\vec{g}\vec{F}) = \nabla\vec{g} \times \vec{F} + \vec{g}(\nabla \times \vec{F})$,

$$\vec{H}_s(\vec{r}) = \frac{1}{4\pi} \iint_S \left\{ \nabla \left(\frac{e^{-jkR}}{R} \right) \times \vec{J}_s(\vec{r}') + \frac{e^{-jkR}}{R} [\nabla \times \vec{J}_s(\vec{r}')] \right\} dS' \quad (7)$$

Since the $\nabla \times$ operation is with respect to the unprimed coordinate $\vec{r}$, $\nabla \times \vec{J}_s(\vec{r}') = 0$.

$$\vec{H}_s(\vec{r}) = \frac{1}{4\pi} \iint_S \nabla \left(\frac{e^{-jkR}}{R} \right) \times \vec{J}_s(\vec{r}') dS' \quad (8)$$

Also

$$\nabla \left(\frac{e^{-jkR}}{R} \right) = -\hat{R} \left(\frac{jkR + 1}{R^2} \right) e^{-jkR} = -\hat{R} \left(\frac{jk}{R} + \frac{1}{R^2} \right) e^{-jkR} \quad (9)$$

In the limit the far field, $|\vec{r}| \rightarrow \infty$, then $R \rightarrow \infty$ and

$$\nabla \left(\frac{e^{-jkR}}{R} \right) \sim -\hat{r} \left(\frac{jk}{R} \right) e^{-jkR} \quad (10)$$

In the far field R can be approximated in the phase as $R \sim r - \vec{r}' \cdot \hat{r}$ and in the amplitude as $R \sim r$. So in the far field,

$$\vec{H}_s(\vec{r}) = \frac{-jk}{4\pi} \iint_S \left(\frac{e^{-jk(r - \vec{r}' \cdot \hat{r})}}{r} \right) \times \vec{J}_s(\vec{r}') dS' \quad (11)$$

Since the integral is over the primed coordinate only,

$$\vec{H}_s(\vec{r}) = \frac{-jk}{4\pi} \left(\frac{e^{-jkr}}{r} \right) \iint_S \hat{r} \times \vec{J}_s(\vec{r}') e^{jk\vec{r}' \cdot \hat{r}} dS' \quad (12)$$

From Maxwell's equations, the scattered electric field is related to the scattered magnetic field as

$$\vec{E}_s(\vec{r}) = \frac{1}{j\omega\epsilon} \nabla \times \vec{H}_s(\vec{r}) \quad (13)$$

If the scattered field is a plane wave,

$$\vec{E}_s(\vec{r}) = \frac{1}{j\omega\epsilon} (-jk) \hat{r} \times \vec{H}_s(\vec{r}) = -\eta \hat{r} \times \vec{H}_s(\vec{r}) \quad (14)$$

where

$$k = \omega \sqrt{\epsilon\mu}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$\omega = 2\pi f$, f = frequency, ε = permittivity, and μ = permeability. Substituting (12) into (14), the scattered electric field in terms of the surface current density is

$$\vec{E}_s(\vec{r}) = \eta \frac{jk}{4\pi} \left(\frac{e^{-jk r}}{r} \right) \iint_S \hat{r} \times \hat{r} \times \vec{J}_s(\vec{r}') e^{jk \vec{r}' \cdot \hat{r}} dS' \quad (15)$$

Plane Wave Scattering from a PEC Surface

If a plane wave, $\vec{E}_i(\vec{r}) = \vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}$, is incident on the PEC surface, where $\vec{E}_0$ is a vector indicating the polarization of the plane wave, and $\vec{k}$ is the vector wave number of the incident plane wave, the incident plane wave creates a surface current on the PEC surface. In the physical optics approximation the surface current density is $\vec{J}_s(\vec{r}') = 2\hat{n} \times \vec{H}_i(\vec{r}')$, so substituting for the surface current density in (15) the physical optics scattered electric field is

$$\vec{E}_s(\vec{r}) = \eta \frac{jk}{4\pi} \left(\frac{e^{-jk r}}{r} \right) \iint_S \hat{r} \times \hat{r} \times 2\hat{n} \times \vec{H}_i(\vec{r}') e^{jk \vec{r}' \cdot \hat{r}} dS' \quad (16)$$

Since the incident field is a plane wave

$$\vec{H}_i(\vec{r}') = \frac{1}{\eta} \hat{k} \times \vec{E}_i(\vec{r}') \quad (17)$$

where

$$\hat{k} = \frac{\vec{k}}{k}$$

Substituting (17) into (16),

$$\vec{E}_s(\vec{r}) = \eta \frac{jk}{4\pi} \left(\frac{e^{-jkr}}{r} \right) \iint_S \hat{r} \times \hat{r} \times \left[2\hat{n} \times \frac{1}{\eta} \hat{k} \times \vec{E}_0 \right] e^{-j\vec{k} \cdot \vec{r}'} e^{jk\vec{r}' \cdot \hat{r}} dS' \quad (18)$$

Limiting the expression to the backscattered fields, then $\hat{r} = -\hat{k}$ and

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \iint_S \hat{k} \times \hat{k} \times [\hat{n} \times \hat{k} \times \vec{E}_0] e^{-j\vec{k} \cdot \vec{r}'} e^{jk\vec{r}' \cdot (-\hat{k})} dS' \quad (19)$$

or

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \iint_S \hat{k} \times \hat{k} \times [\hat{n} \times \hat{k} \times \vec{E}_0] e^{-j2\vec{k} \cdot \vec{r}'} dS' \quad (20)$$

Using the identity $\vec{A} \times \vec{B} \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, then

$$\hat{n} \times \hat{k} \times \vec{E}_0 = (\hat{n} \cdot \vec{E}_0)\hat{k} - (\hat{n} \cdot \hat{k})\vec{E}_0$$

$$\hat{k} \times (\hat{n} \times \hat{k} \times \vec{E}_0) = \hat{k} \times (\hat{n} \cdot \vec{E}_0)\hat{k} - \hat{k} \times (\hat{n} \cdot \hat{k})\vec{E}_0 = -\hat{k} \times (\hat{n} \cdot \hat{k})\vec{E}_0$$

$$\hat{k} \times \hat{k} \times (\hat{n} \times \hat{k} \times \vec{E}_0) = -\hat{k} \times \hat{k} \times (\hat{n} \cdot \hat{k})\vec{E}_0 = (-\hat{k} \cdot (\hat{n} \cdot \hat{k})\vec{E}_0)\hat{k} + (\hat{k} \cdot \hat{k})(\hat{n} \cdot \hat{k})\vec{E}_0$$

Since $\hat{k} \cdot \vec{E}_0 = 0$ for a plane wave,

$$\hat{k} \times \hat{k} \times (\hat{n} \times \hat{k} \times \vec{E}_0) = (\hat{n} \cdot \hat{k})\vec{E}_0 \quad (21)$$

Substituting (21) into (20), the far-field, physical optics, the plane wave backscatter from a PEC surface S is

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \iint_S (\hat{n} \cdot \hat{k})\vec{E}_0 e^{-j2\vec{k} \cdot \vec{r}'} dS' \quad (22)$$

Using (22), the closed-form far-field, physical optics solution for scattering from several simple PEC target shapes can be expressed explicitly.

Scattering from a Rectangular PEC Plate

The physical optics far-field solution for scattering from a flat N -sided polygonal PEC plate can be written in terms of the sum of scattering from the scattering centers at the N vertices. For a flat plate (22) can be written

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 \iint_S e^{-j2\vec{k} \cdot \vec{r}'} dS' \quad (23)$$

In order to evaluate the integral over a flat plate, dropping the prime on the coordinates inside the integral, let

$$\vec{r} = \vec{\rho} = x\hat{x} + y\hat{y} \quad (24)$$

and

$$dS = dx dy \quad (25)$$

where x and y are in the plane of the plate as shown in Figure 1. Following derivation by Gordon [15], let $\vec{w}$ be the vector projection of the unit vector $\hat{k}$ onto the plane of the plate,

$$\vec{w} = \hat{k} - (\hat{k} \cdot \hat{n})\hat{n} \quad (26)$$

$$\vec{w} = w_x\hat{x} + w_y\hat{y} \quad (27)$$

Then for a flat plate

$$\vec{k} \cdot \vec{r} = kxw_x\hat{x} + kyw_y\hat{y} \quad (28)$$

The integral in (23) becomes

$$\iint_S e^{-j2\vec{k} \cdot \vec{r}} dS = \iint_S e^{-j2kw_x x} e^{-j2kw_y y} dx dy \quad (29)$$

If M and L are functions of x and y , then according to Green's Theorem,

$$\iint_S \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy = \int_{\partial S} (L dx + M dy) \quad (30)$$

where ∂S is the boundary of S . Let $M = -w_x e^{-j2kw_x x} e^{-j2kw_y y}$

and $L = w_y e^{-j2kw_x x} e^{-j2kw_y y}$. Then using (30)

$$\begin{aligned} \iint_S e^{-j2kw_x x} e^{-j2kw_y y} dx dy \\ = \frac{1}{j2k(w_x^2 + w_y^2)} \int_{\partial S} (w_y dx - w_x dy) e^{-j2kw_x x - j2kw_y y} \end{aligned} \quad (31)$$

Let $\vec{a}_n$ be the position vectors of the vertices of a 4-sided plate, for $n=1$ to 4. Then

$\overrightarrow{\Delta a_n} = \vec{a}_{n+1} - \vec{a}_n$ is the vector that points along the edge of the plate from $\vec{a}_n$ to $\vec{a}_{n+1}$, as

shown in Figure 2. Using the notation $\vec{a} = (a_x, a_y)^T$ and $\vec{a}^* = (a_y, -a_x)^T$, $\vec{a}^*$ rotates the

vector $\vec{a}$ 90° clockwise as seen from the observation point. $\vec{\rho}(t)$ represents the n th edge of

the surface S :

$$\vec{\rho}(t) = (1-t)\vec{a}_n + t\vec{a}_{n+1} \quad \text{for } t = 0 \text{ to } 1 \quad (32)$$

Figure 2 Definition of Vertex Vectors and Edge Vectors for Rectangular Plate

$$\begin{aligned}
 \int_{\partial S} (w_y dx - w_x dy) e^{-j2k\vec{w} \cdot \vec{\rho}} \\
 &= \sum_{n=1}^N \int_0^1 e^{-j2k\vec{w} \cdot \vec{\rho}(t)} \left(\vec{w}^* \cdot \frac{d\vec{\rho}(t)}{dt} \right) dt \\
 &= \sum_{n=1}^N (\vec{w}^* \cdot \vec{\Delta a}_n) \int_0^1 e^{-j2k\vec{w} \cdot ((1-t)\vec{a}_n + t\vec{a}_{n+1})} dt
 \end{aligned} \tag{33}$$

Combining (33) with (23) for a rectangular plate with edges of lengths a and b , the physical optics scattered field is

$$\vec{E}_s(\vec{k}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{j2k} \frac{ab}{(\vec{w} \cdot \vec{\Delta a}_1)(\vec{w} \cdot \vec{\Delta a}_2)} \left(e^{-j2\vec{k} \cdot \vec{a}_1} - e^{+j2\vec{k} \cdot \vec{a}_2} + e^{+j2\vec{k} \cdot \vec{a}_3} - e^{-j2\vec{k} \cdot \vec{a}_4} \right) \tag{34}$$

Since the form of the field due to a single scattering center at $\vec{a}_p$ is given by

$$\vec{E}_p(\vec{k}) = \left(\frac{e^{-jkr}}{4\pi r} \right) e^{-j2\vec{k} \cdot \vec{a}_p} \tag{35}$$

it appears the backscattered field from the plate is the sum of the fields from four individual scattering centers. The amplitude function of each scattering center depends on frequency, angle of incidence, the normal to the plate, and the vectors along the edges of the plate. Note the frequency dependence of the amplitude in (34) is $1/k$, which is characteristic of corner diffraction in the Geometric Theory of Diffraction (GTD) [4].

In general the scattering centers for the rectangular plate are located at the vertices of the plate, but there are two special cases when the scattering centers are not at the vertices. When the angle of incidence is perpendicular to one of the edges of the plate, (34) appears to become singular. However, (33) can be written

$$\begin{aligned}
& \sum_{n=1}^N (\vec{w}^* \cdot \vec{\Delta a}_n) \int_0^1 e^{-j2k\vec{w} \cdot ((1-t)\vec{a}_n + t\vec{a}_{n+1})} dt \\
&= (\vec{w}^* \cdot \vec{\Delta a}_1) e^{-jk\vec{w} \cdot (\vec{a}_1 + \vec{a}_2)} \frac{\sin(k\vec{w} \cdot \vec{\Delta a}_1)}{(k\vec{w} \cdot \vec{\Delta a}_1)} \\
&+ (\vec{w}^* \cdot \vec{\Delta a}_2) e^{-jk\vec{w} \cdot (\vec{a}_2 + \vec{a}_3)} \frac{\sin(k\vec{w} \cdot \vec{\Delta a}_2)}{(k\vec{w} \cdot \vec{\Delta a}_2)} \\
&+ (\vec{w}^* \cdot \vec{\Delta a}_3) e^{-jk\vec{w} \cdot (\vec{a}_3 + \vec{a}_4)} \frac{\sin(k\vec{w} \cdot \vec{\Delta a}_3)}{(k\vec{w} \cdot \vec{\Delta a}_3)} \\
&+ (\vec{w}^* \cdot \vec{\Delta a}_4) e^{-jk\vec{w} \cdot (\vec{a}_4 + \vec{a}_1)} \frac{\sin(k\vec{w} \cdot \vec{\Delta a}_4)}{(k\vec{w} \cdot \vec{\Delta a}_4)}
\end{aligned} \tag{36}$$

When the plane wave is incident perpendicular to the edges defined by $\vec{\Delta a}_2$ and $\vec{\Delta a}_4$,

$$(\vec{w}^* \cdot \vec{\Delta a}_1) = (\vec{w}^* \cdot \vec{\Delta a}_3) = 0 \tag{37}$$

and

$$\frac{\sin(k\vec{w} \cdot \vec{\Delta a_2})}{(k\vec{w} \cdot \vec{\Delta a_2})} = \frac{\sin(k\vec{w} \cdot \vec{\Delta a_4})}{(k\vec{w} \cdot \vec{\Delta a_4})} = 1 \quad (38)$$

As a result the singularity is removed and (34) can be rewritten as

$$\vec{E}_s(\vec{k}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{w} b \left(e^{-j2\vec{k} \cdot \left(\frac{\vec{a_2} + \vec{a_3}}{2} \right)} - e^{-j2\vec{k} \cdot \left(\frac{\vec{a_1} + \vec{a_4}}{2} \right)} \right) \quad (39)$$

There are two apparent scattering centers located halfway between the vertices and there is no frequency dependence in the amplitude function, characteristic of normal scattering from a finite straight edge.

The other special case is when the angle of incidence is normal, or broadside, to the surface of the plate. In this case, (23) can be easily evaluated since $\vec{k} \cdot \vec{r}' = 0$, and the integral reduces to

$$\vec{E}_s(\vec{k}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) 2jk ab \quad (40)$$

The frequency dependence is now proportional to k as is typical of specular scattering. This expression agrees with the backscattered field of a rectangular PEC plate as a function of frequency and aspect angle as given in terms of radar cross section in Ross [16].

Of particular interest is the frequency dependence of the amplitudes in the transition region where the angle of incidence is approaching broadside. For the case where the angle of incidence is perpendicular to an edge, as the angle of incidence

approaches broadside, the scattering function changes from (39) to (40), and the frequency dependence of the amplitude function changes from k^0 to k . Using the definitions of the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, and $\vec{a}_4$ as given in Figure 2, (39) may also be written

$$\vec{E}_s(\vec{k}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{w} 2jb \sin kwa \quad (41)$$

The Taylor series expansion of the sine function in (41) gives

$$\vec{E}_s(\vec{k}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{w} 2jb \left(kwa - \frac{(kwa)^3}{3!} + \frac{(kwa)^5}{5!} - \dots \right) \quad (42)$$

As the angle of incidence approaches broadside, $w \rightarrow 0$. The higher order terms go to zero faster than the lower order terms, so the scattering function approaches a polynomial in k instead of the sine function in (41). When $w = 0$, the amplitude of the far-field scattering is just a linear function of k . Figure 3 through Figure 6 show that the far-field physical optics scattering function is approximately a polynomial in k for small angles.

The $\left(\frac{e^{-jkr}}{4\pi r} \right)$ term in the physical optics expression for the scattered field is suppressed in all of the following plots and images. In the figures the solid curve is the function given by (41) and the dashed curve is (42) truncated at the k^5 term.

Figure 3 PO scattered field and the Taylor series expansion at aspect angle $\theta=3^\circ$

Figure 4 PO scattered field and the Taylor series expansion at aspect angle $\theta=2^\circ$

Figure 5 PO scattered field and the Taylor series expansion at aspect angle $\theta=1^\circ$

Figure 6 PO scattered field and the Taylor series expansion at aspect angle $\theta=0^\circ$

Alternative Derivation of Scattering from a Rectangular Plate

For the case of a rectangular plate, where $n = 4$ and the sides are equal to a and b the scattering can be written alternatively as follows, using Figure 7. The locations of the corners of the plate are given by the vectors:

$$\vec{a}_1 = \frac{a}{2}\hat{x}' + \frac{b}{2}\hat{y}'; \quad \vec{a}_2 = \frac{a}{2}\hat{x}' - \frac{b}{2}\hat{y}'; \quad \vec{a}_3 = -\frac{a}{2}\hat{x}' - \frac{b}{2}\hat{y}'; \quad \vec{a}_4 = -\frac{a}{2}\hat{x}' + \frac{b}{2}\hat{y}'$$

Figure 7 Geometry and Orientation of Rectangular Plate in Primed Coordinate System

In the primed coordinate system, the plate is in the x' - y' plane, $\hat{n} = \hat{z}'$, and

$$\vec{r}' = x' \hat{x}' + y' \hat{y}'$$

Then (23) becomes

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} e^{-j2\vec{k} \cdot x' \hat{x}'} e^{-j2\vec{k} \cdot y' \hat{y}'} dS' \quad (43)$$

and

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 \left[\frac{e^{-j2\vec{k} \cdot x' \hat{x}'}}{-j2\vec{k} \cdot \hat{x}'} \right]_{-a/2}^{a/2} \left[\frac{e^{-j2\vec{k} \cdot y' \hat{y}'}}{-j2\vec{k} \cdot \hat{y}'} \right]_{-b/2}^{b/2} \quad (44)$$

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{(j2\vec{k} \cdot \hat{x}')(j2\vec{k} \cdot \hat{y}')} \vec{E}_0 \left(e^{-j2\vec{k} \cdot \frac{a}{2} \hat{x}'} - e^{+j2\vec{k} \cdot \frac{a}{2} \hat{x}'} \right) \left(e^{-j2\vec{k} \cdot \frac{b}{2} \hat{y}'} - e^{+j2\vec{k} \cdot \frac{b}{2} \hat{y}'} \right) \quad (45)$$

This expression can be written as a function of a sum of scattering centers at the corners of the rectangular plate

$$\vec{E}_s(\vec{r}) = -j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{(2\vec{k} \cdot \hat{x}')(2\vec{k} \cdot \hat{y}')} \vec{E}_0 \left(e^{-j2\vec{k} \cdot \vec{a}_1} - e^{+j2\vec{k} \cdot \vec{a}_2} + e^{-j2\vec{k} \cdot \vec{a}_3} - e^{+j2\vec{k} \cdot \vec{a}_4} \right) \quad (46)$$

Using the definition of the sine function as a sum of exponentials, (45) can also be written as

$$\vec{E}_s(\vec{r}) = -j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 ab \left[\frac{\sin(\vec{k} \cdot \hat{x}'a)}{(\vec{k} \cdot \hat{x}'a)} \frac{\sin(\vec{k} \cdot \hat{y}'b)}{(\vec{k} \cdot \hat{y}'b)} \right] \quad (47)$$

The result (47) is exactly the same result as (34).

Figure 9, Figure 11, and Figure 13 show the magnitude of the scattered fields for a square PEC plate with edges equal to $a = b = 0.2$ m for several orientations, given by θ_{plate} and ϕ_{plate} in Figure 8. The physical optics fields are calculated from 8 to 12 GHz and the incident aspect angle ranges from -5° to 5° . Figure 10, Figure 12, and Figure 14 show the corresponding synthetic aperture radar SAR images. The SAR imaging process is discussed below in Chapter 3: Radar Imaging.

Figure 8 Orientation of the plate defined by θ_{plate} and ϕ_{plate} .

Figure 9 Normalized Scattered Field, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 20^\circ$.

Figure 10 SAR Image, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 20^\circ$.

Figure 11 Normalized Scattered Field, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 0^\circ$.

Figure 12 SAR Image, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 0^\circ$.

Figure 13 Normalized Scattered Field, Plate orientation $\theta_{plate} = 0^\circ, \phi_{plate} = 0^\circ$.

Figure 14 SAR Image, Plate orientation $\theta_{plate} = 0^\circ, \phi_{plate} = 0^\circ$.

Circular Plate

The physical optics approximation for the far-field backscatter from a circular PEC plate can also be written in closed form [17]. Starting with (23),

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) \iint_S (\hat{n} \cdot \hat{k}) \vec{E}_0 e^{-j2\vec{k} \cdot \vec{r}'} dS' \quad (48)$$

Figure 15 Geometry and Orientation of the Circular Plate in the Primed Coordinate System

In the primed coordinate system, the plate is in the x' - y' plane as shown in Figure 15 with the normal to the plate $\hat{n} = \hat{z}'$, the source point is $\vec{r}' = \rho' \cos\varphi' \hat{x}' + \rho' \sin\varphi' \hat{y}'$, the unit vector in the direction of the observation point is

$$\hat{r} = \sin\theta \cos\varphi \hat{x}' + \sin\theta \sin\varphi \hat{y}' + \cos\theta \hat{z}', \text{ and}$$

$$dS' = \rho' d\varphi' d\rho'$$

If $C = \sin\theta \cos\varphi$ and $D = \sin\theta \sin\varphi$,

$$\hat{r} \cdot \vec{r}' = \rho' C \cos\varphi' \hat{x}' + \rho' D \sin\varphi' \hat{y}'$$

then (48) for a circular flat PEC plate with radius a becomes

$$\vec{E}_s(\vec{r}) = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 \int_0^a \int_0^{2\pi} e^{-j2k\rho'(C \cos\varphi' + D \sin\varphi')} \rho' d\varphi' d\rho' \quad (49)$$

C and D are constants inside the integral. Using the integral

$$\int_0^{2\pi} e^{x \cos \psi + y \sin \psi} d\psi = 2\pi I_0(\sqrt{x^2 + y^2}) \quad (50)$$

where I_0 is the Modified Bessel Function of the 1st Kind and $\sqrt{C^2 + D^2} = \sin \theta$, then

$$\vec{E}_s = j2k \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \vec{E}_0 \int_0^a 2\pi \rho' I_0(j2k\rho' \sin \theta) d\rho' \quad (51)$$

Using the identities

$$I_p(x) = j^{-p} J_p(jx) \quad \text{and} \quad J_0(jx) = J_0(-jx) \quad (52)$$

$$\int x^{p+1} J_p(\alpha x) dx = \frac{1}{\alpha} x^{p+1} J_{p+1}(\alpha x) + \text{Constant} \quad (53)$$

where J_p is the p th Bessel function, the backscattered field from the circular PEC plate of radius a is

$$\vec{E}_s = j2k \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) (\hat{n} \cdot \hat{k}) \frac{\pi a^2}{ka \sin \theta} J_1(2ka \sin \theta) \quad (54)$$

This expression agrees with published radar cross section of a circular plate [17].

For targets with a large radius with respect to the wavelength, $ka \gg 1$. The large argument approximation for the Bessel function is

$$J_p(x) \cong \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4} - \frac{p\pi}{2} \right), \quad \text{when } x \rightarrow \infty \quad (55)$$

Substituting the large argument Bessel function approximation and re-writing the cosine as a sum of exponentials, the solution for the backscattered field takes the form of two scattering centers located on the radius of the plate.

$$\vec{E}_s(\vec{r}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{\sin\theta} \sqrt{\frac{\pi a^2}{2ka \sin\theta}} [(j-1)e^{-j2kas\sin\theta} + (j+1)e^{+j2kas\sin\theta}] \quad (56)$$

Note the coefficients have a frequency dependence of $1/\sqrt{k}$, which is characteristic of curved edge diffraction. [18] If $\vec{a}_1$ and $\vec{a}_2$ are points on opposite edges of the plate in the plane of incidence,

$$\vec{a}_1 = a \cos \varphi \hat{x}' + a \sin \varphi \hat{y}'$$

and

$$\vec{a}_2 = a \cos(\varphi + \pi) \hat{x}' + a \sin(\varphi + \pi) \hat{y}'$$

(56) can be written as two scattering centers with complex scattering coefficients:

$$\vec{E}_s(\vec{r}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \frac{(\hat{n} \cdot \hat{k})}{\sin\theta} \sqrt{\frac{\pi a^2}{2ka \sin\theta}} [(j-1)e^{-j2\vec{k} \cdot \vec{a}_1} + (j+1)e^{-j2\vec{k} \cdot \vec{a}_2}] \quad (57)$$

For a circular plate, a component of $\vec{k}$ is always normal to an edge at two points, except when the plane wave is incident normal to the plate. At broadside the scattered field is similar to the rectangular plate at broadside:

$$\vec{E}_s(\vec{k}) = j2k\vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \pi a^2 \quad (58)$$

The frequency dependence of the scattered electric field is again proportional to k .

Figure 16 and Figure 18 show the magnitude of the scattered field for a circular PEC plate with radius $a = 0.1$ m for two different orientations. Figure 17 and Figure 19 show the corresponding SAR images. The frequency range is 8 to 12 GHz and the angular range of the incident field is -5° to 5° .

Figure 16 Normalized Scattered Field, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 20^\circ$.

Figure 17 SAR Image, Plate orientation $\theta_{plate} = 30^\circ, \phi_{plate} = 20^\circ$.

Figure 18 Normalized Scattered Field, Plate orientation $\theta_{plate} = 0^\circ, \phi_{plate} = 0^\circ$.

Figure 19 SAR Image, Plate orientation $\theta_{plate} = 0^\circ, \phi_{plate} = 0^\circ$.

Chapter 3: Radar Imaging

High resolution radar images are created from measurements of the scattered field as a function of frequency and aspect angle. Synthetic aperture radar (SAR) images are typically created from data sets collected by a moving sensor recording the radar responses from a stationary scene. Inverse SAR (ISAR) images are created from data sets collected by fixing the sensor position and rotating the object or by moving the sensor around a fixed object. Simple radar range profile plots can be created from a Fourier transform of the frequency domain radar data set. 2D or 3D images of the radar response of a target can be created from 2D or 3D Fourier transforms of the measured response.

[19] If the 3D image is $g(\vec{r})$, where $\vec{r}$ is the pixel location in the image, $\vec{k}$ is the vector wavenumber of the incident wave, and $|\vec{k}| = k = 2\pi f/c$, then

$$g(\vec{r}) = \int G(\vec{k}) e^{j2\vec{k} \cdot \vec{r}} d\vec{k} \quad (59)$$

or

$$g(x, y, z) = \iiint G(k_x, k_y, k_z) e^{j2k_x x} e^{j2k_y y} e^{j2k_z z} dk_x dk_y dk_z \quad (60)$$

This is equivalent to a 3D Fourier transform of the measured data set $G(k_x, k_y, k_z)$ collected over a range of frequencies and incidence angles.

For a 2D image in the $y=0$ plane $\vec{r} = x\hat{x} + z\hat{z}$, and (60) simplifies to

$$g(x, 0, z) = \sum_k \sum_{\theta} G(f, \theta) e^{-j2k(x \sin \theta + z \cos \theta)} \quad (61)$$

The range resolution (along z -direction) of the image is

$$\Delta r = \frac{c}{2B}$$

where c is the speed of light and B is the bandwidth of the data ($B = f_{\max} - f_{\min}$). The

cross-range resolution (along x -direction) is

$$\Delta x = \frac{c}{2f \sin \theta}$$

θ = angular aperture

A windowing function can be used to suppress sidelobes in the image. Appending zeros to the data set before the Fourier transform can be used to provide more sample points in the image, but it does not improve the resolution.

The process of forming the radar image can be understood by considering the radar return from a number of individual scattering centers. The total scattered electromagnetic field $E(\vec{k}_m)$ for the m th frequency and aspect angle can be expressed as a superposition of N individual, weighted scattering centers at $\vec{r}_n$.

$$E(\vec{k}_m) = \sum_{n=1}^N g(\vec{r}_n) e^{-j2\vec{k}_m \cdot \vec{r}_n} \quad (62)$$

The image of the range and cross-range distribution of the scattering centers can be formed from the product of the measured data and the conjugate of the phase of a point scatterer located at each pixel $\vec{r}_n$ in the image space. [1] The image is given by $|g(\vec{r}_n)|$, where

$$g(\vec{r}_n) = \sum_{m=1}^M E(\vec{k}_m) e^{j2\vec{k}_m \cdot \vec{r}_n} \quad (63)$$

since, substituting (63) into (62)

$$g(\vec{r}_n) \approx \sum_{m=1}^M \sum_{p=1}^P g(\vec{r}_p) e^{j2\vec{k}_m \cdot (\vec{r}_n - \vec{r}_p)} = \begin{cases} 0 & \text{for } p \neq n \\ g(\vec{r}_n) & \text{for } p = n \end{cases} \quad (64)$$

Images of the scattered fields from PEC plates using (61) are shown in Figure 10, Figure 12, Figure 14, Figure 17, and Figure 19.

Chapter 4: Physical Basis Functions

Physical basis functions consist of a point scatter function with frequency-dependent amplitudes in which the frequency dependences are related to known physical scattering mechanisms. The physical optics solution for the scattering from plates reveals that there are at least four different physical scattering mechanisms present in scattering from PEC objects: corner diffraction, straight edge diffraction, curved edge diffraction, and specular reflection. Each has different frequency dependence in the amplitude term. The backscattered field is the result of specular reflection when the angle of incidence is normal, or broadside, to the plate. When the angle of incidence is broadside to the circular or polygonal plate, the wave vector is perpendicular to an infinite number of points along the edges. The backscatter from polygonal plates is due to corner diffraction when the wave vector is not normal to any edge. The scattering is due to straight edge diffraction when the incidence angle is normal to an edge. For circular plates off broadside the incidence angle is always normal to an edge at two points and the scattering is due to curved edge diffraction.

Frequency Dependent Physical Basis Functions

The physical optics solutions for backscattering from a polygonal plate or a circular disk, shown in (34) or (57), respectively, can be written in the form

$$\vec{E}_s(\vec{r}) = \vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right) \sum_{p=1}^K x_p f_p(\vec{k}) e^{-j2\vec{k} \cdot \vec{r}_p} \quad (65)$$

This expression implies that the scattered field may be sparse or compressible in some basis comprised of physical basis functions $f_p(\vec{k}) e^{-j2\vec{k} \cdot \vec{r}_p}$ if the scattered field is expanded as a function of the vector wave number $\vec{k}$ as

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p f_p(\vec{k}) e^{-j2\vec{k} \cdot \vec{r}_p} \quad (66)$$

The factor $\vec{E}_0 \left(\frac{e^{-jkr}}{4\pi r} \right)$ is suppressed since the measured value is typically the radar cross section σ ,

$$\sigma = \lim_{r \rightarrow \infty} 4\pi r \frac{|\vec{E}_s|^2}{|\vec{E}_i|^2} \quad (67)$$

so these terms cancel. In (65) and (66) x_p are the unknown complex coefficients of the expansion and $\vec{r}_p$ is the vector position of the p th scattering center.

Simple physical basis functions are equivalent to point scatter functions. The simple physical basis functions have the frequency dependence of ideal scattering centers located at $\vec{r}_p$, or straight edge diffraction. $f_p(\vec{k}) = 1$, and the expansion of the scattered field is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p e^{-j2\vec{k} \cdot \vec{r}_p} \quad (68)$$

For a rectangular plate with edges neither parallel nor perpendicular to the incidence angle the scattering mechanism is corner diffraction. From the physical optics scattering solution (34), the frequency dependence of corner diffraction is $f_p(\vec{k}) = 1/k$ and the expansion is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p \frac{1}{k} e^{-j2\vec{k} \cdot \vec{r}_p} \quad (69)$$

The physical optics solution for the circular disk gives the frequency dependence of curved edge diffraction as $f_p(\vec{k}) = 1 / \sqrt{k}$ and the corresponding expansion is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p \frac{1}{\sqrt{k}} e^{-j2\vec{k} \cdot \vec{r}_p} \quad (70)$$

At broadside the scattering is specular reflection and the physical optics solution indicates the frequency dependence is $f_p(\vec{k}) = k$, and the expansion is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p k e^{-j2\vec{k} \cdot \vec{r}_p} \quad (71)$$

If the scattered signal is not accurately represented by a single scattering mechanism or physical basis function, it may be useful to combine the physical basis functions to form a combined physical basis function associated with a single location in the form

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p \left(1 + \frac{a_p}{k} + \frac{b_p}{\sqrt{k}} + c_p k\right) e^{-j2\vec{k} \cdot \vec{r}_p} \quad (72)$$

In this case each basis function will have additional unknown coefficients a_p , b_p , and c_p .

The combined physical basis function may be considered a generalization of the individual physical basis functions. It will be needed for transition regions where the aspect angle is not quite broadside to the plate or an edge, where more than one type of frequency dependence may be required. [18]

Another possible representation uses an over-complete basis which contains more basis functions than the dimensions of the signal. The over-complete basis contains each of the physical basis function types, each with its own location. The expansion is then of the form

$$\hat{\mathbf{y}}_{\mathbf{k}}(\vec{k}) = \sum_{n=1}^N a_n e^{-j2\vec{k} \cdot \vec{r}_n} + \sum_{m=1}^M \frac{b_m}{k} e^{-j2\vec{k} \cdot \vec{r}_m} + \sum_{p=1}^P \frac{c_p}{\sqrt{k}} e^{-j2\vec{k} \cdot \vec{r}_p} + \sum_{q=1}^Q d_q k e^{-j2\vec{k} \cdot \vec{r}_q} \quad (73)$$

And $K = N + M + P + Q$.

Angular Dependent Physical Basis Functions

From (34) or (57) it is not possible to discern the appropriate aspect angle dependence of the physical basis functions without knowing the object shape and orientation. The angular dependence is given by the $\hat{n} \cdot \hat{k}$ and $\vec{w} = \hat{k} - (\hat{n} \cdot \hat{k})\hat{n}$ terms in the physical optics solutions. For a narrow range of aspect angles it may be useful to combine the frequency dependent function with a low-order polynomial in the angle θ_o , where $\theta_o = \theta - \theta_c$, and θ_c is the center of the range of aspect angles of the incident plane waves. Then the generalized expansion is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K x_p \left(1 + \frac{a_p}{k} + \frac{b_p}{\sqrt{k}} + c_p k \right) (1 + d_p \theta_o + g_p \theta_o^2) e^{-j2\vec{k} \cdot \vec{r}_p} \quad (74)$$

There are 5 additional unknown coefficients besides x_p for each term in the expansion.

Polynomial Basis Functions

Non-physical polynomial basis functions are an alternative to the physical basis functions. For polynomials up to order 2 in wave number k and θ_o , the expansion of the scattered field is

$$\tilde{\mathbf{y}}_{\mathbf{K}}(\vec{k}) = \sum_{p=1}^K (c_{0p} + c_{1p}k + c_{2p}k^2) (1 + b_{1p}\theta_o + b_{2p}\theta_o^2) e^{-j2\vec{k} \cdot \vec{r}_p} \quad (75)$$

As with the combined physical basis functions, the polynomial basis requires storage of more than one coefficient per location.

Each of the basis functions and combined basis functions in Table 1 are compared in Chapter 6: Results. The results show that the combined basis functions have a significant effect on compression efficiency.

Table 1 Summary of Physical and Polynomial Basis Functions

Physical Basis Functions	
Simple Scattering Centers or Straight Edge Diffraction	$x_p e^{-j2\vec{k}\cdot\vec{r}_p}$
Corner Diffraction	$x_p \frac{1}{k} e^{-j2\vec{k}\cdot\vec{r}_p}$
Straight Edge Diffraction	$x_p \frac{1}{\sqrt{k}} e^{-j2\vec{k}\cdot\vec{r}_p}$
Specular Reflection	$x_p k e^{-j2\vec{k}\cdot\vec{r}_p}$
Over-complete Set of Physical Basis Functions	$a_n e^{-j2\vec{k}\cdot\vec{r}_n}; \frac{b_m}{k} e^{-j2\vec{k}\cdot\vec{r}_m}; \frac{c_p}{\sqrt{k}} e^{-j2\vec{k}\cdot\vec{r}_p}; d_q k e^{-j2\vec{k}\cdot\vec{r}_q}$
Combined Physical Basis Function	$x_p (1 + \frac{a_p}{k} + \frac{b_p}{\sqrt{k}} + c_p k) e^{-j2\vec{k}\cdot\vec{r}_p}$
Combined Physical Basis Function with Polynomial in Aspect Angle	$x_p (1 + \frac{a_p}{k} + \frac{b_p}{\sqrt{k}} + c_p k) (1 + d_p \theta_o + g_p \theta_o^2) e^{-j2\vec{k}\cdot\vec{r}_p}$
Polynomial Basis Functions	
Polynomial Basis Functions	$(c_{0p} + c_{1p}k + c_{2p}k^2) e^{-j2\vec{k}\cdot\vec{r}_p}$
Polynomial Basis Functions with Aspect Angle	$(c_{0p} + c_{1p}k + c_{2p}k^2) (1 + b_{1p}\theta_o + b_{2p}\theta_o^2) e^{-j2\vec{k}\cdot\vec{r}_p}$

Coefficients of the Sparse Basis Expansion

Once a dictionary of basis functions is selected, the coefficients of the basis functions in (66) must be found. If the scattered field is sparse in the selected basis,

compressed sensing will be useful in reducing the data sampling required to reconstruct the radar signature. The form of the problem is a system of equations represented by

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad (76)$$

where $\mathbf{y}$ is an M by 1 column vector and represents the measured scattered field or the sampled data set. $\mathbf{A}$ is the M by N forward matrix of the selected basis. $\mathbf{A}$ is a mapping between the sampled data set and the sparse coefficients $\mathbf{x}$. $\mathbf{x}$ is the N by 1 column vector of coefficients of the basis functions. In general $\mathbf{y}$, $\mathbf{A}$, and $\mathbf{x}$ are complex. Since in general $M \neq N$, $\mathbf{A}$ is not a square matrix. Solutions must be based on minimizing the error ε ,

$$\varepsilon = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \quad (77)$$

$\|\mathbf{b}\|_2^2$ is the square of the 2-norm of the M by 1 column vector $\mathbf{b}$:

$$\|\mathbf{b}\|_2^2 = \sum_{m=1}^M |b_m|^2 \quad (78)$$

There are many algorithms that can be used to solve for a set of coefficients $\mathbf{x}$ that minimizes the error ε . In this thesis l_0 -minimization techniques based on greedy algorithms are used with the method of least squares to minimize the error and solve for the coefficients in a sparse basis. l_1 -minimization techniques such as basis pursuit and gradient based approaches can be used to find the coefficients [2] [20], but l_1 -minimization is generally more complicated to implement and more computationally expensive than the greedy approaches. Greedy algorithms are generally efficient and easy to implement. They iteratively select the basis function that best correlates to the signal and adds it to the list of selected basis functions. For each iteration the coefficients of the basis functions are solved for using the method of least squares to minimize the total

residual error. The contribution of the selected basis function is removed and the process is repeated until some minimum residual error is obtained. Matching Pursuits (MP) and Orthogonal Matching Pursuits (OMP) are the two greedy algorithms used here, but there are other variations [8] [12] [21].

Method of Least Squares

The method of least squares solves the problem of an over-determined set of linear equations. It solves for the vector $\mathbf{x}$ that minimizes

$$\varepsilon = \|\mathbf{y} - \mathbf{Ax}\|_2^2 = \sum_{m=1}^M |R_m|^2 \quad (79)$$

where

$$R_m = \sum_{n=1}^N (y_m - a_{mn}x_n) \quad (80)$$

by setting the partial derivatives of ε with respect to each x_n equal to zero.

$$\frac{\partial \varepsilon}{\partial x_n} = \frac{\partial}{\partial x_n} \left[\sum_{m=1}^M |R_m|^2 \right] = \frac{\partial}{\partial x_n} \left[\sum_{m=1}^M R_m R_m^* \right] = 0 \quad (81)$$

Substituting (80) into (81) and letting all the terms be complex, (81) can be re-written as

$$\sum_{m=1}^M a_{mn}^* \left[\sum_{l=1}^N (y_m - a_{ml}x_l) \right] = 0 \quad (82)$$

or in matrix notation,

$$\mathbf{A}^H \mathbf{A} \mathbf{x} = \mathbf{A}^H \mathbf{y} \quad (83)$$

$\mathbf{A}^H$ is the hermitian of $\mathbf{A}$, equal to the complex conjugate of transpose of $\mathbf{A}$, i.e.,

$$\mathbf{A}^H = (\mathbf{A}^T)^*$$

(83) can be re-written

$$\mathbf{x} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{y} \quad (84)$$

The n th coefficient that minimizes ε is

$$x_n = (\mathbf{A}^H \mathbf{A})^{-1} \left[\sum_{m=1}^M a_{mn}^* y_m \right] \quad (85)$$

In the case of simple point scatter basis for electromagnetic or radar scattering,

$$a_{mn} = e^{-j2\vec{k}_m \cdot \vec{r}_n} \quad (86)$$

$$y_m = \text{radar signal sampled at wavenumbers } \vec{k}_m \quad (87)$$

$$x_n = \text{coefficient of the } n\text{th basis function in the sparse basis} \quad (88)$$

The sampled scattered electromagnetic field is

$$\mathbf{y} = [E(\vec{k}_1) \ E(\vec{k}_2) \ ... \ E(\vec{k}_M)]^T$$

and the forward matrix is

$$\mathbf{A} = \begin{bmatrix} e^{-j2\vec{k}_1 \cdot \vec{r}_1} & \dots & e^{-j2\vec{k}_1 \cdot \vec{r}_N} \\ \vdots & \ddots & \vdots \\ e^{-j2\vec{k}_M \cdot \vec{r}_1} & \dots & e^{-j2\vec{k}_M \cdot \vec{r}_N} \end{bmatrix}$$

The least squares solution (84) for $\mathbf{x}$ gives the same result as was found for the

SAR/ISAR image using (63) except for the normalization $(\mathbf{A}^H \mathbf{A})^{-1}$:

$$x_n = (\mathbf{A}^H \mathbf{A})^{-1} \left[\sum_{m=1}^M y_m e^{j2\vec{k}_m \cdot \vec{r}_n} \right] = (\mathbf{A}^H \mathbf{A})^{-1} g(\vec{r}_n) \quad (89)$$

Matching Pursuits

The Matching Pursuits algorithm attempts to find the basis functions and corresponding coefficients for the best K -sparse solution. [8] $\mathbf{y}$ is the M by 1 sampled data, $\mathbf{A}$ is the M by N forward matrix in the sparse basis, and $\mathbf{A}_n$ is the n th column of $\mathbf{A}$. The algorithm iteratively finds the n th column of $\mathbf{A}$ that minimizes the residual,

$$\min_n \|\mathbf{y} - \mathbf{A}_n x_n\|_2^2 \quad (90)$$

From the method of least squares, the x_n that minimizes the residual (90) is

$$x_n = \mathbf{A}_n^H \mathbf{y} (\mathbf{A}_n^H \mathbf{A}_n)^{-1} \quad (91)$$

Substituting (91) into (90),

$$\min_n \|\mathbf{y} - \mathbf{A}_n \mathbf{A}_n^H \mathbf{y} (\mathbf{A}_n^H \mathbf{A}_n)^{-1}\|_2^2 \quad (92)$$

$\mathbf{A}_n^H \mathbf{A}_n$ is the same for all n . (92) is equivalent to

$$\max_n \|\mathbf{A}_n^H \mathbf{y}\|_2^2 \quad (93)$$

This process is equivalent to finding the brightest spot in the backprojection (SAR/ISAR) image of the sampled data.

The steps in the MP algorithm to find the K non-zero elements of the N by 1 coefficient vector $\mathbf{x}$ are as follows: [8]

- 1) Initialize the residual $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x} = \vec{0} = N$ by 1 vector of zeros, and $T = [] =$ null vector.
- 2) Repeat the following steps for $p = 1$ to K :
 - a) Find n_p such that $\left\| \mathbf{A}_{n_p}^H \mathbf{r}^p \right\|_2$ is maximum. $\mathbf{A}_{n_p}$ is the n_p th column of $\mathbf{A}$.
 - b) Append the value n_p to the support, $T = \{n_i\}_{i=1}^p$. The support T will be the list of column numbers of $\mathbf{A}$ corresponding to the selected basis functions (the indices of the non-zero elements of $\mathbf{x}$).
 - c) Calculate the value of the n_p th coefficient x_{n_p} from

$$x_{n_p} = \frac{\mathbf{A}_{n_p}^H \mathbf{r}^p}{\mathbf{A}_{n_p}^H \mathbf{A}_{n_p}}$$
 - d) Fill in n_p th element of $\mathbf{x}$:

$$\mathbf{x}(n_p) = x_{n_p}$$
 - e) Calculate the residual. This process removes the contribution of this column of the basis from the sampled data:

$$\mathbf{r}^{p+1} = \mathbf{y} - \mathbf{A}_{n_p} x_{n_p} = \mathbf{y} - \mathbf{A} \mathbf{x}$$
 - f) Set $p = p + 1$. If $p \leq K$ then return to step 2) above.

The resulting sparse representation is $\tilde{\mathbf{y}} = \mathbf{A} \mathbf{x}$. The residual is the difference between the full data $\mathbf{y}$ and $\tilde{\mathbf{y}}$. The key characteristic of MP is that once the coefficient is calculated it is not changed in subsequent iterations. The p^{th} coefficient x_{n_p} is not changed in the $(p + 1)^{\text{th}}$ iteration.

Orthogonal Matching Pursuits

A variation of MP is Orthogonal Matching Pursuits [8]. The key differences between OMP and MP are that in OMP the residual is defined such that it is orthogonal to all of the previously selected basis functions and the values of all of the coefficients $\mathbf{x}$ are recalculated for each iteration. The steps of OMP are as follows: [22]

- 1) Initialize the residual $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \vec{0} = N$ by 1 vector of zeros, and $T = [] =$ null vector.
- 2) Repeat the following steps for $p = 1$ to K :
 - a) Find n_p such that $\left\| \mathbf{A}_{n_p}^H \mathbf{r}^p \right\|_2$ is maximum. $\mathbf{A}_{n_p}$ is the n_p th column of $\mathbf{A}$.
 - b) Append the value n_p to the support, $T = \{n_i\}_{i=1}^p$. The support T will be the list of column numbers or basis functions.
 - c) Calculate the values of p elements of the N by 1 vector $\mathbf{x}_T^p$ where the element indices are defined by the support T . Use the T columns of the pseudoinverse of $\mathbf{A}$ and the sampled data $\mathbf{y}$.
 - i) $\mathbf{x}_T^p = \mathbf{A}_T^+ \mathbf{y} = (\mathbf{A}_T^H \mathbf{A}_T)^{-1} \mathbf{A}_T^H \mathbf{y}$
 - d) Calculate the residual. This process removes the contribution of all selected columns of the basis from the sampled data and guarantees the residual is orthogonal:

$$\mathbf{r}^p = \mathbf{y} - \mathbf{A} \mathbf{x}_T^p$$

- e) Set $p = p + 1$. If $p \leq K$ then return to step 2) above.

Comparison of MP and OMP

Convergence of either the MP or OMP solutions is verified in terms of mean-square error. If the sampled data set is $\mathbf{y}$ and the sparse representation is $\tilde{\mathbf{y}}$, the normalized mean-square error (MSE) is

$$\text{MSE} = \frac{\sqrt{\sum_{i=1}^M |\tilde{y}_i - y_i|^2}}{\sqrt{\sum_{i=1}^M |y_i|^2}} \quad (94)$$

The subscripts indicate the rows of the column vectors $\mathbf{y}$ and $\tilde{\mathbf{y}}$. The MSE is used to evaluate the convergence of the expansions using the coefficients obtained from MP and OMP. Figure 20 shows a comparison of the MSE obtained using MP to the MSE obtained with OMP. The MSE obtained with OMP will always go to zero if a sufficient number of basis functions (scattering centers) are used. MP approaches a minimum, non-zero error, no matter how many basis functions are included. Figure 21 shows that OMP error goes to zero if the number of basis functions equals the number of samples. In this case there were 64 frequency samples in the sampled data set.

A disadvantage of the OMP algorithm is that it recalculates all of the coefficients $\mathbf{x}$ for each iteration, requiring calculation of the pseudoinverse of the forward matrix for each iteration. Figure 22 shows that the values of the coefficients for MP and OMP for a rectangular PEC plate with diffraction at four corners. The values of the OMP coefficients change each time another basis function is added to the expansion. The values of the OMP coefficients when 10 basis functions are used are slightly different from the values when 25 basis functions are used. The values for MP are identical for the first 10 basis functions whether 10 or 25 basis functions are used.

Figure 20 Matching Pursuits and Orthogonal Matching Pursuits for a Rectangular Plate with 4096 samples

Figure 21 Matching Pursuits and Orthogonal Matching Pursuits with 64 samples

Figure 22 Basis function coefficients found using Matching Pursuits and Orthogonal Matching Pursuits. MSE is shown in Figure 23.

Figure 23 MSE using Matching Pursuits and Orthogonal Matching Pursuits for 4096 samples

Modified OMP

If the basis function is a polynomial or a combined physical basis functions, there are multiple coefficients in each basis function, so a modified OMP algorithm is used to find the multiple coefficients for each location. The number of coefficients per basis

function depends on the order of the polynomial or the number of scattering mechanisms included in the combined physical basis function. The forward matrix $\mathbf{A}$ is assembled from several matrices determined by the selected basis.

$$\mathbf{A} = [\mathbf{A00} \ \mathbf{A01} \ \mathbf{A10} \ \dots \ \mathbf{Aij} \ \dots \ \mathbf{AIJ}] = M \text{ by } (I \times J \times N) \text{ array} \quad (95)$$

$\mathbf{A00}$ is M by N array, where

$$[\mathbf{A00}]_{mn} = a00_{mn} = e^{-j2\vec{k}_m \cdot \vec{r}_n} \quad (96)$$

$\mathbf{Aij}$ is M by N array, where

$$[\mathbf{Aij}]_{mn} = aij_{mn} = k_m^{\alpha_i} \theta_{0m}^{\beta_j} e^{-j2\vec{k}_m \cdot \vec{r}_n} \quad (97)$$

The values of α_i and β_j depends on the powers of k_m or θ_{0m} that are included in the basis function, as shown in Table 1. The OMP algorithm is modified as follows:

- 1) Initialize the residual $\mathbf{r}^0 = \mathbf{y}$, $\mathbf{x}^0 = \vec{0}$, and $T = [] = \text{null vector}$.
- 2) Repeat the following steps for $p = 1$ to K :
 - a) The $\mathbf{A00}$ array is used to find the location index n_p such that $\left\| \mathbf{A00}_{n_p}^H \mathbf{r}^p \right\|_2$ is maximum. $\mathbf{A00}_{n_p}$ is the n_p th column of $\mathbf{A}$.
 - b) Append T for the p th iteration
 - i) $T = \{n_i \ (n_i + N) \ (n_i + 2N) \ \dots \ (n_i + (IJ - 1)N)\}_{i=1}^p$

The remaining steps are the same as for OMP:

- c) $\mathbf{x}_T^p = \mathbf{A}_T^+ \mathbf{y} = (\mathbf{A}_T^H \mathbf{A}_T)^{-1} \mathbf{A}_T^H \mathbf{y}$
- d) $\mathbf{r}^p = \mathbf{y} - \mathbf{A} \mathbf{x}_T^p$
- e) Set $p = p + 1$. If $p \leq K$ then return to step 2) above.

Grid Mismatch

A key to obtaining the lowest MSE is finding the locations, $\vec{r}_n$, of the true scattering centers. If $\vec{r}_p$ are the locations on the grid and the true location is not on the grid ($\vec{r}_n \neq \vec{r}_p$) the MSE will never be zero. For a single isolated ideal scattering center and ideal physical optics scattering, the total MSE is entirely due to grid mismatch. If the scattering center is placed at a point on the image grid, the MSE drops to essentially zero using OMP to find the basis function coefficients, as shown in Figure 24. If the scattering center is off the grid, regardless of which type of basis function is used, error is introduced as shown in Figure 25. It also indicates that the grid mismatch error for a scattering center off the grid can be reduced by using a refined peak search algorithm.

Figure 24 Single Scattering Center Located ON the Image Grid

Figure 25 Single Scattering Center Located OFF the Image Grid

Peak Search for Grid Mismatch

The brute force method for reducing the grid mismatch error is to increase the number of points in the grid and reduce the distance between the grid points. The result of increasing the number of points in the grid is illustrated in Figure 26. Of course increasing the number of locations increases the size of the matrix, slows the algorithms, and increases storage requirements. Another possibly more efficient method of reducing the error due to grid mismatch, depending on the number of scattering centers in the target, is to implement a peak search algorithm to find the nearly-exact locations of the largest scattering centers and add just those locations to the grid points. Instead of increasing the resolution uniformly across the image, a search for the peak using a coarse grid is used to find the approximate locations. After finding the approximate peak using the coarse grid, the grid is subdivided around the first coarse peak and space around the first coarse peak is resampled at a higher sampling rate. The coarse grid used for the examples in this thesis is 16 by 16 and the grid spacing is refined four times. The

locations of the four highest image peaks are added to the grid. Unless otherwise noted, a refined peak search is used in the examples here. The result of using a peak search algorithm is shown in Figure 27 for a scattering center off the grid. Figure 26 and Figure 27 correspond to data for single scattering center sampled from 8 to 12 GHz (32 samples) and aspect angles from -5° to 5° (32 samples). The grid used for Figure 27 is 64 by 64.

Figure 26 Single Scattering Center Located OFF the Image Grid

Figure 27 Single Scattering Center Located OFF the Image Grid

Polynomial Basis for Grid Mismatch

Expansion of the scattered signal using a polynomial basis can also correct for grid mismatch, as illustrated in Figure 27. Letting the scattering center be located at exactly R_1 ,

$$y_{exact}(k) = c_1 e^{-2jkR_1} \quad (98)$$

and the grid point R_2 be offset from R_1 by δ ,

$$R_2 = R_1 + \delta \quad (99)$$

then the field from the offset point is

$$y_{off}(k) = c_2 e^{-2jkR_2} = c_2 e^{-2jk(R_1+\delta)} = y_{exact}(k) \frac{c_2}{c_1} e^{-2jk\delta} \quad (100)$$

Then $y_{exact}(k)$ in terms of R_2 is

$$y_{exact}(k) = c_1 e^{-2jkR_2} e^{+2jk\delta} \quad (101)$$

Since δ is small, the Taylor expansion can be used to approximate $e^{+2jk\delta}$.

$$e^{+2jk\delta} \approx 1 + j2k\delta - 2(k\delta)^2 + \dots \quad (102)$$

So

$$y_{exact}(k) \approx c_1 e^{-2jkR_2} (1 + j2k\delta - 2(k\delta)^2 + \dots) \quad (103)$$

This derivation shows that if the scattering center is offset from the grid by a small amount, the exact solution can be approximated by a scattering center on the grid times a polynomial in k , or

$$y_{exact}(k) \approx e^{-2jkR_2} (d_1 + d_2 k + d_3 k^2 + \dots) \quad (104)$$

The values of the coefficients d_i in the polynomial can be found with the modified OMP algorithm described above. The form of (104) motivates the use of polynomials in k and aspect angle θ as basis functions for electromagnetic scattering functions and the result for one case is compared to the peak search in Figure 27. The 2nd order polynomial in k basis function will require two additional coefficients per basis function compared to a simple PBF.

Chapter 5: Compressive Sensing

If a signal is sparse in some basis, compressed sensing theory asserts that the signal can be compressed. Compression in this sense means that the number of measurements required to reproduce the signal within some specified error is reduced. The physical optics solution has been used to show that electromagnetic scattering is sparse in basis of physical basis functions. Once the sparse basis has been identified, compressed sensing theory predicts that random samples of the K -sparse signal can be used to approximate or compress the sampled signal $\mathbf{y}$ with fewer than the number of samples required by the Nyquist criterion provided K is much smaller than the total number of Nyquist samples [2]. The random sampling is non-adaptive and the number of samples is related to the sparseness of the signal, not the size of the dictionary of basis functions.

If a scattered EM signal is sampled at N_f discrete frequency steps Δf over a frequency bandwidth B , [1]

$$B = (N_f - 1)\Delta f \quad (105)$$

where

$$B = f_{max} - f_{min} \quad (106)$$

Nyquist requires that a band-limited signal be sampled in the time domain at a rate greater than or equal to twice the bandwidth (or the maximum frequency) of the signal in

order that no information about the signal is lost, or if Δt is the time between samples in the time domain and f_s is the sampling rate,

$$f_s = \frac{1}{\Delta t} \geq 2B \quad (107)$$

$$\Delta t \leq \frac{1}{2B} \quad (108)$$

For a radar signal, the time sample increment is equivalent to the range increment

$$\Delta r = c\Delta t,$$

$$\Delta r \leq \frac{c}{2B} \quad (109)$$

So if the minimum downrange distance is 0, the maximum downrange distance L is

$$L = (N_f - 1)\Delta r \leq (N_f - 1)\frac{c}{2B} \quad (110)$$

Or the number of uniformly-spaced frequency samples for a maximum downrange distance, or target size L is

$$N_f \geq L \frac{2B}{c} + 1 \approx L \frac{2B}{c} \quad (111)$$

as required by Nyquist criterion.

The maximum cross-range dimension L_θ is [1]

$$L_\theta \leq \frac{c}{2f_{max} \sin(\Delta\theta)} \approx \frac{c}{2f_{max} \Delta\theta} = \frac{c(N_\theta - 1)}{2f_{max} \theta} \quad (112)$$

where θ is the total radian angular aperture. So for a maximum cross-range dimension L_θ , the signal must be sampled at a number of uniformly spaced backscatter angles N_θ ,

$$N_\theta \geq L_\theta \frac{2f_{max}\theta}{c} + 1 \approx L_\theta \frac{2f_{max}\theta}{c} \quad (113)$$

According to compressed sensing theory, if the a frequency-domain radar signal is sparse in the selected basis and the signal is randomly sampled, the K -sparse values of $\mathbf{x}$ can be obtained if the number of samples M is

$$M \geq \mu K \log N_f \quad (114)$$

where μ is the coherence, provided $K \ll N_f$. There is some disagreement over the exact definition of μ , but here it is assumed to be on the order of 1. N_f is the number of samples required by Nyquist. Several examples are given in Chapter 6: Results.

The relationships between the components of the compressed sensing problem are as follows and are illustrated in Figure 28.

Figure 28 Elements of Compressed Sensing

- $\mathbf{f}$ is the full data set, the scattered field in this case, as a function of frequency and angle. $\mathbf{f}$ is an P by 1 vector. Φ selects M (random or uniform) samples of the full data $\mathbf{f}$. Φ is an M by P matrix.
- The sampled data set is $\mathbf{y} = \Phi\mathbf{f}$, an M by 1 vector.
- Ψ is the P by N Linear Mapping Matrix (sparse basis functions) that maps the full data $\mathbf{f}$ to the sparse basis coefficients $\mathbf{x}$. $\mathbf{f} = \Psi\mathbf{x}$
- The forward matrix $\mathbf{A} = \Phi\Psi$, such that $\mathbf{y} = \Phi\mathbf{f} = \Phi\Psi\mathbf{x} = \mathbf{A}\mathbf{x}$. $\mathbf{A}$ is completely defined by the set of basis functions selected and the frequencies and aspect angles at which the scattered field is sampled.
- The coefficients $\mathbf{x}$ in the sparse basis are found by l_1 -norm minimization or l_0 -norm minimization (MP or OMP) as discussed in Chapter 4. $\mathbf{x}$ is an N by 1 vector, but only K elements of the vector $\mathbf{x}$ are non-zero if the signal is K -sparse in the selected basis.
- The sparse representation of the full data $\mathbf{f}$ is $\tilde{\mathbf{y}} = \mathbf{A}\mathbf{x}$.

Once a suitable set of basis functions is found for the radar signature, a basis where the signature is approximately K -sparse, the signal can be compressed. Using one of the sets of physical basis functions discussed above, several cases are examined where the signal is sampled uniformly at the Nyquist rate or better in aspect angle, but sampled either randomly or uniformly at less than the Nyquist rate in frequency. The K coefficients of the expansion in the sparse basis are found and the compressed signal is calculated. The normalized absolute error between the true sampled data set $\mathbf{y}$ and the sparse representation $\tilde{\mathbf{y}}$ is compared as a function of the number of samples used.

$$\text{Normalized Absolute Error} = \frac{\sqrt{\sum_{i=1}^M (\tilde{y}_i - y_i)^2}}{\sqrt{\sum_{i=1}^M (y_i)^2}} \quad (115)$$

In the case of actual measured data sampled at a limited number of frequencies, the true sampled data $\mathbf{y}$ will not be available, so it may be useful to calculate the relative error that compares the sparse representation $\tilde{\mathbf{y}}_M$ to the previous approximation $\tilde{\mathbf{y}}_{M-1}$. $\tilde{\mathbf{y}}_{M-1}$ is found using one fewer samples of the signal than $\tilde{\mathbf{y}}_M$.

$$\text{Normalized Relative Error} = \frac{\sqrt{\sum_{i=1}^M (\tilde{y}_{M_i} - \tilde{y}_{M-1_i})^2}}{\sqrt{\sum_{i=1}^M (\tilde{y}_{M-1_i})^2}} \quad (116)$$

Chapter 6: Results

The primary objective of this thesis is to investigate the efficacy of physical basis functions to provide an improved sparse basis for compression of the representation of scattered EM fields and to compare the residual error resulting from the use of physical basis functions.

The far-field, physical optics solution for the scattered EM field from either a rectangular PEC plate (edges of lengths a and b) or a circular PEC plate (radius of a) is used to demonstrate the use of OMP to find the coefficients $\mathbf{x}$ of the basis functions that minimize the error, $\varepsilon = \|\mathbf{y} - \mathbf{Ax}\|_2^2$, where $\mathbf{y}$ is the sampled scattered field. The scattered field is calculated for Q frequencies and R aspect angles, for a total of $M = QR$ samples. The reconstructed data set, vector $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_M)^T$, is calculated using the scattering coefficients $\mathbf{x} = (x_1, x_2, \dots, x_N)^T$ found from the OMP process and the forward M by N matrix $\mathbf{A}$ is comprised of the basis functions a_{mn} .

$$\tilde{y}_m = \sum_{n=1}^N a_{mn} x_n \quad (117)$$

The MSE vs. the number of basis functions is used to compare the individual physical basis functions, the over-complete basis functions, and the combined physical basis functions. The physical basis functions are also compared to 2nd order polynomial basis functions. The exact forms of the basis functions used are listed in Table 2.

Table 2 Physical and Polynomial Basis Functions

Physical Basis Functions		
Simple Scattering Center or Straight Edge Diffraction	$x_n e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(118)
Corner Diffraction	$x_n \frac{1}{k_m} e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(119)
Curved Edge Diffraction	$x_n \frac{1}{\sqrt{k_m}} e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(120)
Specular Reflection	$x_n k_m e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(121)
Over-complete PBF	$a_n e^{-j2\vec{k}_m \cdot \vec{r}_n}; \frac{b_p}{k_m} e^{-j2\vec{k}_m \cdot \vec{r}_p}; \frac{c_q}{\sqrt{k_m}} e^{-j2\vec{k}_m \cdot \vec{r}_q}; d_s k_m e^{-j2\vec{k}_m \cdot \vec{r}_s}$	(122)
Combined PBF	$x_n (1 + \frac{a_n}{k_m} + \frac{b_n}{\sqrt{k_m}} + c_n k_m) e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(123)
Combined PBF with Polynomial in Aspect Angle	$x_n (1 + \frac{a_n}{k_m} + \frac{b_n}{\sqrt{k_m}} + c_n k_m) (1 + d_n \theta_{om} + g_n \theta_{om}^2) e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(124)
Polynomial Basis Functions		
Polynomial in k_m	$(c_{0n} + c_{1n} k_m + c_{2n} k_m^2) e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(125)
Polynomial in k_m and Aspect Angle	$(c_{0n} + c_{1n} k_m + c_{2n} k_m^2) \times (1 + b_{1n} \theta_{om} + b_{1n} \theta_{om}^2) e^{-j2\vec{k}_m \cdot \vec{r}_n}$	(126)

$\vec{k}_m$ is normalized by replacing it with $\vec{k}_m/k_0$, where $k_0 = 2\pi f_0/c$ and f_0 is the center frequency.

The comparisons are done for rectangular and circular PEC plates at several orientations, including broadside, for several different frequency bandwidths and angular apertures. The orientation of the plate and the direction of the incident wave are defined in Figure 29. The scattering center locations are located on a 2 dimensional grid of points $\vec{r}_p$. The peak search algorithm is used to add four additional locations to the grid to reduce the grid mismatch error. The plate orientations used as examples are illustrated in Figure 30 through Figure 33.

Figure 29 Plate Orientation

Figure 30 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$.

Figure 31 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$.

Figure 32 Near Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$.

Figure 33 Curved Edge Diffraction - Circular Plate, $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$.

Comparison of Physical Basis Functions for Scattered Fields Sampled vs. Frequency

The MSE vs. the number of basis functions for four physical basis functions types for several plate orientations are shown in Figure 34 through Figure 37. Figure 34 through Figure 37 correspond to a single incident aspect angle of 0° and a relatively wide frequency bandwidth of 2 to 18 GHz.

If the plate is tilted and twisted with respect to unprimed coordinate system shown in Figure 29 such that the incident wave vector $\vec{k}$ is neither normal to an edge nor broadside to the plate, the scattering is characterized as corner diffraction. Figure 34 shows the results for such a case. Using OMP to find the coefficients for expansion in each of the four types of PBFs, Figure 34 shows that the k^{-1} PBFs result in the lowest MSE.

If the incident wave vector $\vec{k}$ is normal to an edge, the scattering is characterized as straight edge diffraction. The results for a plate with straight edge diffraction are shown in Figure 35. In this case, as expected, the k^0 PBFs result in the lowest MSE for an expansion in a given number of basis functions.

Figure 34 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle $\theta = 0^\circ$, $\phi = 0^\circ$.

Figure 35 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle $\theta = 0^\circ$, $\phi = 0^\circ$.

If the incident wave vector $\vec{k}$ is broadside to the plate, or parallel to the surface normal of the plate, the backscattering is characterized as specular reflection and the frequency dependence of the scattering is expected to be proportional to k . Figure 36 shows that the k PBFs result is the lowest MSE.

If the PEC plate is a circular disk and the incident wave vector $\vec{k}$ is not broadside to the plate, the scattering is characterized by curved edge diffraction. Figure 37 shows that the MSE is lowest for the $k^{-1/2}$ PBFs, as expected for curved edge diffraction.

Figure 36 Specular Reflection - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Angle $\theta = 0^\circ$, $\phi = 0^\circ$.

Figure 37 Curved Edge Diffraction - Circular Plate, radius $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle $\theta = 0^\circ$, $\phi = 0^\circ$.

When the frequency bandwidth is narrowed to 8 to 12 GHz, the differences between the MSE of the expansions using the four types of PBFs is much smaller, so the choice between the PBFs is less important, except perhaps at broadside. Figure 38

through Figure 41 show the results for the same four examples of corner diffraction, straight edge diffraction, specular reflection, and curved edge diffraction for 8 to 12 GHz.

Figure 38 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 0° .

Figure 39 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 0° .

Figure 40 Specular Reflection - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 0° .

Figure 41 Curved Edge Diffraction - Circular Plate, radius $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 0° .

Frequency Data at or near Broadside

When the incident angle is broadside, the scattering is characterized by specular reflections so the PBF with the k dependence provides the lowest MSE. As the incident angle moves away from broadside the frequency dependence changes from k to k^0 , as long as the incident wave is also perpendicular to an edge. Over a wide frequency band of 2 to 18 GHz, the change occurs between incident angles of 1° and 3° as shown in Figure 42 through Figure 45 for incident angles of 0° , 1° , 3° , and 5° .

Figure 42 Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle 0° .

Figure 43 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle 1° .

Figure 44 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle 3° .

Figure 45 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle 5° .

The results for a frequency band 8 to 12 GHz are shown in Figure 46 through Figure 49.

The transition occurs at a larger aspect angle (farther from broadside) for the narrower frequency bandwidth.

Figure 46 Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 0° .

Figure 47 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 5° .

Figure 48 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 15° .

Figure 49 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle 30° .

Comparison of Physical Basis Functions for Scattered Fields Sampled vs. Frequency and Angle

Scattering that is a function of frequency and aspect angle can be used to form SAR/ISAR images. Figure 50 shows the MSE for a rectangular PEC plate tilted and twisted with respect to the incident plane wave over a frequency band of 2 to 18 GHz and a range of incident aspect angles from -5° to $+5^\circ$. Of the four types of physical basis functions tested, the set with k^{-1} frequency dependence provides the lowest MSE, as expected for corner diffraction. Similarly Figure 51 shows the MSE for a circular PEC plate for the same wide frequency band, narrow aspect angle range case. As expected, the set of physical basis functions with the $k^{-1/2}$ dependence of curved edge diffraction provides the lowest MSE of the four types. For the wave vector perpendicular to a

straight edge the set of physical basis functions with k^0 dependence provides the lowest MSE as shown in Figure 52.

Figure 50 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 51 Curved Edge Diffraction - Circular Plate, radius $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 52 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 53 Near Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 53 through Figure 55 show the results for the near broadside case for increasing angular apertures. For a single aspect angle 0° , the k^0 dependent PBFs provide the lowest MSE, as shown in Figure 36. If the range of aspect angles is -5° to 5° , the result Figure 53 resembles the case for straight edge diffraction with k^0 frequency dependence, but as the angular aperture increases to -10° to 10° and -20° to 20° , the angular dependence changes to k , shown in Figure 54 and Figure 55, respectively.

Figure 54 Near Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -10° to $+10^\circ$ (64 samples).

Figure 55 Near Broadside - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 0^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -20° to $+20^\circ$ (64 samples).

As the frequency bandwidth of the incident radiation is reduced, the advantage of the physical basis functions decreases, as shown in Figure 56 and Figure 57 for 8 to 12 GHz. There is little separation between the MSE using the different types of physical basis functions.

Figure 56 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 57 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Over-complete and Combined Physical Basis Functions

MSE vs. number of basis functions using an over-complete set of physical basis functions (122) and the combined physical basis functions (123) are compared in Figure 58 through Figure 61 for a rectangular plate over the 8 to 12 GHz and the 2 to 18 GHz frequency bands. The MSE using the individual PBFs are also plotted for reference. The lowest MSE for a given number of basis functions is obtained using the combined physical basis functions, however Figure 59 and Figure 61 show that more coefficients are required when using the CPBFs. Fewer scattering center locations are required, but each location requires 3 additional coefficients. For the corner diffraction case the k^{-1} PBFs still provide the lowest MSE if the number of coefficients is less than about 100. The over-complete (OC) basis matches the number of coefficients required by the k^{-1} PBFs, but does not require knowledge of the scattering mechanisms of the target *a priori*.

Figure 58 Corner Diffraction Basis Functions - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 59 Corner Diffraction Coefficients - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 60 Corner Diffraction Basis Functions - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 61 Corner Diffraction Coefficients - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Angular Dependence

Additional significant improvement in the MSE can be achieved by including some angular dependence in the physical basis functions. Figure 66 and Figure 64 compare the over-complete and combined physical basis functions with and without the polynomial in aspect angle for the rectangular plate and circular plate, respectively. The OC basis comprises all four of the individual physical bases. The CPBFs have the frequency dependence of all four scattering mechanisms, but they share the same location, as shown in Table 2. The same OC or CPBFs can be used for both the rectangular and circular plates. The combined physical basis functions provide drastically improved convergence for a wide range of target orientations and result in a very low MSE. However, the CPBFs require more coefficients per basis function as Figure 63 and Figure 65 illustrate. The CPBFs require 3 additional coefficients per basis function and the CPBFs with aspect angle require 5 additional coefficients per basis function.

Figure 62 Basis Functions for Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{\text{plate}} = 30^\circ$, $\phi_{\text{plate}} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$

Figure 63 Coefficients for Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{\text{plate}} = 30^\circ$, $\phi_{\text{plate}} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 64 Basis Functions for Circular Plate, radius $a = 0.1$ m; Plate orientation $\theta_{\text{plate}} = 30^\circ$, $\phi_{\text{plate}} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 65 Coefficients for Circular Plate, radius $a = 0.1$ m; Plate orientation $\theta_{\text{plate}} = 30^\circ$, $\phi_{\text{plate}} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Physical Basis Functions vs. Polynomial Basis Functions

Although the primary focus of this thesis is to define and compare physical basis functions, in the next group of plots the combined physical basis functions (124) are

compared to polynomial basis functions (126), both including a second degree polynomial in aspect angle. Including a polynomial angular dependence can reduce the MSE dramatically when there is more than some minimum number of coefficients. The combined physical basis functions with a polynomial in aspect angle each require five additional coefficients for each basis function or scattering center. The 2nd degree polynomial basis functions in frequency and aspect angle each require four additional coefficients.

For the corner diffraction example, in Figure 66 and Figure 67, the appropriate physical basis function for the corner diffraction of the rectangular plate is k^{-1} . Since there are four physical scattering centers, four basis functions are required before the CPBFs or polynomial basis functions perform better than the k^{-1} PBFs, for a given number of coefficients. Four CPBFs with angle require 24 coefficients. Four 2nd degree polynomial basis functions with angle require 20 coefficients.

Figure 66 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 67 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

For straight edge diffraction the appropriate physical basis is k^0 and there are two scattering centers. Both the CPBF and the polynomial basis include a k^0 term, so both the CPBF and the polynomial basis provide similar MSE and are better than the k^0 dependent basis functions as long as at least two basis functions or at least 10 or 12 coefficients are included, as shown in Figure 68 and Figure 69. A similar situation occurs for a circular plate and curved edge diffraction and the result is shown in Figure 70 and Figure 71.

Figure 68 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 69 Straight Edge Diffraction - Rectangular Plate, $a = b = 0.2$ m; $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 70 Curved Edge Diffraction - Circular Plate, $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 71 Curved Edge Diffraction - Circular Plate, $a = 0.1$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 0^\circ$; Incident Wave: Frequency = 8 to 12 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Physical Basis Functions with Compressed Sensing

The following steps are used to demonstrate the use of physical basis functions to compress a scattered EM signal in a sparse basis.

- Calculate the scattered field for P values:

$$\mathbf{y}(\vec{k}_p) = \text{physical optics scattered field for } \vec{k}_p \quad (127)$$

- Pick a set of physical basis functions on a grid of points $\vec{r}_n$ from Table 2 in which the signal is expected to be sparse.
- Calculate the P by N forward matrix $\mathbf{A}$ from elements:

$$[\mathbf{A}]_{pn} = a_{pn} = f(\vec{k}_p)e^{-j2\vec{k}_p \cdot \vec{r}_n} \quad (128)$$

- Randomly take M samples of $\mathbf{y}(\vec{k}_p)$ to get $\mathbf{y}_{random}(\vec{k}_m)$,

$$\begin{aligned} \mathbf{y}_{random} &= (y(\vec{k}_1), y(\vec{k}_2), \dots, y(\vec{k}_M))^T \\ [\mathbf{A}_{random}]_{mn} &= a_{mn} = f(\vec{k}_m)e^{-j2\vec{k}_m \cdot \vec{r}_n} \end{aligned} \quad (129)$$

- Find the N by 1 vector $\mathbf{x}$ in the sparse basis using OMP to solve

$$\mathbf{y}_{random} = \mathbf{A}_{random}\mathbf{x}$$

- Use $\mathbf{x}$ (N by 1) and $\mathbf{A}$ (P by N) to calculate the sparse representation of $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}} = \mathbf{A}\mathbf{x} \quad (130)$$

- Calculate normalized absolute or relative error as a function of the number of random samples of $\mathbf{y}$ to determine convergence.

Scattering from a rectangular PEC plate from 2 to 18 GHz at a single aspect angle was used to demonstrate the use of a sparse basis to compress a signal. A Nyquist sampling rate requires 64 uniformly spaced frequency samples to properly calculate the downrange profile in this case. For a k^{-1} physical basis with $K = 4$, Figure 72 and Figure 73 show the absolute and relative error in the compressed signal for two different sets of

randomly selected frequencies. The error is plotted as a function of the number of frequency samples used to find the coefficients of the basis. The minimum error is reached after approximately 22 samples in both cases, but in the trial represented in Figure 72 the error increases again for several iterations before settling back to the minimum. The result for random sampling varies slightly because the specific frequencies selected in one trial may be different from the next trial. Figure 74 plots the downrange profiles from the compressed data for 24 randomly sampled frequencies and for 64 uniformly sampled frequencies.

Figure 72 Trial 1: Absolute and relative error vs. the number of random samples used to compress a signal with k^{-1} physical basis functions and sparsity $K = 4$.

Figure 73 Trial 2: Absolute and relative error vs. the number of random samples used to compress a signal with k^{-1} physical basis functions and sparsity $K = 4$.

Figure 74 Downrange profile from compressed data using four coefficients ($K = 4$) in the k^{-1} basis; Nyquist rate required is 64 frequency samples.

In a second example the frequency is sampled randomly between 2 to 18 GHz and the aspect angle is sampled uniformly from -5° to 5° in order to form a SAR image.

Figure 75 shows that a sparse representation is obtained using 4 k^{-1} PBFs and Figure 76 is a SAR image using 64 frequency samples and 64 angle samples. Sampling the scattered field randomly in frequency and uniformly in angle and using OMP to find the coefficients of the 4 k^{-1} PBFs, the error as a function of increasing number of samples is shown in Figure 77. Less than 16 random frequency samples (total less than $16 \times 64 = 1024$) are needed to create the SAR image correctly as shown in Figure 79.

Figure 75 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 76 Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples); Aspect Angle -5° to $+5^\circ$ (64 samples).

Figure 77 Error in Compressed signal for Corner Diffraction - Rectangular Plate, $a = b = 0.2$ m; Plate orientation $\theta_{plate} = 30^\circ$, $\phi_{plate} = 20^\circ$; Incident Wave: Frequency = 2 to 18 GHz (64 samples maximum); Aspect Angle -5° to 5° (64 samples)

Figure 78 SAR Images of Compressed signal for Corner Diffraction using 8, 16, 32, and 64 UNIFORM frequency samples

Figure 79 SAR Images of Compressed signal for Corner Diffraction using 8, 16, 32, and 64 RANDOM frequency samples

Conclusions

The objective of this thesis was to investigate the efficacy of physical basis functions or a combination of physical basis functions to provide an improved sparse basis for backscattered electromagnetic fields as compared to a simple point scatter basis. The closed-form, physical optics solution for the backscatter from flat plates was applied to extract the physical basis functions for the sparse expansion of the backscattered fields.

The results demonstrate that physical basis functions yield a low mean-square error and provide insight into the scattering mechanisms, assuming that the effect of grid mismatch is circumvented. The combined physical basis functions provide at least an order of magnitude reduction in the MSE regardless of which scattering mechanism is present, although more coefficients may be required. An over-complete basis can provide a low MSE, requiring fewer coefficients in many cases, and can be used when the scattering mechanism of the target is not known.

Compressive sensing may be applied to obtain the coefficients of the expansion without measuring the full data set by random sampling of the data set. The number of random samples required is proportional to the sparseness of the signal, not the bandwidth nor the size of the dictionary of basis functions. Convergence was demonstrated by monitoring the relative error as more samples were added. It was observed that using a sparse representation with $K = 4$, significantly fewer random samples of the signal were required to obtain the same absolute error, as compared to the number of uniform samples. The convergence does vary depending on which specific random samples of the signal are selected for a given trial. Some sets of random samples

result in faster convergence than others. In any case the lower the MSE of the sparse representation the faster the convergence. The combined physical basis function converges to a lower error than the other physical basis types. It is expected that the convergence will be further improved by reducing the grid mismatch error, improving the aspect angle dependence, and including additional scattering mechanisms in the combined physical basis functions, although it has not been demonstrated.

Future work on this topic will include further reduction or elimination of the grid mismatch error and investigation into a more physical method of including the aspect angle dependence in the physical basis functions. Additional physical scattering mechanisms, e.g. for singly and doubly curved surfaces and double corner diffraction, may be added to the dictionary of physical basis functions, and the sparse expansion using physical basis functions may be applied to scattering from more complex targets.

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