MATHEMATICAL MODELING AND SIMULATION ANALYSIS
OF HYDRAULIC FRACTURE PROPAGATION
IN THREE-LAYERED PORO-ELASTIC MEDIA

DISSERTATION

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To my wife, Mee-Hye
and
To my daughter, Angela

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NOMENCLATURE

Latin Alphabet Symbols

$A(t)$	Fracture total area
A_p, A_u, A_ℓ	Fracture area in payzone, upper, and lower barriers, respectively. $A(t) = A_p + A_u + A_\ell$
$\dot{a}_n$	Normal fracture propagation velocity
$a(t)$	Elliptic fracture half length
$b_u(t), b_\ell(t)$	Elliptic fracture upper and lower height from the center of borehole
$c(t)$	Elliptic fracture half width ($c=W/2$)
$C_L, \Delta C_L$	Fluid leak-off coefficient and its change in barriers
$C_s, \Delta C_s$	Spurt loss coefficient and its change in barriers
d	Distance between crack tip and fluid front ($d = R - R_f$)
D_η	Viscous dissipation energy rate function
D_q, D_p	Fluid dissipation energy rate function in Lagrangian and variational formulation, respectively. $D_p = \frac{m}{1+m} D_\eta = m D_q$
D_I	Total input power
D_{I2}	Base line power, $\sigma_o \dot{i}_o$ or $\dot{U}_{\sigma_o}$

D_{Ieff}	Effective input power, $D_I - D_{I2}$
D_L	Energy rate due to fluid leak-off
$E(k_j)$	Complete elliptic integral of the 2nd kind
e	Specific strain energy of the elastic reservoir
$f, \underline{f}_g$	Gravitational body force vectors
$K_A[w]$	Boundary integral over the crack surface A
K_I, K_{IC}	Stress Intensity factor and formation fracture toughness
$K(k_j)$	Complete elliptic integral of the 1st kind
$L(t)$	Fracture half length for rectangular crack
L_{ij}	Fluid velocity gradient tensor components
m	Fluid behavior index
$\underline{n}$	Outward normal vector
$G_{cr}, \Delta G_{cr}$	Formation critical energy release rate and its change in barrier layers
H	Payzone total height, $H = h_u + h_\ell$
h_u, h_ℓ	Payzone upper and lower height measured from perforation center
h_p	Perforation placement from center line of the payzone ($h_u = H/2 - h_p$, $h_\ell = H/2 + h_p$)
$i(\underline{x}, t), i_o$	Fracture fluid injection rate
m	Flow behavior index
$p(\underline{x}, t)$	Fracture fluid pressure
$p_e(\underline{x}, t)$	Effective pressure, $p_e(\underline{x}) = p(\underline{x}) - \sigma_o(\underline{x})$

$P_o(t)$	Borehole effective fracture fluid pressure
Q_i	Generalized forces in the Lagrangian eqns
$q_i(\underline{x}, t)$	Components of fluid flow rate vector
$q_L(\underline{x}, t)$	Fluid leak-off rate
$R(t)$	Fracture radius (penny-shaped model)
$R_f(t)$	Radius of fluid front (penny-shaped model)
R_*	Non-dimensionalizing fracture radius
(r, θ, z)	spatial coordinates in cylindrical system
$\underline{q}$	Fracture fluid flux vector
T	Kinetic energy
t	Time
$\underline{u}_s$	Solid displacement vector
U_f	Griffith fracture energy
U_s	Formation strain energy
U_p	Potential energy by effective pressure
(u, v, w)	Displacements in Cartesian coordinates
V	Potential energy
V_f	Fluid control volume
V_F	Current volume in a fracture
V_I	Total injected volume
$v_k \ (k=1,2,3)$	Fluid velocity components
v_{fn}	Prescribed injection fluid velocity
$\underline{v}_s, \underline{v}_f, \underline{v}_L$	Solid, fluid, and fluid leak-off velocities, respectively
$(x, y, z), x_i$	Cartesian coordinate system ($i=1,2,3$)

$\underline{x}$	Position vector, $\underline{x} = (x_1, x_2)$
$w(\underline{x}, t)$	Fracture opening width profile
W, W_k ($k=1, 2$)	Borehole crack opening widths ($W=2c$)
∂A	Crack front
∂A_q	Injection boundary

Greek Alphabet Symbols

α	Pressure-width response variable
β	Fluid lag index, $\beta = R_f/R$
ϵ	Supplementary fluid lag index, $\epsilon = (1-\beta^2)^{1/2}$
ϵ_{ij}	Reservoir strain tensor components
∇, div	Divergences
$\Delta p_j(\underline{x}, t)$	Effective fracture fluid pressure
$\eta_o, \bar{\eta}$	Fluid consistency (viscosity) indices
ψ	Fracture fluid specific dissipation function
γ	Leak-off behavior exponent
$\nu, \Delta\nu$	Poisson's ratio and its change in layers
$\mu, \Delta\mu$	Formation shear modulus and its change in barrier layers
$\hat{\mu}$	Generalized elastic modulus, $\hat{\mu} = \mu/(1-\nu)$
Ω_s, Ω_f	Solid and fluid volumes, respectively
ρ	Normalized radial coordinate

(λ, μ, ν)	Ellipsoidal coordinate system
ϕ	Stress function
σ_{ij}	Stress tensor components
$\sigma_o, \Delta\sigma$	Minimum horizontal in-situ stress and its change in barrier layers
$\hat{\sigma}_{ij}, \sigma_{ij}, \sigma_{ij}^o$	Reservoir total, fracture mediated, and in-situ stress tensor components
τ	Fluid arrival time
τ_{ij}	Fluid deviatoric stress components

CHAPTER I

INTRODUCTION

1.1 General Overview

Hydraulic fracturing plays a pivotal role in the enhancement of oil and gas production rates from low permeability reservoirs. The process of hydraulic fracturing entails the generation of a fracture by pumping treatment fluid blended with special chemicals and proppants into the payzone at high injection rates and pressures to extend and wedge a fracture hydraulically. Fig. 1.1 illustrates a planar vertical hydraulic fracture in a three layered medium. The treatment fluid slurry continues to extend the fracture and concurrently transports the proppants near the fracture tip. After cessation of the injection process, the fracture fluid is withdrawn with the proppants keeping the fracture open, facilitating the flow of the oil/gas hydrocarbons from the low permeable reservoir to the borehole.

Introduced by the industry in 1947, over one million hydraulic fracture treatments have been performed till 1988 with significant enhancements in oil and gas producing rates. Massive hydraulic fracturing treatments

Uncontrollable Parameters : - Payzone Height (H) & Interface - Elastic Moduli (μ, ν) - In situ Stress (σ) - Critical Energy Release Rate (G_{cr})	Controllable Variables : - Fluid Injection Rate (i_o) - Fluid Rheology (η, m, ρ) - Leak-off coefficient (C_L) - Perforation Characteristics

Fig 1.1 Various Hydraulic Fracturing Planar Configurations with Uncontrollable Variables and Controllable Parameters

now exceed one million gallons of fracturing fluid and three million lbm of proppants [1].

It has been reported that fractures are oriented in the vertical plane except for the case at relatively shallow depths (e.g., <2000 ft[610m]) where the fractures are horizontal [2]. In relatively deep formations a fracture tends to grow vertically, because the lateral stresses in the horizontal plane are less than overburden vertical stress. Considering typical target payzones of interest, vertical fractures are considered here.

The availability of field data on hydraulic fracture responses is quite limited mainly due to difficulties in measuring in-situ properties and responses. Therefore, mathematical models and simulators are essential tools for stimulation treatment design and fracture configuration optimization.

The mathematical modeling of hydraulically induced fractures generally entails a coupling between the formation elasticity, fracture fluid flow, fracture fluid constitutive and fracture mechanics equations governing the formation structural responses. This coupling is evidenced through the fluid pressure profile and fracture width with time dependent moving boundary conditions. The effects of fluid leak-off into the poro-elastic formation are considered in association with the fracture fluid mass

conservation equation.

Reservoir property characterization is fundamental to stimulation treatment design and fracture configuration optimization. The hydraulic fracture process quantities can be classified into two categories, namely controllable variables and uncontrollable parameters. Uncontrollable parameters include reservoir permeability, porosity, reservoir fluid properties, in-situ stress, elastic properties, fracture toughness values, and formation geometry. On the other hand, controllable variables include fracture injection treatment rate, volume scheduled, fluid rheology, properties of proppant, wellbore casing, tubing, and perforation location. Since the reservoir permeability and porosity are specified, once the target formation is selected, leak-off can be controlled in a limited way by selecting the proper treatment fluid and additives. For example, the injection of polymer emulsions, which have a low fluid-loss coefficient, generate a longer contained fracture than water and oil-based gels with high fluid-loss coefficients. Although the injection of a very viscous fluid improves fracture opening width and proppant settling, it also causes an undesirably slow return of the fracture fluid after cessation of the treatment stimulation. Moreover, treatments at higher injection

rates are restricted by frictional pressure losses inside the tubing due to a high viscosity. To cope with these difficulties it is necessary to have a higher injection pressure or increase the tubing diameter for reducing the frictional losses. These requirements may be limited by the current technology or associated production costs. Since the various parameters are coupled, it is important to evaluate the relative roles of controllable hydraulic treatment variables and uncontrollable reservoir properties. Hence, mathematical modeling and simulators which accommodate the complexities of computing fracture responses in multi-layered rock formations are needed to perform systematic parametric sensitivity studies for the effective optimization of stimulation treatments.

The unsymmetric three-layered fracture model developed here evolves from a penny-shaped crack in the payzone and becomes elliptic after the crack encounters the upper or lower barriers. The elliptic configuration is influenced by the magnitudes of differentials in the formation mechanical properties (in situ stress, elastic moduli, and fracture toughness contrasts) as well as fracture fluid leak-off rates.

Two allied unsymmetric elliptic fracture models are developed here for fracturing treatment analysis for three-layered rock formations. The first model is based

on the Lagrangian formulation employing pertinent energy components associated with the formation structural responses and fracture fluid flow. The second model is based on a generalized variational principle characterizing the mechanics of the hydraulic fracturing process by introducing energy rate components, which can be deduced from the first law of thermodynamics. The variational concepts play a central role in the interpretation of hydraulic fracture mechanisms and prediction of crack geometries. The variational principle is extended to accommodate the phenomenon of fluid lag. The role of the crack tip closure stress applied on the dry zone and the effective pressure distribution is examined.

The objective of this study is to develop efficient design tools for fracture configuration prediction and parametric sensitivity evaluations for the optimization of hydraulic fracturing stimulation processes based on the parametric sensitivity simulation and energy rate contributions during fracture evolution.

1.2 Literature Review

Since the initial fracturing treatments used for oil/gas stimulation, several hydraulic fracture modeling and design efforts have been conducted. One of the

earliest hydraulic fracture modeling efforts has been made by Zheltov and Kristianovitch [3], who introduced the concept of mobile equilibrium-i.e., slow-moving fracture propagation as a result of hydraulic action. This model consists of a vertical equilibrium crack under plane strain loading induced by the one-dimensional flow of a highly viscous fluid. The model and the concept of equilibrium cracks have been extended by Barenblatt [4]. Two different type of models with a constant height were introduced. A two-dimensional model with a rectangular geometry of constant vertical height, advanced by Perkins and Kern [5] and Nordgren [6], known as the PKN model, was introduced from the premise that the configurations in the horizontal plane and vertical plane, which is perpendicular to the long axis of the fracture, are elliptical shapes, while the configuration in the fracture plane remains rectangular. Another constant height model developed by Geertsma and DeKlerk [7], the KGD model, presumes an approximately elliptical configuration in the horizontal plane and a rectangular shape in the vertical plane. Based on these different geometric assumptions, the fracture length and width were obtained independently. Geertsma and DeKlerk [7], and Perkins and Kern [5] also have corresponding penny-shaped counterparts. These basic models, further developed by Daneshy [8], provide

fundamental information on the principal fracture dimensions and basic parametric sensitivity studies. Comprehensive critiques of these models and associated numerical comparisons have been presented by Geertsma and Haafkens [9]. Applications of these models are limited by their assumptions that the fracture height remains constant, and therefore these models simulate confined fracture growth within the payzone layer. Studies on hydraulic fracture mechanisms in an early stage are also due to Haimson and Fairhurst [10], Seth and Gray [11], Wong and Farmer [12], and Shuck and Advani [13].

Vertical crack growth over the barrier layers was analyzed by Simonson et al [14] and Van Eekelen [15] using the Perkins and Kern [5] constant height fracture model (PKN model). Cleary [16] has presented a pseudo three-dimensional model. He computed a singular integral equation for the elasticity problem associated with the line crack with the lateral flow equations and solved the governing equations numerically using a finite difference method. Incorporation of the effects of elastic-diffusive coupling for hydraulic fracture mechanisms is due to Rice and Cleary [17] and Riuna [18]. Vertical line crack growth in a layered formation has been investigated by Advani et al [19,20,21] using finite element method. In their formulation, the assumed one-dimensional fluid flow

in the horizontal and vertical directions is coupled with fracture opening profile. Non-Newtonian fluid flow including the fluid leak-off into formations was also examined by Settari [22], Dean and Advani [23], Advani et al [24], and Torok and Advani [25]. The height of the vertical cross section was determined by specifying the critical stress intensity factor of the reservoir rock [19,21]. Mendelsohn [26,27], Meyer [28,29], and Bouteica [30,31] also proposed models with a prescribed fracture geometry and variable height. Advani et al [32,33] analyzed hydraulic fracture responses, based on a Lagrangian formulation utilizing the virtual work concept initiated by Biot et al [24], for rectangular (with/without height growth) and penny-shaped models in a uniform medium. The modeling of specialized fracture geometries has been reported by Abe et al [35], Spence and Turcott [36], Wong and Cleary [37], Bouteica [30], and Lin and Keer [38].

General overviews of hydraulic fracture design and treatment technology have been presented by Veatch [39,40], Gidley et al [1], Mendelsohn [26,27], and Waters [41]. A series of laboratory model experiments have been performed by Teufel and Clark [42], Hanson et al [43], Warpinski et al [44], Biot et al [45], and Blanton [46]. Nolte and Smith [47] provided a diagnostic interpretation

of the fracture fluid pressure decline monitored from actual stimulations and also summarized by Nolte and Economides [48].

The classical leak-off representation based on the one dimensional error function solution, reported by Carter [49], indicates that the fluid loss rate is inversely proportional to the square root of the filtration time. Further investigations have been made by Smith [50], Williams [51], Nolte [52], Dean and Advani [23], and Nierode [53]. Cooper et al [54] presented the results of a comprehensive field study comparing those methods. The rate of fluid leak-off into the formation during a hydraulic fracturing treatment is one of the most critical factors involved in determining fracture geometry for a given treatment design.

The planar three-dimensional hydraulic fracturing models can be classified into three categories; lumped-parameter, pseudo 3-D, and generalized 3-D models. Generalized three-dimensional hydraulic fracture models incorporate the three dimensional structural responses of the layered formation with an arbitrary shaped fracture configuration. A general 3-D model incorporating leak-off effects was developed by Settari and Cleary [55], and Lam et al [46]. Also Clifton and Abou-Sayed [57], Palmer and Luiskutty [58], and Advani et al [59,60,61] have developed

general three-dimensional numerical models handling an arbitrary planar fracture which couple the fracture geometry, crack opening, and fluid flow in the fracture plane. The injected fracture fluid is generally idealized to be an incompressible power-law type non-Newtonian fluid.

Energy considerations and associated hydraulic fracture responses have been studied by Perkins and Krech [62] and Shylyapobersky [63]. Virtual work analysis associated with hydraulic fracture propagation, using a Lagrangian formulation, has been developed by Biot et al. [34] and Advani et al [33,64]. The relative contributions of Griffith surface and viscous dissipation energy components have been systematically investigated from a characteristic time concept advanced by Advani et al and Lee et al. [65,66].

Variational techniques associated with hydraulic fracture modeling were studied by Clifton and Abou-Sayed [57]. They employed separate variational principles for the formation structural responses and viscous fracture fluid flow, with the linear elastic fracture mechanics criterion expressed as an auxiliary equation. The subsequent discretizations for the crack width and pressure provide the underpinnings for the numerical solutions (Abou-Sayed et al. [67] and Clifton [68]). A

related variational technique for simulating fracture propagation has also been reported by Toubul et al. [69]. An unified variational principle characterizing the mechanics of the hydraulic fracturing process has been proposed by Advani et al [70]. They demonstrated the applicability of the formulation through a penny-shaped fracture incorporating fluid leak-off effects.

The phenomenon of the fracturing fluid lag has been studied in laboratory experiments [45], field tests [71], and numerical simulators [37,72,73]. Jeffrey [74] proposed a method for the calibration of the fluid lag in PKN type plane strain and penny-shaped fractures. Nison [75] also investigated the leak-off predicting hydraulic fracture propagation driven by gases or liquids

Recent advancements in hydraulic fracturing process modeling include the introduction of heat transfer effects (Biot et al [76], Ben-Naceur et al [77], Clifton et al. [78] and Pak [79]), proppant transport and settling (Novotny [80], Clark and Quadir [81], Acharya [82], and Krishna [83]), multiple fluid injection (Jefferey et al [84], Bhatia et al [85], Nghiem et al [86], and Clifton et al [78]), and multiple fracturing (Clifton et al [78], Jefferey et al [84] and Nghiem et al [86]).

1.3 Research Scope

The main purpose of this investigation is to develop efficient and economic hydraulic fracturing simulators for reliable fracture configuration prediction, optimization and control of stimulation treatment processes through the systematic evaluation of the roles of controllable and uncontrollable parameters. Comprehensive fracture model simulators in three-layered formations are developed based on a Lagrangian and energy rate formulations employing pertinent energy/energy rate components. The time dependent generalized coordinates for the Lagrangian model are defined from the admissible functions for the assumed crack dimensions and pressure profile.

In chapter II, a general elliptic model simulator (ELLIP2D) for hydraulic fracture propagation in unsymmetric three-layered formations is introduced for incorporating differentials in formation and fluid properties, such as contrasts of in-situ stress ($\Delta\sigma$), formation critical energy release rate (ΔG_{cr}), elastic modulus ($\Delta\mu$), Poisson's ratio ($\Delta\nu$), spurt loss (ΔC_{sp}), and leak-off (ΔC_L) in the formation. The roles of fracturing fluid rheology, flow rate, reservoir formation properties, and fluid filtration in the formation are investigated by evaluating fracture propagation in a multi-layered system.

In chapter III, an application of a unified variational principle proposed by Advani et al [70], for unsymmetric elliptic fracture growth in a three-layered formation is introduced. The applicability of this simulator (ERATE2D) is demonstrated by comparing numerical results for elliptic and penny-shaped models including leak-off effects with corresponding responses from the previously developed Lagrangian (ELLIP2D) and finite element methods (HYFRAC3D) and other codes.

In chapter IV, the unified variational principle is extended to accommodate the phenomenon of the fracture fluid lag. The roles of the crack tip closure stress applied on the dry zone and effective borehole pressure are examined. The fracturing fluid lag is investigated for fracture growth of a penny-shaped configuration and the formulations derived from the variational principle are detailed.

In chapter V, the developed simulators are applied to benchmark examples reported in literature for model validation, comparisons, and parametric sensitivity studies. Also numerical results for fluid lag in a penny-shaped model are presented. Validation of the models is also conducted by comparison of the results with other numerical simulators for general symmetric and unsymmetric cases.

In chapter VI, the developed simulators are applied to selected field experiment cases for unsymmetric three-layered formations. Parametric sensitivity studies, placing an emphasis on the role of controllable variables, are conducted in order to demonstrate the efficiency of the simulators as design tools for preliminary fracture configuration prediction and parametric sensitivity evaluations.

In chapter VII, evaluations on the role of hydraulic fracture control parameters are conducted by investigating pertinent energy rate (power) components. The energy rate formulation based on a unified variational principle enables us to monitor how the total input power delivered at the borehole by a pressurized treatment fluid is converted into other power terms during hydraulic fracturing processes. The application of this energy rate formulation provides an efficient design tool which can be used as an indicator for fracture sensitivity, optimization and control.

Finally, a summary of the research contributions and conclusions are presented in chapter VIII.

CHAPTER II
LAGRANGIAN FORMULATION FOR HYDRAULIC FRACTURING
IN THREE-LAYERED FORMATIONS

2.1 Preliminaries

Hydraulic fracture modeling incorporates the coupled effects of the formation elasticity, fracture fluid flow, and fracture mechanics equations. Time explicit solutions and characteristic time concepts using Lagrangian formulations have been introduced by Advani et al [32,33] for rectangular and penny-shaped models.

In this chapter, a Lagrangian formulation employing pertinent energy/energy rate components combined with a virtual work analysis is utilized for developing an unsymmetric elliptic fracture model for three-layered formations. The elliptic model for multi-layered formations presented here, assumes that a fracture, initially a penny-shaped crack, grows and the corresponding half becomes elliptic only when it encounters the upper or lower barriers. Fig. 2.1 illustrates a hydraulically induced elliptic fracture configuration with the associated geometric variables and reservoir properties. The elongated fracture configuration

Fig. 2.1 Unsymmetric Elliptic Model in Three-Layered Formations

is determined by the magnitudes of differentials in the reservoir properties (in situ stresses, elastic moduli, and fracture toughness contrasts) as well as fluid leak-off characteristics.

2.2 Energy Considerations for Hydraulic Fracturing

The relative energy contributions for a hydraulically propagating fracture, with surface area $A(t)$, provide a fundamental basis for evaluation of the process mechanisms and configuration modeling. The relevant energy components taken into consideration are as follows :

Potential Energy Function

The fracture inflation and extension is due to an effective pressure acting on the surface of the crack. The effective pressure, $P_e(\underline{x}, t)$ is defined as

$$P_e(\underline{x}, t) = p(\underline{x}, t) - \sigma_o(y) \quad (2.1)$$

where $p(\underline{x}, t)$ is the fracturing fluid pressure with $\underline{x} = (x_1, x_2)$ on the fracture surface. The minimum horizontal formation in situ stress, $\sigma_o(y)$, which is perpendicular to the plane of the fracture, is the local stress state in a given rock mass at depth. The horizontal principal effective stress components of the local stress state are influenced by the weight of the overburden, tectonic stresses, pore pressure, temperature, rock properties. The in situ stress state in the three layered formations

can be expressed, assuming that the in situ stresses are constant in each layer, in the form

$$\begin{aligned}\sigma(y) &= \sigma_o^p & -h_\ell \leq y \leq h_u \\ \sigma(y) &= \sigma_o^p + \Delta\sigma_u & y > h_u \\ \sigma(y) &= \sigma_o^p + \Delta\sigma_\ell & y < -h_\ell\end{aligned}$$

We note that h_u and h_ℓ are distances from the center of perforation to the upper and lower barrier, respectively. Also the subscripts or superscripts u , ℓ , and p refer to the quantities of upper layer, lower layer, and payzone, respectively.

The potential energy associated with the work done by the effective crack pressure in generating the fracture configuration

$$U_p = \int_{A(t)} p_e(\underline{x}, t) w(\underline{x}, t) \, dA \quad (2.2)$$

where $w(\underline{x}, t)$ is the crack opening width on the fracture surface. Analysis of an unsymmetric elliptic model is conducted by using the fracture half length, $a(t)$, upper fracture height, $b_u(t)$, lower fracture height, $b_\ell(t)$, and fracture opening half width, $c(t)$, as generalized coordinates. The integrals associated with the elliptic crack, for calculation of the energy components, are obtained from the transformation

$$x = \rho a \cos \theta, \quad y = \rho b \sin \theta, \quad dA = \rho ab \, d\rho d\theta \quad (2.3)$$

for $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$.

The crack opening displacement profile is assumed, as a first order approximation, in the elliptical form

$$w(\rho, \theta, t) = 2c(t) (1-\rho^2)^{1/2} \quad (2.4)$$

The potential energy associated with the three-layered unsymmetric formations is given by

$$\begin{aligned} U_p &= \int_{A_o} 2P_o c(1-\rho^2)^{1/2} dA + \int_{A_u} 2c(P_o - \Delta\sigma_u) (1-\rho^2)^{1/2} dA \\ &\quad + \int_{A_\ell} 2c(P_o - \Delta\sigma_\ell) (1-\rho^2)^{1/2} dA \\ &= \sum \frac{1}{2} \left\{ \frac{4\pi}{3} (P_o - \Delta\sigma_j) a b_j c + \Delta\sigma_j \pi h a c \left(1 - \frac{1}{3} (h_j/b_j)^2\right) \right\} \\ &\quad \quad \quad j=u, \ell \end{aligned} \quad (2.5)$$

where Σ implies summation over the upper ($j=u$) and lower layers ($j=\ell$), and $P_o(t)$ denotes borehole effective fracture fluid pressure. The total fracture area is the sum of the area in the payzone (A_p), upper (A_u), and lower (A_ℓ) layers, i.e.

$$A(t) = A_p + A_u + A_\ell .$$

Formation Strain Energy Function

The formation strain energy function corresponding to the crack opening width is

$$U_s = \frac{1}{2} \int_{A(t)} w(\underline{x}, t) \left[\int_{A(t)} K(\underline{x}, \xi) w(\xi, t) dA \right] dA \quad (2.6)$$

where the kernel function $K(\underline{x}, \xi)$ for a multi-layered reservoir is defined by

$$P_e(\underline{x}, t) = \int_{A(t)} K(\underline{x}, \xi) w(\xi, t) dA. \quad (2.7)$$

The formation strain energy in a three-layered unsymmetric formation is obtained from the assumption, proposed by Van Eekelen [15], that the total strain energy associated with the crack in the layered formation is the sum of contributions from the fracture in each material, i.e.

$$U_s = U_s^p + U_s^u + U_s^\ell. \quad (2.8)$$

The contribution of the payzone can be approximated by the elliptic fracture with the fracture height modified by

$$b_j^* = \frac{\hat{\mu}_p}{\hat{\mu}_j} b_j + h_j \left(1 - \frac{\hat{\mu}_p}{\hat{\mu}_j}\right) \quad j=u, \ell \quad (2.9)$$

where $\hat{\mu}_j = \mu_j / (1 - \nu_j)$. j=p, u, \ell

The contribution of the barrier layers can be approximated via the redefined elastic modulus given by

$$\hat{\mu}_j^* = [\hat{\mu}_p \hat{\mu}_j \frac{1 + (b_j - h_j)/2h_j}{1 + \hat{\mu}_p (b_j - h_j)/2\hat{\mu}_j h_j}]^{1/2} \quad j=u, \ell \quad (2.10)$$

The total strain energy stored in an unsymmetric elliptic crack for the three-layered formation is

$$U_s = \sum \left\{ \frac{2\pi}{3} a c^2 \hat{\mu}_p^* E(k_j^*) (A_{pj}^*/A_{Tj}^*) + \frac{2\pi}{3} a c^2 \hat{\mu}_j^* E(k_j) (A_{bj}/A_{Tj}) \right\} \quad j=u, \ell \quad (2.11)$$

where $k_j = (1 - b_j^2/a^2)^{1/2}$ and $k_j^* = (1 - b_j^{*2}/a^2)^{1/2}$. j=u, \ell

We note that A_{Tj} , A_{pj} and A_{bj} designate the upper(j=u)/lower(j=\ell) half fracture area, the area in the

payzone and the area in the barriers, respectively. The components, associated with the fracture height and modified height (b_j and b_j^* , respectively), are in the form

$$A_{pj}^*/A_{Tj}^* = 1 - \frac{2}{\pi} [\sin^{-1}\{1 - (h_j/b_j^*)^2\}^{1/2} + h_j/b_j^*\{1 - (h_j/b_j^*)^2\}^{1/2}]$$

$$\text{and } A_{bj}/A_{Tj} = \frac{2}{\pi} [\sin^{-1}\{1 - (h_j/b_j)^2\}^{1/2} + h_j/b_j\{1 - (h_j/b_j)^2\}^{1/2}]$$

Equation (2.11) can be rewritten as

$$U_s = \sum \frac{1}{2} \left\{ \frac{2\pi}{3} ac^2 \hat{\mu}_p E(k_j) \left[1 - \frac{2}{\pi} [\sin^{-1}\{1 - (h_j/b_j)^2\}^{1/2} - h_j/b_j^*\{1 - (h_j/b_j^*)^2\}^{1/2}] \right] + \frac{2\pi}{3} ac^2 \hat{\mu}_j^* E(k_j) \frac{2}{\pi} [\sin^{-1}\{1 - (h_j/b_j)^2\}^{1/2} - h_j/b_j\{1 - (h_j/b_j)^2\}^{1/2}] \right\} \quad j=u, \ell \quad (2.12)$$

where the complete elliptic integral of the second kind, $E(k_j)$, is

$$E(k_j) = \int_0^{\pi} \sqrt{1 - k_j^2 \sin^2 \theta} \, d\theta.$$

Griffith Surface Energy Function

Since the formation layers are assumed to be predominantly brittle, most efforts to understand the behavior of crack equilibrium and growth in rocks have relied on linear elastic fracture mechanics theories as of Griffith's theory. The Griffith fracture surface energy

for crack propagation can be expressed as

$$\begin{aligned} U_f &= \int_{A(t)} G_{cr} dA \\ &= G_{cr} A_p + \sum (G_{cr} + \Delta G_{cr}^j) A_j \quad j = u, \ell \end{aligned} \quad (2.13)$$

where G_{cr} is the critical energy release rate of the reservoir rock. The Griffith fracture surface energy for a three-layered formation is

$$\begin{aligned} U_f &= \sum \frac{1}{2} \left\{ G_{cr} \pi a b_j + \Delta G_{cr}^j [2 a b_j \sin^{-1} \{1 - (h_j/b_j)^2\}^{1/2} \right. \\ &\quad \left. - 2 a h_j \{1 - (h_j/b_j)^2\}^{1/2} \right\} \quad j = u, \ell \end{aligned} \quad (2.14)$$

Fluid Dissipation Energy Rate Function

The viscous fluid dissipation energy rate can be defined by

$$D_q = \int_{V_f} \psi(L) d\Omega \quad (2.15)$$

where V_f is the fluid control volume, ψ is the dissipation function, and L is the velocity gradient tensor defined by

$$L_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \quad (2.16)$$

For a power law non-Newtonian fracture fluid, the relation between the dissipation function and the shear stress is

$$\tau_{ij} = \frac{\partial \psi}{\partial L_{ij}} = \eta_o |L|^{m-1} L_{ij} \quad (2.17)$$

where $|L| = (L_{ij} L_{ij})^{1/2} = \{\text{tr}(L^T L)\}^{1/2}$

Hence, the fluid dissipation energy rate can be written as

$$D_q = \frac{\eta_o}{1+m} \int_{V_f} (L_{ij}L_{ij})^{(1+m)/2} d\Omega \quad (2.18)$$

(i,j = 1,2,3)

We now evaluate the above integral for laminar Hele-Shaw flow in a cavity with width $w = 2c$, by assuming fluid velocity components (v_i) as

$$v_i \approx \frac{2m+1}{m+1} \left[1 - \left(\frac{x_3}{w/2} \right)^{(1+m)/m} \right] \frac{q_i}{w}, \quad (i=1,2)$$

and $v_3 \approx 0$ (2.19)

where $q_i = \int_{-w/2}^{w/2} v_i dx_3$, $i=1,2$, the average flow rate.

The resulting two-dimensional form of equation (2.18) is derived as

$$\begin{aligned} D_q &= \frac{\eta_o}{1+m} \int_{A(t)} \left[\int_{-w/2}^{w/2} (L_{ij}L_{ij})^{(1+m)/2} dx_3 \right] dA \\ &= \frac{2^{m+1}}{1+m} \int_{A(t)} \eta_o \left[\frac{2m+1}{m} \right]^m |q_j(\underline{x}, t)|^{m+1} w(\underline{x}, t)^{-2m-1} dA \end{aligned} \quad (2.20)$$

where η_o and m are the fluid consistency index and flow behavior index, respectively, $q_j(\underline{x}, t)$ are the two-dimensional flow rate vector components, and

$$|q_j(\underline{x}, t)| = [q_1(\underline{x}, t)^2 + q_2(\underline{x}, t)^2]^{1/2}.$$

The pressure gradient-flow rate relations for a power law non-Newtonian fracture fluid, corresponding to equation (2.19), are assumed to have the form

$$\partial p(\underline{x}, t) / \partial x_i = f(\underline{x}, t) q_i(\underline{x}, t) \quad (2.21)$$

with

$$f(\underline{x}, t) = -\eta_o \left[\frac{2m+1}{m} \right]^m 2^{m+1} |q_j(\underline{x}, t)|^{m-1} w(\underline{x}, t)^{-2m-1}.$$

From equation (2.21), equation (2.20) can be alternatively

expressed in the form

$$D_q = - \frac{1}{1+m} \int_{A(t)} \partial p(\underline{x}, t) / \partial x_i q_i(\underline{x}, t) dA \quad (i=1,2) \quad (2.22)$$

Since the flow in a fracture is influenced by the fluid leak-off into the formation, and the magnitude of the fluid leak-off coefficients can vary across the layers, the flow rate vector components should be updated at each position as the fracture propagates into the upper/lower layers.

The viscous fluid dissipation energy rate including leak-off effects in the layers is

$$D_q = \sum \left\{ \frac{2}{1+m} \left(\frac{2m+1}{2m} \right)^m \frac{\eta_o a b_j}{c^{2m+1}} \int_{\rho_b}^1 \int_0^\pi \frac{ \{ q_\rho(\rho, \theta_j, t) \}^{m+1}}{(1-\rho^2)^{(2m+1)/2}} \rho d\theta d\rho \right\} \\ j = u, \ell \quad (2.23)$$

The local flow rate, $q_\rho(\rho, \theta, t)$, derived from the mass conservation equation including fluid leak-off effects, is

$$q_\rho^j(\rho, \theta_j, t) = \frac{1}{\rho} (\sin^2 \theta_j / a^2 + \cos^2 \theta_j / b_j^2)^{1/2} \left[\frac{2}{3} \frac{\dot{a} b_j c (1-\rho^2)^{3/2}}{ab_j} \right. \\ \left. + \left\{ 2c (1-\rho^2)^{1/2} \rho^2 + 2\rho^2 \left[\frac{C_L(\rho, \theta_j)}{1-\gamma} (t - \tau(\rho))^{1-\gamma} \right. \right. \right. \\ \left. \left. + C_S(\rho, \theta_j) \right] \right\} (\dot{a} b_j \cos^2 \theta_j + \dot{a} b_j \sin^2 \theta_j) \\ \left. + \frac{\dot{a} b_j}{ab_j} \int_\rho^1 \left[\frac{2C_L^j(\rho, \theta_j)}{1-\gamma} (t - \tau(\rho))^{1-\gamma} + C_S^j(\rho, \theta_j) \right] \rho d\rho \right] \quad (2.24)$$

where C_L and C_S denote the fluid leak-off and spurt loss coefficients, respectively. The quantity $\tau(\rho)$ is the elapsed arrival time that the position, ρ , has been

exposed to the fracturing fluid and γ is the exponent of leak-off coefficient. We note that leak-off and spurt loss coefficients are the functions of the position, $\underline{x}$, so that the appropriate coefficients should be applied after the fracture tip penetrates the barriers. Therefore

- i) $c_L^j(\rho, \theta_j) = c_L^p$ and $c_S^j(\rho, \theta_j) = c_S^p$
 if $|\theta_j| \leq \theta_j^*$, where $\theta_j^* \equiv \sin^{-1}(h_j/b_j)$, $j=u, \ell$
- ii) $c_L^j(\rho, \theta_j) = c_L^p$ and $c_S^j(\rho, \theta_j) = c_S^p$
 if $|\theta_j| > \theta_j^*$ and $\rho < \rho_j^*$, where $\rho_j^* \equiv \frac{h_j}{b_j \sin \theta_j^*}$ $j=u, \ell$
- iii) $c_L^j(\rho, \theta_j) = c_L^p + \Delta c_L^j$ and $c_S^j(\rho, \theta_j) = c_S^p + \Delta c_S^j$
 if $|\theta_j| > \theta_j^*$ and $\rho > \rho_j^*$ $j=u, \ell$

Fluid leak-off effects and derivation of the local flow rate, $q_\rho(\rho, \theta_j, t)$, are detailed in section 2.5.

2.3 Governing Equations

The governing equations for fracture propagation can now be derived by using the classical form of Lagrange's equations for non-conservative systems with n degrees of freedom

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{Y}_i} \right) - \frac{\partial L}{\partial Y_i} = F_i \quad i=1, 2, \dots, n \quad (2.25)$$

where $L = T - V$ is the Lagrangian function, and T and V denote the kinetic and potential energy of the system, respectively. The generalized coordinates, Y_i ,

representing the principal evolution dimensions, are the length ($Y_1=L$) and fracture width ($Y_2=W$) for rectangular fracture configuration with confined height growth, or the radius ($Y_1=R$) and fracture width ($Y_2=W$) of the penny-shaped configuration, or the major semi-axes ($Y_1=a$), upper ($Y_2=b_u$) and lower ($Y_3=b_l$) minor semi-axes of an elliptic and the fracture half-width ($Y_4=c$) for the unsymmetric elliptical configuration.

Since the hydraulic fracture propagation process is assumed to be quasi-static, the kinetic energy of the system is ignored. Consequently, the governing equation (2.25) reduces to

$$\frac{\partial V}{\partial Y_i} = F_i \quad i=1,2,\dots,n \quad (2.26)$$

The generalized forces, F_i , can be expressed in terms of the dissipative forces due to the viscous fracturing fluid which are derivable from a dissipation function, D_q , and other generalized forces, Q_i as follows :

$$F_i = - \frac{\partial L}{\partial Y_i} + Q_i \quad i=1,2,\dots,n \quad (2.27)$$

The generalized forces, Q_i are derived from a virtual work principle. The virtual work done by the generalized forces, Q_i can be divided into two components as

$$dU = dU_p + dU_s \quad (2.28)$$

where dU_p is the work done by the fluid pressure in opening the fracture surface and dU_s is the work done in increasing the fracture surface.

Therefore, the resulting Lagrange's equations are

$$\partial U_s / \partial y_i + \partial U_f / \partial y_i + \partial D_q / \partial \dot{y}_i = \partial U_p / \partial y_i \quad (2.29)$$

$i=1, 2, \dots, n$

For an incompressible fluid, the flow rate satisfies the mass conservation equation

$$\partial q_i(\underline{x}, t) / \partial x_i + \partial w(\underline{x}, t) / \partial t + q_L(\underline{x}, t) - i_o(\underline{x}, t) = 0 \quad (2.30)$$

where $q_L(\underline{x}, t)$ represents the fluid leak-off into the reservoir and $i_o(\underline{x}, t)$ denotes the injection of the fracturing fluid into the reservoir.

The governing equations (2.29) with $Y_1=a$, $Y_2=b_u$, $Y_3=b_\ell$, and $Y_4=W$ can now be expressed in the form

$$\begin{aligned} f_1 = 0 = & \sum \frac{1}{2} \left\{ -\frac{4\pi}{3} (P_o - \Delta\sigma_j) b_j c - \Delta\sigma_j 2\pi h_j c \{1 - \frac{1}{3}(h_j/b_j)^2\} \right. \\ & + \Delta G_{cr}^j [2b_j \sin^{-1}\{1 - (h_j/b_j)^2\}^{1/2} - 2h_j \{1 - (h_j/b_j)^2\}^{1/2}] \\ & + G_{cr} \pi b_j + \frac{2\pi}{3} \hat{\mu}_p c^2 [E(k_j^*) - b_j^{*2}/(a^2 - b_j^{*2}) \{K(k_j^*) - E(k_j^*)\}] \\ & [1 - \frac{2}{\pi} \sin^{-1}\{1 - (h_j/2b_j^*)^2\}^{1/2} + 2h_j/\pi b_j^* \{1 - (h_j/b_j^*)^2\}^{1/2}] \\ & + \frac{2\pi}{3} \hat{\mu}_j^* c^2 [E(k_j) - b_j^2/(a^2 - b_j^2) \{K(k_j) - E(k_j)\}] \\ & [\frac{2}{\pi} \sin^{-1}\{1 - (h_j/b_j)^2\}^{1/2} - 2h_j/\pi b_j \{1 - (h_j/b_j)^2\}^{1/2}] \\ & + \aleph \frac{\eta_o a b_j^2}{c^{2m}} \int_{\rho_b}^1 \int_0^{\pi} (1 - \rho^2 + 3\rho^2 \cos^2 \theta) \left\{ \frac{\cos^2 \theta}{b_j^2} + \frac{\sin^2 \theta}{a^2} \right\}^{(1+m)/2} \\ & \left. \frac{(\dot{a} b_j c (1 - \rho^2) + 3c \rho^2 (\dot{a} b_j \cos^2 \theta + \dot{a} b_j \sin^2 \theta))^m}{\rho^m (1 - \rho^2)^{m/2}} d\theta d\rho \right\} \quad (2.31) \end{aligned}$$

where $N = \frac{8}{3^{m+1}} \frac{(2m+1)^m}{m^m}$ and the complete elliptic integrals of the first and second kind are denoted by $K(k_j)$ and $E(k_j)$ ($j = u, \ell$), respectively.

$$\begin{aligned}
 f_2 = 0 = & \frac{1}{2} \left\{ -\frac{4\pi}{3} (P_o - \Delta\sigma_u)ac - \Delta\sigma_u \frac{4\pi}{3} ac h_u^3/b_u^3 \right. \\
 & + G\pi a + \Delta G_{cr}^u [2a \sin^{-1}\{1 - (h_u/b_u)^2\}^{1/2} \\
 & + 2ah_u/b_u \{1 - (h_u/b_u)^2\}^{1/2}] + \frac{2\pi}{3} \hat{\mu}_p \frac{\hat{G}_p}{\hat{G}_u} ac^2 \left[-4h_u \right. \\
 & E(k_u^*) \{1 - (h_u/b_u^*)^2\}^{1/2} / \pi b_u^{*2} + b_u^* \frac{K(k_u^*) - E(k_u^*)}{a^2 - b_u^{*2}} \\
 & \left. [1 - \frac{2}{\pi} \sin^{-1}\{1 - (h_u/b_u^*)^2\}^{1/2} + 2h_u/\pi b_u^* \{1 - (h_u/b_u^*)^2\}^{1/2}] \right] \\
 & + \frac{2\pi}{3} \hat{\mu}_u^* ac^2 \left[4h_u E(k_u) / \pi b_u^2 \{1 - (h_u/b_u)^2\}^{1/2} \right. \\
 & + b_u \frac{K(k_u) - E(k_u)}{a^2 - b_u^2} [\frac{2}{\pi} \sin^{-1}\{1 - (h_u/b_u)^2\}^{1/2} \\
 & \left. - 2h_u/\pi b_u \{1 - (h_u/b_u)^2\}^{1/2}] \right] \\
 & + \frac{\pi}{3} \hat{\mu}_p (\hat{\mu}_u - \hat{\mu}_p) / \hat{\mu}_u^* \frac{ac^2 E(k_u)}{2h_u \{1 + \hat{\mu}_p (b_u - h_u) / b_u \hat{\mu}_u\}^2} \\
 & [\frac{2}{\pi} \sin^{-1}\{1 - (h_u/b_u)^2\}^{1/2} - 2h_u/\pi b_u \{1 - (h_u/b_u)^2\}^{1/2}] \\
 & + N \frac{\eta_o a^2 b_u}{c^{2m}} \int_{\rho_b}^1 \int_0^{\pi/2} (1 - \rho^2 + 3\rho^2 \sin^2 \theta) \left\{ \frac{\cos^2 \theta}{b_u^2} + \frac{\sin^2 \theta}{a^2} \right\}^{(1+m)/2} \\
 & \left. \frac{\{ab_u c(1 - \rho^2) + 3c\rho^2 (\dot{a}b_u \cos^2 \theta + \dot{a}b_u \sin^2 \theta)\}^m}{\rho^m (1 - \rho^2)^{m/2}} d\theta d\rho \right\} \quad (2.32)
 \end{aligned}$$

$$\begin{aligned}
f_3 = 0 = & \frac{1}{2} \left\{ -\frac{4\pi}{3} (P_o - \Delta\sigma_\ell) ac - \Delta\sigma_\ell \frac{4\pi}{3} ac h_\ell^3/b_\ell^3 \right. \\
& + G_{cr} \pi a + \Delta G_{cr}^\ell [2a \cdot \sin^{-1}\{1 - (h_\ell/b_\ell)^2\}^{1/2} \\
& + ah/b_\ell \{1 - (h_\ell/b_\ell)^2\}^{1/2}] + \frac{2\pi}{3} \hat{\mu}_p \frac{\hat{\mu}_p}{\hat{\mu}_\ell} ac^2 \left[-4h_\ell \right. \\
& E(k_\ell^*) \{1 - (h_\ell/b_\ell^*)^2\}^{1/2} / \pi b_\ell^{*2} + b_\ell^* \frac{K(k_\ell^*) - E(k_\ell^*)}{a^2 - b_\ell^{*2}} \\
& \left. [1 - \frac{2}{\pi} \sin^{-1}\{1 - (h_\ell/b_\ell^*)^2\}^{1/2} + 2h_\ell/\pi b_\ell^* \{1 - (h_\ell/b_\ell^*)^2\}^{1/2}] \right] \\
& + \frac{2\pi}{3} \hat{\mu}_\ell^* ac^2 \left[4h_\ell E(k_\ell)/\pi b_\ell^2 \{1 - (h_\ell/b_\ell)^2\}^{1/2} \right. \\
& + b_\ell \frac{K(k_\ell) - E(k_\ell)}{a^2 - b_\ell^2} \left[\frac{2}{\pi} \sin^{-1}\{1 - (h_\ell/b_\ell)^2\}^{1/2} \right. \\
& \left. \left. - 2h_\ell/\pi b_\ell \{1 - (h_\ell/b_\ell)^2\}^{1/2} \right] \right] \\
& + \frac{\pi}{3} \hat{\mu}_p (\hat{\mu}_\ell - \hat{\mu}_p) / \hat{\mu}_\ell^* \frac{ac^2 E(k_\ell)}{2h_\ell \{1 + \hat{\mu}_p (b_\ell - h_\ell)/b_\ell \hat{\mu}_\ell\}^2} \\
& \left[\frac{2}{\pi} \sin^{-1}\{1 - (h_\ell/b_\ell)^2\}^{1/2} - 2h_\ell/\pi b_\ell \{1 - (h_\ell/b_\ell)^2\}^{1/2} \right] \\
& + \frac{\eta_o a^2 b_\ell}{c^{2m}} \int_{\rho_b}^1 \int_0^{\frac{\pi}{2}} (1 - \rho^2 + 3\rho^2 \sin^2 \theta) \left\{ \frac{\cos^2 \theta}{b_\ell^2} + \frac{\sin^2 \theta}{a^2} \right\}^{(1+m)/2} \\
& \left. \frac{\{\dot{a}b_\ell \dot{c}(1 - \rho^2) + 3c\rho^2 (\dot{a}b_\ell \cos^2 \theta + \dot{a}b_\ell \sin^2 \theta)\}^m}{\rho^m (1 - \rho^2)^{m/2}} d\theta d\rho \right\} \quad (2.33)
\end{aligned}$$

$$\begin{aligned}
f_4 = 0 = & \sum \frac{1}{2} \left\{ -\frac{4\pi}{3} (P_o - \Delta\sigma_j) ab_j c - \Delta\sigma_j 2\pi ha \{1 - \frac{1}{3}(h_j/b_j)^2\} \right. \\
& + \frac{4\pi}{3} \hat{\mu}_p ac E(k_j^*) [1 - \frac{2}{\pi} \sin^{-1}\{1 - (h_j/b_j^*)^2\}^{1/2} \\
& \left. + 2h_j/\pi b_j^* \{1 - (h_j/b_j^*)^2\}^{1/2}] + \frac{4\pi}{3} \hat{\mu}_j^* ac E(k_j) \right\}
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{2}{\pi} \sin^{-1} \{1 - (h_j/b_j)^2\}^{1/2} - 2h_j/\pi b_j \{1 - (h_j/b_j)^2\}^{1/2} \right] \\
& + N \frac{\eta_o a^2 b_j^2}{c^{2m+1}} \int_{\rho_b}^1 \int_0^{\pi/2} \frac{\{ \dot{a} b_j c (1-\rho^2) + 3c \rho^2 (\dot{a} b_j \cos^2 \theta + \dot{a} b_j \sin^2 \theta) \}^m}{\rho^m (1-\rho^2)^{m/2}} \\
& (1-\rho^2) \left\{ \frac{\cos^2 \theta}{b_j^2} + \frac{\sin^2 \theta}{a^2} \right\}^{(1+m)/2} d\theta d\rho \} \quad (2.34)
\end{aligned}$$

where $j = u, \ell$.

The mass conservation equation (2.30) reduces to

$$f_5 = 0 = \sum_{j=u, \ell} 2\pi a b_j c/3 - \int_0^t i_o(t) dt + V_L(t) + V_S(t) \quad (2.35)$$

$$\text{where } V_L(t) = \int_0^t \int_{A(t)} q_L(\underline{x}) dA ds$$

$$\text{and } V_S(t) = \int_{A(t)} 2 C_s(\underline{x}) dA .$$

We note that V_L and V_S are volumes lost due to the effects of fluid leak-off and spurt loss, respectively. The mass conservation equation indicates that the total injected volume is equal to the sum of the occupied fracture fluid volume and the fluid volume lost due to the fluid leak-off and spurt loss. The integrations for each volume components are detailed in the chapter 2.5.

2.4 Solution Procedure

For the presented multi-layered system, a numerical solution procedure is devised so that equations (2.31) through (2.35) are simultaneously satisfied i.e. $\underline{f}^{t+\Delta t} = \underline{0}$

at time $t+\Delta t$. The procedure begins with the known solution vector $\underline{x}^t$ at time t given by

$$\underline{x}^t = \{x_1, x_2, x_3, x_4, x_5\}^t = \{a(t), b_u(t), b_\ell(t), c(t), P_o(t)\} . \quad (2.36)$$

Using the two point implicit time stepping scheme, the time rate is expressed in terms of the unknown solution vector $\underline{x}^{t+\Delta t}$ by

$$\dot{\underline{x}} = (\underline{x}^{t+\Delta t} - \underline{x}^t) / \Delta t . \quad (2.37)$$

The solution after the n^{th} Newton-Raphson iteration scheme can be written in the incremental form as

$$\underline{x}_{n+1}^{t+\Delta t} = \underline{x}_n^{t+\Delta t} + \Delta \underline{x}_n \quad (2.38)$$

where the solution increments are computed from

$$\Delta \underline{x}_n = - [\partial \underline{f} / \partial \underline{x}]_{\underline{x}=\underline{x}_n}^{-1} \underline{f}_n \quad (2.39)$$

along with the starting solution $\underline{x}_1^{t+\Delta t} = \underline{x}^t$. The numerical procedure begins with an initial condition for a specified penny shaped closed form solution, detailed in Appendix A, at time $t=\Delta t$. For relatively a small initial time, the dissipation dominant solution provides reasonable initial approximation in most cases. During the iteration, the gradient matrix computation in equation (2.35) is carried out numerically (i.e. $\partial \underline{f} / \partial \underline{x} = \{\underline{f}(\underline{x} + \Delta \underline{x}) - \underline{f}(\underline{x})\} / \Delta \underline{x}$ for small $\Delta \underline{x}$) for program simplicity. The iteration is terminated when convergence, expressed in terms of relative error, $|\Delta \underline{x}_n| / |\underline{x}_n^{t+\Delta t}|$, is obtained within a specified tolerance range. In the unsymmetric elliptic

fracture model, the following criteria is employed to terminate the iteration:

$$\text{Error} = \sum_{n=1}^5 \left| \frac{\Delta x_n}{x_n} \right| = \text{Tolerance} \leq 0.05\%$$

The energy rate terms associated with the time increment are then computed and the solution vector is updated for the subsequent time step solution.

2.5 Fluid Leak-off Effects

The fluid leak-off rate is one of the major factors influencing the fracture configuration. The fluid volume lost during the treatment determines the hydraulic fracture process efficiency, i.e. the ratio of fracture volume to the total volume injected. The fluid leak-off rate is directly governed by the fracturing fluid leak-off coefficient, which has been defined by a combination of three types of linear flow mechanisms encountered during a fracturing treatment. The three types of linear flow mechanisms are the effluent viscosity and relative permeability effects, C_v , reservoir fluid viscosity/compressibility effects, C_c , and the wall-building coefficient, C_w . The first two coefficients can be calculated from reservoir data and fracturing fluid viscosity. The third coefficient is derived from the leak-off data for fluid-loss additives, which must be

determined experimentally. The simplest method used by the industry today is to assume that the wall-building coefficient, C_w , will dominate and thus to use that value in the fracture configuration calculations. Therefore we assume that the rate of fluid loss through an incremental area, dA , at time t is expressed by the traditionally accepted, Carter relation as

$$q_L = \frac{2C_L}{[t - \tau(\underline{x})]^\gamma} \quad (2.40)$$

where C_L is the overall leak-off coefficient and $\tau(\underline{x})$ is the elapsed arrival time that the position, $\underline{x}$, has been exposed to fluid. Practically the exponent of leak-off coefficient, $\gamma < 1$, is assigned the value of 0.5. The mass balance equation (2.35) for an incompressible fracturing fluid requires that the total volume injected (V_I) is equal to the fracture volume (V_F) plus the volume of fluid lost to the formation leak-off (V_L) and spurt loss (V_S). Therefore

$$\begin{aligned} V_I(t) &= \int_0^t i_o(t) dt \\ &= V_F(t) + V_L(t) + V_S(t) \\ &= \int_{A(t)} w(\underline{x}) dA + \int_0^t \int_{A(t)} q_L(\underline{x}) dA dt + \int_{A(t)} 2C_S(\underline{x}) dA \end{aligned} \quad (2.41)$$

$$\text{where } V_L(t) = \int_0^t \int_{A(t)} q_L(\underline{x}) dA ds$$

$$= \sum 2 \int_0^{\pi/2} \int_0^1 \int_{\tau}^t \frac{2C_L(\underline{x})}{[s - \tau(\underline{x})]^\gamma} ds |J_j| d\rho d\theta$$

$$\text{and } v_s(t) = \sum 2 \int_0^{\pi/2} \int_0^1 2 C_s(\rho, \theta_j) \rho ab_j d\rho d\theta$$

with the Jacobian, $|J_j| = \rho ab_j$. We assume that the flow vector is normal to the local elliptical fracture contour.

The incremental volumes bounded by an area enclosed by

$\theta_j \leq \hat{\theta} \leq \theta_j + d\theta$ and $\rho \leq \hat{\rho} \leq 1$ are expressed as

$$\Delta v_F(\rho, \theta_j, t) = \int_{\theta_j}^{\theta_j + d\theta} \int_{\rho}^1 2c (1 - \rho^2)^{1/2} \rho ab_j d\rho d\theta$$

$$\Delta v_L(\rho, \theta_j, t) = \int_{\theta_j}^{\theta_j + d\theta} \int_{\rho}^1 \frac{2C_L(\rho, \theta_j)}{1-\gamma} [t - \tau(\rho)]^{1-\gamma} \rho ab_j d\rho d\theta$$

$$\Delta v_s(\rho, \theta_j, t) = \int_{\theta_j}^{\theta_j + d\theta} \int_{\rho}^1 2C_s \rho ab_j d\rho d\theta$$

$$j = u, \ell \quad (2.42)$$

where the subscripts, u and ℓ , denote components for the upper and lower layers, respectively.

From the mass conservation equation, the local fluid flow rate can be derived as

$$\begin{aligned} q_\rho(\rho, \theta_j, t) & \rho (\sin^2 \theta_j / a^2 + \cos^2 \theta_j / b_j^2)^{-1/2} \\ &= \frac{d}{dt} \left\{ \frac{2ab_j c}{3} (1 - \rho^2)^{3/2} \right. \\ & \quad \left. + \int_{\rho}^1 \left[\frac{2C_L(\rho, \theta_j)}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} + C_s(\rho, \theta_j) \right] \rho ab_j d\rho \right\} \end{aligned} \quad (2.43)$$

The last term in eqn (2.43) can be expressed as

$$\begin{aligned}
1) \quad dV_1(\rho, \theta_j, t) &= \frac{1}{2} ab_j(1-\rho^2) \left[\frac{2c_L^p}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} + c_s^p \right] \\
&\quad \text{if } |\theta_j| \leq \theta_j^*, \text{ where } \theta_j^* = \sin^{-1}(h_j/b_j) \\
2) \quad dV_1(\rho, \theta_j, t) &= \frac{1}{2} ab(1-\rho^2) \left[\frac{2c_L^p}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} + c_s^p \right] \\
&\quad + \frac{1}{2} ab_j(1-\rho^{*2}) \left[\frac{2\Delta c_L(\theta_j)}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} + \Delta c_s(\theta_j) \right] \\
&\quad \text{if } |\theta_j| > \theta_j^* \text{ and } \rho < \rho^*, \text{ where } \rho^* = \frac{h_j}{b_j \sin \theta} \\
3) \quad dV_1(\rho, \theta_j, t) &= \frac{1}{2} ab_j(1-\rho^2) \left[\frac{2\Delta c_L(\theta_j)}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} \right. \\
&\quad \left. + \Delta c_s(\theta_j) \right] \\
&\quad \text{if } |\theta_j| > \theta_j^* \text{ and } \rho > \rho^*.
\end{aligned}$$

Fluid leak-off and spurt loss coefficients in the payzone are denoted by c_L^p and c_s^p , respectively.

The fluid leak-off and spurt loss coefficients are

$$c_L(\rho, \theta_j) = c_L^p + \Delta c_L(\rho, \theta_j)$$

$$c_s(\rho, \theta_j) = c_s^p + \Delta c_s(\rho, \theta_j)$$

where $\Delta c_L(\rho, \theta_j) = \Delta c_L^u$

$$\Delta c_s(\rho, \theta_j) = \Delta c_s^u \quad \text{if } \theta^* \leq \theta_j \leq \pi \text{ and } \rho > \rho^*$$

$$\Delta c_L(\rho, \theta_j) = \Delta c_L^\ell$$

$$\Delta c_s(\rho, \theta_j) = \Delta c_s^\ell \quad \text{if } -\pi \leq \theta_j \leq -\theta^* \text{ and } \rho > \rho^*$$

Therefore the local flow rate, $q_\rho(\rho, \theta_j)$ may be summarized as follows:

$$\begin{aligned}
q_{\rho}(\rho, \theta_j) = \frac{1}{\rho} \left(\frac{\sin^2 \theta_j}{a^2} + \frac{\cos^2 \theta_j}{b^2} \right)^{1/2} & \left[\frac{2}{3} \dot{a} b_j c(1-\rho^2+3\rho^2 \cos^2 \theta_j) \cdot \right. \\
& (1-\rho^2)^{1/2} + \frac{2}{3} \dot{a} b_j c(1-\rho^2+3\rho^2 \sin^2 \theta_j) \cdot (1-\rho^2)^{1/2} \\
& + \frac{2}{3} \dot{a} b_j \dot{c}(1-\rho^2)^{3/2} + c_{1s}^*(\rho, \theta_j) \{ \dot{a} b_j (1-\rho^2+2\rho^2 \cos^2 \theta_j) \\
& + \dot{a} b_j (1-\rho^2+2\rho^2 \sin^2 \theta_j) \} + \left. \frac{2c_L(\rho, \theta_j)}{[t - \tau(\rho)]^\gamma} \dot{a} b_j (1-\rho^2) \right] \\
& j = u, \ell \quad (2.44)
\end{aligned}$$

where $c_{1s}^*(\rho, \theta_j) \equiv \left[\frac{2c_L(\rho, \theta_j)}{1-\gamma} \{t - \tau(\rho)\}^{1-\gamma} + c_s(\rho, \theta_j) \right]$.

During the numerical integration of the dissipation energy rate function, coefficients of leak-off and spurt loss as well as the fluid arrival time are evaluated and assigned the proper values. The fluid arrival time is updated and interpolated for the current locations from the values obtained from the previous solutions.

CHAPTER III

APPLICATION OF VARIATIONAL PRINCIPLE TO
HYDRAULIC FRACTURING IN THREE-LAYERED FORMATIONS

3.1 Preliminaries

In this chapter, the energy rate formulation for an elliptical hydraulic fracturing model (ERATE2D) is introduced. The formulation is based on the generalized variational principle, proposed by Advani et al [70], characterizing the mechanics of the hydraulic fracturing process. The variational formulation coupling formation structural, fracture mechanics, and fracture fluid flow responses, has been verified through time-explicit solution comparisons with Lagrangian and finite element methods for penny-shaped models. This methodology is extended for the mathematical modeling of elliptical crack growth in three-layered formations. One of the advantages of this formulation when compared to the Lagrangian energy method is that the fluid leak-off effects are uncoupled from the fluid flow equation. Therefore, computation of the local flow rate, $q_{\rho}(\rho, \theta, t)$, is not needed at every iteration step. The energy rate formulation for elliptical crack model and associated fracture response

evaluation are presented. Numerical procedures along with benchmark model comparisons are also detailed.

3.2 Energy Rate Components

The variational formulation employing pertinent energy rate contributions for an elliptical crack model, utilizing the same geometry and notations introduced in Chapter II, provides a more effective approach for the evaluation of the process mechanisms and configuration modeling. This method provides an alternative methodology for performing parametric sensitivity studies for an elliptical model in multi-layered system.

3.2.1 Summary of General Functional Derivation

A general form of an energy rate functional, coupling the elastic reservoir structural, linear elastic fracture mechanics and viscous incompressible fracturing fluid, has been derived by Advani et al [70]. Prior to the derivation of appropriate equations for an elliptic crack in unsymmetric three-layered formations, the variational function will be reviewed. The beauty of this variational function is that an entire set of Euler equations (field equations) and boundary conditions are derived from the functional after taking the first variation. These equations deduced from the functional present the

equilibrium equation for the reservoir, fluid linear momentum equations, fracturing fluid incompressibility condition, fracture propagation criterion, and associated boundary conditions. This functional, obtained from the principle of virtual work, can be shown to be [70]

$$\begin{aligned}
 F(\underline{v}_s, \underline{v}_f, p, \dot{a}_n) = & \frac{d}{dt} \int_{\Omega_s} e \, dV + \int_{\Omega_f} \psi \, dV \\
 & + \int_{\Omega_f} v_{fi} (p_{,i} - f_{gi}) \, dV - 2 \int_A p \, v_L \, dA - 2 \int_A p \, v_{si} n_i \, dA \\
 & + \frac{d}{dt} \int_A G_{cr} \, dA - \int_{\partial A_q} p v_{fn} \, dA \quad (3.1)
 \end{aligned}$$

where e is the specific strain energy for the reservoir and ψ is the fracture fluid dissipation energy rate. We denote that $\underline{v}_s$, $\underline{v}_f$ and v_L are solid, fluid, and fluid leak-off velocities, respectively.

For the case of no fluid lag during hydraulic fracturing processes the solid fracture boundary coincides with the fracture fluid boundary. Therefore

$$|\partial \Omega_s| = |\partial \Omega_f| = 2|A|$$

$\Omega_s(t)$ and $\Omega_f(t)$ denotes the volumes of the reservoir (excluding the fracture) and the fluid (occupying the entire fracture), respectively.

For the vertical hydraulic fracturing problem, the equation (3.1) can be rewritten in the following simplified form :

$$\begin{aligned}
F(\dot{w}, p, \dot{a}_n) = & \int_A \dot{w} \{K_A[w] - p + \sigma_o\} dA - \int_A p q_L dA \\
& + \int_{\partial A} \dot{a}_n \left\{ \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} + G_{cr} \right\} dS - \int_{\partial A_q} p Q_n dS \\
& - \frac{m}{1+m} (\bar{\eta})^{-\frac{1}{m}} \int_A w^{\frac{2m+1}{m}} \{(\nabla p - \underline{f}_g) \cdot (\nabla p - \underline{f}_g)\}^{\frac{1+m}{2m}} dA \quad (3.2)
\end{aligned}$$

where $\bar{\eta} = \eta_o 2^{m+1} (\frac{2m+1}{m})^m$. The boundary integral corresponding to formation elasticity for the crack surface A is represented by $K_A[w]$. We also denote that $\underline{f}_g$ is a gravitational body force vector and Q_n is a fracture fluid injection rate, respectively.

The resulting field equations, obtained from the first variation of the velocity-dependent and pressure terms are

$$\delta \dot{w}: K_A[w] = p - \sigma_o \quad \text{in } A \quad (3.3)$$

$$\delta p: \nabla \cdot \underline{q} + \dot{w} + q_L = 0 \quad \text{in } A \quad (3.4)$$

$$\underline{q} \cdot \underline{n} = Q_n \quad \text{along } \partial A_q \quad (3.5)$$

$$\delta \dot{a}_n: G_{cr} = - \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} \quad \text{along } \partial A \quad (3.6)$$

where

$$\underline{q} = - (\bar{\eta})^{-1/m} \{w^{(2+1/m)} |\nabla p - \underline{f}_g|^{(1-m)/m} (\nabla p - \underline{f}_g)\} \quad (3.7)$$

The derivation of a set of resulting field equations from the first variation of a functional is detailed in the next chapter for a case where the separation of fracture and treatment fluid during stimulation exists. The energy rate functional obtained from the principle of

virtual work is extended in chapter IV to accommodate the phenomenon of fluid lag in a hydraulically induced fracture.

3.2.2 Energy Rate Conservation Law

Prior to the application of the energy rate functional in an elliptic fracture model in a three layered formation, an energy rate conservation law governing the fracture fluid control volume and reservoir elasticity will be presented to understand the roles of pertinent energy rate components in the hydraulic fracturing processes. The derivation of energy rate conservation enables us to monitor pertinent energy rate components so that design, control and optimization of hydraulic fracturing stimulation can be implemented through numerical simulations.

Fracturing Fluid Control Volume

The equilibrium equation for the fracturing fluid is

$$(-p \delta_{ij} + \tau_{ij})_{,j} + \rho f_i = 0 \quad \text{in } \Omega_f \quad (3.8)$$

Multiplying by v_{fi} , we re-write this equation as

$$\int_{\Omega_f} [(-p \delta_{ij} + \tau_{ij})_{,j} v_{fi} + \rho f_i v_{fi}] dV = 0 \quad (3.9)$$

or alternatively as

$$\int_{\partial\Omega_f} (-p \delta_{ij} + \tau_{ij}) v_{fi} n_j dA$$

$$- \int_{\Omega_f} (-p \delta_{ij} + \tau_{ij}) v_{fi,j} dA + \int_{\Omega_f} \rho f_i v_{fi} dV = 0 \quad (3.10)$$

Since $v_{i,i} = 0$ for incompressible fluid, we have

$$\int_{\Omega_f} p \delta_{ij} v_{fi,j} dA = 0 \quad (3.11)$$

Assuming that the contribution of fluid deviatoric stress components (τ_{ij}) on the boundary $(\partial\Omega_f)$ is negligible and

$\partial\Omega_f = \partial A_q + 2A_f$, we get

$$\int_{\partial\Omega_f} (-p \delta_{ij} + \tau_{ij}) v_{fi} n_j dA = \int_{\partial A_q} -p v_n dA + 2 \int_{A_f} -p v_{f3} dA \quad (3.12)$$

We note that the superposition form of the fluid velocity (v_{f3}) normal to the crack surface is represented by the sum of the fracture velocity (v_{s3}) normal to the crack surface and fluid leak-off velocity as

$$\begin{aligned} 2 v_{f3} &= 2 v_{s3} + q_L \\ &= \dot{w} + q_L \end{aligned} \quad (3.13)$$

Therefore the energy rate conservation from eqn. (3.12) has the form

$$\begin{aligned} \int_{\partial A_q} -p v_n dA &= \int_{A_f} p \dot{w} dA + \int_{A_f} p q_L dA \\ &\quad + \int_{\Omega_f} \tau_{ij} L_{ij} dV - \int_{\Omega_f} \rho f_i v_{fi} dV \end{aligned} \quad (3.14)$$

Assuming laminar Hele-Shaw flow behavior in a cavity and using the power-law type non-Newtonian fluid constitutive relation as

$$\tau_{ij} = \eta_o |L|^{m-1} L_{ij} , \quad (3.15)$$

we have

$$\begin{aligned} D_\eta &\equiv \int_{\Omega_f} \tau_{ij} L_{ij} dV = \int_{\Omega_f} \eta_o |L|^{m+1} dV \\ &= \bar{\eta} \int_{A_f} \frac{|q|^{m+1}}{w^{2m+1}} dA . \end{aligned} \quad (3.16)$$

Equation (3.16) can be rewritten from the relationship in eqn (3.6) as

$$\begin{aligned} D_\eta &= \bar{\eta} \int_{A_f} \frac{|q|^{m-1}}{w^{2m+1}} \mathbf{q} \cdot \mathbf{q} dA \\ &= - \int_{A_f} (\nabla p - \mathbf{f}_g) \cdot \mathbf{q} dA \end{aligned} \quad (3.17)$$

Substituting eqn (3.16) into eqn (3.14), we obtain

$$\begin{aligned} \int_{\partial\Omega_q} -p v_n dA &= \int_{A_f} p \dot{w} dA + \int_{A_f} p q_L dA \\ &+ \bar{\eta} \int_{A_f} \frac{|q|^{m+1}}{w^{2m+1}} dA - \int_{A_f} \rho \mathbf{f} \cdot \mathbf{q} dV \end{aligned} \quad (3.18)$$

Reservoir Formation Solid Control Volume

The hydraulic fracturing fluid and reservoir responses are coupled via the fluid pressure acting on the fracture surface. Since the power is transformed on the crack surface by the fluid pressure, we have

$$\begin{aligned} \int_{A_s} p \dot{w} dA &= \frac{d}{dt} \left[\int_{A_s} \frac{1}{2} w K[w] dA + \int_{A_s} \sigma_o w dA \right] \\ &+ \frac{d}{dt} \int_{A_s} G_{cr} dA \end{aligned} \quad (3.19)$$

Coupled Fluid-Solid Energy Rate Principle

Substituting the power term governing reservoir behavior in eqn (3.19) into eqn (3.18), the overall coupled fluid-solid energy rate principle can be expressed for the case of no fluid lag ($A = A_f = A_s$) in the form

$$\begin{aligned} \int_{\partial\Omega_q} -p \, v_n \, dA &= \int_A p \, q_L \, dA + \bar{\eta} \int_A \frac{|q|^{m+1}}{w^{2m+1}} \, dA \\ &- \int_A \rho \underline{f} \cdot \underline{q} \, dV + \frac{d}{dt} \left[\int_A \frac{1}{2} w \, K[w] \, dA + \int_A \sigma_o w \, dA \right] \\ &+ \frac{d}{dt} \int_A G_{cr} \, dA \end{aligned} \quad (3.20)$$

or alternatively as

$$D_I = D_L + D_\eta - D_{bf} + \frac{d}{dt}[U_s^*] + \dot{U}_f \quad (3.21)$$

where D_{bf} denotes energy rate term done by body force.

We note that the equation (3.20) holds when the Euler equations deduced from the variational functional, eqn (3.3 - 3.7), are satisfied. The total input power (D_I) is converted into other energy rate components, associated with leak-off (D_L), dissipation (D_η), body force (D_{bf}), strain energy ($\frac{d}{dt}[U_s^*]$) and fracture energy ($\dot{U}_f$) contributions. It is possible to observe changes of energy rate components during fracturing simulation in a systematic way. Therefore, parametric sensitivity and energy rate transformations can be investigated to evaluate the roles of controllable hydraulic treatment

variables and uncontrollable reservoir property characterization parameters, detailed in chapter VII.

3.2.3 Energy Rate Components for Unsymmetric Elliptic Fracture Geometry

In the energy rate formulation (ERATE2D) it is assumed that fracture opening width profile consists of two components, which is different from the previous approach in chapter II. The assumed pressure profile for an unsymmetric elliptic fracture is

$$\begin{aligned}\Delta p(\underline{x}, t) &= p(\underline{x}, t) - \sigma_o(\underline{x}) \\ &= P_o(t) \{1 - \hat{a}_j(t, \theta) \rho^2\} - \Delta\sigma(\underline{x})\end{aligned}\quad (3.22)$$

where $\Delta\sigma(\underline{x}) = \sigma_o(\underline{x}) - \sigma_{op}$. The pressure profile parameters in the upper/lower half of an elliptic crack, $\hat{a}_j(t, \theta)$, which are θ dependent variables is defined as

$$\begin{aligned}\hat{a}_j(t, \theta) &\equiv a_p(t) \cos^2 \theta + a_j(t) \sin^2 \theta \\ (j=u : 0 \leq \theta \leq \frac{\pi}{2}, j=\ell : -\frac{\pi}{2} \leq \theta \leq 0)\end{aligned}$$

and ρ is the normalized radial coordinate. The quantity a_p is the pressure profile parameter along horizontal direction (x) in the payzone while a_u and a_ℓ are the pressure profile parameters along vertical direction (y), respectively. We note that $\Delta p(\underline{x}, t)$ is discontinuous across the interfaces while the fracture fluid pressure, $p(\underline{x}, t)$, is continuous in the fracture.

The pertinent energy rate components for an elliptic

crack in an unsymmetric three-layered formation are obtained, as in the previous chapter, from the coordinate transformation

$$x = \rho a \cos \theta, \quad y = \rho b \sin \theta, \quad dA = \rho ab \, d\rho d\theta.$$

Therefore the effective pressure profiles in the upper and lower halves of the unsymmetric elliptic fracture, illustrated in Fig. 3.1, can be rewritten as

$$\begin{aligned} \Delta p_u &= P_o(t) \{1 - \alpha_p \rho^2 \cos^2 \theta - \alpha_u \rho^2 \sin^2 \theta\} & 0 \leq \theta \leq \frac{\pi}{2} \\ \Delta p_\ell &= P_o(t) \{1 - \alpha_p \rho^2 \cos^2 \theta - \alpha_\ell \rho^2 \sin^2 \theta\} & -\frac{\pi}{2} \leq \theta \leq 0 \end{aligned} \quad (3.23)$$

The assumed two term fracture opening width profile corresponding to the effective pressure distribution is

$$w(x, t) = W_1(t) (1 - \rho^2)^{1/2} + W_2(t) (1 - \rho^2)^{3/2}. \quad (3.24)$$

The first term in eq. (3.24) represents the crack singular behavior while the second term corresponds to the equilibrium crack growth condition by Barenblatt [87], satisfying a zero slope at the crack front ($\rho=1$). Shah and Kobayashi [88], and Raju [89] have proposed solutions for the crack opening displacements for an embedded elliptic crack in an infinite solid under arbitrary tractions. The crack opening width rate is given by

$$\begin{aligned} \dot{w} &= \dot{W}_1 (1 - \rho^2)^{1/2} + \dot{W}_2 (1 - \rho^2)^{3/2} \\ &+ \rho^2 \left(\frac{\dot{a}}{a} \cos^2 \theta + \frac{\dot{b}}{b} \sin^2 \theta \right) [W_1 (1 - \rho^2)^{-1/2} + 3W_2 (1 - \rho^2)^{1/2}] \end{aligned} \quad (3.25)$$

$$\Delta p_j(t, \rho, \theta) = P_o(t) (1 - \alpha_p \rho^2 \cos^2 \theta - \alpha_j \rho^2 \sin^2 \theta) \quad (j=u, j=\ell)$$

Fig. 3.1 Effective Pressure Profile of ERATE2D

The energy rate components obtained from the unified variational principle are as follows :

Formation strain energy rate function

$$\dot{U}_s = \int_A \dot{w} K_A[w] dA \quad (3.26)$$

$$\begin{aligned} &= \sum \int_{A_j} \dot{w} \frac{E(k_j)}{2b_j} \left[\frac{\mu}{(1-\nu)} (w_1 + \frac{3}{2} w_2 - \frac{9}{4} w_2 \rho^2) \right] dA \\ &= \sum \frac{E(k_j)}{2b_j} \left[(w_1 + \frac{3}{2} w_2) \dot{w}_1 \left(\frac{\pi}{3} ab_j \right) \{ \bar{\mu} + \Delta \bar{\mu}_j R_1(\xi_j) \} \right. \\ &\quad + (w_1 + \frac{3}{2} w_2) \dot{w}_2 \left(\frac{\pi}{3} ab_j \right) \{ \bar{\mu} + \Delta \bar{\mu}_j R_2(\xi_j) \} \\ &\quad + (w_1 + \frac{3}{2} w_2) \left[w_1 \frac{\pi}{3} (\dot{a}b_j + a\dot{b}_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_8(\xi_j) \} \right. \\ &\quad + 3w_2 \frac{\pi}{15} (\dot{a}b_j + a\dot{b}_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_9(\xi_j) \} \left. \right] \\ &\quad - \frac{9}{4} \dot{w}_1 w_2 \frac{2\pi}{15} (ab_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_4(\xi_j) \} \\ &\quad - \frac{9}{4} \dot{w}_2 w_2 \frac{2\pi}{35} (ab_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_5(\xi_j) \} \\ &\quad - \frac{9}{4} w_2 \left[w_1 \frac{4\pi}{15} (\dot{a}b_j + a\dot{b}_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_{10}(\xi_j) \} \right. \\ &\quad \left. + 3w_2 \frac{4\pi}{105} (\dot{a}b_j + a\dot{b}_j) \{ \bar{\mu} + \Delta \bar{\mu}_j R_{11}(\xi_j) \} \right] \quad (3.27) \\ &\quad j = u, \ell \end{aligned}$$

where $\xi_j = \frac{h_j}{b_j}$ ($0 < \xi_j < 1$). We define $R_k(\xi_j)$ and $g_k(\xi_j)$ as the parameters representing ratios of the integral functions, detailed in Appendix D, as follows

$$\begin{aligned}
R_1(\xi_j) &\equiv \frac{2}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} Z(\theta, \xi_j)^{3/2} d\theta \\
R_2(\xi_j) &\equiv \frac{2}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} Z(\theta, \xi_j)^{5/2} d\theta \\
R_4(\xi_j) &\equiv \frac{15}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} \left[\frac{1}{3} Z(\theta, \xi_j)^{3/2} - \frac{1}{5} Z(\theta, \xi_j)^{5/2} \right] d\theta \\
R_5(\xi_j) &\equiv \frac{35}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} \left[\frac{1}{5} Z(\theta, \xi_j)^{5/2} - \frac{1}{7} Z(\theta, \xi_j)^{7/2} \right] d\theta \\
R_8(\xi_j) &\equiv 2[\dot{a}b_j g_1(\xi_j) + a\dot{b}_j g_2(\xi_j)] / [\frac{\pi}{3}(\dot{a}b_j + a\dot{b}_j)] \\
R_9(\xi_j) &\equiv 2[\dot{a}b_j g_3(\xi_j) + a\dot{b}_j g_4(\xi_j)] / [\frac{\pi}{15}(\dot{a}b_j + a\dot{b}_j)] \\
R_{10}(\xi_j) &\equiv 2[\dot{a}b_j g_5(\xi_j) + a\dot{b}_j g_6(\xi_j)] / [\frac{4\pi}{15}(\dot{a}b_j + a\dot{b}_j)] \\
R_{11}(\xi_j) &\equiv 2[\dot{a}b_j g_7(\xi_j) + a\dot{b}_j g_8(\xi_j)] / [\frac{\pi}{105}(\dot{a}b_j + a\dot{b}_j)]
\end{aligned} \tag{3.28}$$

where $Z(\theta, \xi_j) = 1 - \zeta_j(\theta, \xi_j)$ and $\zeta_j = \frac{\xi_j}{\sin \theta}$

$$\begin{aligned}
g_1(\xi_j) &\equiv \int_{\theta_j^*}^{\frac{\pi}{2}} \cos^2 \theta \left[Z(\theta, \xi_j)^{1/2} - \frac{1}{3} Z(\theta, \xi_j)^{3/2} \right] d\theta \\
g_2(\xi_j) &\equiv \int_{\theta_j^*}^{\frac{\pi}{2}} \sin^2 \theta \left[Z(\theta, \xi_j)^{1/2} - \frac{1}{3} Z(\theta, \xi_j)^{3/2} \right] d\theta \\
g_3(\xi_j) &\equiv \int_{\theta_j^*}^{\frac{\pi}{2}} \cos^2 \theta \left[\frac{1}{3} Z(\theta, \xi_j)^{1/2} - \frac{1}{5} Z(\theta, \xi_j)^{5/2} \right] d\theta \\
g_4(\xi_j) &\equiv \int_{\theta_j^*}^{\frac{\pi}{2}} \sin^2 \theta \left[\frac{1}{3} Z(\theta, \xi_j)^{1/2} - \frac{1}{5} Z(\theta, \xi_j)^{5/2} \right] d\theta
\end{aligned}$$

$$\begin{aligned}
g_5(\xi_j) &\equiv \int_{\theta_j^*}^{\pi} \cos^2 \theta \left[z(\theta, \xi_j)^{1/2} - \frac{2}{3} z(\theta, \xi_j)^{3/2} + \frac{1}{5} z(\theta, \xi_j)^{5/2} \right] d\theta \\
g_6(\xi_j) &\equiv \int_{\theta_j^*}^{\pi} \sin^2 \theta \left[z(\theta, \xi_j)^{1/2} - \frac{2}{3} z(\theta, \xi_j)^{3/2} + \frac{1}{5} z(\theta, \xi_j)^{5/2} \right] d\theta \\
g_7(\xi_j) &\equiv \int_{\theta_j^*}^{\pi} \cos^2 \theta \left[\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{2}{5} z(\theta, \xi_j)^{5/2} + \frac{1}{7} z(\theta, \xi_j)^{7/2} \right] d\theta \\
g_8(\xi_j) &\equiv \int_{\theta_j^*}^{\pi} \sin^2 \theta \left[\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{2}{5} z(\theta, \xi_j)^{5/2} + \frac{1}{7} z(\theta, \xi_j)^{7/2} \right] d\theta
\end{aligned} \tag{3.29}$$

where $\theta_j^* = \sin^{-1}(\xi_j)$.

As a special case of the unsymmetric elliptic fracture, the compatible polynomial expression for the crack opening width profile corresponding to the effective pressure for a penny-shaped fracture, detailed in Appendix C, can be expressed as

$$W_1(t) = \frac{4(1-\nu)}{\pi\mu} P_O(t) R(t) \left[1 - \frac{2}{3} a(t) \right] \tag{3.30a}$$

$$W_2(t) = \frac{16(1-\nu)}{9\pi\mu} P_O(t) R(t) a(t) \tag{3.30b}$$

The boundary integral $K_A[w]$ for a penny-shaped fracture has the form

$$K_A[w] = \frac{\pi\mu}{4(1-\nu)} \frac{1}{R} \left(W_1 + \frac{3}{2} W_2 - \frac{9}{4} W_2 \rho^2 \right) \tag{3.31}$$

Therefore formation strain energy rate for a penny-shaped crack is written as

$$\dot{U}_S = \frac{\pi^2 \mu R}{2(1-\nu)} \left[\dot{W}_1 \left(\frac{1}{3} W_1 + \frac{1}{5} W_2 \right) + \dot{W}_2 \left(\frac{1}{5} W_1 + \frac{6}{35} W_2 \right) \right]$$

$$+ \frac{\pi^2 \dot{\mu} R}{2(1-\nu)} \left(\frac{2}{3} \dot{W}_1^2 + \frac{1}{5} \dot{W}_1 \dot{W}_2 + \frac{3}{35} \dot{W}_2^2 \right) \quad (3.32)$$

Potential energy rate function

$$\begin{aligned} \dot{U}_p &= \int_A \dot{w} \{ p - \sigma_o(\underline{x}) \} dA \\ &= \int_A \dot{w} (p - \sigma_o) dA - \sum \int_{A_j} \dot{w} \Delta \sigma_j dA \\ &= \int_A \dot{w} P_o \{ 1 - \hat{a}(\theta) \rho^2 \} dA - \sum \Delta \sigma_j \int_{A_j} \dot{w} dA \end{aligned} \quad (3.33)$$

Therefore the potential energy rate function for an unsymmetric fracture can be expressed as

$$\begin{aligned} \dot{U}_p &= \sum \left[\pi a b_j P_o \left(\frac{1}{3} \dot{W}_1 + \frac{1}{5} \dot{W}_2 \right) + \pi P_o \left(\frac{1}{3} \dot{W}_1 + \frac{1}{5} \dot{W}_2 \right) (\dot{a} b_j + a \dot{b}_j) \right. \\ &\quad - \pi a b_j P_o (\alpha_p + \alpha_j) \left(\frac{1}{15} \dot{W}_1 + \frac{1}{35} \dot{W}_2 \right) \\ &\quad - \pi P_o \{ \alpha_p (3 \dot{a} b_j + a \dot{b}_j) + \alpha_j (\dot{a} b_j + 3 a \dot{b}_j) \} \left(\frac{1}{15} \dot{W}_1 + \frac{1}{35} \dot{W}_2 \right) \\ &\quad - \Delta \sigma_j \left[\dot{W}_1 \left(\frac{\pi}{3} a b_j \right) R_1(\xi_j) + \dot{W}_2 \left(\frac{\pi}{5} a b_j \right) R_2(\xi_j) \right. \\ &\quad \left. \left. + 2 \dot{W}_1 \{ \dot{a} b_j g_1(\xi_j) + a \dot{b}_j g_2(\xi_j) \} + 6 \dot{W}_1 \{ \dot{a} b_j g_3(\xi_j) + a \dot{b}_j g_4(\xi_j) \} \right] \right] \\ &\quad j=u, \ell \end{aligned} \quad (3.34)$$

Griffith surface energy rate function

$$\begin{aligned} \frac{d}{dt} U_f &= \dot{U}_f = \frac{d}{dt} \int_{A(t)} G_{cr} dA \\ &= \frac{d}{dt} \left[G_{cr} A_o + \sum (G_{cr} + \Delta G_{cr}^j) A_j \right] \quad j=u, \ell \end{aligned} \quad (3.35)$$

where G_{cr} is the critical energy release rate of the

reservoir rock. The Griffith fracture surface energy rate for a three-layered formation

$$\dot{U}_f = \sum_{j=u, \ell} \left[\frac{\pi}{2} \dot{a} b_j \left\{ G_{cr} + \Delta G_{cr}^j [\Psi_1(b_j) - \Psi_2(b_j)] \right\} + \frac{\pi}{2} \dot{a} b_j \left\{ G_{cr} + \Delta G_{cr}^j [\Psi_1(b_j) + \Psi_2(b_j)] \right\} \right] \quad (3.36)$$

$$\text{where } \Psi_1(b_j) = \frac{2}{\pi} \sin^{-1} \left[1 - \left(\frac{h_j}{b_j} \right)^2 \right]^{1/2}$$

$$\text{and } \Psi_2(b_j) = \frac{2}{\pi} \frac{h_j}{b_j} \left[1 - \left(\frac{h_j}{b_j} \right)^2 \right]^{1/2}.$$

Fluid dissipation energy function

$$\begin{aligned} D_p &= - \frac{1}{1+m} \int_{A(t)} \partial p(\underline{x}, t) / \partial x_i q_i(\underline{x}, t) dA \quad (i=1, 2) \\ &= \frac{m}{1+m} (\bar{\eta})^{-\frac{1}{m}} \int_A w^{\frac{2m+1}{m}} \{ (\nabla p - \underline{f}_g) \cdot (\nabla p - \underline{f}_g) \}^{\frac{m+1}{2m}} dA \end{aligned} \quad (3.37)$$

where $\bar{\eta} = \eta_0 2^{m+1} \left(\frac{2m+1}{m} \right)^m$. From the relationship in equations (2.21), (3.16-17) and (3.37), the fluid pressure gradient dissipation energy function can be expressed as

$$\begin{aligned} D_p &= \frac{m}{1+m} D_\eta \\ &= - \frac{\bar{\eta}}{1+m} \int_A \frac{|q|^{1+m}}{w^{2m+1}} dA - \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) dA \end{aligned} \quad (3.38)$$

Since $p_j(\rho, \theta, t) = P_0(t) [1 - \hat{a}_j(t, \theta) \rho^2] + \sigma_0$ or

$$p_j(x, y, t) = P_0(t) [1 - a_p(x/a)^2 - a_j(y/b_j)^2] + \sigma_0,$$

the pressure gradient component in the dissipation energy function is given in the absence of gravitational body

force by

$$(\nabla_p \cdot \nabla_p)_j^{\frac{m+1}{2m}} = (2P_o)^{\frac{m+1}{m}} \left[\left(\alpha_p \frac{x}{a} \right)^2 + \left(\alpha_j \frac{y}{b_j} \right)^2 \right]^{\frac{m+1}{2m}} \quad \text{or}$$

$$(\nabla_p \cdot \nabla_p)_j^{\frac{m+1}{2m}} = (2P_o)^{\frac{m+1}{m}} \left[\left(\alpha_p \frac{\cos \theta}{a} \right)^2 + \left(\alpha_j \frac{\sin \theta}{b_j} \right)^2 \right]^{\frac{m+1}{2m}}$$

For an unsymmetric fracture, the pressure gradient dissipation energy function is

$$D_p = \sum \frac{m^2}{(1+m)(2m+1)} \eta_o^{-\frac{1}{m}} P_o^{\frac{m+1}{m}} N_o(a, b_j, W_1, W_2, m) \quad (3.39)$$

where

$$N_o(a, b_j, W_1, W_2, m) \equiv 2 \int_0^{\frac{\pi}{2}} \int_0^1 [\rho(1-\rho^2)]^{1/2} \{W_1 + W_2(1-\rho^2)\}^{\frac{2m+1}{m}} \left[\left(\alpha_p \frac{\cos \theta}{a} \right)^2 + \left(\alpha_j \frac{\sin \theta}{b_j} \right)^2 \right]^{\frac{m+1}{2m}} ab_j \, d\rho d\theta$$

Energy release rate integral function

$$\dot{U}_\Gamma = - \int_{\partial A} G \dot{a}_n \, dS \quad (3.40)$$

Since the energy release rate, G , can be rewritten in the form

$$G = \frac{(1-\nu)}{2\mu} K_I^2 \quad (3.41)$$

from the assumption that the crack tip experiences plane strain, mode I loading conditions, the fracture opening, $w(d)$, at a small distance d inside the fracture measured normal to the leading edge of the fracture can be expressed as

$$w(d) = \frac{4(1-\nu)}{\mu} K_I \sqrt{\frac{d}{2\pi}} \quad (3.42)$$

Near the edge of the fracture, the above equation yields

$$w \nabla w \cdot \underline{n} = - \frac{4(1-\nu)^2}{\pi \mu^2} K_I^2 \quad (3.43)$$

where $\underline{n}$ is a unit normal vector along the direction of fracture propagation.

The energy release rate integral in eqn (3.40) can be expressed as a function of the current crack configuration and displacement in the form

$$\dot{U}_I = \int_{\partial A} \left\{ \frac{\pi \mu}{8(1-\nu)} w \nabla w \cdot \underline{n} \right\} \dot{a}_n \, dS \quad (3.44)$$

A unit normal vector along the direction of fracture propagation in an elliptic fracture model is

$$\begin{aligned} \underline{n} &= (n_x, n_y) \\ &= \left(\frac{b_j \cos \theta}{\sqrt{a^2 \sin^2 \theta + b_j^2 \cos^2 \theta}}, \frac{a \sin \theta}{\sqrt{a^2 \sin^2 \theta + b_j^2 \cos^2 \theta}} \right) \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} w \nabla w \cdot \underline{n} &= w \left(\frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \right) \Big|_{\rho=1} \\ &= -w_1^2 \left(\frac{\cos \theta}{a} n_x + \frac{\sin \theta}{b_j} n_y \right) \quad j=u, \ell \end{aligned} \quad (3.46)$$

Fracture propagation velocity at the front edge, $\dot{a}_n$, normal to the elliptic fracture front is

$$\begin{aligned} \dot{a}_n &= \dot{a} \cos \theta n_x + \dot{b}_j \sin \theta n_y \\ \text{and } dS_j &= \sqrt{a^2 \sin^2 \theta + b_j^2 \cos^2 \theta} \, d\theta \quad j=u, \ell \end{aligned} \quad (3.47)$$

Therefore, the energy release rate integral function for an unsymmetric elliptic fracture can be rewritten as

$$\dot{U}_I = \sum \left[-\frac{\pi}{4} \frac{\mu}{(1-\nu)} w_1^2 \int_0^{\frac{\pi}{2}} \left(\frac{\dot{a}}{a} \cos^2 \theta + \frac{\dot{b}_j}{b_j} \sin^2 \theta \right) \frac{a \sqrt{\sin^2 \theta + (b_j/a)^2 \cos^2 \theta}}{d\theta} \right] \quad j=u, \ell \quad (3.48)$$

Energy rate associated with fluid injection

$$D_I = \int_{\partial A_q} p Q_n dA = -i_o (P_o + \sigma_o) \quad (3.49)$$

Dissipation energy rate due to fluid leak-off

$$\begin{aligned} D_L &= \int_A p q_L dA \\ &= \int_A \frac{2C_L(\underline{x})}{\{t-\tau(\rho)\}^\gamma} [P_o(1-\alpha\rho^2) + \sigma_o] dA \end{aligned} \quad (3.50)$$

where fluid leak-off rate, $q_L(\underline{x}, t) = \frac{2C_L(\underline{x})}{\{t-\tau(\rho)\}^\gamma}$.

Therefore

$$\begin{aligned} D_L &= \sum \left[4ab_j C_L \int_0^{\frac{\pi}{2}} \int_0^1 \frac{[P_o\{1-\hat{\alpha}_j(\theta)\rho^2\} + \sigma_o]}{\{t-\tau(\rho)\}^\gamma} \rho d\rho d\theta \right. \\ &\quad \left. + 4ab_j \Delta C_L^j \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \frac{[P_o\{1-\hat{\alpha}_j(\theta)\rho^2\} + \sigma_o]}{\{t-\tau(\rho)\}^\gamma} \rho d\rho d\theta \right] \\ &\quad j=u, \ell \end{aligned} \quad (3.51)$$

where $\theta_j^* = \sin^{-1}(\zeta_j)$ and $\zeta_j = \frac{h_j}{b_j \sin \theta}$.

3.3 Governing Equations and Solution Procedures

A functional in terms of pertinent energy rate components with nine generalized coordinates can be rewritten by plugging these energy rate components

detailed in the equations (3-27,33,35,38,40,49,50) into eqn (3.2) as

$$\begin{aligned} F &= F(\dot{W}_1, \dot{W}_2, \dot{a}, \dot{b}_u, \dot{b}_\ell, P, \alpha_p, \alpha_u, \alpha_\ell) \\ &= \dot{U}_s - \dot{U}_p - D_p - D_L - D_I + \dot{U}_\Gamma + \dot{U}_f \end{aligned} \quad (3.52)$$

A set of non-linear governing equations derived from a variational function will be solved numerically for the three-layered system discussed above. The generalized coordinates, Y_i , representing specialized fracture response variables, are the fracture half length ($a(t)$), upper height and lower height measured from the center of perforation ($b_u(t)$, $b_\ell(t)$), fracture opening width components ($W_1(t)$, $W_2(t)$), and the pressure profile parameters ($\alpha_p(t)$, $\alpha_u(t)$, $\alpha_\ell(t)$).

Taking the first variation of equation (3.52) after substituting the appropriate energy rate selected components, we obtain the following governing equations :

$$\begin{aligned} \delta \dot{W}_1: F(1) \equiv 0 &= \sum \left[\frac{E(k_j)}{2b_j} \left[(W_1 + \frac{3}{2}W_2) \left(\frac{\pi}{3}ab_j \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_1(\xi_j) \} \right. \right. \\ &\quad \left. \left. - \frac{9}{4} W_2 \frac{2\pi}{15}(ab_j) \{ \hat{\mu} + \Delta \hat{\mu}_j R_4(\xi_j) \} \right] \right. \\ &\quad \left. + \left[\pi ab_j P_o \left\{ \frac{1}{3} - \frac{1}{15}(\alpha_p + \alpha_j) \right\} + \Delta \sigma_j \left(\frac{\pi}{3}ab_j \right) R_1(\xi_j) \right] \right] \\ &\quad j=u, \ell \end{aligned} \quad (3.53)$$

$$\begin{aligned} \delta \dot{W}_2: F(2) \equiv 0 &= \sum \left[\frac{E(k_j)}{2b_j} \left[(W_1 + \frac{3}{2}W_2) \left(\frac{\pi}{5}ab_j \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_2(\xi_j) \} \right. \right. \\ &\quad \left. \left. - \frac{9}{4} W_2 \frac{2\pi}{35}(ab_j) \{ \hat{\mu} + \Delta \hat{\mu}_j R_5(\xi_j) \} \right] \right] \end{aligned}$$

$$+ \left[\pi a b_j P_o \left(\frac{1}{5} - \frac{1}{35} (\alpha_p + \alpha_j) \right) + \Delta \sigma_j \left(\frac{\pi a b_j}{5} \right) R_2(\xi_j) \right] \Bigg]_{j=u, \ell} \quad (3.54)$$

$$\begin{aligned} \dot{\delta a} : F(3) \equiv 0 = \sum \Bigg[& \frac{E(k_j)}{2b_j} (W_1 + \frac{3}{2}W_2) \left[W_1 \left(\frac{\pi b_j}{3} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_8(\xi_j) \} \right. \\ & \left. + W_2 \left(\frac{\pi b_j}{5} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_9(\xi_j) \} \right] \\ & - \frac{9}{4} W_2 \left[W_1 \left(\frac{4\pi b_j}{15} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_{10}(\xi_j) \} \right. \\ & \left. + W_2 \left(\frac{4\pi b_j}{35} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_{11}(\xi_j) \} \right] \\ & - \left[\pi P_o b_j \left(\frac{1}{3}W_1 + \frac{1}{5}W_2 \right) - \pi P_o b_j (3\alpha_p + \alpha_j) \left(\frac{1}{15}W_1 + \frac{1}{35}W_2 \right) \right. \\ & \left. - \Delta \sigma_j b_j \{ 2W_1 g_1(\xi_j) + 6W_1 g_3(\xi_j) \} \right] \\ & + \frac{\pi}{2} b_j \left\{ G_{cr} + \Delta G_{cr}^j \frac{2}{\pi} \left[\sin^{-1} \{ 1 - (h_j/b_j)^2 \}^{1/2} \right. \right. \\ & \left. \left. - (h_j/b_j) \{ 1 - (h_j/b_j)^2 \}^{1/2} \right] \right\} \\ & - \frac{\pi}{4} \frac{\mu}{(1-\nu)} W_1^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta \sqrt{\sin^2 \theta + (b_j/a)^2 \cos^2 \theta} d\theta \Bigg]_{j=u, \ell} \quad (3.55) \end{aligned}$$

$$\begin{aligned} \dot{\delta b_j} : F(i) \equiv 0 = \sum \Bigg[& \frac{E(k_j)}{2b_j} (W_1 + \frac{3}{2}W_2) \left[W_1 \left(\frac{\pi a}{3} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_8(\xi_j) \} \right. \\ & \left. + W_2 \left(\frac{\pi a}{5} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_9(\xi_j) \} \right] \\ & - \frac{9}{4} W_2 \left[W_1 \left(\frac{4\pi a}{15} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_{10}(\xi_j) \} \right. \\ & \left. + W_2 \left(\frac{4\pi a}{35} \right) \{ \hat{\mu} + \Delta \hat{\mu}_j R_{11}(\xi_j) \} \right] \\ & - \left[\pi P_o a \left(\frac{1}{3}W_1 + \frac{1}{5}W_2 \right) - \pi P_o a (\alpha_p + 3\alpha_j) \left(\frac{1}{15}W_1 + \frac{1}{35}W_2 \right) \right. \end{aligned}$$

$$\begin{aligned}
& - \Delta \sigma_j a \{ 2W_1 g_1(\xi_j) + 6W_1 g_3(\xi_j) \}] \\
& + \frac{\pi}{2} a \left\{ G_{cr} + \Delta G_{cr}^j \frac{2}{\pi} [\sin^{-1} \{ 1 - (h_j/b_j)^2 \}^{1/2} \right. \\
& \quad \left. + (h_j/b_j) \{ 1 - (h_j/b_j)^2 \}^{1/2}] \right\} \\
& - \frac{\pi}{4} \frac{\mu}{(1-\nu)} W_1^2 \left(\frac{a}{b_j} \right) \int_0^{\frac{\pi}{2}} \sin^2 \theta \sqrt{\sin^2 \theta + (b_j/a)^2 \cos^2 \theta} d\theta \Bigg]
\end{aligned}$$

$$i=4 : j=u : \quad (3.56)$$

$$i=5 : j=\ell : \quad (3.57)$$

$$\begin{aligned}
\delta P_o : \sum & \left[[\pi a b_j (\frac{1}{3}\dot{W}_1 + \frac{1}{5}\dot{W}_2) + \pi (\dot{a}b_j + a\dot{b}_j) (\frac{1}{3}W_1 + \frac{1}{5}W_2) \right. \\
& - \pi a b_j (\alpha_p + \alpha_j) (\frac{1}{15}\dot{W}_1 + \frac{1}{35}\dot{W}_2) \\
& - \pi \{ (3\alpha_p + \alpha_j) \dot{a}b_j + (\alpha_p + 3\alpha_j) a\dot{b}_j \} (\frac{1}{15}W_1 + \frac{1}{35}W_2)] \\
& + \frac{m}{(2m+1)} \eta_o^{-\frac{1}{m}} P_o^{\frac{1}{m}} N_o(a, b_j, W_1, W_2, m) \\
& + [2\pi a b_j C_L \int_0^1 \frac{\{ 1 - \frac{1}{2}(\alpha_p + \alpha_j) \rho^2 \}}{\{ t - \tau(\rho) \}^\gamma} \rho d\rho \\
& \quad + 4a b_j \Delta C_L^j \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \frac{\{ 1 - \hat{a}_j(\theta) \rho^2 \}}{\{ t - \tau(\rho) \}^\gamma} \rho d\rho d\theta] \Bigg] \\
& - i_o = 0 \qquad \qquad \qquad j=u, \ell \qquad \qquad (3.58)
\end{aligned}$$

$$\begin{aligned}
\delta \alpha_p : F(7) \equiv 0 = \sum & \left[[\pi a b_j (\frac{1}{15}\dot{W}_1 + \frac{1}{35}\dot{W}_2) \right. \\
& + \pi (3\dot{a}b_j + a\dot{b}_j) (\frac{1}{15}W_1 + \frac{1}{35}W_2)] \\
& - \frac{m}{(2m+1)} \eta_o^{-\frac{1}{m}} P_o^{\frac{1}{m}} \alpha_p I_p(a, b_j, W_1, W_2, m)
\end{aligned}$$

$$\begin{aligned}
& + \left[\pi a b_j C_L \int_0^1 \frac{a_p \rho^3}{\{t - \tau(\rho)\}^\gamma} d\rho \right. \\
& \left. + 4 a b_j \Delta C_L^j \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \frac{\cos^2 \theta \rho^3}{\{t - \tau(\rho)\}^\gamma} d\rho d\theta \right] \quad j=u, \ell \quad (3.59)
\end{aligned}$$

where

$$\begin{aligned}
I_p(a, b_j, w_1, w_2, m) & \equiv 2 \int_0^{\frac{\pi}{2}} \int_0^1 [\rho(1-\rho^2)^{1/2} \{w_1 + w_2(1-\rho^2)\}]^{\frac{2m+1}{m}} \\
& \quad \left(\frac{\cos \theta}{a}\right)^2 \left[\left(a_p \frac{\cos \theta}{a}\right)^2 + \left(a_j \frac{\sin \theta}{b_j}\right)^2 \right]^{\frac{1-m}{2m}} a b_j d\rho d\theta
\end{aligned}$$

$$\begin{aligned}
\delta a_j : F(k) & \equiv 0 = \left[\pi a b_j P_O \left(\frac{1}{15} \dot{w}_1 + \frac{1}{35} \dot{w}_2 \right) \right. \\
& + \pi (3 \dot{a} b_j + a \dot{b}_j) \left(\frac{1}{15} w_1 + \frac{1}{35} w_2 \right) \left. \right] \\
& - \frac{m}{(2m+1)} \eta_O^{-\frac{1}{m}} P_O^{\frac{1}{m}} a_j I_j(a, b_j, w_1, w_2, m) \\
& + \left[\pi a b_j C_L \int_0^1 \frac{a_j \rho^3}{\{t - \tau(\rho)\}^\gamma} d\rho \right. \\
& \left. + 4 a b_j \Delta C_L^j \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \frac{\sin^2 \theta \rho^3}{\{t - \tau(\rho)\}^\gamma} d\rho d\theta \right] \quad j=u : k=8 : (3.60)
\end{aligned}$$

$$j=\ell : k=9 : (3.61)$$

where

$$\begin{aligned}
I_j(a, b_j, w_1, w_2, m) & \equiv 2 \int_0^{\frac{\pi}{2}} \int_0^1 [\rho(1-\rho^2)^{1/2} \{w_1 + w_2(1-\rho^2)\}]^{\frac{2m+1}{m}} \\
& \quad \left(\frac{\sin \theta}{b_j}\right)^2 \left[\left(a_p \frac{\cos \theta}{a}\right)^2 + \left(a_j \frac{\sin \theta}{b_j}\right)^2 \right]^{\frac{1-m}{2m}} a b_j d\rho d\theta
\end{aligned}$$

The mass balance equation can be derived by combining equations (3.58-61) as

$$\sum \left[\frac{d}{dt} [\pi ab_j (\frac{1}{3}w_1 + \frac{1}{5}w_2)] + 2\pi ab_j c_L \int_0^1 \frac{\rho}{\{t-\tau(\rho)\}^\gamma} d\rho + 4ab_j \Delta c_L^j \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \frac{\rho}{\{t-\tau(\rho)\}^\gamma} d\rho d\theta \right] - i_o = 0 \quad (3.62)$$

The ninth equation is obtained from eqn (3.62) by taking the time increment, Δt , at a certain time, is

$$F(6) \equiv 0 = \sum \left[\{ \pi ab_j (\frac{1}{3}w_1 + \frac{1}{5}w_2) - v_{fp}^j \} + [\frac{2\pi}{1-\gamma} ab_j c_L \int_0^1 \{t-\tau(\rho)\}^{1-\gamma} \rho d\rho - v_{Lp}^j] + \{ \frac{4}{1-\gamma} ab_j \Delta c_L^j J_Y(\xi_j) - v_{dlp}^j \} \right] - i_o \Delta t \quad j=u, \ell \quad (3.63)$$

where $J_Y(\xi_j) = \int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\zeta_j}^1 \{t-\tau(\rho)\}^{1-\gamma} \rho d\rho d\theta$

$$v_{fp}^j = \pi ab_j \left(\frac{1}{3}w_1^* + \frac{1}{5}w_2^* \right)$$

$$v_{Lp}^j = \frac{2\pi}{1-\gamma} a^* b_j^* c_L \int_0^1 \{t-\tau(\rho)\}^{-\gamma} \rho d\rho$$

$$v_{dlp}^j = \frac{4}{1-\gamma} a^* b_j^* \Delta c_L^j J_Y(\xi_j^*) .$$

The quantities, w_1^* , w_2^* , a^* , b_j^* , and ξ_j^* , denote the solution values at the previous step during the numerical calculations.

Solution Procedures

The numerical solution scheme is the same as in the previous chapter with a difference in the selected generalized coordinates. The unknown variables for the unsymmetric elliptical fracture model, $a(t)$, $b_u(t)$, $b_\ell(t)$, $w_1(t)$, $w_2(t)$, $\alpha_p(t)$, $\alpha_u(t)$, $\alpha_\ell(t)$, and $P_o(t)$, are obtained by satisfying the nine governing non-linear differential equations (3.53-57,59-61,63) simultaneously using the same convergency criterion in Section 2.4.

3.4 Special Cases

In the absence of differential terms and leak-off, solutions obtained from the nine governing differential equations are identical to the time-explicit forms in Appendix A derived from Lagrange's method [66]. It is noteworthy that the fracture pressure profile in this approach varies during the simulation while that in the Lagrangian formulation remains constant. Some special cases can be obtained from the formulation of unsymmetric elliptic model in three-layered formations as follows :

i) Penny-Shaped Fracture without Leak-off

For a penny-shaped fracture without leak-off, isotropic formation properties are prescribed as

$$\Delta\mu_j = \Delta\sigma_j = \Delta G_{cr}^j = \Delta C_L^j = \Delta C_S^j = 0, \text{ and } C_L = 0$$

The solutions corresponding to negligible fracture energy ($G_{cr}=0$) can be obtained from the principal hydraulic fracturing response variables

$$\begin{aligned} \alpha_p &= \alpha_u = \alpha_\ell = 3/2, \\ W_1 &= 0, \quad W_2 = \frac{8(1-\nu)}{3\pi\mu} P_o R \\ i_o &= \frac{d}{dt} \left(\frac{2\pi}{5} W_2 R^2 \right) \end{aligned} \quad (3.64a)$$

in the form

$$R(t) = C_1(m) \left[\frac{i_o^{m+2} \hat{\mu}}{\eta_o} \right]^{\frac{1}{3(m+2)}} t^{\frac{2(m+1)}{3(m+2)}} \quad (3.64b)$$

$$W(t) = W_2(t) = C_2(m) \left[\frac{i_o^{m+2} \eta_o^2}{\hat{\mu}^2} \right]^{\frac{1}{3(m+2)}} t^{\frac{2-m}{3(m+2)}} \quad (3.64c)$$

$$P_o(t) = C_3(m) \left[\frac{\hat{\mu}^{m+1} \eta_o}{\mu^2} \right]^{\frac{1}{m+2}} t^{\frac{-m}{m+2}} \quad (3.64d)$$

where $\hat{\mu}$ is the generalized elastic modulus ($\hat{\mu} = \mu/(1-\nu)$) and $C_i(m)$ ($i=1,2,3$) are the constants in Appendix A.3.

When the fracture fluid dissipation can be neglected ($\eta_o=0$), the principal hydraulic fracturing response variables and corresponding solutions are given by

$$\begin{aligned} \alpha_p &= \alpha_u = \alpha_\ell = 0 \\ W_1 &= \frac{4(1-\nu)}{\pi\mu} P_o R, \quad W_2 = 0, \\ i_o &= \frac{d}{dt} \left(\frac{2\pi}{3} W_1 R^2 \right) \end{aligned} \quad (3.65a)$$

$$R(t) = 0.617 \left[\frac{\hat{\mu} i_o^2}{G_{cr}} \right]^{1/5} t^{2/5} \quad (3.65b)$$

$$W(t) = W_1(t) = 1.254 \left[\frac{G_{cr}^2 i_o}{\hat{\mu}^2} \right]^{1/5} t^{1/5} \quad (3.65c)$$

$$P_o(t) = 1.595 \left[\frac{\hat{\mu}^2 G_{cr}^3}{i_o} \right]^{1/5} t^{-1/5} \quad (3.65d)$$

The response variables and corresponding solutions are the same as those obtained from the Lagrangian formulation detailed in Appendix A.1.

ii) **PKN Model without Leak-off**

For PKN cases without leak-off, $b = H/2$, $\dot{b} = 0$ and $C_L = 0$. The solutions corresponding to negligible fracture energy can be obtained by the similar manner in the previous cases for a penny-shaped model. The response variables and corresponding solutions are

$$\begin{aligned} \alpha_p &= 3/2, & \alpha_u &= \alpha_\ell = 0 \\ W_1 &= 0, & W_2 &= \frac{2}{3} \frac{(1-\nu)}{\mu E(k)} P_o H \\ i_o &= \frac{d}{dt} \left(\frac{\pi}{5} a H W_2 \right) \end{aligned} \quad (3.66a)$$

$$a(t) = A_1(m) \left[\frac{\hat{\mu} i_o^{m+2}}{\eta_o H^{m+3}} \right] \frac{1}{2m+3} t^{\frac{2m+2}{2m+3}} \quad (3.66b)$$

$$W(t) = W_2(t) = A_2(m) \left[\frac{i_o^{m+1} \eta_o}{H^m \hat{\mu}} \right] \frac{1}{2m+3} t^{\frac{1}{2m+3}} \quad (3.66c)$$

$$P_o(t) = A_3(m) \left[\frac{\hat{\mu}^{2m+2} i_o^{m+2} \eta_o}{H^{3m+3}} \right] \frac{1}{2m+3} t^{\frac{1}{2m+3}} \quad (3.66d)$$

where $A_i(m)$ ($i=1,2,3$) are the constants for dissipation dominant solutions detailed in Appendix A.3.

When the fracture fluid dissipation can be neglected, the principal hydraulic fracturing response variables and corresponding solutions are in the form

$$\alpha_p = \alpha_u = \alpha_\ell = 0$$

$$W_1 = \frac{(1-\nu)}{\mu E(k)} P_0 H, \quad W_2 = 0,$$

$$i_o = \frac{d}{dt} \left(\frac{\pi}{3} a H W_1 \right) \quad (3.67a)$$

$$a(t) = 0.551 \left[\frac{\hat{\mu} i_o^2}{G_{cr} H^3} \right]^{1/2} t \quad (3.67b)$$

$$W(t) = W_1(t) = 1.732 \left[\frac{H G_{cr}}{\hat{\mu}} \right]^{1/2} \quad (3.67c)$$

$$P_0(t) = 1.732 \left[\frac{\hat{\mu} G_{cr}}{H} \right]^{1/2} \quad (3.67d)$$

iii) **General Case of A Penny-Shaped Fracture**

$$\text{For } \Delta\mu_j = \Delta\sigma_j = \Delta G_{cr}^j = \Delta C_L^j = \Delta C_S^j = 0,$$

$$\alpha_p = \alpha_u = \alpha_\ell \quad \text{and} \quad R = a = b_u = b_\ell \quad (3.68)$$

iv) **Symmetric Three-Layered Formation Case**

$$\text{For } \Delta\mu_u = \Delta\mu_\ell, \Delta\sigma_u = \Delta\sigma_\ell, \Delta G_{cr}^u = \Delta G_{cr}^\ell,$$

$$\Delta C_L^u = \Delta C_L^\ell, \text{ and } \Delta C_S^u = \Delta C_S^\ell,$$

$$\alpha_u = \alpha_\ell \quad \text{and} \quad b_u = b_\ell \quad (3.69)$$

v) **Unsymmetric Three-Layered formation Case**

$$\text{For } \Delta\mu_u \neq \Delta\mu_\ell, \Delta\sigma_u \neq \Delta\sigma_\ell, \Delta G_{cr}^u \neq \Delta G_{cr}^\ell,$$

$$\Delta C_L^u \neq \Delta C_L^\ell, \text{ and } \Delta C_S^u \neq \Delta C_S^\ell,$$

$$\alpha_u \neq \alpha_\ell \quad \text{and} \quad b_u \neq b_\ell \quad (3.70)$$

CHAPTER IV
ANALYSIS OF FLUID LAG PHENOMENON
IN HYDRAULIC FRACTURE PROPAGATION

4.1 Introduction

The phenomenon of fracturing fluid lag refers to a lag between the fluid front and the fracture tip. It has been shown that hydraulic fractures often propagate even when the fracturing fluid does not reach the tip of the fracture through laboratory tests by Biot [45], field measurement data presented by Warpinski [71], and numerical solutions by Wiles et al. [72], Nilson [73], and Wong et al. [37]. The combined effect of fluid lag and fracture toughness on hydraulic fracture propagation by using the weight function method has been reported by Jeffrey [74]. Hydraulic fracture phenomena incorporating the effects of fluid lag can be analyzed by modifying the variational formulations, detailed in chapter III. The locations of fracturing fluid as well as fracture front and corresponding pressure distribution of injection fluid during the fracturing process are predicted for a penny-shaped fracture model based on the variational principle, introduced in this chapter. The role of the

effective fracture tip closure stress and fracture fluid pressure distribution is investigated in chapter V.

4.2 Formulation of an Energy Rate Functional

The energy rate functional in equation (3.1) can be modified to analyze fluid lag phenomena in a hydraulically induced fracture. Fig. 4.1 shows a hydraulic fracture with fluid lag, where the volumes of the reservoir and the fluid (occupying a part of the fracture) are denoted by $\Omega_s(t)$ and $\Omega_f(t)$, respectively. The entire fracture surface area, the surface area filled by fracture fluid, and the near tip fluid-lag region are designated by $A_s(t)$, $A_f(t)$, and $A_d(t)$, respectively.

The energy rate functional in eqn.(3.1) with additional generalized coordinates and contributions from the fluid lag can be written as

$$\begin{aligned}
 F(\underline{v}_s, \underline{v}_f, p, \dot{a}_n, \dot{b}_n) = & \frac{d}{dt} \int_{\Omega_s} e \, dV + \int_{\Omega_f} \psi \, dV \\
 & + \int_{\Omega_f} v_{fi} (p_i - f_{gi}) \, dV - 2 \int_{A_f} p \, v_L \, dA \\
 & - 2 \int_{A_f} p \, v_{si} n_i \, dA + \frac{d}{dt} \int_{A_s} G_{cr} \, dA - \int_{\partial A_q} p \, v_{fn} \, dA \\
 & - 2 \int_{A_d} p_w \, v_{si} n_i \, dA - \int_{A_w} (p - p_w) \, \dot{b}_n \, dA \quad (4.1)
 \end{aligned}$$

where $\dot{a}_n$ and $\dot{b}_n$ represent the normal crack front and fluid front propagation velocity, respectively. The fluid-lag region, $A_d(t)$, is assumed to be at reservoir pressure, p_w ,

as shown in Fig. 4.1.

Taking the first variation of the functional (eq.4.1) we have

$$\begin{aligned}
 \delta F = & \int_{\Omega_s} \hat{\sigma}_{ij} \delta \dot{\epsilon}_{ij} dv + \int_{\Omega_f} \frac{\partial \psi}{\partial L_{ij}} \delta L_{ij} dv \\
 & - \delta \int_{\Omega_f} (p_{,i} - \rho f_i) v_{fi} dv + 2 \delta \int_{A_f} (v_L + v_{fi} n_i) p dA \\
 & + \int_{\partial A_s} (G_{cr} - G) \delta \dot{a}_n dS + \int_{\partial A_q} v_{fn} p dA \\
 & - 2 \delta \int_{A_d} p_w v_{si} n_i dA - \delta \int_{A_w} (p - p_w) \dot{b}_n dA = 0 \quad (4.2)
 \end{aligned}$$

Using Green's theorem, the following identities can be derived as

$$\begin{aligned}
 \int_{\Omega_s} \sigma_{ij} \delta \dot{\epsilon}_{ij} dv &= \int_{\partial \Omega_s} \sigma_{ij} n_j \delta v_{si} dA - \int_{\Omega_s} \sigma_{ij,j} \delta v_{si} dv \\
 \int_{\Omega_f} \tau_{ij} \delta L_{ij} dv &= \int_{\partial \Omega_f} \tau_{ij} n_j \delta v_{fi} dA - \int_{\Omega_f} \tau_{ij,j} \delta v_{fi} dv \\
 \int_{\Omega_f} \delta p_{,i} v_{fi} dv &= \int_{\partial \Omega_f} v_{fi} n_i \delta p dv - \int_{\Omega_f} v_{fi,i} \delta p dA
 \end{aligned} \quad (4.3)$$

The boundary areas for the crack surfaces and fluid surfaces are represented by

$$\begin{aligned}
 |\partial \Omega_f| &= 2|A_f| + |A_w| + |\partial A_q| \\
 \text{and } |\partial \Omega_s| &= 2|A_s| = 2|A_f| + 2|A_d|
 \end{aligned} \quad (4.4)$$

Therefore equation (4.2) can now be rewritten in the form

$$\begin{aligned}
 \delta F = & - \int_{\Omega_s} \sigma_{ij} \delta v_{si} dv + \int_{\Omega_f} (p_{,i} - \rho f_i - \tau_{ij,j}) \delta v_{fi} dv \\
 & - \int_{\Omega_f} v_{fi,i} \delta p dv + \int_{\partial A_s} (G_{cr} - G) \delta \dot{a}_n dS
 \end{aligned}$$

Fig. 4.1 Hydraulic Fracturing with Fluid Lag

$$\begin{aligned}
& + 2 \int_{A_f} (v_{fi} n_i - v_{si} n_i - v_L) \delta p \, dA \\
& + 2 \int_{A_f} (\tau_{ij} n_j - p n_i) \delta v_{fi} \, dA + 2 \int_{A_s} (\hat{\sigma}_{ij} \hat{n}_j + \sigma_{ij}^0 \hat{n}_j) \delta v_{si} \, dA \\
& - 2 \int_{A_d} p_w n_i \delta v_{si} \, dA + \int_{\partial A_q} \tau_{ij} n_j \delta v_{fi} \, dA \\
& - \int_{\partial A_q} (v_{fn} - v_{fi} n_i) \delta p \, dA - \int_{A_w} (p - p_w) \delta \dot{b}_n \, dA \\
& + \int_{A_w} \tau_{ij} n_j \delta v_{fi} \, dA + \int_{A_w} (v_{fi} n_i - \dot{b}_n) \delta p \, dA = 0 \quad (4.5)
\end{aligned}$$

where $\hat{n}$ and $\underline{n}$ are the unit vector normal to crack surfaces and fluid surfaces, respectively. We assume that the values of $\tau_{ij} n_j$ are negligible along the surface A and at the injection boundary, ∂A_q . The resulting Euler equations are obtained by setting the first variation of eq.(4.5) to zero, and these governing equations are :

Equilibrium equations for the reservoir

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega_s \quad (4.6)$$

Fluid linear momentum equations for the fracturing fluid

$$-p_{,i} + \tau_{ij,j} + f_{gi} = 0 \quad \text{in } \Omega_f \quad (4.7)$$

Fracturing fluid incompressibility equation

$$v_{fi,i} = 0 \quad \text{in } \Omega_f \quad (4.8)$$

Reservoir fracture propagation criterion

$$G = G_{cr} \quad \text{on } \partial A_s \quad (4.9)$$

Injection flow condition for the fracturing fluid

$$v_{fi} n_i = v_{fn} \quad \text{on } \partial A_q \quad (4.10)$$

The fracturing fluid and reservoir velocity relations on the fluid surface are

$$v_{fi} n_i = v_{si} n_i + v_L \quad \text{on } A_f \quad (4.11)$$

$$v_{fi} n_i = \dot{b}_n \quad \text{on } A_w \quad (4.12)$$

We consider the boundary conditions from eq.(4.5) given by

$$\begin{aligned} & \int_{A_f} (\hat{\sigma}_{ij} \hat{n}_j + \sigma_{ij}^o \hat{n}_j) \delta v_{si} dA + \int_{A_f} (\tau_{ij} n_j - p n_i) \delta v_{fi} dA \\ & + \int_{A_d} (\hat{\sigma}_{ij} \hat{n}_j + \sigma_{ij}^o \hat{n}_j) \delta v_{si} dA - \int_{A_d} p_w n_i \delta v_{si} dA \\ & - \int_{A_w} (p - p_w) \delta \dot{b}_n dA + \int_{A_w} \tau_{ij} n_j \delta v_{fi} dA = 0 \end{aligned} \quad (4.13)$$

Since $v_{fi} = v_{si}$ and $\hat{n}_i = -n_i$ in the fluid filled region, A_f , the traction boundary conditions are obtained from equation (4.13) are

$$\sigma_{ij} n_j + (p - \sigma_o) n_i - \tau_{ij} n_j = 0 \quad \text{on } A_f \quad (4.14)$$

$$\sigma_{ij} n_j + (p_w - \sigma_o) n_i = 0 \quad \text{on } A_d \quad (4.15)$$

$$(p - p_w) n_i - \tau_{ij} n_j = 0 \quad \text{on } A_w \quad (4.16)$$

The stationary value of the functional given by eq. (4.1) yields the necessary field equations and boundary conditions for investigating the overall hydraulic fracture responses including fluid lag.

The functional can be further simplified using the identical assumptions for the derivation of equation (3.2) from (3.1) in terms of five generalized coordinates as follows :

$$\begin{aligned}
F(\dot{w}, \underline{q}, p, \dot{a}_n, \dot{b}_n) = & \int_{A_s} \dot{w} \{K_A[w] + \sigma_o\} dA + \frac{\bar{\eta}}{1+m} \int_{A_f} \frac{|\underline{q}|^{1+m}}{w^{2m+1}} dA \\
& + \int_{A_f} \underline{q} \cdot (\nabla p - \underline{f}_g) dA + \int_{\partial A_s} \dot{a}_n \left\{ \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} + G_{cr} \right\} dS \\
& - \int_{A_f} p q_L dA - \int_{\partial A_q} p Q_n dS - \int_{A_f} \dot{w} p dA - \int_{A_d} \dot{w} p_w dA \\
& - \int_{A_w} (p - p_w) \dot{b}_n dA \quad (4.17)
\end{aligned}$$

After eliminating the flux, $\underline{q}$, from the generalized coordinates, a simplified form of the functional can be derived as

$$\begin{aligned}
F(\dot{w}, p, \dot{a}_n, \dot{b}_n) = & \int_{A_f} \dot{w} \{K_A[w] - p + \sigma_o\} dA - \int_{A_f} p q_L dA \\
& + \int_{A_d} \dot{w} \{K_A[w] - p_w + \sigma_o\} dA - \int_{A_w} (p - p_w) \dot{b}_n dA \\
& + \int_{\partial A_s} \dot{a}_n \left\{ \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} + G_{cr} \right\} dS - \int_{\partial A_q} p Q_n dS \\
& - \frac{m}{1+m} (\bar{\eta})^{\frac{-1}{m}} \int_{A_f} w^{\frac{2m+1}{m}} \{(\nabla p - \underline{f}_g) \cdot (\nabla p - \underline{f}_g)\}^{\frac{1+m}{2m}} dA \quad (4.18)
\end{aligned}$$

Taking the first variation on equation (4.18),

$$\begin{aligned}
\delta F(\dot{w}, p, \dot{a}_n, \dot{b}_n) = & \int_{A_f} \delta \dot{w} \{K_A[w] - p + \sigma_o\} dA - \int_{A_f} \dot{w} \delta p dA \\
& + \int_{A_d} \delta \dot{w} \{K_A[w] - p_w + \sigma_o\} dA - \int_{A_f} q_L \delta p dA \\
& - \int_{\partial A_q} Q_n \delta p dS + \int_{\partial A_s} \dot{a}_n \left\{ \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} + G_{cr} \right\} dS \\
& - \int_{A_w} (p - p_w) \delta \dot{b}_n dA + \int_{A_w} \dot{b}_n \delta p dA \\
& + (\bar{\eta})^{\frac{-1}{m}} \int_{A_f} \nabla \cdot \left\{ w^{\frac{2m+1}{m}} |\nabla p - \underline{f}_g|^{\frac{1-m}{m}} (\nabla p - \underline{f}_g) \right\} \delta p dA
\end{aligned}$$

$$- (\bar{\eta})^{\frac{-1}{m}} \int_{\partial A_q + \partial A_w} w^{\frac{2m+1}{m}} |\nabla p - \underline{f}_g|^{\frac{1-m}{m}} (\nabla p - \underline{f}_g) \cdot \underline{n} \, \delta p \, dS \quad (4.19)$$

Since $\underline{q} = (\bar{\eta})^{\frac{-1}{m}} w^{\frac{2m+1}{m}} |\nabla p - \underline{f}_g|^{\frac{1-m}{m}} (\nabla p - \underline{f}_g)$,

$$\begin{aligned} (\bar{\eta})^{\frac{-1}{m}} \int_{\partial A_w} w^{\frac{2m+1}{m}} |\nabla p - \underline{f}_g|^{\frac{1-m}{m}} (\nabla p - \underline{f}_g) \cdot \underline{n} \, \delta p \, dS &= \int_{\partial A_w} \underline{q} \cdot \underline{n} \, \delta p \, dS \\ &= \int_{A_w} \underline{v}_f \cdot \underline{n} \, \delta p \, dA \end{aligned} \quad (4.20)$$

The Euler equations and boundary conditions, resulting from (4.19) after applying the variational procedure and divergence theorem, are

$$\delta \dot{w}: K_A[w] = p - \sigma_0 \quad \text{in } A_f \quad (4.21a)$$

$$K_A[w] = p_w - \sigma_0 \quad \text{in } A_d \quad (4.21b)$$

$$\begin{aligned} \delta p: & - (\bar{\eta})^{-1/m} \nabla \cdot \{ w^{(2+1/m)} |\nabla p - \underline{f}_g|^{(1-m)/m} (\nabla p - \underline{f}_g) \} \\ & + \dot{w} + q_L = 0 \quad \text{in } A_f \end{aligned} \quad (4.22)$$

$$\begin{aligned} Q_n &= - (\bar{\eta})^{-1/m} \{ w^{(2+1/m)} |\nabla p - \underline{f}_g|^{(1-m)/m} (\nabla p - \underline{f}_g) \} \cdot \underline{n} \\ & \quad \text{along } \partial A_q \end{aligned} \quad (4.23)$$

$$\dot{b}_n = \underline{v}_f \cdot \underline{n} \quad \text{in } A_w \quad (4.24)$$

$$\delta \dot{a}_n: G_{cr} = - \frac{\pi \mu}{8(1-\nu)} w \nabla w \cdot \underline{n} \quad \text{along } \partial A \quad (4.25)$$

$$\delta \dot{b}_n: p = p_w \quad \text{in } A_w \quad (4.26)$$

The normal fracture front and fluid propagation velocities for a penny-shaped fracture with fluid lag are represented by $\dot{R}(\dot{a}_n)$ and $\dot{R}_f(\dot{b}_n)$, respectively.

4.3 Fluid Lag in a Penny-Shaped Hydraulic Fracture

Using the same crack opening width profile as in chapter III, the pertinent energy rate components for a penny-shaped fracture with fluid lag effects can be written as follows :

$$\begin{aligned}\dot{U}_s = \int_{A_s} \dot{w} K_A[w] dA &= \frac{\pi^2 \mu R}{2(1-\nu)} \left\{ \dot{w}_1 \left(\frac{1}{3} w_1 + \frac{1}{5} w_2 \right) + \dot{w}_2 \left(\frac{1}{5} w_1 + \frac{6}{35} w_2 \right) \right\} \\ &+ \frac{\pi^2 \mu R}{2(1-\nu)} \left(\frac{2}{3} w_1^2 + \frac{1}{5} w_1 w_2 + \frac{3}{35} w_2^2 \right)\end{aligned}\quad (4.27)$$

For a penny-shaped fracture model with fluid lag, the formation strain energy rate function ($\dot{U}_p$) is

$$\begin{aligned}\dot{U}_p &= \int_{A_f} \dot{w}(p - \sigma_o) dA + \int_{A_d} \dot{w}(p_w - \sigma_o) dA \\ &+ \int_{A_w} \dot{R}_f(p - p_w) dA\end{aligned}\quad (4.28)$$

where

$$\begin{aligned}\int_{A_f} \dot{w}(p - \sigma_o) dA &= \int_{A_f} \dot{w} P(1 - a\rho^2) dA \\ &= 2\pi R P \left[\dot{w}_1 R \{M_1(\beta) - aM_4(\beta)\} + \dot{w}_2 R \{M_2(\beta) - aM_5(\beta)\} \right. \\ &\quad \left. + w_1 \dot{R} \{M_3(\beta) - aM_6(\beta)\} + 3w_2 \dot{R} \{M_4(\beta) - aM_7(\beta)\} \right], \\ \int_{A_d} \dot{w}(p_w - \sigma_o) dA &= 2\pi R (p_w - \sigma_o) \left[\dot{w}_1 R \left\{ \frac{1}{3} - M_1(\beta) \right\} \right. \\ &\quad \left. + \dot{w}_2 R \left\{ \frac{1}{5} - M_2(\beta) \right\} + w_1 \dot{R} \left\{ \frac{2}{3} - M_3(\beta) \right\} + 3w_2 \dot{R} \left\{ \frac{2}{15} - M_4(\beta) \right\} \right],\end{aligned}$$

$$\text{and } \int_{A_w} \dot{R}_f (p - p_w) dA = 2\pi R_f \dot{R}_f \{p (1 - \alpha\beta^2) + (\sigma_o - p_w)\} [W_1(1-\beta^2)^{1/2} + W_2(1-\beta^2)^{3/2}] .$$

where the quantities, $M_i(\beta)$ are detailed in Appendix D.

The fluid dissipation energy rate (D_p) is

$$\begin{aligned} D_p &= \frac{m}{1+m} (\bar{\eta})^{-1/m} \int_{A_f} w^{\frac{2m+1}{m}} \{(\nabla p - \underline{f}_g) \cdot (\nabla p - \underline{f}_g)\}^{\frac{m+1}{2m}} dA \\ &= \frac{2\pi m^2}{(1+m)(2m+1)} (\eta_o)^{-1/m} R^{(m-1)/m} (\alpha P)^{(m+1)/m} I_{\beta}^*(W_1, W_2, m, \beta) \end{aligned} \quad (4.29)$$

where

$$I_{\beta}^*(W_1, W_2, m, \beta) = \int_0^{\beta} [\rho(1-\rho^2)^{1/2} \{W_1 + W_2(1-\rho^2)\}]^{\frac{2m+1}{m}} d\rho .$$

Additional energy rate components in eq.(4.18) for a penny shaped crack can be obtained in the form

$$\begin{aligned} \dot{U}_{\Gamma} &= \int_{\partial A_s} \dot{a}_n \left\{ \frac{\pi\mu}{8(1-\nu)} w \nabla w \cdot \underline{n} \right\} dS \\ &= - \frac{\pi^2 \mu}{4(1-\nu)} \dot{R} W_1^2 \end{aligned} \quad (4.30)$$

$$\dot{U}_f = \int_{\partial A_s} G_{cr} \dot{a}_n dS = 2\pi G_{cr} \dot{R} R \quad (4.31)$$

$$D_I = \int_{\partial A_q} p Q_n dA = - i_o P_o \quad (4.32)$$

$$D_L = \int_{A_f} p q_L dA = 4\pi C_L R^2 P_o J_{\beta}(\alpha, \beta) \quad (4.33)$$

where

$$J_{\beta}(\alpha, \beta) = \int_0^{\beta} \frac{1-\alpha\rho^2}{\{t-\tau(\rho)\}^{\gamma}} \rho d\rho .$$

The functional in eq.(4.18) for fluid lag case can be rewritten in the form

$$F(\dot{W}_1, \dot{W}_2, P, \alpha, \dot{R}, \dot{R}_f) = \dot{U}_s - \dot{U}_p - D_p - D_L - D_I + \dot{U}_\Gamma + \dot{U}_f \quad (4.34)$$

The following governing equations are obtained from the first variation of equation (4.34)

$$\begin{aligned} \delta \dot{W}_1: f(1) &\equiv \frac{\pi^2 \mu R}{2(1-\nu)} \left(\frac{1}{3} W_1 + \frac{1}{5} W_2 \right) - 2\pi R^2 P \{ M_1(\beta) - \alpha M_4(\beta) \} \\ &- 2\pi R^2 (p_w - \sigma_o) \left\{ \frac{1}{3} - M_1(\beta) \right\} = 0 \end{aligned} \quad (4.35)$$

$$\begin{aligned} \delta \dot{W}_2: f(2) &\equiv \frac{\pi^2 \mu R}{2(1-\nu)} \left(\frac{1}{5} W_1 + \frac{6}{35} W_2 \right) - 2\pi R^2 P \{ M_2(\beta) - \alpha M_5(\beta) \} \\ &- 2\pi R^2 (p_w - \sigma_o) \left\{ \frac{1}{5} - M_2(\beta) \right\} = 0 \end{aligned} \quad (4.36)$$

$$\begin{aligned} \delta R: f(3) &\equiv \frac{\pi^2 \mu}{2(1-\nu)} \left(\frac{2}{3} W_1^2 + \frac{1}{5} W_1 W_2 + \frac{3}{35} W_2^2 \right) - \frac{\pi^2 \mu}{4(1-\nu)} W_1^2 + 2\pi G_{cr} R \\ &- 2\pi R P [W_1 \{ M_3(\beta) - \alpha M_6(\beta) \} + 3W_2 \{ M_4(\beta) - \alpha M_7(\beta) \}] \\ &- 2\pi R (p_w - \sigma_o) [W_1 \left\{ \frac{2}{3} - M_3(\beta) \right\} + 3W_2 \left\{ \frac{2}{15} - M_4(\beta) \right\}] \\ &= 0 \end{aligned} \quad (4.37)$$

$$\begin{aligned} \delta P: &- 2\pi R^2 [\dot{W}_1 \{ M_1(\beta) - \alpha M_4(\beta) \} + \dot{W}_2 \{ M_2(\beta) - \alpha M_5(\beta) \}] \\ &- 2\pi R \dot{R} [W_1 \{ M_3(\beta) - \alpha M_6(\beta) \} + 3W_2 \{ M_4(\beta) - \alpha M_7(\beta) \}] \\ &- \frac{2\pi m}{2m+1} (\eta_o)^{-1/m} R^{(m-1)/m} \alpha^{(m+1)/m} P^{1/m} I(W_1, W_2, m, \beta) \\ &- 2\pi R_f \dot{R}_f (1-\alpha\beta^2) [W_1 (1-\beta^2)^{1/2} + W_2 (1-\beta^2)^{3/2}] \\ &- 4\pi C_L R^2 J(\alpha, \beta) + i_o = 0 \end{aligned} \quad (4.38)$$

$$\begin{aligned}
\delta\alpha: \quad f(4) \equiv & 2\pi R^2 \{ \dot{W}_1 M_4(\beta) + \dot{W}_2 M_5(\beta) \} \\
& + 2\pi R \dot{R} \{ W_1 M_6(\beta) + 3W_2 M_7(\beta) \} + 4\pi C_L R^2 J^*_{\gamma}(\beta) \\
& - \frac{2\pi m}{2m+1} (\eta_0)^{-1/m} R^{-1/m} \alpha^{1/m} P^{(m+1)/m} I^*_{\beta}(W_1, W_2, m, \beta) \\
& + 2\pi \beta^2 R_f \dot{R}_f [W_1 (1-\beta^2)^{1/2} + W_2 (1-\beta^2)^{3/2}] = 0 \quad (4.39)
\end{aligned}$$

$$\text{where } J^*_{\gamma}(\beta) = \int_0^{\beta} \frac{\rho^3}{\{t-\tau(\rho)\}^{\gamma}} d\rho.$$

$$\begin{aligned}
\delta \dot{R}_f: \quad f(5) \equiv & \{P(1-\alpha\beta^2) + (\sigma_0 - p_w)\} [W_1 (1-\beta^2)^{1/2} + W_2 (1-\beta^2)^{3/2}] \\
& = 0 \quad (4.40)
\end{aligned}$$

From equations (4.38) and (4.39), the following equation is derived :

$$\begin{aligned}
& \pi R^2 [\{ \dot{W}_1 M_1(\beta) + \dot{W}_2 M_2(\beta) \} + \{ W_1 \beta \dot{\beta} (1-\beta^2)^{1/2} + W_2 \beta \dot{\beta} (1-\beta^2)^{3/2} \}] \\
& + 2\pi R \dot{R} [\{ W_1 M_3(\beta) + 3W_2 M_4(\beta) \} + \{ W_1 \beta^2 (1-\beta^2)^{1/2} + W_2 \beta^2 (1-\beta^2)^{3/2} \}] \\
& - i_0 + 4\pi C_L R^2 \int_0^{\beta} \{t-\tau(\rho)\}^{-\gamma} \rho d\rho = 0 \quad (4.41)
\end{aligned}$$

The above equation represents the mass balance and can be simplified in the form

$$\begin{aligned}
f(6) \equiv & \frac{d}{dt} [2\pi R^2 \{ W_1 M_1(\beta) + W_2 M_2(\beta) \}] - i_0 \\
& + 4\pi C_L R^2 \int_0^{\beta} \{t-\tau(\rho)\}^{-\gamma} \rho d\rho = 0 \quad (4.42)
\end{aligned}$$

4.4 Fluid Leak-off in Variational Energy Rate Approach

As mentioned in the previous chapter, the fluid leak-off effect in the variational energy rate is

separated from the dissipation energy rate term so that the fluid leak-off contributions can be calculated directly using simpler integral forms without the evaluation of flow rate at every integration point in the Lagrangian approach.

We consider a penny-shaped fracture model with fluid lag and fluid leak-off into a formation. The dissipation energy rate term (eq. 4.33) due to fluid leak-off has the form

$$\begin{aligned} D_L &= \int_{A_f(t)} p q_L dA \\ &= 4\pi C_L R^2 P J_\beta(\alpha, \beta) \end{aligned} \quad (4.43)$$

where

$$J_\beta(\alpha, \beta) \equiv \int_0^\beta \frac{1-\alpha\rho^2}{\{t-\tau(\rho)\}^\gamma} \rho d\rho. \quad (4.44)$$

The fluid front, $R_f(t)$, and fluid front arrival time, $\tau(\rho)$, are monitored and saved at every solution steps so that $\tau(\rho)$ at the next step can be linearly interpolated from previous solutions.

The solution procedure is exactly the same as in chapter III and the six equations (4.35-37,39,40,42) are to be satisfied simultaneously i.e. $\underline{f}^{t+\Delta t} = \underline{0}$ at time $t+\Delta t$. The procedure starts with the known solution vector $\underline{x}^t$ at time t given by

$$\begin{aligned} \underline{x}^t &= \{x_1, x_2, x_3, x_4, x_5, x_6\} \\ &= \{W_1(t), W_2(t), R(t), R_f(t), P(t), \alpha(t)\} \end{aligned} \quad (4.45)$$

The Newton-Raphson iteration scheme is also used for the solutions and time explicit closed form solutions are utilized for the initial guess. As previous model cases, the iteration is terminated if the error falls in a tolerance:

$$\text{Error} = \sum_{n=1}^6 \left| \frac{\Delta x_n}{x_n} \right| = \text{Tolerance} \leq 0.05\% .$$

The energy rate terms associated with the time increment are computed and the solution vector is updated for the subsequent time step solution as detailed in chapter II.

CHAPTER V
FRACTURE PROPAGATION SIMULATION
AND MODEL COMPARISONS

To demonstrate the validity of the developed models, several benchmark comparisons with published results are presented in this chapter. Parametric sensitivity studies are performed for symmetric and unsymmetric three-layered cases with leak-off effects. Also the roles of the effective fracture tip closure stress and fracture fluid pressure distribution are examined for the investigation of fluid lag phenomenon in hydraulic fracturing processes by a model expanded from the variational formulation.

5.1 Benchmark Model Results and Comparisons

Selected benchmark cases and reported responses for penny-shaped and PKN type models are evaluated prior to the application to cases in multi-layered formations to demonstrate the validity of the developed models. Penny-shaped fractures as well as PKN type evolutions are special cases of the general three-layered elliptic model. Elliptic fracture growth in the payzone is examined to check its asymptotic behavior relative to constant height

results from one-dimensional FEM and PKN closed form models.

5.1.1 Benchmark Penny-Shaped Models

As a first step, fracture responses from ELLIP2D are compared with those obtained by other simulators. The input data for a penny-shaped model, including leak-off and spurt loss effects, is revealed in Table 5.1. Fracture radius, opening width, and bottom-hole treatment pressure are compared with a general three-dimensional simulator, HYFRAC3D [61], and time explicit solutions detailed in Appendix A. The characteristic time concepts associated with the closed form solutions, introduced by Lee et al [66], provide a measure to separate the roles of Griffith fracture energy and fracture fluid dissipation energy contributions during the evolution of a hydraulic fracture. The closed form solutions can be derived explicitly from the discrete Lagrangian equations by neglecting one of those energy components, Griffith surface energy ($\partial U_f / \partial Y_k$) or the fracture fluid dissipation energy ($\partial D_q / \partial \dot{Y}_k$), for rectangular (PKN & CGDD), penny-shaped, and elliptic fracture models in an isotropic medium without leak-off effects. The elliptic fracture model in a homogeneous medium in Appendix A is an extension of the penny-shaped model. It is assumed that

Table 5.1 Input Data for Penny-Shaped Models

	Cases		
	Case A [*]	TT-A	TT-G
Formation Properties			
Shear modulus (GPa)	2.0	0.215	2.15
Poisson's ratio	0.2	0.2	0.2
Payzone Height (m)	$\infty / 20^{\dagger}$	∞	∞
In-situ stress contrast ($\Delta\sigma$)	$0^{\dagger} / \infty$	0	0
Energy release rate (Pa-m)	200	2240	224
Fluid Properties			
Behavior index, m	1.0	0.39	0.39
Consistency index (Pa-min ^m)	1.6×10^{-4}	1.164	1.164
Leak-off coefficient (m/ $\sqrt{\text{min}}$)	0.0	2×10^{-4}	2×10^{-4}
Spurt loss (m)	0.0	0.0021	0.0021
Injection Scheme			
Injection rate (m ³ /min)	5.0	6.34	6.34
Total time (min)	100	44	40

[†] : Non-dimensionalizing dimensions in Fig. 5.1 & 5.2

$W_{*} = 2 \text{ mm}$, $R_{*} = 50 \text{ m}$, $P_{O}^{*} = 0.1 \text{ Mpa}$ (penny-shaped model)

[‡] : Fracture contained in payzone (constant height model)
by large in situ stress contrast ($\Delta\sigma \rightarrow \infty$)

the vertical stiffness is dominant in an elongated crack and the flow is one directional parallel to the semi-major axis of ellipse. The two sets of solution, designated as the dissipation energy dominant (negligible fracture energy) and fracture energy dominant (negligible dissipation energy) solutions, provide the initial guess for the ELLIP2D and ERATE2D models. Fracture opening width and bottom-hole treatment comparisons for a penny-shaped case (Case A^{*}), illustrated in Fig. 5.1 and 5.2, demonstrates the shift in the governing influence regimes and good agreements between simulators. For the case of large in situ stress contrasts in the barrier layer (a variation of Case A^{*}), the vertical growth of the fracture is inhibited, as expected. The resulting fracture responses are compared with constant height PKN type solutions. Considering the inherent differences in the model geometries (elliptic versus rectangular), the results are reasonable. The computed elliptical versus PKN rectangular fracture half length, borehole fracture width, and bottom-hole pressure values, for an injection duration of 55.5 minutes, are 1462 m vs 1115 m, 9.06 mm vs 11.01 mm, and 1.49 MPa vs 1.26 MPa, respectively. We note that, despite the larger discrepancy in the fracture length, the geometry induced difference in the fracture area between the two models is only 3%. Two more cases (TT-A & TT-G)

Fig. 5.1 Comparisons of Fracture Opening Width Evolutions for Penny-Shaped Model (case A*)

Fig. 5.2 Comparisons of Bottom-Hole Treatment Pressure versus Fracture Radius for Penny-Shaped Model (case A^{*})

are examined for the verification of results for Terra-Tek model cases [67]. These cases have identical fracture fluid properties except for differential formation material properties. The induced fracture height, opening width, and borehole fracture fluid pressure, summarized in Table 5.2, were compared, using the HYFRAC3D [61], Terra Tek [90], Lagrangian (ELLIP2D in Chapter II), and Variational (ERATE2D in Chapter III) models. It is noteworthy that there are differences in the modeling assumptions. The fracture opening width derived from the variational principle is in two component form, one representing the crack singular behavior and the other corresponding to the equilibrium condition for crack growth. These dual components of fracture opening width are obtained independently balancing pertinent energy rate components during the hydraulic fracturing process. In the Lagrangian formulation (ELLIP2D), the cross sectional view of a crack perpendicular to the semi-major axis of elliptic model remains elliptic and fluid leak-off effects are incorporated by updating the local flow rate vectors from the conservation of mass. In the variational formulation (ERATE2D), the overall energy rate conservation by the variational principle is considered, which can be deduced from the first law of thermodynamics. Therefore the dissipative fluid loss is calculated from

**Table 5.2 Summary of Penny-Shaped Model Result Comparisons
Including Leak-off and Spurt Loss Effects**

	Cases	
	TT-A	TT-G
Total Height	(m)	(m)
HYFRAC3D	112.1	184.8
Terra Tek	117.0	201.7
Lagrangian ¹	109.8/105.4	157.0/140.8
Variational ²	111.6/ NA	177.4/ NA
Half Length	(m)	(m)
HYFRAC3D	56.1	92.4
Terra Tek	55.7	86.1
Lagrangian	54.9/52.7	78.5/70.4
Variational	55.8/ NA	88.7/ NA
Borehole Opening	(mm)	(mm)
HYFRAC3D	38.1	14.3
Terra Tek	35.6	12.7
Lagrangian	38.5/39.2	14.1/15.9
Variational	38.9/ NA	13.7/ NA
Effect. Pressure	(Mpa)	(Mpa)
HYFRAC3D	0.15	0.53
Terra Tek	0.14	0.45
Lagrangian	0.168/0.176	0.529/0.594
Variational	0.156/ NA	0.436/ NA

¹ : ELLIP2D   ² : ERATE2D Model

* L/LS where L : Leak-off effect without spurt loss
LS : Leak-off and spurt loss

the energy rate conservation law governing the fracture fluid and reservoir elasticity. Spurt loss represents as additional fluid leak-off effect which is proportional to the area exposed to treatment fluid and is not influenced by the exposure time to treatment fluid and the spurt loss is not directly included in the variational formulation. The fluid leak-off, which is a function of permeability, gel concentration, pressure, shear, temperature and time, is lumped approximately to match the fluid loss under actual conditions.

5.1.2 Benchmark PKN Models

To demonstrate the relative differences of the developed models (ELLIP2D and ERATE2D) with respect to the PKN type model solutions, input data for Case A* are selected excluding the leak-off effects. Since the fracture is contained only within the payzone, we take around half of total injection time for a penny-shaped model. Fig. 5.3 illustrates the fracture opening width profiles and overall fracture configurations of different models at the cessation of stimulation ($t=55.5$ min). The fracture dimensions should be interpreted carefully by considering the initial approximations in each models. The elliptic crack models (ELLIP2D and ERATE2D) obviously give much longer lengths than the PKN model because of the

Fig. 5.3 Fracture Opening Width Profiles and Fracture Configurations for PKN ELLIP2D and ERATE2D Models

fracture profile definitions. The comparisons in the half length and opening width between the two elliptic models also reflect the differences in the model assumption. It is noteworthy that the representation of pressure profiles and corresponding opening width of the ERATE2D varies with elapsed time while those of ELLIP2D remain the same during simulations. The figure also demonstrates that opening width profiles for ERATE2D vary with respect to the values of critical energy release rates. In a formation where considerably large critical energy release rate is involved (e.g. $G_{cr}=2000$ Pa-m), the fracture half length is much lower and the crack opening width is higher than the case with negligible critical energy release rate ($G_{cr}=0$). The ERATE2D model yields a higher effective borehole pressure than the other cases (1.407 Mpa with $G_{cr}=0$ and 1.437 Mpa with $G_{cr}=2000$ Pa-m) compared to the one-dimensional FEM solution (1.292 Mpa with $G_{cr}=0$). Since both simulators satisfy the mass balance equation exactly, the total volume for these cases must be the same with no-leak-off. In the numerical calculations for PKN type solutions the variables assigned for heights (b_u and b_ℓ) are set to be constant ($H/2$). Therefore the ERATE2D and ELLIP2D models have seven and three unknowns respectively for this case. The response for the elliptic models are shown to agree well with the asymptotic

responses obtained from the closed form solution in Appendix A. Generally, the responses from the different PKN type models give fair agreement.

5.2 Three-Layered Models with Fluid Leak-off:

Comparisons and Parametric Sensitivity Studies

As a first step to evaluating fracture evolution in multi-layered formations, the effects of elastic modulus and in situ stress contrasts are examined through symmetric three-layered cases. The combined effects including differentials of formation properties and fluid leak-off characteristics are investigated through general unsymmetric model cases to check the role of uncontrollable parameters and controllable variables on the fracture evolution. Parametric sensitivity studies are applied to some selected field tests in the next chapter.

5.2.1 Symmetric Three-Layered Models

The influence of elastic modulus and in situ stress contrasts is shown in Figures 5.4 through 5.7 for the data given in Table 5.3. The numerical experiment values listed in this table assume a elastic modulus ratio ($\hat{\mu}_2/\hat{\mu}_1$) ranging from 1 to 100 with a payzone height $h = 20$ m. The

Table 5.3 Input Data for Elastic Modulus and In Situ Stress Contrast Cases

	Case B	Case C [†]
	Elastic modulus contrast case	in situ stress contrast case
Shear modulus (μ_1)	16.67 GPa	16.55 GPa (2.4E6 psi)
$\hat{\mu}_2/\hat{\mu}_1^*$	1 - 100	1
Poisson's ratio (ν)	0.2	0.25
Consistency index (η_o)	1.67E-3 Pa-min	0.31 Pa- $\sqrt{\text{min}}$ (4.48E-5 psi- $\sqrt{\text{min}}$)
Behavior index (m)	1.0	0.5
Injection rate (i_o)	5.0 m ³ /min	3.18 m ³ /min (20 bbl/min)
Critical Energy Release Rate (G_{cr})	0.24 Pa-m	4.38 Pa-m (2.5E-2 psi-in)
Payzone Height (h)	20 m	45.72 m (150 ft)
In-Situ Stress Contrast ($\Delta\sigma$)	0 MPa	0 - 6.89 MPa (0 - 1000 psi)

* : Subscripts, 1 and 2 denotes quantities in payzone and barriers, respectively.

† : British units used in Meyer's publication [28].

Fig. 5.4 Comparisons of Normalized Fracture Length versus Height in a Symmetric Three-Layered Elliptic Crack Model with Elastic Modulus Contrasts and No Leak-off

Fig. 5.5 Fracture Evolution for Elastic Modulus Contrast
Cases ($\hat{\mu}_2/\hat{\mu}_1 = 2$ and 10) with No Leak-off

Fig. 5.6 Comparisons of Normalized Fracture Length versus Height in a Symmetric Three-Layered Elliptic Crack Model with In-situ Stress Contrasts and No Leak-off

Fig. 5.7 Fracture Evolution for In-situ Stress Contrast Cases ($\Delta\sigma = 100$ and 250 psi) with No Leak-off

computed fracture geometries, fracture extent versus height, are illustrated in Fig. 5.4.

The "order-of-magnitude" estimations of the fracture geometries from Van Eekelen [15], given by the equation,

$$2a/h = \left[1 + \frac{12}{19} (\hat{\mu}_2/\hat{\mu}_1) \{ \log(2b/h) + \frac{1}{4} (3 + \hat{\mu}_1/\hat{\mu}_2) (2b/h - 1) \} \right]^{1/2} \quad (5.1)$$

are also shown for comparison. Approximation (5.1) clearly underestimates the effectiveness of the elastic modulus contrast on the fracture containment. However, considering the range of representative elastic modulus values for payzone and barrier layers, the elastic modulus contrast alone does not act as a strong fracture containment barrier. For example, with a modulus ratio $\hat{\mu}_2/\hat{\mu}_1 = 10$ (Fig. 5.5), the fracture advances significantly into the barrier layers ($2b/h=2$) when the fracture length is five times the payzone height ($2a/h=5$).

The effects of in situ stress contrast are also evaluated by investigating a series of numerical examples (Table 5.3) reported by Meyer [28]. Distinctly different from the elastic modulus contrast results, in situ stress differentials provide a strong barrier to vertical crack penetration. For example, with the specified fracture parameters, a 1.72 MPa (250 psi) differential in situ stress significantly restricts barrier layer penetration (Fig. 5.6). For an in situ stress contrast of 6.89 MPa

(500 psi), the fracture is virtually contained within the payzone. Fracture evolutions for selected cases of elastic modulus and in situ stress contrasts are revealed in Figs. 5.5 and 5.7, respectively.

A set of verification tests for a symmetric three-layered elliptic crack model is conducted by comparing the results from the developed model with the responses reported by Bouteica [91], Dougherty and Abou-Sayed [92], and Abou-Sayed et al. [67]. The pertinent input data are given in Table 5.4 and the corresponding results are presented in Table 5.5. An extra set of results is also presented for an "equivalent" volume with no leak-off effects. Comparison of the results shows that the leak-off effect may not be replaced by compensating the volume in the simulation.

Parametric sensitivity investigations associated with a symmetric three-layered model with leak-off are shown in Table 5.7, using the input data shown in Table 5.6. The final fracture geometry, effective pressure and volume efficiency are compared with the results reported by Terra tek [57], Bouteica [30], and Meyer [28]. Based upon the benchmark case of penny-shaped fracture growth (Case $\hat{A}$) in a homogeneous medium, different sets of stress contrast over the formations with a payzone thickness of $H = 223$ ft are studied. Also fracture responses for different

Table 5.4 Input Data for Cases VI and ℓ

	Unit	Case VI	Case ℓ
Shear modulus (μ)	GPa	16.67	2.19
Poisson's ratio ratio (ν)	-	0.2	0.2
Consistency index (η_0)	Pa-min ^m	1.67E-3	5.58E-2
Behavior index (m)	-	1.0	0.75
Leak-off Coefficient	m/min ^{1/2}	3.03E-4	4.95E-5
Spurt Loss Coefficient	m	0	1.0E-3
Injection rate (i_0)	m ³ /min	5.0	6.36
Injected Volume	m ³	1000	268
Fracture Volume	m ³	300	223
Critical Energy Release Rate (G_{cr})	Pa-m	0.24	229.4
Payzone Height (H)	m	45	68
In situ Stress Contrast ($\Delta\sigma$)	MPa	8.4	3.5

Table 5.5 Model Comparisons for Cases VI and ℓ

CASE VI

	Length a (m)	Height 2b (m)	Width 2c (mm)	Eff. Press. P_o (MPa)
Dougherty & Abou-Sayed [92]	628.5	60.7	9.4	4.04
Bouteca [30]	537.3	59.4	9.7	3.74
ELLIP2D ¹	559.6	57.5	10.4	4.71
ELLIP2D ²	697.1	52.2	7.9	4.39

¹ : Leak-off effect included, Efficiency = 35.1%
(30.0% efficiency estimated in the input data)

² : No leak-off effect
(only 30% volume injected to match the fracture volume)

CASE ℓ

	Length a (m)	Height 2b (m)	Width 2c (mm)	Eff. Press. P_o (MPa)
Abou-Sayed et al. [92]	134.7	73.6	22.9	0.97
Bouteca [30]	148.7	71.8	22.3	1.03
ELLIP2D ³	145.9	74.2	20.7	1.05
ELLIP2D ⁴	161.6	72.3	18.2	1.03

³ : Leak-off included, Efficiency = 87.8% (83% estimated)

⁴ : No leak-off, (only 83% volume injected)

Table 5.6 Summary of Input Data for Case $\hat{A}$ - $\hat{L}$ in Ref.[28]
(Default Values are those of Case $\hat{A}$)

	Case $\hat{A}$	Case $\hat{E}$	Case $\hat{G}$	Case $\hat{I}$	Case $\hat{L}$
Reservoir Mechanical Properties (Uncontrollable Variables)					
Young's Modulus (psi)	7.5E4				
Poisson's ratio	0.2				
Stress contrast (psi)	0	100.0		100.0	500.0
K_{IC} (psi·in ^{1/2})	1000				
Fracturing Fluid Data (Controllable variables)					
Flow Rate (bbl/min)	40.0				
Total Volume (gal)	72000	68000	67000	66000	71000
m, Behavior index	0.39			0.75	0.75
η_o (lb _f ·s ^m /ft ²)	0.12			0.07	0.07
Spurt loss (gal/ft ²)	0.025				
Fluid Loss (ft/min ^{1/2})	1.625E-4				
Payzone Data					
Payzone Thickness (ft)	N/A	223.0	N/A	223.0	223.0

Table 5.7 Result Comparisons with Ref.[28] for Case $\hat{A} - \hat{L}$

Case #	[†] Model	Length (a:ft)	Height (2b:ft)	Max.Width (W:2c:in)	Eff.Press. (ΔP :psi)	a/b	Efficiency (%)
Case $\hat{A}$ ($\Delta\sigma = 0$)	A	184.0	368.0	1.51	23.9	1.000	93.0
	B	185.7	390.0	1.40	20.0	0.952	N/A
	C	189.3	378.0	1.49	21.8	1.002	N/A
	E	188.9	377.7	1.62	32.2/23.3	1.000	N/A
Case $\hat{E}$ ($\Delta\sigma = 100$)	A	262.7	231.6	1.60	29.7	2.269	93.6
	B	190.5	251.7	1.97	34.0	1.514	N/A
	C	263.1	218.2	1.86	30.5	2.412	N/A
	E	232.7	226.7	1.92	32.2/34.8	2.053	N/A
Case $\hat{G}$ ($\Delta\sigma = 0$)	A	280.3	560.6	0.54	73.3	1.000	82.8
	B	286.9	672.6	0.50	55.0	0.853	N/A
	C	294.2	614.2	0.60	72.5	0.958	N/A
	E	327.5	655.0	0.57	95.9/56.9	1.000	N/A

[†] A : ELLIP2D

B : TERRAFRAC (Terra Tek model)

C : FRANK (IFP model) E : MFRAC-II

Table 5.7 (continued)

Case #	[†] Model	Length (a:ft)	Height (2b:ft)	Max.Width (W:2c:in)	Eff.Press. (ΔP:psi)	a/b	Efficiency (%)
Case I (Δσ = 100)	A	443.6	313.2	0.62	107.6	2.833	84.5
	B	426.8	501.3	0.75	92.0	1.703	N/A
	C	368.1	430.4	0.69	100.0	1.711	N/A
	D	430.0	361.0	1.32	163.0	2.382	N/A
	E	385.9	396.8	0.75	95.9/87.6	1.945	N/A
Case L (Δσ = 500)	A	499.9	242.2	0.78	148.4	4.128	87.2
	B	441.8	241.4	0.90	138.0	3.660	N/A
	C	561.4	237.1	1.00	142.1	4.736	N/A
	D	454.5	270.0	1.35	169.0	3.367	N/A
	E	476.3	241.2	0.93	150.1	3.949	N/A

[†] A : ELLIP2D

B : TERRAFRAC (Terra Tek model)

C : FRANK (IFP model)

D : ADVANI

E : MFRAC-II

treatment fluid rheological properties are computed. Additional parametric sensitivity comparisons, using the cases in Table 5.8 presented by Bouteica [30], are listed in Table 5.9. The baseline values are characterized by the isotropic case (Case3). Based on the benchmark case (Case0), the parameters are switched sequentially to study their role on fracture geometry and containment. The comparison of cases from Case0 to Case3 shows that in situ stress contrast over the formations plays a major role in containing the fracture. The decreased elastic moduli with stress contrast (Case4: $\mu = 4.E4$ Mpa and $\Delta\sigma = 10$ Mpa) yields higher crack opening width, decreased effective borehole pressure and lower penetration into barriers compared to Case0. The higher injection flow rate (Case7) produces increased fracture length, height, opening width and effective borehole pressure due to better fracture volume efficiency. Since the fluid leak-off effect is a function of time, the efficiency is directly influenced by the injection rate. When the viscosity of treatment fluid increases (Case5 & Case6), fracture height, opening width, effective borehole pressure and efficiency increase while fracture length decrease. Cases 8 and 9 demonstrate that the ratio of containment in the payzone decreases when the payzone thickness decreases, although they have the same elastic modulus and in situ stress contrasts. Case10

Table 5.8 Reservoir and Fracturing Fluid Data in Ref. [30]
(Default Values are those of Case 0)

	Case0	Case1	Case2	Case3	Case4	Case5	Case6	Case7	Case8	Case9	Case10
Reservoir Mechanical Properties (Uncontrollable Variables)											
Young's Modulus (Mpa)	4.E4				2.E4						
Poisson's ratio	0.2										
Stress contrast (Mpa)	10	5	2	0							
K_{IC} (Mpa·m ^{1/2})	0.5										10
Fracturing Fluid Data (Controllable Variables)											
Flow Rate (m ³ /min)	2							5			
Total Volume (m ³)	300										
m, Behavior index	1										
η_o (mPa·s ^m)	100					200	10				
Spurt loss (cm ³ /cm)	0										
Fluid Loss (cm/s ^{1/2})	0.004										
Payzone Data											
Payzone Thickness (m)	50								20	10	
Perforated Interval (m)	50								20	10	

Table 5.9 Result Comparisons with Ref.[30] for Symmetric Cases

Case #	* Model	Length (a:m)	Height (2b:m)	2b/h	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Efficiency (%)
Case 0	A	210.6	55.2	1.11	7.3	3.19	29.8
	B	254.3	53.0	1.06	6.5	2.41	18.3
Case 1	A	188.6	62.3	1.25	7.1	2.93	29.1
	B	219.0	63.9	1.28	5.8	2.18	17.1
Case 2	A	129.1	94.6	1.89	6.3	2.26	26.7
	B	119.0	122.2	2.44	4.4	1.79	13.5
Case 3	A	79.0	157.9	3.16	5.7	1.30	24.8
	B	84.4	176.8	3.54	3.5	1.00	11.4
Case 4	A	210.9	52.8	1.06	8.5	1.90	33.0
	B	261.7	50.0	1.00	8.4	1.54	23.8
Case10	A	208.1	55.0	1.10	7.8	3.35	31.3
	B	244.9	50.6	1.01	7.0	2.67	26.8

* : A denotes ELLIP2D model

B is Bouteuca's model presented in Ref.[30]

Table 5.9 (continued)

Case #	* Model	Length (a:m)	Height (2b:m)	2b/h	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Efficiency (%)
Case 0	A	210.6	55.2	1.11	7.3	3.19	29.8
	B	254.3	53.0	1.06	6.5	2.41	18.3
Case 5	A	199.5	56.2	1.12	8.3	3.64	32.6
	B	245.8	54.4	1.09	7.3	2.71	20.8
Case 6	A	243.0	53.1	1.06	4.8	2.05	21.4
	B	290.9	50.0	1.0	4.4	1.63	13.2
Case 7	A	261.3	56.8	1.14	8.5	3.80	44.3
	B	326.6	56.3	1.13	8.2	3.0	32.2
Case 8	A	358.1	31.9	1.60	7.9	7.54	31.7
	B	379.3	35.6	1.78	6.7	5.97	19.5
Case 9	A	102.9	131.1	13.1	4.0	10.0	18.7
	B	173.7	81.59	8.2	5.	9.38	15.4

* : A denotes ELLIP2D model

B is Bouteica's model presented in Ref. [30]

Table 5.9 (continued)

Case No.	Length (a:m)	Height (2b:m)	2b/h	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Efficiency (%)
Case 0	210.6	55.2	1.11	7.3	3.19	29.8
Case 1	188.6	62.3	1.25	7.1	2.93	29.1
Case 2	129.1	94.6	1.89	6.3	2.26	26.7
Case 3	79.0	157.9	3.16	5.7	1.30	24.8
Case 4	210.9	52.8	1.06	8.5	1.90	33.0
Case 5	199.5	56.2	1.12	8.3	3.64	32.6
Case 6	243.0	53.1	1.06	4.8	2.05	21.4
Case 7	261.3	56.8	1.14	8.5	3.80	44.3
Case 8	358.1	31.9	1.60	7.9	7.54	31.7
Case 9	102.9	131.1	13.10	4.0	10.02	18.7
Case 10	208.1	55.0	1.10	7.8	3.35	31.3

shows that the fracture configuration is not significantly influenced by the fracture toughness in this case where the elastic modulus is relatively high. Obviously, the fracture responses are influenced by the effect of combined parameters and the sensitivities are coupled in a complex manner. Additionally, time-explicit closed form solutions and associated characteristic time concepts can serve as diagnostic tool for isotropic models with specialized geometries. The presented responses for penny-shaped and symmetric three-layered models presented in Table 5.3 to 5.9 are in general agreement with reported results.

The execution of one iteration in the Lagrangian model (ELLIP2D) takes about 0.3-0.4 seconds on the CRAY-YMP system versus 2.7-3.6 seconds on the VAX 8550 system. Since six to nine iterations are needed for one solution step of an unsymmetric model case, it takes a total execution time of about 40 seconds on the CRAY-YMP (6 minutes on VAX 8550) assuming that ten different step solutions for an unsymmetric three-layered case are needed. Depending on the required solution accuracy, the CPU time may be shortened greatly by reducing the integration points for the calculation of the dissipation energy rate function.

5.2.2 Unsymmetric Three-Layered Models

To demonstrate the capabilities of the developed Lagrangian model (ELLIP2D) for unsymmetric multi-layered formations, previously documented symmetric cases [64], Cases A through L in Table 5.10-1, are extended based on the benchmark penny shaped case (Case M in Table 5.10-2) to include the effects of symmetric leak-off differentials (Case N), unsymmetric in situ stress contrast (Case O), and symmetric/unsymmetric in situ stress contrast (Case P/Case Q). The results for the symmetric models are compared and a summary of the responses for the symmetric (Case A through M) and unsymmetric (Cases N,O,P,Q) models are provided in Table 5.11. As a graphic illustration of the response bounds, Figure 5.8 compares the vertical fracture configurations for the symmetric model (Case J in Table 5.10-1) with the unsymmetric in situ stress contrast model (Case O in Table 5.10-2). These cases are further compared with the symmetric high in situ stress contrast model (Case L in Table 5.10-1) in Figure 5.9.

As expected, the upper height (b_u) and lower height (b_l) growth agrees with the responses for Cases J and L, respectively. Also, the bottom hole treatment pressure responses for Case O are within the thresholds defined by the symmetric Cases J and L, as shown in Figure 5.10. The

Table 5.10-1. Reservoir and Fracturing Fluid Data for Symmetric Cases (A-L)
(Default Values are those of Case A)

	Case A	Case G	Case H	Case I	Case J	Case L
Reservoir Mechanical Properties (Uncontrollable Variables)						
Young's Modulus (Mpa)	525	5250	5250	5250	5250	5250
Poisson's ratio	0.2					
Stress contrast (Mpa)	0		0.7	0.7	0.7	3.5
K_{IC} (Mpa·m ^{1/2})	1.12		11.12			
Fracturing Fluid Data (Controllable Variables)						
Flow Rate (m ³ /min)	6.36					
Total Volume (m ³)	250					
m, Behavior index	0.39				0.75	0.75
η_o (mPa·s ^m)	5745.6				3351	3351
Spurt loss (cm ³ /cm)	0.1					
Fluid Loss (cm/s ^{1/2})	6.39E-4					
Payzone Data						
Payzone Thickness (m)	-		68	68	68	68
Perforated Interval (m)	0	55				

Table 5.10-2. Reservoir and Fracturing Fluid Data for Unsymmetric Cases (M-Q)
(Default Values are those of Case M)

	Case M	Case N	Case O	Case P	Case Q
Reservoir Mechanical Properties (Uncontrollable Variables)					
E_u/E_o (Mpa)	5250/5250/5250				
Poisson's ratio	0.2/0.2/0.2				
Stress, $\Delta\sigma_u/\Delta\sigma_\ell$ (Mpa)	0/0		0.7/3.5	3.5/3.5	3.5/0.7
K_{IC} (Mpa·m ^{1/2})	1.12/1.12/1.12				
Fracturing Fluid Data (Controllable Variables)					
Flow Rate (m ³ /min)	6.36				
Total Volume (m ³)	250				
m, Behavior index	0.75				
η_o (mPa·s ^m)	3351.				
Spurt Loss, C_s (cm ³ /cm)	0.1/0.1/0.1				
Fluid Loss, C_L (cm/s ^{1/2})	$C_L/C_L/C_L$	$0/C_L/0$			
Payzone Data					
Payzone Thickness (m)	-	68	68	68	68
Perforation Placement(m)	Center		-11.33	-11.33	-11.33

* $C_L = 6.39E-4$

Table 5.11 Result Comparisons with Ref. [30] for Case A - Q

Case #	[†] Model	Length (a:m)	Height (2b:m)	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Total V. (m ³)	Efficiency (%)
Case A	A	54.2	108.0	38.1	0.17	250.0	93.1
	B	56.6	118.9	35.6	0.14	272.0	90.5
	C	57.7	115.2	37.8	0.15	275.4	90.1
Case G	A	83.2	166.5	14.4	0.52	250.0	83.6
	B	87.5	205.0	12.7	0.39	253.6	74.1
	C	89.7	187.2	15.2	0.50	255.8	71.6
Case H	A	56.3	112.5	34.9	1.36	250.0	92.5
	B	57.0	112.8	32.5	1.18	274.4	87.2
	C	57.7	119.2	34.4	1.27	274.7	89.7

[†] A : ELLIP2D

B : TERRAFRAC (Terra Tek model)

C : FRANK (IFP model)

Table 5.11 (continued)

Case #	† Model	Length (a:m)	Height (2b:m)	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Total V. (m ³)	Efficiency (%)
Case I	A	136.0	95.4	15.6	0.75	250.0	74.5
	B	130.0	152.8	19.1	0.64	249.8	80.0
	C	112.2	131.2	17.6	0.69	250.8	74.7
	D	131.0	110.0	33.4	1.14	-	-
Case J	A	107.8	104.9	17.5	0.86	250.0	86.7
	B	115.1	157.1	20.3	0.77	242.2	84.0
	C	100.2	135.2	18.7	0.76	242.4	76.1
Case L	A	138.6	74.0	20.4	1.04	250.0	87.8
	B	134.7	73.6	22.9	0.97	268.7	83.0
	C	171.1	72.3	25.3	0.98	267.6	80.0
	D	138.5	82.3	34.3	1.18	-	-

† A : ELLIP2D

B : TERRAFRAC (Terra Tek model)

C : FRANK (IFP model)

D : ADVANI

Table 5.11 (continued)

Case #	Length (a:m)	Upper/lower Heights (b _u :m)	Max.Width (W:2c:mm)	Eff.Press. (ΔP :Mpa)	Efficiency (%)
Case A	54.0	54.0	38.1	0.17	93.1
Case G	83.2	83.2	14.4	0.52	83.6
Case H	56.3	56.3	34.9	1.36	92.5
Case I	129.6	48.2	16.3	0.76	85.2
Case J	107.8	52.5	18.3	0.86	86.7
Case L	138.6	37.0	20.4	1.04	87.8
Case M	78.6	78.6	16.5	0.63	85.5
Case N	84.0	85.2	14.3	0.51	85.9
Case O	119.2	54.5	19.1	0.92	87.2
Case P	138.7	49.1	20.5	1.04	87.8
Case Q	118.5	48.7	19.0	0.93	87.1

Fig. 5.8 Comparisons between Symmetric and Unsymmetric Models with in situ Stress Contrasts

Fig. 5.9 Normalized Fracture Height versus Fracture Length Growth for Different in situ Stress Contrast Cases

Fig. 5.10 Pressure versus Time for Cases (M,J,L & O)

penny-shaped model pressure response (Case M), included for comparison, presumably serves as a lower bound. Case N shows vertically elongated fracture configuration because fluid leak-off is restricted in the barriers. Figure 5.11 illustrates the fracture configurations with lowered perforation placement for cases with symmetric and unsymmetric in situ stress contrast. Case Q, with an unsymmetric stress contrast, tends to propagate similar to a symmetric case. On the other hand, Case P, with a symmetric in situ stress contrast, propagates unsymmetrically.

The unsymmetric three layered model is an economical tool for parametric sensitivity as well as controllable variable versus uncontrollable parameter studies of fracture responses in three-layered formation.

5.3 Contribution of Pertinent Energy Rate Components in the Hydraulic Fracturing Processes

The use of the variational methodologies detailed in chapter III can facilitate the computation of pertinent energy rate components (available power at the borehole (D_{Ieff}), time rate of change of strain energy in the formation ($d[U_s^*]/dt$), fluid pressure gradient dissipation energy rate in the crack (D_η), leak-off dissipation energy rate (D_{L1}), and Griffith fracture energy rate ($\dot{U}_f$)).

Stimulation treatments can be designed on the basis of the pertinent energy ratios i.e. D_η/D_{Ieff} , D_{L1}/D_{Ieff} , and $\dot{U}_f/D_{\text{Ieff}}$. These ratios serve as diagnostic indicators for fracture design and are fundamentally aligned with the characteristic time concepts. Hydraulic fracture investigations on the pertinent energy rate components have been conducted for three previously reported examples [66], shown in Table 5.12. According to the characteristic time concepts, proposed by Lee et al [66], the time ranges observed (say $t = 0$ to 10^3 [min]) fall into three categories by the characteristics (i) the dissipation energy dominant regime (fracture energy negligible), (ii) fracture energy dominant regime (dissipation energy negligible), and (iii) the transition regime wherein the regimes change from one regime to the other. Fig. 5.12 - 5.14 demonstrate the changes in each pertinent energy rate component with elapsed time. Considering the characteristic time for each example, the asymptotic behavior of hydraulic fracturing responses, provided by the time-explicit closed form solutions can also be verified. The pertinent energy components i.e. E_η , E_{L1} , U_{s1} and U_f for Example 2 are monitored in Fig. 5.15. The energy components are defined in the form

$$E_\eta \equiv \int_0^t D_\eta(t) dt, \quad E_{L1} \equiv \int_0^t D_{L1}(t) dt,$$

Fig. 5.11 Fracture Configurations Corresponding to Lowered Perforation Placement (With Symmetric and Unsymmetric in situ Stress Contrasts)

Table 5.12 Lists of Parameters for Selected Examples

	Examples		
	Ex.1 [93]	Ex.2 [38]	Ex.3 [†] [66]
Formation Properties			
Shear modulus (GPa)	4.14	2.07	2.0
Poisson's ratio	0.25	0.2	0.2
Energy release rate (Pa-m)	150	2101	200
Fluid Parameters			
Behavior index, m	1.0	1.0	1.0
Consistency index (Pa-min ^{m})	1.67×10^{-3}	1.67×10^{-3}	1.6×10^{-4}
Fluid leak-off Coef. (m/ $\sqrt{\text{min}}$)	5×10^{-5}	5×10^{-5}	5×10^{-5}
Injection rate (m ³ /min)	2.39	3.93	5.0
Characteristic Values			
Time, τ (min)	1819	6.7×10^{-3}	1.0
Radius, R_* (m)	979	3.85	50.
Opening Width, W_* (m)	5.16×10^{-3}	1.77×10^{-3}	2.00×10^{-3}
Pressure, P_O^* (Mpa)	0.29	1.19	0.10

[†] : Example 3 is identical to Case A^{*} in Table 5.1 except fluid leak-off coefficient.

Fig. 5.12 Contributions of Energy Rate Components in a Penny-Shaped Fracture with Leak-off (Example 1 in Table 5.12)

Fig. 5.13 Contributions of Energy Rate Components in a Penny-Shaped Fracture with Leak-off (Example 2 in Table 5.12)

Fig. 5.14 Contributions of Energy Rate Components in a Penny-Shaped Fracture with Leak-off (Example 3 in Table 5.12)

Fig. 5.15 Contributions of Energy Components
in a Penny-Shaped Fracture with Leak-off
(Example 2 in Table 5.12)

$$U_{s1} \equiv \int_0^t [dU_{s1}/dt] dt \quad \text{and} \quad U_f \equiv \int_0^t [dU_f/dt] dt .$$

It is noteworthy that the pressure distribution in the fracture and corresponding fracture opening width profile vary during the fracturing simulation conducted by the energy rate formulation (ERATE2D). The pressure and fracture opening profile changes for Example 3 are revealed in Fig. 5.16. The computed pressure-width response variable, $\alpha(t)$, decreases monotonically from 1.22 at the early stage ($t=5 \times 10^{-4}$ min) to 0.11 at the cessation of the simulation ($t=1000$ min). Fig. 5.17 shows the relationship between the pressure distribution in a penny-shaped fracture and corresponding fracture opening width profile, derived in Appendix C.

The extreme values of α are 0 and 1.5 for the zero dissipation energy and negligible fracture energy solutions, respectively. Systematic studies on these energy component ratios apparently can serve as diagnostic indicators for fracture design and optimization.

5.4 Fluid Lag During Hydraulic Fracture Evolution

The developed simulator for fluid lag, detailed in chapter IV, is examined for a penny-shaped model case (case TT-G) previously discussed in chapter 5.1. The fluid lag index, β ($\beta = R_f/R$), is computed through the entire stimulation (expanded up to $t=250$ min) to

Fig. 5.16 Pressure and Fracture Opening Width Profiles
at Initial and Final Stages
(Example 3 in Table 5.12)

Fig. 5.17 Fracture Opening Width Profiles Associated with Pressure Distributions in a Penny-Shaped Fracture

investigate the relationship between the fracture front (R) and fluid front (R_f), shown in Fig. 5.18. It is obvious that the fluid lag is significantly influenced by the effective crack tip closure stress, $\sigma_o - P_w$. Fig. 5.19 shows that a penny-shaped fracture becomes flatter and has a larger radius while the inner fluid front has a smaller radius as the effective crack tip closure stress increases. It is noteworthy that the maximum opening width, $W(t)$ increases though $W_1(t)$ becomes smaller when the crack tip closure stress increases. The total fracture opening width is the summation of two components, W_1 and W_2 . Since $W_1(t)$ represents the crack singular behavior while $W_2(t)$ corresponds to the equilibrium crack growth condition satisfying a zero slope at the crack tip, the above phenomenon indicates that an increase of the crack tip closure stress produces a fracture opening profile with bigger opening width and sharper crack tip due to an increased W_2 value. Fig. 5.19 also illustrates that an increased crack tip closure stress yields a decreased fracture front, increased fluid front, and higher fracture opening width. It also shows that no fluid lag exists after approximately 210 minutes for case e. The non-dimensionalizing radius (R_*) and fracture opening width (W_*) in Fig. 5.19 are 250 m and 0.5 mm, respectively. Fig. 5.20 illustrates the effective

Fig. 5.18 Fluid Lag Index, β ($\beta = R_f/R$) versus Time
 (Case TT-G in Table 5.1)

Fig. 5.19 Normalized Fracture Front, Fluid Front, and Fracture Opening Width in Case of Fluid Lag (Case TT-G in Table 5.1, Normalizing Dimensions : $R_* = 250$ m and $W_* = 0.5$ mm)

pressure profile versus the elapsed time and the location of fluid front with respect to the dimension of the fracture front. Pressure profiles at the initial stage ($t=0.05$ min) are observed for cases with and without fluid lag. The principal dimensions at $t=0.05$ min are as follows :

1) Case A at $t=0.05$ (min) with fluid lag

$$W = 3.59 \times 10^{-3} \text{ (m)}, R = 8.275 \text{ (m)}, R_f = 7.089 \text{ (m)},$$

$$P_o = 1.369 \text{ (Mpa)} \text{ and } \beta (=R_f/R) = 0.857$$

2) Case B at $t=0.05$ (min) without fluid lag

$$W = 3.70 \times 10^{-3} \text{ (m)}, R = 7.272 \text{ (m)}, P_o = 1.432 \text{ (Mpa)}$$

$$\text{and } \alpha = 1.122$$

Case B at $t=0.1$ (min) without fluid lag

$$W = 3.95 \times 10^{-3} \text{ (m)}, R = 9.733 \text{ (m)}, P_o = 1.130 \text{ (Mpa)}$$

$$\text{and } \alpha = 1.086.$$

At a certain time ($t=0.05$ [min]), the case with no fluid lag has a higher borehole pressure than a case with fluid lag, producing a larger fracture front radius. To compare the magnitudes of borehole pressure with the same fracture radius, the dimensions for no fluid lag case have been obtained by interpolation of two given values. The borehole pressure and pressure-width response variable (β) with the same fracture length ($R=8.275$ [m]) in case of no fluid lag are 1.309 (Mpa) and 1.107, respectively. Therefore the borehole pressure with fluid lag is higher

Fig. 5.20 Pressure Distribution with respect to ρ
 (Case TT-G in Table 5.1 , $\rho=x/R(t)$)

than that with no fluid lag when they have the same fracture front radius.

It is assumed that hydraulic fracture propagates when the stress intensity factor associated with loading conditions exceeds the fracture toughness of the rock formation. Jeffrey [74] analyzed the fluid lag phenomenon in simpler geometry, such as rectangular and penny-shaped models, by assuming that the pressure is constant within a fracture. The closed form expressions for fluid lag are an extension of his results for various pressure distributions and detailed in Appendix B.

Figures 5.22 and 23 demonstrate the sensitivities of the fluid lag responses due to the variation of effective closure tip stress applied on dry zone and fracture toughness, respectively, corresponding to selected pressure profiles, illustrated in Fig. 5.21. These results show that the variation of pressure profile in a fracture gives significant change of fluid lag distance. However, the uniform fluid filled pressure distribution case provides a limiting bound for the fluid lag distance. In the energy rate formulation wherein pressure distribution is a variable, the distance of fluid lag, detailed in Appendix B, can be expressed as

$$d = d(R, \hat{\sigma}, \hat{K}, t) = R(t) \left(1 - \sqrt{1 - \{\epsilon(\hat{\sigma}, \hat{K}, t)\}^2} \right)$$

Fig. 5.21 Pressure Profiles in a Fracture with Fluid Lag

Fig. 5.22 Fluid Lag Distance versus Effective Fracture Tip Closure Stress with Different Pressure Profiles in a Fracture

Fig. 5.23 Fluid Lag Distance versus Fracture Toughness with Different Pressure Profiles in a Fracture

$$\text{where } \epsilon = \frac{(1-2\hat{\sigma}-3\hat{K}) + \sqrt{(1-2\hat{\sigma}-3\hat{K})(9+6\hat{\sigma}-3\hat{K})}}{4(\hat{\sigma}+1)}$$

$$\hat{K}(t) \equiv \frac{\sqrt{\pi}}{2} \frac{K_{IC}}{P_o(t)\sqrt{R(t)}} \quad \text{and} \quad \hat{\sigma}(t) \equiv \frac{(\sigma_o - P_w)}{P_o(t)}$$

It is obvious that the non-dimensionalized parameters, $\hat{K}$ and $\hat{\sigma}$, play major roles in the determination of fluid lag. The non-dimensionalized parameters are restricted by necessary conditions in order to yield a physical solution. The necessary inequalities describing the range of parameters for the pressure distribution are given by

$$0 < \hat{K}(t) < \frac{1}{3} \quad \text{and} \quad 0 < \hat{\sigma}(t) < \frac{1}{2}.$$

These conditions can serve as diagnostic indicators for the occurrence of fluid lag for a penny-shaped fracture. For example, if the effective borehole pressure, $P_o(t)$, drops below $\frac{1}{2}(\sigma_o - P_w)$, it is not possible to have fluid lag with the pressure distribution, $p(\rho) = P_o(1-\alpha\rho^2)$.

Considering that values of crack tip closure stress in the field are higher than those presented in the figures, it is concluded that the effects of fluid lag become negligible within a relatively short time interval for a penny-shaped fracture. However, significant fluid lag for an elongated fracture in multi-layered formation or 3-D rectangular (PKN model) fracture, can cause significant excess pressures.

CHAPTER VI

FIELD APPLICATIONS

In this chapter, the developed models (ELLIP2D and ERATE2D) are applied to simulate the Staged Field Experiment [SFE No.3 Cases #5 (Newtonian) and #6 (Non-Newtonian)] for unsymmetric three-layered formations [16]. These simulations demonstrate the overall versatility of the codes . A set of parametric sensitivity studies is presented with an emphasis on the roles of controllable variables in the given three-layered formation.

6.1 Staged Field Experiment Simulation

Currently the industry is far from reaching a consensus on hydraulic fracturing theory and modeling. Although several comprehensive hydraulic fracturing simulators are available for multi-layered formations, there exist considerable discrepancies in the responses computed from hydraulic fracturing simulators and their capabilities. Moreover, discrepancies of input data for simulations make it more difficult to compare the response results logically. Based on the guidelines set forth by

the committee of the Fracture Propagation Modeling Forum 1991 [94], a set of benchmark data has been provided for the comparisons of available simulator responses and sensitivity studies. Table 6.1 reveals the input data for the unsymmetric three-layered cases (Cases 5 and 6).

Prior to conducting simulations for the unsymmetric three-layered formation, benchmark tests for a single layered formation have been evaluated using ELLIP2D and ERATE2D with the same input data of the payzone in Table 6.1. Figures 6.1 and 6.2 show the normalized principal dimensions for the Newtonian (SFE No.3 Case 5) and non-Newtonian fluid benchmark cases (SFE No.3 Case 6), respectively. These simulators yield good agreement for the fracture opening width. ELLIP2D yields a shorter fracture radius, higher borehole pressure and better fracture efficiency. Since the opening profiles of ELLIP2D always remain elliptic in shape during the entire simulation while the profiles vary in the ERATE2D model, the simulator, ELLIP2D, generally gives higher efficiency although it has a shorter fracture radius. As expected, the injection of a non-Newtonian fluid produces the same trends. It is believed that the ERATE2D model is more rigorous than the ELLIP2D simulator, since ERATE2D accommodates changes in the pressure and opening width profiles during the simulation processes. Moreover,

Table 6.1 Input Data for SFE No.3 Cases 5 and 6

	Case 5 and Case 6	
Formation Properties Shear modulus (GPa) Poisson's ratio In situ Stress (MPa) Energy release rate (Pa-m) Payzone height (m)	Upper/Payzone/Lower 24.218/24.218/24.218 0.21/0.21/0.21 49.299/39.302/50.678 78.78/78.78/78.78 51.51 (170ft)	
Fluid Properties Consistency index (Pa-min ^m) Behavior index, m Leak-off coefficient (m/√min) Spurt loss (m)	Case 5	Case 6
	3.333x10 ⁻³ 1.0 0.76x10 ⁻⁴ 0.0	0.37089 0.5 0.76x10 ⁻⁴ 0.0
Injection Scheme Injection rate (m ³ /min) Total time (min)	7.9494 (50BPM) 200	

- * Benchmark tests for a penny shaped model also use the same input data of the payzone for Case 5 (Newtonian) and Case 6 (Non-Newtonian fluid case).

Fig. 6.1 Comparisons of Principal Fracture Responses for a Penny-Shaped Model by ELLIP2D and ERATE2D (Newtonian Fluid Benchmark Case, Normalized Dimensions : R_* = 1000 [ft], W_* = 1 [in], P_* = 500 [psi])

Fig. 6.2 Comparisons of Principal Fracture Responses for a Penny-Shaped Model by ELLIP2D and ERATE2D (Non-Newtonian Fluid Benchmark Case, Normalized Dimensions : $R_*=1000$ [ft], $W_*=1$ [in], $P_*=500$ [psi])

ERATE2D handles fluid leak-off effects without updating the local flow rate for the calculation of the dissipation energy and mass balance equations.

The unsymmetric fracture responses, listed in Tables 6.2 and 6.3, are monitored at 25 minute intervals. The predicted fracture response results obtained from the presented unsymmetric elliptic model simulators (ELLIP2D and ERATE2D), are compared with a generalized 3-D simulator (HYFRAC3D). The presented responses of ELLIP2D for unsymmetric three-layered models in Tables 6.2 and 6.3 are in general agreement with the generalized three-dimensional simulator (HYFRAC3D) except values of fracture length. The discrepancy in length increases as a fracture get highly contained. Results from the ERATE2D are also compared with ELLIP2D and HYFRAC3D. The overall response trends are generally comparable. In particular, ERATE2D gives a higher length and lower height and effective pressure values when compared with ELLIP2D and HYFRAC3D results. The lower pressure build up obtained from the ERATE2D model results in a lower penetration into the barriers where relatively high differential in situ stresses exist. It is noteworthy that the fracture pressure profile in the ERATE2D model varies during the simulation while that in the ELLIP2D model remains constant. Throughout the parametric sensitivity studies

presented in the previous chapter, it is demonstrated that in situ stress contrasts play an important role in the fracture configuration evolution. The fracture sensitivity configuration to in situ stress differentials in the ERATE2D model appears to be much higher than that exhibited in other models. The variations of the pressure-width response variables (α_p , α_u and α_ℓ), are monitored during the simulations. Initially, the same values for these variables are used. After the fracture hits barrier layers, these variables are assumed to behave independently. The pressure profile of the ERATE2D along the vertical direction is flat (pressure is constant along y direction) when the fracture becomes highly contained. The phenomenon is also observed in the PKN benchmark tests, detailed in chapter 5.1.2. As a graphic illustration of the response bounds, Figures 6.3 and 6.4 illustrate the ELLIP2D fracture configurations computed for the Newtonian and non-Newtonian fluid cases, respectively, evaluated at 25 minute intervals and compared with HYFRAC3D responses. The dotted line fracture configurations from the HYFRAC3D code are not monitored during the regular time steps, and the outer line presents the fracture configuration at $t=200$ min. The ELLIP2D model simulator gives a more confined geometry with a longer length, shorter height, and larger width,

Table 6.2 Principal Dimensions of SFE No.3 Case 5
(Newtonian Fluid Case)

Time (min)	†	Half Length (a:ft)	Frac. Height (b _u :ft)	Frac. Height (b _ℓ :ft)	Average Width (W:in)	Effect. Pressure (ΔP:psi)
25	A	865.1	143.9	136.9	0.176	889.2
	B	654.2	124.2	115.8	0.218	884.7
50	A	1304.1	155.8	146.5	0.209	990.1
	B	1068.5	143.3	130.6	0.220	971.6
75	A	1661.3	163.8	152.7	0.230	1046.8
	B	1303.0	158.0	140.5	0.245	1039.6
100	A	1961.1	170.0	157.4	0.245	1089.2
	B	1501.4	168.8	147.0	0.248	1047.7
125	A	2225.4	175.4	161.4	0.259	1123.9
	B	1635.1	176.7	151.0	0.265	1077.2
150	A	2462.8	180.1	165.0	0.271	1152.7
	B	1777.7	185.2	154.6	0.265	1089.9
175	A	2677.8	184.3	168.2	0.281	1176.2
	B	1982.5	192.7	159.4	0.238	1112.0
200	A	2879.0	188.3	171.2	0.290	1197.0
	B	2088.9	194.9	161.7	0.250	1112.6

† A : ELLIP2D

B : HYFRAC3D

Table 6.2 (continued)

Time (min)	★	Half Length (a:ft)	Frac. Height (b _u :ft)	Frac. Height (b _ℓ :ft)	Average Width (W:in)	Effect. Pressure (ΔP:psi)
		Pressure Response Var. a _p	Press. Response Var. a _u	Press. Response Var. a _ℓ	Max. Width (W ₁ :in)	Max. Width (W ₂ :in)
25	C	1039.9	145.0	138.9	0.146	626.9
		1.602	3.1E-2	2.9E-2	0.017	0.335
50	C	1689.0	152.2	143.9	0.164	685.2
		1.600	1.3E-2	1.2E-2	0.015	0.386
75	C	2232.2	155.1	147.0	0.176	722.7
		1.595	7.7E-3	6.9E-3	0.013	0.417
100	C	2714.2	158.0	149.4	0.184	750.4
		1.590	5.4E-3	4.8E-3	0.013	0.440
125	C	3155.1	160.4	151.2	0.191	772.5
		1.586	4.1E-3	3.6E-3	0.012	0.458
150	C	3563.8	162.3	152.8	0.197	790.7
		1.582	3.3E-3	2.9E-3	0.011	0.473
175	C	3948.4	164.1	154.2	0.202	806.3
		1.578	2.7E-3	2.4E-3	0.011	0.487
200	C	4312.4	165.6	155.3	0.206	819.8
		1.574	2.3E-3	2.0E-3	0.010	0.498

★ C : ERATE2D

**Table 6.3 Principal Dimensions of SFE No.3 Case 6
(Non-Newtonian Fluid Case)**

Time (min)	†	Half Length (a:ft)	Frac. Height (b _u :ft)	Frac. Height (b _ℓ :ft)	Average Width (W:in)	Effect. Pressure (ΔP:psi)
25	A	831.5	152.1	143.8	0.174	901.5
	B	904.8	131.4	117.1	0.165	864.8
50	A	1306.4	162.5	151.8	0.199	970.4
	B	1322.8	157.1	138.7	0.191	995.1
75	A	1708.9	169.1	156.8	0.213	1005.4
	B	1615.9	169.0	146.3	0.198	1017.4
100	A	2058.7	173.7	160.3	0.225	1032.8
	B	1789.9	178.5	150.7	0.218	1048.2
125	A	2373.7	177.5	163.1	0.234	1055.7
	B	2017.8	196.3	164.8	0.206	1105.6
150	A	2662.4	180.8	165.6	0.242	1074.2
	B	2188.2	201.6	170.5	0.213	1104.3
175	A	2929.6	183.7	167.7	0.250	1090.3
	B	2380.9	225.7	180.4	0.216	1142.3
200	A	3181.0	186.3	169.7	0.256	1104.0
	B	2424.2	244.0	190.5	0.211	1170.9

† A : ELLIP2D

B : HYFRAC3D

Table 6.3 (continued)

Time (min)	★	Half Length (a:ft)	Frac. Height (b _u :ft)	Frac. Height (b _ℓ :ft)	Average Width (W:in)	Effect. Pressure (ΔP:psi)
		Pressure Response Var. α _p	Press. Response Var. α _u	Press. Response Var. α _ℓ	Max. Width (W ₁ :in)	Max. Width (W ₂ :in)
25	C	1030.1	144.7	138.6	0.147	633.5
		1.600	5.5E-2	5.0E-2	0.017	0.338
50	C	1608.9	153.1	145.5	0.171	711.2
		1.596	2.6E-2	2.4E-2	0.014	0.403
75	C	2079.2	158.5	149.8	0.186	761.3
		1.588	1.7E-2	1.5E-2	0.012	0.446
100	C	2487.9	162.6	153.1	0.198	798.5
		1.580	1.3E-2	1.1E-2	0.011	0.477
125	C	2855.5	166.0	155.7	0.208	828.2
		1.573	1.0E-2	9.0E-3	0.011	0.502
150	C	3193.5	169.0	158.0	0.216	852.9
		1.566	8.4E-3	7.4E-3	0.010	0.524
175	C	3507.2	171.6	160.0	0.223	874.1
		1.560	7.2E-3	6.3E-3	0.010	0.542
200	C	3802.2	173.9	161.7	0.230	892.6
		1.554	6.3E-3	5.4E-3	0.009	0.559

★ C : ERATE2D

Fig. 6.3 Predicted Fracture Geometry Evolution
 (SFE No.3 Case 5 : Newtonian Case)

Fig. 6.4 Predicted Fracture Geometry Evolution
(SFE No.3 Case 6 : Non-Newtonian Case)

than the HYFRAC3D model. The normalized fracture dimensions obtained from ELLIP2D and HYFRAC3D are illustrated in Figures 6.5 and 6.6 for the Newtonian and non-Newtonian cases, respectively. At the first time station ($t=25\text{min}$), the non-Newtonian fluid case gives a larger fracture height, a slightly reduced half length and fracture opening width, and increased effective borehole pressure when compared to the Newtonian case because of the viscosity effect. HYFRAC3D gives higher lengths and lower pressures at the first time station. There exists a significant discrepancy among the three simulators (ELLIP2D, ERATE2D and HYFRAC3D) in the magnitudes of the fracture length and height because of the inherent assumptions. ELLIP2D and HYFRAC3D yield higher fracture lengths for the non-Newtonian case, while the ERATE2D model gives a smaller fracture length in comparison to the Newtonian case. However it is observed that the non-Newtonian fluid induces greater penetrations in the barrier layers and, as expected, an increased fracture opening width than the Newtonian case for the ERATE2D model. One of the limitations the ELLIP2D model is that fracture opening width profile and corresponding pressure distribution profile in a fracture remain constant, even for the three-layered formation case with in situ stress contrasts. Although the opening width profile varies with

Fig. 6.5 Normalized Fracture Responses versus Time
 (SFE No.3 Case 5 : Newtonian Fluid Case, by
 ELLIP2D, Normalized Dimension : $a_*= 4000$ [ft],
 $b_*= 250$ [ft], $P_*= 3000$ [psi], $W_*= 1$ [in])

Fig. 6.6 Normalized Fracture Responses versus Time
 (SFE No.3 Case 6 : Non-Newtonian Fluid Case, by
 ELLIP2D, Normalized Dimensions : $a_*= 4000$ [ft],
 $b_*= 250$ [ft], $P_*= 3000$ [psi], $W_*= 1$ [in])

the dual components corresponding to the crack singular behavior and the equilibrium crack growth condition in the ERATE2D model, the crack opening profile does not change as it encounters a relatively high in situ stress contrasts in a limited portion of the total fracture area. For more accurate modeling, it is advisable to include an additional generalized coordinate to accommodate the opening profile variations along the x and y directions and associated effective pressure profile variation.

Modification for in situ Stress Contrast Cases

As indicated, although the ELLIP2D model gives reasonable overall results for an unsymmetric three-layered formation, it generally yields a larger fracture length and smaller height for cases involving in situ stress contrasts. Obviously, the fracture opening profile associated with large in situ stress differentials departs significantly from the elliptic shape. The formation potential energy can be rewritten as

$$\begin{aligned}
 U_p &= \int_A P_e(\underline{x}, t) w(\underline{x}, t) \, dA \\
 &= \int_A P_e w(\underline{x}, t) \, dA - \Delta\sigma_u \int_{A_u} w(\underline{x}, t) \, dA - \Delta\sigma_\ell \int_{A_\ell} w(\underline{x}, t) \, dA \\
 &= P_o V_T - \Delta\sigma_u V_{bu} - \Delta\sigma_\ell V_{b\ell} \quad .
 \end{aligned}$$

Since the fracture volumes in the barriers are highly overestimated for this formulation, the fracture tends to

grow laterally rather than across the barriers.

For a realistic simulation, the fracture opening profile should be modified in accordance with in situ stress contrasts. The compensation for the case with in situ differentials can be accomplished as follows :

1) Method I

In this method, the potential energy term is compensated by obtaining P_{eq} from the equivalent stress intensity for the in situ stress contrast case i.e.

$$P_{eq}^j = \frac{K}{\sqrt{\pi b}} = P_o \left[1 - \frac{\Delta\sigma_j}{P_o} \left\{ 1 - \frac{2}{\pi} \sin^{-1}(h_j/b_j) \right\} \right]$$

Therefore the potential energy term associated with the equivalent pressure, P_{eq} , is

$$\begin{aligned} U_p &= \sum \int_{A_j} P_{eq}^j w(\underline{x}, t) dA \\ &= \sum \int_{A_j} P_o w_{eq}^j dA \end{aligned}$$

$$\text{where } w_{eq}^j = w(\underline{x}, t) \left[1 - \frac{\Delta\sigma_j}{P_o} \left\{ 1 - \frac{2}{\pi} \sin^{-1}(h_j/b_j) \right\} \right]$$

2) Method II

The crack opening profile is selected to consist of two components as in Van Eekelen's work [15]. The first component is assumed to be associated with the effective pressure, P_o , and the second component is assumed to result from the differential in situ stress, $-\Delta\sigma_j$, prevalent in the barriers. From the continuity condition

at the interface between the payzone and barriers, the compensated width can be expressed as

$$c_j^* = c_j \frac{b_j}{h_j} \frac{(1 - \frac{\Delta\sigma_j}{P_o})}{\sqrt{1 + ((\frac{b_j}{h_j})^2 - 1)(1 - \frac{\Delta\sigma_j}{P_o})^2}}$$

3) Method III

The pressure profile for the three-layered formation models is considerably different from that for the isotropic and axisymmetric models due to in situ stress differentials. The opening profile in the barriers is taken to approximate a realistic opening profile incorporating in situ stress contrast, i.e.

$$c_j^* = c_j [1 - \frac{\Delta\sigma_j}{P_o} (\frac{h_j}{b_j})^{1/2}] .$$

Consequently the opening width profiles are represented by dual quantities, where two components present quantities in the payzone and barriers, respectively. This approximation for the fracture opening width profile in the presence of in situ stress contrasts is similar to Van Eekelen's approach [15] with the two different opening width profiles satisfying the continuity condition at interfaces. The local flow rate component change due to the change of opening width profile affects the calculation of dissipation energy components. However, considering that the portion of the volume affected in the

total volume and the magnitude of local flow rate around the tip of the fracture is relatively small, it is assumed that the influence to the dissipation energy component due to the change of width profile is negligible. This assumption provides no problems with convergence, because the potential energy terms due to the high in situ stress contrast is involved in the numerical calculation in a smooth way (no sudden jump across the interface).

After careful consideration of the possible methods for accommodating in situ stress contrasts, method III is selected for improving the results derived from ELLIP2D for cases with in situ stress contrast.

Generally ELLIP2D gives realistic agreements when compared to those for HYFRAC3D with high in situ stress contrasts. Based on these simulation results, further parametric sensitivity studies are conducted with an emphasis on the controllable variables. As a first step, injection rate sensitivities are examined in the next section.

6.2 Parametric Sensitivity Studies

Among the various controllable variables, the injection rate plays a central role in the determination of the fracture configuration. Figures 6.7-12 illustrate the role of injection fluid flow rate on the behavior of

Fig. 6.7 Normalized Half Length versus Normalized Volume
 (SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

Fig. 6.8 Normalized Upper Height versus Normalized Volume
(SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

Fig. 6.9 Normalized Lower Height versus Normalized Volume
 (SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

Fig. 6.10 Normalized Maximum Fracture Width versus Normalized Volume (SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

Fig. 6.11 Normalized Net Fracture Fluid Pressure
 versus Normalized Volume (SFE No.3 Case 6 :
 Non-Newtonian Case by ELLIP2D)

Fig. 6.12 Fluid Efficiency versus Normalized Volume
 (SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

fracture propagation. The same amount of injection fluid is injected for each case with a different injection rate. To check the overall trend with respect to the injection rate, fluid volume and the principal fracture dimensions are normalized. The normalized volume denotes the ratio of the injected volume with respect to the total scheduled volume. The fracture dimensions are non-dimensionalized by using $a_* = 4000\text{ft}$, $b_* = 250\text{ft}$, $w_* = 4\text{in}$, and $P_o^* = 3000\text{psi}$, respectively. The higher pumping rates yield a better efficiency due to lower leak-off volumes into the formation (Fig. 6.12). The higher injection rate tends to produce monotonically increased upper and lower fracture heights, as well as pressure. It is noteworthy that the fracture length, shown in Fig. 6.7, which is closely related to the effectiveness of the stimulation treatment, behaves differently in comparison to the other response trends. If the leak-off differentials in the layers exist, it is obviously impossible to predict the optimum injection rate without numerical simulations. Since the effectiveness of the stimulation processes is directly related to the generated effective area, where proppants can be embedded to prop the fracture open after injection, it may be assumed that the fractured area in payzone can be represented as the effective area. The normalized effective area, A_{eff}^* , defined as a fractured area in

payzone divided by the payzone area parameter, H^2 , is introduced to highlight this effect (Fig 6.13). These results show that the selected injection rate ($i_o = 50$ BPM) gives reasonable results in comparison to other pumping rates. However, the injection rate may be reduced to 25 BPM (curve 3 in Fig. 6.13) to have maximum effectiveness with respect to the area generated in the payzone at the cessation of stimulation.

Additional parametric sensitivity studies have been conducted to check the effects of viscosity and fluid leak-off. Figures 6.14-15 illustrate the effect of fluid viscosity on the fracture height and opening width for the three-layered case SFE #3 Case5 (Newtonian). As mentioned previously, a low viscosity fluid generally yields a higher fracture length and lower borehole pressure build-up in the stimulation. Although the pressure build-up causes the fracture to break into the upper/lower barriers to some extent, significant in situ stress differentials over the barriers force the fracture to be contained in the payzone. Therefore, appropriate selection of treatment fluid may be decided by the in situ stress characteristics of the reservoir formation. Figures 6.16 - 6.18 illustrates the effect of fluid leak-off into the formations. The fluid leak-off plays an important role to the fracture evolution configuration.

Fig. 6.13 Normalized Effective Area versus Normalized Volume (SFE No.3 Case 6 : Non-Newtonian Case by ELLIP2D)

Fig. 6.14 Normalized Fracture Half Length (a) versus Time
 (Sensitivity Studies on Fracture Fluid Viscosity
 SFE No.3 Case 5 : Newtonian Case by ELLIP2D)

Fig. 6.15 Normalized Fracture Height (b_u) versus Time
 (Sensitivity Studies on Fracture Fluid Viscosity
 SFE No.3 Case 5 : Newtonian Case by ELLIP2D)

Fig. 6.16 Normalized Fracture Half Length (a) versus Time
 (Sensitivity Studies on Fluid Leak-off Effects
 SFE No.3 Case 5 : Newtonian Case by ELLIP2D)

Fig. 6.17 Normalized Fracture Height (b_u) versus Time
(Sensitivity Studies on Fluid Leak-off Effects
SFE No.3 Case 5 : Newtonian Case by ELLIP2D)

Fig. 6.18 Fracture Fluid Efficiency versus Time
(Sensitivity Studies on Fluid Leak-off Effects
SFE No.3 Case 5 : Newtonian Case by ELLIP2D)

The fracture length is considerably influenced by the fluid leak-off, but relatively less dominated by its differentials in case of high in situ stress contrasts because the fracture is highly contained in the payzone and the area exposed in the barriers is comparatively small (Fig. 6.16). The fracture penetration into the barrier layers is directly influenced by the leak-off differentials. The cases with leak-off differentials (3 : $0/C_L/0$ and 4 : $0/C_L/C_L$ in Fig. 6.17) apparently give increased fracture heights (b_u) in comparison to the case without leak-off differentials (5 : $C_L/C_L/C_L$). It is obvious that the fracture efficiency is considerably dominated by the magnitude of fluid leak-off (Fig. 6.18). These plots provide information on the role of fluid leak-off and leak-off coefficient differentials in the barriers.

A series of sensitivity studies has been conducted using ELLIP2D for the three-layered formations. It is demonstrated the simulator is capable of handling fracture simulations involving layer contrasts. ELLIP2D also has a capability of incorporating time dependent injection rates. Pertinent power components during the fracture evolution are discussed in the next chapter. Since the hydraulic fracturing simulator introduced in chapter III enables us to observe how the total input power driven by

the injection of fracturing fluid is converted into other energy rate components during the process, the characteristics of hydraulic fracturing for a given formation and roles of uncontrollable parameters and controllable variables can be systematically investigated by tracking the input power transformation.

Comparison of Computation Efficiencies

The execution of one iteration using ELLIP2D for the cases presented in the previous chapter, with layer differentials, takes about 0.3-0.4 seconds on the CRAY-YMP system versus 2.7-3.6 seconds on the VAX 8550 system. Since six to nine iterations are needed for one solution step of an unsymmetric model, the total execution time is about 40 seconds on the CRAY-YMP (6 minutes on VAX 8550) assuming that ten different step solutions for an unsymmetric three-layered case are needed. For the case of the ERATE2D, to achieve good convergency, relatively large amount of steps are necessary in obtaining solutions for SFE #3 cases. Currently 78 steps are used and 349 total iterations are required for the case with in situ stress contrasts. The total execution for each case takes about 32 seconds on the CRAY-YMP. Therefore 0.09 seconds are needed for an iteration and 4.47 iterations are required for a step in an average sense. The total CPU

time is determined from the number of differential quantities involved and the required accuracy of the numerical calculations in the simulations. It is found that ERATE2D requires considerable CPU time if differential leak-off or elastic moduli are involved because additional numerical integrations for the calculation involving the differential values are necessary for each iteration. As far as computational time efficiency, it is desirable to use these models based on the complexity and purpose of stimulations. Throughout the numerical simulation, ERATE2D also gives additional information about the power components for each step. Considering the overall performance, ERATE2D is a superior simulator. Depending on the required solution accuracy, the CPU time may be shortened greatly by reducing the number of integration points for the numerical calculation of dissipation energy rate functions for both ELLIP2D and ERATE2D models.

CHAPTER VII

DESIGN GUIDELINES FOR HYDRAULIC FRACTURING

7.1 Evaluation of Hydraulic Fracture Control Parameters

For the optimum design and control of hydraulic fracturing, it is important to evaluate the relative roles of uncontrollable parameters and controllable variables in the stimulation processes. The variational formulations, detailed in chapter III and IV, provide an explicit form for energy rate components, which can also be expressed in terms of the first law of thermodynamics. The energy rate conservation law governing the fracture fluid control volume and reservoir elasticity enables us to monitor pertinent energy rate components associated with selected uncontrollable parameters (reservoir mechanical properties) and controllable variables (injection rate, fracture fluid rheological properties including the control of fluid leak-off, and perforation placement) during the hydraulic fracture configuration evolution. Systematic investigations on transformation of the total input power provided by injection of the treatment fluid and its distribution into other forms of power components during the process as a function of these uncontrollable

parameters and controllable variables provide valuable information on configuration control and optimum design of fracturing treatment. The energy rate components from the fracture fluid control volume energy rate principle, introduced in section 3.2.2, can be rewritten from the equation (3.18) after neglecting the body force contribution as

$$D_I = D_f + D_L + D_\eta \quad (7.1)$$

$$\text{where } D_I = \int_{\partial A_q} p q_n ds$$

$$D_f = \int_A p \dot{w} dA$$

$$D_L = \int_A p q_L dA$$

$$\text{and } D_\eta = - \int_A \underline{g} \cdot (\nabla p - \underline{f}) dA .$$

The total power input (D_I) derived from the pressurized injection of the treatment fluid is the sum of the energy rate components associated with the formation opening due to the fluid pressure distribution (D_f), power loss due to fluid leak-off into the formations (D_L), and fracturing fluid dissipation (D_η).

The power component due to the fluid pressure induced fracture opening (D_f), transmitted by the crack surface, is further transformed into strain energy component ($\frac{d}{dt}U_s^*$) and the fracture surface energy rate ($\dot{U}_f$) expended for the crack propagation. Therefore the reservoir energy rate conservation principle governing the formation

structural responses is given by

$$\begin{aligned}
 D_f &= \left\{ \frac{d}{dt} \dot{U}_s^* \right\} + \dot{U}_f \\
 &= \left\{ \frac{d}{dt} \left[\int_A \frac{1}{2} w K_A[w] dA + \int_A \sigma_o(\underline{x}) w dA \right] \right\} + \frac{d}{dt} \int_A G_{cr} dA \\
 &= \left\{ \int_A \dot{w} K_A[w] dA - \frac{d}{dt} \int_{\partial A} G \dot{a}_n dS + \int_A \sigma_o(\underline{x}) \dot{w} dA \right\} \\
 &\quad + \frac{d}{dt} \int_A G_{cr} dA \\
 &= \left\{ \dot{U}_s + \dot{U}_\Gamma + \dot{U}_{\sigma_o}^* \right\} + \dot{U}_f \tag{7.2}
 \end{aligned}$$

$$\begin{aligned}
 \text{where } \dot{U}_{\sigma_o}^* &\equiv \int_A \sigma_o(\underline{x}) \dot{w} dA \\
 &= \sigma_o \int_A \dot{w} dA + \sum \Delta \sigma_o^j \int_{A_j} \dot{w} dA \\
 &\equiv \dot{U}_{\sigma_o} + \dot{U}_{\Delta \sigma_o} .
 \end{aligned}$$

We note that $\dot{U}_{\sigma_o}$ and $\dot{U}_{\Delta \sigma_o}$ are the energy rate components associated with the payzone minimum in situ stress and stress differentials in the barriers, respectively.

The total input power balance equation, obtained by combining eqns (7.1) and (7.2), is

$$D_I = \left\{ \dot{U}_s + \dot{U}_\Gamma + \dot{U}_{\sigma_o} + \dot{U}_{\Delta \sigma_o} \right\} + \dot{U}_f + D_\eta + D_L \tag{7.3}$$

The pressure gradient dissipation term can be divided into two components, comprising of the borehole effective pressure and the minimum in situ stress terms as follows :

$$D_L = \int_A p(\underline{x}) q_L(\underline{x}) dA$$

$$\begin{aligned}
&= \int_A P_O (1 - \alpha \rho^2) q_L(\underline{x}) \, dA + \sigma_O \int_A q_L(\underline{x}) \, dA \\
&\equiv D_{L1} + D_{L2}
\end{aligned} \tag{7.4}$$

Similarly, the total instantaneous input power can be also expressed in the form

$$\begin{aligned}
D_I &= \int_{\partial A_q} p \, q_n \, ds = P_O i_O + \sigma_O i_O \\
&\equiv D_{Ieff} + D_{I2}
\end{aligned} \tag{7.5}$$

where $D_{Ieff} = P_O i_O$ represents the power available for transformation into other forms of energy rate during the fracturing stimulation and $D_{I2} = \sigma_O i_O$ denotes the base line power required to negate payzone in situ stress effects.

Therefore the total input power can be expressed in terms of six power components in the form

$$D_I = \left\{ \frac{dU_{s1}}{dt} + \frac{dU_{s2}}{dt} \right\} + \dot{U}_f + D_\eta + D_{L1} + D_{L2} \tag{7.6}$$

where $\frac{dU_{s1}}{dt} = \dot{U}_s + \dot{U}_\Gamma + \dot{U}_{\Delta\sigma_O}$

$$= \int_A \dot{w} \, K_A[w] \, dA - \frac{d}{dt} \int_{\partial A} G \, \dot{a}_n \, dS + \sum \Delta\sigma_O^j \int_{A_j} \dot{w} \, dA$$

and $\frac{dU_{s2}}{dt} \equiv \dot{U}_{\sigma_O}$.

Normalization of equation (7.6) with respect to D_I provides a useful basis for assessing pertinent energy rate components, i.e.

$$\frac{\left[\frac{dU_{s1}}{dt} + \frac{dU_{s2}}{dt} \right]}{D_I} + \frac{\dot{U}_f}{D_I} + \frac{D_\eta}{D_I} + \frac{[D_{L1} + D_{L2}]}{D_I} = 1 \quad (7.7)$$

Since $\sigma_o i_o = \sigma_o \int_A \dot{w} dA + \sigma_o \int_A q_L(\underline{x}) dA$, D_{I2} may be rewritten as $D_{I2} = \frac{dU_{s2}}{dt} + D_{L2}$, which is equivalent to the volume conservation equation multiplied by the payzone in situ stress as a weight. The available effective power can then be found from equation (7.6)

$$D_{Ieff} = \frac{dU_{s1}}{dt} + \dot{U}_f + D_\eta + D_{L1} \quad (7.8)$$

Equation (7.8) can now be normalized with respect to the available effective power, D_{Ieff} , to give

$$\frac{\frac{dU_{s1}}{dt}}{D_{Ieff}} + \frac{\dot{U}_f}{D_{Ieff}} + \frac{D_\eta}{D_{Ieff}} + \frac{D_{L1}}{D_{Ieff}} = 1 \quad (7.9)$$

Numerical results are presented in the subsequent sections to illustrate the variation of the energy rate components during the hydraulic fracturing treatment. The energy rate component versus time trends demonstrate the role of uncontrollable parameters and controllable variables.

Explicit asymptotic solutions in the absence of fluid leak-off effects obtained by Advani et al [70] predict the distribution of energy rates for the bounding cases of dissipation dominant and fracture dominant regimes for a penny shaped fracture. Prior to complex numerical

simulations, the two explicit asymptotic solutions for penny-shaped and PKN models in the absence of fluid leak-off are briefly presented below.

7.1.1 Penny-Shaped Fracture Model: No Leak-off

For a penny-shaped fracture model, the pertinent energy rate terms can be simplified as

$$D_I = P_o i_o + \sigma_o i_o \quad (7.10)$$

$$\begin{aligned} \frac{dU_s^*}{dt} = \frac{d}{dt} \left[\frac{\pi^2 \mu R}{4(1-\nu)} \left\{ W_1 \left(\frac{1}{3} W_1 + \frac{1}{5} W_2 \right) + W_2 \left(\frac{1}{5} W_1 + \frac{6}{35} W_2 \right) \right\} \right. \\ \left. + \sigma_o 2\pi R^2 \left(\frac{1}{3} W_1 + \frac{1}{5} W_2 \right) \right] \quad (7.11) \end{aligned}$$

where the dual components of the crack opening width, W_1 and W_2 , are defined for a penny-shaped fracture from eqn (3.24) and

$$\frac{dU_f}{dt} = \frac{d}{dt} \left[G_{cr} \pi R^2 \right] \quad (7.12)$$

$$D_\eta = \frac{2\pi m}{2m+1} (\eta_o)^{\frac{-1}{m}} R^{\frac{m-1}{m}} (\alpha P)^{\frac{m+1}{m}} I(W_1, W_2, m) \quad (7.13)$$

where $I(W_1, W_2, m) \equiv \int_0^1 [\rho(1-\rho^2)^{1/2} \{W_1 + W_2(1-\rho^2)\}]^{(2m+1)/m} d\rho$.

i) Penny-Shaped Fracture with Negligible Dissipation

Energy and No Leak-off

Since $C_L = 0$, $\eta_o = 0$, $\alpha = 0$, and $W_2 = 0$, from eqns. (7.1 and 7.2) we obtain

$$D_I = \frac{dU_s^*}{dt} + \dot{U}_f \quad (7.14)$$

In the case of no leak-off, $i_o = \frac{2\pi}{3} \frac{d}{dt} \left[R^2 W_1 \right]$ and

$$\begin{aligned} \frac{dU_s^*}{dt} &= \frac{d}{dt} \left[\frac{\pi^2 \mu R}{4(1-\nu)} \frac{1}{3} W_1^2 + \sigma_o i_o \right] \\ &= \frac{d}{dt} \left[\frac{2\pi}{3} G_{cr} R^2 \right] + \sigma_o i_o \end{aligned} \quad (7.15)$$

$$\frac{dU_f}{dt} = \frac{d}{dt} \left[G_{cr} \pi R^2 \right] \quad (7.16)$$

Since $\frac{dU_{s1}}{dt} = \frac{dU_s^*}{dt} - \sigma_o i_o$, the energy rate ratio can be expressed as

$$\frac{dU_f}{dt} / \frac{dU_{s1}}{dt} = 3/2 \quad (7.17)$$

This result reveals that the percentages of $\frac{dU_f}{dt}$ and $\frac{dU_{s1}}{dt}$ with respect to the effective input power (D_{Ieff}) for this case are approximately 60% and 40%, respectively.

ii) Penny-Shaped Fracture with Negligible Fracture Energy and No Leak-off

Since $C_L = 0$, $G_{cr} = 0$, $\alpha = 3/2$, and $W_1 = 0$, W_2 and i_o can be rewritten by utilizing relationships obtained from equations (3.30 and 3.62), as

$$W_2 = \frac{8(1-\nu)}{3\pi\mu} P_o R \quad (7.18)$$

$$i_o = \frac{d}{dt} \left(\frac{2\pi}{5} W_2 R^2 \right) \quad (7.19)$$

we obtain the energy rate ratio by utilizing the closed form solutions (eqns 3.64a-d) for a penny-shaped fracture model, also detailed in Appendix A.1 and A.3 given by

$$D_\eta / \frac{dU_{s1}}{dt} = \frac{10+7m}{4} \quad (7.20)$$

where m is the flow behavior index.

The ratio in (7.20) is maximum when the fluid is Newtonian ($m=1$), with a value of 4.25. Therefore the percentages of D_η and $\frac{dU_{s1}}{dt}$ with respect to the effective input power (D_{Ieff}) for this case are 81 % and 19 %, respectively.

7.1.2 PKN Model : No Leak-off

For a PKN model, the pertinent energy rate terms can also be simplified as in the penny-shaped fracture cases in the form

$$\frac{dU_{s1}}{dt} = \frac{\pi\mu}{(1-\nu)} \left[a \left\{ \dot{W}_1 \left(\frac{1}{3} W_1 + \frac{1}{5} W_2 \right) + \dot{W}_2 \left(\frac{1}{5} W_1 + \frac{6}{35} W_2 \right) \right\} + \dot{a} \left(\frac{1}{6} W_1^2 + \frac{1}{10} W_1 W_2 + \frac{6}{35} W_2^2 \right) \right] \quad (7.21)$$

$$\frac{dU_f}{dt} = \frac{d}{dt} \left[\frac{\pi}{2} G_{cr} Ha \right] \quad (7.22)$$

$$D_\eta = \frac{m}{2(2m+1)} \eta_o^{\frac{-1}{m}} (aP)^{\frac{m+1}{m}} H a^{\frac{-1}{m}} N_1(m) I(W_1, W_2, m) \quad (7.23)$$

where $N_1(m) \equiv 4 \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{1+m}{m}} d\theta$. The PKN model for the comparison of energy rate ratio is assumed to be highly elongated ($a \gg H$).

i) **PKN Model with Negligible Dissipation Energy and No Leak-off**

The pertinent energy rate components for a PKN model with negligible dissipation energy and no leak-off can be simplified as

$$\frac{dU_{s1}}{dt} = \frac{d}{dt} \left[\frac{\pi\mu}{(1-\nu)} \frac{1}{6} a W_1^2 \right] \quad (7.24)$$

$$\frac{dU_f}{dt} = \frac{d}{dt} \left[G_{cr} \frac{\pi}{2} Ha \right] \quad (7.25)$$

since $i_o = \frac{d}{dt} \left[\frac{\pi}{3} Ha W_1 \right]$ and $W_1^2 = 3G_{cr} H \frac{(1-\nu)}{\mu}$ for this case, the energy rate ratio is given by

$$\frac{dU_f}{dt} / \frac{dU_{s1}}{dt} = 1 \quad (7.26)$$

It reveals that two energy rate components $\left(\frac{dU_{s1}}{dt} \right)$ and $\left(\frac{dU_f}{dt} \right)$ transformed from the effective input power (D_{Ieff}) remain equal during the hydraulic fracturing process.

**ii) PKN Model with Negligible Fracture Energy
and No Leak-off**

For the penny-shaped model case, $C_L = 0$, $G_{cr} = 0$, $W_1 = 0$, $\alpha_p = 3/2$. The pressure-width response variables along vertical directions, α_u and α_ℓ , are assumed to be zero for PKN model cases when a fracture is highly elongated. The assumption is validated through numerical simulations by ERATE2D and also demonstrated in the following section by a case with high in situ stress contrasts (Fig. 7.10). Since W_2 and i_o can be rewritten from the relationships obtained from equations (3.30 and 3.62), as

$$W_2 = \frac{2}{3} \frac{(1-\nu)}{\mu} P_o H \quad (7.27)$$

$$i_o = \frac{d}{dt} \left(\frac{\pi}{5} Ha W_2 \right) \quad (7.28)$$

we obtain the energy rate ratio by utilizing the closed

form solutions for a PKN model through equations (3.66b-d) in the form

$$D_{\eta} / \frac{dU_{s1}}{dt} = I_s(m) [A_1(m)]^{\frac{-(1+m)}{m}} [A_2(m)]^{\frac{1}{m}} [A_3(m)]^{\frac{1+m}{m}} \quad (7.29)$$

$$\text{where } I_s(m) \equiv \frac{35m}{6\pi(m+3)(2m+1)} \left(\frac{3}{2}\right)^{(1+m)/m} N_1(m)N_2(m)$$

$$\text{and } N_2(m) \equiv \int_0^1 [\rho(1-\rho^2)^{1/2}]^{(2m+1)/m} d\rho.$$

The ratio in (7.29) is 0.675 in case of Newtonian fluid case ($m=1$). Therefore the percentages of D_{η} and $\frac{dU_{s1}}{dt}$ with respect to the effective input power are 40 % and 60 %, respectively.

Since the explicit solutions are available only for limiting cases, the numerical simulator presented here can serve as an efficient design tool for general cases, where both fracture as well as dissipation energy components including leak-off are not negligible.

7.1.3 Relationship Between Fluid Dissipation Functionals and Energy Rate Components

It is noteworthy that the expression for the fluid dissipation term in the Lagrangian formulation is different from that in the energy rate formulation because of their basic definitions, similar to energy and complementary energy formulations in the theory of elasticity for non-Hookean materials. The dissipation

energy term, D_q , in the Lagrangian formulation can be written as

$$D_q = - \frac{1}{1+m} \int_{A(t)} \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA \quad (7.30)$$

Meanwhile the energy rate dissipation function in the fracture fluid control volume energy rate principle in chapter III, denoted as D_η , is given by

$$D_\eta = \bar{\eta} \int_A \frac{|\underline{q}|^{1+m}}{w^{2m+1}} \, dA = - \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA \quad (7.31)$$

The pressure gradient dissipation used in the functional for the variational formulation also can be rewritten as

$$\begin{aligned} D_p &= - \frac{\bar{\eta}}{1+m} \int_A \frac{|\underline{q}|^{1+m}}{w^{2m+1}} \, dA - \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA \\ &= \frac{1}{1+m} \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA - \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA \\ &= - \frac{m}{1+m} \int_A \underline{q} \cdot (\nabla p - \underline{f}_g) \, dA \\ &= \frac{m}{1+m} D_\eta \end{aligned} \quad (7.32)$$

Therefore, the pertinent relationships between these dissipation energy rate terms can be summarized in the form

$$D_p = m D_q = \frac{m}{1+m} D_\eta \quad (7.33)$$

7.2 Evaluation of Energy Rate Components: Penny-Shaped Fracture Benchmarking

For designing an efficient fracture treatment, it is important to know the relative dominance of the pertinent

energy during selected simulations. As detailed in chapter V, fracture propagation regimes can be characterized as follows :

i) fluid dissipation dominant regime, ii) fracture energy dominant regime, or iii) combined fracture energy and fluid dissipation regimes.

An efficient methodology for checking the governing regime(s) can be implemented through the characteristic time concepts, proposed by Lee et al [66]. An alternative but more general approach is the use of the ERATE2D model for representing the energy rate components. Design methodologies can then be based on criteria delineating the preferred energy rate distribution.

Prior to conducting simulations to evaluate the energy rate components for three layered formations and associated design criteria, some benchmarking is performed for a penny-shaped model. Figure 7.1 shows the energy rate components normalized with respect to the input power for a penny-shaped fracture, using input data given in Table 6.1. In this case (Newtonian fluid case), the dissipation energy ratio, D_η/D_{Ieff} , is dominant throughout the entire treatment and is about 80 % of the effective power while the fracture energy rate remains negligible (around 1 %) during the entire stimulation. Obviously, an

Fig. 7.1 Effective Energy Rate Component Time Evolution for a Penny-shaped Benchmark Model Corresponding to SFE No.3 Case 5

increase in the other rate components can be accomplished primarily through the fluid dissipation power component by selecting a fluid with a lower viscosity so that the efficiency of the stimulation treatment is increased. Fig. 7.2 (Newtonian fluid case) quantitatively demonstrates this observation with a lower fluid viscosity fluid showing a decreased D_{η}/D_{Ieff} but at the expense of an increased D_{L1}/D_{Ieff} . Since the reduction of fluid viscosity generally yields a higher fracture radius and lower fracture width, the fluid leak-off increases due to the increased area exposure.

The leak-off coefficient sensitivity of the energy rate components is monitored in Fig. 7.3. Fracture responses, especially fracture radius, are significantly influenced by the fluid leak-off (Fig. 7.4). Fluid loss behavior is determined by the type and quantity of gelling agent and additives against fluid loss, permeability, porosity of the formation, and pressure differential between the fracture and the formation. Leak-off is also influenced by the formation fluid viscosity and temperature. It has been found that a hydrocarbon phase (e.g. 5% diesel) can reduce fluid loss significantly in tight, low-permeable formations [95]. Although permeability and porosity of the target formation are

Fig. 7.2 Effective Energy Rate Component Changes due to Fluid Viscosity Variation (Penny-shaped Benchmark Model for SFE No.3 Case 5)

Fig. 7.3 Effective Energy Rate Component Changes due to Fluid Leak-off Variation (Penny-shaped Benchmark Model for SFE No.3 Case 5)

Fig. 7.4 Normalized Fracture Radius versus Time
(Penny-shaped Benchmark Model for SFE No.3
Case 5)

specified as uncontrollable parameters, the selection of the appropriate fluid combinations, such as foam, can significantly control the leak-off effects.

One of the advantages of the ERATE2D model when compared to other simulators is that it monitors not only the fracture evolution but also the temporal characteristics of pertinent energy rates and related indices.

Some benchmarking through parametric sensitivity studies for a penny-shaped model is performed. The principal dimensions at the cessation of the stimulation ($t=200\text{min}$) are shown in Tables 7.1-1 through 7.1-5. These tables provide a quick overview of the overall sensitivity studies. It is also possible to distinguish the dominance of parameters and their degree of sensitivity. Table 7.1-1 illustrates the fracture response variation with respect to changes in the fluid leak-off coefficient. The fracture responses resulting from the variation in leak-off behavior exponent and viscosity change in Newtonian fluid are shown in Table 7.1-2. Although the leak-off behavior exponent, γ , is conventionally taken as 0.5, additional investigations on the appropriate value of this exponent are needed for improved simulations involving a variety of treatment fluids such as polymer emulsion, water and oil based gels including different

Table 7.1-1 Benchmark Tests for Parametric Sensitivity Studies
(Effects of Fluid Leak-off, C_L)

Case No.	Case Descriptions	Fracture Length (R:ft)	Maximum Width (W:in)	Effect. Pressure (ΔP :psi)	ΔP -Width Response Vari. (a)	Efficiency (%)
1	Newtonian Benchmark	924.6	0.2853	126.3	1.300	51.2
2	Non-Newton. Benchmark	866.8	0.3701	177.7	1.353	56.7
3	$C_L = 0$ (Newtonian)	1240.0	0.3074	100.9	1.284	100.0
4	$C_L = 0$ (Non-Newton)	1090.0	0.4127	157.6	1.352	100.0
5	$C_L = C_L^*/4$ (Non-Newton)	1055.0	0.4065	160.4	1.353	92.2
6	$C_L = C_L^*/2$ (Non-Newton)	963.2	0.3894	168.2	1.353	73.7
2	$C_L = C_L^*$ (Non-Newton)	866.8	0.3701	177.7	1.353	56.7
7	$C_L = 2C_L^*$ (Non-Newton)	731.0	0.3402	193.7	1.353	37.1
8	$C_L = 4C_L^*$ (Non-Newton)	575.3	0.3019	218.4	1.353	20.4

* Principal dimensions are at the cessation of stimulation ($t = 200$ min).

Table 7.1-2 Benchmark Tests for Parametric Sensitivity Studies
(Effects of Leak-off Behavior Exponent [γ] and Viscosity [η])

Case No.	Case Descriptions	Fracture Length (R:ft)	Maximum Width (W:in)	Effect. Pressure (ΔP :psi)	ΔP -Width Response Vari. (a)	Efficiency (%)
9	$\gamma = 0.2$ (Non-Newton)	641.2	0.3132	203.1	1.351	26.3
2	$\gamma = 0.5$ (Non-Newton)	866.8	0.3701	177.7	1.353	56.7
10	$\gamma = 0.8$ (Non-Newton)	943.1	0.3824	168.7	1.352	69.4
11	$\eta = \eta_o/20$ (Newtonian)	1045.0	0.1397	50.4	1.029	36.2
12	$\eta = \eta_o/2$ (Newtonian)	956.5	0.2419	102.1	1.257	47.5
1	$\eta = \eta_o$ (Newtonian)	924.6	0.2853	126.3	1.300	51.2
13	$\eta = 2\eta_o$ (Newtonian)	890.6	0.3362	156.2	1.335	54.9

* Principal dimensions are at the cessation of stimulation (t = 200 min).

Table 7.1-3 Benchmark Tests for Parametric Sensitivity Studies
(Effects of Injection Flow Rate [i] and Flow Behavior Index [m])

Case No.	Case Descriptions	Fracture Length (R:ft)	Maximum Width (W:in)	Effect. Pressure (ΔP :psi)	ΔP -Width Response Vari. (α)	Efficiency (%)
14	$i = i_o/8$ (Non-Newton)	363.1	0.1708	191.9	1.290	38.0
15	$i = i_o/4$ (non-Newton)	488.9	0.2218	186.5	1.314	44.2
16	$i = i_o/2$ (Non-Newton)	653.4	0.2870	181.7	1.335	50.5
2	$i = i_o$ (non-Newton)	866.8	0.3701	177.7	1.353	56.7
17	$i = 2i_o$ (Non-Newton)	1142.0	0.4755	174.2	1.370	62.6
18	$i = 4i_o$ (Non-Newton)	1495.0	0.6091	171.3	1.384	68.2
19	$i = 8i_o$ (Non-Newton)	1945.0	0.7781	168.8	1.397	73.2
20	$m = 0.2$ (Non-Newton)	1008.0	0.1729	66.8	1.138	39.8
2	$m = 0.5$ (Non-Newton)	866.8	0.3701	177.7	1.353	56.7
21	$m = 0.8$ (Non-Newton)	742.3	0.6396	366.7	1.423	69.0

* Principal dimensions are at the cessation of stimulation ($t = 200$ min).

Table 7.1-4 Benchmark Tests for Parametric Sensitivity Studies
(Effects of Generalized Elastic Modulus, $\hat{\mu} = \mu/(1-\nu)$)

Case No.	Case Descriptions	Fracture Length (R:ft)	Maximum Width (W:in)	Effect. Pressure (ΔP :psi)	ΔP -Width Response Vari. (α)	Efficiency (%)
22	$\hat{\mu} = \hat{\mu}_O/8$ (Non-Newton)	708.8	0.6742	48.5	1.290	71.4
23	$\hat{\mu} = \hat{\mu}_O/4$ (non-Newton)	761.8	0.5531	74.6	1.314	66.9
24	$\hat{\mu} = \hat{\mu}_O/2$ (Non-Newton)	814.8	0.4529	115.0	1.335	61.9
2	$\hat{\mu} = \hat{\mu}_O$ (non-Newton)	866.8	0.3701	177.7	1.353	56.7
25	$\hat{\mu} = 2\hat{\mu}_O$ (Non-Newton)	917.1	0.3017	275.2	1.370	51.3
26	$\hat{\mu} = 4\hat{\mu}_O$ (Non-Newton)	964.7	0.2453	427.5	1.384	45.7
27	$\hat{\mu} = 8\hat{\mu}_O$ (Non-Newton)	1009.0	0.1989	665.7	1.397	40.3
28	$\hat{\mu} = 100\hat{\mu}_O$ (Non-Newton)	1135.0	0.0907	3413.0	1.433	22.8

* Principal dimensions are at the cessation of stimulation ($t = 200$ min).

Table 7.1-5 Benchmark Tests for Parametric Sensitivity Studies
(Effects of Formation Critical Energy Release Rate, G_{Cr})

Case No.	Case Descriptions	Fracture Length (R:ft)	Maximum Width (W:in)	Effect. Pressure (ΔP :psi)	ΔP -Width Response Vari. (α)	Efficiency (%)
29	$G_{Cr} = G_{Cr}^*/4$ (Non-New.)	877.7	0.3692	179.3	1.428	55.5
30	$G_{Cr} = G_{Cr}^*/2$ (Non-New.)	873.2	0.3695	178.6	1.397	56.0
2	$G_{Cr} = G_{Cr}^*$ (Non-New.)	866.8	0.3701	177.7	1.353	56.7
31	$G_{Cr} = 2G_{Cr}^*$ (Non-New.)	857.9	0.3711	176.5	1.290	57.6
32	$G_{Cr} = 4G_{Cr}^*$ (Non-New.)	845.6	0.3733	175.1	1.199	58.9
33	$G_{Cr} = 10G_{Cr}^*$ (Non-New.)	821.4	0.3807	174.1	1.014	61.3
34	$G_{Cr} = 50G_{Cr}^*$ (Non-New.)	741.8	0.4429	194.2	0.474	68.6
35	$G_{Cr} = 100G_{Cr}^*$ (Non-New.)	687.6	0.5218	233.6	0.247	73.1

* Principal dimensions are at the cessation of stimulation ($t = 200$ min).

additives. As observed in chapter V, the fracture geometry is influenced considerably by changes in fluid injection rate (Table 7.1-3). Table 7.1-4 and 7.1-5 demonstrate the role of the formation critical energy release rate and elastic moduli. Relatively large variations in their values are necessary to significantly influence the fracture responses.

For the SFE Cases 5 & 6, the magnitude of the minimum in situ stress is 5700 psi (39.302 Mpa). Considering the borehole effective pressure initially ($t=0.01$ min) is near 1400 psi, the ratio of effective input power, D_{Ieff} , with respect to the total input power is about 20%. For the case of a penny-shaped fracture the effective pressure drops monotonically and the effective input power component is only 3% of the total power at the cessation of stimulation treatment ($t=200$ min).

Fig. 7.5 illustrates the magnitudes of each power component during the treatment. The various power components are normalized with respect to the effective input power ($P_o i_o$) in the upper plot and the strain energy rate (dU_{s2}/dt) and fluid leak-off (D_{L2}) associated with base line power ($i_o \sigma_o$) are shown in the lower plot. It is noteworthy that in the upper plot the values are normalized with respect to $P_o(t)$ while in the lower plot

Fig. 7.5 Base Line and Available Rate Component Evolution for a Penny-shaped Benchmark Model Corresponding to SFE No.3 Case 5

the denominator is invariant. These results show that about 70% of the instantaneous input power is dissipated through leak-off ($D_{L1} + D_{L2}$) at the cessation of stimulation. Based on the information obtained from the benchmark penny-shaped cases, more general three layered cases can be investigated.

7.3 Hydraulic Fracture Design Evaluations **in Three-Layered Reservoirs**

In this section, a set of simulations is presented to demonstrate the capability of the ERATE2D model as an efficient diagnostic tool for the optimum fracture design for three-layered formations. Selected design guidelines are also discussed.

From the previous sensitivity studies, it has been demonstrated that in situ stress contrasts play a major role in governing fracture geometry. It is also clear that the role of fluid leak-off is quite important. Optimization of the hydraulic fracturing process can be achieved by appropriately selecting the controllable variables, such as fluid rheology, leak-off, injection rate, and perforation placement relative to the uncontrollable reservoir properties and in situ stress field. In this optimization process, the same input data in ERATE2D are used as in SFE No.3 except for the

differential in situ stresses. The summary of input data is shown in Table 7.2. Simulations to permit changes in perforation characteristics for the case with unsymmetric in situ stress contrasts are also performed. The simulated unsymmetric case assumes $\Delta\sigma_u = \Delta\sigma_u^*/2$ and $\Delta\sigma_\ell = \Delta\sigma_u^*$ with $\Delta\sigma_u^* = 9.997$ Mpa (1450.0 psi). Non-Newtonian cases (Case a-g) as well as Newtonian cases (Case h-m) are investigated by changing the controllable variables to determine the desirable combinations of the variables for fracture optimization. The fracture responses at the cessation of the stimulation are shown in Table 7.3. The same amount of treatment fluid (10,000 barrels) is injected for each case, with different injection rates in some cases (double for Case d and k, half for Cases e-g and j).

The fracture geometry and fracture efficiency are monitored relative to the selected stimulation variables. However, the fracture efficiency generally may not be a sound indicator. For a three-layered formation, it is necessary to maximize the productive area or volume in the payzone. Therefore our efforts are focused on the optimization or maximization of productive payzone fracture areas and volumes.

We define the normalized effective fracture area (A_{eff}^*) and normalized effective fracture volume (V_{eff}^*) in

Table 7.2 Input Data for Case a-m

Case # [*] †	Injection Fluid Type Newtonian/Non-Newtonian (SFE #3 Case 5/Case 6)	Fluid Consistency η [Pa·min] ^m	Injection Rate i [BPM]	Fluid Leak-off C_L [m/ $\sqrt{\text{min}}$]	Perforation Placement, h _p (+:up, -:down)
Case a	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o$	$C_L=C_{Lo}$	0
Case b	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o$	$C_L=C_{Lo}$	-H/6
Case c	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o$	$C_L=C_{Lo}$	H/6
Case d	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o^{*2}$	$C_L=C_{Lo}$	H/6
Case e	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o/2$	$C_L=C_{Lo}$	H/6
Case f	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o/2$	$C_L=C_{Lo}^{*1.2}$	H/6
Case g	Non-Newtonian m=0.5	$\eta=\eta_{\text{non}}$	$i=i_o/2$	$C_L=C_{Lo}^{*0.8}$	H/6
Case h	Newtonian m=1.0	$\eta=\eta_o$	$i=i_o$	$C_L=C_{Lo}$	0
Case i	Newtonian m=1.0	$\eta=\eta_o$	$i=i_o$	$C_L=C_{Lo}$	H/6
Case j	Newtonian m=1.0	$\eta=\eta_o$	$i=i_o/2$	$C_L=C_{Lo}$	H/6
Case k	Newtonian m=1.0	$\eta=\eta_o$	$i=i_o^{*2}$	$C_L=C_{Lo}$	H/6
Case l	Newtonian m=1.0	$\eta=\eta_o/2$	$i=i_o$	$C_L=C_{Lo}$	H/6
Case m	Newtonian m=1.0	$\eta=\eta_o^{*2}$	$i=i_o$	$C_L=C_{Lo}$	H/6

* : The same input data used as in SFE #3 except $\Delta\sigma_u=\Delta\sigma_u^*/2$, $\Delta\sigma_\ell=\Delta\sigma_u^*$ where $\Delta\sigma_u^*=1449.9$ psi.

† : $\eta_{\text{non}} = 0.37089$ [Pa·min]^m (Non-Newtonian), $\eta_o = 3.333 \times 10^{-3}$ [Pa·min] (Newtonian),

$i_o = 50$ [BPM], $C_{Lo} = 0.76 \times 10^{-4}$ [m/ $\sqrt{\text{min}}$], and $H = 170$ [ft].

Table 7.3 Summary of Responses for Case a-m

Case #	Half Length (a:ft)	Heights Upper (b _u :ft)	Lower (b _l :ft)	Max. Width (W:in)	W ₁ (in)	W ₂ (in)	Eff. Press. (ΔP:psi)	Frac. Eff. (%)	Vol. Eff.	Area Eff.
Case a	2800.1	312.4	155.8	0.530	0.011	0.519	716.6	65.6	0.394	31.9
Case b	2624.6	407.5	95.2	0.522	0.011	0.511	690.4	65.1	0.368	30.0
Case c	3086.8	201.6	219.6	0.541	0.010	0.531	759.0	66.5	0.438	35.0
Case d	2848.5	245.0	226.6	0.593	0.009	0.583	795.8	75.2	0.448	32.4
Case e	3176.5	170.8	211.8	0.485	0.011	0.474	711.1	55.8	0.398	35.9
Case f	2987.2	166.5	210.4	0.475	0.011	0.464	702.4	50.7	0.366	33.7
Case g	3390.4	175.6	213.3	0.496	0.011	0.485	720.1	61.8	0.435	38.3
Case h	3414.5	269.6	150.8	0.473	0.011	0.462	668.1	64.4	0.423	38.7
Case i	3702.4	171.0	212.0	0.485	0.011	0.474	709.2	65.1	0.464	41.8
Case j	3854.8	145.1	202.1	0.417	0.013	0.404	607.4	53.1	0.407	43.3
Case k	3320.8	211.2	221.4	0.553	0.010	0.543	765.7	75.0	0.482	37.7
Case l	4357.7	148.5	150.2	0.428	0.012	0.416	652.0	62.4	0.474	49.0
Case m	3075.3	205.8	220.4	0.545	0.010	0.535	760.7	67.5	0.440	34.9

the payzone as follows :

$$\begin{aligned}
 A_{\text{eff}}^* &\equiv A_{\text{mid}}/H^2 \\
 &= \sum_{j=u, \ell} \left[\frac{\pi}{2} ab_j - ab_j \sin^{-1}\{1 - (\xi_j)^2\}^{1/2} \right. \\
 &\quad \left. + h_j/b_j \{1 - (\xi_j)^2\}^{1/2} \right] / H^2
 \end{aligned} \tag{7.34}$$

$$\begin{aligned}
 \text{and } V_{\text{eff}}^* &\equiv V_{\text{Fmid}}/V_I \\
 &= \sum_{j=u, \ell} \left[\frac{\pi}{3} ab_j W_1\{1 - R_1(\xi_j)\} \right. \\
 &\quad \left. + \frac{\pi}{5} ab_j W_2\{1 - R_2(\xi_j)\} \right] / (i_o t)
 \end{aligned} \tag{7.35}$$

where $\xi_j = h_j/b_j$, A_{mid} and V_{Fmid} denote the fracture area and volume in the payzone. The ratios of the volumes in the barriers (upper and lower) to the total volume, $R_1(\xi_j)$ and $R_2(\xi_j)$ are defined in chapter III. The fracture dimension and efficiency optimization process with respect to the achieving maximum effective fracture areas using ELLIP2D is already demonstrated in chapter VI.

Figures 7.6-8 show the fracture efficiency, normalized effective area and effective volume, respectively, for the non-Newtonian fluid cases.

Changes in perforation placement (h_p) for Cases a-c do not give significant changes with respect to the fracture efficiency, but give remarkable changes with respect to the normalized effective area/volume efficiency. In particular, Case c with an elevated perforation placement ($h_p = H/6$) a fracture efficiency of

Fig. 7.6 Fracture Efficiency versus Normalized Volume
(Non-Newtonian, Three-Layered Cases a-g)

Fig. 7.7 Normalized Effective Area versus Normalized Volume (Non-Newtonian, Three-Layered Cases a-g)

Fig. 7.8 Effective Volume Efficiency vs. Normalized Volume (Non-Newtonian, Three-Layered Cases a-g)

66.4% at $t=200$ [min], while Case a ($h_p = 0$) and Case b ($h_p = -H/6$) yield fracture efficiencies of 65.6 % and 65.1 %, respectively. However, the normalized effective area (area in the payzone divided by H^2) and effective volume (volume in the payzone divided by the injected volume) are 35.03 and 43.8% for case c, 31.87 and 39.4% for Case a and 29.96 and 36.8% for Case b, respectively. Appropriate perforation placement (Case c) gives significant improvements in the defined normalized effective area and effective volume efficiencies, respectively.

Figure 7.9 graphically demonstrates the differences in the response behavior for the unsymmetric in situ stress contrast cases. Since in situ stress contrasts play a major role in governing fracture geometry, perforation placement changes are important in the optimization process for the unsymmetric in situ stress contrast cases. The growth of a fracture is governed by energy conservations and the perforation placement plays a key role in the delivery of the injected power and fracture evolution. The numerical simulations indicate that early penetration into the upper barrier by placing the perforation placement upwards, where a smaller differential in situ stress exists, forces the fracture to grow more symmetrically. How far the placement should be shifted from the center line depends on the in situ stress

Fig. 7.9 Fracture Evolution Configurations with Perforation Placement and Unsymmetric In Situ Stress Differentials (at Time = 0.134, 5, 10, 20, 40, 60, 80 and 100 [min])

and leak-off differentials as well as the fluid rheology, scheduled injection rate and duration. The perforation placement and other variables can be optimized through iterative design steps until maximum efficiencies and economies are realized.

An increased injection rate in Case d ($i=i_o*2$ with $i_o = 50$ BPM) apparently yields a better fracture efficiency than Case e ($i=i_o/2$) with increased borehole effective pressure. However, the increased injection rate in Case d yields lower normalized effective area and higher normalized effective volume than Case e. This indicates that an increased injection rate produces less containment in the payzone and a higher opening width with a primary contribution from W_2 .

As previously indicated, the fluid leak-off is also an important parameter. The effect of fluid leak-off is examined for non-Newtonian cases for a relatively slow pumping rate and elevated perforation placement through Case f and g (Figures 7.6-8). The reduced leak-off significantly improves the fracture efficiency, normalized effective area and volume. Since these cases fall into the dissipation dominant regime, the advantage derived from the reduced leak-off should be carefully examined relative to the disadvantages arising from the changes in the rheological properties.

The pressure-width response variables (α_p , α_u , and α_ℓ) are plotted for Case c in Figure 7.10. This figure demonstrates the responses of the pressure-width variables when the fracture penetrates the upper and lower barriers. The time lag in the plots occurs because of the selected perforation placement ($h_p = H/6$). The response variables, initially have the same values ($\alpha_p = \alpha_u = \alpha_\ell = 1.375$) until the fracture contacts the upper barrier first and then decreases as the fracture is contained. After the fracture is highly contained due to the in situ stress differentials, the upper and lower pressure-width parameters along the vertical direction become zero ($\alpha_u = \alpha_\ell \approx 0$), while α_p remains constant ($\alpha_p \approx 1.4$) throughout the entire simulation. This pressure distribution affects not only the pressure gradient dissipation function, but also the fluid leak-off energy rate and formation strain energy rate function.

An additional set of parametric sensitivity studies is performed for the Newtonian cases (Cases h-m) with variations in injection rate and fluid viscosity. Figures 7.11-13 illustrate fracture efficiency, normalized effective area and effective volume for the Newtonian fluid cases. Using the same perforation placement, Case i (Newtonian fluid case) has a higher normalized effective area and volume than Case c (Non-Newtonian), although

Fig. 7.10 Pressure-width response variables (α_p , α_u & α_l) versus Time for Case c

Fig. 7.11 Fracture Efficiency versus Normalized Volume (Newtonian, Three-Layered Cases h-m)

Fig. 7.12 Normalized Effective Area vs. Normalized Volume
(Newtonian, Three-Layered Cases h-m)

Fig. 7.13 Effective Volume Efficiency vs. Normalized Volume (Newtonian, Three-Layered Cases h-m)

Case i has a lower fracture efficiency than Case c. Apparently non-Newtonian fluids give higher fracture penetration into the barriers than the Newtonian cases.

An increased injection rate in Case k ($i=i_o*2$) apparently gives better fracture efficiency, normalized effective volume and a decreased effective area than in Case j ($i=i_o/2$), as in non-Newtonian cases. The increased Newtonian fluid viscosity case (Case m) yields a higher efficiency than Case l, but it gives a lower normalized effective volume and area. From the fracture configuration, we recognize that a highly viscous fluid yields an increased borehole pressure and a higher penetration into barriers. The selection of a highly viscous fluid is recommended for improved proppant transport if high in situ stress differentials exist. Although Case m gives a higher fracture efficiency than Case l, the fracture geometry (Case m) has a greater the non-productive area (barriers), which results in a decreased effective area and volume.

The representation of each energy rate component during the process and its variation with respect to various parameters enables us to understand the role of controllable variables and uncontrollable parameters. The normalized volumes provide valuable information for hydraulic fracture design. Six energy rate components,

detailed in eqn (7.7), are plotted (bold lines) for the non-Newtonian, three-layered case (SFE No.3 Case 6) in Fig. 7.14. The dotted lines denote the responses of the energy rate terms in a homogeneous medium with an initial overlap in the responses. The strain energy rate terms are much higher when the fracture is contained in the payzone. For the three layered case, after the fracture is contained, D_{L1}/D_I (symbol 1) increases due to the pressure build up, while D_{L2}/D_I (symbol L) decreases.

The energy rate ratios with respect to the instantaneous input power at the cessation of the treatment ($t=200$ min) are as follows :

- i) for SFE NO.3 Case 6 (non-Newtonian response with differential in situ stress)

$$D_{\eta}/D_I = 0.232, \quad (dU_{s1}/dt)/D_I = 0.198, \quad D_{L1}/D_I = 0.000 \\ (dU_f/dt)/D_I = 0.005, \quad (dU_{s2}/dt)/D_I = 0.559, \quad D_{L2}/D_I = 0.006$$

- ii) for a penny-shaped case

$$D_{\eta}/D_I = 0.012, \quad (dU_{s1}/dt)/D_I = 0.004, \quad D_{L1}/D_I = 0.002 \\ (dU_f/dt)/D_I = 0.000, \quad (dU_{s2}/dt)/D_I = 0.476, \quad D_{L2}/D_I = 0.506.$$

In general, the dissipation (D_{η}/D_I) and strain energy rate $((dU_{s1}/dt)/D_I)$ ratios tend to increase for higher in situ stress contrasts and the total leak-off energy rate ratio $((D_{L1} + D_{L2})/D_I)$ decreases. Figures 7.15-17 illustrate the changes in energy rate components due to differential

Fig. 7.14 Energy Rate Components versus Time for SFE No.3
Case 6 (Non-Newtonian, Three-Layered Case)

Fig. 7.15 Comparisons of Energy Rate Components between Case a and SFE #3 Case 6 (In-situ Stress Effect)

Fig. 7.16 Comparisons of Energy Rate Components between Case f and Case g (Leak-off Effect)

Fig. 7.17 Comparisons of Energy Rate Component between Case j and Case k (Injection Rate Effect)

in situ stress, leak-off, and injection rate, respectively. The variations of the energy rate components due to a change in the in situ stress differential are illustrated in Fig. 7.15 by comparing the SFE No.3 Case 6 with Case a in Table 7.2. Due to the reduced in situ stress differential for Case a, the fracture length (height) is decreased (increased) as listed in Table 6.3 and 7.3. Therefore the dissipation energy rate component for Case a decreases when compared to the results corresponding to Case 6. This difference in the dissipation causes a substantial change in the leak-off component, while the stored strain energy terms remain unchanged. Figures 7.16 and 7.17 illustrate the energy component trend sensitivity due to the leak-off and injection rate variations. The higher leak-off coefficient (Case f in Fig.7.16) obviously requires a large amount of available instantaneous energy rate. This expenditure is compensated by the reduced strain energy rate, while the high dissipation remains the same. This indicates that the increase in leak-off components in the dissipation dominant regime yields an unfavorable fracture geometry as well as unfavorable energy component responses since maximizing the strain energy is usually optimum. A reversal of the energy rate component responses is shown in Fig. 7.17. The increase in the injection rate produces

a smaller percentage of leak-off energy rate drained from the system. The additional available energy rate is converted to the strain energy rate while the dissipation energy rate increase is limited. This trend creates preferred volume responses, unless the borehole pressure build-up enhances the vertical fracture growth and thereby reduces the normalized effective area and productive volume efficiency.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

Two hydraulic fracture models are developed for simulating fracture configuration evolutions in three-layered reservoirs. The first model (ELLIP2D), detailed in chapter II, is developed based on a Lagrangian formulation incorporating pertinent energy components associated with the formation structural responses and fracture fluid flow. The treatment fluid is assumed to be an incompressible power-law type non-Newtonian fluid and fluid pressure distribution in a fracture is uniform. The generalized elliptic model simulators for the hydraulic fracturing in unsymmetric three-layered formations incorporate formation and fluid property differentials, such as in-situ stress ($\Delta\sigma$), formation critical energy release rate (ΔG_{cr}), elastic modulus ($\Delta\mu$), Poisson's ratio ($\Delta\nu$), spurt loss (ΔC_{sp}), and leak-off (ΔC_L) contrasts.

The second model (ERATE2D), detailed in chapter III, is developed based on a generalized variational principle, introducing an energy rate related functional. The variational formulations, detailed in chapter III and IV, provide an explicit form of energy rate conservation,

which can also be expressed in terms of the first law of thermodynamics. The variational concepts play a central role in the interpretation of hydraulic fracture mechanisms and prediction of crack geometries. Systematic investigations on the transformation of the total input power provided by the injection of treatment fluid into various components are conducted to examine the role of uncontrollable parameters and controllable variables.

The ERATE2D model is modified to accommodate the phenomenon of fluid lag. The locations of fracturing fluid as well as the fracture front and corresponding pressure distribution of injection fluid during the fracturing process are predicted for a penny-shaped fracture model based on the variational principle. The crack tip closure stress applied on the dry zone and effective pressure are shown to be primary factors in controlling the magnitude of fluid lag. It is obvious that the non-dimensionalized fracture toughness, $\hat{K}$, and in situ stress parameter, $\hat{\sigma}$, play major roles in the determination of fluid lag. It is demonstrated that these non-dimensionalized parameters are bounded by necessary conditions in order to yield a finite fluid lag length. The inequality expression, detailed in chapter V, can serve as a diagnostic indicator for the occurrence of fluid lag in a penny-shaped fracture. It is concluded

that the effects of fluid lag are negligible after a relatively short time interval for a penny-shaped fracture. However, the excess pressure for contained fractures in multi-layered formation or PKN model cases may be a result of fluid lag.

Since penny-shaped fracture growth in a homogeneous medium and contained elongated fractures (PKN type) are special cases of the unsymmetric elliptic fracture models in three-layered formations, selected benchmark and validation tests are completed for these models. The predicted benchmark model responses for constant height and penny-shaped models, are in good agreement with represented results of the time explicit closed form solutions (Appendix A) and FEM numerical solutions.

Following the benchmark test verification, the developed simulators are applied to selected field examples (SFE No.3 Case 5 and 6) for comparisons and parametric sensitivity studies depicting the role of controllable variables and uncontrollable parameters. Comparisons with published results and associated parametric sensitivity studies for symmetric and unsymmetric three-layered cases with leak-off effects reveal the general validity of the developed models in a multi-layered formation. The roles of fracturing fluid rheology, flow rate, and reservoir formation properties

are investigated by evaluating fracture propagation in a three-layered system. A modest discrepancy exists in the results for three-layered cases with high in situ stress contrasts among ELLIP2D, ERATE2D, and HYFRAC3D. This disagreement results from basic modeling assumptions, such as fracture opening profile and vertical fracture configuration. The overall results show that these comprehensive simulators for multi-layered reservoirs with differentials of elastic moduli, in situ stresses, fracture toughness, and fluid leak-off effects, can serve as efficient design tools for preliminary fracture configuration prediction and parametric sensitivity evaluations. It is obvious that fracture containment is primarily governed by differentials of reservoir properties. Considering the range of representative elastic modulus values for payzone and barrier layers, the elastic modulus and fracture toughness contrasts do not act as strong fracture containment barriers.

Some guidelines for fracture configuration design, control, and optimization procedure are presented in chapter VII through systematic parametric sensitivity studies and delineation of pertinent energy rate components by means of ERATE2D model. The role of controllable variables in the treatment is emphasized in the optimum design process. Fracture efficiency,

normalized effective area and volume are used as optimization indices to evaluate the effectiveness of treatments.

The following generic guidelines for hydraulic fracture configuration design, control, and optimization are proposed:

- 1) Penny-shaped and PKN asymptotic model representations are good benchmarks for estimating the response trends as a function of hydraulic fracture process parameters.
- 2) Influence regime evaluations using the characteristic concept serve as reasonable trend predictors in terms of controllable variables and uncontrollable parameters.
- 3) Large fluid leak-off coefficients make the fracture responses less sensitive to changes in the other fracture parameters (in terms of energy efficiency).
- 4) In situ stress contrasts play a significant role in governing fracture vertical penetration with fracture fluid viscosity and injection flow rate having a secondary influence. The effects of elastic moduli contrast are also secondary and are primarily reflected in the fracture width-effective pressure relation and resulting dissipation energy rate variation.
- 5) Perforation placement in relation to the prevalent in situ field can be an effective design tool in governing fracture evolution. The payzone fracture effective area

and volume efficiencies are more realistic measures for optimum fracture design than the traditional fracture efficiency definition in the case of multi-layered media. Parametric sensitivity configuration studies delineating the role of fracture fluid rheology injection rate, and leak-off are useful for the optimal stimulation treatment design.

6) A higher injection rate, for the same treatment volume size, reduces leak-off but decreases fracture containment in the payzone. Similarly, a higher fluid viscosity increases fracture width and fracture vertical penetration. An increased injection rate raises overall volume efficiency. However the normalized effective area may be reduced in the process. This result is directly obtained from the fact that the increased injection produces higher borehole pressure build-up and enhances vertical fracture growth.

7) Even if leak-off behavior is not totally controllable, reduced leak-off drastically improves the overall efficiency, effective area as well as effective volume. Reducing the leak-off behavior effect should be measured against the drawbacks realized from the fluid rheology change, such as possible effects on proppant transport.

8) The energy rate efficiency of the hydraulic fracture process evaluated using effective bottom hole treatment

pressure (BHTP) or wellhead treatment pressure (WHTP) values is an important diagnostic measure. The available hydraulic horsepower ($i_o P_o$) at the wellhead or bottom hole is converted into dissipative, strain storage, leak-off, and fracture surface energy rates during the fracture evolution. Transfer of fracture fluid leak-off energy to reservoir strain energy, for example, should be optimized during the dissipation dominant fracture evolution.

9) The three-layered models obtained by smearing layers suffice in most cases and require minimum computational effort for parametric sensitivity and design studies.

Compared to complex three-dimensional simulators, the presented simulators have advantages in some respects despite some inherent disadvantages regarding the elliptical fracture configurations in a three layered formation. The implementation of parametric sensitivity studies and systematic design investigations for optimum fracture design through general three dimensional simulators are more cumbersome and expensive.

The compact information provided by ERATE2D for the principal response variables (a , b_u , b_ℓ , W_1 , W_2 , P_o , α_p , α_u and α_ℓ), optimization indices (fracture efficiency, normalized effective volume (V_{eff}^*) and effective area (A_{eff}^*)) as well as pertinent energy rate components ($\dot{U}_f$, $d[U_{s1}]/dt$, $d[U_{s2}]/dt$, D_η , D_{L1} and D_{L2}) derived from the

input power at each time step solutions, provide significant design information. The energy rate conservation law governing the fracture fluid control volume and reservoir elasticity enables us to monitor pertinent energy rate components corresponding to selected uncontrollable parameters (reservoir mechanical properties) and controllable variables (injection rate, fracture fluid rheological properties including the control of fluid leak-off, and perforation placement) during hydraulic fracturing processes.

In order to take advantage of these simulators as a systematic design tool and to extend the methodology to the formulation of a more generalized hydraulic fracturing, the following recommendations for further research are proposed:

- 1) Inclusion of fracture fluid thermal effects during the hydraulic fracturing process is recommended. Thermal effects on fracturing fluid rheology which influence the dissipation properties of the fluid, proppant settling characteristics and fluid filtration to the formation are important.
- 2) Simulations incorporating multiple injection fluids, proppant transport and settling effects are desirable. Lee et al [96] have proposed the applicability of a fixed grid method for multi-phase nonlinear diffusion problems

with moving boundaries, demonstrating its validation through some cases like solidification and injection molding problems. Analysis of multiple fracture fluids can be implemented through this approach. The tracking of moving boundaries/interfaces is a challenging numerical task. Localized self-adaptive meshing around moving boundaries/interfaces is recommended for efficient implementation.

3) Since the developed simulators have a capability to simulate hydraulic fracturing treatments with a time varying injection rate, optimum design by sequentially controlling injection rates and proppant characteristics during the treatment process are recommended.

4) Finally, the assumed unsymmetric elliptical geometry can be generalized by introducing additional degrees of freedom to give more flexibility in defining different fracture evolution configurations, corresponding pressure profiles, and control parameters.

APPENDIX A

TIME EXPLICIT SOLUTIONS FOR CONSTANT HEIGHT, PENNY-SHAPED AND ELLIPTICAL MODELS

Table A.1 Time-Explicit Solutions for Constant Height, Penny-Shaped and Elliptical Models

Model	Half-Length (L,a)	Borehole Width (V,2c)	Borehole Height (2b)	Effective Pressure (P ₀)
PKN MODEL	(D) $A_1 \left[\frac{\hat{\mu}_{i0}^{m+2}}{\eta_0 H^{m+3}} \right] \frac{1}{t^{2m+3}}$	$A_2 \left[\frac{i_0^{m+1} \eta_0}{H^m \mu} \right] \frac{1}{t^{2m+3}}$	2b = H = constant	$A_3 \left[\frac{\hat{\mu}_{i0}^{2m+2} \eta_0}{H^{3m+3}} \right] \frac{1}{t^{2m+3}}$
	(F) $0.415 \left[\frac{\hat{\mu}_{i0}^{2m+2}}{G_{cr} H^3} \right] t$	$1.954 \left[\frac{HG_{cr}}{\mu} \right] \frac{1}{t^2}$	2b = H = constant	$1.954 \left[\frac{\hat{\mu}_{i0}^{2m+2}}{H} \right] \frac{1}{t^2}$
CGDD MODEL	(D) $B_1 \left[\frac{\hat{\mu}_{i0}^{m+2}}{\eta_0 H^{m+2}} \right] \frac{1}{t^{m+2}}$	$B_2 \left[\frac{i_0^{m+2} \eta_0}{H^{m+2} \mu} \right] \frac{1}{t^{m+2}}$	2b = H = constant	$B_3 \left[\frac{\hat{\mu}_{i0}^{m+1} \eta_0}{H^{m+2}} \right] \frac{1}{t^{m+2}}$
	(F) $0.542 \left[\frac{\hat{\mu}_{i0}^{2m+2}}{G_{cr} H^2} \right] \frac{1}{t^{m+2}}$	$1.175 \left[\frac{G_{cr} i_0}{\mu H} \right] \frac{1}{t^{m+2}}$	2b = H = constant	$1.084 \left[\frac{\hat{\mu}_{i0}^{2m+2} h}{i_0} \right] \frac{1}{t^{m+2}}$
PENNY- SHAPED MODEL	(D) $C_1 \left[\frac{\hat{\mu}_{i0}^{m+2}}{\eta_0} \right] \frac{1}{t^{2m+2}}$	$C_2 \left[\frac{i_0^{m+2} \eta_0}{\mu} \right] \frac{1}{t^{2m+2}}$	$2C_1 \left[\frac{\hat{\mu}_{i0}^{m+2}}{\eta_0} \right] \frac{1}{t^{2m+2}}$	$C_3 \left[\frac{\hat{\mu}_{i0}^{m+1} \eta_0}{H^{m+2}} \right] \frac{1}{t^{m+2}}$
	(F) $0.617 \left[\frac{\hat{\mu}_{i0}^{2m+2}}{G_{cr}} \right] \frac{1}{t^{2m+2}}$	$1.254 \left[\frac{G_{cr}^2 i_0}{\mu^2} \right] \frac{1}{t^{2m+2}}$	2b = 2a	$1.595 \left[\frac{\hat{\mu}_{i0}^{2m+2} G_{cr}}{i_0} \right] \frac{1}{t^{2m+2}}$
ELLIPTIC FRACTURE MODEL	(D) $D_1 \left[\frac{\hat{\mu}_{i0}^{2m+2}}{\eta_0 (2b)^{m+3}} \right] \frac{1}{t^{2m+2}}$	$D_2 \left[\frac{i_0^{m+1} \eta_0}{\mu (2b)^m} \right] \frac{1}{t^{2m+2}}$	$D_3 \left[\frac{\hat{\mu}_{i0}^{2m+1} \eta_0}{G_{cr}^{2m+3}} \right] \frac{1}{t^{2m+2}}$	$D_4 \left[\frac{\hat{\mu}_{i0}^{2m+2} \eta_0}{(2b)^{3m+3}} \right] \frac{1}{t^{2m+2}}$

$\hat{\mu} = \frac{\mu}{1-\nu}$; A_j, B_j, C_j, D_j ($j=1, \dots, 4$): Functions of m , as listed in Table A.2 and A.3.

(D): Solutions in rows correspond to (dissipation) energy dominant solutions

(F): Solutions in rows correspond to (fracture)

Table A.2 Fracture Model Solution Constants Versus Flow Behavior Indices

★ m	PKN MODEL			CGDD MODEL			PENNY-SHAPED MODEL			ELLIPTICAL MODEL			
	L	W	P _O	L	W	P _O	a	2c	P _O	a	2c	2b	P _O
	A ₁ (m)	A ₂ (m)	A ₃ (m)	B ₁ (m)	B ₂ (m)	B ₃ (m)	C ₁ (m)	C ₂ (m)	C ₃ (m)	D ₁ (m)	D ₂ (m)	D ₃ (m)	D ₄ (m)
0.1	0.543	1.493	1.493	0.642	0.992	0.773	0.684	1.019	2.183	0.670	1.425	1.949	2.013
0.2	0.517	1.568	1.568	0.622	1.024	0.823	0.667	1.073	2.370	0.650	1.470	1.891	2.074
0.3	0.499	1.624	1.624	0.607	1.048	0.863	0.654	1.117	2.530	0.663	1.509	1.840	2.118
0.4	0.485	1.670	1.670	0.596	1.068	0.896	0.643	1.156	2.676	0.619	1.544	1.799	2.152
0.5	0.474	1.711	1.711	0.586	1.086	0.926	0.633	1.193	2.811	0.606	1.576	1.765	2.179
0.6	0.464	1.748	1.748	0.578	1.101	0.953	0.623	1.229	2.940	0.594	1.608	1.737	2.201
0.7	0.455	1.783	1.783	0.571	1.116	0.978	0.615	1.263	3.062	0.583	1.637	1.714	2.220
0.8	0.446	1.817	1.817	0.564	1.129	1.001	0.607	1.297	3.179	0.573	1.666	1.693	2.234
0.9	0.438	1.849	1.849	0.558	1.141	1.023	0.599	1.332	3.289	0.564	1.694	1.675	2.246
1.0	0.431	1.882	1.882	0.552	1.153	1.044	0.591	1.366	3.394	0.555	1.721	1.660	2.256

★ : Constants are obtained through the Lagragian approach with constant pressure distribution.

Table A.3. Fracture Model Solution Constants for Dissipation Dominant Solutions

★ m	PKN MODEL			PENNY-SHAPED MODEL		
	a	W	P _o	R	W	P _o
	A ₁ (m)	A ₂ (m)	A ₃ (m)	C ₁ (m)	C ₂ (m)	C ₃ (m)
0.1	1.040	1.531	2.296	0.797	1.252	1.851
0.2	0.956	1.664	2.497	0.774	1.327	2.020
0.3	0.901	1.767	2.650	0.758	1.385	2.153
0.4	0.860	1.851	2.776	0.745	1.434	2.267
0.5	0.828	1.923	2.885	0.734	1.476	2.369
0.6	0.801	1.987	2.980	0.725	1.515	2.461
0.7	0.779	2.043	3.065	0.717	1.550	2.548
0.8	0.760	2.094	3.142	0.709	1.582	2.628
0.9	0.743	2.141	3.212	0.702	1.613	2.705
1.0	0.729	2.184	3.276	0.696	1.641	2.777

★ : Constants are obtained for the dissipation dominant solutions through the variational approach with variable pressure distribution.

APPENDIX B
CLOSED FORM EXPRESSIONS ON THE FLUID LAG
IN HYDRAULIC FRACTURING PROCESS
ASSOCIATED WITH DIFFERENT PRESSURE PROFILES

The fluid lag in hydraulic fracturing process is directly influenced by the pressure distribution in a fracture, fracture toughness and effective fracture tip closure stress. Jeffrey [74] derived an analytical expression for the fluid lag based on the assumption that a fracture hydraulically induced develops if the loading on the fracture exceeds the fracture toughness of the rock formation. The pressure distribution of injection fluid during the fracturing process remains as a variable in the formulation of a fracture model based on the variational principle, detailed in Chapter III, since the pressure-width response variable, $\alpha(t)$, varies during the process. Therefore it is important to evaluate fluid lag associated with different pressure distribution profiles through the fracture domain to understand the behavior of fluid lag in hydraulic fracturing process. The evaluation of fluid lag in penny-shaped and PKN models with constant pressure profiles has been presented by Jeffrey [74].

The stress intensity factor of a penny-shaped fracture under axisymmetric normal loading, $p(r)$, is

$$K_I = \int_0^R \frac{2}{\sqrt{\pi R}} \frac{r}{\sqrt{R^2 - r^2}} p(r) dr \quad (B.1)$$

along with the fracture mechanics criterion

$$K_I = K_{IC} \quad (B.2)$$

1. Case A

When the pressure distribution in a fracture is expressed as

$$p(\rho) = \begin{cases} P_O(1 - \alpha\rho^2) & 0 < \rho < \beta \\ -(\sigma_O - P_w) & \beta \leq \rho \leq 1 \end{cases} \quad (B.3)$$

where $\rho = r/R$ and $\beta = R_f/R$. The mode I stress intensity factor, K_I , becomes

$$K_I = 2\sqrt{\frac{R}{\pi}} P_O \left[1 - \sqrt{1 - \beta^2} - \alpha \left\{ \frac{2}{3} - \sqrt{1 - \beta^2} + \frac{1}{3}(1 - \beta^2)^{3/2} \right\} \right] \\ - 2\sqrt{\frac{R}{\pi}} (\sigma_O - P_w) \sqrt{1 - \beta^2} \quad (B.4)$$

It is assumed that the induced fracture grows when the stress intensity factor in (B.4) exceeds the formation fracture toughness.

$$K_{IC} = 2\sqrt{\frac{R}{\pi}} P_O \left[1 - \epsilon - \alpha \left(\frac{2}{3} - \epsilon + \frac{1}{3}\epsilon^3 \right) - \frac{(\sigma_O - P_w)}{P_O} \epsilon \right] \quad (B.5)$$

where $\epsilon \equiv \sqrt{1 - \beta^2}$.

From the continuity of pressure distribution near the fluid front, $\rho = \beta$,

$$P_o(1-\alpha\beta^2) = -(\sigma_o - P_w)$$

where α can be expressed in terms of ϵ as

$$\alpha = \frac{1}{1-\epsilon^2} \left(1 + \frac{\sigma_o - P_w}{P_o} \right) \quad (B.6)$$

Substituting eqn (B.6) into eqn (B.5)

$$\frac{\sqrt{\pi}}{2} \frac{K_{IC}}{P_o \sqrt{R}} = \frac{1}{3(1+\epsilon)} \left[(1-\epsilon)(1+2\epsilon) - \frac{(\sigma_o - P_w)}{P_o} 2(\epsilon^2 + \epsilon + 1) \right] \quad (B.7)$$

Equation (B.7) yields a quadratic equation with respect to ϵ in the form

$$2(1+\hat{\sigma})\epsilon^2 + (2\hat{\sigma} + 3\hat{K} - 1)\epsilon + (2\hat{\sigma} + 3\hat{K} - 1) = 0 \quad (B.8)$$

where $\hat{K} \equiv \frac{\sqrt{\pi}}{2} \frac{K_{IC}}{P_o \sqrt{R}}$ and $\hat{\sigma} \equiv \frac{(\sigma_o - P_w)}{P_o}$.

The necessary condition so that the eqn (B.8) have at least one solution within $0 < \epsilon < 1$, can be written as

$$-\frac{1}{2} < \frac{2\hat{\sigma} + 3\hat{K} - 1}{2(\hat{\sigma} + 1)} < 0 \quad (B.9)$$

Since $\hat{K}$ and $\hat{\sigma}$ are positive, eqn (B.9) can be summarized as follows :

$$0 < \frac{K_{IC}}{P_o \sqrt{R}} < \frac{2}{3\sqrt{\pi}} \quad \text{and} \quad 0 < \frac{\sigma_o - P_w}{P_o} < \frac{1}{2}$$

The corresponding solution is given by

$$\epsilon = \frac{(1-2\hat{\sigma}-3\hat{K}) + \sqrt{(1-2\hat{\sigma}-3\hat{K})(9+6\hat{\sigma}-3\hat{K})}}{4(\hat{\sigma}+1)} \quad (B.10)$$

The distance of fluid lag, d , measured from the crack tip can be expressed in the form

$$d = d(R, \hat{\sigma}, \hat{K})$$

$$= R \left(1 - \sqrt{1 - \{\epsilon(\hat{\sigma}, \hat{K})\}^2} \right) \quad (\text{B.11})$$

2. Case B

When the pressure distribution in a fracture is expressed as

$$p(\rho) = \begin{cases} P_o \sqrt{1 - \left(\frac{\rho}{\beta}\right)^2} & 0 < \rho < \beta \\ -(\sigma_o - P_w) & \beta \leq \rho \leq 1 \end{cases} \quad (\text{B.12})$$

Similarly, the fracture mechanics criterion yields

$$K_{IC} = 2\sqrt{\frac{R}{\pi}} P_o \left[N(\beta) - \frac{(\sigma_o - P_w)}{P_o} \epsilon \right] \quad (\text{B.13})$$

where $N(\beta) = \int_{\beta}^1 \frac{\sqrt{1 - \left(\frac{\rho}{\beta}\right)^2}}{\sqrt{1 - \rho^2}} \rho \, d\rho$. The necessary conditions

can be written as

$$0 < \frac{K_{IC}}{P_o \sqrt{R}} < \frac{2}{\sqrt{\pi}} N(\beta) \quad \text{and} \quad 0 < \frac{\sigma_o - P_w}{P_o} < \frac{N(\beta)}{\sqrt{1 - \beta^2}} \quad (\text{B.14})$$

The corresponding solution can be obtained numerically from eqn (B.13).

3. Case C

When the pressure distribution in a fracture is expressed as

$$p(\rho) = \begin{cases} \{ P_o + (\sigma_o - P_w) \} \sqrt{1 - \left(\frac{\rho}{\beta}\right)^2} - (\sigma_o - P_w) & 0 < \rho < \beta \\ -(\sigma_o - P_w) & \beta \leq \rho \leq 1 \end{cases} \quad (\text{B.15})$$

The fracture mechanics criterion yields

$$K_{IC} = 2\sqrt{\frac{R}{\pi}} P_o [(1+\hat{\sigma}) N(\beta) - \hat{\sigma} \epsilon] \quad (B.16)$$

where $N(\beta) = \int_{\beta}^1 \frac{\sqrt{1-(\frac{\rho}{\beta})^2}}{\sqrt{1-\rho^2}} \rho \, d\rho$. The necessary conditions can be written as

$$0 < \frac{K_{IC}}{P_o \sqrt{R}} < \frac{2}{\sqrt{\pi}} \{N(\beta) - \epsilon\} \text{ and } 0 < \frac{\sigma_o - P_w}{P_o} < \frac{N(\beta)}{N(\beta) - \epsilon} \quad (B.17)$$

The corresponding solution can be obtained numerically from eqn (B.16).

4. Case D

When the pressure distribution is written as

$$p(\rho) = \begin{cases} P_o & 0 < \rho < \beta \\ -(\sigma_o - P_w) & \beta \leq \rho \leq 1 \end{cases} \quad (B.18)$$

The fracture mechanics criterion yields

$$K_{IC} = 2\sqrt{\frac{R}{\pi}} P_o \left[1 - \epsilon - \frac{(\sigma_o - P_w)}{P_o} \epsilon \right] \quad (B.19)$$

along with the condition given by

$$0 < \frac{K_{IC}}{P_o \sqrt{R}} < \frac{2}{\sqrt{\pi}} \text{ and } 0 < \frac{\sigma_o - P_w}{P_o}$$

The corresponding solution fluid front location can be written as

$$\epsilon = \left\{ \frac{1-\hat{K}}{1+\hat{\sigma}} \right\}$$

and $d = R \left[1 - \sqrt{1 - \left\{ \frac{1-\hat{K}}{1+\hat{\sigma}} \right\}^2} \right] \quad (B.20)$

APPENDIX C

DEFINITION OF MATHEMATICAL FUNCTIONS

1) Definition of Functions, $R_k(\xi_j)$, in Chapter III

$$R_k(\xi_j) \equiv \frac{\int_{A_{bj}} f_k(\rho) dA}{\int_{A_{mj}} f_k(\rho) dA} \quad (j=u, \ell \text{ and } k=1, \dots, 7)$$

where

$$\begin{aligned} f_1(\rho) &\equiv (1-\rho^2)^{1/2} \\ f_2(\rho) &\equiv (1-\rho^2)^{3/2} \\ f_3(\rho) &\equiv \rho^2(1-\rho^2)^{-1/2} \\ f_4(\rho) &\equiv \rho^2(1-\rho^2)^{1/2} \\ f_5(\rho) &\equiv \rho^2(1-\rho^2)^{3/2} \\ f_6(\rho) &\equiv \rho(1-\rho^2)^{-1/2} \\ f_7(\rho) &\equiv \rho^4(1-\rho^2)^{1/2} . \end{aligned}$$

A_{pj} and A_{bj} denote area of the upper($j=u$)/lower($j=\ell$) half of an elliptic crack model and area only in upper/lower barriers, respectively.

$R_k(\xi_j)$ for an unsymmetric elliptic crack model can be expressed as

$$R_k(\xi_j) = \frac{\int_{\theta_j^*}^{\frac{\pi}{2}} \int_{\eta_j}^1 f_k(\rho) \rho_{ab} d\rho d\theta}{\int_0^{\frac{\pi}{2}} \int_0^1 f_k(\rho) \rho_{ab} d\rho d\theta} \quad (j=u, \ell \text{ and } k=1, \dots, 7)$$

where $\theta_j^* = \sin^{-1}(\xi_j)$, $\xi_j = \frac{h_j}{b_j}$ and $\eta_j(\theta, \xi_j) = \xi_j / \sin \theta$.

The function, $R_k(\xi_j)$ and $g_k(\xi_j)$ can be simplified as follows :

$$R_1(\xi_j) = \frac{2}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} z(\theta, \xi_j)^{3/2} d\theta$$

$$R_2(\xi_j) = \frac{2}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} z(\theta, \xi_j)^{5/2} d\theta$$

$$R_3(\xi_j) = \frac{3}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} [z(\theta, \xi_j)^{1/2} - \frac{1}{3} z(\theta, \xi_j)^{3/2}] d\theta$$

$$R_4(\xi_j) = \frac{15}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} [\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$R_5(\xi_j) = \frac{35}{\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} [\frac{1}{5} z(\theta, \xi_j)^{5/2} - \frac{1}{7} z(\theta, \xi_j)^{7/2}] d\theta$$

$$R_6(\xi_j) = \frac{15}{4\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} [z(\theta, \xi_j)^{1/2} - \frac{2}{3} z(\theta, \xi_j)^{3/2} + \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$R_7(\xi_j) = \frac{105}{4\pi} \int_{\theta_j^*}^{\frac{\pi}{2}} [\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{2}{5} z(\theta, \xi_j)^{5/2} + \frac{1}{7} z(\theta, \xi_j)^{7/2}] d\theta$$

$$R_8(\xi_j) = 2 [\dot{a}b g_1(\xi_j) + a\dot{b} g_2(\xi_j)] / [\frac{\pi}{3} (\dot{a}b + a\dot{b})]$$

$$R_9(\xi_j) = 2 [\dot{a}b g_3(\xi_j) + a\dot{b} g_4(\xi_j)] / [\frac{\pi}{15} (\dot{a}b + a\dot{b})]$$

$$R_{10}(\xi_j) = 2 [\dot{a}b g_5(\xi_j) + a\dot{b} g_6(\xi_j)] / [\frac{4\pi}{15} (\dot{a}b + a\dot{b})]$$

$$R_{11}(\xi_j) = 2 [\dot{a}b g_7(\xi_j) + a\dot{b} g_8(\xi_j)] / [\frac{\pi}{105} (\dot{a}b + a\dot{b})]$$

and

$$g_1(\xi_j) = \int_{\theta_j^*}^{\pi/2} \cos^2 \theta [z(\theta, \xi_j)^{1/2} - \frac{1}{3} z(\theta, \xi_j)^{3/2}] d\theta$$

$$g_2(\xi_j) = \int_{\theta_j^*}^{\pi/2} \sin^2 \theta [z(\theta, \xi_j)^{1/2} - \frac{1}{3} z(\theta, \xi_j)^{3/2}] d\theta$$

$$g_3(\xi_j) = \int_{\theta_j^*}^{\pi/2} \cos^2 \theta [\frac{1}{3} z(\theta, \xi_j)^{1/2} - \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$g_4(\xi_j) = \int_{\theta_j^*}^{\pi/2} \sin^2 \theta [\frac{1}{3} z(\theta, \xi_j)^{1/2} - \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$g_5(\xi_j) = \int_{\theta_j^*}^{\pi/2} \cos^2 \theta [z(\theta, \xi_j)^{1/2} - \frac{2}{3} z(\theta, \xi_j)^{3/2} + \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$g_6(\xi_j) = \int_{\theta_j^*}^{\pi/2} \sin^2 \theta [z(\theta, \xi_j)^{1/2} - \frac{2}{3} z(\theta, \xi_j)^{3/2} + \frac{1}{5} z(\theta, \xi_j)^{5/2}] d\theta$$

$$g_7(\xi_j) = \int_{\theta_j^*}^{\pi/2} \cos^2 \theta [\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{2}{5} z(\theta, \xi_j)^{5/2} + \frac{1}{7} z(\theta, \xi_j)^{7/2}] d\theta$$

$$g_8(\xi_j) = \int_{\theta_j^*}^{\pi/2} \sin^2 \theta [\frac{1}{3} z(\theta, \xi_j)^{3/2} - \frac{2}{5} z(\theta, \xi_j)^{5/2} + \frac{1}{7} z(\theta, \xi_j)^{7/2}] d\theta$$

where $z(\theta, \xi_j) \equiv 1 - \eta_j^2(\theta, \xi_j)$.

2) Definition of Functions, $M_k(\beta)$, in Chapter IV

$$\begin{aligned}
 M_1(\beta) &= \int_0^\beta \rho(1-\rho^2)^{1/2} d\rho \\
 &= \frac{1}{3} - \frac{1}{3} \cos^3(\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_2(\beta) &= \int_0^\beta \rho(1-\rho^2)^{3/2} d\rho \\
 &= \frac{1}{5} - \frac{1}{5} \cos^5(\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_3(\beta) &= \int_0^\beta \rho^3(1-\rho^2)^{-1/2} d\rho \\
 &= \frac{2}{3} - \cos(\sin^{-1}\beta) + \frac{1}{3} \cos^3(\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_4(\beta) &= \int_0^\beta \rho^3(1-\rho^2)^{1/2} d\rho \\
 &= \frac{2}{15} - \frac{1}{3} \cos^3(\sin^{-1}\beta) + \frac{1}{5} \cos^5(\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_5(\beta) &= \int_0^\beta \rho^3(1-\rho^2)^{3/2} d\rho \\
 &= \frac{2}{35} - \frac{1}{5} \cos^5(\sin^{-1}\beta) + \frac{1}{7} \cos^7(\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_6(\beta) &= \int_0^\beta \rho^5(1-\rho^2)^{-1/2} d\rho \\
 &= \frac{8}{15} - \frac{5}{8} \cos(\sin^{-1}\beta) + \frac{5}{48} \cos(3\sin^{-1}\beta) \\
 &\quad - \frac{1}{80} \cos(5\sin^{-1}\beta)
 \end{aligned}$$

$$\begin{aligned}
 M_7(\beta) &= \int_0^\beta \rho^5(1-\rho^2)^{1/2} d\rho \\
 &= \frac{8}{105} - \frac{1}{7} \sin^4(\sin^{-1}\beta) \cos^3(\sin^{-1}\beta) \\
 &\quad - \frac{4}{21} \cos^3(\sin^{-1}\beta) + \frac{4}{35} \cos^5(\sin^{-1}\beta)
 \end{aligned}$$

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