SORTING IN HETEROGENEOUS CONTESTS

DISSERTATION

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To My Parents

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Chapter I

INTRODUCTION

This dissertation analyzes the efficiency aspects of contests as a compensation scheme when workers are heterogeneous. Our focus is on the sorting ability of contests. When it is not possible to sort out agents efficiently, the second-best solution is sought. Recently, a body of literature has developed considering the use of rank-order among competitors as a criterion for compensation (Lazear and Rosen, 1981; Green and Stokey, 1983; Nalebuff and Stiglitz, 1984; Holmstrom, 1982; Malcomson, 1984; Carmichael, 1983; O'Keefe, Viscusi and Zeckhauser, 1984; Rosen, 1986; Bull, Schotter and Weigelt, 1987). Although neo-classical economics suggests that a worker is paid according to the value of his marginal product, in reality there are at least many anecdotal or casual experiences in which a worker is paid relative to the performance of his immediate competitors. Players or teams are rewarded entirely by their rankings in many sports. In an internal labor market, the worker who is the best among his peers is promoted. The tenure system in academics is another example. In a rank-order payment scheme, agents compete for a given winning and losing prize. The incentive lies in the difference between winning and

losing prizes. A contest is one of the simplest possible cases among compensation schemes which are based on relative performance among agents, in that only rank-order among agents matters in compensation.

The main concern of the literature on economic contests has been to characterize the situations under which contests are superior to linear piece rates as a compensation scheme. When it is costly to monitor agents' efforts (inputs) and outputs, compensation based on individual output observed by the employer, as in a linear piece rate system, is not always best. Given stochastic disturbances in production in which there are common disturbances affecting the outputs of all agents, the rank-order payment scheme performs better than the linear piece rate system since the common disturbances do not affect the ranking of outputs of agents, while the linear piece rate system is subject to any disturbances. Secondly, if the agents cannot verify the principal's (firm's) observation of output, then the principal can provide the proper incentive to agents by committing himself to fixed prizes and awarding them based upon rank-order of the agents' output rather than awarding agents on the basis of the principal's incomplete observation of output. Hence contests can be used to prevent 'moral hazard' on the part of employers. These incentive properties of contests are established under the assumption of a homogeneous population.

There are at least two reasons why economists should be interested in contests. The first is that the analysis of contests can be viewed as a first step toward the study of the multi-agent, single-principal

problem. Since it is difficult to characterize optimal compensation rules in a principal-agent problem with many agents, one approach is to study a simple compensation scheme for many agents like contests. The second reason is that contests may be able to provide an alternative explanation to certain widely-perceived empirical phenomena. For instance, various explanations for promotion and related internal labor market phenomena have been offered based on supervision, bargaining cost, job-specificity or implicit contracts. However, contests have the edge in its explanatory power over the other theories¹.

The great majority of the existing literature assumes a homogeneous worker population in constructing models of contests. The real world, however, is inherently heterogeneous in worker types, and information dissemination is generally asymmetric among various economic agents. If the identity of the contestants is known, contests in a heterogeneous population can be changed easily into contests in a homogeneous population using the handicap system demonstrated by Lazear and Rosen. When the identity of the agents unknown, things are not simple. First, the heterogeneity of the contestants itself may be determined in the labor market. As a first step in considering heterogeneous contests when the identity of worker type is not known, it is necessary to determine whether contests can screen and differentiate among workers of different abilities. In fact, sorting among workers is an important aspect of labor contract arrangements. The solution to this sorting problem provides an understanding of how

agents and firms behave when they know there are different types of agents.

An exception to the existing literature is Lazear and Rosen who examine whether contests can separate heterogeneous workers when workers' identities are not known to others workers and firms. They find that less able workers always prefer to participate in contests designed for more able workers. If their analysis is correct, it is not possible to have efficient sorting. Even if workers separate by types into different contests, the workers' effort incentives will be disturbed due to the "rat race" phenomena under asymmetric information. It is not unreasonable to imagine that less able workers will enter contests designed for more able workers if the ability difference between worker types is large. Even the losing prize in contests for more able workers can be larger than the winning prize in contests for less able workers. None the less their result is significant since it applies no matter how disparate the abilities of the types of workers.

O'Keefe, Viscusi and Zeckhauser are not satisfied with the result of Lazear and Rosen. They demonstrate that firms can induce workers to self-select into the appropriate contests and maintain efficient effort levels if firms are able to choose any level of monitoring precision. By increasing the prize spread of contests for more able workers, the firm prevents less able workers from participating in contests for more able workers. Reducing the precision of monitoring at the same time, the firm can keep the effort level of more able workers efficient. A similar argument can be offered for contests of less able workers.

Unfortunately, this method may not be enough to separate workers if there is a large ability difference between types of workers. O'Keefe, Viscusi and Zeckhauser explain that this occurs because there is a limit to increasing the prize spread. Increasing the prize spread under the constraint that the sum of the prizes is less than or equal to total output may require the losing prize be negative to deter the less-able type of workers from the contest. This is difficult to implement because of legal and institutional reasons. Also, there is no guarantee that the monitoring precision level in each tournament is optimal when monitoring costs are included.

Guasch and Weiss (1980, 1982) investigate schemes where a firm can sort workers using hiring standards, corresponding contingency wages and application fees. Under such schemes, workers are compared to a defined standard, whereas workers are compared to one another in contests. Their works are similar to the contest literature since the welfare of workers is determined by the probability of passing the standard (or winning the contest) and the corresponding contingency wages (similar to the winning and losing prizes). But there is a major difference in modelling. The effort levels of workers are invariant to hiring standards and the wage structure in Guasch and Weiss, while the effort levels of workers in contests depend on the opponent type and wage structure.

In this paper, we return to an elementary principal-agent framework (Lazear-Rosen framework) and demonstrate that there are cases in which efficient sorting occurs among heterogeneous workers via

contests. Existing models fail to show that the probability of winning can play a decisive role in determining whether to participate in a contest. Contests in the presence of heterogeneous population are complex because of the interaction between different types of competitors. In homogeneous contests, however, things can be simplified by focusing on a symmetric equilibrium. The analysis in this paper is limited to a risk-neutral, separable utility function for contestants which is used both in Lazear and Rosen, and O'Keefe et al. This particular functional form allows the analysis to be tractable. A key element of the model is a prize space on which both the firm's iso-profit curve and the workers' indifference curves can be drawn. This diagrammatic approach is the same as that used by Miyazaki(1977), Rothschild and Stiglitz(1976) and others.

We consider a situation where the productivity of each type of worker is different depending on the firms employing the workers. The firms must pay workers sufficient wages to satisfy their reservation utilities, which vary by the type of worker. There are many firms so that the wage decision of one firm does not affect the reservation utilities of workers. Hence, each firm prefers one or several types of workers to other types.

The set-up of this situation is achieved by extending the Lazear-Rosen model. In their work, the impact of markets external to the firm is summarized by the binding zero-profit condition. Given the zero-profit constraint, the firm maximizes the utilities of its workers. In this paper, the firm maximizes its profit given the reservation

utilities of workers are satisfied. This simply amounts to choosing a different point on the contract curve. This facilitates an easier illustration of how differences in reservation utilities between different types of workers dictate the firm's design of contests. Another way to interpret this model is that we imbue the Lazear-Rosen model with more flexibility lost by limiting to a certain class of utility function.

Contests can sort workers by attracting only the desired type(s) of worker to the firm. We assume there are two types of workers, more and less able. More able workers are defined by a lower level of disutility than less able workers incurred for doing the same amount of work. If the difference in reservation utilities is small compared to what the difference in incentive to work implies, the firm favors more able workers. Less able workers may choose not to join the contests for more able workers if the probability of winning is low for the less able, and the prizes, which have been determined according to the reservation utility of more able workers, are not significantly higher than the reservation utilities of the less able workers. If the difference in reservation utilities is large compared to what the difference in incentives to work implies, the firm prefers less able workers. More able workers have no incentive to participate if the probability of winning is not increased sufficiently to compensate for the lower winning prize which the profit maximizing firm determined from the reservation utility of less able workers.

When the difference in reservation utilities of workers is such that the firm is indifferent to the type of worker when maximizing profit, it is argued that there is an efficient sorting since homogeneous contests for less able workers are still feasible. More able workers do not have the incentive to join such contests.

Mixed contests are also investigated. In mixed contests, the firm and workers suspect there is more than one type of worker in the contest. Mixed contests are used to check adverse selection in the analysis. The expected utility of the unintended type of worker can be derived by imagining a mixed contest since the unintended type of worker expects to compete against different types of worker as in mixed contests. Also we make it clear that a pooling equilibrium (mixed contests in equilibrium) can exist. This result is not well-documented in the literature. It is much more difficult to design a separating homogeneous contest which dominates mixed contests in certain situations. This occurs because homogeneous contests for less favored type of worker may lower profits relative to mixed contests. This difficulty is evidenced by the presence of adverse selection. However, mixed contests are not efficient because it is difficult to provide the proper incentive for two different types of workers under one prize structure.

We also investigate the second-best contests when efficient separation is not possible. There are two methods of preventing contamination of homogeneous contests by workers of another type. The first is to penalize them by increasing the prize difference and the

second is to subsidize them by opening the contests and offering prize levels higher than what their productivities warrant. Mixed contests are also a legitimate candidate for the second-best contests. Among mixed and separating contests, profit-maximizing contests are derived. Given the difference in work incentives between types, when the difference in reservation utilities is small, separating contests provide higher profit than mixed contests. This occurs because the separating cost is small in that the firm need not change the prizes from the optimal ones. When the difference in reservation utilities is large, mixed contests provide higher profit than separating contests. When the differences in reservation utilities reflect more closely the difference in work-incentives within the firm, mixed contests resemble homogeneous contests. The analysis indicates that separation is feasible in a broader range of situations if the firm manipulates the prize level and its structure. The composition and similarity of the contestants play important roles.

In the remainder of the paper, we present an analysis of contests in a homogeneous population which serves as an illustration of the model and as an integral part of the analysis in the heterogeneous case. Next, the analysis of contests in the presence of a heterogeneous population is considered. Finally, we summarize the result and offer concluding remarks.

CHAPTER II

CONTESTS IN A GROUP OF HOMOGENEOUS WORKERS

We first consider contests between homogeneous workers. They serve as an illustration of the model and because they are the simplest kind of contest the basic structure is revealed in its clearest form. More importantly, they are an integral part in the analysis of sorting in contests in the presence of heterogeneous workers. When there are several types of workers, the firm may be able to sort workers according to their type by offering differential self-selection incentives for the different worker types. When the firm can sort out workers completely according to type, the result of contests among homogeneous workers turns out to be very useful in understanding the self-selection incentives of the firm's contest design. We use the word 'separating' to indicate the situation in which the firm can sort out worker types using contests. If the firm cannot sort out the situation by contests, we use the term 'mixed' or 'pooling'.

We now consider the profit-maximizing firm's problem of designing a contest for a homogeneous group of workers. The basic setup for analyzing homogeneous contests follows the Lazear-Rosen model, but with an added emphasis on the role of the reservation utilities of

workers. For simplicity, and without loss of generality, we assume that there are only two workers that behave non-cooperatively with each other. The worker's effort determines the output produced according to a known relationship. The output is observed with error by the firm, but the effort supplied by each worker is known only to the worker. Workers compete for given prizes. The worker with the larger output, as observed by the firm, takes the winning prize. The firm chooses the prizes knowing the workers' behavior and the reservation utility of the workers, which must be satisfied to keep workers with the firm.

The firm is assumed to have the following stochastic, constant returns to scale production function for each worker i ,

$$q_i = e_i + \nu_i, \quad i = 1, 2.$$

where q_i is the output of worker i , e_i is the effort level of worker i and ν_i is a random variable with mean zero. Labor is the only input. The randomness results from inaccuracy in monitoring. ν_i is i.i.d. for all workers. In the principal-agent literature, it is also common to assume that the stochastic nature of production causes a randomness in output apart from the input. However, we opt for the interpretation of monitoring error in explaining the random factor in output since one can define the type of worker by their ability to adjust to the stochastic nature of production. In the next section analyzing heterogeneous contests, we assume that the random factor is same regardless of the types of workers involved. Total output is the sum

of the outputs of the two workers, $Q = q_1 + q_2$. This form of specification for the production function assumes all workers are equally productive. The only difference in output among the workers other than the randomness, ν_1 , comes solely from the workers' willingness to supply effort, e . Later, when a heterogeneous population is considered, we allow the incentive to supply e to differ among the different worker types. The differential incentive comes from the difference in the marginal rate of substitution between e and Income among the workers.

Workers are assumed to have the following separable expected utility function that is risk-neutral in Income, EU :

$$EU = Pr \cdot Y_w + (1-Pr)Y_l - C(e). \quad (1)$$

Y_w and Y_l are the winning and losing prizes and Pr and $(1-Pr)$ are the probabilities of winning and losing, respectively. $C(\cdot)$ is the disutility function of effort, e . The disutility function of effort, $C(e)$, is twice differentiable and the first and second derivatives are both positive. The probability of winning for the first worker is,

$$\begin{aligned} Pr &= Pr(y_1 > y_2) = Pr(e_1 + \nu_1 > e_2 + \nu_2) = Pr(e_1 - e_2 > \nu_2 - \nu_1) \\ &= Pr(e_1 - e_2 > \xi) = G(e_1 - e_2) \end{aligned} \quad (2)$$

where the subscripts 1 and 2 indicate the competing first and second workers and $\xi = \nu_2 - \nu_1$. $G(e_1 - e_2)$ is the cumulative distribution

function of ξ and $g(\cdot)$ is the probability density function of ξ , $dG=g$. ξ is a new random variable with mean zero because the means of ν_1 and ν_2 are each zero. Since it is assumed that ν is i.i.d., the derivation of the pdf of ξ is in principle straightforward. For a general formula, see Chapter 4 of Hogg and Craig. It is assumed that $g(\cdot)$ is known to the firms and workers. The difference in effort levels and the probability distribution function $G(\cdot)$ determine the probability of winning. Given the prizes, Y_w and Y_l , workers determine their effort levels to achieve their maximum expected utilities. Assuming an interior solution, the first-order and second-order conditions are²

$$\frac{\partial Pr}{\partial e_i} [Y_w - Y_l] - C'(e_i) = 0 \quad (3)$$

and

$$\frac{\partial^2 Pr}{\partial e_i^2} [Y_w - Y_l] - C''(e_i) < 0, \quad i = 1, 2.$$

In calculating the change in the probability of winning, we assume Nash behavior. Each player chooses an effort level, given the effort choice made by the opponent. Hence player 1 takes e_2 as given in determining his effort level³. Therefore, we know from (2) that for the first worker,

$$\frac{\partial Pr}{\partial e_1} = \frac{\partial G(e_1 - e_2)}{\partial e_1} = g(e_1 - e_2)$$

and from (3)

$$g(e_1 - e_2) [Y_w - Y_l] - C_1'(e_1) = 0, \quad (4)$$

where $g(e_1 - e_2)$ shows the probability density at e_1 given e_2 , a parameter in this case. The equation (4) shows how the effort level is determined to maximize the worker's utility. At this point, the marginal disutility of effort is equal to the increase in expected income. However, there need not be an equilibrium level of effort. If the competition between workers is very severe, it is conceivable that one worker backs away from the competition and there may be no equilibrium. To illustrate, suppose the variance of the pdf of ξ is small so that a worker can increase his probability of winning significantly by supplying a little more effort. Then, the other worker has the same incentive to supply more effort. Both workers will increase their effort. However, it is conceivable that a worker might decrease his effort in response to an increase in effort by the opponent. This occurs because the worker may expect a higher level of utility with the loser's prize and lower effort supply than if the effort level were increased. In this case, equilibrium may not exist since the worker with the larger effort level can lower his input and still receive the winning prize. In the remainder of this section, we only consider the case where the existence of an equilibrium effort level is guaranteed.

The firm is risk neutral and seeks to maximize sum of outputs minus total payments to the contestants, $\Pi = (e_1 + e_2) + (\nu_1 + \nu_2) - (Y_w + Y_l)$. The price of output is normalized to one and does not

depend on the level of production of the firm. The prizes and disutility of effort are measured according to the market value of a unit of output (effort). The firm's expected profit is therefore,

$$E\Pi = e_1 + e_2 - (Y_w + Y_l). \quad (5)$$

Formally, the firm's problem is to choose Y_w and Y_l to maximize profit under the constraint of providing the reservation utilities of the workers, $\bar{U}$,

$$\begin{aligned} \text{Max}_{Y_w, Y_l} \quad & E\Pi = e_1 + e_2 - (Y_w + Y_l) \\ \text{s.t.} \quad & e_l \in \underset{e_l}{\text{argmax}} \quad EU_l = [Pr \cdot Y_w + (1-Pr)Y_l - C(e_l)], \\ & \text{Max}_{e_l} EU_l = [Pr \cdot Y_w + (1-Pr)Y_l - C(e_l)] \geq \bar{U}, \quad l=1,2. \end{aligned}$$

We limit our analysis to a symmetric equilibrium following the literature.⁴ In a symmetric equilibrium, the effort level of the homogeneous agents is the same. The probability of winning in a symmetric equilibrium is one half for each of the two contestants given that $e_1 = e_2$,

$$Pr = G(e_1 - e_2) = G(0) = \frac{1}{2}, \quad (2)'$$

and the first order condition becomes

$$g(0) (Y_w - Y_l) = C'(e). \quad (4)'$$

(4)' indicates that the utility maximizing level of effort depends solely on the difference between the winning and losing prizes, given the marginal disutility function of effort, $C'(e)$, and $g(0)$. This has been observed by Lazear and Rosen, and others. It comes from the simple form of the utility function, separable and risk-neutral. If either separability or the risk-neutral assumption is violated, then (4)' is not possible. A nonseparable utility function implies that $C'(e)$ itself is affected by the level of expected income. The risk aversion assumption implies that the marginal utility of income changes as the absolute level of the prize changes.

We can rewrite the expected utility and profit function using (2)' and (4)' for the homogeneous case as

$$EU = \frac{1}{2} (Y_w + Y_l) - C(e) \quad (1)'$$

and

$$E\Pi = 2e - (Y_w + Y_l), \quad (5)'$$

where $e = C'^{-1}(g(0) (Y_w - Y_l))$ by (4)'. (1)' and (5)' suggest that both the expected utility and profit are determined by the losing prize, Y_l , and the difference of the prizes, $Y_w - Y_l$. Thus the firm's constrained profit maximization problem can be solved graphically in a two dimensional space, with $Y_w - Y_l$ on the horizontal axis and Y_l on the

vertical axis. Note that any point in this prize space $(Y_w - Y_l, Y_l)$ represents a unique prize structure, (Y_w, Y_l) .

To solve diagrammatically the maximization problem of the firm, we draw indifference and iso-profit curves in the $(Y_w - Y_l, Y_l)$ space. Let $Y_w - Y_l = X$ and $Y_l = Z$. The indifference curve with reservation utility level $\bar{U}$ in the (X, Z) space derived from (1)' is

$$Z_{\bar{U}} = -Pr \cdot X + C(e) + \bar{U} = -\frac{1}{2}X + C[C'^{-1}(g(0)(Y_w - Y_l))] + \bar{U} \quad (6)$$

and its slope is

$$\frac{dZ}{dX}|_{\bar{U}} = -Pr + C'(e) \cdot \frac{\partial e}{\partial X} = -\frac{1}{2} + g(0)X \frac{\partial e}{\partial X}. \quad (7)$$

$Z_{\bar{U}}$ indicates the losing prize level Z (hence, prize level) necessary to provide utility level $\bar{U}$ for each X . The slope of Z with respect to X can be interpreted similarly. The second equalities in (6) and (7) follow from (4)'. Given X , a larger Z implies a higher utility level, holding other things constant. Note that $\frac{\partial e}{\partial X} > 0$ as $C''(e) > 0$ for an interior solution of effort level. See Figure 1. Figure 1 shows that a larger X is associated with a larger e in equilibrium. $\frac{\partial^2 Z}{\partial X^2}|_{\bar{U}}$ can be positive or negative depending on the value of $C'''(e)$ because $\frac{\partial^2 e}{\partial X^2}$ is determined by the change in the slope of $C'(e)$. The slope of the indifference curve is independent of the level of utility. Given X , the slope of utility function does not depend on the level of Z . This occurs because the utility function is linear with respect to income

because of the assumptions of risk-neutrality and separability of the utility function.

The indifference map is different from the individualistic indifference map that frequently appears in textbooks describing the microeconomics of consumer choice. The space in which our indifference curve is drawn is not in terms of effort and income. Our indifference curve is similar to that of the indirect utility function drawn in the space of prizes. In addition, the slope and level of the indifference curve depends on the opponent's effort level. Therefore, for the purpose of our equilibrium analysis, we frequently make recourse to drawing one worker's indifference curve given the equilibrium input of the opponent. In this sense, one may call this an equilibrium indifference curve. Also, note that the slope represents the marginal rate of substitution (MRS) between work effort and the prize level, not the expected compensation. Depending on probability of winning, the MRS varies even though it represents the same level of utility.

An iso-profit curve with profit level Π is derived from (5)',

$$Z_{\Pi} = -\frac{X}{2} + C'^{-1}(g(0)(Y_w - Y_l)) - \frac{\Pi}{2} \quad (8)$$

and its slope is

$$\frac{dZ}{dX}|_{\Pi} = -\frac{1}{2} + \frac{\partial e}{\partial X}. \quad (9)$$

Z_{Π} represents the level of the prizes which achieves the profit level Π for each X . Given X , a smaller Z implies a larger profit. The slope of the iso-profit curve indicates the increase in prize level(Z)

necessary to balance any increase in output when the prize spread(X) is increased. This slope does not depend on the level of profit, similar to the analysis of the indifference curve. The profit-maximizing firm will choose the combination (X, Z) where its iso-profit curve is tangent to the indifference curve representing the worker's reservation utility level. The slopes of the two curves are equal only at $X = \frac{1}{g(0)}$, as long as $\frac{\partial e}{\partial X} \neq 0$, the case of an interior equilibrium. It follows that $\frac{dZ}{dX}|_{\bar{u}} - \frac{dZ}{dX}|_{\Pi} = \frac{\partial e}{\partial X} \cdot (g(0)X - 1)$. Hence, if $X > \frac{1}{g(0)}$, then $\frac{dZ}{dX}|_{\bar{u}} > \frac{dZ}{dX}|_{\Pi}$ and if $X < \frac{1}{g(0)}$, then $\frac{dZ}{dX}|_{\bar{u}} < \frac{dZ}{dX}|_{\Pi}$. Also, $\frac{d^2Z}{dX^2}|_{\bar{u}} > \frac{d^2Z}{dX^2}|_{\Pi}$ at $X = \frac{1}{g(0)}$. This means that the iso-profit curve is tangent to the indifference curve from below. Hence, the sufficient condition for optimum is satisfied. Z is determined from the constraint on reservation utility $\bar{u}$. We state this result as a proposition.

Proposition 1. If there is an equilibrium prize in a homogeneous contest, it is unique.⁵

Remark 1. The invariance of the slopes of the indifference curve and the iso-profit curve with respect to Z implies that the utility-maximizing solution for X is invariant to the level of reservation utility.

Definition. A contest is called efficient if the marginal disutility and the value of marginal product of each contestant are equal at the equilibrium level of effort.

Proposition 2. Homogeneous contests are efficient.⁶

Homogeneous contests are efficient since the marginal disutility of the contestants, $C'(e)$, at $X = \frac{1}{g(0)}$, is equal to 1, the normalized price of additional effort. Under the risk-neutral and separable utility function we assumed, the equilibrium effort supply of the contestants is efficient in the contests in a homogeneous population. An example is drawn in Figure 2. However, in general, it is not true that contests are efficient if a more general utility function is assumed (See Appendix A). Since the slope of the indifference curve is affected by the degree of risk aversion, the profit maximizing X and Z differ from those found for the risk neutral and separable case, whose solution is efficient. Also, note that the optimal prize difference between winning and losing is the same, $X = \frac{1}{g(0)}$, no matter how productive or unproductive contestants are so long as the distribution of the random variable is same. Of course, the effort levels are all different. Again, this follows from the simple form of the utility function. This simple solution of X , under the separable and risk-neutral utility function, allows the analysis in the presence of heterogeneous population to be tractable in the following section.

CHAPTER III

SORTING IN HETEROGENEOUS CONTESTS

Since we are interested in the sorting ability of contests in a heterogeneous population, we consider a situation where workers know their own type, but the firm does not know the type of each individual worker. There are two possible situations which depend on the assumption that is made about the workers' knowledge of the types of their opponents. The first possibility is that workers know the opponents' type and the second is that the opponents' type is unknown. Under either assumption, we can analyze the sorting of workers. However we address the second situation only. It is more likely that if the firm cannot discern worker types, then neither can the workers. Furthermore, the first situation can be considered a special and degenerate case of the second situation we address in which the probability of facing a particular type of worker is equal to one and the probability of competing against the other type of worker is zero. Also, it is more difficult to sustain contests between different types of workers because both the less-able and more-able type of worker tend to reduce their effort levels when they know with certainty that the opponent is of the other type.

There are only two types of workers, say 'a' and 'b'. All workers are the same, except in their incentive to supply work effort. The functional form of the utility function is the same for each type, $EU_l = Pr_l \cdot Y_w + (1-Pr_l)Y_l - C_l(e)$, $l = a, b$. We distinguish the types of workers by assuming $C_b'(e) > C_a'(e)$ for any level of effort, $e > 0$, and $C_a(0) = C_b(0) = 0$. This assumption is stronger than assuming that $C_b(e) > C_a(e)$ for $e > 0$ and $C_a(0) = C_b(0) = 0$. But under the stronger assumption, we obtain clearer results on separation. The proportions of a-type workers and b-type workers in the labor pool are assumed to be α and $1-\alpha$ respectively, where $0 < \alpha < 1$. α is known to all parties involved.

We assume an economy where the productivity difference between the two types of workers varies between firms.⁷ There are economy wide reservation utilities for each type of worker. Hence, a certain type of worker is preferred by each firm. When the firm announces that its contests are for a certain type of worker, if only the workers of the designated type can satisfy their reservation utilities by participating, can these contests survive as homogeneous contests. Thus it is crucial to understand how the workers of a particular type decide whether to participate in contests designed for the other type. To calculate the expected utility of a worker in a contest for the other type, we analyze mixed contests because participating in a contest designed for another worker type is similar to participating in a mixed contest. In mixed contests, the firm offers prizes expecting

both types of worker to join and workers understand that the opponent can be of another type.

3.1 Mixed Contests⁸

In mixed contests, the probabilities of winning for each type of worker are given by

$$\begin{aligned} Pr_a^m &= \alpha \cdot G(e_a - e_a) + (1-\alpha)G(e_a - e_b) \\ &= \alpha \cdot G(0) + (1-\alpha)G(e_a - e_b), \end{aligned} \quad (10)$$

$$\begin{aligned} Pr_b^m &= (1-\alpha)G(e_b - e_b) + \alpha \cdot G(e_b - e_a) \\ &= (1-\alpha)G(0) + \alpha \cdot G(e_b - e_a). \end{aligned} \quad (11)$$

The superscript m will represent mixed contests hereafter. For simplicity, it is once again assumed that the monitoring technique is not dependent on the worker type, so that the pdf of ν_a is the same as that for ν_b . It follows trivially that ξ_a and ξ_b have the same pdf. For simplicity we write $\xi_a = \xi_b = \xi$.

Again, we are interpreting ν not as a random state of nature affecting production, but strictly as measurement error. A given worker's response to the opponent in general depends on the pdf of both his own and the opponent's white noise in the observation of their outputs. To illustrate, suppose that the given worker's random term ν has a very tight distribution. In that case, a b -type worker, might decline contests vis-a-vis an a -type opponent. On the other hand if

this worker's random term is more diffused, he might perceive a better chance of winning and might be willing to engage in such contests. Generally, given the opponent's effort level, the worker's expected utility maximizing response in e is not neutral to a mean preserving spread in ν . Essentially, this says that a given worker's reaction function to the opponent's effort input is a function of the pdf of ν . Since both contestants calculate their respective reaction functions similarly, the equilibrium outcome, which is the intersection of the two reaction curves, is dependent on the pdfs of ν_a and ν_b . But assuming different pdfs for ν_a and ν_b will not change our analysis below since the distributions of observed outputs for a- and b-type workers (q_a and q_b) are different even if the same pdf is assumed for both ν_a and ν_b . The means of q_a and q_b are different. Since the probability of winning depends on q_a and q_b , the distributions of q_a and q_b are what really matter. For both types of workers, the probability of facing an a-type worker is α and the probability of facing a b-type worker is $1-\alpha$. Since both a- and b-type of workers can get the reservation utilities each in mixed contests, the proportion of applicants for mixed contests will be the same as that of the labor pool. e_a and e_b in (10) and (11) represent the equilibrium effort levels of a-type and b-type workers. $G(e_a - e_b)$ is the probability of winning for a-type workers when they face b-type workers. A similar interpretation can be given for $G(e_b - e_a)$.

We derive the utility maximizing rule of effort supply for each type of worker given the prizes Y_w and Y_l as we did in (4). Assuming interior solutions,

$$C_a'(e_a) = [\alpha \cdot g(0) + (1-\alpha)g(e_a - e_b)] (Y_w - Y_l), \quad (12)$$

$$C_b'(e_b) = [(1-\alpha)g(0) + \alpha \cdot g(e_b - e_a)] (Y_w - Y_l). \quad (13)$$

The second-order conditions are assumed to be satisfied. Workers supply effort until the marginal disutility of work equals the marginal increase in utility. (12) and (13) imply that $(Y_w - Y_l)$ alone determines the effort level for both types of workers for a given $C_a'(\cdot)$, $C_b'(\cdot)$, α and $g(\cdot)$, but in a more complex way than in the homogeneous worker case. In homogeneous contests, all we need to know about $g(\cdot)$ is $g(0)$ because the effort levels of workers are the same in equilibrium. But in mixed contests, since $e_a - e_b$ can be any number depending on X in equilibrium, we need to know $g(\cdot)$ fully. This means that for each level of $X = Y_w - Y_l$, the coefficient which relates X and $C'(e)$ is a function of X . This contrasts with the homogeneous case where the coefficient was a constant of $g(0)$ for all X , as shown in Figure 1. It is not true that $e_a > e_b$ always. If the right-hand side of (12) and (13) are the same for a given X , we know that $e_a > e_b$ and $Pr_a^m > Pr_b^m$. But it is possible that the right-hand side of (13) is larger than that of (12). For instance, this could occur if α is small and $g(0) > g(e_a - e_b)$. In this case, $e_b > e_a$ and $Pr_b^m > Pr_a^m$. It is also not true that the utility-maximizing effort level of a worker in a mixed contest will

always increase as X increases, as was found in homogeneous contests under an arbitrary pdf of $g(\cdot)$. The adverse effect on $g(e_a - e_b)$ and $g(e_b - e_a)$ may dominate the positive effect of an increased X .

The equilibrium indifference curves and their slopes are derived in a manner similar to the homogeneous case,

$$Z_a^m = -Pr_a^m \cdot X + C_a(e_a) + U_a, \quad (14)$$

$$Z_b^m = -Pr_b^m \cdot X + C_b(e_b) + U_b, \quad (15)$$

$$\frac{dZ_a^m}{dX} = -Pr_a^m - \frac{\partial Pr_a^m}{\partial X} \cdot X + C_a'(e_a) \cdot \frac{\partial e_a}{\partial X}, \quad (16)$$

$$\frac{dZ_b^m}{dX} = -Pr_b^m - \frac{\partial Pr_b^m}{\partial X} \cdot X + C_b'(e_b) \cdot \frac{\partial e_b}{\partial X}. \quad (17)$$

The equilibrium iso-profit curve and its slope are

$$Z_{\Pi}^m = -\frac{X}{2} + \alpha \cdot e_a + (1-\alpha)e_b - \frac{\Pi}{2}, \quad (18)$$

$$\frac{dZ_{\Pi}^m}{dX} = -\frac{1}{2} + \alpha \cdot \frac{\partial e_a}{\partial X} + (1-\alpha) \frac{\partial e_b}{\partial X}. \quad (19)$$

e_a and e_b are determined in (12) and (13), first-order conditions.

Formally, the firm's problem is

$$\begin{array}{ll} \text{Max}_{X,Z} & \Pi = 2\alpha e_a + 2(1-\alpha)e_b - (X + 2Z) \end{array}$$

$$\text{s.t. } e_l \in \underset{e_l}{\operatorname{argmax}} EU_l = [Pr_l^m \cdot X + Z - C_l(e_l)],$$

$$\underset{e_l}{\operatorname{Max}} EU_l = [Pr_l^m \cdot X + Z - C_l(e_l)] \geq \bar{U}_l, \quad l=a,b.$$

From (14) and (15), an effective constraint facing the firm can be traced out which guarantees that a-type workers receive $\bar{U}_a$ and b-type workers $\bar{U}_b$. In principle, one can find profit-maximizing prize using this constraint and the iso-profit curve. In section 3.4, we find solutions for a certain class of mixed contests.

3.2 Efficient Separating Contests

With the format of mixed contests in hand, we are in a position to determine whether efficient separation is possible. The disutility function together with the production function of the firm determines the utility-maximizing effort level, given X and $g(\cdot)$, and hence the output level of the firm. The reservation utility level determines the prize level, $Z=Y_1$. When $(\bar{U}_a - \bar{U}_b)$, the difference in reservation utilities of the two types of workers, is larger than what the work-incentive difference of the two types of workers warrants, it is more profitable for the firm to offer contests for a-type workers only. And when the difference in reservation utilities of the two types of workers is smaller than what the work-incentive difference of the two types of workers warrants, it is more profitable for the firm to offer contests for b-type workers only.

But the firm with imperfect information about worker type must take into account the possibility that workers from the unintended group will enter the contests and create an adverse selection problem. For the purpose of equilibrium analysis, it is innocuous to presume

that the probability of competing against an a- or b-type worker is the same as the composition of the labor pool, as was true in mixed contests when the unintended type of workers joins the offered contests.⁹ Hence, we use (14) and (15) in mixed contests in calculating the expected utility of joining the contests for the other type of worker. If the utility of a worker from the unintended type in the contest is smaller than the reservation utility of their type, there is no adverse selection by that type of worker in equilibrium.

With the information given so far, we can be more precise on efficient separating contests. We first find the firm's choice of worker-type by comparing the profits generated by a-type and b-type homogeneous contests. Since the profits of a-type and b-type contests are given by, $\Pi_a^h = 2e_a^h - (x_a^h + 2z_a^h)$ and $\Pi_b^h = 2e_b^h - (x_b^h + 2z_b^h)$ in (5), the difference in profits is

$$\begin{aligned}\Pi_a^h - \Pi_b^h &= 2(e_a^h - e_b^h) - (x_a^h - x_b^h) - 2(z_a^h - z_b^h) \\ &= 2(e_a^h - e_b^h) - 2(C_a(e_a^h) + \bar{U}_a - C_b(e_b^h) - \bar{U}_b).\end{aligned}\quad (20)$$

e_a^h, x_a^h, z_a^h are the equilibrium values in the homogeneous contests for a-type workers. e_b^h, x_b^h, z_b^h are defined similarly for b-type workers. The second equality in (20) comes from the assumption that $x_a^h = x_b^h = \frac{1}{g(0)}$, and the previous results that $\Pr_a^h = \Pr_b^h = \frac{1}{2}$, $z_a^h = -\Pr_a^h \cdot x_a^h + C_a(e_a^h) + \bar{U}_a$, and $z_b^h = -\Pr_b^h \cdot x_b^h + C_b(e_b^h) + \bar{U}_b$. Changing the assumption that $x_a^h = x_b^h = \frac{1}{g(0)}$ does not affect the analysis below in any

significant way. Hence, the choice of worker-type by the firm depends on $(\bar{U}_a - \bar{U}_b)$, according to (20). If

$$0 \leq (\bar{U}_a - \bar{U}_b) \leq (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h)), \quad (21)$$

then the firm chooses an a-type-worker contest, and if

$$(\bar{U}_a - \bar{U}_b) \geq (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h)), \quad (22)$$

then the firm offers an b-type-worker contest.

The firm selects the worker type which requires the lower wage per unit of output produced. $(\bar{U}_a - \bar{U}_b)$ is assumed to be larger than or equal to zero. Note that $\Pi_a^h, \Pi_b^h \geq 0$ must hold in equilibrium. Throughout the section, non-negative profits of the contests are assumed whenever we claim the existence of an equilibrium of the contests. This is not the final solution. The existence of adverse selection must be checked. Therefore, we calculate the expected utility of the undesired type of worker when they participate in a contest for the desired type of worker.

First, consider the case where the firm offers contests for a-type workers using (21). To see whether b-type of workers join the contests, we calculate,

$$\begin{aligned} U_b^m &= Pr_b^m \cdot X_a^h + Z_a^h - C_b(e_b) \\ &= Pr_b^m \cdot X_a^h - Pr_a^h \cdot X_a^h + C_a(e_a^h) + \bar{U}_a - C_b(e_b). \end{aligned}$$

If a b-type worker joins an a-type contest and expects a higher utility level than what could be expected in a b-type contest, then every b-type worker will join a-type contests in equilibrium. Hence Pr_b^m should reflect the distribution of a- and b-type workers in the labor pool. Since we are interested in sorting, it is appropriate to assume the effort level of a-type is e_a^h and see the maximum utility of a b-type worker knowing e_a^h . A b-type worker will choose $e_b=0$, the corner solution, if $U_b^m(0) \geq U_b^m(e_b^m)$, and will choose $e_b = e_b^m$ if $U_b^m(0) \leq U_b^m(e_b^m)$. e_b^m is derived from the first order condition in (13) given that $e_a = e_a^h$. A zero effort level does not mean zero literally. A better interpretation is that the worker supplies the minimum effort level required to stay in the contests. Assuming that $G(e_b - e_a) = G(-e_a) = 0$, the Pr_b^m at $e_b = 0$ is $(1-\alpha)G(0) = \frac{1-\alpha}{2}$, by (11). Even if the probability of winning by b-type workers when competing against a-type workers is zero, the overall probability of winning for b-type workers is not zero because of the participation by many b-type workers. Also note that assuming zero effort at the corner solution is not inconsistent with the rest of the model even in the unique equilibrium case. To have an interior solution for effort level by b-type workers, EU_b must be concave unimodally with respect to e_b . Then, $\frac{\partial EU_b}{\partial e_b} \Big|_{e_b=0} = (1-\alpha)g(0) + \alpha g(-e_a^h) - C_b'(0) > 0$ must hold. Even if $g(-e_a^h) = 0$, since $(1-\alpha)g(0) > 0$, $\frac{\partial EU_b}{\partial e_b} \Big|_{e_b=0}$ can be positive and, assuming zero effort is justified. In the case of multiple equilibria, there is no restriction on

$\frac{\partial EU_b}{\partial e_b} \Big|_{e_b=0}$ since the objective function EU_b can have a convex portion.

The effort supply of b-type workers is

$$e_b = e_b^m \text{ if } \alpha \cdot G(e_b^m - e_a^h) \cdot \chi_a^h \geq C_b(e_b^m), \text{ and} \quad (23)$$

$$e_b = 0 \text{ if } \alpha \cdot G(e_b^m - e_a^h) \cdot \chi_a^h \leq C_b(e_b^m). \quad (24)$$

(23) and (24) mean that the expected gain in utility must be larger than or equal to total disutility of supplying the effort. If b-type workers do not expect some positive probability of winning against a-type workers, $G(e_b^m - e_a^h)$, for an interior solution, b-type workers select the corner solution and supply zero effort. Hence, it is conceivable that if there is a sufficient difference in the disutility level for effort between a- and b-type of workers, then b-type workers will supply zero effort.

We examine the possibility that $\bar{U}_b \geq U_b^m$. Let us consider the case described by (23) first. In this case, $(\bar{U}_b - U_b^m) = \bar{U}_b - \bar{U}_a + (Pr_a^h - Pr_b^m) \cdot \chi_a^h + C_b(e_b^m) - C_a(e_a^h)$. Given (21), if

$$(\bar{U}_a - \bar{U}_b) \leq (Pr_a^h - Pr_b^m) \cdot \chi_a^h + C_b(e_b^m) - C_a(e_a^h), \quad (25)$$

then there can be a separating equilibrium since $(\bar{U}_a - \bar{U}_b)$ is assumed non-negative. After replacing $(\bar{U}_a - \bar{U}_b)$ with $z_a^h - C_a(e_a^h) - z_b^h + C_b(e_b^h)$ using (6), (25) can be written as

$$(Z_a^h + Pr_b^m \cdot x_a^h) - (Z_b^h + Pr_b^h \cdot x_b^h) \leq C_b(e_b^m) - C_b(e_b^h). \quad (26)$$

The left-hand and right-hand sides of (26) show the change in expected gain and the change in disutility when b-type workers enter contests for a-type workers. If the reward is less than the increased cost in effort, or if the loss is larger than the reduction in effort, then b-type workers have no incentive to participate in contests designed for a-type workers and there is separation.

Lemma 1. A sufficient condition for deterring b-type workers from entering a-type contests is given by (26).

Next a necessary condition is derived. Let us assume that $e_b^m = e_b^h$ so that the right-hand side of (26) zero. If the left-hand side is positive at $e_b^m = e_b^h$, (26) cannot be satisfied because b-type workers can choose $e_b^m = e_b^h$ and be better off in a-type contests than in b-type contests. However, if the right-hand side of (26) is negative at $e_b^m = e_b^h$, (26) can be satisfied.

Lemma 2. A necessary condition for preventing b-type workers from attempting an a-type contest is for (26) to be satisfied at $e_b^m = e_b^h$.

Only if the reduced probability of winning dominates the effect of the increased prize level of a-type contests at the same effort level e_b^h , is there a possibility of efficient separation.

The sufficient and necessary conditions are meaningful only if they are feasible. Since there are many parameters from six independent sources in this model, α , pdf $g(\cdot)$, $\bar{U}_a$, $\bar{U}_b$, $C_a(\cdot)$ and $C_b(\cdot)$, several cases are possible. It is important to note that the utility levels of the contestants are determined by both the prize level and the probability of winning in the compensation scheme of the contests. The higher prize level in an a-type contest induces b-type workers to join, but the lower probability of winning discourages the participation of b-type workers. Depending on the values of the parameters, either effect can dominate the other. An example is given in the next section where (26) is satisfied. With this example, we conclude that there can be a separating equilibrium. Such separating contests are efficient because these contests are among a homogeneous group of workers, a-type workers.

For the second case, using (24), $(\bar{U}_b - U_b^m) = \bar{U}_b - \bar{U}_a + (Pr_a^h - \frac{1-\alpha}{2}) \cdot X_a^h - C_a(e_a^h)$. Hence, if

$$(\bar{U}_a - \bar{U}_b) \leq (Pr_a^h - \frac{1-\alpha}{2}) \cdot X_a^h - C_a(e_a^h), \quad (25)'$$

there can be a separating equilibrium. (25)' can be rewritten, as we did to (26), to give

$$(Z_a^h + \frac{1-\alpha}{2} \cdot X_a^h) - (Z_b^h + Pr_b^h \cdot X_b^h) \leq -C_b(e_b^h). \quad (26)'$$

(25)' and (26)' indicate that the case of a corner solution can be incorporated into the previous analysis of interior solutions as a particular case. We state our analysis thus far as a proposition.

Proposition 3. When $0 \leq (\bar{U}_a - \bar{U}_b) \leq (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h))$, the firm offers contests designed for a-type workers. b-type workers do not attempt to join the a-type contests if $(\bar{U}_a - \bar{U}_b) \leq (Pr_a^h - Pr_b^m) \cdot X_a^h + C_b(e_b^m) - C_a(e_a^h)$. For any $(\bar{U}_a - \bar{U}_b)$ satisfying both inequalities for the firm and the b-type worker, there exists an efficient separating equilibrium.

Note that whether Π_b^h is positive or negative is immaterial as long as the firm offers a-type contests. Whether b-type of workers achieve their reservation utility in this firm or in other firms is not relevant. It is interesting to look at the problem from the worker's perspective. The firm offers a-type contests when a-type workers are more fit than b-type workers for this firm. In these contests, the probability of winning for b-type workers is smaller than that in other firms. Since the firm sets prizes just large enough to satisfy the reservation utility of a-type workers, b-type workers have significantly less incentive to participate in the contests of the firm where the productivity difference between a- and b-types is larger than in the contests of other firms offering a-type contests. Hence, given the difference in reservation utilities, separation is likely for the firm where the productivity difference is large.

In the case of (22), the firm offers contests for b-type workers. Given x_b^h and z_b^h , U_a^m at these prize level is given by

$$U_a^m = Pr_a^m \cdot x_b^h + z_b^h - C_a(e_a^m) = Pr_a^m \cdot x_b^h - Pr_b^h \cdot x_b^h + C_b(e_b^h) - C_a(e_a^m) + \bar{U}_b.$$

Since a-type workers are more able than b-type workers, $e_a = 0$ cannot be optimal. If

$$(\bar{U}_a - \bar{U}_b) \geq (Pr_a^m - Pr_b^h) \cdot x_b^h + C_b(e_b^h) - C_a(e_a^m), \quad (27)$$

then $\bar{U}_a \geq U_a^m$ and workers separate. Replacing $(\bar{U}_a - \bar{U}_b)$ with (6) in (27), the inequality above can be written as

$$(z_b^h + Pr_a^m \cdot x_a^h) - (z_a^h + Pr_a^h \cdot x_a^h) \leq C_a(e_a^m) - C_a(e_a^h). \quad (28)$$

a-type workers do not have the incentive to participate contests for b-type workers if the value of reduction in effort is not worth the loss in expected income, or if the gain in the reward is smaller than the cost of increasing effort.

Lemma 3. A sufficient condition that provides that a-type workers do not attempt b-type contests is given by (28).

Lemma 4. A necessary condition that a-type workers do not attempt b-type contests is that (28) is satisfied at $e_a^m = e_a^h$.

One can give the same kind of interpretation to Lemmas 3 and 4 as was given to Lemmas 1 and 2. Since the firm offers b-type worker contests when $(\bar{U}_a - \bar{U}_b)$ is larger than or equal to some positive number according to (22), there is always a case for separation. Unless the difference in ability between types of worker is small, there is not much incentive for an a-type worker to participate in contests for b-type workers since the loss in the expected income outweighs the reduction in work effort.

From an a-type worker's point of view, while the prize is just big enough to satisfy b-type workers, the probability of winning is not high generally since this is a firm in which b-type workers fit. Given the difference in reservation utilities, it is not attractive for a-type workers to join b-type contests where the productivity difference is smaller than in other firms offering b-type contests.

Proposition 4. When $(\bar{U}_a - \bar{U}_b) \geq (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h))$, the firm offers contests designed for b-type workers. If $(\bar{U}_a - \bar{U}_b) \geq (Pr_a^m - Pr_b^h) \cdot x_b^h + C_b(e_b^h) - C_a(e_a^m)$, then a-type workers have no incentive to join such b-type contests. For a $(\bar{U}_a - \bar{U}_b)$ satisfying both inequalities, there exists an efficient separating equilibrium.

Again, whether Π_a^h is positive or negative is immaterial, as long as the firm offers a-type contests, because a-type workers still enjoy the reservation utility, $\bar{U}_a$, in the economy. Now we consider the case where

$$(\bar{U}_a - \bar{U}_b) = (e_a^h - e_b^h) - (c_a(e_a^h) - c_b(e_b^h)). \quad (29)$$

This is the situation where offering two separate contests, one designed for each type of worker, is consistent with the profit-maximizing behavior of the firm. It corresponds to the Lazear-Rosen case. When the firm earns the same profit from homogeneous contests of either type of worker and tries to offer contests for both types of worker, there is adverse selection by less able workers, similar to the work of Lazear and Rosen. We prove a slightly generalized version of the Lazear-Rosen case by following their approach. The firm's profit need not be zero. Also, it is emphasized that the proof depends on the assumption of the risk-neutral and separable utility function.

We make an additional assumption that $g(\cdot)$ is symmetric around zero and $g(\cdot)$ has its largest value at zero. Assume the contests are separate, and consider the expected return, R_l , generated by participating in contests $l=a,b$ with an arbitrary effort level e . Then $R_l(e) = Z_l^h + \text{Pr}^l \cdot X_l^h = Z_l^h + G(e - e_l^h) \cdot X_l^h$. Pr^l is the probability of winning in contest l . Then $R_l(e_l^h) = Z_l^h + G(0) \cdot X_l^h = C_l(e_l^h) + \bar{U}_l$ since the profit maximizing firm provides the minimum required utility using

(6). The slopes of these expected returns are $R_l' = \frac{\partial R_l}{\partial e} = g(e - e_l^h) \cdot X_l^h > 0$. The slope is steepest when $e = e_l^h$ and the slope equals 1 since $X_a^h = X_b^h = \frac{1}{g(0)}$. Furthermore, $R_b'[e - (e_a^h - e_b^h)] = R_a'(e)$. Note that $e_a^h > e_b^h$. $R_a(e_a^h) - R_b(e_b^h) = C_a(e_a^h) + \bar{U}_a - C_b(e_b^h) - \bar{U}_b = (e_a^h - e_b^h)$ by (29). Therefore $R_b(e)$ is a pure displacement of $R_a(e)$ (see Figure 3) and the revenue functions never cross. b-type workers can always achieve higher expected utility

by joining a-type-worker contests. Note that this proof is not robust to changing assumptions. Unless the revenue function is an exact displacement of another, the proof is not correct. In particular, note that this proof hinges upon $x_a^h = x_b^h$. Without this equality, $R_b(e)$ is not a displacement of $R_a(e)$. $x_a^h = x_b^h$ when the utility function is risk-neutral and separable even though $g(\cdot)$ is same for both types of workers. For more general functional forms, the existence of adverse selection is uncertain. Also, this proof reveals that if the firm offers contests for b-type-workers, a-type-workers will not join the contests.

Proposition 5. When $(\bar{U}_a - \bar{U}_b) = (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h))$, that is, the firm is indifferent between a-type-worker contests and b-type-worker contests, if the firm offers two separate homogeneous contests for each type of worker, then b-type workers always prefer a-type contests to their own contests and create an adverse selection problem.

Remark 2. If the utility function of the contestants is either risk-averse or non-separable, Proposition 5 may not hold.

Remark 3. When $(\bar{U}_a - \bar{U}_b) = (e_a^h - e_b^h) - (C_a(e_a^h) - C_b(e_b^h))$, the firm can induce an efficient separation by offering contests for b-type workers.

In this model, the firm is not a monopsonist of labor inputs and $\bar{U}_a$ and $\bar{U}_b$ are, in a sense, the equilibrium reservation utilities of a- and b-type workers. We do not worry about employment of a-type workers since somewhere in the economy a-type workers are preferred and employed.

3.3 An Example

The pdf of $g(\cdot)$ is assumed to be uniform with the support of $g(\cdot)$, $(-\frac{1}{2}\delta, \frac{1}{2}\delta)$. $g(\cdot)$ is $\frac{1}{\delta}$ for all points in the domain. Quadratic disutility functions are given for each type of worker as $C_a(e) = \frac{k_a}{2} e^2$ and $C_b(e) = \frac{k_b}{2} e^2$ with $k_b > k_a$. The proportions of a-type workers and b-type workers are α and $(1-\alpha)$ respectively with $0 < \alpha < 1$. Then, the indifference and iso-profit curves for a-type and b-type homogeneous contests are,

$$z_a^h = -\frac{x}{2} + \frac{1}{2\delta^2 k_a} x^2 + \bar{U}_a,$$

$$z_b^h = -\frac{x}{2} + \frac{1}{2\delta^2 k_b} x^2 + \bar{U}_b,$$

$$z_a^\Pi = -\frac{x}{2} + \frac{x}{\delta k_a} - \frac{\Pi}{2},$$

$$z_b^\Pi = -\frac{x}{2} + \frac{x}{\delta k_b} - \frac{\Pi}{2}.$$

The solution values include $x_a^h = x_b^h = \delta$, $e_a^h = \frac{1}{k_a}$, $e_b^h = \frac{1}{k_b}$, $z_a^h = -\frac{\delta}{2} + \frac{1}{2k_a} + \bar{u}_a$, $z_b^h = -\frac{\delta}{2} + \frac{1}{2k_b} + \bar{u}_b$, $\Pi_a^h = \frac{1}{k_a} - 2\bar{u}_a$ and $\Pi_b^h = \frac{1}{k_b} - 2\bar{u}_b$. Hence $\Pi_a^h \geq \Pi_b^h$ when $(\bar{u}_a - \bar{u}_b) \leq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right)$. For this range of $(\bar{u}_a - \bar{u}_b)$, the

firm offers a-type contests. The expected utility of a b-type worker in a-type contests, i.e., U_b^m at (x_a^h, z_a^h) , is $\left(\frac{1}{2} - \alpha\right) \left(\frac{1}{k_a} - \frac{1}{k_b}\right) + \bar{u}_a$. Then $\bar{u}_b \geq U_b^m$ when $(\bar{u}_a - \bar{u}_b) \leq \left(\alpha - \frac{1}{2}\right) \left(\frac{1}{k_a} - \frac{1}{k_b}\right)$. If $\alpha > \frac{1}{2}$, it is

possible to have separation. It is more likely to have separation when α is larger. A larger α means there is a lower probability of winning for each b-type worker and prevents b-type workers from joining a-type contests. Note also that the closer $\bar{u}_b$ is to $\bar{u}_a$, the more likely separation is. The outside opportunity plays an important role. When $(\bar{u}_a - \bar{u}_b) \geq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b}\right)$, the firm offers b-type worker contests. U_a^m at (x_b^h, z_b^h) is $\frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b}\right) + \bar{u}_b$. Hence, for $(\bar{u}_a - \bar{u}_b) \geq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b}\right)$, $U_a^m \leq \bar{u}_a$. a-type workers have no incentive to participate in b-type contests. More generally, given a particular quadratic disutility function, it can be proven that (22) implies (27) if $g(\cdot)$ is uniform.

It is interesting to see why it is much easier to have separation in b-type contests than in a-type contests. First, note that in this example $e_a^h = e_a^m$ and $e_b^h = e_b^m$ so that the disutility level is the same whether a worker participates in contests for his type or the other worker type for both types of workers. On the other hand, the gain for an a-type worker in b-type contests is $(Pr_a^m - \frac{1}{2})x - (z_a^h - z_b^h)$ and that of

a b-type worker in a-type contests is $(Pr_b^m - \frac{1}{2})X + (Z_a^h - Z_b^h)$. By simple calculation, we get $(Pr_a^m - \frac{1}{2})X = (1-\alpha)(e_a^h - e_b^h)$ and $(Pr_b^m - \frac{1}{2})X = -\alpha(e_a^h - e_b^h)$. Note also that $(e_a^h - e_b^h) = \frac{1}{k_a} - \frac{1}{k_b}$ and $Z_a^h - Z_b^h = \frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b}) + \bar{U}_a - \bar{U}_b$. In b-type contests, $(\bar{U}_a - \bar{U}_b) \geq \frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b})$. Hence $(Pr_a^m - \frac{1}{2})X - (Z_a^h - Z_b^h) < 0$. That is, the gain in the probability of winning for a-type workers in b-type contests is not sufficient to compensate for the loss in the prize level. In a-type contests, $(\bar{U}_a - \bar{U}_b) \geq \frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b})$. Hence $(Pr_b^m - \frac{1}{2})X + (Z_a^h - Z_b^h) = (\frac{1}{2} - \alpha)(\frac{1}{k_a} - \frac{1}{k_b}) + (\bar{U}_a - \bar{U}_b)$, which can be either positive or negative depending on $(\bar{U}_a - \bar{U}_b)$. We summarize this section in Table 1. (A) corresponds to Proposition 3 and (B) corresponds to Proposition 4. Note that in (A), a limit on the possible values for α is required to have separation.

Table 1
Efficient Separating Contests

<u>Range of α</u>	<u>Range of $(\bar{U}_a - \bar{U}_b)$</u>	<u>Type of Contest</u>
(A) $\frac{1}{2} < \alpha < 1$	$0 \leq (\bar{U}_a - \bar{U}_b) \leq (\alpha - \frac{1}{2})(\frac{1}{k_a} - \frac{1}{k_b})$	a type
(B) $0 < \alpha < 1$	$\frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b}) \leq (\bar{U}_a - \bar{U}_b)$	b type

3.4 Existence and Efficiency of Mixed Contests

Since it is not always possible to have efficient separating homogeneous contests, there remains the question whether mixed or pooling contests are efficient. However, before one argues about the efficiency of mixed contests, the existence of mixed contests needs to be proven. As mentioned in the last part of section 3.1, the firm can find the optimal prize structure in a way similar to homogeneous contests. But unlike the homogeneous contests, the maximum profit may occur on either the indifference curve of Z_a^m , that of Z_b^m or at the intersection of Z_a^m and Z_b^m . Depending on which case is considered, the profit-maximizing X is found by equating (16) and (19), (17) and (19), or (14) and (15). The real question concerning the existence of a pooling equilibrium is whether there exists a prize structure which dominates the pooled equilibrium. In the literature of sorting, it is well known that there always exists a set of prizes (wages) which upsets the pooling equilibrium (see for instance, Rothschild and Stiglitz). The intuition is that one can take away a certain type of worker by offering a contract (prizes) arbitrarily close to the pooling one and increase the firm's profit. One can offer prizes which satisfy the reservation utility of one type, but the other type does not obtain their reservation utility. The firm can increase its profit with this prize structure over the prize structure in which both types of workers can have the reservation utilities. Hence, a pooling equilibrium is not sustainable. However, in contests, because of adverse selection,

there does not exist a set of prizes with which the firm can attract only one type of worker in particular situations. This was demonstrated in Propositions 3 and 4. Note that the firm is limited in choosing its worker-type because there is usually a favored type for the firm in our model, unlike other sorting models. It is likely to be more profitable to have mixed contests than to have homogeneous contests of less favored workers because mixed contests include the favored type of worker. In chapter IV, it is shown that mixed contests can be profit-maximizing in certain situations.

For both types of workers to be efficient, the marginal disutilities of each worker type must be equal to the value of marginal product, $C_a'(e_a) = C_b'(e_b) = 1$. To have equality of the marginal disutilities of each worker type, the right-hand sides of (12) and (13) must be satisfied, $\alpha g(0) + (1-\alpha) g(e_a - e_b) = (1-\alpha) g(0) + \alpha g(e_b - e_a)$. The equality of the two equations is satisfied only if $g(\cdot)$ is uniform, or $g(\cdot)$ is symmetric and $\alpha = \frac{1}{2}$. If $g(\cdot)$ is uniformly distributed, the proportions of α and $1-\alpha$ become meaningless. If $\alpha = \frac{1}{2}$, since $g(e_a - e_b) = g(e_b - e_a)$ by the symmetry of $g(\cdot)$, the equality of the right-hand sides of (12) and (13) are achieved again. These results are well observed by Lazear and Rosen. However, it is not yet known if there exists an efficient mixed contest even under the conditions of either a uniform pdf of $g(\cdot)$, or symmetric $g(\cdot)$ and $\alpha = \frac{1}{2}$. Since it is intuitively clear that it is difficult to provide the proper incentives for both types of worker with one prize structure, we consider under what circumstances, if any exist, is it possible to have

efficient mixed contests when $g(\cdot)$ is uniform or symmetric $g(\cdot)$ and $\alpha = \frac{1}{2}$. While Lazear and Rosen find a necessary condition for efficient mixed contests, we investigate a sufficient condition for efficient mixed contests.

First, let us assume that $g(\cdot) = \frac{1}{\delta}$, a constant for the whole support of $g(\cdot)$. Then (16) and (17) become

$$\frac{dZ^m}{dX_a} = -(\alpha G(0) + (1-\alpha) G(e_a - e_b)) + \frac{1}{\delta} X \left((1-\alpha) \frac{\partial e_b}{\partial X} + \alpha \frac{\partial e_a}{\partial X} \right) \quad (16)'$$

$$\frac{dZ^m}{dX_b} = -((1-\alpha) G(0) + \alpha G(e_b - e_a)) + \frac{1}{\delta} X \left((1-\alpha) \frac{\partial e_b}{\partial X} + \alpha \frac{\partial e_a}{\partial X} \right) \quad (17)'$$

Note that $\frac{dZ^m}{dX_b} - \frac{dZ^m}{dX_a} = G(e_a - e_b) - G(0) > 0$. $e_a > e_b$ because $C_a'(e) < C_b'(e)$ in (12) and (13). To avoid unnecessary complication, it is assumed that $C_l(\cdot)$ is quadratic so that $C_l''(\cdot) > 0$ and $C_l'''(\cdot) = 0$ for any effort level. That is, $C_l'(\cdot)$ is linear for $l=a,b$. Therefore, $\frac{\partial e_a}{\partial X} > \frac{\partial e_b}{\partial X}$ and $\frac{\partial^2 e_a}{\partial X^2} = \frac{\partial^2 e_b}{\partial X^2} = 0$ for all positive values of X and the iso-profit curve is linear in X by (19). Only a positively sloped iso-profit line is considered since we have assumed a non-negative profit and a positive losing prize.

To check for a sufficient condition, we differentiate the indifference curves once again,

$$\frac{d^2 Z}{dX^2}_a^m = \frac{1}{\delta} \left(\left(2 \frac{\partial e_b}{\partial X} - \frac{\partial e_a}{\partial X} \right) + 2\alpha \left(\frac{\partial e_a}{\partial X} - \frac{\partial e_b}{\partial X} \right) \right), \text{ and}$$

$$\frac{d^2 Z}{dX^2}_b^m = \frac{1}{\delta} \left(2\alpha \left(\frac{\partial e_a}{\partial X} - \frac{\partial e_b}{\partial X} \right) + \frac{\partial e_b}{\partial X} \right).$$

$\frac{d^2 Z}{dX^2}_b^m$ is always positive under our assumptions. $\frac{d^2 Z}{dX^2}_a^m$ is positive if $\alpha \geq$

$\frac{1}{2}$. But if $\alpha < \frac{1}{2}$, $\frac{d^2 Z}{dX^2}_a^m$ is positive when the productivity gap between

a-type and b-type workers is not very large. When α is nearly zero,

$\frac{d^2 Z}{dX^2}_a^m$ is positive unless a-type workers are not more than twice as

productive as b-type workers. Since $\frac{\partial^2 e_a}{\partial X^2} = \frac{\partial^2 e_b}{\partial X^2} = 0$ and $C_a(0) = C_b(0)$

$= 0$, $\frac{\partial e_a}{\partial X} = 2 \frac{\partial e_b}{\partial X}$ implies that an a-type worker supplies twice as large

an effort as the effort put forth by a b-type worker. We assume $\frac{d^2 Z}{dX^2}_a^m >$

0 for all X .

Depending on the size of the difference $\bar{U}_a - \bar{U}_b$, the solutions can be characterized in three different ways. The size of the difference in reservation utilities changes the contour of the constraints since the prizes must be designed to attract both types of worker. See Figure 4. If $(\bar{U}_a - \bar{U}_b)$ is relatively small, the point of tangency occurs between Z_b^m and Π (Figure 4A). In this case, the expected utility of an a-type worker is larger than his reservation utility. If

$(\bar{U}_a - \bar{U}_b)$ is relatively large, it is between Z_a^m and Π (Figure 4C). The expected utility of a b-type worker is larger than the reservation utility. For points between, the profit maximizing point is the intersection of Z_a^m and Z_b^m (Figure 4B). Let x_a^m and x_b^m be the points where Z_a^m and Π , and Z_b^m and Π are tangent, respectively. Depending on the size of $(\bar{U}_a - \bar{U}_b)$, any point x_m^m between x_a^m and x_b^m can be the profit-maximizing prize spread. x_a^m and x_b^m are easily calculated,

$$x_a^m = \delta \left(1 + \frac{\alpha G(0) + (1-\alpha) G(e_a - e_b) - \frac{1}{2}}{(1-\alpha) \frac{\partial e_b}{\partial X} + \alpha \frac{\partial e_a}{\partial X}} \right) \text{ and}$$

$$x_b^m = \delta \left(1 - \frac{\frac{1}{2} - (1-\alpha) G(0) - \alpha G(e_b - e_a)}{(1-\alpha) \frac{\partial e_b}{\partial X} + \alpha \frac{\partial e_a}{\partial X}} \right).$$

Note that $x_a^m > \delta$ and $x_b^m < \delta$. The efficient level of X can be derived from (12) and (13). $C_a'(e_a) = C_b'(e_b) = 1 = \frac{1}{\delta} X$. At $X = \delta$, the marginal disutility of effort is equal to the value of marginal effort. For this value of X , there are corresponding Z , U_a^m and U_b^m . Since X and Z are the same for both types of worker,

$$\begin{aligned} \bar{U}_a - \bar{U}_b &= (Pr_a^m - Pr_b^m) X - (C_a(e_a^m) - C_b(e_b^m)) \\ &= (G(e_a^m - e_b^m) - G(0)) \delta - (C_a(e_a^m) - C_b(e_b^m)) \\ &= (e_a^m - e_b^m) - (C_a(e_a^m) - C_b(e_b^m)). \end{aligned} \tag{30}$$

(10) and (11) are used to obtain the second equality. The third equality follows from the uniform distribution assumption. Comparing this result with equation (29), it is clear that the difference in reservation utilities at $X = \delta$ is exactly that which makes the firm indifferent to the type of worker. Mixed contests result in efficiency under the assumption that $g(\cdot)$ is uniform, only when $(\bar{U}_a - \bar{U}_b)$ is such that the firm makes same profit with either a-type or b-type homogeneous contests. Recall that the profit of the firm depends on the productivity and the reservation utility of the workers. In sum, at only one value of $(\bar{U}_a - \bar{U}_b)$ are mixed contests efficient when $g(\cdot)$ is a uniform pdf. But note that even this value of $(\bar{U}_a - \bar{U}_b)$ is not meaningful. Remark 3 noted that at this value of $(\bar{U}_a - \bar{U}_b)$, the firm can have efficient separating homogeneous contests by offering contests for b-type workers.

Thus far, we considered the case of a unique equilibrium. But if the assumptions on the convexity of the indifference and iso-profit curves are relaxed, so that $C_l'''(\cdot) = 0$, $l=a,b$ and $2 \frac{\partial e_b}{\partial X} > \frac{\partial e_a}{\partial X}$ no longer hold, then the indifference and iso-profit curves may have a concave portion. This allows the possibility of multiple equilibria. The existence of multiple equilibria does not change the basic point, however. Another tangency between effective constraint on reservation utilities and the iso-profit curve is irrelevant, for there is only one corresponding $(\bar{U}_a - \bar{U}_b)$ for an equilibrium at $X = \delta$, the efficient prize difference.

Next we consider the case of $\alpha = \frac{1}{2}$. In this case, the slopes of Indifference curves are,

$$\frac{dZ^m}{dX_a} = -\frac{1}{2} (G(0) + G(e_a - e_b)) + \frac{1}{2} X (g(e_a - e_b) \frac{\partial e_b}{\partial X} + g(0) \frac{\partial e_a}{\partial X}), \text{ and}$$

$$\frac{dZ^m}{dX_b} = -\frac{1}{2} (G(0) + G(e_b - e_a)) + \frac{1}{2} X (g(e_b - e_a) \frac{\partial e_a}{\partial X} + g(0) \frac{\partial e_b}{\partial X}).$$

This gives us $\frac{dZ^m}{dX_b} - \frac{dZ^m}{dX_a} = (G(0) + G(e_b - e_a)) - \frac{X}{2} (g(0) - g(e_b - e_a))$
 $(\frac{\partial e_a}{\partial X} - \frac{\partial e_b}{\partial X})$. It is not clear which Indifference curve is steeper.

We solve X_a and X_b as before by equating $\frac{dZ^m}{dX_a}$ and $\frac{dZ^m}{dX_b}$ with $\frac{dZ^m}{dX_{II}}$. This yields,

$$X_a^m = \frac{\frac{\partial e_a}{\partial X} + \frac{\partial e_b}{\partial X} + G(0) + G(e_a - e_b) - 1}{g(0) \frac{\partial e_a}{\partial X} + g(e_a - e_b) \frac{\partial e_b}{\partial X}} \text{ and}$$

$$X_b^m = \frac{\frac{\partial e_a}{\partial X} + \frac{\partial e_b}{\partial X} + G(0) + G(e_b - e_a) - 1}{g(0) \frac{\partial e_b}{\partial X} + g(e_b - e_a) \frac{\partial e_a}{\partial X}}.$$

On the other hand, to be efficient, X must be $\frac{2}{g(0) + g(e_a - e_b)}$ since $C'_a(e_a) = C'_b(e_b) = 1$. Denote this value as X^* . It is straightforward to show that $(X_a^m - X^*)(X_b^m - X^*) < 0$. This implies that X^* is between X_a^m and X_b^m as in the uniform distribution case. If $X_a^m < X_b^m$, then the firm maximizes its profit at $X = X_a^m$ and there cannot be efficient mixed contests. If $X_b^m < X_a^m$, then

$$\begin{aligned}\bar{U}_a - \bar{U}_b &= (Pr_a^m - Pr_b^m) X^* - (C_a(e_a^m) - C_b(e_b^m)) \\ &= (G(e_a^m - e_b^m) - G(0)) \frac{2}{g(0) + g(e_a - e_b)} - (C_a(e_a^m) - C_b(e_b^m)).\end{aligned}\quad (31)$$

$(G(e_a^m - e_b^m) - G(0)) \frac{2}{g(0) + g(e_a - e_b)}$ is not equal to $e_a^m - e_b^m$, as we found in previous uniform distribution case, except for a very small class of particular probability distributions. For instance, this could occur if $g''(\cdot) = 0$. As in the case of uniform $g(\cdot)$, mixed contests are efficient only for one value of $(\bar{U}_a - \bar{U}_b)$ when $\alpha = \frac{1}{2}$. It should now be clear that mixed contests are efficient only at one value of $(\bar{U}_a - \bar{U}_b)$ for the cases of $\alpha = \frac{1}{2}$ and a uniform $g(\cdot)$, so we can conclude that mixed contests are not efficient. We have stronger result on the inefficiency of mixed contests than Lazear and Rosen provide. Their conclusion on the inefficiency of mixed contests is based on the observation that $\alpha = \frac{1}{2}$ or that a uniform distribution on $g(\cdot)$ is required for efficiency. Our conclusion is based on the observation that mixed contests are inefficient except possibly when $\alpha = \frac{1}{2}$ and the difference $(\bar{U}_a - \bar{U}_b)$ satisfies (31).

Proposition 6. Mixed contests are not efficient.

CHAPTER IV

CONTESTS AS A SECOND-BEST SOLUTION

It is now clear that the firm cannot have the first-best outcome (efficient sorting) in all situations so the question remains as to what the firm should do in these situations. In this section, we find the profit-maximizing contests when the firm cannot separate agents efficiently. Mixed contests are natural candidates. But there are other possibilities. Lazear and Rosen suggest the possibility that a firm can prevent the less-able type of agent from participating in contests designed for the more-able type of worker by penalizing the less-able type of agent by increasing the prize gap between winning and losing. On the other hand, Miyazaki suggests that by subsidizing the less-able type of agent, a firm can separate the more-able type from the less-able workers although the analysis is presented in a different context. We would particularly like to determine when these separating contests are preferred to mixed contests. Since the emphasis of this section is to compare the different kinds of contests, strong assumptions are required. Throughout this section we assume a uniform distribution for the difference in the observed effort levels of agents and quadratic disutility functions, as we have previously. In fact, we

use the previous example again. The only difference is that this time we limit the difference between the two types by requiring that $1 > k = \frac{k_a}{k_b} > \frac{1}{2}$. With this limitation on k , the analysis which follows is quite complete. Note also that it is more likely that the firm can discern worker types when the quality-differences among contestants are very large.

4.1 Separating Contests

4.1.1 The Lazear-Rosen Case

Lazear and Rosen, and others envision that a firm can separate workers by increasing the prize gap offered in the contests of more-able type workers. We first consider this possibility and later compare it with another separating contest. Increasing the prize gap reduces the expected utility of the less-able type of worker. This occurs because the less-able type worker has a small probability of winning. As before, it is much easier to see this result with a diagram. See Figure 5. When $U_b^m > \bar{U}_b$, b-type workers have an incentive to participate in a-type worker contests. By increasing the prize gap until $U_b^m = \bar{U}_b$, which means shifting Z_b^m to $Z_b^{m'}$, the contamination by b-type workers is prevented. The profit from this contest becomes $\Pi_a^L = 2\theta_a^L - (X_a^L + 2Z_a^L)$ where the superscript L represents solution of the Lazear-Rosen case. However, there is no guarantee that the slope of Z_b^m

is steeper than the slope of z_a^h . If the slope of z_a^h is steeper than that of z_b^m , increasing the prize gap will increase $U_b^m - \bar{U}_b$ and will not separate workers according to type. See Figure 6. In this case, the increased prize gap increases the expected utility of b-type workers in a-type contests. This result follows when the probability of winning is not small for b-type workers. Note that z_a^h (prize level) is also rising as $\frac{d^2 z}{dX^2} > 0$. In this example, when $\alpha < \frac{1}{2}$, the slope of z_b^m is flatter than that of z_a^h . One can compute the actual values of Π_a^L , x_a^L , e_a^L and z_a^L by equating U_b^m at x_a^L to $\bar{U}_b$. Then, $x_a^L = \delta \left(\frac{2}{2\alpha-1} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (\bar{U}_a - \bar{U}_b) \right)^{\frac{1}{2}}$, $e_a^L = \frac{x_a^L}{k_a \cdot \delta}$, $z_a^L = -\frac{x_a^L}{2} + \frac{x_a^{L^2}}{2\delta^2 k_a} + \bar{U}_a$ and $\Pi_a^L = \frac{2k_b}{(2\alpha-1)(k_b-k_a)} \bar{U}_b - \frac{2(2\alpha k_b - (2\alpha-1)k_a)}{(2\alpha-1)(k_b-k_a)} \bar{U}_a + \frac{2}{k_a} \left(\left(\frac{2}{2\alpha-1} \right) \left(\frac{k_a k_b}{k_b - k_a} \right) (\bar{U}_a - \bar{U}_b) \right)^{\frac{1}{2}}$. Note that $\frac{1}{2} (\Pi_a^L + \Pi_b^h)$ is dominated by or is equal to Π_b^h or Π_a^L . If $\Pi_a^L \geq \Pi_b^h$, then $\Pi_a^L \geq \frac{1}{2} (\Pi_a^L + \Pi_b^h)$ and if $\Pi_a^L \leq \Pi_b^h$, then $\Pi_b^h \geq \frac{1}{2} (\Pi_a^L + \Pi_b^h)$. Hence, the firm offers only a-type contests in this case. We will use the notation Π , the profit of the contest, to represent the type of the contest if the contexts used are clear hereafter. Π^L will be used to represent the contest and profit of the Lazear-Rosen case.

4.1.2 The Miyazaki Case

There is another way to prevent b-type workers from contaminating a-type worker contests. If the prize level is increased in b-type worker contests until the higher expected utility for b-type workers in their own contests is equal to the utility in the a-type worker contests, then $U_b^h = U_b^m$. Again, it is convenient to present this graphically. See Figure 7. As in the Lazear and Rosen case, it is necessary for Z_b^m to have a steeper slope than Z_a^h . U_b^m must be smaller than $\bar{U}_a$. Remember that X_b^h is the same at each level of U_b . Z_b^h is derived from the equality $U_b = U_b^m$ at X_a^h and Z_a^h . The per contest profit is $\Pi S = \frac{1}{2} (\Pi_a^h + \Pi_b^h - 2(U_b^m - \bar{U}_b)) = \frac{\alpha}{k_a} + \frac{1-\alpha}{k_b} - 2\bar{U}_a$. We denote this contest and profit by ΠS . However, there is possible adverse selection by a-type workers in this modified b-type contest. To check this possibility, U_a^m at X_b^h and $Z_b^{h'}$ is compared to $\bar{U}_a$. $Z_b^{h'} = Z_b^h + U_b^m - \bar{U}_b = \frac{1}{2k_b} - \frac{\delta}{2} - (\alpha-1) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) + \bar{U}_a$ and $X_b^h = \delta$. Then $U_a^m = Z_a^m + Pr_a^m \cdot X_a^m - C_a(e_a^m) = Z_b^{h'} + Pr_a^m \cdot \delta - \frac{1}{2k_a}$. But $Pr_a^m \cdot \delta = \frac{\delta}{2} + (1-\alpha) \left(\frac{1}{k_a} - \frac{1}{k_b} \right)$. Hence $U_a^m = \bar{U}_a - (2\alpha-1) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) < \bar{U}_a$ because $\alpha > \frac{1}{2}$. No adverse selection by a-type workers exists in this case.

We are ready to compare the Lazear-Rosen case and the Miyazaki case. A straightforward calculation shows that $\Pi L \geq \Pi S$ when $(\bar{U}_a - \bar{U}_b) \leq US$ and $\Pi S \geq \Pi L$ when $(\bar{U}_a - \bar{U}_b) \geq US$ where $US = \frac{(2\alpha-1)}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (2-\alpha-(1-\alpha)k$

$\frac{1}{2} + 2((1-\alpha)(1-k))^2$ (for the derivation, see Appendix C1). Recall that $k = \frac{k_a}{k_b}$. Since we consider the range of $(\bar{U}_a - \bar{U}_b)$ which satisfies the inequality, $(\alpha - \frac{1}{2})(\frac{1}{k_a} - \frac{1}{k_b}) \leq (\bar{U}_a - \bar{U}_b) \leq \frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b})$, the range where efficient sorting is not possible (see Table 1), we must examine how US fits into this inequality. We can show that $US > (\alpha - \frac{1}{2})(\frac{1}{k_a} - \frac{1}{k_b}) = Q$ for $\frac{1}{2} < \alpha < 1$, for the relevant range of α . But US can be smaller than $\frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b}) = R$. We use Q and R to indicate the relevant range of the reservation utility difference between worker types. We find that approximately, if $\alpha < \frac{13}{16}$, then $US < R$. If $\alpha > \frac{13}{16}$, $US < R$ for $\text{Max}\{\frac{1}{2}, \frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1-\alpha)(2\alpha-1)}\} \leq k < 1$ (see Appendix C1). In sum, if $\frac{1}{2} < k \leq \text{Max}\{\frac{1}{2}, \frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1-\alpha)(2\alpha-1)}\}$, $\Pi_L \geq \Pi_S$ for $Q \leq (\bar{U}_a - \bar{U}_b) \leq R$. If $\text{Max}\{\frac{1}{2}, \frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1-\alpha)(2\alpha-1)}\} \leq k < 1$, $\Pi_L \geq \Pi_S$ for $Q \leq (\bar{U}_a - \bar{U}_b) \leq US$ and $\Pi_S \geq \Pi_L$ for $US \leq (\bar{U}_a - \bar{U}_b) \leq R$. Again, it is easier to consider this graphically. In Figure 8, we draw a line showing $US = R$ in the parameter space of k and α . If the two types are different (small k) and the proportion of a-type workers is large (large α), then US is larger than R and only Lazear-Rosen separating contests are relevant. There is room for the Miyazaki case only if the two types are close (large k) and the proportion of a-type workers is small (a small α , but still greater than $\frac{1}{2}$). In the remainder of this section, the same

pattern appears repeatedly. Each of the dividing lines have a similar shape. To achieve the equality, a larger α accompanies a larger k . This is intuitively clear since the similarity of the types of worker (or competitiveness) dilutes the effect of the increased proportion of a-type workers.

4.2 Mixed Contests

Under the assumptions we have made in this section, the solution for a mixed contest is (refer to section 3.4, Existence and Efficiency of Mixed Contests),

$$\text{If } 0 \leq (\bar{U}_a - \bar{U}_b) \leq UM, \quad x_b^m = \delta \left(\frac{\alpha k_b + (1-\alpha)k_a}{2\alpha k_b - (2\alpha-1)k_a} \right) \quad \text{and} \quad \Pi_b^m = \left(\frac{1}{k_a k_b} \right) \frac{(\alpha k_b + (1-\alpha)k_a)^2}{2\alpha k_b - (2\alpha-1)k_a} - 2\bar{U}_b,$$

$$\text{If } UM \leq (\bar{U}_a - \bar{U}_b) \leq UA, \quad x_m^m = \delta \left(\frac{2k_a k_b}{k_b - k_a} (\bar{U}_a - \bar{U}_b) \right)^{\frac{1}{2}} \quad \text{and} \quad \Pi_m^m = - \frac{2(2\alpha k_b - (2\alpha-1)k_a)}{k_b - k_a} \bar{U}_a + \frac{2(2(1-\alpha)k_a + (2\alpha-1)k_b)}{k_b - k_a} \bar{U}_b + 2((1-\alpha)k_a + \alpha k_b) \left(\frac{2(\bar{U}_a - \bar{U}_b)}{k_a k_b (k_b - k_a)} \right)^{\frac{1}{2}} \quad \text{and}$$

$$\text{If } (\bar{U}_a - \bar{U}_b) \geq UA, \quad x_a^m = \delta \frac{\alpha k_b + (1-\alpha)k_a}{(2\alpha-1)k_b + 2(1-\alpha)k_a}, \quad \Pi_a^m = \left(\frac{1}{k_a k_b} \right) \left(\frac{\alpha k_b + (1-\alpha)k_a}{2(1-\alpha)k_a + (2\alpha-1)k_b} \right)^2 - 2\bar{U}_a,$$

$$\text{where } UM = \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left(\frac{\alpha + (1-\alpha)k}{2\alpha - (2\alpha-1)k} \right)^2 \text{ and } UA = \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left(\frac{\alpha + (1-\alpha)k}{(2\alpha-1) + 2(1-\alpha)k} \right)^2. \text{ Again, this result need to be compared to the}$$

Inequality, $Q \leq (\bar{U}_a - \bar{U}_b) \leq R$. $UA > R$ because $(1-\alpha)k_a + \alpha k_b > 2(1-\alpha)k_a + (2\alpha-1)k_b$ for any $0 < \alpha < 1$. Depending on the values of α and k , UM can be larger than or smaller than Q . More precisely, for a given α , $K = \frac{-(8\alpha^3 - 9\alpha^2 + 3\alpha) + \alpha\sqrt{2\alpha-1}}{-8\alpha^3 + 13\alpha^2 - 8\alpha + 2}$ and determines the boundary (Appendix C2). We

show this in a diagram. See Figure 9. Note that $\frac{\partial UM}{\partial \alpha} < 0$ and $\frac{\partial Q}{\partial \alpha} > 0$.

k approaches 1 as α approaches 1 so that Q remains larger than UM . In sum,

$$\text{If } \frac{1}{2} \leq k \leq \text{Max} \left\{ \frac{1}{2}, \frac{-(8\alpha^3 - 9\alpha^2 + 3\alpha) + \alpha\sqrt{2\alpha-1}}{-8\alpha^3 + 13\alpha^2 - 8\alpha + 2} \right\}, \quad x^* = x_m^m \text{ for } Q \leq (\bar{U}_a - \bar{U}_b) \leq R \text{ as } Q \geq UM \text{ and if } \text{Max} \left\{ \frac{1}{2}, \frac{-(8\alpha^3 - 9\alpha^2 + 3\alpha) + \alpha\sqrt{2\alpha-1}}{-8\alpha^3 + 13\alpha^2 - 8\alpha + 2} \right\} \leq k < 1, \quad x^* = x_b^m \text{ for } Q \leq (\bar{U}_a - \bar{U}_b) \leq UM \text{ and } x^* = x_m^m \text{ for } UM \leq (\bar{U}_a - \bar{U}_b) \leq R \text{ as } UM \geq Q.$$

Note that $UM > Q$ for all $\alpha < \frac{1}{2}$. We denote the contest and profit of Π_b^m and Π_m^m as ΠB and ΠM .

4.3 Second-Best Contests

To solve for the profit-maximizing contests, we combine separating contests and mixed contests. Four possible situations are examined. See Table 2. Other cases are excluded because $US > Q$ or because both $UM > Q$ and $R > US$ cannot be satisfied.

Case 1

We must first define the feasible range of α and k which satisfy the inequalities $UM < Q < R < US$. We know the region for which $Q > UM$ and $US > R$. This is drawn in Figure 10. We next consider which kind of contest results in a larger profit in the feasible range of the parameter space. It is not difficult to find the range of $(\bar{U}_a - \bar{U}_b)$ where $\Pi_L \geq \Pi_M$ and $\Pi_M \geq \Pi_L$. $\Pi_L \geq \Pi_M$ for $(\bar{U}_a - \bar{U}_b) \leq A$ and $\Pi_M \geq \Pi_L$ for $(\bar{U}_a - \bar{U}_b) \geq A$, where $A = (\alpha - \frac{1}{2})(\frac{1}{k_a} - \frac{1}{k_b}) \left(\frac{(1-\alpha\sqrt{2\alpha-1})-(1-\alpha)k\sqrt{2\alpha-1}}{(1-\alpha)(2\alpha-(2\alpha-1)k)} \right)^2$. As before, it is important to compare this A with Q and R . It can be shown that $A > Q$ and $R > A$ for all relevant α , $\frac{1}{2} < \alpha < 1$ (Appendix C3). Hence, for the parameter values in the region I, the second-best contests are, Π_L for $Q \leq (\bar{U}_a - \bar{U}_b) \leq A$ and Π_M for $A \leq (\bar{U}_a - \bar{U}_b) \leq R$.

Table 2
Four Cases In Ordering of Q,R,UM,US

(Case 1)

(Case 2)

(Case 3)

(Case 4)

Case 2

The feasible region is derived as in Case 1. The inequalities to be satisfied are $UM < Q < US < R$. Figure 11 shows the feasible region for this case. $IIS \geq IIM$ when $(\bar{U}_a - \bar{U}_b) \leq B$ and $IIM \geq IIS$ when $(\bar{U}_a - \bar{U}_b) \geq B$

$$\text{where } B = \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \frac{((1-\alpha)k + \alpha)}{(2(1-\alpha)k + (2\alpha-1))^2} \{1 - 2\sqrt{(1-\alpha)(1-k)((1-\alpha)k + \alpha)}\}$$

(Appendix C4). We must compare B with US and R. Actual calculations show that $US > B$ for the relevant region. Hence only IIM is relevant in the region between US and R. We consider whether $A > US$ or $US > A$ to determine the relevancy of IIM in the region between Q and US. Refer to Table 2. It is impossible to solve for when $A > US$ in closed form. Computer analysis shows that $US > A$ in region II. Hence, the following results are realized: IIL for $Q \leq (\bar{U}_a - \bar{U}_b) \leq A$ and IIM for $A \leq (\bar{U}_a - \bar{U}_b) \leq R$.

Case 3

In this case, another constraint needs to be introduced into the analysis, namely $US = UM$. See Table 2. As in the other instances, it is not possible to have an explicit solution of the equation, $US = UM$. We draw the feasible range in the parameter space for this case, region III. See Figure 12. $IIL \geq IIB$ when $(\bar{U}_a - \bar{U}_b) \leq C$ and $IIB \geq IIL$ when $(\bar{U}_a - \bar{U}_b) \geq C$

$\geq C$ where $C = \frac{(\alpha - \frac{1}{2})}{k_a k_b} \left(\frac{1}{(2\alpha - (2\alpha - 1)k)^2} \right) \{ (2 - \alpha^2) - 2\alpha(1 - \alpha)k - (1 - \alpha)^2 k^2 + 2[(1 - \alpha^2) - 2\alpha(1 - \alpha)k - (1 - \alpha)^2 k^2]^{\frac{1}{2}} \}$ (Appendix C5). Again we compare c to Q and UM . It turns out that $C > UM > Q$ for all relevant values of α . Hence, $\Pi_L > \Pi_B$ for $Q \leq (\bar{U}_a - \bar{U}_b) \leq UM$. Next we compare A with UM in the feasible range of region III. The computer work suggests that $A = UM$ occurs somewhere in the middle of region III. For the portion of the region where $A > UM$, which we will call region III-a, $\Pi_L \geq \Pi_M$ for $UM \leq (\bar{U}_a - \bar{U}_b) \leq A$ and $\Pi_M \geq \Pi_L$ for $A \leq (\bar{U}_a - \bar{U}_b) \leq US$. For the region of $UM > A$, region III-b, $\Pi_L \geq \Pi_M$ for $UM \leq (\bar{U}_a - \bar{U}_b) \leq US$. Also note that $\Pi_M \geq \Pi_S$ for $US \leq (\bar{U}_a - \bar{U}_b) \leq R$ as in case 2.

Case 4

We first note that $\Pi_S \geq \Pi_B$ when $(\bar{U}_a - \bar{U}_b) \leq D$ and $\Pi_B \geq \Pi_S$ when $(\bar{U}_a - \bar{U}_b) \geq D$, where $D = \frac{\alpha}{2} \frac{k_b - k_a}{k_a k_b} \frac{(1 - \alpha)k + \alpha}{(2\alpha - (2\alpha - 1)k)}$ (Appendix C6). It turns out that the points which satisfy $US = D$ lie in region IV. But a simple calculation shows that $UM > D$ for the feasible range of α and k . Note also, $C > US$ as $C > UM > US$ and $UM > B$ as $UM > US > B$ for all relevant α . Refer to Table 2. Recall that C is the dividing value between Π_L and Π_B and B is the dividing value between Π_S and Π_M . See Figure 13. For the region where $D < US$, which we shall call region IV-a, the solution is Π_L for $Q \leq (\bar{U}_a - \bar{U}_b) \leq US$, Π_B for $US \leq (\bar{U}_a - \bar{U}_b) \leq UM$ and Π_M for $UM \leq (\bar{U}_a - \bar{U}_b) \leq R$. In region IV-b where $D > US$, the solution is Π_L

for $Q \leq (\bar{U}_a - \bar{U}_b) \leq US$, IIS for $US \leq (\bar{U}_a - \bar{U}_b) \leq D$, IIB for $D \leq (\bar{U}_a - \bar{U}_b) \leq UM$ and IIM for $UM \leq (\bar{U}_a - \bar{U}_b) \leq R$.

Four types of solutions emerge depending on the parameter values of α and k . In the region I, II and III-a, one pattern appears. In the other regions, each solution is different. It is noteworthy that in all situations IIL and IIM are a part of the solution. When α is relatively large and there is small difference between Q and R , only IIL and IIM are possible. As α becomes smaller, there is room for IIB and IIS. Also the trend is that when $(\bar{U}_a - \bar{U}_b)$ is closer to Q , the separating contests (IIL or IIS) are optimal. This observation enhances the credibility of contests as a sorting compensation (price) mechanism. Although there might not be a large range of $(\bar{U}_a - \bar{U}_b)$ which provides an efficient sorting, if we include the second-best solution, separating contests can be significant. When $(\bar{U}_a - \bar{U}_b)$ is closer to R , mixed contests are optimal. When the reservation utilities are not very different, it pays to separate workers rather than to pool them. This is clear intuitively. While the productivity difference is $\frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b})$, if the reservation utilities imply a wage difference that is much smaller than $\frac{1}{2}(\frac{1}{k_a} - \frac{1}{k_b})$, then it is more profitable to maintain separating contests even at some cost.

Proposition 7. Assuming a uniform distribution on $g(\cdot)$, quadratic disutility functions for a- and b-type workers and an interior solution for the contests, the second-best contests are:

$$\text{for } \alpha \leq \frac{1}{2}, 0 \leq (\bar{U}_a - \bar{U}_b) \leq UM \quad \Pi B,$$

$$UM \leq (\bar{U}_a - \bar{U}_b) \leq R \quad \Pi M;$$

$$\text{for } \alpha > \frac{1}{2}, \text{ In Region A } Q \leq (\bar{U}_a - \bar{U}_b) \leq A \quad \Pi L,$$

$$A \leq (\bar{U}_a - \bar{U}_b) \leq R \quad \Pi M;$$

$$\text{In Region B } Q \leq (\bar{U}_a - \bar{U}_b) \leq US \quad \Pi L,$$

$$US \leq (\bar{U}_a - \bar{U}_b) \leq R \quad \Pi M;$$

$$\text{In Region C } Q \leq (\bar{U}_a - \bar{U}_b) \leq US \quad \Pi L,$$

$$US \leq (\bar{U}_a - \bar{U}_b) \leq UM \quad \Pi B,$$

$$UM \leq (\bar{U}_a - \bar{U}_b) \leq R \quad \Pi M;$$

$$\text{In Region D } Q \leq (\bar{U}_a - \bar{U}_b) \leq US \quad \Pi L,$$

$$US \leq (\bar{U}_a - \bar{U}_b) \leq D \quad \Pi S,$$

$$D \leq (\bar{U}_a - \bar{U}_b) \leq UM \quad \Pi B,$$

$$UM \leq (\bar{U}_a - \bar{U}_b) \leq R \quad \Pi M.$$

CHAPTER V

CONCLUDING REMARKS

This paper considers the sorting property of rank-order contests as a compensation scheme in the presence of a heterogeneous population where it is costly to identify the type of each individual agent. A simple model is developed where the prevailing reservation utilities for each type of worker is known and the productivities of each type of worker vary with the firms. Depending on the differences in reservation utilities and the incentives to work among the types of workers, one type of worker is most preferred by the firm. It is shown that there can be an efficient sorting if the agents are rewarded according to the rank-order of the competitors. This occurs because both the probability of winning and the level of prizes determine the expected return in contests.

To separate, for a less-able type worker, the effect of the reduced probability of winning should dominate the effect of the increased level of prizes, and for a more-able type worker, the effect of lower prize level should dominate the effect of the improved probability of winning. If the less-able type is near to the chosen type and the more-able type is apart from the chosen one in ability,

self selection seems more likely. Also we note that when the firm chooses a worker type that fits its needs particularly well, this tends to reduce the probability of winning by other types of worker.

We identify the Lazear-Rosen case of adverse selection as a special case in our model and show that their proof requires rather stringent assumptions.

In establishing the result, a diagrammatic approach is taken. This diagrammatic approach enables us to extend the model of Lazear and Rosen. In turn, this extended framework is essential to establish our result.

To the extent that sorting workers improves productivity due to complementarities in production or some other reason, the efficiency comparison in the homogeneous case between contests and a linear piece rate needs to be adjusted in case of contests in a heterogeneous group of workers, since such contests can sort out workers.

In the literature on sorting among heterogeneous agents whose identities are unknown, there has been much emphasis on the non-existence of efficient separation (Akerlof, 1976; Riley, 1975; Rothschild and Stiglitz, 1976; and Spence, 1974). However, in the comparison between contests and a linear piece rate compensation scheme, the emphasis should be given to the existence of efficient separation. Even if it does not always occur, separation is often possible in contests, but there is no separation in a linear piece rate system.

One interesting point on mixed (or pooling) contests is that, unlike the result found in the sorting literature, there can exist a pooling equilibrium even though it is inefficient. This occurs because the alternative to a mixed contest is limited to a homogeneous contest of the favored type of worker. In certain situations, one cannot take away a certain type of worker from the mixed contests and still expect a positive profit since it is difficult to offer a set of prizes that satisfies the desired type of workers without inviting the other type of workers as well.

We also investigate the second-best contests when there is adverse selection. The proportion (α) and similarity in incentive to work (k) of the types of workers in the population and the difference in reservation utilities ($\bar{U}_a - \bar{U}_b$) determine the optimal contests. The firm wants to separate the different types of workers when the work-incentive difference in the firm is much larger than what the reservation utility difference implies. The firm chooses to pool different types of workers when the work-incentive difference in the firm is not much larger than what the reservation utility difference implies. Since separating contests are always a part of the second-best solution, the investigation enhances the usefulness of contests as a sorting price mechanism.

ENDNOTES

1. Malcomson finds the following phenomena consistent with his model: (1) hierarchical wage structures, (2) wage rates that rise with seniority and experience more than productivity, (3) the variance of earnings increasing with experience and (4) wage rates attached to jobs rather than individuals, with differentials set by administrative procedures rather than by reference to external market wages.

2. An interior solution requires $EU \geq Y_1 - C(0)$. If workers choose to set their efforts at zero and collect the bottom prize, then the contests cannot be sustained.

3. For example, see Carmichael pp 54-5 for the justification of assuming Nash behavior in a tournament situation. The most serious criticism of Nash behavior is that it requires that players have knowledge of the other players' strategies. One justification for the assumption is to understand the full-information Nash equilibrium as the outcome of an adjustment process. Since each player can observe output and his own effort, the value of the marginal product of one's own effort would become available in an ongoing working situation. When successive observations of output leave each agent no reason to believe that his marginal product has changed, this will yield a Nash equilibrium since knowledge of one's marginal product essentially gives the worker knowledge of the other player's effort levels.

4. It is conceivable that both workers have no incentive to change their effort levels even though the effort levels are different under a particular $g(\cdot)$. Hence, there may still be an asymmetric equilibrium. An asymmetric equilibrium is possible, but it is arbitrary and less plausible.

5. In Guasch and Weiss (1980, 1982), there are multiple equilibria since the effort level does not depend on the wage difference. If $Y_1 - Y_2 = X$ is an equilibrium, then $Y_1 - Y_2 = 2X$ is also an equilibrium as long as the mean of the prize is the same, since the effort level is the same whether $Y_1 - Y_2$ is X or $2X$.

6. This is a well known result in the literature. For completeness, we include it here.

7. One can think of the production function of firm i as $y_i = k_i \cdot e_i + \nu_i$, $k_i > 0$. Without loss of generality, we assume that the production function of the firm considered in the paper is $y = e + \nu$, i.e., $k=1$.

8. At this point in the analysis, we do not need the existence of mixed contests. Mixed contests are used to do some mental experiments to determine which contest each worker participates in.

9. When the firm wants to have contests of a certain type, workers from the unintended group have two alternatives. They can either join contests designed for the other type of worker or not join the contests and settle with their reservation utility. The question is what proportion of the unintended type of worker participate in the contests of the firm. This is important because the degree of participation of the unintended type affects the expected utility of the unintended type of worker in the contests through the probability of winning. If some portion of the unintended type find it utility-maximizing to participate in the contests, other workers of the unintended type will follow. It is also possible that when a portion of the unintended type participate in the offered contests, workers of the unintended type find it attractive. However, when every worker tries for the offered contests, the offered contests become unattractive for the unintended type of workers.

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APPENDIX A

Figure 1

Unique Relationship between X and Effort Level

Figure 2

Existence and Uniqueness of Equilibrium Prize Structure in Homogeneous Contests

Figure 3

Adverse Selection When the Firm Offers Two Separate Homogeneous Contests

Figure 4

(4A)
Solutions in Mixed Contests

Figure 4

(4B)

Solutions In Mixed Contests

Figure 4

(4C)

Solutions In Mixed Contests

Figure 5

Lazear-Rosen Separating Contests ($\frac{1}{2} < \alpha < 1$)

Figure 6

The Case of No Separation ($0 < \alpha < \frac{1}{2}$)

Figure 7

Miyazaki Separating Contests ($\frac{1}{2} < \alpha < 1$)

Figure 8

$$R = US$$

Figure 9

$$Q = UM$$

Figure 10

Region I

Figure 11

Region II

Figure 12

Region III

Figure 13

Region IV

Figure 14
Region A,B,C,D

APPENDIX B

If the utility function is not risk-neutral, $EU = Pr \cdot U(Y_w) + (1 - Pr) \cdot U(Y_l) - C(e)$. Since the probability of winning is $\frac{1}{2}$ in

equilibrium,

$$EU = \frac{1}{2} [U(Y_w) + U(Y_l)] - C(e). \quad (1)$$

The first-order condition is

$$g(0) \cdot [U(Y_w) - U(Y_l)] = C'(e). \quad (2)$$

To find $\frac{dZ}{dX}$, we change (1) to

$$U(Y_w) + U(Y_l) = 2(\bar{U} + C(e)) \quad (3)$$

Differentiate each side of (3) with respect to X ,

$$U'(Y_w) \cdot \frac{\partial Y_w}{\partial X} + U'(Y_l) \cdot \frac{\partial Y_l}{\partial X} = 2 \cdot C'(e) \cdot \frac{\partial e}{\partial X}. \quad (4)$$

Since $Y_w = Y_l + (Y_w - Y_l) = Z + X$, (4) can be arranged as

$$U'(Y_w) \cdot \left(\frac{dZ}{dX} + 1 \right) + U'(Y_l) \cdot \frac{dZ}{dX} = 2 \cdot [g(0) \cdot (U(Y_w) - U(Y_l))] \cdot \frac{\partial e}{\partial X}. \quad \text{We solve for}$$

$$\frac{dZ}{dX} \cdot \frac{dZ}{dX} \Big|_{\bar{U}} = \frac{-U'(Y_w)}{U'(Y_w) + U'(Y_l)} + \frac{2 \cdot g(0) [U(Y_w) - U(Y_l)]}{U'(Y_w) + U'(Y_l)} \cdot \frac{\partial e}{\partial X}. \quad \text{To have } \frac{dZ}{dX} \Big|_{\bar{U}} =$$

$$\frac{dZ}{dX} \Big|_{II} \cdot \frac{-U'(Y_w)}{U'(Y_w) + U'(Y_l)} + \frac{2 \cdot g(0) [U(Y_w) - U(Y_l)]}{U'(Y_w) + U'(Y_l)} \cdot \frac{\partial e}{\partial X} = -\frac{1}{2} + \frac{\partial e}{\partial X}. \quad \text{Then } \frac{\partial e}{\partial X} =$$

$$\frac{U'(Y_w) - U'(Y_l)}{2[U'(Y_w) + U'(Y_l)]} \cdot [2g(0) \cdot \frac{(U(Y_w) - U(Y_l))}{U'(Y_w) + U'(Y_l)} - 1]^{-1} \text{ at the profit-maximizing}$$

level of X .

To be efficient X has to be chosen such that $U(Y_w) - U(Y_l) = \frac{1}{g(0)}$.

$\frac{\partial e}{\partial X}$ at efficient level of X is $\frac{U'(Y_w) - U'(Y_l)}{2[2 - U'(Y_w) - U'(Y_l)]}$. Hence, it is clear

that $\frac{\partial e}{\partial X}$ at profit-maximizing level and $\frac{\partial e}{\partial X}$ at efficient level are not

same generally which, in turn, means different X for profit maximization and efficiency.

APPENDIX C

Appendix C1

$$\Pi_L - \Pi_S = \frac{2k_b}{(2\alpha-1)(k_b-k_a)} (\bar{U}_a - \bar{U}_b) - \frac{2}{k_a} \left(\left(\frac{2}{2\alpha-1} \right) \left(\frac{k_a k_b}{k_b - k_a} \right) (\bar{U}_a - \bar{U}_b) \right)^{\frac{1}{2}} + \frac{\alpha}{k_a} + \frac{1-\alpha}{k_b}.$$

$$\Pi_S \geq \Pi_L, \text{ if } (\bar{U}_a - \bar{U}_b) \geq \frac{(2\alpha-1)}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (2-\alpha-(1-\alpha)k + 2((1-\alpha)(1-k))^{\frac{1}{2}}) \quad (US)$$

$$\text{or } (\bar{U}_a - \bar{U}_b) \leq \frac{(2\alpha-1)}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (2-\alpha-(1-\alpha)k - 2((1-\alpha)(1-k))^{\frac{1}{2}}) \quad (US'). \quad \text{But}$$

$$\text{there is the constraint that } Q = \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \leq (\bar{U}_a - \bar{U}_b) \leq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right). \quad \text{But } US' < Q \text{ and } US > Q.$$

$$US - R = \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (-(3-2\alpha)(1-\alpha) - (2\alpha-1)(1-\alpha)k + 2(2\alpha-1)[(1-\alpha)(1-k)]^{\frac{1}{2}}).$$

Let $-(3-2\alpha)(1-\alpha) - (2\alpha-1)(1-\alpha)k = \text{left}$ and $2(2\alpha-1)[(1-\alpha)(1-k)]^{\frac{1}{2}} = \text{right}$. Square each term and solve for k . Then,

$$\text{right} \geq \text{left} \text{ if } \frac{1}{2} < k \leq \frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1 - \alpha)(2\alpha - 1)}, \text{ and left} \geq \text{right} \text{ if } k \geq \frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1 - \alpha)(2\alpha - 1)}.$$

$$\frac{-(2\alpha^2 - \alpha + 1) + 2\sqrt{2\alpha - 1}}{(1 - \alpha)(2\alpha - 1)} \geq \frac{1}{2} \text{ if approximately, } \alpha > \frac{13}{16}.$$

Appendix C2

$UM - Q = \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left(\frac{1}{2} \left(\frac{(1-\alpha)k+\alpha}{2\alpha-(2\alpha-1)k} \right)^2 - \left(\alpha - \frac{1}{2} \right) \right)$. Solve this for k by putting
 to 0. The equation we solve is $(-8\alpha^3 + 13\alpha^2 - 8\alpha + 2)k^2 + 2(8\alpha^3 - 9\alpha^2 + 3\alpha)k + \alpha^2(-8\alpha + 5) = 0$.

Appendix C3

$$\begin{aligned} \Pi_L - \Pi_M &= 2(\bar{U}_a - \bar{U}_b)^{\frac{1}{2}} \left\{ \frac{-2(1-\alpha) [2\alpha(1-k)+k]}{(2\alpha-1)(1-k)} (\bar{U}_a - \bar{U}_b)^{\frac{1}{2}} + \left[\frac{2}{(2\alpha-1)k_a(1-k)} \right]^{\frac{1}{2}} \right. \\ &\quad \left. [1 - ((1-\alpha)k + \alpha)(2\alpha-1)^{\frac{1}{2}}] \right\}. \quad \Pi_L \geq \Pi_M, \text{ if } 0 \leq (\bar{U}_a - \bar{U}_b) \leq \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \\ &\quad \left(\frac{(1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1}}{(1-\alpha)(2\alpha-(2\alpha-1)k)} \right)^2 (= A) \text{ and } \Pi_M \geq \Pi_L, \text{ if } \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \\ &\quad \left(\frac{(1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1}}{(1-\alpha)(2\alpha-(2\alpha-1)k)} \right)^2 \leq (\bar{U}_a - \bar{U}_b) \leq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right). \end{aligned}$$

Proof of A > Q

$$\begin{aligned} A - Q &= \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left\{ \frac{(1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1}}{(1-\alpha)(2\alpha-(2\alpha-1)k)} - 1 \right\}^2. \text{ But } (1- \\ &\quad \alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1} - (1-\alpha)(2\alpha-(2\alpha-1)k) > 0 \text{ because } (1-\alpha\sqrt{2\alpha-1}) - \\ &\quad 2\alpha(1-\alpha) > (1-\alpha)(\sqrt{2\alpha-1} - (2\alpha-1)). \text{ Hence } A > Q. \end{aligned}$$

Proof of A < R

$$\begin{aligned} R - A &= \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) (1-(2\alpha-1)) \left\{ \frac{(1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1}}{(1-\alpha)(2\alpha-(2\alpha-1)k)} \right\}^2. \text{ The} \\ &\quad \text{numerator of the large right bracket can be arranged to } (1-\alpha)^2(2\alpha-(2\alpha-1)k)^2 \\ &\quad - (2\alpha-1) [(1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)k\sqrt{2\alpha-1}]^2. \text{ This is in turn, } [(1-\alpha)(2\alpha-(2\alpha-1)k)]^2 \\ &\quad - [\sqrt{2\alpha-1} (1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)(2\alpha-1)k]^2. \text{ But } [(1-\alpha)(2\alpha-(2\alpha-1)k)] \\ &\quad - [\sqrt{2\alpha-1} (1-\alpha\sqrt{2\alpha-1}) - (1-\alpha)(2\alpha-1)k] > 0 \text{ for } \frac{1}{2} < \alpha < 1. \end{aligned}$$

Appendix C4

$$\begin{aligned}
\Pi_S - \Pi_M &= \frac{2[(2-2\alpha)k + (2\alpha-1)]}{(1-k)} (\bar{U}_a - \bar{U}_b) - 2[(1-\alpha)k + \alpha] \left[\frac{2(\bar{U}_a - \bar{U}_b)}{k_a(1-k)} \right]^{\frac{1}{2}} + \frac{\alpha}{k_a} \\
&+ \frac{1-\alpha}{k_b}. \quad \Pi_S \geq \Pi_M \quad \text{if} \quad (\bar{U}_a - \bar{U}_b) \leq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \frac{((1-\alpha)k + \alpha)}{(2(1-\alpha)k + (2\alpha-1))^2} \{1 - \\
&2\sqrt{(1-\alpha)(1-k)((1-\alpha)k + \alpha)}\} \quad (=B) \quad \text{or} \quad (\bar{U}_a - \bar{U}_b) \geq \frac{1}{2} \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \frac{((1-\alpha)k + \alpha)}{(2(1-\alpha)k + (2\alpha-1))^2} \\
&\{1 + 2\sqrt{(1-\alpha)(1-k)((1-\alpha)k + \alpha)}\} \quad (=B'). \quad \text{But } B' - R \geq 0. \quad B' - R = \frac{1}{2} \left(\frac{1}{k_a} - \right. \\
&\left. \frac{1}{k_b} \right) \frac{((1-\alpha)k + \alpha)}{(2(1-\alpha)k + (2\alpha-1))^2} \{1 + 2\sqrt{(1-\alpha)(1-k)((1-\alpha)k + \alpha)}\} - 1. \quad \text{Since} \\
&\sqrt{(1-\alpha)(1-k)((1-\alpha)k + \alpha)} \geq (1-\alpha)(1-k) \text{ as } k > \frac{1}{2}, \text{ the numerator in the large} \\
&\text{bracket is bigger than } ((1-\alpha)k + \alpha)(1 + 2(1-\alpha)(1-k)) - (2(1-\alpha)k + (2\alpha-1))^2 \\
&= -6(1-\alpha)^2 k^2 - (1-\alpha)(12\alpha-7)k - (2\alpha^2 - 7\alpha + 1) > 0 \text{ for } \frac{1}{2} < k < 1. \quad \text{Hence } B' \\
&> R.
\end{aligned}$$

Appendix C5

$$\Pi_L - \Pi_B = \frac{2}{(2\alpha-1)(1-k)} \bar{U}_b + 2\bar{U}_b - \frac{2(2\alpha-(2\alpha-1)k)}{(2\alpha-1)(1-k)} \bar{U}_a + 2 \left(\frac{2}{2\alpha-1} \frac{1}{1-k} \frac{1}{k_a} (\bar{U}_a - \bar{U}_b) \right)^2 - \frac{((1-\alpha)k + \alpha)^2}{k_a(2\alpha-(2\alpha-1)k)}.$$

$$\Pi_L \geq \Pi_B \text{ for } \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left(\frac{1}{(2\alpha-(2\alpha-1)k)^2} \right) \{ (2-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2 - 2[(1-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2]^{\frac{1}{2}} \} (=C') \leq (\bar{U}_a - \bar{U}_b) \leq \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left(\frac{1}{(2\alpha-(2\alpha-1)k)^2} \right) \{ (2-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2 + 2[(1-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2]^{\frac{1}{2}} \} (=C).$$

$$\text{Claim } C' < Q. \quad Q - C' = \left(\alpha - \frac{1}{2} \right) \left(\frac{1}{k_a} - \frac{1}{k_b} \right) \left[1 - \left(\frac{1}{(2\alpha-(2\alpha-1)k)^2} \right) \{ (2-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2 - 2[(1-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2]^{\frac{1}{2}} \} \right].$$

The numerator of the bracket [] can be arranged as $(5\alpha^2 - 6\alpha + 2)k^2 -$

$$2\alpha(5\alpha-3)k + (5\alpha^2-2) + 2[(1-\alpha^2) - 2\alpha(1-\alpha)k - (1-\alpha)^2 k^2]^{\frac{1}{2}}.$$

It is hard to solve explicitly. Computer work shows this numerator is positive for the feasible range of parameter space. Hence $Q > C'$. One can see that

$Q > C'$ implies that $C > Q$ from the equation form of the numerator above.

Appendix C6

$$\begin{aligned} \Pi_S - \Pi_B &= \frac{\alpha}{2} \frac{k_b - k_a}{k_a k_b} \frac{(1-\alpha)k + \alpha}{(2\alpha - (2\alpha-1)k)} - 2(\bar{U}_a - \bar{U}_b). \quad \text{Hence } \Pi_S \geq \Pi_B \text{ if } (\bar{U}_a - \bar{U}_b) \\ &\leq \frac{\alpha}{2} \frac{k_b - k_a}{k_a k_b} \frac{(1-\alpha)k + \alpha}{(2\alpha - (2\alpha-1)k)} \quad (=D). \end{aligned}$$