VIBRATIONS OF CANTILEVER BEAMS
WITH VARIOUS INITIAL CONDITIONS

A Thesis

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To my Mother and Father
Nuha and Isam Sonalla

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CHAPTER I

INTRODUCTION

1.1 GENERAL

A beam is a structural element having one dimension, length, many times greater than its two other dimensions, width and depth. It may be a straight beam or a curved beam. The discussion here will be limited to straight cantilever beams.

The vibration characteristics of beams has an important role in the design of structures that are subjected to vibration. How large is the maximum dynamic bending stress for a cantilever beam for various initial conditions as compared to the maximum initial static bending stress is an important practical stand point. Further, is the subsequent vibratory motion periodic after the release of the initial static load? How fast does the cantilever beam come back to its original static deflection shape after the release of the initial static load? How do higher frequencies influence the motion of the beam when it is displaced from equilibrium?

1.2 STATEMENT OF THE PROBLEM

The cantilever beam shown in Figure 1 is of length ℓ with constant cross sectional area. It is made of a homogenous material. The cross-section has two axes of symmetry.

Figure 1. A cantilever beam

The beam is subjected to various initial static loads. After the release of the initial load, the subsequent vibratory motion is studied. The vibrational motion is assumed to take place strictly in the z-direction, with a displacement component $w = w(x,t)$.

The usual strength of materials assumptions will be made here for the transverse displacements of the beam. This makes the problem both mathematically and physically one-dimensional; that is, w depends upon x , but not upon y or z . Thus, the motion of the beam is completely determined by the motion of its centerline, and w is defined more precisely to be the displacement of the centerline away from its straight equilibrium position.

Further, the classical Euler-Bernoulli beam theory, which assumes that plane cross-sections normal to the centerline remain plane and normal during deformation, is valid here.

1.3 DIFFERENTIAL EQUATION OF FREE VIBRATION

For simplicity, the beam is assumed to undergo small amplitude vibrations. It is made of a homogenous, elastic, material and has a constant cross section. For free vibration, the differential equation of motion is (Ref. 1)

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (1.1)$$

where A is the cross-sectional area, ρ is the mass per unit volume and EI is the flexural rigidity (E is Young's modulus of the material, and I is the second moment of the cross-sectional area with respect to the neutral axis).

In Chapter 2 the free vibration response of a cantilever beam due to arbitrary initial conditions using the eigenfunction superposition is discussed. The characteristic frequency equation is derived as a solution of (1.1). Subsequently, the free vibration frequencies and

mode shapes are found. Evaluation of free vibration displacement coefficients is done in subsequent chapters for various initial conditions.

1.4 INITIAL CONDITIONS

The response of the beam is evaluated in Chapters 3,5,6 and 7 for an initial end moment, initial point load at the end, initial uniformly distributed load and an initial cubic deflection shape, respectively. In all of the above initial conditions, the beam is displaced initially in a shape resembling its first mode. In Chapter 8 the response of the beam for an initial shape similar to the second mode is discussed. Numerical analysis and round off error related to eigenvalues, eigenvectors and vibration displacement coefficients is presented in Chapter 4. Finally, a summary of results is given in Chapter 9.

CHAPTER II
FREE VIBRATION RESPONSE OF A CANTILEVER
BEAM DUE TO ARBITRARY INITIAL CONDITIONS

2.1 SOLUTION OF THE DIFFERENTIAL EQUATION OF FREE VIBRATION

Using the separation of variables procedure to solve the differential equation of motion (1.1), the solution may be written as

$$w(x,t) = XT \quad (2.1)$$

where $X = X(x)$ and $T = T(t)$. Substituting (2.1) into (1.1) yields

$$EI X^{IV}T + \rho AXT'' = 0 \quad (2.2)$$

where $X^{IV} = d^4X/dx^4$. Dividing through (2.2) by XT and collecting variables in single terms, (2.2) may be rewritten as

$$\frac{X^{IV}}{X} = - \left(\frac{\rho A}{EI} \right) \frac{T''}{T} \quad (2.3)$$

Thus, the variables have been separated, and each side of (2.3) must equal a constant, say α^4 . One therefore may write

$$X^{IV} - \alpha^4 X = 0 \quad (2.4a)$$

$$T'' + \omega^2 T = 0 \quad (2.4b)$$

$$\text{where } \omega^2 = \frac{EI\alpha^4}{\rho A} \quad (2.4c)$$

To solve (2.4a) rewrite it first in operator form as

$$(\psi^4 - \alpha^4)X = 0 \quad (2.5)$$

where $\psi = d/dx$, $\psi^2 = d^2/dx^2$, etc. Equation (2.5) may be factored into

$$(\psi^2 + \alpha^2)(\psi^2 - \alpha^2)X = 0 \quad (2.6)$$

Because (2.6) is a linear differential equation, its solution is the sum of the solutions obtained from the separate equations

$$(\psi^2 + \alpha^2)X = 0 \quad (2.7a)$$

$$(\psi^2 - \alpha^2)X = 0 \quad (2.7b)$$

Assuming α^2 to be positive (2.7a) obviously has a solution in terms of trigonometric functions, and the solution to (2.7b) may be written in terms of either exponential or hyperbolic functions. The latter are usually preferred for they are either even or odd for symmetry or anti-symmetry of the beam configuration. Thus, the solution to (2.4a) becomes

$$X(x) = C_1 \sin \alpha x + C_2 \cos \alpha x + C_3 \sinh \alpha x + C_4 \cosh \alpha x \quad (2.8)$$

where C_1, C_2, C_3, C_4 are constants of integration, and where $\sinh$ and $\cosh$ are hyperbolic sine and cosine functions, respectively. Equation (2.6) shows that the solution for α^2 negative is the same as (2.8). Solutions for $\alpha^2 = 0$ will be trivial for the cantilever beam, which may be shown by applying the boundary conditions.

The solution to (2.4b) is the sinusoidal behavior in time expected in a free undamped vibration problem:

$$T(t) = A \sin \omega t + B \cos \omega t \quad (2.9)$$

where A and B are additional constants of integration. Substituting (2.8) and (2.9) into (2.1) one obtains

$$w(x,t) = X(x)(A \sin \omega t + B \cos \omega t) \quad (2.10)$$

There are eight combined constants of integration. These are all determined from the boundary conditions and initial conditions for our particular problem.

In equation (2.10) ω is the circular frequency of free vibration in radians/sec and, from (2.4c), takes the following form

$$\omega = \alpha^2 \sqrt{\frac{EI}{\rho A}} \quad (2.11)$$

Multiplying by ℓ^2 and rearranging one obtains

$$(\alpha \ell)^2 = \omega \ell^2 \sqrt{\frac{\rho A}{EI}} \quad (2.12)$$

Taking the square root of both sides yields

$$\alpha \ell = \left[\omega \ell^2 \sqrt{\frac{\rho A}{EI}} \right]^{1/2} \quad (2.13)$$

The $\alpha \ell$ in (2.13) are the eigenvalues of the problem. That is, they are the "proper values" which if chosen, permits nontrivial solutions satisfying both the differential equation and boundary conditions, all of which are homogeneous. The $(\alpha \ell)^2$ are the non-dimensional frequency parameters.

Further, when the constants in equation (2.8) are evaluated from the boundary conditions, the eigenfunction is obtained. For each $\alpha \ell$

determined, there exists one eigenfunction. This gives the displaced shape of the beam vibrating in this mode and, in the usual terminology, is called the mode shape.

2.2 BOUNDARY CONDITIONS. THE EIGENFUNCTION

The physical conditions which exist at the two ends of the beam must be expressed mathematically as boundary conditions. When these conditions are applied to equation (2.1) the eigenfunction is obtained. The corresponding boundary conditions here are listed below:

At the clamped end, deflection and slope are equal to zero:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad (2.14)$$

At the free end, moment and shear are equal to zero:

$$M = 0 \rightarrow \frac{\partial^2 w}{\partial x^2} = 0 \quad (2.15)$$

$$V = 0 \rightarrow \frac{\partial^3 w}{\partial x^3} = 0$$

From equations (2.1) and (2.14) one obtains

$$\begin{aligned} X(0) &= 0 \\ X'(0) &= 0 \end{aligned} \quad (2.16)$$

Substituting equations (2.8) into (2.16) yields

$$X(0) = C_2 + C_4 = 0 \rightarrow C_4 = -C_2 \quad (2.17a)$$

and

$$X'(0) = C_1 + C_3 = 0 \rightarrow C_3 = -C_1 \quad (2.17b)$$

From equations (2.15) it follows

$$X''(\ell) = 0 \quad (2.18)$$

$$X'''(\ell) = 0$$

Substituting equations (2.8) into (2.18) yields

$$X''(\ell) = -C_1 \sin \alpha \ell - C_2 \cos \alpha \ell + C_3 \sinh \alpha \ell + C_4 \cosh \alpha \ell = 0 \quad (2.19a)$$

$$X'''(\ell) = -C_1 \cos \alpha \ell + C_2 \sin \alpha \ell + C_3 \cosh \alpha \ell + C_4 \sinh \alpha \ell = 0 \quad (2.19b)$$

Substituting equations (2.17) into equations (2.19) yields

$$C_3[\sin \alpha \ell + \sinh \alpha \ell] + C_4[\cos \alpha \ell + \cosh \alpha \ell] = 0 \quad (2.20a)$$

$$C_3[\cos \alpha \ell + \cosh \alpha \ell] + C_4[-\sin \alpha \ell + \sinh \alpha \ell] = 0 \quad (2.20b)$$

Rearranging and dividing equations (2.20) by C_4 , the ratio C_3/C_4 becomes

$$\frac{C_3}{C_4} = \frac{\cos \alpha \ell + \cosh \alpha \ell}{\sin \alpha \ell + \sinh \alpha \ell} = \frac{\sin \alpha \ell - \sinh \alpha \ell}{\cos \alpha \ell + \cosh \alpha \ell} \quad (2.21)$$

To obtain the eigenfunction from (2.8), C_4 may be factored out to obtain

$$X(x) = C_4[\cosh \alpha x - \cos \alpha x + \frac{C_3}{C_4}(\sinh \alpha x - \sin \alpha x)] \quad (2.22)$$

Substituting (2.21) into (2.22) yields

$$X(x) = C_4[(\cosh \alpha x - \cos \alpha x) - \gamma(\sinh \alpha x - \sin \alpha x)] \quad (2.23)$$

where $\gamma = -\frac{C_3}{C_4}$ and is the eigenvector and where C_4 is determined from the initial conditions.

Omitting the coefficient C_4 , equation (2.23) may be written as

$$X_m(x) = (\cosh \alpha_m x - \cos \alpha_m x) - \gamma_m(\sinh \alpha_m x - \sin \alpha_m x) \quad (2.24)$$

where the subscript m has been added to identify the m^{th} mode shape. Equation (2.24) is therefore the eigenfunction.

Summarizing the above results and from equation (2.10) and (2.24) it may be said that the general solution for the free undamped vibration displacement is expressed as

$$w(x,t) = \sum_{m=1}^{\infty} X_m(x) (A_m \sin \omega_m t + B_m \cos \omega_m t) \quad (2.25)$$

where ω_m is the natural frequency of the m^{th} mode with an amplitude determined by A_m and B_m . Thus the motion is represented by the superposition of free vibration modes, each vibrating at its own frequency.

2.3 CHARACTERISTIC FREQUENCY EQUATION AND EIGENVALUES

The free vibration frequencies and mode shapes for the beam are determined from equations (2.20):

$$\begin{bmatrix} \sin \alpha l + \sinh \alpha l & \cos \alpha l + \cosh \alpha l \\ \cos \alpha l + \cosh \alpha l & -\sin \alpha l + \sinh \alpha l \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution ($C_3 \neq 0$, $C_4 \neq 0$), setting the determinant of the coefficient matrix equal to zero, expanding it, and using some identities, yields the characteristic frequency equation:

$$\cos \alpha_m \ell \cosh \alpha_m \ell + 1 = 0 \quad (2.26)$$

Solving for the roots of (2.26), which are the eigenvalues, one obtains:

$$\alpha_m \ell = 1.87510410, 4.69409113, 7.85475743, 10.99554074, 14.13716839,$$

The corresponding eigenvectors are, from equation (2.21):

$$\gamma_m = 0.7340955, 1.01846644, 0.99922450, 1.00003355, 0.99999855$$

Substituting these results into the eigenfunction, equation (2.24), one obtains the mode shapes corresponding to the first five natural frequencies. A plot of the first five modes is shown in Figure 2.1.

Continuing, from (2.12), the frequency parameters, $\omega_m \ell^2 \sqrt{\rho A / EI}$, are = 3.51601539, 22.03449154, 61.69721428, 120.9019162, 199.8595301

hence $\omega_2 = C_1 \omega_1$

$$\omega_3 = C_2 \omega_1$$

$$\omega_4 = C_3 \omega_1$$

$$\omega_5 = C_4 \omega_1$$

Figure 2.1. A superposition of the first five mode shapes for a cantilever beam.

where $C_1 = 6.26689281$

$$C_2 = 17.54748131$$

$$C_3 = 34.38609944$$

$$C_4 = 56.93028045$$

Therefore the frequencies are not integer multiples of the first fundamental frequency.

2.4 INITIAL CONDITIONS

Amplitudes A_m and B_m in (2.25) need to be evaluated using initial conditions. Suppose the beam is given initial displacement and velocity of arbitrary forms, $f(x)$ and $g(x)$, respectively, which are consistent with the boundary conditions at the ends; i.e.,

$$w(x,0) = f(x) \quad (2.27a)$$

$$\frac{\partial w}{\partial t}(x,0) = g(x) \quad (2.27b)$$

But, from (2.25),

$$w(x,0) = \sum_{m=1}^{\infty} B_m X_m \quad (2.28a)$$

$$\frac{\partial w}{\partial t}(x,0) = \sum_{m=1}^{\infty} A_m \omega_m X_m \quad (2.28b)$$

Therefore,

$$f(x) = \sum_{m=1}^{\infty} B_m X_m \quad (2.29a)$$

$$g(x) = \sum_{m=1}^{\infty} A_m \omega_m X_m \quad (2.29b)$$

Multiply both sides of (2.29a) by X_n , where X_n represents the n^{th} eigenfunction, and integrate over the length of the beam; i.e.,

$$\int_0^{\ell} f(x) X_n dx = \int_0^{\ell} \sum_{m=1}^{\infty} B_m X_m X_n dx \quad (2.30)$$

Interchanging the order of summation and integration on the R.H.S. of (2.30), and considering the orthogonality of X_m and X_n , (Ref. 1) it is seen that all terms of the infinite sum on the R.H.S. of (2.30) vanish, except when $m = n$. Thus, (2.30) yields

$$B_m = \frac{\int_0^{\ell} f(x) X_m(x) dx}{\int_0^{\ell} X_m^2(x) dx} \quad (2.31)$$

Following the same procedure, one obtains from (2.29b)

$$A_m = \frac{\frac{1}{\omega_m} \int_0^{\ell} g(x) X_m(x) dx}{\int_0^{\ell} X_m^2(x) dx} \quad (2.32)$$

Equations (2.31) and (2.32) are formulas which may be straightforwardly used to determine the amplitude components A_m and B_m of each mode as the beam responds to initially imposed displacements and/or velocities. In this paper discussion will be limited to the special case when the beam is given an initial displacement shape and released from rest; then $g(x) = 0$ and $A_m = 0$. Conversely, if the beam were in its straight, equilibrium position and subjected to an impulsive loading of short duration, the subsequent free vibration motion would begin with $f(x) = 0$ and $B_m = 0$

CHAPTER III

INITIAL END MOMENT

3.1 STATIC DISPLACEMENT SHAPE

The cantilever beam shown in Figure 3.1 is initially deformed by a statically applied end moment M_0 . suddenly the moment is released.

Figure 3.1 A cantilever beam with initial static end moment.

To study the behavior of the beam, it is desirable to make a plot of $\delta(t)/\Delta$ versus $\frac{\omega_1 t}{2\pi}$, where $\delta(t)$ and Δ are the vibratory and static displacements of the free end, respectively. The first natural frequency of free vibration is ω_1 .

It is first necessary to determine the static displacement shape of the beam, from which it is released from rest. From the equation of motion in the special case of equilibrium;

$$EI \frac{d^4 w}{dx^4} - q = 0 \quad (3.1)$$

where q is the intensity of distributed load. Integrating, and knowing that the shear is zero everywhere, yields:

$$EI \frac{d^3 w}{dx^3} = C_0 = 0 \quad (3.2)$$

Integrating, and setting the moment everywhere equal to M_0 ; yields:

$$EI \frac{d^2 w}{dx^2} = C_1 = M_0 \quad (3.3)$$

Integrating, and knowing that the slope is zero at the clamped end, yields:

$$EI \frac{dw}{dx} = M_0 x \quad (3.4)$$

Integrating, and knowing that the deflection is zero at the clamped end, yields:

$$EIw = M_0 \frac{x^2}{2} \quad (3.5)$$

The initial static displacement is therefore

$$w_0(x) = f(x) = \frac{M_0}{2EI} x^2 \quad (3.6)$$

From (3.6) the initial static displacement at the free end is:

$$w_0(\ell) = \Delta = \frac{M_0}{2EI} \ell^2 \quad (3.7)$$

3.2 VIBRATION AMPLITUDE COEFFICIENTS

In Chapter 2, expressions for vibration amplitude coefficients A_m and B_m were derived in equations (2.31) and (2.32). It was further

deduced that in the present case, the coefficient A_m is zero, since there is no impact. At this stage, the eigenfunction, X_m , is derived in (2.24) and the initial static displacement, $f(x)$, is obtained in (3.6). Substituting (2.24) and (3.6) into (2.31) yields:

$$B_m = \frac{\frac{M_o}{2EI} \int_0^{\ell} x^2 [\cosh \alpha_m x - \cos \alpha_m x - \gamma_m (\sinh \alpha_m x - \sin \alpha_m x)] dx}{\int_0^{\ell} [\cosh \alpha_m x - \cos \alpha_m x - \gamma_m (\sinh \alpha_m x - \sin \alpha_m x)]^2 dx} \quad (3.8)$$

Evaluation of integrals for the numerator and denominator of (3.8) is rather complicated. To get the exact answers, considerable integration by parts is required for the numerator integrals, and many integral terms must be evaluated for the denominator.

3.3 EVALUATION OF INTEGRALS AND HENCE VIBRATION AMPLITUDE COEFFICIENTS

Since evaluating the integrals is the crux of the problem, this section will be devoted to exactly carrying out the integrals by parts.

Expanding the integrals for the denominator, yields:

$$I_1 = \int_0^{\ell} \cosh^2 \alpha_m x \, dx = \frac{\ell}{2} + \frac{\sinh 2\alpha_m \ell}{4\alpha_m}$$

$$I_2 = \int_0^{\ell} \cosh \alpha_m x \cos \alpha_m x \, dx = \frac{1}{2\alpha_m} [\cosh \alpha_m \ell \sin \alpha_m \ell + \sinh \alpha_m \ell \cos \alpha_m \ell]$$

$$I_3 = \int_0^{\ell} \cos^2 \alpha_m x \, dx = \frac{\ell}{2} + \frac{\sin 2\alpha_m \ell}{4\alpha_m}$$

$$I_4 = \int_0^{\ell} \cosh \alpha_m x \sinh \alpha_m x \, dx = \frac{1}{2\alpha_m} [\cosh^2 \alpha_m \ell - 1]$$

$$I_5 = \int_0^{\ell} \cosh \alpha_m x \sin \alpha_m x \, dx = \frac{1}{2\alpha_m} [\sinh \alpha_m \ell \sin \alpha_m \ell - \cosh \alpha_m \ell \cos \alpha_m \ell + 1]$$

$$I_6 = \int_0^{\ell} \cos \alpha_m x \sinh \alpha_m x \, dx = \frac{1}{2\alpha_m} [\cosh \alpha_m \ell \cos \alpha_m \ell + \sinh \alpha_m \ell \sin \alpha_m \ell - 1]$$

$$I_7 = \int_0^{\ell} \cos \alpha_m x \sin \alpha_m x \, dx = \frac{1}{4\alpha_m} [1 - \cos \alpha_m \ell]$$

$$I_8 = \int_0^{\ell} \sinh^2 \alpha_m x \, dx = \frac{\sinh 2\alpha_m \ell}{4\alpha_m} - \frac{\ell}{2}$$

$$I_9 = \int_0^{\ell} \sinh \alpha_m x \sin \alpha_m x \, dx = \frac{1}{2\alpha_m} [\cosh \alpha_m \ell \sin \alpha_m \ell - \sinh \alpha_m \ell \cos \alpha_m \ell]$$

$$I_{10} = \int_0^{\ell} \sin^2 \alpha_m x \, dx = \frac{\ell}{2} - \frac{\sin 2\alpha_m \ell}{4\alpha_m}$$

Expanding the integrals for the numerator, yields:

$$I_{11} = \int_0^{\ell} x^2 \cosh \alpha_m x \, dx = \frac{\ell^2}{\alpha_m} \sinh \alpha_m \ell - \frac{2\ell}{\alpha_m^2} \cosh \alpha_m \ell + \frac{2}{\alpha_m^3} \sinh \alpha_m \ell$$

$$I_{12} = \int_0^{\ell} x^2 \cos \alpha_m x \, dx = \frac{2\ell}{\alpha_m^2} \cos \alpha_m \ell + \frac{\ell^2}{\alpha_m} \sin \alpha_m \ell - \frac{2}{\alpha_m^3} \sin \alpha_m \ell$$

$$I_{13} = \int_0^{\ell} x^2 \sinh \alpha_m x \, dx = \frac{\ell^2}{\alpha_m} \cosh \alpha_m \ell - \frac{2\ell}{\alpha_m^2} \sinh \alpha_m \ell + \frac{2}{\alpha_m^3} \cosh \alpha_m \ell - \frac{2}{\alpha_m^3}$$

$$I_{14} = \int_0^{\ell} x^2 \sin \alpha_m x \, dx = \frac{2\ell}{\alpha_m^2} \sin \alpha_m \ell - \frac{\ell^2}{\alpha_m} \cos \alpha_m \ell + \frac{2}{\alpha_m^3} \cos \alpha_m \ell - \frac{2}{\alpha_m^3}$$

Substituting the above integrals in (3.8) yields

$$B_m = \frac{\frac{M_o}{2EI} [I_{11} - I_{12} + \gamma_m(-I_{13} + I_{14})]}{[I_1 - 2I_2 + I_3 + 2\gamma_m(-I_4 + I_5 + I_6 - I_7) + \gamma_m^2(I_8 - 2I_9 + I_{10})]} \quad (3.9)$$

Calculations for the fourteen integrals for the first five natural frequencies are shown in Table 3.1. Substituting for the integrals in (3.9) yields the first five amplitude coefficients:

$$B_1 / \frac{M_o \ell^2}{2EI} = 0.44538586$$

$$B_2 / \frac{M_o \ell^2}{2EI} = -0.03938044$$

$$B_3 / \frac{M_o \ell^2}{2EI} = 0.00821649$$

$$B_4 / \frac{M_o \ell^2}{2EI} = -0.00297822$$

$$B_5 / \frac{M_o \ell^2}{2EI} = 0.00153090$$

It is noticed that B_1/Δ where Δ is defined in (3.7) is by far the largest coefficient, and so the response of the first mode strongly dominates the ensuing motion. This is expected for the parabolic initial shape which approximates the first vibration mode shape reasonably close. It is also seen that, while the first mode dominates the response, the higher frequency modes make small contributions to the motion. Consequently, the first five modes represent the subsequent motion with sufficient accuracy.

3.4 FREE VIBRATION DISPLACEMENT

Since the amplitude coefficient A_m is zero, equation (2.25) for the free undamped vibration displacement reduces to

Table 3.1. Integrals used in Equation 3.9 for the first five eigenfunctions

I_i	$\alpha_1 \ell$	$\alpha_2 \ell$	$\alpha_3 \ell$	$\alpha_4 \ell$	$\alpha_5 \ell$
I_1	3.333604209 ℓ	318.6223954 ℓ	105763.2787 ℓ	40392316.06 ℓ	1.682426208E10 ℓ
I_2	0.594640242 ℓ	-5.92712932 ℓ	81.98755202 ℓ	-1355.31628 ℓ	24393.33444 ℓ
I_3	423773279 ℓ	0.501948595 ℓ	0.499950616 ℓ	0.500001526 ℓ	0.499999948 ℓ
I_4	2.70341315 ℓ	318.0691413 ℓ	105762.7468 ℓ	40392315.53 ℓ	1.682426208E10 ℓ
I_5	1.343334963 ℓ	-5.60662207 ℓ	82.17849302 ℓ	-1355.17984 ℓ	24393.44047 ℓ
I_6	0.276727445 ℓ	-6.03268967 ℓ	81.92387162 ℓ	-1355.36175 ℓ	24393.29907 ℓ
I_7	0.242711903 ℓ	0.106481233 ℓ	0.063655651 ℓ	0.045473 ℓ	0.035367761 ℓ
I_8	2.333604209 ℓ	317.6223954 ℓ	105762.2787 ℓ	40392315.06 ℓ	1.682426208E10 ℓ
I_9	1.103441184 ℓ	-5.71413117 ℓ	82.114862 ℓ	-1355.22532 ℓ	24393.4051 ℓ
I_{10}	0.57622672 ℓ	0.498051404 ℓ	0.500049383 ℓ	0.499998474 ℓ	0.500000051 ℓ
I_{11}	0.765584367 ℓ^3	7.737106302 ℓ^3	127.6377975 ℓ^3	2262.354676 ℓ^3	42373.03876 ℓ^3
I_{12}	0.048942987 ℓ^3	-0.19532571 ℓ^3	0.123184368 ℓ^3	-0.08944205 ℓ^3	0.070027656 ℓ^3
I_{13}	0.677743362 ℓ^3	7.720726346 ℓ^3	127.6337341 ℓ^3	2262.353173 ℓ^3	42373.03805 ℓ^3
I_{14}	0.308233542 ℓ^3	-0.10654393 ℓ^3	0.028384957 ℓ^3	-0.01804379 ℓ^3	0.009299277 ℓ^3

$$w(x, t) = \sum_{m=1}^{\infty} B_m \cos \omega_m t X_m(x) \quad (3.10)$$

Substituting for $X_m(x)$ yields,

$$w\left(\frac{x}{\ell}, t\right) = \sum_{m=1}^{\infty} B_m \cos \omega_m t [\cosh \alpha_m \ell \frac{x}{\ell} - \cos \alpha_m \ell \frac{x}{\ell} - \gamma_m (\sinh \alpha_m \ell \frac{x}{\ell} - \sin \alpha_m \ell \frac{x}{\ell})] \quad (3.11)$$

using the nondimensional coordinate $\xi = \frac{x}{\ell}$, (3.11) becomes

$$w(\xi, t) = \sum_{m=1}^{\infty} B_m \cos \omega_m t [\cosh \alpha_m \ell \xi - \cos \alpha_m \ell \xi - \gamma_m (\sinh \alpha_m \ell \xi - \sin \alpha_m \ell \xi)] \quad (3.12)$$

Expanding and substituting for the first five amplitude coefficients, yields:

$$\begin{aligned} \frac{w(\xi, t)}{\Delta} = & 0.44538586 \cos \omega_1 t X_1(\xi) - 0.03938044 \cos \omega_2 t X_2(\xi) \\ & + 0.00821649 \cos \omega_3 t X_3(\xi) - 0.00297822 \cos \omega_4 t X_4(\xi) \\ & + 0.00153090 \cos \omega_5 t X_5(\xi) \end{aligned} \quad (3.13)$$

$$\text{where } X_m(\xi) = \cosh \alpha_m \ell \xi - \cos \alpha_m \ell \xi - \gamma_m (\sinh \alpha_m \ell \xi - \sin \alpha_m \ell \xi)$$

For one cycle, $\omega_1 t$ will vary from zero to 2π , i.e.,

$$0 \leq \omega_1 t \leq 2\pi$$

Dividing through out by 2π yields,

$$0 \leq \frac{\omega_1 t}{2\pi} \leq 1$$

It is useful to express the motion in (3.13) in terms of the first fundamental frequency. Equation (3.13) thus becomes:

$$\begin{aligned} \frac{w(\xi, t)}{\Delta} = & 0.44538586 \cos \left(\frac{\omega_1 t}{2\pi} \right) 2\pi X_1(\xi) \\ & -0.03938044 \cos \left(\frac{\omega_1 t}{2\pi} \right) 2\pi C_1 X_2(\xi) + 0.00821649 \cos \left(\frac{\omega_1 t}{2\pi} \right) 2\pi C_2 X_3(\xi) \\ & -0.00297822 \cos \left(\frac{\omega_1 t}{2\pi} \right) 2\pi C_3 X_4(\xi) + 0.00153090 \cos \left(\frac{\omega_1 t}{2\pi} \right) 2\pi C_4 X_5(\xi) \end{aligned} \quad (3.14)$$

where

$$C_1 = \omega_2/\omega_1 = 6.26689281$$

$$C_2 = \omega_3/\omega_1 = 17.54748131$$

$$C_3 = \omega_4/\omega_1 = 34.38609944$$

$$C_4 = \omega_5/\omega_1 = 56.93028045$$

Further, the period, T_1 for the first vibration frequency, ω_1 , is the reciprocal of the cyclic frequency, i.e.,

$$T_1 = \frac{2\pi}{\omega_1} \quad (3.15)$$

Substituting (3.15) into (3.14) yields;

$$\begin{aligned} \frac{w(\xi, t)}{\Delta} = & 0.44538586 \cos \frac{t}{T_1} (2\pi) X_1(\xi) \\ & -0.03938044 \cos \frac{t}{T_1} (2\pi C_1) X_2(\xi) + 0.00821649 \cos \frac{t}{T_1} (2\pi C_2) X_3(\xi) \\ & -0.00297822 \cos \frac{t}{T_1} (2\pi C_3) X_4(\xi) + 0.00153090 \cos \frac{t}{T_1} (2\pi C_4) X_5(\xi) \end{aligned}$$

$$-0.00297822 \cos \frac{t}{T_1} (2\pi C_3) X_4 (\xi) + 0.00153090 \cos \frac{t}{T_1} (2\pi C_4) X_5 (\xi) \quad (3.16)$$

where for one cycle $0 \leq \frac{t}{T_1} \leq 1$ and for three cycles $0 \leq \frac{t}{T_1} \leq 3$ and so on.

Figure 3.2 below is a plot of the non-dimensional displacement, δ/Δ , versus the non-dimensional time, t/T_1 , in cycles for the first cycle of the response of the displacement at the free end. It is seen that while the first mode dominates the response, the higher frequency modes make small contributions to the motion which disturb its smoothness. Further it is observed that the curve peaks at 0.96 cycle rather than one cycle. The motion, thus, is not periodic. Figure 3.3 shows free end deflection for the first three cycles. The amplitude peaks in a different fashion in the vicinity of one, two and three cycles. Differences are also noticeable at $\frac{1}{2}$, $1 \frac{1}{2}$ and $2 \frac{1}{2}$ cycles. This can also be seen clearly in Figure 3.4 for the first six cycles. It is interesting to note that at $t/T_1 = 4$, the amplitude has almost the exact same pattern as that in the vicinity of $t/T_1 = 0$. Further, from $t/T_1 = 4$ to $t/T_1 = 6$, the beam follows nearly the same pattern as that from the $t/T_1 = 0$ to $t/T_1 = 2$, but with different amplitude. It can be, therefore, concluded that the motion is nearly periodic, with a quasi-periodicity of $4T_1$.

If the beam were initially displaced in exactly the first mode shape, it would vibrate subsequently in that single mode. No other modes would be involved and the motion would, therefore, be periodic. Since an infinite number of modes is excited, then the super imposed motion can no longer be periodic unless the frequencies are integer multiples of the first one.

Continuing, and investigating the response of the beam at $\xi = 0.8$, in Figure 3.5, the displacement becomes much smoother. This may look strange, but can be explained rather simply. The nodal point for

Figure 3.2. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial end moment (one cycle).

Figure 3.3. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial end moment (three cycles).

Figure 3.4. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial end moment (six cycles).

Figure 3.5. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial end moment (three cycles).

the second mode shape is in a very close proximity to $\xi = 0.8$. In fact it is at $\xi = 0.7834$. This mode makes very little contribution to the motion in the vicinity of its nodal point. As was seen earlier, contributions of the other modes is very weak. The motion, however, is still not periodic. It will not be periodic at any point on the beam.

In Figure 3.6 the response at $\xi = 0.6$ is investigated. The motion becomes disturbed again since the second mode again contributes significantly. Contributions from higher modes become relatively stronger and disturb its smoothness. At time equals zero, the amplitude at the free end is almost 0.35. At one cycle and two cycles, the amplitude becomes larger and is a little over 0.45. This response did not occur at $\xi = 1.0$ and 0.8 where subsequent amplitudes were almost the same or less than the amplitude at $t = 0$.

At $\xi = 0.4$ the response is investigated in Figure 3.7. The motion has become more disturbed. Again, amplitudes at one and two cycles are higher than the static amplitude. Here higher frequencies have strong contributions to the motion, and force the response out of the first vibration mode. The resulting motion is the most disturbed compared to $\xi = 1.0$, 0.8 and 0.6.

The response at $\xi = 0.2$ is shown in Figure 3.8. The motion is completely disturbed with hardly any smooth lines. The amplitude is small since it is close to the clamped end.

It is important to note that conservation of energy still holds even though the motion is not periodic.

3.5 SHAPE OF THE BEAM AT SUBSEQUENT INSTANTS

Figure 3.9 depicts the parabolic shape of the deflected beam with the initial end moment. Figure 3.10 shows the shape of the beam during one cycle of the first frequency for various time intervals. At time equal to zero, i.e. before the release of the applied end moment, the normalized non-dimensional tip displacement is equal to

Figure 3.6. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial end moment (three cycles).

Figure 3.7. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial end moment (three cycles).

Figure 3.8. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial end moment (three cycles).

Figure 3.9. Initial displaced shape for an applied end moment

Figure 3.10. Shape of the beam during the first cycle of motion for various time intervals for an initial end moment.

one. At time equal to a quarter of a cycle, the beam almost returns back to its equilibrium position. At this instant, however, it does not become a straight horizontal beam as would be expected if the motion was periodic. The beam here has a shallow curved shape. At one half cycle, the beam continues with its motion past its equilibrium position to a downward deflection having considerably less displacement than the initial value. Continuing, at three quarters of the first cycle, the beam returns back to its quarter of a cycle position, but with a different shape. The beam at this instant is almost a straight line and is very close to the equilibrium position. Finally at the end of the first cycle, the beam continues with its motion past the equilibrium position and almost goes back to its original deflected shape. At this instant, however, the beam is not at its original 1.0 displacement at the free end, but rather 0.86. Further, it is interesting to note that at quarter and three quarters of a cycle and at zero cycle and one cycle the different beam shapes intersect very close to $\xi = 0.78$ which is the nodal point of the second mode. This is another indication of why the motion at $\xi = 0.8$ in Figure 3.5 became smooth.

A superposition of zero, one, two and three cycle instants of time is shown in Figure 3.11. The third cycle has the closest shape to the instant when $t = 0$. The beam thus does not go back to its original deflected shape since the motion is not periodic.

Figure 3.11. A superposition of the initial shape of the beam and its shape at one, two and three cycles for an initial end moment.

CHAPTER IV

NUMERICAL ANALYSIS

4.1 NUMERICAL INTEGRATION

From Chapter 3, it is seen that calculating the integrals to evaluate amplitude coefficients, B_m , is the crux of the problem. Calculating the integrals of (3.9) was a tedious task even though the exact solution was obtained. To study the response of the beam for other initial conditions, it is much easier to do the integrations numerically. This is because other initial conditions will result in integrations that are even more complicated.

In order to do the integrations numerically, it is necessary to solve for the roots of the frequency equation (2.26) using double precision. Numerical integrations are then carried out using the new eigenvalues and their corresponding eigenvectors.

Further, the accuracy of the numerical integrations is established by comparing the results with the exact integrations. Exact integrations, however, are all carried out by using the new eigenvalues and their corresponding eigenvectors with double precision.

4.2 EIGENVALUES AND EIGENVECTORS

A comparison of eigenvalues and eigenvectors calculated by single and double precision is presented in Tables 4.1 and 4.2, respectively. Eigenvalues agree up to the seventh or eighth decimal place except for the first value, which agrees up to the sixth decimal. The error is very small and is of no significance. The discrepancy between eigenvectors calculated by single and double precision is higher. The error, however, is small and has no significance in the present work.

Table 4.1: Eigenvalues calculated by single and double precision

Mode Number	$\alpha_m \ell$ (eigenvalue)		
	Single Precision	Double Precision	% Error
1	1.87510410	1.87510407	0.00000160
2	4.69409113	4.69409113	0
3	7.85475743	7.85475744	0.00000013
4	10.99554074	10.99554073	0.00000009
5	14.13716839	14.13716839	0

Table 4.2: Eigenvectors calculated by single and double precision

Mode Number	γ_m (eigenvector coefficient)		
	Single Precision	Double Precision	% Error
1	0.73409550	0.73409551	0.00000136
2	1.01846644	1.01846732	0.00008640
3	0.99922450	0.99922450	0
4	1.00003355	1.00000006	0.00334900
5	0.99999855	0.99999970	0.00011500

4.3 EIGENFUNCTION AMPLITUDE COEFFICIENTS

Using the eigenvalues and eigenvectors obtained by double precision, the eigenfunction amplitude coefficients are calculated using exact integration with double precision. Table 4.3 compares eigenfunction amplitude coefficients, B_m/Δ , calculated using exact

integration with both single and double precision. At first glance, it is noticed that the error here is much larger than the error in Tables 4.1 and 4.2. It ranges from 2.2×10^{-6} to 8.1 percent for the fifth amplitude coefficient.

Table 4.3: Eigenfunction amplitude coefficients calculated by single and double precision using the exact integration.

Mode Number	B_m/Δ (eigenfunction amplitude coefficients)		
	Single Precision	Double Precision	% Error
1	0.44538586	0.44538587	0.00000225
2	-0.03938044	-0.03938697	0.01657909
3	0.00821649	0.00824755	0.37659669
4	-0.00297822	-0.00300902	1.02358908
5	0.00153090	0.00141570	8.13731722

The error is larger for higher amplitude coefficient orders. This is not surprising, and can be reasoned rather simply. In Table 3.1, it is observed that higher frequencies result in larger integral values as in I_1 , whereas they result in smaller integral values as in I_{14} . Otherwise, the value of the integral remains more or less the same as in I_3 . When these integrals are added for the first frequency as per equation (3.9), the resulting arithmetic error is very small since the added numbers are close to each other. For higher frequencies, very large numbers are added to very small numbers resulting in a much higher arithmetic error due to roundoff. Double precision calculations reduce this error since 16 decimal places are carried out.

Next, numerical integrations are carried out by the Gaussian Quadrature Formula (Ref. 2) with 48 integration points. A comparison of eigenfunction amplitude coefficients, B_m/Δ , is obtained by both integration methods is given in Table 4.4

Table 4.4: Eigenfunction amplitude coefficients obtained by exact and numerical integration.

Mode Number	B_m/Δ (eigenfunction amplitude coefficients)		
	Exact Integration	Numerical Integration	% Error
1	0.44538587	0.44538587	0
2	-0.03938697	-0.03938697	0
3	0.00824755	0.00824755	0
4	-0.00300902	-0.00300902	0
5	0.00141570	0.00141570	0

To the eighth decimal point, coefficients obtained by numerical integration are seen to be the same as those obtained by exact integration. Round off error is negligible. The computer program written for numerical integration is, therefore, accurate enough to yield the exact coefficients.

In conclusion, when the automated integration is used to evaluate the amplitude coefficients for other initial conditions, it is expected to yield results that are accurate up to the eighth decimal.

In the subsequent chapters, the response of the beam will be studied for initial point load, initial uniformly distributed load, initial cubic deflection shape and initial shape similar to the second mode in Chapters 5, 6 7 and 8 respectively. In all of the above mentioned chapters, the automated numerical integration will be used to calculate the eigenfunction amplitude coefficients since its accuracy has been established.

CHAPTER V
INITIAL POINT LOAD AT THE END

5.1 STATIC DISPLACEMENT SHAPE

The cantilever beam shown in Figure 5.1 is initially deformed by a statically applied point load, P , at the end. Suddenly the load is released.

Figure 5.1 A cantilever beam with initial static end force

Again, to study the behavior of the beam, it is necessary to make a plot of $\delta(t)/\Delta$ versus $\frac{\omega_1 t}{2\pi}$, where $\delta(t)$ and Δ are the vibratory and static displacements of the free end, respectively. The first natural frequency of free vibration is ω_1 .

To determine the initial static displacement shape of the beam, one proceeds from the equation of motion in the special case of equilibrium;

$$EI \frac{d^4 w}{dx^4} = q = 0 \quad (5.1)$$

Where, q , is the intensity of distributed load as defined in Chapter 3. Integrating, and knowing that the shear is equal to P ; yields:

$$EI \frac{d^3 w}{dx^3} = -P \quad (5.2)$$

Integrating, and setting the moment at the clamped end equal to, $P\ell$; yields:

$$EI \frac{d^2 w}{dx^2} = P\ell - Px \quad (5.3)$$

Integrating, and knowing that the slope is zero at the clamped end, yields:

$$EI \frac{dw}{dx} = P\ell x - P\frac{x^2}{2} \quad (5.4)$$

Integrating, and knowing that the deflection is zero at the clamped end, yields:

$$EIw = P\ell\frac{x^2}{2} - P\frac{x^3}{6} \quad (5.5)$$

Rearranging, the initial static displacement becomes

$$w_o(x) = f(x) = \frac{P}{3EI} (3x^2\ell/2 - x^3/2) \quad (5.6)$$

From (5.6) the initial static displacement at the free end is

$$w_o(\ell) = \Delta = \frac{P\ell^3}{3EI} \quad (5.7)$$

5.2 VIBRATION AMPLITUDE COEFFICIENTS

In order to evaluate the amplitude coefficient, B_m , (5.6) is substituted into (2.31). Thus,

$$B_m = \frac{\frac{P}{3EI} \int_0^\ell (3x^2\ell/2 - x^3/2) X_m(x) dx}{\int_0^\ell X_m^2(x) dx} \quad (5.8)$$

where $X_m(x)$ is defined in (2.24). Evaluating the integrals of (5.8) for the first five natural frequencies, using the numerical integration developed in Chapter 4, yields the following vibration amplitude coefficients:

$$B_1/\frac{P\ell^3}{3EI} = 0.48534409$$

$$B_2/\frac{P\ell^3}{3EI} = -0.01235791$$

$$B_3/\frac{P\ell^3}{3EI} = 0.00157623$$

$$B_4/\frac{P\ell^3}{3EI} = -0.00041047$$

$$B_5/\frac{P\ell^3}{3EI} = 0.00015021$$

Again, it is noticed that B_1/Δ , where Δ is defined in (5.7), is by far the largest coefficient, and so the response of the first mode strongly dominates the ensuing motion. It is also seen that higher frequency modes have lesser influence on the response than the initial end moment discussed in Chapter 3. Deflection due to the initial point load at the end thus approximates the first vibration mode closer than the initial end moment.

5.3 FREE VIBRATION DISPLACEMENT

The general expression for free undamped vibration displacement is given by (3.10). Substituting for B_m and using $\xi = x/\ell$, and expanding using the same analysis given in Chapter 3, the expression becomes

$$\begin{aligned} \frac{w}{\Delta}(\xi, t) = & 0.48534409 \cos \frac{t}{T_1} (2\pi) X_1(\xi) \\ & - 0.01235791 \cos \frac{t}{T_1} (2\pi C_1) X_2(\xi) + 0.00157623 \cos \frac{t}{T_1} (2\pi C_2) X_3(\xi) \\ & - 0.00041047 \cos \frac{t}{T_1} (2\pi C_3) X_4(\xi) + 0.00015021 \cos \frac{t}{T_1} (2\pi C_4) X_5(\xi) \end{aligned} \quad (5.9)$$

Figure 5.2 below is a plot of the non-dimensional displacement, δ/Δ , versus the non-dimensional time, t/T_1 , in cycles for the first three cycles of the response of the first mode at the free end. It is apparent how little influence higher frequencies have on the motion since the curve is rather smooth and resembles a cosine curve closely. Free end deflection is shown in Figure 5.3 for six cycles. Again, the peak amplitude have significantly different shapes in the first, second and third cycle. The peak amplitude at the fourth cycle is similar to the one at the beginning. Similarly the fifth is similar to the first, the sixth peak amplitude is similar to the second, and so on.

Investigating the response at $\xi = 0.8$ is shown in Figure 5.4. The deflection curve is very smooth and has virtually the same peak amplitude at all three cycles. Even though the motion may look periodic, it is not. The response is smooth since the chosen point on the beam is close to the nodal point of the second mode shape at $\xi = 0.78$. This nodal point dominates the response.

At $\xi = 0.6$ the response becomes disturbed because higher frequencies have stronger contribution. This is shown in Figure 5.5.

Figure 5.2. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial point load at the end (three cycles).

Figure 5.3. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial point load at the end (six cycles).

Figure 5.4. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial point load at the end (three cycles).

Figure 5.5. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial point load at the end (three cycles).

The peak amplitudes are not equal anymore. This can be noticed when the peaks at zero and three cycles are compared. Moving farther away from the nodal point, the response becomes more disturbed, as in Figure 5.6 for $\xi = 0.4$. While the vibration amplitude is smaller, the motion becomes jerky. The response at $\xi = 0.2$ in Figure 5.7 is similar to the response at $\xi = 0.4$ but with a smaller amplitude. The contribution of higher frequency modes is more apparent at this location.

5.4 SHAPE OF THE BEAM AT SUBSEQUENT INSTANTS

Figure 5.8 shows the initial shape of the beam before the release of the initial point load. It has a cubic deflection shape which approximates the first mode very closely. The shape of the beam during the first cycle of motion for various time intervals is shown in Figure 5.9. Before the release of the initial point load, the end of the beam has a normalized nondimensional displacement of 1.0. At one quarter of a cycle after the release of the initial load, the beam tends to go back to its original equilibrium position, but not quite. It is still not a horizontal straight line as in the case of a periodic motion. It has a very shallow curvature. The beam continues with its motion past the horizontal position to a downward deflection at one half cycle. The end deflection here is not -1.0, but rather -0.95. As the beam continues with its motion, it goes back to its horizontal position at three quarters of a cycle. Here the beam is very nearly straight and has a reflection a little less than zero at the tip. As the beam continues with its motion, the tip goes back to a tip deflection of 0.96 instead of one, as expected. It should be mentioned that the different beam shapes at zero and one cycle, one fourth and three fourth cycle intersect close to $\xi = 0.78$ which is the nodal point of the second mode.

A superposition of one, two, three cycles and the initial deflected shape is shown in Figure 5.10. While all cycles have

Figure 5.6. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial point load at the end (three cycles).

Figure 5.7. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial point load at the end (three cycles).

Figure 5.8. Initial displaced shape for an applied point load at the end.

Figure 5.9. Shape of the beam during the first cycle of motion for various time intervals for an initial point load at the end.

Figure 5.10. A superposition of the initial shape of the beam and its shape at one, two and three cycles for an initial point load at the end.

similar shapes, the third cycle is the closest to the initial shape. Further it can be seen that at the end of the first and third cycle, the shape of the beam is almost overlapping.

CHAPTER VI

INITIAL UNIFORMLY DISTRIBUTED LOAD

6.1 STATIC DISPLACEMENT SHAPE

The cantilever beam shown in Figure 6.1 is initially deformed by a uniformly distributed load, q . Suddenly the load is released.

Figure 6.1 A cantilever beam with initial static uniformly distributed load

As in the previous analysis, the objective is to make a plot of δ/Δ versus $\frac{\omega_1 t}{2\pi}$, where δ and Δ are the vibratory and static displacements of the free end, respectively and ω_1 is the first natural frequency of free vibration. Proceeding with the equation of motion in the special case of equilibrium, to determine the initial static displacement;

$$EI \frac{d^4 w}{dx^4} = q \quad (6.1)$$

Integrating, and knowing that the shear is equal to zero at the free end; yields

$$EI \frac{d^3 w}{dx^3} = qx - q\ell \quad (6.2)$$

Integrating, and knowing that the moment is zero at the free end; yields

$$EI \frac{d^2 w}{dx^2} = qx^2/2 - q\ell x + q\ell^2/2 \quad (6.3)$$

Integrating, and since the slope is zero at the clamped end, yields

$$EI \frac{dw}{dx} = qx^3/6 - q\ell x^2/2 + q\ell^2/2x \quad (6.4)$$

Integrating, and since the deflection is zero at the clamped end, yields

$$EIw = q \frac{x^4}{24} - q\ell \frac{x^3}{6} + q\ell^2 \frac{x^2}{4} \quad (6.5)$$

Rearranging, the initial static displacement becomes:

$$w_o(x) = f(x) = \frac{q}{8EI} (x^4/3 - 4x^3\ell/3 + 2x^2\ell^2) \quad (6.6)$$

From (6.6), the initial static displacement at the free end is

$$w_o(x) = \Delta = \frac{q\ell^4}{8EI} \quad (6.7)$$

6.2 VIBRATION AMPLITUDE COEFFICIENTS

Next (6.7) is substituted into (2.31) to calculate the amplitude coefficients, B_m . Thus

$$B_m = \frac{\frac{q}{8EI} \int_0^{\ell} (x^4/3 - 4x^3\ell/3 + 2x^2\ell^2) X_m(x) dx}{\int_0^{\ell} X_m^2(x) dx} \quad (6.8)$$

where $X_m(x)$ is defined in (2.24). Evaluating the integrals of (6.8) for the first five natural frequencies, using the numerical integration procedure developed in Chapter 4, yields the following vibration amplitude coefficients:

$$B_1/\frac{q\ell^4}{8EI} = 0.50669389$$

$$B_2/\frac{q\ell^4}{8EI} = 0.00715006$$

$$B_3/\frac{q\ell^4}{8EI} = 0.00053471$$

$$B_4/\frac{q\ell^4}{8EI} = 0.00009955$$

$$B_5/\frac{q\ell^4}{8EI} = 0.00002833$$

Here, it is noticed how high is the first amplitude coefficient, B_1/Δ , where Δ is defined in (6.7), compared to the second coefficient. The first coefficient is about seventy times larger than the second one and nine hundred forty-seven times larger than the third one. The ensuing motion will be strongly dominated by the first mode. Higher frequency modes will have very little influence. This influence will not even be noticed. The subsequent motion will strongly resemble the periodic motion that would be encountered if the beam were excited in its first mode. Thus, the initial uniformly distributed load approximates the first vibration mode closer than the end moment and the point load at the end.

6.3 FREE VIBRATION DISPLACEMENT

Following the same analysis in Chapters 3 and 5, and expanding the general expression for free undamped vibration displacement (3.10), we obtain

$$\begin{aligned}
 \frac{w}{\Delta}(\xi, t) = & 0.50669389 \cos \frac{t}{T_1} (2\pi) X_1(\xi) \\
 & + 0.00715006 \cos \frac{t}{T_1} (2\pi C_1) X_2(\xi) + 0.00053471 \cos \frac{t}{T_1} (2\pi C_2) X_3(\xi) \\
 & + 0.00009955 \cos \frac{t}{T_1} (2\pi C_3) X_4(\xi) + 0.00002833 \cos \frac{t}{T_1} (2\pi C_4) X_5(\xi)
 \end{aligned}
 \tag{6.9}$$

where ξ_1 , T_1 , C_i , $X_m(\xi)$ are as defined in the previous chapters.

A plot of the nondimensional displacement, δ/Δ , versus the nondimensional time, t/T_1 , of the response of the first mode at the free end is shown in Figure 6.2. The response is very smooth and is almost periodic. Higher frequencies hardly have any influence in driving the motion out of its first mode. Therefore the disturbance of the smoothness is very little. Further, the response strongly resembles the sinusoidal motion which would be obtained if the beam were excited exactly in its first mode. The difference here is that the response peaks at slightly higher amplitudes than the initial amplitude at zero time. This is because the motion is not periodic. Table 6.1 lists peak response values for $\xi = 1.0, 0.8, 0.6, 0.4$ and 0.2 .

Figure 6.2. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial uniformly distributed load (three cycles).

Table 6.1: Peak response values for $\xi = 1.0, 0.8, 0.6, 0.4$ and 0.2 at zero, one, two and three cycles.

ξ	Nondimensional Displacement w/Δ			
	Time in Cycles t/T_1			
	0	1	2	3
1.0	1.00001466	1.01408216	1.02826014	1.00814784
0.8	0.73386043	0.73557254	0.73581597	0.73520196
0.6	0.47520572	0.46698435	0.45866503	0.47023873
0.4	0.24319584	0.23140066	0.22383897	0.23562579
0.2	0.06986508	0.06357260	0.06109710	0.06576069

At $\xi = 0.8$ the response is even smoother since it is close to the nodal point of the second mode. This is shown in Figure 6.3 where the motion looks exactly like a sinusoidal wave since peak amplitudes are equal at all cycles.

The response at $\xi = 0.6$ is shown in Figure 6.4 for the first three cycles. Again, the response becomes disturbed because of the influence of higher frequency modes. Peak amplitudes are not only unequal but also have different patterns.

Moving farther away from the free end, the response at $\xi = 0.4$ for the first three cycles is shown in Figure 6.5. The disturbance due to higher frequency modes is more obvious, and the motion is not smooth anymore. The behavior here is the same as the behavior at $\xi = 0.6$ as far as different peak amplitude values and different patterns.

Finally the response at $\xi = 0.2$ is shown in Figure 6.6. A shallow disturbed motion can be seen. It is not jerky but it is not smooth either. Peak amplitudes are unequal and have similar patterns.

Figure 6.3. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial uniformly distributed load (three cycles).

Figure 6.4. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial uniformly distributed load (three cycles).

Figure 6.5. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial uniformly distributed load (three cycles).

Figure 6.6. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial uniformly distributed load (three cycles).

6.4 SHAPE OF THE BEAM AT SUBSEQUENT INSTANTS

Figure 6.7 shows the shape of the beam before the release of the initial uniformly distributed load. It is a fourth degree function having a strong curvature at the clamped end, and a straight free end.

The shape of the beam for the first cycle, of the first frequency, during various time intervals is shown in Figure 6.8. The nondimensional displacement at the tip of the beam before the release of the initial load is 1.0. After the release of the load, the beam proceeds with its motion towards the horizontal equilibrium position. At one quarter of a cycle, the beam becomes essentially straight and horizontal. At this instant, however, the free end does not quite have a zero displacement. The value of w_m/Δ is 0.012. Continuing with its motion past the equilibrium position and deflecting downwards, the free end passes the -1.0 deflection value. At one half cycle, the beam has a free end deflection of $w_m/\Delta = -1.022$. As the beam continues with its motion, it goes back to its equilibrium position at three quarters of a cycle. At this instant, the beam shape nearly overlaps with its position at one quarter of a cycle. As the beam continues with its motion, the free end passes the initial deflected value of 1.0. Finally, at the end of the first cycle, the free end deflection is $w_m/\Delta = 1.014$.

A superposition of the beam shape at the end of the first, second, third cycle and the initial shape is shown in Figure 6.9. Again the first and third cycle are almost overlapping. The second cycle has the largest free end deflection. The shape of the beam at the end of the third cycle is the closest to the initial deflected shape. In conclusion it can be seen that the beam never goes back to its initial deflected shape since the motion is not periodic.

Figure 6.7. Initial displaced shape for a uniformly distributed load.

Figure 6.8. Shape of the beam during the first cycle of motion for various time intervals for an initial uniformly distributed load.

Figure 6.9. A superposition of the initial shape of the beam and its shape at one, two and three cycles for an initial uniformly distributed load.

CHAPTER VII

INITIAL CUBIC DISPLACEMENT SHAPE

7.1 STATIC DISPLACEMENT

The cantilever beam shown in Figure 7.1 is initially given a simple cubic displacement shape with a combination of end moment and end force. Suddenly, applied loads are released.

Figure 7.1: A cantilever beam with initial cubic deflection shape.

The initial static displacement shape is defined as

$$w_0(x) = f(x) = x^3 \quad (7.1)$$

This is accomplished by the superposition of initial force and moment at the free end. From (7.1) the initial static displacement at the free end is

$$w_0(l) = \Delta = l^3 \quad (7.2)$$

At first glance, this case resembles the first free vibration mode. However, its curvature is a maximum at the free end and zero at the

wall, whereas the first mode has maximum curvature at the wall and zero at the free end. This should bring the second and higher modes more strongly into the subsequent motion. It is, therefore, necessary to get more than five terms, say ten, to obtain an accurate response. Solving the characteristic frequency equation (2.26) for an additional five terms, yields the following eigenvalues:

$$\alpha_m \ell = 17.27875953, 20.42035225, 23.56194490, 26.70353755, \\ 29.84513021$$

The corresponding eigenvectors are: $\gamma_m = 1.00000006, 1.00000000, 1.00000000, 1.00000000, 1.00000000$.

7.2 VIBRATION AMPLITUDE COEFFICIENTS

The coefficients are obtained by substituting (7.1) into (2.31) yielding

$$B_m = \frac{\int_0^\ell x^3 X(x) \, dx}{\int_0^\ell X_m^2(x) \, dx} \quad (7.3)$$

where $X_m(x)$ is defined in (2.24). Evaluating the integrals of (7.3) for the first ten natural frequencies, using the numerical integration developed in Chapter 4, yields the following vibration coefficients:

$$B_1/\ell^3 = 0.36546942$$

$$B_2/\ell^3 = -0.09344507$$

$$B_3/\ell^3 = 0.02159017$$

$$B_4/\ell^3 = - 0.00820611$$

$$B_5/\ell^3 = 0.00394668$$

$$B_6/\ell^3 = - 0.00219236$$

$$B_7/\ell^3 = 0.00153171$$

In evaluating the eighth coefficient, both the denominator and numerator of (7.3) were found to be zero. Thus calculating B_8 can only be done by employing L'Hospital's Rule, which requires taking the derivative of the integrals, and involves considerable effort. Since the value of B_8 will be small, it would have very little influence on the response. The subsequent motion of the beam is, therefore, studied for the first seven terms.

Observing the coefficients, we see that B_1/Δ , where Δ is defined in (7.2), is the largest coefficient, but not by far. It is only four times larger than B_2/Δ and seventeen times larger than B_3/Δ . Even though the response is dominated by the first mode, higher modes have strong influence in driving the response out of its first mode. The subsequent motion will be disturbed strongly by higher modes and, thus, will not be smooth.

7.3 FREE VIBRATION DISPLACEMENT

Expanding the general expression for free undamped vibration displacement (3.10) for the first seven terms, we obtain

$$\begin{aligned} \frac{w}{\Delta} (\xi, t) = & 0.36546942 \cos \frac{t}{T_1} (2\pi) X_1 (\xi) \\ & - 0.09344507 \cos \frac{t}{T_1} (2\pi C_1) X_2 (\xi) + 0.02159017 \cos \frac{t}{T_1} (2\pi C_2) X_3 (\xi) \end{aligned}$$

$$\begin{aligned}
& - 0.00820611 \cos \frac{t}{T_1} (2\pi C_3) X_4 (\xi) + 0.00394668 \cos \frac{t}{T_1} (2\pi C_4) X_5 (\xi) \\
& - 0.00219236 \cos \frac{t}{T_1} (2\pi C_5) X_6 (\xi) + 0.00153171 \cos \frac{t}{T_1} (2\pi C_6) X_7 (\xi)
\end{aligned} \tag{7.4}$$

where ξ , T_1 , $X_m(\xi)$ are as defined in the previous chapters, and

$$C_1 = \omega_2/\omega_1 = 6.26689281$$

$$C_2 = \omega_3/\omega_1 = 17.54748131$$

$$C_3 = \omega_4/\omega_1 = 34.38609944$$

$$C_4 = \omega_5/\omega_1 = 56.93028045$$

$$C_5 = \omega_6/\omega_1 = 84.91303312$$

$$C_6 = \omega_7/\omega_1 = 118.59754300$$

A plot of the nondimensional displacement, δ/Δ , versus the nondimensional time, t/T_1 , of the response of the first mode at the free end is shown in Figure 7.2 for three cycles. The motion is very jerky. Higher modes come in very strongly and disturb the smoothness of the curve. Each of the first three cycles has a different peak amplitude and shape. Free end deflection for the first six cycles is shown in Figure 7.3. Again, the fourth cycle and the zeroeth cycle have the same pattern. Similarly, the fifth and first, the sixth and second, and so on.

The response at $\xi = 0.8$ is shown in Figure 7.4 for three cycles. The motion is smoother. This is due to the effect of the nodal point of the second mode at $\xi = 0.78$ which virtually eliminates the

Figure 7.2. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial cubic displacement shape (three cycles).

Figure 7.3. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial cubic displacement shape (six cycles).

Figure 7.4. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial cubic displacement shape (three cycles).

contribution of the second mode. The effect of the other modes, however, is apparent in disturbing the smoothness.

The response at $\xi = 0.6$ is shown in Figure 7.5 for three cycles. Contributions from higher modes is strong again. The motion is jerky. It is also interesting to note that the first, second and third cycles have higher amplitudes than the zeroeth cycle.

The response at $\xi = 0.4$ is given in Figure 7.6 for three cycles. The motion is very jerky. The disturbance is strong. As we move closer to the wall, the response becomes even more disturbed. Higher modes tend to drive the motion out of its first mode. The response at $\xi = 0.4$ for six cycles is given in Figure 7.7. At first glance, the response looks similar in all cycles. However, for the first three cycles the pattern is different. The fourth cycle follows a similar pattern as the zeroeth cycle. Similarly the fifth and first, the sixth and second and so on.

The response for $\xi = 0.2$ is shown in Figure 7.8. A very disturbed motion with a small amplitude is noticed. Since an infinite number of frequencies is excited, the subsequent motion is disturbed. Further, since at $\xi = 0.2$ the vibration amplitude is small, the disturbance results in a motion that is very jerky.

7.4 SHAPE OF THE BEAM AT SUBSEQUENT INSTANTS

The initial cubic deflected shape is shown in Figure 7.9. The linearized curvature (i.e., for small slopes) is maximum at the free end and zero at the wall. The first mode, on the other hand, has maximum curvature at the wall and zero curvature at the free end. Figure 7.10 shows the shape of the beam during the first cycle at various time intervals. At time zero, i.e., before the release of the applied loads, the displacement at the tip of the beam is 1.0. After the release of the loads, the beam deflects downwards with an opposite curvature. At one quarter of a cycle, it has a strong negative curvature over most of the beam length, with a negative deflection at

Figure 7.5. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial cubic displacement shape (three cycles).

Figure 7.6. Nondimensional displacement versus nondimensional time in cycles at $\zeta=0.4$ for an initial cubic displacement shape (three cycles).

Figure 7.7. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial cubic displacement shape (six cycles).

Figure 7.8. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial cubic displacement shape (three cycles).

Figure 7.9. Initial displaced cubic shape.

Figure 7.10. Shape of the beam during the first cycle of motion for various time intervals for an initial cubic displacement shape.

the free end. As the beam proceeds with its motion, it continues to deflect downwards. Continuing, at one half cycle, the free end has a negative deflection of 0.6. As the motion continues, the beam reflects upwards. At three quarters of a cycle, it almost comes back to its horizontal equilibrium position. This is the closest the beam can get to this position during the first cycle. In other words, it is taking the beam three quarters of a cycle to come back, as close as it can get, to its equilibrium position. As the motion continues with an upwards deflection, the beam acquires a curvature that is positive. At the one cycle instant the shape of the beam is far from the initial shape. The free end has a deflection value less than 0.7, and the curvature of the beam is opposite. The motion is, far from being periodic.

A superposition of the initial shape of the beam and its shape at one, two and three cycles is shown in Figure 7.11. Again, the shape at the third cycle is the closest to the initial shape. At all instants of time greater than zero, however, the beam has zero curvature at the free end, unlike the initial shape which has a maximum curvature at the free end. The shape of the beam at both the first and second cycles has an inflection point since it has a double curvature. At the second cycle, the beam has a very strong curvature at the wall. The first cycle also has a stronger curvature at the wall than the initial shape.

From above it is apparent that when an infinite number of modes is excited, the subsequent motion will not be periodic.

Figure 7.11. A superposition of the initial shape of the beam and its shape at one, two and three cycles for an initial cubic displacement shape.

CHAPTER VIII
INITIAL SHAPE SIMILAR TO SECOND MODE

8.1 STATIC DISPLACEMENT

The cantilever beam shown in Figure 8.1 is initially given a static displacement which is predominantly the second free vibration mode. This initial shape can be obtained by an end force and moment.

Figure 8.1 A cantilever beam with initial displacement similar to the second mode.

In order to obtain the above displacement shape, the following displacement function

$$w(x) = C_2 x^2 + C_3 x^3 = C_2 \left(x^2 + \frac{C_3}{C_2} x^3 \right) \quad (8.1)$$

is chosen. This satisfies the boundary conditions of zero deflection and slope at the clamped end, i.e., $w(0) = w'(0) = 0$. Next, the function in (8.1) is forced to have a deflection shape similar to the second mode. This is done by choosing C_2 and C_3 such that C_3/C_2 gives

the correct location of the nodal point of the second mode. In nondimensional form, the eigenfunction (2.24) becomes

$$X_m(\xi) = \cosh \alpha_m \ell(\xi) - \cos \alpha_m \ell(\xi) - \gamma_m [\sinh \alpha_m \ell(\xi) - \sin \alpha_m \ell(\xi)] \quad (8.2)$$

Substituting for $\alpha_2 \ell$ and γ_2 in (8.2), setting $X_2(\xi) = 0$, and solving for ξ , yields the nodal point location, $\xi = 0.78344455$. Continuing with forcing the function to come close to the second mode, the following constraint conditions are applied to (8.1) in nondimensional form:

$$W(\xi) = C_2(\xi^2 + \frac{C_3}{C_2} \xi^3) \quad (8.3)$$

constraint conditions:

$$W(0.78344455) = 0$$

$$W(1.0) = 1.0$$

The above yields the following function

$$W(\xi) = 4.61775495 \xi^3 - 3.61775495 \xi^2 \quad (8.4)$$

A plot of the forced function (8.4) versus the second mode shape is given in Figure 8.2. The forced shape is reasonably close to the second mode, but with a weaker curvature at the wall and a stronger curvature at the free end. This will bring the first and higher modes more strongly into the response.

8.2 VIBRATION AMPLITUDE COEFFICIENTS

In order to calculate the coefficients, the function (8.4) is rewritten in the form

Figure 8.2. A superposition of the shape of the forced function and the second mode shape.

$$W(x) = 4.61775495x^3 - 3.61775495x^2 \quad (8.5)$$

Substituting (8.5) into (2.31) yields

$$B_m = \frac{\int_0^l (4.61775495x^3 - 3.61775495x^2) X_m(x) \, dx}{\int_0^l X_m^2(x) \, dx} \quad (8.6)$$

where $X_m(x)$ is defined in (2.24). Evaluating the integrals of (8.6) for the first ten natural frequencies, using the numerical integration developed in Chapter 4, yields the following vibration coefficients:

$$B_1 = 0.07635131$$

$$B_2 = -0.28901405$$

$$B_3 = 0.06986053$$

$$B_4 = -0.02700790$$

$$B_5 = 0.01310316$$

$$B_6 = -0.00731756$$

$$B_7 = 0.00513084$$

Again, the denominator and numerator of (8.6) are both zero for B_8 . Taking the derivative of the integrals is the only way to evaluate B_8 . Since its value is small, it would have very little influence on the response. The subsequent motion is, thus, studied for the first seven terms.

Looking at the amplitude coefficients, B_m , it is apparent that B_2 is the dominant one, but it is not strongly dominant. It is 3.78 times larger than B_1 , 4.14 times larger than B_3 and 10.70 times larger than B_4 . Other modes also enter significantly into the motion. Even though the response is dominated by the second mode, the subsequent motion will be strongly disturbed.

8.3 FREE VIBRATION DISPLACEMENT

Expanding the general expression for free undamped vibration displacement (3.10) for the first seven terms, the following is obtained:

$$\begin{aligned}
 w(\xi, t) = & 0.07635131 \cos \frac{t}{T_1} (2\pi) X_1(\xi) \\
 & -0.28901405 \cos \frac{t}{T_1} (2\pi C_1) X_2(\xi) + 0.06986053 \cos \frac{t}{T_1} (2\pi C_2) X_3(\xi) \\
 & -0.02700790 \cos \frac{t}{T_1} (2\pi C_3) X_4(\xi) + 0.01310316 \cos \frac{t}{T_1} (2\pi C_4) X_5(\xi) \\
 & -0.00731756 \cos \frac{t}{T_1} (2\pi C_5) X_6(\xi) + 0.00513084 \cos \frac{t}{T_1} (2\pi C_6) X_7(\xi)
 \end{aligned} \tag{8.7}$$

where ξ , T_1 , $X_m(\xi)$ and C_i (where $i = 1, 2, \dots, 6$) are as defined in Chapter 7.

A plot of the nondimensional displacement, δ/Δ , versus the nondimensional time, t/T_1 , of the response in one cycle of the first frequency at the free end is shown in Figure 8.3. As expected, there are six peaks in one cycle of the first mode. This is because the second frequency, ω_2 , is 6.26 times the first frequency, ω_1 . Thus the period of the second mode is equal to the time that elapses between two adjacent peaks. The motion is jerky since other modes are also driving the response out of its second mode. The response for three

Figure 8.3. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial shape similar to the second mode (one cycle).

cycles of the first mode at the free end is shown in Figure 8.4. There is some resemblance of a pattern to the motion. The fifth peak of the second frequency has a similar shape as the zeroeth peak. The sixth peak has the same pattern as the first. The seventh peak of the second frequency is similar to the second peak of the first frequency cycle, and so on. All peaks, however, have different amplitudes. Thus the motion almost repeats itself, but with a different amplitude, and with almost the same pattern. It is apparent that the motion is not periodic. The response for six cycles is shown in Figure 8.5.

The response at $\xi = 0.8$ for three cycles of the first frequency is shown in Figure 8.6. The amplitude of the response is small. This is because the function (8.1) was forced to be zero at $\xi = 0.78$, the nodal point of the second mode shape. The influence of other modes, however, is strong in driving the response. The response for six cycles of the first frequency is shown in Figure 8.7. It is quite obvious that at $\xi = 0.8$ the response tends to behave like a disturbed sinusoidal wave of the first mode response contribution.

The response at $\xi = 0.6$ for three cycles is shown in Figure 8.8. The motion is relatively smooth compared to the free end motion. This can be seen by comparing periods of the second mode. Other modes have a weaker effect in disturbing the smoothness of these periods. The response for six cycles is shown in Figure 8.9. Again the motion can be enclosed by an envelope, since it is similar to the free end motion but a little smoother.

The response at $\xi = 0.4$ is shown in Figure 8.10 for three cycles. The motion is a lot smoother. The second mode here is the dominant mode. Influence of other modes is much weaker. The response for six cycles is shown in Figure 8.11. The big variation in amplitudes that was apparent at $\xi = 1.0$ and 0.6 has diminished. Peak amplitudes here are not the same, but they are not much different either. When an envelope is drawn, the variation of its amplitude is little.

Figure 8.4. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial displacement shape similar to second mode (3 cycles).

Figure 8.5. Nondimensional displacement versus nondimensional time in cycles at the free end for an initial displacement shape similar to second mode (6 cycles).

Figure 8.6. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial displacement shape similar to second mode (three cycles).

Figure 8.7. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.8$ for an initial displacement shape similar to second mode (six cycles).

Figure 8.8. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial displacement shape similar to second mode (three cycles).

Figure 8.9. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.6$ for an initial displacement shape similar to second mode (six cycles).

Figure 8.10. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial displacement shape similar to second mode (three cycles).

Figure 8.11. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.4$ for an initial displacement shape similar to second mode (six cycles).

The response at $\xi = 0.2$ is shown in Figure 8.12 for three cycles. The motion is extremely jerky. Disturbances from the higher modes are strong. There is a pattern to the motion. The eleventh peak of the first cycle has a shape similar to that of the first peak. The twelfth peak, or the sixth peak of the second cycle has a shape similar to that of the second peak. The thirteenth is similar to the third, and so on. Thus the motion almost repeats itself, but with different amplitudes and a rather similar shape. The response for six cycles is shown in Figure 8.13. It is seen that for small amplitudes of vibration, the variation is rather large.

From the previous plots, it is quite apparent that the motion is not periodic. Since an infinite number of modes is excited, the subsequent motion is disturbed. The motion, however, is predominantly a second mode vibration.

8.4 SHAPE OF THE BEAM AT SUBSEQUENT INSTANTS

Figure 8.14 shows the shape of the beam during the first cycle at various time intervals ($t/T_2 = 0, 1/4, 1/2, 3/4, 1$). At time zero the initial shape resembles the second mode as discussed earlier. After the release of the applied loads, the beam deflects downwards. At one quarter of a cycle, the beam is near the horizontal equilibrium position, but has a double curvature. If the motion were periodic, the beam would have been straight and horizontal. As the beam continues to deflect downwards, its shape becomes almost a mirror image of its original shape, however, its free end deflection is much less than unity. As the beam continues with its motion, it deflects upwards. At three quarters of a cycle, it goes back to the nearly horizontal position with double curvature which is opposite to the one quarter of a cycle double curvature. Finally, when the first cycle is over, the beam obtains a shape, which at first glance, looks similar to the initial shape. The initial shape, however, has a stronger curvature both at the wall and the free end.

Figure 8.12. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial displacement shape similar to second mode (three cycles).

Figure 8.13. Nondimensional displacement versus nondimensional time in cycles at $\xi=0.2$ for an initial displacement shape similar to second mode (six cycles).

Figure 8.14. Shape of the beam during the first cycle of motion for various time intervals for an initial displacement shape similar to second mode.

A superposition of the shape of the beam at zero, one, two and three cycles is shown in Figure 8.15. The shape of the beam at zero and three cycles has similar curvature at the wall. The free end deflection, however, is very different in both. It has a much lesser value at three cycles. At two cycles, the beam has a strong curvature at the wall. The free end deflection at one cycle is the closest to the initial deflection but has a much weaker curvature. It is apparent that the motion is not periodic.

Figure 8.15. A superposition of the initial shape of the beam and its shape at one, two and three cycles for an initial displacement shape similar to second mode.

CHAPTER IX

SUMMARY OF RESULTS

9.1 GENERAL

When a cantilever beam is given an initial displacement using a point load, moment, uniform distributed load or any combination, an infinite number of modes will be excited after the release of the load. The subsequent motion will, thus, be disturbed and will not be periodic. If the beam were excited exactly in its first or second mode, the response would be sinusoidal and thus periodic.

When the initial shape of the beam resembles a certain mode shape, the subsequent motion is predominantly that certain initial free vibration mode. The initial static displacement is, thus, the predominant vibration mode. How jerky or disturbed the response is depends on how different the initial static shape is from the free vibration mode shape. When the cantilever is given an initial static displacement with a uniform distributed load, the response is very smooth. This is because the initial uniform load approximates the first mode very closely. Higher modes come into the motion very weakly. An initial cubic shape, on the other hand, is a lot different from the first mode due to its curvature. Higher modes enter strongly into the subsequent motion, resulting in a jerky response. Further, for the cantilever beam, the natural frequencies are not integer multiples of the first one. This is why the superimposed motion can no longer be periodic.

9.2 FIRST FREE VIBRATION MODE

When the initial static shape approximates the first free vibration mode reasonably closely, the response is predominantly the

first mode. In the subsequent motion, the beam tends to follow the same deflection pattern as the zeroeth cycle starting from the fourth cycle. Thus, from the fourth cycle and onwards, the deflection pattern nearly repeats itself.

Another interesting point is that the response at $\xi = 0.8$ is much smoother than at any other point on the beam. This is due to the nodal point of the second free vibration mode which occurs at $\xi = 0.78$. This point smoothens the motion or rather, dominates the subsequent response. As we go away from the node point, the response becomes disturbed. At $\xi = 0.4$ the motion is too jerky and at $\xi = 0.2$ the motion is also very jerky, but has a smaller amplitude.

In conclusion, it is important to note the principal of conservation of energy throughout the beam length is still valid.

9.3 SECOND FREE VIBRATION MODE

Similarly, when the initial static shape approximates the second free vibration mode reasonably closely, the response is predominantly the second mode. In one cycle of the first mode, there are six cycles of the second mode since the second frequency, ω_2 , is 6.26 times larger than the first frequency, ω_1 . The subsequent motion follows a certain pattern and tends to repeat itself.

Further, unlike initial displacements similar to the first free vibration mode, the response at the free end and at $\xi = 0.6$ and 0.4 is relatively smooth compared to that at $\xi = 0.2$. The motion at $\xi = 0.2$ is jerky.

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