THE TWISTED TENSOR L-FUNCTION OF GSP(4)

DISSERTATION

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By

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ABSTRACT

We construct an integral representation for the “twisted tensor” L -function of a globally generic cuspidal automorphic representation of GSp_4 over a number field. We prove that the integral is Eulerian, i.e., has an infinite product expansion. We compute the unramified integrals and show by way of a branching result (from GL_4 to Sp_4) that these integrals calculate the correct local L -factor. This gives a new proof of the analogous identity in D. Jiang’s thesis. Finally, we show all the local integrals are absolutely convergent in a right half-plane and that they are nonvanishing for appropriate choice of data. We close with some remarks about poles of our global integral and possible future applications to period integrals and quadratic base change for GSp_4 .

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CHAPTER 1

INTRODUCTION

The L -function of an automorphic representation, though defined in terms of essentially local data, carries a great deal of global information about the representation. For example, the presence of a pole of some order at $s = 1$ may indicate that the representation in question is a functorial lift from an automorphic representation on another group. Several such cases have been established; see [25] for a recent example. It is our purpose here to lay the groundwork for a new case, that of the twisted tensor L -function for GSp_4 as it relates to detecting quadratic base change lifts. We do this by constructing an integral representation for this L -function.

Given an integral representation of an L -function, one obtains analytic information about the L -function in the following way. The L -function depends on a complex variable s , and this dependence enters the integral by way of some function in the integrand, usually an Eisenstein series. The integral will usually be convergent in a right half-plane (i.e. for $\mathrm{Re}(s)$ sufficiently large), and then the analytic properties of the Eisenstein series propagates to the L -function.

Though our construction is one obvious analogue to D. Jiang's thesis [24], its origins (as well as those of Jiang's work) lie in the theory of integral representations of L -functions for $\mathrm{GL}_n \times \mathrm{GL}_n$. We will briefly recall this and several other related

constructions. For the theory on $\mathrm{GL}_n \times \mathrm{GL}_n$, the author greatly benefited from the exposition in [6], where he first learned this material.

The theory of integral representations for $\mathrm{GL}_n \times \mathrm{GL}_m$ was developed by H. Jacquet, I. Piatetski-Shapiro and J. Shalika in a series of papers ([17, 18, 23, 22, 19, 21]; see also [7]); here we consider the case $m = n$. Let π and π' be smooth cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, where $\mathbb{A}$ is the adèle ring of a number field F . Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ be cusp forms. Then there is an Eisenstein series $E_n(g, s)$ on $\mathrm{GL}_n(\mathbb{A})$, given explicitly in terms of a theta series, such that the integral

$$I(s, \varphi, \varphi', E) = \int_{Z_n(\mathbb{A})\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)E_n(g, s) dg \quad (*)$$

represents the product (or Rankin-Selberg) L -function $L(s, \pi \times \pi')$. (See [6, Lec. 9], [7].)

Jiang's construction enters the picture through an “accidental isomorphism” and the theory of seesaw dual reductive pairs [30]. The original motivation for his work comes from questions of functoriality and theta liftings; see the introduction to [24] for a discussion of this. The integral representation arises as follows. Consider the integral $(*)$ in the case $n = 4$. Since $\mathrm{PGL}_4 \simeq \mathrm{GO}_{3,3}$, this can be thought of as an integral of cusp forms and Eisenstein series on $\mathrm{GO}_{3,3}$:

$$\int_{Z_{3,3}(\mathbb{A})\mathrm{GO}_{3,3}(F)\backslash\mathrm{GO}_{3,3}(\mathbb{A})} \phi(g)\phi'(g)E_{3,3}(g, s) dg.$$

Now this integral can be interpreted as half of a seesaw identity (in the sense of [30]),

where the dual reductive pairs in question are given by the diagram

$$\begin{array}{ccc}
\mathrm{GO}_{3,3}(F) \times \mathrm{GO}_{3,3}(F) & & \mathrm{GSp}_8(F) \\
\uparrow & \swarrow \text{dotted} & \uparrow \\
\mathrm{GO}_{3,3}(F) & \tilde{G}_0(F) = (\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ &
\end{array}$$

where the left-hand inclusion is the diagonal embedding and the right-hand inclusion is the doubling embedding (see [35] and Section 2.1 of this paper), and

$$(\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ = \{(g_1, g_2) \mid g_1, g_2 \in \mathrm{GSp}_4(F), \lambda(g_1) = \lambda(g_2)\},$$

where λ is the similitude character. The seesaw identity then reads

$$\begin{aligned}
\int_{Z_{3,3}(\mathbb{A}) \mathrm{GO}_{3,3}(F) \backslash \mathrm{GO}_{3,3}(\mathbb{A})} \phi(g) \phi'(g) E_{3,3}(g, s) dg \\
= \int_{Z(\mathbb{A}) \tilde{G}_0(F) \backslash \tilde{G}_0(\mathbb{A})} \tilde{\phi}(g_1) \tilde{\phi}'(g_2) \tilde{E}_8((g_1, g_2), s) dg,
\end{aligned}$$

where $\tilde{\phi}$ and $\tilde{\phi}'$ are the theta lifts of ϕ and ϕ' , respectively, from $\mathrm{GO}_{3,3}(\mathbb{A})$ to $\mathrm{GSp}_4(\mathbb{A})$, and $\tilde{E}_8$ is the theta lift of the Eisenstein series $E_{3,3}$ from $\mathrm{GO}_{3,3}(\mathbb{A})$ to $\mathrm{GSp}_8(\mathbb{A})$, given by the regularized theta integral of S. Kudla and S. Rallis [31]. Using the appropriate choice for $\tilde{E}_8$, Jiang shows this integral represents the Rankin-Selberg L -function for the product of two (generic) cuspidal automorphic representations of GSp_4 . These relations can be loosely summarized by the diagram:

$$\begin{array}{ccccc}
\mathrm{GL}_4(F) \times \mathrm{GL}_4(F) & \xrightarrow{\sim} & \mathrm{GO}_{3,3}(F) \times \mathrm{GO}_{3,3}(F) & & \mathrm{GSp}_8(F) \\
\uparrow \text{Jacquet, P-S,} & & \uparrow & \swarrow \text{dotted} & \uparrow \text{Jiang} \\
\mathrm{GL}_4(F) & \xrightarrow{\sim} & \mathrm{GO}_{3,3}(F) & & \tilde{G}_0(F)
\end{array} \tag{1.1}$$

One natural generalization of these construction is to consider $G(F) \times G(F)$ as $G(F \oplus F)$, and ask whether the construction can be repeated when the quadratic algebra

$F \oplus F$ is replaced by a quadratic extension E of F . The answer for $G = \mathrm{GL}_n$ is affirmative and was given by Y. Flicker in [8]. His integral is analogous to (*):

$$I(s, \varphi, E_n) = \int_{Z_n(\mathbb{A}_F) \mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F)} \varphi(g) E_n(g, s) dg, \quad (**)$$

where φ is now a cuspidal automorphic form on $\mathrm{GL}_n(\mathbb{A}_E)$ and E_n is the same Eisenstein series as in (*). Flicker shows that the integral represents the “twisted tensor” L -function (in some contexts, also called the Asai L -function) for the cuspidal representation containing φ .

Can the seesaw construction be repeated in this context to give another integral representation, this time on $\mathrm{GSp}_4(E)$? The point of this dissertation is to answer in the affirmative. The situation for E/F a quadratic extension can be represented by the diagram

$$\begin{array}{ccccc} \mathrm{GL}_4(E) & \xrightarrow{\sim} & \mathrm{GO}_{3,3}(E) & & \mathrm{GSp}_8(F) \\ \uparrow \text{Flicker} & & \uparrow & \swarrow & \uparrow \text{This work} \\ \mathrm{GL}_4(F) & \xrightarrow{\sim} & \mathrm{GO}_{3,3}(F) & \searrow & G_0(F) = \mathrm{GSp}_4(E)^\circ \end{array} \quad (1.2)$$

where $\mathrm{GSp}_4(E)^\circ = \{g \in \mathrm{GSp}_4(E) \mid \lambda(g) \in F^\times\}$.

Now we will briefly describe the contents of this work. In the second chapter, after giving the notation and basic setup for our integral, we carry out an orbit decomposition and establish a technical result on cusp forms to prove that the integral is Eulerian. The third chapter is the calculation of the unramified integrals and identification of these with the correct local L -factor. The method of this identification is new: Jiang uses the Kostant-Rallis algorithm in [24], but here we use a branching rule from GL_4 to Sp_4 , which gives a new proof of Jiang’s identity for places v of F

which are split in E . Finally, the fourth chapter carries out some analyses of the local integrals; we show these are absolutely convergent in a right half-plane and that they are nonvanishing for appropriate choice of data. Then we close with some remarks about poles of our global integral and possible future applications.

CHAPTER 2

GLOBAL INTEGRAL

2.1 Preliminaries

Let us first give the main notations of the work.

Let E/F be a quadratic extension of number fields generated by an algebraic integer θ with $\theta^2 \in F$, and let $\mathbb{A} \stackrel{\text{def}}{=} \mathbb{A}_F$ be the adèle ring of F . Our construction involves the groups of symplectic similitudes,

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid gJ_{2n} {}^t g = \lambda J_{2n}\},$$

where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $\lambda = \lambda(g)$ is the similitude character. We get embeddings between these groups by considering a “twisted” variation of the usual doubling method of Piatetski-Shapiro and Rallis [35].

Let V be a $2n$ -dimensional vector space over E with nondegenerate symplectic form $\langle \cdot, \cdot \rangle$, and let $W = \mathrm{Res}_{E/F} V$ with form $\langle\langle x, y \rangle\rangle = \frac{1}{2} \mathrm{Tr}_{E/F} \langle x, y \rangle$, which is also symplectic. Denote the similitude groups of these forms by $\mathrm{GSp}(V)$ and $\mathrm{GSp}(W)$, respectively. Notice that if $g \in \mathrm{GSp}(V)$ has similitude $\lambda(g) \in F$, then for $v, w \in V$,

$$\begin{aligned} \langle\langle vg, wg \rangle\rangle &= \mathrm{Tr}_{E/F} \langle vg, wg \rangle = \mathrm{Tr}_{E/F} (\lambda(g) \langle v, w \rangle) \\ &= \lambda(g) \langle\langle v, w \rangle\rangle, \end{aligned}$$

so we get an embedding $\mathrm{GSp}(V)^\circ \hookrightarrow \mathrm{GSp}(W)$, where $^\circ$ denotes the subgroup of elements whose similitudes lie in F . After fixing a basis for V and W this becomes $\mathrm{GSp}_{2n}(E)^\circ \hookrightarrow \mathrm{GSp}_{4n}(F)$.

We consider the case $n = 2$. Namely, V is 4-dimensional over E , with nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ and standard basis $\{e_1, e_2, e'_1, e'_2\}$, so that $\langle e_i, e'_j \rangle = \delta_{ij}$ and $\langle e_i, e_j \rangle = \langle e'_i, e'_j \rangle = 0$. Let $W = \mathrm{Res}_{E/F} V$ with form $\langle\langle x, y \rangle\rangle = \frac{1}{2} \mathrm{Tr}_{E/F} \langle x, y \rangle$. Denote

$$G_0(F) \stackrel{\mathrm{def}}{=} \mathrm{GSp}_4(E)^\circ = \{g \in \mathrm{GSp}_4(E) \mid \lambda(g) \in F\}. \quad (2.1)$$

(Alternatively, G_0 can be described as a fibre product:

$$G_0 = \mathrm{Res}_{E/F} \mathrm{GSp}_4 \times_{\mathrm{Res}_{E/F} \mathbb{G}_m} \mathbb{G}_m \simeq (\mathrm{Res}_{E/F} \mathrm{Sp}_4) \rtimes \mathbb{G}_m,$$

where $\mathrm{GSp}_4 = \mathrm{Sp}_4 \rtimes \mathbb{G}_m$.) For later reference, we remark that G_0 is quasisplit over F and denote the center of G_0 by Z_0 . Z will denote the center of GSp_4 . Our embedding is $\iota: G_0(F) \hookrightarrow \mathrm{GSp}_8(F)$. Explicitly, fix the basis $\{e_1, \theta e_1, e_2, \theta e_2, e'_1, \theta^{-1} e'_1, e'_2, \theta^{-1} e'_2\}$ for W . Then ι is given by

$$\iota: (\alpha_{ij}) \mapsto \begin{pmatrix} 1 & & & \\ & T_\theta^{-1} & & \\ & & m(\alpha_{ij}) & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & T_\theta & & \\ & & & 1 \end{pmatrix}, \quad (2.2)$$

where

$$\alpha = a + a'\theta \mapsto m(\alpha) = \begin{pmatrix} a & a' \\ a'\theta^2 & a \end{pmatrix}$$

is the usual embedding $E \hookrightarrow \mathrm{GL}_2(F)$, and

$$T_\theta = \mathrm{diag}(1, \theta^2, 1, \theta^2).$$

For convenience, we will identify an element of $G_0(F)$ with its image in $\mathrm{GSp}_8(F)$.

In the classical doubling method [35], one takes cusp forms φ_1 and φ_2 on $\mathrm{Sp}_{2n}(\mathbb{A})$ and integrates against a Siegel-type Eisenstein series on $\mathrm{Sp}_{4n}(\mathbb{A})$ over the diagonally embedded copy of $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{Sp}_{2n}(\mathbb{A})$. (For the similitude case, one can only embed the subgroup

$$(\mathrm{GSp}_{2n}(\mathbb{A}) \times \mathrm{GSp}_{2n}(\mathbb{A}))^\circ = \{(g_1, g_2) | g_1, g_2 \in \mathrm{GSp}_{2n}(\mathbb{A}), \lambda(g_1) = \lambda(g_2)\}$$

into $\mathrm{GSp}_{4n}(\mathbb{A})$.) In our setup, we will take a cusp form φ on $\mathrm{GSp}_4(\mathbb{A}_E)$ and a non-Siegel Eisenstein series on $\mathrm{GSp}_8(\mathbb{A}_F)$. (See the introduction for an explanation of this.)

Namely, $P_8 \subseteq \mathrm{GSp}_8$ is the maximal parabolic subgroup stabilizing the isotropic 3-plane spanned by $\{e'_1, \theta^{-1}e'_1, e'_2\}$:

$$P_8 = \left\{ g = \begin{pmatrix} A & & & \\ & x_1 & & x_2 \\ & & A' & \\ & x_3 & & x_4 \end{pmatrix} \begin{pmatrix} I_3 & u & w & v \\ & 1 & t_v & \\ & & I_3 & \\ & & & -t_u & 1 \end{pmatrix} \middle| \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}_2; A \in \mathrm{GL}_3; \right. \\ \left. A' = (x_1x_4 - x_2x_3)^t A^{-1}; u, v \in \mathrm{Mat}_{3,1}; w \in \mathrm{Mat}_{3,3}; w + v^t u \in \mathrm{Sym}_3 \right\} \quad (2.3)$$

The modulus character of P_8 at an element g as above is

$$\delta_{P_8}(g) = (\det A)^6 \lambda(g)^{-9},$$

and $\lambda(g) = x_1x_4 - x_2x_3$. (By convention we write $\delta_{P_8}(g)^s = |(\det A)^6 \lambda(g)^{-9}|^s$ for complex s .) Form the normalized induced representation

$$\mathrm{Ind}_{P_8(\mathbb{A})}^{\mathrm{GSp}_8(\mathbb{A})}(\delta_{P_8}^{s/6}) = \{f: \mathrm{GSp}_8(\mathbb{A}) \rightarrow \mathbb{C} \text{ smooth} \mid f(pg) = \delta_{P_8}(p)^{\frac{1}{2}} \delta_{P_8}(p)^{\frac{s}{6}} f(g)\},$$

and let $f_s \in \text{Ind}_{P_8(\mathbb{A})}^{\text{GSp}_8(\mathbb{A})}(\delta_{P_8}^{s/6})$ be a flat section, i.e., the restriction of f_s to the maximal compact subgroup of $\text{GSp}_8(\mathbb{A})$ is independent of s . Construct the Eisenstein series

$$E(g, s, f_s) = \sum_{\gamma \in P_8(F) \backslash \text{GSp}_8(F)} f_s(\gamma g).$$

This is identical to the Eisenstein series used by Jiang [24, II.2.2]. We must “normalize” this Eisenstein series in the following way. The constant term of the Eisenstein series is a sum of intertwining operators between various induced representations. Each of these operators is defined for $\text{Re}(s) \gg 0$ and has a “denominator” in the sense that, after multiplying the operator by a certain (computable) product of zeta functions (the “denominator”), the operator has meromorphic continuation to the whole plane with a finite number of poles. The normalizing factor of the Eisenstein series is defined to be the LCM of these “denominators.” Jiang computes the normalizing factor for $E(g, s, f_s)$ in [24], chapter III, Theorem 4.0.2 and preceding discussion. We only use the case $n = 4$.

Lemma 2.1.1 ([24]). $d_G(s) = \prod_{v \notin S} \zeta_v(2s + 2) \zeta_v(s + 1) \zeta_v(s + 2) \zeta_v(s + 3)$ is the normalizing factor for the Eisenstein series $E(g, s, f_s)$, where S is the set of “bad” primes of F , to be defined later.

We define the normalized Eisenstein series $E^*(g, s, f_s)$ by

$$E^*(g, s, f_s) = d_G(s) E(g, s, f_s). \tag{2.4}$$

Proposition 2.1.2 ([33, 1]). *The series $E^*(g, s, f_s)$ converges absolutely for $\text{Re}(s)$ sufficiently large, has meromorphic continuation to the complex plane with finitely many poles, and satisfies a functional equation relating its values at s and $-s$.*

Let π be a smooth irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_E)$ which is globally generic and has central character $\omega_\pi \equiv 1$, and let V_π be a realization of π in the space of cusp forms on $\mathrm{GSp}_4(\mathbb{A}_E)$. By the embedding ι , we may consider the restriction of $E(g, s, f_s)$ to G_0 . We will examine the integral

$$I(s, \varphi, f_s) = \int_{Z(\mathbb{A})G_0(F) \backslash G_0(\mathbb{A})} \varphi(g) E(g, s, f_s) dg. \quad (2.5)$$

Since $\omega_\pi = 1$, this is well-defined. It is absolutely convergent by the moderate growth of Eisenstein series and the rapid decay of cusp forms (modulo the center). In order to understand this integral, we need a few auxiliary results.

Remark 2.1.3. We don't use the full center Z_0 of G_0 in the integral, but it is easy to check that

$$Z(\mathbb{A})Z_0(F) \backslash Z_0(\mathbb{A}) \simeq \mathbb{A}_F^\times E_0^\times \backslash \mathbb{A}_{E,0}^\times$$

with $E_0 = \{\alpha \in E \mid \alpha^2 \in F\}$. Now $\mathbb{A}_F^\times \simeq \mathbb{A}_F^1 \times \mathbb{R}_+^\times$ and similarly for E , and the natural embedding $\mathbb{A}_F^\times \hookrightarrow \mathbb{A}_E^\times$ is the map

$$\begin{aligned} \mathbb{A}_F^1 \times \mathbb{R}_+^\times &\hookrightarrow \mathbb{A}_E^1 \times \mathbb{R}_+^\times, \\ ((\alpha_v)_{v \in \Sigma_F}, r) &\mapsto ((\alpha_w)_{w \in \Sigma_E}, r^2), \end{aligned}$$

where as usual $\alpha_w = \alpha_v$ whenever w lies over v . So $\mathbb{A}_F^\times E_0^\times \backslash \mathbb{A}_{E,0}^\times \simeq \mathbb{A}_F^1 E_0^\times \backslash \mathbb{A}_{E,0}^1$, and $E_0^\times \backslash \mathbb{A}_{E,0}^1 \hookrightarrow E^\times \backslash \mathbb{A}_E^1$, which is compact.

2.2 Roots, Weights and Subgroups

We will consider various subgroups of $\mathrm{GSp}_4(E)$ when analyzing our global integral. To clarify their structure, we start with root data.

Lemma 2.2.1. *Let $h = h(t_1, \dots, t_{2n})$ denote the diagonal matrix in GSp_{2n} with given entries from upper-left to lower-right, so that $t_{n+i} = \lambda t_i^{-1}$ for $1 \leq i \leq n$. Let T_{2n} be the subgroup of all such diagonal matrices, which is a maximal (split) torus in GSp_{2n} . Let B_{2n} be the Borel subgroup of GSp_{2n} with Levi factor T_{2n} and unipotent radical equal to*

$$N^{B_{2n}} = \left\{ \begin{pmatrix} A & \\ & \lambda {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} \middle| A \in \mathrm{GL}_n \text{ upper triangular unipotent, } B \in \mathrm{Sym}_n \right\},$$

where “unipotent” denotes matrices with all diagonal entries equal to 1. Define characters ϵ_i of T_{2n} by

$$\epsilon_i(h) = h^{\epsilon_i} \stackrel{\mathrm{def}}{=} t_i, \quad 1 \leq i \leq 2n.$$

The based root system of $(\mathrm{GSp}_{2n}, B_{2n}, T_{2n})$ is $(X^*(T_4), \Phi, \Delta)$, where

$$X^*(T_{2n}) = \bigoplus_{i=1}^{2n} \mathbb{Z} \epsilon_i / \langle \epsilon_i + \epsilon_{n+i} - \epsilon_{i+1} - \epsilon_{n+i+1} : 1 \leq i < n \rangle,$$

$$\Phi = \{ \pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i - \epsilon_{j+n}) : 1 \leq i \leq j \leq n \} - \{0\} \quad (\text{roots})$$

$$\Phi^+ = \{ \epsilon_i - \epsilon_j, \epsilon_i - \epsilon_{j+n} : 1 \leq i \leq j \leq n \} - \{0\} \quad (\text{positive roots})$$

$$\Delta = \{ \alpha_1, \dots, \alpha_n \}, \quad (\text{basis})$$

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad 1 \leq i \leq n-1,$$

$$\alpha_n = \epsilon_n - \epsilon_{2n}.$$

Let $P_4 \subseteq \mathrm{GSp}_4$ be the Klingen parabolic subgroup

$$P_4 = \left\{ g = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \in \mathrm{GSp}_4 \right\}. \quad (2.6)$$

It has Levi subgroup isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$. We will consider two “ F -forms” $Q(F), Q'(F)$ of $P_4(E)$, given as subgroups of $\mathrm{GSp}_4(E)$ by

$$Q(F) = \left\{ \begin{pmatrix} e & x & w\theta & s'\theta \\ & a & s\theta & b\theta \\ & & e' & \\ c\theta & x' & & d \end{pmatrix} \middle| a, \dots, x' \in F, ad - bc\theta^2 = ee' = \lambda(g) \right\} \quad (2.7)$$

and

$$Q'(F) = \left\{ \begin{pmatrix} e & x(1+\theta) & w(1+\theta) & s' \\ & a & s & b(1-\theta) \\ & & e' & \\ c(1+\theta) & x'(1+\theta) & & d \end{pmatrix} \middle| a, \dots, x' \in F, \right. \\ \left. ad - bc(1 - \theta^2) = ee' = \lambda(g) \right\}; \quad (2.8)$$

these are intersections of (maximal) parabolic subgroups of $\mathrm{GSp}_8(F)$ with $G_0(F)$. Q and Q' should be thought of as analogues of the subgroups

$$P_4^\Delta = \{(g, g) : g \in P_4\} \subseteq \mathrm{GSp}_4 \times \mathrm{GSp}_4,$$

$$P_4^1 = \{(g, 1) : g \in P_4\} \subseteq \mathrm{GSp}_4 \times \mathrm{GSp}_4$$

under the usual doubling map $(\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ \hookrightarrow \mathrm{GSp}_8(F)$. They are “complementary” in the sense that if $Y(E)$ is a root subgroup of $P_4(E)$, then $(Y(E) \cap Q(F)) \cdot (Y(E) \cap Q'(F)) = Y(E)$. Finally, define $\gamma_0 \in \mathrm{GSp}_8(F)$ by

$$\gamma_0 = \begin{pmatrix} 1 & & & & 0 & & & \\ & 0 & 1 & & 0 & 0 & 0 & \\ & & 0 & & 1 & 0 & 1 & \\ 0 & & & & & 1 & 0 & 0 \\ & 0 & & 0 & & 0 & 1 & \\ & & 0 & -1 & & 0 & & 0 \\ -1 & & & & & & & 0 \end{pmatrix}; \quad (2.9)$$

then γ_0 represents an element of the Weyl group of $\mathrm{GSp}_8(F)$ of order 3.

For parabolic subgroups $P = P_8, P_4, Q, Q'$ above, denote their respective Levi subgroups by M^P , their *maximal* unipotent subgroups by N^P and their unipotent radicals by U^P . (So, for example, $N^{P_4} = N^{B_4} = U^{B_4}$ in the notation of Lemma 2.2.1.)

Similarly the opposite unipotent subgroups will be $\overline{N}^P, \overline{U}^P$. The unipotent radicals U^P for $P = P_4, Q, Q'$ are Heisenberg groups, and we denote their centers by X_2^P ; in fact, if X_2 denotes the root subgroup of GSp_4 corresponding to the highest root $2\alpha_1 + \alpha_2$, then $X_2^P = X_2 \cap P$. Moreover, X_2^P is normalized by P :

$$\begin{pmatrix} e & & & \\ & a & b & \\ & & 1 & \\ & c & d & \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} e & & & \\ & a & b & \\ & & 1 & \\ & c & d & \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ex & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where $e = ad - bc$. One can check directly that

$$Q(F)X_2^{Q'}(F) = G_0(F) \cap \gamma_0^{-1}P_8(F)\gamma_0, \quad (2.10)$$

which we will need for the proof of Prop. 2.3.1. For convenience we often denote by $e(p)$ and $e'(p)$ the $(1, 1)$ - and $(3, 3)$ -entries of an element of P_4, Q or Q' , so that $e(p)e'(p) = \lambda(p)$. We will also frequently make use (without mention) of the fact that for $P = P_4, Q, Q'$, we can decompose M^P as a commuting product of subgroups

$$M^P = Z \cdot \widetilde{M}^P,$$

where

$$\widetilde{M}^P = \{m \in M^P \mid e'(m) = 1\}. \quad (2.11)$$

Lastly we note two facts which will be needed for the unramified calculation.

Proposition 2.2.2 ([2], 1.2.2(5)). *The complex dual group of GSp_4 is $\mathrm{GSp}_4(\mathbb{C})$.*

Proposition 2.2.3. *Every finite-dimensional representation of $\mathrm{GSp}_4(\mathbb{C})$ is of the form $\lambda^s \otimes \tau_{(m,n)}$, where λ is the similitude factor, $s \in \mathbb{C}$, and $\tau_{(m,n)}$ is the (finite-dimensional) representation of $\mathrm{Sp}_4(\mathbb{C})$ of highest weight $(m,n) = m\omega_1 + n(\omega_1 + \omega_2)$.*

2.3 Orbit Decomposition

We need to know the orbits of $G_0(F)$ on $X = P_8(F) \setminus \mathrm{GSp}_8(F)$. We may think of the latter as the variety of isotropic F -subspaces L which are 3-dimensional in $W = \mathrm{Res}_{E/F} V$. For such an isotropic F -subspace L , we can consider the number of independent E -lines contained in L . Equivalently, we may determine the dimension of $E \cdot L$, the E -subspace of V generated by L . Let $d(L) = \dim_E(E \cdot L)$.

Proposition 2.3.1. *$d(L)$ is a complete invariant for $G_0(F)$ -orbits in X , and there are two such orbits, corresponding to $d(L) = 2$ and $d(L) = 3$. The former is negligible in the sense of [35] and the latter has stabilizer $Q(F)X_2^{Q'}(F)$ (up to conjugacy).*

Proof. Let $X^{(i)}$ be the collection of $L \in X$ with $d(L) = i$, so that X is a union of the sets $X^{(i)}$, $i = 1, 2, 3$. Since E/F is quadratic, any $L \in X^{(1)}$ would have F -dimension 2, so $X^{(1)}$ is empty.

Set $L_1 = F\theta e_2 + Fe'_1 + Fe'_2 \in X^{(3)}$. For $L' \in X^{(3)}$, we may write $L' = Fv_1 + Fv_2 + Fv_3$, where $\langle v_1, v_2 \rangle = \theta$ and $\langle v_3, L' \rangle = 0$. To see this, take any basis $\{v_1, v_2, v_3\}$ of L' . Then $\langle\langle v_i, v_j \rangle\rangle = 0$ for $1 \leq i, j \leq 3$. But $\dim_E V = 4$, so $E \cdot L'$ cannot be isotropic. Hence $\langle v_i, v_j \rangle \neq 0$ for some i, j ; we may assume $i = 1, j = 2$. Since $\langle\langle v_1, v_2 \rangle\rangle = 0$, $\langle v_1, v_2 \rangle \in F^\times \cdot \theta$. By scaling v_2 , we can assume $\langle v_1, v_2 \rangle = \theta$. Similarly, $\langle v_1, v_3 \rangle = a\theta$

and $\langle v_2, v_3 \rangle = b\theta$, for $a, b \in F$. Replacing v_3 by $v_3 - av_2 + bv_1$ now gives what we want.

Now define a linear map $g_1 : L' \rightarrow L_1$ by

$$v_1 g_1 = \theta e_2,$$

$$v_2 g_1 = e'_2,$$

$$v_3 g_1 = e'_1.$$

By construction, $\langle v_i g_1, v_j g_1 \rangle = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq 3$. Now the Witt extension theorem shows g_1 extends to $g' \in \text{Sp}_4(E) \subseteq G_0$. Thus G_0 is transitive on $X^{(3)}$.

A similar argument applies to $X^{(2)}$. Let $L_0 = Fe'_1 + F\theta^{-1}e'_1 + Fe'_2 \in X^{(2)}$. If $L'' \in X^{(2)}$, there is a $0 \neq v \in L''$ such that $\theta v \in L''$ also. Set $v_1 = v$, $v_2 = \theta^{-1}v$ and extend to a basis $\{v_1, v_2, v_3\}$ of L'' over F . It is easy to see that $E \cdot L''$ is isotropic. If not, then $\langle v, v_3 \rangle \neq 0$. Since $\langle\langle v, v_3 \rangle\rangle = 0$, we can assume (after scaling v_3) that $\langle v, v_3 \rangle = \langle v_1, v_3 \rangle = \theta$. Then $\langle v_2, v_3 \rangle = \langle \theta^{-1}v, v_3 \rangle = 1$, so $\langle\langle v_2, v_3 \rangle\rangle = 1$, a contradiction.

Let $g_0 : L'' \rightarrow L_0$ be the linear map defined by

$$v_1 g_0 = e'_1,$$

$$v_2 g_0 = \theta^{-1}e'_1,$$

$$v_3 g_0 = e'_2.$$

Then as before g_0 extends to a $g'' \in \text{Sp}_4(E) \subseteq G_0$, so G_0 is transitive on $X^{(2)}$.

Finally, it is clear that $X^{(2)}$ and $X^{(3)}$ are disjoint, since any $g \in G_0(F)$ is invertible and commutes with multiplication by elements of E . Moreover, as was noted above

$L' \in X^{(3)}$ implies that $E \cdot L$ is not (totally) isotropic, while $L'' \in X^{(2)}$ implies $E \cdot L''$ is isotropic. Thus the orbits are exactly $X^{(2)}$ and $X^{(3)}$.

Now $L_0 \in X^{(2)}$ has stabilizer $G_0(F) \cap P_8(F)$, which is (isomorphic to) the subgroup

$$\left\{ \begin{pmatrix} * & * & * & * \\ & a_{22} & * & * \\ & & * & \\ & & * & d_{22} \end{pmatrix} : a_{22}, d_{22} \in F \right\} \subseteq \mathrm{GSp}_4(E)$$

which has unipotent radical $U^{B_4}(E)$, where B_4 is the Borel subgroup of GSp_4 . Hence $X^{(2)}$ is negligible.

For $X^{(3)}$, we can write $L_1 = L_0 \gamma_0$, where γ_0 is given by (2.9). Then the stabilizer of $L_1 \in X^{(3)}$ is $G_0 \cap \gamma_0^{-1} P_8(F) \gamma_0 = Q(F) X_2^{Q'}(F)$ as in (2.10). $\square$

2.4 A Lemma on Cusp Forms

For the unfolding of our integral, we need a way to introduce a “Whittaker expansion” for a cusp form φ on $\mathrm{GSp}_4(\mathbb{A}_E)$. We will follow Jiang’s lead [24, sec. 2.2.1].

For our purposes, let $\psi_\circ$ be a fixed, non-degenerate additive character of $\mathbb{A}$, trivial on F , and $\psi: N^{B_4}(\mathbb{A}_E) \rightarrow \mathbb{C}^\times$ be given by

$$\psi \begin{pmatrix} 1 & x & * & * \\ & 1 & * & y \\ & & 1 & \\ & & -x & 1 \end{pmatrix} = \psi_\circ \left(\frac{1}{2} \mathrm{Tr}_{E/F}(\theta x + y) \right),$$

so that $\psi|_{N^Q(\mathbb{A})} \equiv 1$. ψ is generic in the sense that it is not identically zero on either simple root subgroup of GSp_4 .

Recall that a cuspidal automorphic representation (π, V_π) of $\mathrm{GSp}_4(\mathbb{A}_E)$ is said to be ψ -generic (or simply *generic*) if there is a nonzero linear functional $\Lambda: V_\pi \rightarrow \mathbb{C}$

satisfying $\Lambda(\pi(n)\varphi) = \psi(n)\Lambda(\varphi)$ for $n \in N^{B_4}(\mathbb{A}_E)$ and $\varphi \in V_\pi$. Note that if π is ψ -generic, it will be ψ' -generic for any other generic character ψ' of $N^{B_4}(\mathbb{A}_E)$; see [26, pp. 381-2]. For $\varphi \in V_\pi$, the *Whittaker function* associated to φ and ψ is defined to be

$$W_{\varphi, \psi}(g) = \int_{N^{B_4}(E) \backslash N^{B_4}(\mathbb{A}_E)} \varphi(ug) \psi^{-1}(u) du.$$

π is said to be *globally ψ -generic* if $W_{\varphi, \psi} \neq 0$ for some $\varphi \in V_\pi$. As above, such π will be globally ψ' -generic for every generic ψ' , so we simply say π is *globally generic*.

Consider the parabolic subgroup P_4 . It has as a normal subgroup the root subgroup $X_2 = X_2^{P_4}$, which is the center of N^{P_4} . Taking the quotient by this and the center Z of GSp_4 , we have a natural projection onto the (upper triangular) mirabolic subgroup P'_3 in GL_3 :

$$\begin{aligned} \rho: P_4(\mathbb{A}_E) &\rightarrow P'_3(\mathbb{A}_E), \\ z \begin{pmatrix} e & & & \\ & a & b & \\ & & 1 & \\ & c & d & \end{pmatrix} \begin{pmatrix} 1 & x & w & s \\ & 1 & s & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} &\mapsto \begin{pmatrix} a & b\theta & \\ c\theta & d & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & s\theta & \\ & 1 & -x & \\ & & 1 \end{pmatrix}. \end{aligned} \quad (2.12)$$

Note that we have decomposed the Levi of P_4 as Z times a subgroup isomorphic to GL_2 and as above, $e = ad - bc$.

Lemma 2.4.1. (i). ρ is surjective with kernel $X_2 \cdot Z$, where Z is the center of GSp_4 .

(ii). $\rho^{-1}(P'_3(F)) = Z(E)X_2(E) \cdot Q(F)$ and $\rho^{-1}(N_3(F)) = Z(E)X_2(E) \cdot N^Q(F)$.

Proof. Follows directly from the definitions of ρ , Q and N^Q . □

Following Jiang, for a cusp form φ of π we introduce the function

$$\varphi^*(g) = \int_{X_2(E) \backslash X_2(\mathbb{A}_E)} \varphi(xg) \, dx. \quad (2.13)$$

Remark 2.4.2. (i). The idea to use such an integration comes from [11] and is also found in [10].

(ii). If π is generic and $\varphi^* \equiv 0$ for all $\varphi \in V_\pi$, π is called “hypercuspidal” by Gelbart and Piatetski-Shapiro. By (2.16) and (2.21) below, we have

$$\varphi^*(g) = \sum_{\tilde{\alpha} \in Z(E)N^{P_4}(E) \backslash P_4(E)} W_{\varphi, \psi}(\tilde{\alpha}g)$$

so if $\varphi^* \equiv 0$ for all $\varphi \in V_\pi$, then $W_{\varphi, \psi} \equiv 0$ for all φ and ψ , which implies π is not globally generic. So a “hypercuspidal” representation in this context is one which is generic but not globally generic. It is not known whether such representations exist (cf. [26], p. 383).

When φ^* is restricted to $P_4(\mathbb{A}_E)$, it has a natural push-forward under ρ , namely

$$\begin{cases} \phi: P'_3(\mathbb{A}_E) \rightarrow \mathbb{C} \\ \phi(r) = \varphi^*(\rho^{-1}(r)g) \end{cases} \quad (2.14)$$

for $g \in \mathrm{GSp}_4(\mathbb{A}_E)$. Since π has trivial central character and by the previous lemma, this is a well-defined function on $P'_3(\mathbb{A}_E)$ which is clearly left-invariant under $P'_3(E)$ and is in fact a cuspidal automorphic form on $P'_3(\mathbb{A}_E) \subseteq \mathrm{GL}_3(\mathbb{A}_E)$. We will see below that ϕ is generic if and only if φ is. We give an expansion for ϕ in terms of its Whittaker functions, following Shalika [38]; see [36]. The argument is due to Jacquet-Langlands [15] for GL_2 and independently to Piatetski-Shapiro ([37]; see [36,

p. 210]) and Shalika for GL_n , $n \geq 3$. First, define a character ψ' on the maximal unipotent subgroup $N_3(\mathbb{A}_E) \subseteq \mathrm{GL}_3(\mathbb{A}_E)$ by

$$\psi' \begin{pmatrix} 1 & y & z \\ & 1 & x \\ & & 1 \end{pmatrix} = \psi \circ \left(\frac{1}{2} \mathrm{Tr}_{E/F}(\theta(x+y)) \right), \quad (2.15)$$

so that for $n \in N^{P_4}(\mathbb{A}_E)$ we have $\psi(n) = \psi'(\rho(n))$, and $\psi'|_{N_3(\mathbb{A}_F)} = \psi|_{N^Q(\mathbb{A}_F)} \equiv 1$.

Then the Whittaker expansion for ϕ is

$$\phi(r) = \sum_{\alpha \in N_3(E) \setminus P'_3(E)} W_{\phi, \psi'}(\alpha r), \quad (2.16)$$

where

$$W_{\phi, \psi'}(r) = \int_{N_3(E) \setminus N_3(\mathbb{A}_E)} \phi(nr) \psi'^{-1}(n) \, dn. \quad (2.17)$$

We want to modify this expansion to include only the F -rational terms.

Lemma 2.4.3. *Let ϕ be a cusp form on $\mathrm{GL}_n(\mathbb{A}_E)$ or on the mirabolic subgroup $P'_n(\mathbb{A}_E)$ and χ a nondegenerate character on the maximal unipotent subgroup $N_n(\mathbb{A}_E)$ of $\mathrm{GL}_n(\mathbb{A}_E)$ which is trivial on $N_n(E)$ and $N_n(\mathbb{A}_F)$. Then*

$$\int_{N_n(F) \setminus N_n(\mathbb{A}_F)} \phi(ng) \, dn = \sum_{\alpha \in N_{n-1}(F) \setminus \mathrm{GL}_{n-1}(F)} W_{\phi, \chi} \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \right).$$

Remark 2.4.4. This result is also used by Flicker [8] to unfold his global integral into an Euler product.

Proof of Lemma 2.4.3. This is the argument given in [38] with a minor modification; we give it here for completeness. The case $n = 1$ is trivial, so assume $n \geq 2$. Let

P'_n be the mirabolic (upper triangular) subgroup of GL_n and let V_n be its unipotent radical,

$$P'_n = \left\{ \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\} \quad V_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \vdots \\ & & 1 & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\}.$$

Define, for $x \in \mathrm{GL}_{n-1}(\mathbb{A}_E)$,

$$F_g(x) = \int_{V_n(E) \backslash V_n(\mathbb{A}_E)} \phi \left(v \begin{pmatrix} x & \\ & 1 \end{pmatrix} g \right) \chi^{-1}(v) dv$$

so that for $\alpha \in \mathrm{GL}_{n-1}(E)$

$$\begin{aligned} F_g(\alpha) &= \int_{V_n(E) \backslash V_n(\mathbb{A}_E)} \phi \left(v \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \right) \chi^{-1}(v) dv \\ &= \int_{V_n(E) \backslash V_n(\mathbb{A}_E)} \phi(vg) \chi^{-1} \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} v \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}^{-1} \right) dv \\ &= \int_{V_n(E) \backslash V_n(\mathbb{A}_E)} \phi(vg) \chi_\alpha^{-1}(v) dv, \end{aligned}$$

so by basic Fourier analysis we get

$$\phi(vg) = \sum_{\alpha \in P'_{n-1}(E) \backslash \mathrm{GL}_{n-1}(E)} F_g(\alpha) \chi_\alpha(v).$$

Now take $v \in V_n(\mathbb{A}_F)$ and integrate:

$$\int_{V_n(F) \backslash V_n(\mathbb{A}_F)} \phi(vg) dv = \sum_{\alpha \in P'_{n-1}(E) \backslash \mathrm{GL}_{n-1}(E)} F_g(\alpha) \int_{V_n(F) \backslash V_n(\mathbb{A}_F)} \chi_\alpha(v) dv. \quad (2.18)$$

Now the integral on the right is nonzero if and only if $\chi_\alpha|_{\mathbb{A}_F} \equiv 1$. But by definition of χ_α , this holds if and only if the last row of α has all entries in F .

Now look at the Bruhat-type decomposition in GL_m ,

$$\mathrm{GL}_m = \coprod_{w \in W_{P'_m} \setminus W_m} P'_m w B_m,$$

where B_m is the upper triangular Borel subgroup of GL_m , W_m is the Weyl group of GL_m and $W_{P'_m}$ is the Weyl group for P'_m . The quotient $W_{P'_m} \setminus W_m$ has order m and is represented by elements $w_j, 1 \leq j \leq m$, where left multiplication by w_j exchanges the j -th and m -th rows. Let $\mathcal{O}_j(E) = P'_m(E) \setminus P'_m(E)w_j B_m(E)$, so we can choose as a representative of $\alpha \in \mathcal{O}_j(E)$ an element of the form

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & t \end{pmatrix} w_j u,$$

for $t \in E^\times$ and $u \in N_m(E)$, the unipotent radical of $B_m(E)$. But it is easy to see that we can decompose $u = u' u''$, where $u' \in N_m(E) \cap w_j^{-1} P'_m(E) w_j$ and u'' has the

form

$$u'' = \begin{pmatrix} I_j & B \\ & I_{m-j} \end{pmatrix},$$

I_j = the $j \times j$ identity matrix,

$$B = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ b_1 & \dots & b_{m-j} \end{pmatrix} \in \text{Mat}_{j \times (m-j)}(E).$$

So $\alpha \in \mathcal{O}_j(E)$ has a representative with entries 1 and 0 above the last row, and last row entries $(0, \dots, 0, t, tb_1, \dots, tb_{m-j})$. But the α -summand in the right-hand side of (2.18) is nonzero only if the last row of α is in F . So we may assume that α itself is in $\mathcal{O}_j(F)$. That is, the sum in (2.18) may be taken over $\alpha \in P'_{n-1}(F) \setminus \text{GL}_{n-1}(F)$. Continuing inductively now gives the result. $\square$

Applying the lemma to our cusp form ϕ gives

$$\int_{N_3(F) \backslash N_3(\mathbb{A}_F)} \phi(nr) \, dn = \sum_{\alpha \in N_3(F) \backslash P'_3(F)} W_{\phi, \psi'}(\alpha r), \quad (2.19)$$

where we have used the standard fact $N_{n-1}(F) \setminus \text{GL}_{n-1}(F) = N_n(F) \setminus P'_n(F)$. Using the definition of ϕ from (2.14) and taking $r = 1$, the integral becomes

$$\begin{aligned} \int_{N_3(F) \backslash N_3(\mathbb{A}_F)} \phi(n) \, dn &= \int_{N_3(F) \backslash N_3(\mathbb{A}_F)} \varphi^*(\rho^{-1}(n)g) \, dn \\ &= \int_{N^Q(F)X_2^Q(\mathbb{A}_F) \backslash N^Q(\mathbb{A}_F)} \varphi^*(ug) \, du, \end{aligned} \quad (2.20)$$

whereas the Whittaker function $W_{\phi, \psi'}$ is just

$$\begin{aligned}
W_{\phi, \psi'}(\alpha) &= \int_{N_3(E) \backslash N_3(\mathbb{A}_E)} \phi(n\alpha) \psi'^{-1}(n) \, dn \\
&= \int_{N_3(E) \backslash N_3(\mathbb{A}_E)} \int_{X_2(E) \backslash X_2(\mathbb{A}_E)} \varphi(x\rho^{-1}(n\alpha)g) \psi'^{-1}(n) \, dx \, dn \\
&= \int_{N^{P_4}(E) \backslash N^{P_4}(\mathbb{A}_E)} \varphi(u\tilde{\alpha}g) \psi^{-1}(u) \, du = W_{\varphi, \psi}(\tilde{\alpha}g),
\end{aligned} \tag{2.21}$$

where $\tilde{\alpha}$ is any preimage of α under ρ . Using these, (2.19) becomes

$$\int_{N^Q(F)X_2^Q(\mathbb{A}_F) \backslash N^Q(\mathbb{A}_F)} \varphi^*(ug) \, du = \sum_{\tilde{\alpha} \in Z(F)N^Q(F) \backslash Q(F)} W_{\varphi, \psi}(\tilde{\alpha}g). \tag{2.22}$$

2.5 The Global Integral

Proposition 2.5.1 (Main identity).

$$I(s, \varphi, f_s) = \int_{\mathcal{D}(\mathbb{A})} f_s(\gamma_0 g) W_{\varphi, \psi}(g) \, dg \tag{2.23}$$

where $I(s, \varphi, f_s)$ is given by (2.5) and $\mathcal{D}(\mathbb{A}) = Z(\mathbb{A})X_2^{Q'}(\mathbb{A})N^Q(\mathbb{A}) \backslash G_0(\mathbb{A})$.

Proof.

$$\begin{aligned}
I(s, \varphi, f_s) &= \int_{Z(\mathbb{A})G_0(F) \backslash G_0(\mathbb{A})} \varphi(g) E(g, s, f_s) \, dg \\
&= \int_{Z(\mathbb{A})G_0(F) \backslash G_0(\mathbb{A})} \varphi(g) \sum_{\gamma \in P_8(F) \backslash \text{GSp}_8(F)} f_s(\gamma g) \, dg \\
&= \sum_{\gamma \in P_8(F) \backslash \text{GSp}_8(F)/G_0(F)} \int_{Z(\mathbb{A})G_0^\gamma(F) \backslash G_0(\mathbb{A})} \varphi(g) f_s(\gamma g) \, dg.
\end{aligned} \tag{2.24}$$

Since we have “unfolded” the Eisenstein series, each integral on the right is now convergent for $\text{Re}(s)$ sufficiently large. By Prop. 2.3.1, the sum is over two orbits,

and one is negligible. The non-negligible orbit is represented by γ_0 as in (2.9), and the stabilizer of this orbit is $G_0^{\gamma_0}(F) = Q(F)X_2^Q(F)$ by (2.10). Then the integral is just

$$I(s, \varphi, f_s) = \int_{Z(\mathbb{A})Q(F)X_2^{Q'}(F)\backslash G_0(\mathbb{A})} \varphi(g) f_s(\gamma_0 g) dg. \quad (2.25)$$

Now we follow Jiang's lead in projecting our cusp form onto the mirabolic in GL_3 :

$$\begin{aligned} I(s, \varphi, f_s) &= \int_{Z(\mathbb{A})Q(F)X_2(\mathbb{A}_E)\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) \int_{X_2(E)\backslash X_2(\mathbb{A}_E)} \varphi(xg) dx dg \\ &= \int_{Z(\mathbb{A})X_2(\mathbb{A}_E)Q(F)\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) \varphi^*(g) dg \\ &= \int_{Z(\mathbb{A})X_2(\mathbb{A}_E)N^Q(\mathbb{A})Q(F)\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) \int_{N^Q(F)X_2^Q(\mathbb{A})\backslash N^Q(\mathbb{A})} \varphi^*(ug) du dg \\ &= \int_{Z(\mathbb{A})X_2(\mathbb{A}_E)N^Q(\mathbb{A})Q(F)\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) \sum_{\alpha \in Z(F)N^Q(F)\backslash Q(F)} W_{\varphi, \psi}(\alpha g) dg \quad (\text{by (2.22)}) \\ &= \int_{Z(\mathbb{A})X_2(\mathbb{A}_E)N^Q(\mathbb{A})\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) W_{\varphi, \psi}(g) dg, \end{aligned}$$

since $\gamma_0 \alpha \gamma_0^{-1} \in P_8(F)$. Finally, since $X_2(\mathbb{A}_E) = X_2^Q(\mathbb{A})X_2^{Q'}(\mathbb{A})$ and $X_2^Q(\mathbb{A}) \subseteq N^Q(\mathbb{A})$, this last integral is equal to

$$\int_{Z(\mathbb{A})X_2^{Q'}(\mathbb{A})N^Q(\mathbb{A})\backslash G_0(\mathbb{A})} f_s(\gamma_0 g) W_{\varphi, \psi}(g) dg,$$

as desired. $\square$

Fix a tensor product decomposition $\pi \simeq \otimes'_v \pi_v$. By the local uniqueness of Whittaker models, if we choose the section f_s to be factorizable and $\varphi \simeq \otimes_v \xi_v$ to correspond to a pure tensor under this isomorphism, the right-hand integral in (2.23) factors as

$$\begin{aligned} \int_{\mathcal{D}(\mathbb{A})} f_s(\gamma_0 g) W_{\varphi, \psi}(g) dg &= \prod_{v \in \Sigma_F} \int_{\mathcal{D}(F_v)} f_{s,v}(\gamma_0 g_v) W_{\xi_v, \psi_v}(g_v) dg_v \\ &= \prod_{v \in \Sigma_F} \Psi_v(s, W_{\xi_v, \psi_v}, f_{s,v}), \end{aligned} \quad (2.26)$$

with Σ_F the set of places of F .

Proposition 2.5.2. *The infinite product on the right converges for $\operatorname{Re}(s) \gg 0$.*

Proof. By Thm. 3.1.14 and Thm. 3.2.5, we have

$$\Psi_v^*(s, W_v, f_{s,v}) \stackrel{\text{def}}{=} d_{G,v}(s) \Psi_v(s, W_v, f_{s,v}) = L_v(s, \pi_v, \rho_{\text{TT}})$$

for almost all (i.e., all but finitely many) places v of F . Let $S \subseteq \Sigma_F$ be the finite set of places v for which this equality does *not* hold. (See the next section for a more explicit definition of S .) Then we can write

$$\begin{aligned} \prod_{v \in \Sigma_F} \Psi_v^*(s, W_v, f_{s,v}) &= \prod_{v \in S} \Psi_v^*(s, W_v, f_{s,v}) \cdot \prod_{v \notin S} \Psi_v^*(s, W_v, f_{s,v}) \\ &= \prod_{v \in S} \Psi_v^*(s, W_v, f_{s,v}) \cdot \prod_{v \notin S} L_v(s, \pi_v, \rho_{\text{TT}}). \end{aligned}$$

The product over $v \notin S$ is absolutely convergent for $\operatorname{Re}(s) \gg 0$ by [32, p. 45]. Each integral $\Psi_v^*(s, W_v, f_{s,v})$ is absolutely convergent for $\operatorname{Re}(s) \gg 0$ by Prop. 4.1.7 and Prop. 4.1.8. Since S is a finite set, the product over $v \in S$ is also absolutely convergent for $\operatorname{Re}(s) \gg 0$. $\square$

We will show that at almost all places this local integral computes the local twisted tensor L -factor, after suitable normalization of the Eisenstein series. Thus our (normalized) global integral represents the partial twisted tensor L -function.

CHAPTER 3

UNRAMIFIED COMPUTATION

Let $S \subset \Sigma_F$ be a finite set of places of F such that for $v \notin S$, v is finite, unramified in E , of residue characteristic not 2, such that θ_v is a unit and such that π_v and ψ_v° are also unramified. The place v is either inert or split in E . If v is split, we will show that the local integral agrees with Jiang's local unramified integral in [24]. We will also give a different proof that this integral calculates the local L -factor, based on branching laws. (Jiang's proof uses the Kostant-Rallis algorithm [29].) However, since the combinatorics in the split case are slightly more involved, we will do the inert case first.

3.1 Inert Case

Suppose v is inert in E . As is customary, we omit the subscript v henceforth. Fix a uniformizer ϖ for F (hence for E), let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers of F and $\mathfrak{p} = \mathfrak{p}_F$ be the maximal ideal in $\mathcal{O}$, and denote by $q = q_F$ the order of the residue field $\mathcal{O}/\mathfrak{p}$ of F , so $q_E = q_F^2$. Take f_s° and W° to be the spherical section and spherical Whittaker function, respectively, normalized so that $f_s^\circ(e) = W^\circ(e) = 1$. This section will closely parallel section II.3.2 of [24].

We have the Iwasawa decomposition $\mathrm{GSp}_4(E) = N^{B_4}(E)T_4(E)K_4$, where $K_4 = \mathrm{GSp}_4(\mathcal{O}_E)$. By restricting similitudes to lie in F , this becomes

$$G_0(F) = N_0(F)T_0(F)K_0,$$

where as sets $N_0(F) = N^{B_4}(E)$. Note that $T_0(F)$ is not split over F ! Also, using the above notation we have $X_2^{Q'}(F)N^Q(F)\backslash N_0(F) \simeq X_2^{Q'}(F)\backslash N^{Q'}(F)$. Using this decomposition, the local integral becomes

$$\begin{aligned} \Psi(s, W^\circ, f_s^\circ) &= \int_{\mathcal{D}(F)} f_s^\circ(\gamma_0 g) W^\circ(g) dg \\ &= \int_{K_0 \cap Z(F) \backslash K_0} \int_{Z(F) \backslash T_0(F)} \\ &\quad \cdot \int_{X_2^{Q'}(F)N^Q(F) \backslash N_0(F)} f_s^\circ(\gamma_0 ntk) W^\circ(ntk) \delta_{B_0}^{-1}(t) dn dt dk \\ &= \int_{Z(F) \backslash T_0(F)} W^\circ(t) \delta_{B_0}^{-1}(t) \int_{X_2^{Q'}(F) \backslash N^{Q'}(F)} f_s^\circ(\gamma_0 nt) \psi(n) dn dt, \end{aligned}$$

where δ_{B_0} is the modulus of the Borel subgroup $B_0 \subseteq G_0$. It is easy to see that $\delta_{B_0} = \delta_{B_4}^2$ is the square of the modulus character for the Borel of GSp_4 .

We first evaluate the inner integral $I(t, f_s^\circ, \psi)$ by reducing it to a product of one-variable integrals, and then we will use the formula of Casselman-Shalika [5] to compute the (non-normalized) unramified integral.

3.1.1 The Inner Integral $I(t, f_s^\circ, \psi)$

Denote by

$$h(a, b, c, d), \quad a, b, c, d \in E \tag{3.1}$$

and

$$h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8), \quad a_1, \dots, a_8 \in F \quad (3.2)$$

the diagonal matrices in $\mathrm{GSp}_4(E)$ and $\mathrm{GSp}_8(F)$, respectively, with given entries from upper-left to lower-right. Set $K' = K'(F) = \mathrm{GSp}_8(\mathcal{O}_F)$. For a root α , let χ_α denote the corresponding root subgroup (so $t \cdot \chi_\alpha(x) \cdot t^{-1} = \chi_\alpha(\alpha(t)x)$). Define

$$\begin{aligned} \chi(x, y, w, u, v) &= \chi_{-\epsilon_2-\epsilon_4}(x) \chi_{-2\epsilon_3}(y) \chi_{-\epsilon_3-\epsilon_4}(w) \chi_{-\epsilon_1-\epsilon_4}(u) \chi_{-\epsilon_2-\epsilon_3}(v) \\ &= \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & u & & & & \\ & & & & 1 & & & \\ & v & & x & & 1 & & \\ & & v & y & w & & 1 & \\ u & x & w & & & & & 1 \end{pmatrix} \end{aligned} \quad (3.3)$$

Set $\chi(x) = \chi(x, 0, 0, 0, 0)$, $\chi(x, y) = \chi(x, y, 0, 0, 0)$, etc. (Our $\chi(x, y, w, u, v)$ is denoted $\chi(w, y, x, u, v)$ in [24].)

Lemma 3.1.1. *We have the following:*

- (i). $h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)^{-1} \chi(x, y, w, u, v) h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$
 $= \chi(a_4 a_6^{-1} x, a_3 a_7^{-1} y, a_4 a_7^{-1} w, a_4 a_5^{-1} u, a_3 a_6^{-1} v);$
- (ii). $\delta_{P_8}(h(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)) = \left(\frac{a_1^2 a_2^2}{a_3 a_7^3}\right)^3 = \left(\frac{a_1^2 a_3^2}{a_2 a_6^3}\right)^3 = \left(\frac{a_2^2 a_3^2}{a_1 a_5^3}\right)^3;$
- (iii). If $|v| > 1$, $\chi_{-\epsilon_2-\epsilon_3}(v) = h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1) \chi_{\epsilon_2+\epsilon_3}(v) k_v$, for some $k_v \in K'$;
- (iv). $\chi(x, y, w, u) \chi_{\epsilon_2+\epsilon_3}(v) = p_1(x, y, w, v) \chi(x, y, w - vxy, u)$, where $p_1(x, y, w, v) \in P_8(F)$ satisfies $\delta_{P_8}(p_1(x, y, w, v)) = 1$;
- (v). If $|u| > 1$, $\chi_{-\epsilon_1-\epsilon_4}(u) = h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u) \chi_{\epsilon_1+\epsilon_4}(u) k_u$, for some $k_u \in K'$;

(vi). $\chi(x, y, w)\chi_{\epsilon_1+\epsilon_4}(u) = p_2(x, w, u)\chi(x, y, w)$, where $p_2(x, w, u) \in P_8(F)$ satisfies

$$\delta_{P_8}(p_2(x, w, u)) = 1;$$

(vii). If $|x| > 1$, $\chi_{-\epsilon_2-\epsilon_4}(x) = h(1, x^{-1}, 1, x^{-1}, 1, x, 1, x)\chi_{\epsilon_2+\epsilon_4}(x)k_x$, for some $k_x \in K'$;

(viii). If $|y| > 1$, $\chi_{-2\epsilon_3}(y) = h(1, 1, y^{-1}, 1, 1, 1, y, 1)\chi_{2\epsilon_3}(y)k_y$, for some $k_y \in K'$.

(ix). $\chi(x, 0, w)\chi_{2\epsilon_3}(y) = p_3(y, w)\chi(x, 0, w)$, where $p_3(y, w) \in P_8(F)$ satisfies

$$\delta_{P_8}(p_3(y, w)) = 1;$$

(x). $\chi(x)\chi_{\epsilon_3+\epsilon_4}(w) = p_4(x, w)\chi(x)$, where $p_4(x, w) \in P_8(F)$ satisfies $\delta_{P_8}(p_4(x, w)) = 1$.

Proof. All but the last two statements are found in [24]; all are standard computations. □

Lemma 3.1.2. For $t = h(a, b, \lambda a^{-1}, \lambda b^{-1})$ and $a = a_1 + a_2\theta, b = b_1 + b_2\theta$, write $\mathbb{N}a = \mathbb{N}_{E/F}(a)$ and similar for $\mathbb{N}b$. Then

$$\begin{aligned} I(t, f_s^\circ, \psi) &= |a_1|^{-s-1} |b_1|^{-2s-2} |\mathbb{N}a|^{s+3} |\mathbb{N}b|^{s+2} |\lambda|^{-\frac{s+7}{2}} \\ &\cdot \int_{F^3} f_s^\circ \left(\chi(x, y, w, -\frac{a_2\theta^2}{a_1}, -\frac{b_2\theta^2}{b_1}) \right) \psi_\circ \left(-\frac{a_1b_1}{\mathbb{N}b}x - \frac{b_1^2}{\lambda\theta^2}y \right) dx dy dw. \end{aligned} \quad (3.4)$$

Proof. By direct calculation, we have $\gamma_0 nt = n' p \bar{n} \gamma_0$, where $n', p \in P_8(F)$ and n' is unipotent, and $\bar{n} \in \overline{U}^P(F)$. Since $\gamma_0 \in K_0(F)$ we will have $f_s^\circ(\gamma_0 nt) = f_s^\circ(p \bar{n}) = \delta_{P_8}(p) f_s^\circ(\bar{n})$. If n is of the form

$$n = \begin{pmatrix} 1 & x(1+\theta) & w(1+\theta) & s' \\ & 1 & s & y(1-\theta) \\ & & 1 & \\ & -x(1+\theta) & & 1 \end{pmatrix}$$

with $s' = s + xy(1 - \theta^2)$, then $\bar{n} = \chi \left(-\frac{x\theta^2\mathbb{N}b}{a_1b_1}, -\frac{\lambda\theta^2y}{b_1^2}, \frac{\lambda\theta^2s''}{a_1b_1^2}, -\frac{a_2\theta^2}{a_1}, -\frac{b_2\theta^2}{b_1} \right)$ and $\psi(n) = \psi_\circ(\theta^2x + y)$, where $s'' = -b_1s + (b_1 + b_2)xy\theta^2$. Since $\delta_{P_8}(p) = \left(\frac{\mathbb{N}a^2\mathbb{N}b^2}{a_1^2b_1^4\lambda} \right)^3$, the integral becomes

$$\begin{aligned} I(t, f_s^\circ, \psi) &= \int_{F^3} \delta_{P_8}(p)^{\frac{s+3}{6}} f_s^\circ \left(\chi \left(-\frac{x\theta^2\mathbb{N}b}{a_1b_1}, -\frac{\lambda\theta^2y}{b_1^2}, \frac{\lambda\theta^2s''}{a_1b_1^2}, -\frac{a_2\theta^2}{a_1}, -\frac{b_2\theta^2}{b_1} \right) \right) \\ &\quad \cdot \psi_\circ(\theta^2x + y) dx dy ds \\ &= \left| \frac{\mathbb{N}a^2\mathbb{N}b^2}{a_1^2b_1^4\lambda} \right|^{\frac{s+3}{2}} \int_{F^3} f_s^\circ \left(\chi \left(x, y, w, -\frac{a_2\theta^2}{a_1}, -\frac{b_2\theta^2}{b_1} \right) \right) \\ &\quad \cdot \psi_\circ \left(-\frac{a_1b_1}{\mathbb{N}b}x - \frac{b_1^2}{\lambda\theta^2}y \right) \left| \frac{a_1b_1}{\theta^2\mathbb{N}b} \right| dx \left| \frac{b_1^2}{\lambda\theta^2} \right| dy \left| \frac{a_1b_1}{\lambda\theta^2} \right| dw \end{aligned}$$

after the change of variables $x \rightarrow -\frac{a_1b_1}{\theta^2\mathbb{N}b}x$, $y \rightarrow -\frac{b_1^2}{\lambda\theta^2}y$ and $s = -\frac{a_1b_1}{\lambda\theta^2} \left(w - \frac{b_1(b_1+b_2)}{\mathbb{N}b}xy \right)$.

Simplifying now gives the result.

More explicitly, the images of n and t in $\mathrm{GSp}_8(F)$ are

$$n = \begin{pmatrix} 1 & x & x & w & \theta^2w & s' & 0 \\ & 1 & \theta^2x & x & \theta^2w & \theta^2w & 0 \\ & & 1 & s & 0 & y & -\theta^2y \\ & & & 1 & 0 & s\theta^2 & -\theta^2y \\ & & & & 1 & -\theta^2y & \theta^2y \\ & & & & & 1 & \\ & & & & & & -x \\ & & & & & & -\theta^2x \\ & & & & & & 1 \\ & & & & & & -x \\ & & & & & & -x \\ & & & & & & & 1 \end{pmatrix}$$

and

$$t = \begin{pmatrix} a_1 & a_2 & & & & & & \\ a_2\theta^2 & a_1 & & & & & & \\ & & b_1 & b_2 & & & & \\ & & b_2\theta^2 & b_1 & & & & \\ & & & & \frac{a_1\lambda}{\mathbb{N}a} & -\frac{a_2\lambda\theta^2}{\mathbb{N}a} & & \\ & & & & -\frac{a_2\lambda}{\mathbb{N}a} & \frac{a_1\lambda}{\mathbb{N}a} & & \\ & & & & & & \frac{b_1\lambda}{\mathbb{N}b} & -\frac{b_2\lambda\theta^2}{\mathbb{N}b} \\ & & & & & & -\frac{b_2\lambda}{\mathbb{N}b} & \frac{b_1\lambda}{\mathbb{N}b} \end{pmatrix},$$

and we will have

$$n' = \begin{pmatrix} 1 & x & -\theta^2xy & \theta^2(w+x^2y) & w & s' & -x & 0 \\ 0 & 1 & -\theta^2y & \theta^2xy & s & y & 0 & 0 \\ 0 & 0 & 1 & -x & -x & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & \\ & & & & -x & 1 & 0 & \\ & & & & 0 & \theta^2y & 1 & \\ & & & -\theta^2(w+x^2y-xs') & -\theta^2w & 0 & x & 1 \end{pmatrix},$$

$$p = \begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix},$$

$$\begin{aligned}
A(p) &= \begin{pmatrix} \frac{\mathbb{N}a}{a_1} & -\frac{a_2\theta^2x\mathbb{N}b}{a_1b_1} & \frac{a_2\lambda\theta^2s''}{a_1b_1^2} & 0 \\ 0 & \frac{\mathbb{N}b}{b_1} & -\frac{b_2\lambda\theta^2y}{b_1^2} & \frac{b_2\lambda\theta^2s''}{a_1b_1^2} \\ 0 & 0 & \frac{\lambda}{b_1} & -\frac{b_2\lambda\theta^2x}{a_1b_1} \\ 0 & 0 & 0 & \frac{\lambda}{a_1} \end{pmatrix}, \\
B(p) &= \begin{pmatrix} 0 & & & -a_2 \\ & 0 & -b_2 & \\ & -\frac{b_2\lambda}{\mathbb{N}b} & 0 & \\ -\frac{a_2\lambda}{\mathbb{N}a} & & & 0 \end{pmatrix}, \\
C(p) &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & -\frac{b_2\lambda\theta^4x(b_1(s'-xy)-s'')}{a_1b_1^2} \end{pmatrix}, \\
D(p) &= \begin{pmatrix} \frac{a_1\lambda}{\mathbb{N}a} & 0 & 0 & 0 \\ \frac{a_2\lambda\theta^2x}{\mathbb{N}a} & \frac{b_1\lambda}{\mathbb{N}b} & 0 & 0 \\ \frac{a_2\lambda\theta^2(s'-xy)}{\mathbb{N}a} & \frac{b_2\lambda\theta^2y}{\mathbb{N}b} & b_1 & 0 \\ 0 & \frac{b_2\lambda\theta^2(s'-xy)}{\mathbb{N}b} & b_2\theta^2x & a_1 \end{pmatrix},
\end{aligned}$$

and

$$\bar{n} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & 0 & 0 & -\frac{a_2\theta^2}{a_1} & 1 \\ & 0 & 0 & -\frac{b_2\theta^2}{b_1} & -\frac{x\theta^2\mathbb{N}b}{a_1b_1} & 1 \\ & 0 & -\frac{b_2\theta^2}{b_1} & -\frac{\lambda\theta^2y}{b_1^2} & \frac{\lambda\theta^2s''}{a_1b_1^2} & 1 \\ -\frac{a_2\theta^2}{a_1} & -\frac{x\theta^2\mathbb{N}b}{a_1b_1} & \frac{\lambda\theta^2s''}{a_1b_1^2} & 0 & & 1 \end{pmatrix}.$$

□

So it suffices to calculate the integral

$$I(u, v, \alpha, \beta; f_s^\circ, \psi_\circ) = \int_{F^3} f_s^\circ(\chi(x, y, w, u, v)) \psi_\circ(\alpha x + \beta y) dx dy dw.$$

This is exactly the integral found in [24], II.3.2, equation (41). We give details of this calculation below.

Lemma 3.1.3.

$$I(u, v, \alpha, \beta; f_s^\circ, \psi_\circ) = \begin{cases} I(0, 0, \alpha, \beta; f_s^\circ, \psi_\circ) & \text{if } |u| \leq 1, |v| \leq 1, \\ |v|^{-2s-2} I(0, 0, v\alpha, v^2\beta; f_s^\circ, \psi_\circ) & \text{if } |u| \leq 1, |v| > 1, \\ |u|^{-s-1} I(0, 0, u\alpha, \beta; f_s^\circ, \psi_\circ) & \text{if } |u| > 1, |v| \leq 1, \\ |u|^{-s-1} |v|^{-2s-2} I(0, 0, uv\alpha, v^2\beta; f_s^\circ, \psi_\circ) & \text{if } |u| > 1, |v| > 1. \end{cases}$$

Proof. (i). If $|u|, |v| \leq 1$, then $\chi(x, y, w, u, v) = \chi(x, y, w)\chi_{-\epsilon_1-\epsilon_4}(u)\chi_{-\epsilon_2-\epsilon_3}(v)$, and the last two terms on the right are in $K' = \mathrm{GSp}_8(\mathcal{O})$.

(ii). If $|u| \leq 1$ and $|v| > 1$, then Lemma 3.1.1 gives

$$\begin{aligned}
\chi(x, y, w, u, v) &= \chi(x, y, w)\chi_{-\epsilon_2-\epsilon_3}(v)k_u \\
&= \chi(x, y, w)h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1)\chi_{\epsilon_2+\epsilon_3}(v)k_vk_u \\
&= h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1)\chi(v^{-1}x, v^{-2}y, v^{-1}w)\chi_{\epsilon_2+\epsilon_3}(v)k_vk_u \\
&= h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1)p_1(x, y, w, v)\chi(v^{-1}x, v^{-2}y, v^{-1}w - v^{-2}xy)k_uk_v.
\end{aligned}$$

for some $k_u, k_v \in K'$. So the integral becomes

$$\begin{aligned}
I(u, v, \alpha, \beta; f_s^\circ, \psi_\circ) &= \int_{F^3} f_s^\circ(\chi(x, y, w, u, v))\psi_\circ(\alpha x + \beta y) \, dx \, dy \, dw \\
&= \int_{F^3} |(v^{-2})^6|^{\frac{s}{6}+\frac{1}{2}} f_s^\circ(\chi(v^{-1}x, v^{-2}y, v^{-1}w - v^{-2}xy))\psi_\circ(\alpha x + \beta y) \, dx \, dy \, dw \\
&= |v|^{-2s-6} \int_{F^3} f_s^\circ(\chi(x, y, w))\psi_\circ(\alpha vx + \beta v^2y) |v| \, dx \, |v^2| \, dy \, |v| \, dw \\
&= |v|^{-2s-2} I(0, 0, v\alpha, v^2\beta; f_s^\circ, \psi_\circ).
\end{aligned}$$

(iii). If $|u| > 1$ and $|v| \leq 1$, we have

$$\begin{aligned}
\chi(x, y, w, u, v) &= \chi(x, y, w)\chi_{-\epsilon_1-\epsilon_4}(u)k_v \\
&= \chi(x, y, w)h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u)\chi_{\epsilon_1+\epsilon_4}(u)k_uk_v \\
&= h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u)\chi(u^{-1}x, y, u^{-1}w)\chi_{\epsilon_1+\epsilon_4}(u)k_uk_v \\
&= h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u)p_2(x, y, w, u)\chi(u^{-1}x, y, u^{-1}w)k_uk_v,
\end{aligned}$$

for $k_u, k_v \in K'$, giving the integral

$$\begin{aligned}
& I(u, v, \alpha, \beta; f_s^\circ, \psi_\circ) \\
&= \int_{F^3} |(u^{-1})^6|^{\frac{s}{6} + \frac{1}{2}} f_s^\circ(\chi(u^{-1}x, y, u^{-1}w)) \psi_\circ(\alpha x + \beta y) \, dx \, dy \, dw \\
&= |u|^{-s-3} \int_{F^3} f_s^\circ(\chi(x, y, w)) \psi_\circ(\alpha ux + \beta y) |u| \, dx \, dy \, |u| \, dw \\
&= |u|^{-s-1} I(0, 0, u\alpha, \beta; f_s^\circ, \psi_\circ).
\end{aligned}$$

(iv). If $|u|, |v| > 1$, we will have

$$\begin{aligned}
\chi(x, y, w, u, v) &= \chi(x, y, w, u) \chi_{-\epsilon_2 - \epsilon_3}(v) \\
&= \chi(x, y, w, u) h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1) \chi_{\epsilon_2 + \epsilon_3}(v) k_v \\
&= h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1) \chi(v^{-1}x, v^{-2}y, v^{-1}w, u) \chi_{\epsilon_2 + \epsilon_3}(v) k_v \\
&= h(1, v^{-1}, v^{-1}, 1, 1, v, v, 1) p_3(x, y, w, v) \\
&\quad \cdot \chi(v^{-1}x, v^{-2}y, v^{-1}w - v^{-2}xy, u) k_v,
\end{aligned}$$

and

$$\begin{aligned}
\chi(v^{-1}x, v^{-2}y, v^{-1}s - v^{-2}xy, u) &= \chi(v^{-1}x, v^{-2}y, v^{-1}s - v^{-2}xy) \chi_{-\epsilon_1 - \epsilon_4}(u) \\
&= \chi(v^{-1}x, v^{-2}y, v^{-1}s - v^{-2}xy) h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u) \chi_{\epsilon_1 + \epsilon_4}(u) k_u \\
&= h(u^{-1}, 1, 1, u^{-1}, u, 1, 1, u) p_4(x, y, w, u) \\
&\quad \cdot \chi((uv)^{-1}x, v^{-2}y, (uv)^{-1}(w - v^{-1}xy)) k_u,
\end{aligned}$$

for $k_u, k_v \in K'$, so the integral is

$$\begin{aligned}
I(u, v, \alpha, \beta; f_s^\circ, \psi_\circ) &= |uv^2|^{-s-3} \\
&\cdot \int_{F^3} f_s^\circ(\chi((uv)^{-1}x, v^{-2}y, (uv)^{-1}(w - v^{-1}xy))) \psi_\circ(\alpha x + \beta y) \, dx \, dy \, dw \\
&= |uv^2|^{-s-3} \int_{F^3} f_s^\circ(\chi(x, y, w)) \psi_\circ(\alpha uvx + \beta v^2y) |uv| \, dx \, |v^2| \, dy \, |uv| \, dw \\
&= |u|^{-s-1} |v|^{-2s-2} I(0, 0, uv\alpha, v^2\beta; f_s^\circ, \psi_\circ).
\end{aligned}$$

□

Finally we compute the integral $I(\alpha, \beta) = I(0, 0, \alpha, \beta; f_s^\circ, \psi_\circ)$.

Lemma 3.1.4.

$$\begin{aligned}
I(\alpha, \beta) &= \left[\int_{|x| \leq 1} \psi_\circ(\alpha x) \, dx + \int_{|x| > 1} |x|^{-s-2} \psi_\circ(\alpha x) \, dx \right] \\
&\cdot \left[\int_{|y| \leq 1} \psi_\circ(\beta y) \, dy + \int_{|y| > 1} |y|^{-s-2} \psi_\circ(\beta y) \, dy \right] \left[1 + \int_{|w| > 1} |w|^{-s-3} \, dw \right].
\end{aligned}$$

Proof. $I(\alpha, \beta)$ is absolutely convergent for $\text{Re}(s)$ large, so fix s in the domain of absolute convergence. Then we may freely change the order of integration since F^3 is abelian. So first change the order of integration to $dy \, dw \, dx$ and consider the integration in y . The integral over $\{|y| \leq 1\}$ is just

$$\int_{|y| \leq 1} f_s^\circ(\chi(x, 0, w)) \psi_\circ(\beta y) \, dy = f_s^\circ(\chi(x, 0, w)) \int_{|y| \leq 1} \psi_\circ(\beta y) \, dy,$$

while the integral over $\{|y| > 1\}$ becomes

$$\begin{aligned}
&\int_{|y| > 1} f_s^\circ(\chi(x, 0, w) h(1, y^{-1}, 1, 1, 1, y, 1, 1) \chi_{2\epsilon_3}(y) k_y) \psi_\circ(\beta y) \, dy \\
&= \int_{|y| > 1} |y|^{-s-3} f_s^\circ(\chi(x, 0, y^{-1}w) \chi_{2\epsilon_3}(y)) \psi_\circ(\beta y) \, dy \\
&= \int_{|y| > 1} |y|^{-s-3} f_s^\circ(\chi(x, 0, y^{-1}w)) \psi_\circ(\beta y) \, dy,
\end{aligned}$$

where we used Lemma 3.1.1(viii),(i),(ii),(ix). Now exchange the y - and w -integration and change variables $w \rightarrow yw$. This introduces a factor of $|y|$, so $I(\alpha, \beta)$ becomes

$$\begin{aligned} I(\alpha, \beta) &= \int_F \psi_\circ(\alpha x) \left[\int_{|y| \leq 1} \psi_\circ(\beta y) \left(\int_F f_s^\circ(\chi(x, 0, w)) dw \right) dy \right. \\ &\quad \left. + \int_{|y| > 1} |y|^{-s-3} \psi_\circ(\beta y) \left(\int_F f_s^\circ(\chi(x, 0, y^{-1}(yw))) |y| dw \right) dy \right] dx \\ &= \left[\int_{|y| \leq 1} \psi_\circ(\beta y) dy + \int_{|y| > 1} |y|^{-s-2} \psi_\circ(\beta y) dy \right] \left[\int_{F^2} f_s^\circ(\chi(x, 0, w)) \psi_\circ(\alpha x) dx dw \right]. \end{aligned}$$

The other two integrations are worked out similarly. $\square$

Lemma 3.1.5. *Let $\tilde{\psi}$ be an additive character of F with conductor $\mathcal{O}$, $\alpha \in F$ with $\text{ord}_v(\alpha) = a$ and $s \in \mathbb{C}$. Then for $\text{Re}(s) \gg 0$,*

$$(i). \int_{|x| > 1} |x|^{-s-2} \tilde{\psi}(\alpha x) dx = -1 + (1 - q^{-s-2}) \frac{1 - q^{-(a+1)(s+1)}}{1 - q^{-s-1}}, \text{ if } a \geq 0, \text{ and is } 0 \text{ otherwise;}$$

$$(ii). 1 + \int_{|w| > 1} |w|^{-s-3} dw = \frac{1 - q^{-s-3}}{1 - q^{-s-2}}.$$

Proof. Standard calculations. $\square$

Using Lemma 3.1.5, it is easy to see that if $A = \text{ord}_v \alpha, B = \text{ord}_v \beta$, then

$$I(\alpha, \beta) = \frac{(1 - q^{-s-3})(1 - q^{-s-2})}{(1 - q^{-s-1})^2} (1 - q^{-(s+1)(A+1)}) (1 - q^{-(s+1)(B+1)}) \quad (3.5)$$

for $|\alpha|, |\beta| \leq 1$ and $I(\alpha, \beta)$ is zero otherwise.

We also note for later reference that we can calculate $I(0, 0, \alpha, \beta; f_s^\circ, 1)$ for $\psi_\circ = 1$: the integrals $\int_{\mathcal{O}} \psi_\circ(\alpha x) dx$ and $\int_{\mathcal{O}} \psi_\circ(\beta y) dy$ are both 1, and so the x -integrals become

$$1 + \int_{|x| > 1} |x|^{-s-2} dx = \frac{1 - q^{-s-2}}{1 - q^{-s-1}}$$

by Lemma 3.1.5(ii), and the y -integrals are identical. The result, after cancellation, is

$$I(0, 0, \alpha, \beta; f_s^\circ, 1) = \frac{(1 - q^{-s-2})(1 - q^{-s-3})}{(1 - q^{-s-1})^2} = \frac{\zeta_v(s+1)^2}{\zeta_v(s+2)\zeta_v(s+3)}. \quad (3.6)$$

Now we will fix a parametrization of $Z(F) \backslash T_0(F)$ by taking as a set of representatives elements

$$t = t'(a_1, a_2, b_1, b_2) \stackrel{\text{def}}{=} h(a_1(1 + a_2\theta), b_1(1 + b_2\theta), \lambda_{a,b}(1 - a_2\theta), 1 - b_2\theta), \quad (3.7)$$

where

$$\lambda_{a,b} = \frac{b_1(1 - b_2^2\theta^2)}{a_1(1 - a_2^2\theta^2)}. \quad (3.8)$$

(Note that this actually parametrizes the open subset of elements $h(a, b, c, d)$ in $T_0(F)$ with $d = d_1 + d_2\theta$ and $d_1 \neq 0$, but its complement has measure zero, so we may use this parametrization in our integral.)

For convenience, define a function $\delta: \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$\delta(m) = \begin{cases} 0 & \text{if } m > 0, \\ m & \text{if } m \leq 0. \end{cases}$$

Then we have the following

Lemma 3.1.6. *For $t = t'(a_1, a_2, b_1, b_2)$ as above and $m_i = \text{ord}_v a_i, n_i = \text{ord}_v b_i$, $\delta_{B_4}(t) = q^{-3n_1 - 4(m_1 + \delta(m_2) - n_1 - \delta(n_2))}$.*

Assembling the above results gives us the inner integral:

Proposition 3.1.7. *For $t = t'(a_1, a_2, b_1, b_2)$ as above, let $m_i = \text{ord}_v a_i, n_i = \text{ord}_v b_i$ for $i = 1, 2$. Then*

$$I(t, f_s^\circ, \psi) = \zeta_v(s+1)^2 \zeta_v(s+2)^{-1} \zeta_v(s+3)^{-1} q^{-(s+5)(m_1+\delta(m_2)-\delta(n_2))+\left(\frac{s+3}{2}\right)n_1} \\ \cdot \left(1 - q^{-(s+1)(1+n_1)}\right) \left(1 - q^{-(s+1)(1+m_1+\delta(m_2)-n_1-\delta(n_2))}\right),$$

if $m_1 + \delta(m_2) - \delta(n_2) \geq n_1 \geq 0$,

and is equal to 0 otherwise. More explicitly,

$$I(t, f_s^\circ, \psi) = \zeta_v(s+1)^2 \zeta_v(s+2)^{-1} \zeta_v(s+3)^{-1} q^{-(s+5)m_1+\left(\frac{s+3}{2}\right)n_1} \left(1 - q^{-(s+1)(1+n_1)}\right) \\ \cdot \left\{ \begin{array}{ll} \left(1 - q^{-(s+1)(1+m_1-n_1)}\right) & \text{if } m_2 \geq 0, n_2 \geq 0, m_1 \geq n_1 \geq 0, \\ q^{(s+5)n_2} \left(1 - q^{-(s+1)(1+m_1-n_1-n_2)}\right) & \text{if } m_2 \geq 0, n_2 < 0, m_1 - n_2 \geq n_1 \geq 0, \\ q^{-(s+5)m_2} \left(1 - q^{-(s+1)(1+m_1+m_2-n_1)}\right) & \text{if } m_2 < 0, n_2 \geq 0, m_1 + m_2 \geq n_1 \geq 0, \\ q^{-(s+5)(m_2-n_2)} \left(1 - q^{-(s+1)(1+m_1+m_2-n_1-n_2)}\right) & \text{if } m_2 < 0, n_2 < 0, m_1 + m_2 - n_2 \geq n_1 \geq 0, \\ 0 & \text{otherwise.} \end{array} \right.$$

Proof. Change variables $a_2 \rightarrow a_1 a_2, b_2 \rightarrow b_1 b_2$ in (3.4) and set $a = 1 + a_2 \theta, b = 1 + b_2 \theta$, so that $\mathbb{N}a \rightarrow a_1^2 \mathbb{N}a, \mathbb{N}b \rightarrow b_1^2 \mathbb{N}b, \lambda \rightarrow b_1 \mathbb{N}b$; then by Lemma 3.1.2 and Lemma 3.1.3

we have

$$\begin{aligned}
I(t, f_s^\circ, \psi_\circ) &= |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} |\mathbb{N}a|^{s+3} |\mathbb{N}b|^{\frac{s-3}{2}} \\
&\quad \cdot I\left(-a_2\theta^2, -b_2\theta^2, -\frac{a_1}{b_1\mathbb{N}b}, -\frac{b_1}{\theta^2\mathbb{N}b}; f_s^\circ, \psi_\circ\right) \\
&= C \begin{cases} I\left(-\frac{a_1}{b_1\mathbb{N}b}, -\frac{b_1}{\theta^2\mathbb{N}b}\right) & \text{if } m_2 \geq 0, n_2 \geq 0, \\ |b_2|^{-2s-2} I\left((-b_2\theta^2)\left(-\frac{a_1}{b_1\mathbb{N}b}\right), (-b_2\theta^2)^2\left(-\frac{b_1}{\theta^2\mathbb{N}b}\right)\right) & \text{if } m_2 \geq 0, n_2 < 0, \\ |a_2|^{-s-1} I\left((-a_2\theta^2)\left(-\frac{a_1}{b_1\mathbb{N}b}\right), -\frac{b_1}{\theta^2\mathbb{N}b}\right) & \text{if } m_2 < 0, n_2 \geq 0, \\ |a_2|^{-s-1} |b_2|^{-2s-2} I\left((a_2b_2\theta^4)\left(-\frac{a_1}{b_1\mathbb{N}b}\right), (-b_2\theta^2)^2\left(-\frac{b_1}{\theta^2\mathbb{N}b}\right)\right) & \text{if } m_2 < 0, n_2 < 0. \end{cases}
\end{aligned}$$

where $C = C(a_i, b_i; s) = |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} |\mathbb{N}a|^{s+3} |\mathbb{N}b|^{\frac{s-3}{2}}$. Observing that $|\mathbb{N}a| = 1$ if $|a_2| \leq 1$ and $|\mathbb{N}a| = |a_2|^2$ if $|a_2| > 1$ gives

$$C = |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} \max\{1, |a_2|\}^{2s+6} \max\{1, |b_2|\}^{s-3}$$

and so the coefficient of $I(\alpha, \beta)$ in each case is

$$\begin{aligned}
&C \cdot \max\{1, |a_2|\}^{-s-1} \max\{1, |b_2|\}^{-2s-2} \\
&= |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} \max\{1, |a_2|\}^{s+5} \max\{1, |b_2|\}^{-s-5} \\
&= q^{-(s+5)m_1 + (\frac{s+3}{2})n_1 - \delta(m_2)(s+5) + \delta(n_2)(s+5)}.
\end{aligned}$$

Now applying (3.5) gives the result. $\square$

Remark 3.1.8. Note that Lemma 3.1.2 and Lemma 3.1.3 make no assumption on ψ

or $\psi_\circ$: it is only in Lemma 3.1.4 that $\psi_\circ$ is assumed to have conductor $\mathcal{O}$. If we take $\psi_\circ$ and hence ψ to be trivial and use (3.6), following the proof above gives

$$\begin{aligned}
I(t, f_s^\circ, 1) &= |\theta|^{-6} C(a_i, b_i; s) \max\{1, |a_2|\}^{-s-1} \max\{1, |b_2|\}^{-2s-2} \frac{\zeta_v(s+1)^2}{\zeta_v(s+2)\zeta_v(s+3)} \\
&= |\theta|^{-6} |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} \max\{1, |a_2|\}^{s+5} \max\{1, |b_2|\}^{-s-5} \frac{\zeta_v(s+1)^2}{\zeta_v(s+2)\zeta_v(s+3)} \\
&= |\theta|^{-6} |a_1|^{s+5} |b_1|^{-\frac{s+3}{2}} |\mathbb{N}a|^{\frac{s+5}{2}} |\mathbb{N}b|^{-\frac{s+5}{2}} \frac{\zeta_v(s+1)^2}{\zeta_v(s+2)\zeta_v(s+3)}
\end{aligned} \tag{3.9}$$

where we have inserted $|\theta|$ for clarity.

3.1.2 The Unramified Integral

We need the following formula for the unramified Whittaker function on $\mathrm{GSp}_{4/E}$, found in Casselman-Shalika [5]. Since we are dealing with a split group, we may use the simpler formula found in Bump's survey [3]; see also [24], Thm. I.3.3.1.

Theorem 3.1.9 ([5]). *Let $t = t'(a_1, a_2, b_1, b_2)$ as above, with $m_i = \mathrm{ord}_v a_i, n_i = \mathrm{ord}_v b_i$ for $i = 1, 2$. The spherical Whittaker function $W^\circ = W_E^\circ$ for π is given by*

$$W^\circ(t) = \begin{cases} \delta_E^{\frac{1}{2}}(t) \mathrm{tr} \tau_{(n_1, m_1 + \delta(m_2) - n_1 - \delta(n_2))}(z_\pi) & \text{if } m_1 + \delta(m_2) - \delta(n_2) \geq n_1 \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

where δ_E is the modulus character of the Borel subgroup $B_4(E)$ in $\mathrm{GSp}_4(E)$ and $\mathrm{tr} \tau_{(m,n)}(z_\pi)$ is the character of the representation $\tau_{(m,n)}$ of $\mathrm{Sp}_4(\mathbb{C})$ evaluated at the Satake parameter z_π of π , with $\tau_{(m,n)}$ as in Prop. 2.2.3.

Finally, we can evaluate the unramified integral.

Theorem 3.1.10. *The normalized local integral is*

$$\begin{aligned} \Psi^*(s, W^\circ, f_s^\circ) &= d_{G,v}(s) \Psi(s, W^\circ, f_s^\circ) \\ &= \frac{1}{(1-X^2)^3(1-X^4)} \sum_{k,l \geq 0} \text{tr } \tau_{(k,l)}(z_\pi) X^{k+2l} (1-(X^2)^{k+1}) (1-(X^2)^{l+1}), \end{aligned} \quad (3.10)$$

where $X = q^{-\frac{s+1}{2}}$ and $d_{G,v}(s) = \zeta_v(2s+2)\zeta_v(s+1)\zeta_v(s+2)\zeta_v(s+3)$ is the local component of the normalizing factor for the Eisenstein series $E(g, s, f_s)$.

Proof. We break up the integral over $Z(F) \setminus T_0(F)$ into four pieces, corresponding to the cases in Prop. 3.1.7. For $\eta_1, \eta_2 = \pm$, set

$$T_{\eta_1, \eta_2} = \{t = t'(a_1, a_2, b_1, b_2) \mid \text{sgn ord}_v a_2 = \eta_1, \text{sgn ord}_v b_2 = \eta_2\},$$

with the convention that $\text{sgn } 0 = +$. Then $T_{\eta_1, \eta_2} \simeq \mathbb{Z}^2 \times K_{\eta_1, \eta_2}$, where $K_{\eta_1, \eta_2} \subseteq T_0(\mathcal{O})$ and the $\mathbb{Z}^2$ -part is given by the v -adic orders of the simple roots. In fact, by setting $\bar{c} = \varpi^{\text{ord}_v c}$ for $c \in F$, we see that $t \in T_{\eta_1, \eta_2}$ has one of the following forms (up to an element of K_{η_1, η_2}):

$$\begin{aligned} (\eta_1, \eta_2) = (+, +) &\quad \Rightarrow \quad t = h(\bar{a}_1, \bar{b}_1, \bar{a}_1^{-1} \bar{b}_1, 1); \\ (\eta_1, \eta_2) = (+, -) &\quad \Rightarrow \quad t = h(\bar{a}_1, \bar{b}_1 \bar{b}_2, \bar{a}_1^{-1} \bar{b}_1 \bar{b}_2^2, \bar{b}_2); \\ (\eta_1, \eta_2) = (-, +) &\quad \Rightarrow \quad t = h(\bar{a}_1 \bar{a}_2, \bar{b}_1, (\bar{a}_1 \bar{a}_2)^{-1} \bar{b}_1, 1); \\ (\eta_1, \eta_2) = (-, -) &\quad \Rightarrow \quad t = h(\bar{a}_1 \bar{a}_2, \bar{b}_1 \bar{b}_2, (\bar{a}_1 \bar{a}_2)^{-1} \bar{b}_1 \bar{b}_2^2, \bar{b}_2). \end{aligned}$$

By considering the variants of $t'(a_1, a_2, b_1, b_2)$ given by

$$\begin{aligned} t'_1(a_1, a_2, b_1, b_2) &= h(a_1(a_2 + \theta), b_1(1 + b_2\theta), \lambda_{a,b}^{(1)}(a_2 - \theta), 1 - b_2\theta), \\ t'_2(a_1, a_2, b_1, b_2) &= h(a_1(1 + a_2\theta), b_1(b_2 + \theta), \lambda_{a,b}^{(2)}(1 - a_2\theta), b_2 - \theta), \\ t'_{1,2}(a_1, a_2, b_1, b_2) &= h(a_1(a_2 + \theta), b_1(b_2 + \theta), \lambda_{a,b}^{(1,2)}(a_2 - \theta), b_2 - \theta), \end{aligned}$$

with similitudes defined appropriately, the K_{η_1, η_2} may be described as follows:

$$\begin{aligned} K_{++} &= \{t'(u_1, a_2, v_1, b_2) \mid a_2, b_2 \in \mathcal{O}, u_1, v_1 \in \mathcal{O}^\times\}, \\ K_{+-} &= \{t'_2(u_1, a_2, v_1, b_2) \mid a_2, b_2 \in \mathcal{O}, u_1, v_1 \in \mathcal{O}^\times\}, \\ K_{-+} &= \{t'_1(u_1, a_2, v_1, b_2) \mid a_2, b_2 \in \mathcal{O}, u_1, v_1 \in \mathcal{O}^\times\}, \\ K_{--} &= \{t'_{1,2}(u_1, a_2, v_1, b_2) \mid a_2, b_2 \in \mathcal{O}, u_1, v_1 \in \mathcal{O}^\times\}. \end{aligned}$$

Normalize the measure on $Z(F) \backslash T_0(F)$ so that $\text{vol}(Z(F) \backslash Z(F)T_0(\mathcal{O})) = 1$. Then we easily compute that

$$\begin{aligned} \text{vol}(Z(F) \backslash Z(F)K_{++}) &= \frac{q^2}{(q+1)^2}, & \text{vol}(Z(F) \backslash Z(F)K_{--}) &= \frac{1}{(q+1)^2}, \\ \text{vol}(Z(F) \backslash Z(F)K_{+-}) &= \text{vol}(Z(F) \backslash Z(F)K_{-+}) = \frac{q}{(q+1)^2}. \end{aligned}$$

Since $\delta_{B_0}^{-1}(t)\delta_E^{\frac{1}{2}}(t) = \delta_{B_4}^{-2}(t)\delta_{B_4}(t) = \delta_{B_4}^{-1}(t)$, using Prop. 3.1.7 and Theorem 3.1.9 gives

$$\begin{aligned}
\Psi(s, W^\circ, f_s^\circ) &= \zeta_v(s+1)^2 \zeta_v(s+2)^{-1} \zeta_v(s+3)^{-1} \\
&\cdot \left[\text{vol}(K_{++}) \sum_{\substack{m_1-n_1 \geq 0 \\ n_1 \geq 0}} \text{tr } \tau_{(n_1, m_1-n_1)}(z_\pi) P_1(m_1, m_2, n_1, n_2, X) \right. \\
&+ \text{vol}(K_{-+}) \sum_{\substack{m_1+m_2-n_1 \geq 0 \\ n_1 \geq 0}} \text{tr } \tau_{(n_1, m_1+m_2-n_1)}(z_\pi) P_2(m_1, m_2, n_1, n_2, X) \\
&+ \text{vol}(K_{+-}) \sum_{\substack{m_1-n_1-n_2 \geq 0 \\ n_1 \geq 0}} \text{tr } \tau_{(n_1, m_1-n_1-n_2)}(z_\pi) P_3(m_1, m_2, n_1, n_2, X) \\
&\left. + \text{vol}(K_{--}) \sum_{\substack{m_1+m_2-n_1-n_2 \geq 0 \\ n_1 \geq 0}} \text{tr } \tau_{(n_1, m_1+m_2-n_1-n_2)}(z_\pi) P_4(m_1, m_2, n_1, n_2, X) \right],
\end{aligned}$$

where $X = q^{-\frac{s+1}{2}}$ and

$$\begin{aligned}
P_1(m_1, m_2, n_1, n_2, X) &= X^{2m_1-n_1} (1 - X^{2(1+n_1)}) \\
&\cdot (1 - X^{2(1+m_1-n_1)}) \\
P_2(m_1, m_2, n_1, n_2, X) &= X^{2m_1+2m_2-n_1} (1 - X^{2(1+n_1)}) \\
&\cdot (1 - X^{2(1+m_1+m_2-n_1)}) \\
P_3(m_1, m_2, n_1, n_2, X) &= X^{2m_1-2n_2-n_1} (1 - X^{2(1+n_1)}) \\
&\cdot (1 - X^{2(1+m_1-n_1-n_2)}) \\
P_4(m_1, m_2, n_1, n_2, X) &= X^{2m_1+2m_2-n_1-2n_2} (1 - X^{2(1+n_1)}) \\
&\cdot (1 - X^{2(1+m_1+m_2-n_1-n_2)}).
\end{aligned}$$

Now we substitute the volumes and change variables $n_1 \rightarrow k, m_1+\delta(m_2)-n_1-\delta(n_2) \rightarrow$

l , noting that all four sums collapse to the single sum in (3.10). Finally, multiplying by the normalizing factor $d_{G,v}(s)$ gives us what we want. $\square$

3.1.3 Local L -factor

We need to identify our local integral with a local Langlands L -factor, in this case that associated with the twisted tensor representation defined as follows.

As in [2], the dual group of GSp_4 is $\mathrm{GSp}_4(\mathbb{C})$, and if Γ_F is the absolute Galois group of F , then

$$\begin{aligned} {}^L G_0 &= (\mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C}))^\circ \rtimes \Gamma_F \\ &= ((\mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C}))^\circ \rtimes \Gamma_{E/F}) \times \Gamma_E, \end{aligned} \tag{3.11}$$

where $\Gamma_{E/F} = \Gamma_F/\Gamma_E$. The superscript $^\circ$ here denotes the similitude of each factor is the same. This follows from the general discussion of L -groups of groups arising from restriction of scalars in [2, I.4,5]. We also find that if $\mathrm{GL}_{4/E}$ is the group GL_4 considered as a group over E and $H = \mathrm{Res}_{E/F} \mathrm{GL}_{4/E}$, then

$${}^L H = (\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})) \rtimes \Gamma_{E/F}.$$

Let (ρ, V) be the defining representation for $\mathrm{Sp}_4(\mathbb{C})$, and extend ρ to $\mathrm{GSp}_4(\mathbb{C})$ trivially on similitudes. That is, write $g \in \mathrm{GSp}_4(\mathbb{C})$ as

$$g = g' \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$$

for $g' \in \mathrm{Sp}_4(\mathbb{C})$ and define $\rho(g) = \rho(g')$. Then $\rho: \mathrm{GSp}_4(\mathbb{C}) \rightarrow \mathrm{GL}_4(\mathbb{C})$. Let σ generate

the group $\Gamma_{E/F}$. Then the twisted tensor representation $\rho'_{\text{TT}} : {}^L H \rightarrow \text{GL}(V \otimes V) \simeq \text{GL}_{16}(\mathbb{C})$ is given by

$$\rho'_{\text{TT}}(g_1, g_2) : x \otimes y \mapsto xg_1 \otimes yg_2,$$

$$\rho'_{\text{TT}}(\sigma) : x \otimes y \mapsto y \otimes x,$$

for $x, y \in V$ and $g_1, g_2 \in \text{GL}_4(\mathbb{C})$. We define the *twisted tensor representation* of GSp_4 by the composition

$$\rho_{\text{TT}} : \text{GSp}_4(\mathbb{C}) \xrightarrow{\rho} \text{GL}_4(\mathbb{C}) \xrightarrow{\rho'_{\text{TT}}} \text{GL}_{16}(\mathbb{C})$$

so that

$$\rho_{\text{TT}}(g_1, g_2) : x \otimes y \mapsto \rho(g_1)x \otimes \rho(g_2)y, \quad (3.12)$$

$$\rho_{\text{TT}}(\sigma) : x \otimes y \mapsto y \otimes x, \quad (3.13)$$

for $(g_1, g_2) \in (\text{GSp}_4(\mathbb{C}) \times \text{GSp}_4(\mathbb{C}))^\circ$.

Since the group $G = \text{GSp}_4(E)$ is split over E , its L -group is ${}^L G = \text{GSp}_4(\mathbb{C}) \times \Gamma_E$. The Satake parameter for π is a conjugacy class which can be represented by a semisimple element $z' = \text{diag}(z_1, \dots, z_4)$. The L -homomorphism

$$r_{E/F} : \begin{cases} {}^L G = \text{GSp}_4(\mathbb{C}) \times \Gamma_E \rightarrow {}^L G_0, \\ g \times \tau \mapsto (g, 1) \rtimes \iota(\tau), \end{cases}$$

where $\iota : \Gamma_E \hookrightarrow \Gamma_F$ is the natural inclusion, arises from restriction of scalars. Then the representative of z' in ${}^L G_0$ is $z = r_{E/F}(z') \rtimes \sigma$. It is an easy exercise to check that

$$L(s, \pi, \rho_{\text{TT}}) = \det(I - \rho_{\text{TT}}(z)q^{-s})^{-1} = \prod_{i=1}^4 (1 - z_i q^{-s})^{-1} \prod_{1 \leq i < j \leq 4} (1 - z_i z_j q^{-2s})^{-1}. \quad (3.14)$$

The method of identifying this local L -factor with the unramified integral is as follows. First, in [34] we find a formula that relates the right-hand side of (3.14) to classical Schur functions, which are the characters of finite-dimensional representations of $\mathrm{GL}_4(\mathbb{C})$. Our motivation for using this is [8, p. 305]. Second, we use a branching formula of Koike-Terada [28] to relate characters of $\mathrm{GL}_4(\mathbb{C})$ to those of $\mathrm{Sp}_4(\mathbb{C})$. This latter formula involves sums of Littlewood-Richardson coefficients. Finally, we evaluate these coefficients and relate them to the rational functions found in Prop. 3.1.10, which will give the desired equality. We should point out that this approach through restriction of characters first came to our attention through a note in [13] pointing to work of Sundaram [42, 41], which we consulted first and from which we greatly benefitted. Ultimately [28] was more easily adaptable to our work here.

Let us begin by recalling some definitions from [34].

A *partition* λ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots$ and such that there is an r satisfying $\lambda_k = 0$ for all $k > r$. The smallest such r is the *length* of λ , denoted $\ell(\lambda)$. The nonzero λ_i are the *parts* of λ , and the sum of the parts of λ is the *size* of λ , written $|\lambda|$. If $|\lambda| = n$, we say λ is a *partition of n* and write $\lambda \vdash n$. A partition is called *even* if all its parts are even.

Given a partition λ of length at most n , the *Schur function* in n variables associated to λ is a certain symmetric polynomial $s_\lambda = s_\lambda(x_1, \dots, x_n)$ with integral coefficients. We do not need the precise definition here; see [34], p. 40. What we need to know is that s_λ is exactly the character $\chi_{\mathrm{GL}(n)}(\lambda)$ of the finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ , restricted to the maximal torus. Then we find in

[34], p. 76, the formula

$$\sum_{\lambda} s_{\lambda}(x_1, \dots, x_n) = \prod_{i=1}^n (1 - x_i)^{-1} \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^{-1}, \quad (3.15)$$

where the sum is over partitions λ of length at most n . So if Π is an irreducible admissible spherical representation of $\mathrm{GL}_n(F)$ with Satake parameter $z_{\Pi} = \mathrm{diag}(z_1, \dots, z_n)$, then since s_{λ} is homogeneous of weight $|\lambda|$,

$$L(s, \Pi, \rho'_{\mathrm{TT}}) = \sum_{\lambda: \ell(\lambda) \leq n} \chi_{\mathrm{GL}(n)}(\lambda)(z_{\Pi}) q^{-s|\lambda|}. \quad (3.16)$$

So we need to know how the characters $\chi_{\mathrm{GL}(4)}$ decompose when restricted to $\mathrm{Sp}_4(\mathbb{C})$. For this, we have a result of Koike and Terada [28]. First, some more definitions.

A *Young diagram* is a (finite) collection of boxes arranged in left-justified rows, with the lengths of the rows weakly decreasing top to bottom. The list of the row lengths of such a diagram forms a partition, say λ , and the diagram is said to be of *shape* λ . Listing the column lengths of a Young diagram of shape λ gives another partition denoted λ' and called the *transpose* of λ . Explicitly, $\lambda'_i = \#\{j | \lambda_j \geq i\}$. For partitions λ and μ , we write $\mu \subseteq \lambda$ when $\mu_i \leq \lambda_i$ for all i , and $\lambda + \mu$ is the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.

A *Young tableau* (or simply *tableau*) is a filling of a Young diagram with the symbols $\{1, 2, \dots, n\}$ (with the usual ordering), where $n \leq |\lambda|$ and λ is the shape of the diagram. A tableau is called *column-strict* (or *semistandard*) if the entries are weakly increasing along the rows and strictly increasing along the columns; a semistandard tableau which has entries strictly increasing along rows is called *standard*. The *weight* of a tableau is the sequence which records the number of times each entry appears in the tableau. Thus, for example, all standard tableau have weight $(1, 1, \dots, 1)$.

Figure 3.1: A Young diagram of shape $(6, 4, 4, 1)$

1	2	2	4	5	8
3	5	5	5		
6	6	7	8		
8					

Figure 3.2: A column-strict Young tableau

Given a pair of partitions λ, μ with $\lambda \supseteq \mu$, the *skew diagram of skew-shape* λ/μ is obtained by removing the diagram of shape μ from the diagram of shape λ . We have the analogous notions of *skew tableau* and semistandard and standard skew tableau. The *word* of a skew tableau is the string obtained by listing the entries of the tableau from right to left and top to bottom. We may also define the *weight* of a word to be the sequence giving the number of times each symbol appears in the word, with

			1	1	1
		1	2		
2	2	3	3		
4					

Figure 3.3: Skew tableau of skew-shape $(6, 4, 4, 1)/(3, 2)$

entries in increasing order. A word in $\{1, 2, \dots, n\}$ is called a *lattice permutation* if, in any initial segment of the word, the symbol i appears at least as often as the symbol $i + 1$. So, for example, the word of the tableau in Figure 3.3 is 1112133224, and this is not a lattice permutation since the initial segment 1112133 has more 3's than 2's. Note that a word is a lattice permutation if and only if the weight of every initial segment is a partition.

For a partition μ with $\ell(\mu) \leq n$, let V^μ be the finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight μ . Then the *Littlewood-Richardson coefficients* $c_{\mu,\beta}^\lambda$ are defined by

$$V^\mu \otimes V^\beta = \sum_{\lambda} c_{\mu,\beta}^\lambda V^\lambda. \quad (3.17)$$

Note that this implies $c_{\mu,\beta}^\lambda = c_{\beta,\mu}^\lambda$. The Littlewood-Richardson rule ([34], p. 143) gives an alternate characterization of these coefficients in terms of tableaux, namely

For partitions λ, μ, β , the coefficient $c_{\mu,\beta}^\lambda$ is the number of column-strict

tableaux of skew-shape λ/μ and weight β whose word is a lattice permutation.

From this characterization, we see that $c_{\mu,\beta}^\lambda = 0$ unless $\lambda \supseteq \mu$, $\lambda \supseteq \beta$ and $|\lambda| = |\mu| + |\beta|$. Now we may state the result of Koike-Terada.

Theorem 3.1.11 ([28], p. 504–506). *Let λ be a partition of length at most $2n$. Then*

$$\text{Res}_{\text{Sp}(2n)}^{\text{GL}(2n)} \chi_{\text{GL}(2n)}(\lambda) = \sum_{\mu} \sum_{\beta' \text{ even}} c_{\mu,\beta'}^\lambda \tilde{\chi}(\mu), \quad (3.18)$$

where the sum is over all μ and β' (with β' even), $\tilde{\chi}(\mu)$ is 0 or $\pm \chi_{\text{Sp}(2n)}(\tilde{\mu})$, $\tilde{\mu}$ is a partition derived from μ , and both are determined as follows. Set $\mu' = (\mu'_1, \dots, \mu'_l)$.

(i). If $\mu'_i - i + \mu'_j - j = 2n$ for any $1 \leq i, j \leq n$, then $\tilde{\chi}(\mu) = 0$.

(ii). Otherwise, set

$$k_i = \begin{cases} \mu'_i & \text{if } \mu'_i - i < n; \\ 2n + 2i - \mu'_i & \text{if } \mu'_i - i > n. \end{cases}$$

Let s be the number of indices i such that $\mu'_i - i > n$, and let $t_i = k_i - (i - 1)$.

By hypothesis, the t_i are distinct, so there is a permutation σ such that $t_{\sigma(1)} > t_{\sigma(2)} > \dots > t_{\sigma(l)}$. Set $\tilde{\mu}'_i = t_{\sigma(i)} + (i - 1) = k_{\sigma(i)} + i - \sigma(i)$. Define $\tilde{\mu}$ to be the partition whose transpose is $\tilde{\mu}' = (\tilde{\mu}'_1, \dots, \tilde{\mu}'_l)$. Then we have

$$\tilde{\chi}(\mu) = (-1)^s \text{sgn } \sigma \chi_{\text{Sp}(2n)}(\tilde{\mu}). \quad (3.19)$$

In particular, $\tilde{\chi}(\mu) = \chi_{\text{Sp}(2n)}(\mu)$ if $\ell(\mu) \leq n$. □

In our case $n = 2$, so all partitions appearing in (3.18) with nonzero coefficients will have length at most 4. Applying the above to $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ shows that

$$\tilde{\chi}(\mu) = \begin{cases} \chi_{\mathrm{Sp}(4)}((\mu_1, \mu_2)) & \text{if } \mu_3 = \mu_4 = 0; \\ -\chi_{\mathrm{Sp}(4)}((\mu_1, \mu_2)) & \text{if } \mu_3 = \mu_4 = 1; \\ 0 & \text{otherwise.} \end{cases} \quad (3.20)$$

Hence we may rewrite (3.18) as

$$\mathrm{Res}_{\mathrm{Sp}(4)}^{\mathrm{GL}(4)} \chi_{\mathrm{GL}(4)}(\lambda) = \sum_{\mu: \ell(\mu) \leq 2} \chi_{\mathrm{Sp}(4)}(\mu) \sum_{\beta' \text{ even}} (c_{\mu, \beta}^{\lambda} - c_{\mu^+, \beta}^{\lambda}), \quad (3.21)$$

where $\mu^+ = \mu + (0, 0, 1, 1) = (\mu_1, \mu_2, 1, 1)$.

So we need to understand the coefficients $c_{\mu, \beta}^{\lambda}$ and their sums over β with β' even.

Proposition 3.1.12. *Let λ, μ, β be partitions such that $\lambda \supseteq \mu, \lambda \supseteq \beta, |\lambda| = |\mu| + |\beta|, \ell(\lambda) \leq 4$ and β' is even (i.e., $\beta_1 = \beta_2$ and $\beta_3 = \beta_4$). Set $L_i = \mu_i + \beta_i - \lambda_i$, $1 \leq i \leq 4$, so that $\sum L_i = 0$.*

(i). *If $\mu_i = \mu_j$ or if $\lambda_i = \lambda_j$ for any $i \neq j$, then $c_{\mu, \beta}^{\lambda} \leq 1$.*

(ii). *Suppose $\mu_3 = \mu_4$. Then $c_{\mu, \beta}^{\lambda} = 1$ if and only if all of the following hold:*

$$(a). \quad 0 \leq -L_4 \leq L_2 \leq \mu_2 - \mu_3;$$

$$(b). \quad 0 \leq L_1 + L_4 \leq \mu_1 - \mu_2;$$

$$(c). \quad -L_3 \leq \beta_2 - \beta_4.$$

This will be proven in Section 3.3. As for the sums of these coefficients, the following result gives what we need.

Proposition 3.1.13. *Given a nonnegative integer M and a partition $\mu = (\mu_1, \mu_2)$ of length at most two, define*

$$B_{\mu, M} = \sum_{\substack{\lambda \vdash M \\ \ell(\lambda) \leq 4}} \sum_{\beta' \text{ even}} (c_{\mu, \beta}^{\lambda} - c_{\mu^+, \beta}^{\lambda}), \quad (3.22)$$

where $\mu^+ = \mu + (0, 0, 1, 1) = (\mu_1, \mu_2, 1, 1)$. Then we have

$$\sum_{N=0}^{\infty} B_{\mu, |\mu|+2N} X^N = \frac{(1 - X^{1+\mu_2})(1 - X^{1+\mu_1-\mu_2})}{(1 - X)^3(1 - X^2)} \quad (3.23)$$

as functions in X .

This will also be proved in Section 3.3. Note also that

$$B_{\mu, M} = 0 \text{ for } M \leq |\mu| \text{ or if } M - |\mu| \text{ is odd,} \quad (3.24)$$

since $B_{\mu, M}$ involves only those β with β' even, and since $c_{\mu, \beta}^{\lambda} = 0$ unless $|\lambda| - |\mu| = M - |\mu| = |\beta|$, which is certainly even if β' is. With these facts in hand, we can now prove

Theorem 3.1.14. *Let $s' = (s + 1)/2$. Then*

$$\Psi^*(s, W^{\circ}, f_s^{\circ}) = L(s', \pi, \rho_{\text{TT}}).$$

Proof. In the notation of Theorem 3.1.10, $\text{tr } \tau_{(k, l)}$ is the character of the finite-dimensional representation of $\text{Sp}_4(\mathbb{C})$ of highest weight $k\omega_1 + l(\omega_1 + \omega_2) = (k + l)\omega_1 + l\omega_2$ and so corresponds to the partition $\mu = (k + l, l)$. Thus if z_{π} is the Satake parameter

for π , thought of as a semisimple element in $\mathrm{Sp}_4(\mathbb{C})$ (since ρ_{TT} is trivial on similitudes), $\mathrm{tr} \tau_{(k,l)}(z_\pi) = \chi_{\mathrm{Sp}(4)}((k+l, l))(z_\pi) = \chi_{\mathrm{Sp}(4)}((k+l, l))$ (we will often suppress the z_π for brevity). Hence (3.10) becomes

$$\begin{aligned} \Psi^*(s, W^\circ, f_s^\circ) &= \sum_{k,l \geq 0} \chi_{\mathrm{Sp}(4)}((k+l, l)) X^{k+2l} \frac{(1 - X^{2(k+1)})(1 - X^{2(l+1)})}{(1 - X^2)^3(1 - X^4)} \\ &= \sum_{\mu: \ell(\mu) \leq 2} \chi_{\mathrm{Sp}(4)}(\mu) X^{|\mu|} \frac{(1 - X^{2(1+\mu_1-\mu_2)})(1 - X^{2(1+\mu_2)})}{(1 - X^2)^3(1 - X^4)} \\ &= \sum_{\mu: \ell(\mu) \leq 2} \chi_{\mathrm{Sp}(4)}(\mu) X^{|\mu|} \left(\sum_{N=0}^{\infty} B_{\mu, |\mu|+2N} X^{2N} \right). \end{aligned}$$

Now the inner sum is, by (3.24),

$$\begin{aligned} \sum_{N=0}^{\infty} B_{\mu, |\mu|+2N} X^{2N} &= \sum_{N=0}^{\infty} B_{\mu, |\mu|+N} X^N = \sum_{N=|\mu|}^{\infty} B_{\mu, N} X^{N-|\mu|} \\ &= \sum_{N=|\mu|}^{\infty} \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq 4}} \sum_{\beta' \text{ even}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) X^{|\lambda|-|\mu|} \\ &= \sum_{\substack{\lambda: \lambda \supseteq \mu \\ \ell(\lambda) \leq 4}} \sum_{\beta' \text{ even}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) X^{|\lambda|-|\mu|}, \end{aligned}$$

so substituting into the above and exchanging summation of λ and μ gives

$$\begin{aligned} \Psi^*(s, W^\circ, f_s^\circ) &= \sum_{\mu: \ell(\mu) \leq 2} \chi_{\mathrm{Sp}(4)}(\mu) X^{|\mu|} \sum_{\substack{\lambda: \lambda \supseteq \mu \\ \ell(\lambda) \leq 4}} \sum_{\beta' \text{ even}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) X^{|\lambda|-|\mu|} \\ &= \sum_{\lambda: \ell(\lambda) \leq 4} X^{|\lambda|} \sum_{\substack{\mu: \mu \subseteq \lambda \\ \ell(\mu) \leq 2}} \chi_{\mathrm{Sp}(4)}(\mu)(z_\pi) \sum_{\beta' \text{ even}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) \\ &= \sum_{\lambda: \ell(\lambda) \leq 4} \chi_{\mathrm{GL}(4)}(\lambda)(z_\pi) X^{|\lambda|}, \end{aligned}$$

the last equality following from (3.21). Recall $X = q^{-s'}$ and $s' = (s+1)/2$. Therefore the last line is just

$$\sum_{\lambda: \ell(\lambda) \leq 4} \chi_{\text{GL}(4)}(\lambda)(z_\pi) q^{-s'|\lambda|} = L(s', \pi, \rho_{\text{TT}})$$

as we wanted. □

3.2 Split case

Now suppose v is split in E , so we have $E_v \simeq F_v \oplus F_v$. We show two things here: first, that our integral agrees with the local integral found in [24]; second, that this integral actually calculates the correct L -factor. As mentioned above, Jiang accomplishes this in his thesis by using the Kostant-Rallis algorithm. We give a new proof using the branching laws stated in the previous section. We wish to thank Professor Jiang for the suggestion to carry out this computation.

In order to clarify what follows, we make the above isomorphism explicit. Since v splits in E , we have

$$E_v \stackrel{\text{def}}{=} F_v \otimes_F E \xrightarrow[\sim]{r} F_v \oplus F_v. \quad (3.25)$$

Since E_v is spanned by the elements $a \otimes 1$ for $a \in F_v$ and $1 \otimes \theta$, the isomorphism r can be described by

$$r: \begin{cases} a \otimes 1 \mapsto (a, a), & a \in F_v, \\ 1 \otimes \theta \mapsto (\theta_v, -\theta_v), \end{cases}$$

since $\theta_v \in F$. If we denote the usual conjugation mappings by $\alpha \mapsto \bar{\alpha}$, i.e.,

$$\begin{aligned}\alpha = a + b\theta &\mapsto \bar{\alpha} = a - b\theta, & \alpha \in E_v, \\ \beta = (a, b) &\mapsto \bar{\beta} = (b, a), & \beta \in F_v \oplus F_v,\end{aligned}$$

then the isomorphism r is compatible with these conjugations:

$$\begin{aligned}\overline{r(a + b\theta)} &= \overline{(a + b\theta_v, a - b\theta_v)} = (a - b\theta_v, a + b\theta_v) \\ &= r(a - b\theta) = r(\overline{a + b\theta}).\end{aligned}$$

So we may drop the subscript v and unambiguously refer to “conjugation with respect to θ ,” bearing in mind that under $E \simeq F \oplus F$ we have $\theta \leftrightarrow (\theta, -\theta)$.

3.2.1 The Split Integral

In this subsection only, we allow F to be archimedean as well, so we consider the cases F nonarchimedean, $F = \mathbb{R}$ and $F = \mathbb{C}$. We will treat all cases simultaneously and note any additional arguments as they are needed.

Since E/F is split, $\mathrm{GSp}_4(E) \simeq \mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F)$. More explicitly, consider the commutative diagram

$$\begin{array}{ccc} G_0(F) = \mathrm{GSp}_4(E)^\circ & \xrightarrow[\sim]{\sigma} & \tilde{G}_0(F) = (\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ \mathrm{GSp}_8(F) & \xrightarrow[\sim]{\tau} & \mathrm{GSp}_8(F) \end{array}$$

where

$$(\mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F))^\circ = \{(g_1, g_2) | g_i \in \mathrm{GSp}_4(F), \lambda(g_1) = \lambda(g_2)\}$$

and ι_1 is defined by (2.2), ι_2 is the map

$$\iota_2\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\right) = \begin{pmatrix} A & B & -B' \\ C & A' & D \\ -C' & D' & \end{pmatrix} \quad (3.26)$$

and σ and τ are isomorphisms defined by

$$\sigma : \begin{cases} \mathrm{GSp}_4(E) \rightarrow \mathrm{GSp}_4(F) \times \mathrm{GSp}_4(F), \\ g \mapsto \left(g, \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \bar{g} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}\right) \end{cases}$$

and

$$\tau(h) = C_\theta^{-1} h C_\theta,$$

where

$$C_\theta = \begin{pmatrix} Y_\theta & \\ & T_\theta^{-1} Y_\theta \end{pmatrix},$$

$$Y_\theta = \begin{pmatrix} 1 & & \\ \theta & 1 & \\ & \theta & 1 \end{pmatrix}, \quad T_\theta = \mathrm{diag}(1, \theta^2, 1, \theta^2)$$

and $g \mapsto \bar{g}$ is conjugation with respect to θ . Note that since θ is a unit, $C_\theta \in \mathrm{GSp}_8(\mathcal{O})$.

(If F is archimedean, it is clear that $C_\theta \in \mathrm{O}_8(F)$.) It is easy to check that

$$(\tau \circ \iota_1)(g) = (\iota_2 \circ \sigma)(g), \quad \text{for } g \in G_0(F).$$

Now π is an irreducible admissible representation of $\mathrm{GSp}_4(E)$, so it decomposes under σ as $\pi \simeq \pi \circ \sigma^{-1} = \pi_1 \otimes \pi_2$, where π_1, π_2 are irreducible admissible representations of $\mathrm{GSp}_4(F)$, and if π is spherical, π_1, π_2 will be as well. Let W_1°, W_2° be the spherical

Whittaker functions for π_1, π_2 respectively, normalized so that $W_1^\circ(1) = 1 = W_2^\circ(1)$.

Then we have a similar decomposition for the spherical Whittaker function W° of π :

$$W^\circ(\sigma^{-1}(g_1, g_2)) = W_1^\circ(g_1)W_2^\circ(g_2).$$

Jiang's [24, I.2.2.1] local, unramified integral is

$$\tilde{\Psi}(s, W_1^\circ, W_2^\circ, f_s^\circ) = \int_{\tilde{\mathcal{D}}(F)} f_s^\circ(\tilde{\gamma}_0 \iota_2(g_1, g_2)) W_1^\circ(g_1) W_2^\circ(g_2) d(g_1, g_2) \quad (3.27)$$

where

$$\tilde{\mathcal{D}}(F) = Z(F)N^{P_4^\Delta}(F)(X_2^{P_4}(F) \times I_4) \backslash \tilde{G}_0(F), \quad (3.28)$$

$$P_4^\Delta(F) = \{(g, g) | g \in P_4(F)\}, \quad (3.29)$$

and

$$\tilde{\gamma}_0 = \begin{pmatrix} 1 & & & 0 & & \\ & 1 & & 0 & & \\ & & 0 & & 0 & 1 \\ 0 & & & 1 & & \\ & 0 & & & 1 & \\ & 1 & -1 & & 0 & 1 \\ & & & & & 0 \end{pmatrix}. \quad (3.30)$$

Recall our local integral is

$$\Psi(s, W^\circ, f_s^\circ) = \int_{\mathcal{D}(F)} f_s(\gamma_0 \iota_1(g)) W^\circ(g) dg \quad (3.31)$$

where

$$\mathcal{D}(F) = Z(F)N^Q(F)X_2^{Q'}(F) \backslash G_0(F) \quad (3.32)$$

and γ_0 is given by (2.9). To see this integral is the same as (3.27), we first note that

$Q(F) \subseteq \{(\begin{smallmatrix} A & B\theta \\ C\theta & D \end{smallmatrix})\}$, so that

$$\sigma(Q(F)) \subseteq \left\{ \left(\begin{pmatrix} A & B\theta \\ C\theta & D \end{pmatrix}, \begin{pmatrix} A & B\theta \\ C\theta & D \end{pmatrix} \right) \right\}.$$

Since we clearly have $\sigma(Q(F)) \subseteq P_4(F) \times P_4(F)$, it follows that

$$\sigma(Q(F)) = P_4^\Delta(F) \Rightarrow \sigma(N^Q(F)) = N^{P_4^\Delta}(F).$$

It is equally clear that $\sigma(X_2^Q(F)) = X_2^{P_4^\Delta}(F) \stackrel{\text{def}}{=} \{(w, w) \mid w \in X_2^{P_4}(F)\}$. For $\alpha \in E$, let

$$w(\alpha) = \chi_{2\epsilon_2}(\alpha) = \begin{pmatrix} 1 & \alpha \\ & 1 \\ & & 1 \end{pmatrix} \in X_2^{P_4}(E);$$

then we may write

$$\begin{aligned} \sigma(w(\alpha)) &= (w(\alpha), w(-\bar{\alpha})) \\ &= (w(-\bar{\alpha}), w(-\bar{\alpha})) \cdot (w(\alpha + \bar{\alpha}), 1) \\ &\in X_2^{P_4^\Delta}(F) \cdot (X_2^{P_4}(F) \times I_4), \end{aligned}$$

so that

$$\sigma(X_2^{P_4}(E)) = X_2^{P_4^\Delta}(F) \cdot (X_2^{P_4}(F) \times I_4)$$

and

$$\sigma(N^Q(F)X_2^{Q'}(F)) = N^{P_4^\Delta}(F) \cdot (X_2^{P_4}(F) \times I_4),$$

which gives

$$\sigma(\mathcal{D}(F)) = \tilde{\mathcal{D}}(F).$$

Also note that $\gamma_0 C_\theta = p_\theta \tilde{\gamma}_0$, where

$$p_\theta = \begin{pmatrix} 2 & & & & & & & \\ & 2 & & & & & & \\ & & -2\theta^{-1} & & & & & \\ & & & -2\theta^{-1} & & & & \\ & & & & \theta^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & -\theta^{-1} \\ & & & & & & & & -\theta^{-1} \end{pmatrix}$$

is in $P_8(F)$, and $\delta_{P_8}(p_\theta) = 2^9\theta^{-6}$ is a unit since θ is a unit and the residue characteristic of F is not 2. Hence $f_s^\circ(p_\theta) = 1$.

Finally, we normalize the measures on $\mathcal{D}(F)$ and $\tilde{\mathcal{D}}(F)$ compatibly:

$$\int_{\mathcal{D}(F)} \phi(g) dg = \int_{\tilde{\mathcal{D}}(F)} \phi(\sigma^{-1}(g_1, g_2)) d(g_1, g_2)$$

for absolutely integrable functions ϕ on $\mathcal{D}(F)$.

Proposition 3.2.1. *Notation as above, we have*

$$\Psi(s, W^\circ, f_s^\circ) = \tilde{\Psi}(s, W_1^\circ, W_2^\circ, f_s^\circ).$$

Proof.

$$\begin{aligned} \Psi(s, W^\circ, f_s^\circ) &= \int_{\mathcal{D}(F)} f_s^\circ(\gamma_0 \iota_1(g)) W^\circ(g) dg \\ &= \int_{\sigma(\mathcal{D}(F))} f_s^\circ(\gamma_0 \iota_1 \circ \sigma^{-1}(g_1, g_2)) W^\circ(\sigma^{-1}(g_1, g_2)) d(g_1, g_2) \\ &= \int_{\tilde{\mathcal{D}}(F)} f_s^\circ(\gamma_0 \tau^{-1} \circ \iota_2(g_1, g_2)) W_1^\circ(g_1) W_2^\circ(g_2) d(g_1, g_2) \\ &= \int_{\tilde{\mathcal{D}}(F)} f_s^\circ(\gamma_0 C_\theta \iota_2(g_1, g_2) C_\theta^{-1}) W_1^\circ(g_1) W_2^\circ(g_2) d(g_1, g_2) \\ &= \int_{\tilde{\mathcal{D}}(F)} f_s^\circ(p_\theta \tilde{\gamma}_0 \iota_2(g_1, g_2)) W_1^\circ(g_1) W_2^\circ(g_2) d(g_1, g_2) \\ &= \int_{\tilde{\mathcal{D}}(F)} f_s^\circ(\tilde{\gamma}_0 \iota_2(g_1, g_2)) W_1^\circ(g_1) W_2^\circ(g_2) d(g_1, g_2) \\ &= \tilde{\Psi}(s, W_1^\circ, W_2^\circ, f_s^\circ). \end{aligned}$$

□

Jiang calculates the integral in [24].

Theorem 3.2.2 ([24], I.3.3.2). *Let $X = q^{-\frac{s+1}{2}}$. Then the normalized unramified [split] local integral is*

$$\begin{aligned}
\Psi^*(s, W_1^\circ, W_2^\circ, f_s^\circ) &= d_{G,v}(s) \Psi(s, W_1^\circ, W_2^\circ, f_s^\circ) \\
&= \frac{1}{(1-X^2)^3(1-X^4)} \\
&\quad \cdot \left\{ \sum_{\substack{k_1+k_2 \geq l_1+l_2 \geq k_1 \\ 2l_1 \geq k_1 \geq l_1}} \text{tr } \tau_{(k_1, k_2 | 2l_1 - k_1, l_2)}(x_{\pi_1}, y_{\pi_2}) P_1(k_1, k_2, l_1, l_2; X) \right. \\
&\quad + \sum_{\substack{k_1+k_2 \geq l_1+l_2 \\ l_1 > k_1}} \text{tr } \tau_{(k_1, k_2 | 2l_1 - k_1, l_2)}(x_{\pi_1}, y_{\pi_2}) P_2(k_1, k_2, l_1, l_2; X) \\
&\quad + \sum_{\substack{k_1+k_2 < l_1+l_2 \\ 2l_1 \geq k_1 \geq l_1}} \text{tr } \tau_{(k_1, k_2 | 2l_1 - k_1, l_2)}(x_{\pi_1}, y_{\pi_2}) P_3(k_1, k_2, l_1, l_2; X) \\
&\quad \left. + \sum_{\substack{l_1 \leq k_1+k_2 < l_1+l_2 \\ l_1 > k_1}} \text{tr } \tau_{(k_1, k_2 | 2l_1 - k_1, l_2)}(x_{\pi_1}, y_{\pi_2}) P_4(k_1, k_2, l_1, l_2; X) \right\}, \tag{3.33}
\end{aligned}$$

where $d_{G,v}(s) = \zeta_v(2s+2)\zeta_v(s+1)\zeta_v(s+2)\zeta_v(s+3)$ is the local component of the normalizing factor for the Eisenstein series, $\text{tr } \tau_{(a_1, a_2 | b_1, b_2)}(x_{\pi_1}, y_{\pi_2})$ is the character of the tensor product of the finite-dimensional representations of $\text{Sp}_4(\mathbb{C})$ of highest weights $a_1\omega_1 + a_2(\omega_1 + \omega_2)$ and $b_1\omega_1 + b_2(\omega_1 + \omega_2)$, evaluated at the Satake parameter $(x_{\pi_1}, y_{\pi_2}) \in (\text{GSp}_4(\mathbb{C}) \times \text{GSp}_4(\mathbb{C}))^\circ$ for $\pi_1 \otimes \pi_2$, and

$$\begin{aligned}
P_1(k_1, k_2, l_1, l_2, X) &= X^{3k_1+2k_2-2l_1} (1 - X^{2(1+l_1+l_2-k_1)}) (1 - X^{2(1+2l_1-k_1)}) \\
P_2(k_1, k_2, l_1, l_2, X) &= X^{2k_2-k_1+2l_1} (1 - X^{2(1+l_2)}) (1 - X^{2(1+k_1)}) \\
P_3(k_1, k_2, l_1, l_2, X) &= X^{k_1+2l_2} (1 - X^{2(1+k_2)}) (1 - X^{2(1+2l_1-k_1)}) \\
P_4(k_1, k_2, l_1, l_2, X) &= X^{4l_1+2l_2-3k_1} (1 - X^{2(1+k_1+k_2-l_1)}) (1 - X^{2(1+k_1)}).
\end{aligned}$$

It will be useful in the next section to rephrase (3.33) in terms of partitions. With all notation as in the previous theorem, set

$$\mu = (\mu_1, \mu_2) = (k_1 + k_2, k_2), \quad \nu = (\nu_1, \nu_2) = (2l_1 - k_1 + l_2, l_2),$$

so that μ and ν are partitions, provided $2l_1 \geq k_1$. Let $\chi_{\mathrm{Sp}(4)}(\mu)$ be the character of the finite-dimensional representation of $\mathrm{Sp}_4(\mathbb{C})$ with highest weight $\mu_1\omega_1 + \mu_2\omega_2 = k_1\omega_1 + k_2(\omega_1 + \omega_2)$, and similarly for $\chi_{\mathrm{Sp}(4)}(\nu)$. So if the Satake parameter for $\pi_1 \otimes \pi_2$ is represented by the semisimple element $(x_{\pi_1}, y_{\pi_2}) \in (\mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C}))^\circ$, then

$$\mathrm{tr} \tau_{(k_1, k_2 | 2l_1 - k_1, l_2)}(x_{\pi_1}, y_{\pi_2}) = \chi_{\mathrm{Sp}(4)}(\mu)(x_{\pi_1}) \chi_{\mathrm{Sp}(4)}(\nu)(y_{\pi_2}).$$

Define $\delta_\mu = \mu_1 - \mu_2$, $\delta_\nu = \nu_1 - \nu_2$ and

$$\begin{aligned} M_1 &= \max\{|\mu|, |\nu|\}, & M_2 &= \min\{|\mu|, |\nu|\}, \\ m_1 &= \max\{\delta_\mu, \delta_\nu\}, & m_2 &= \min\{\delta_\mu, \delta_\nu\}. \end{aligned}$$

Substituting μ, ν into (3.33) yields

$$\begin{aligned} \Psi^*(s, W_1^\circ, W_2^\circ, f_s^\circ) &= \frac{1}{(1 - X^2)^3(1 - X^4)} \\ &\quad \cdot \sum_{i=1}^4 \sum_{\mu, \nu \in \mathcal{A}_i} \chi_{\mathrm{Sp}(4)}(\mu)(x_{\pi_1}) \chi_{\mathrm{Sp}(4)}(\nu)(y_{\pi_2}) P_i(\mu, \nu; X), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \{\mu, \nu \text{ partitions} : |\mu| \geq |\nu| \geq \delta_\mu \geq \delta_\nu\} \\ \mathcal{A}_2 &= \{\mu, \nu \text{ partitions} : |\mu| \geq |\nu| \geq \delta_\nu > \delta_\mu\} \\ \mathcal{A}_3 &= \{\mu, \nu \text{ partitions} : |\nu| > |\mu| \geq \delta_\mu \geq \delta_\nu\} \\ \mathcal{A}_4 &= \{\mu, \nu \text{ partitions} : |\nu| > |\mu| \geq \delta_\nu > \delta_\mu\} \end{aligned}$$

(which are disjoint sets) and

$$\begin{aligned}
P_1(\mu, \nu; X) &= X^{|\mu|+\delta_\mu-\delta_\nu} (1 - X^{2(1+\frac{1}{2}(|\nu|-\delta_\mu))}) (1 - X^{2(1+\delta_\nu)}) \\
P_2(\mu, \nu; X) &= X^{|\mu|-\delta_\mu+\delta_\nu} (1 - X^{2(1+\frac{1}{2}(|\nu|-\delta_\nu))}) (1 - X^{2(1+\delta_\mu)}) \\
P_3(\mu, \nu; X) &= X^{|\nu|+\delta_\mu-\delta_\nu} (1 - X^{2(1+\frac{1}{2}(|\mu|-\delta_\mu))}) (1 - X^{2(1+\delta_\nu)}) \\
P_4(\mu, \nu; X) &= X^{|\nu|-\delta_\mu+\delta_\nu} (1 - X^{2(1+\frac{1}{2}(|\mu|-\delta_\nu))}) (1 - X^{2(1+\delta_\mu)}).
\end{aligned}$$

So in terms of M_1, M_2, m_1, m_2 , we have

$$\Psi^*(s, W_1^\circ, W_2^\circ, f_s^\circ) = \sum_{\substack{\mu, \nu: M_2 \geq m_1 \\ \ell(\mu), \ell(\nu) \leq 2}} \chi_{\mathrm{Sp}(4)}(\mu)(x_{\pi_1}) \chi_{\mathrm{Sp}(4)}(\nu)(y_{\pi_2}) X^{M_1} P(\mu, \nu; X^2), \quad (3.34)$$

where

$$P(\mu, \nu; X) = \frac{X^{\frac{1}{2}(m_1-m_2)} (1 - X^{1+\frac{1}{2}(M_2-m_1)}) (1 - X^{1+m_2})}{(1 - X)^3 (1 - X^2)}. \quad (3.35)$$

3.2.2 Local L -factor

We now identify the integral (3.34) with the local (split, unramified) L -factor for π .

The method is completely analogous to that used in the inert case.

Since E/F is split, the L -group of G_0 is also split:

$${}^L G_0 = (\mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C}))^\circ \times \Gamma_F$$

where, as before, Γ_F is the absolute Galois group of F .

Our representation π also decomposes as a tensor product $\pi = \pi_1 \otimes \pi_2$, where π_1 and π_2 are irreducible admissible generic representations of $\mathrm{GSp}_4(F)$, the product of whose central characters is trivial. Since the Frobenius element is trivial here, the

Satake parameter for π will be a semisimple conjugacy class in ${}^L G_0$ represented, say, by

$$(x_{\pi_1}, y_{\pi_2}) \in (\mathrm{GSp}_4(\mathbb{C}) \times \mathrm{GSp}_4(\mathbb{C}))^\circ. \quad (3.36)$$

The representation ρ_{TT} is now the tensor product $\rho \otimes \rho$ as defined by (3.12), where ρ is the standard representation of $\mathrm{Sp}_4(\mathbb{C})$, extended trivially on similitudes. So the local L -factor is just

$$L(s, \pi_1 \times \pi_2, \rho \otimes \rho) = \prod_{1 \leq i, j \leq 4} (1 - x_i y_j q^{-s})^{-1}. \quad (3.37)$$

In [34], we find on p. 63 the identity

$$\sum_{\lambda: \ell(\lambda) \leq n} s_\lambda(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n) = \prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j)^{-1}. \quad (3.38)$$

Take $n = 4$ and recall that s_λ is homogeneous of degree $|\lambda|$ (so $s_\lambda(q^{-s}y_1, \dots, q^{-s}y_4) = q^{-s|\lambda|} s_\lambda(y_1, \dots, y_4)$, for example). Let $x_{\pi_1} = (x_1, \dots, x_4)$ and $y_{\pi_2} = (y_1, \dots, y_4)$ be the Satake parameters for π_1 and π_2 , respectively. Then writing (3.38) in terms of characters yields

$$L(s, \pi_1 \times \pi_2, \rho \otimes \rho) = \sum_{\lambda: \ell(\lambda) \leq 4} \chi_{\mathrm{GL}(4)}(\lambda)(x_{\pi_1}) \chi_{\mathrm{GL}(4)}(\lambda)(y_{\pi_2}) q^{-s|\lambda|}. \quad (3.39)$$

So we may again apply our branching rule (3.21) to relate the L -factor to characters of $\mathrm{Sp}_4(\mathbb{C})$, and via a combinatorial fact to the integral above. The relevant identity here is the following.

Proposition 3.2.3. *Given a nonnegative integer N and partitions $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ of length at most two, set $\delta_\mu = \mu_1 - \mu_2$, $\delta_\nu = \nu_1 - \nu_2$ and*

$$M_1 = \max\{|\mu|, |\nu|\}, \quad M_2 = \min\{|\mu|, |\nu|\},$$

$$m_1 = \max\{\delta_\nu, \delta_\mu\}, \quad m_2 = \min\{\delta_\mu, \delta_\nu\},$$

and define

$$D_{\mu, \nu, N} = \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq 4}} \sum_{\substack{\beta' \text{ even} \\ \alpha' \text{ even}}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) (c_{\nu, \alpha}^\lambda - c_{\nu^+, \alpha}^\lambda). \quad (3.40)$$

Then we have

$$\sum_{N=0}^{\infty} D_{\mu, \nu, 2N+M_1} X^N = \frac{X^{\frac{1}{2}(m_1-m_2)} (1 - X^{1+m_2}) (1 - X^{1+\frac{1}{2}(M_2-m_1)})}{(1 - X)^3 (1 - X^2)} \quad (3.41)$$

as functions in X .

Remark 3.2.4. (i). It is easy to see from the definition that $D_{\mu, \nu, N} \neq 0$ implies $N - |\mu|$ and $N - |\nu|$ are both even and $N \geq |\mu|, |\nu|$. Hence $D_{\mu, \nu, 2N+M_1}$ is zero unless $M_1 \equiv M_2 \pmod{2}$. Since $|\mu| \equiv \delta_\mu \pmod{2}$ and the analogous statement holds for ν , we actually have $D_{\mu, \nu, 2N+M_1} = 0$ unless $M_1 \equiv M_2 \equiv m_1 \equiv m_2 \pmod{2}$. So the statement (3.41) is well-defined, that is, the exponents on the right-hand side are all integral.

(ii). We will show later that $D_{\mu, \nu, 2N+M_1} = 0$ for $M_2 < m_1$ (see (3.51)) and that

$$D_{\mu, \nu, 2N+M_1} = 0 \text{ for } N < \frac{1}{2}(m_1 - m_2) \text{ (see (3.52)).}$$

(iii). The right-hand side of (3.41) is exactly $P(\mu, \nu; X)$ as in (3.35).

The proof will be given in Section 3.3. Now we may state the main result.

Theorem 3.2.5. *Let $s' = (s + 1)/2$. Then*

$$\Psi^*(s, W_1^\circ, W_2^\circ, f_s^\circ) = L(s', \pi_1 \times \pi_2, \rho \otimes \rho).$$

Proof. By (3.41), (3.35), Remark 3.2.4(i) and definition of $D_{\mu, \nu, 2N+M_1}$,

$$\begin{aligned} P(\mu, \nu; X^2) &= \sum_{N=0}^{\infty} D_{\mu, \nu, 2N+M_1} X^{2N} \\ &= \sum_{N=0}^{\infty} D_{\mu, \nu, N+M_1} X^N = \sum_{N=M_1}^{\infty} D_{\mu, \nu, N} X^{N-M_1} \\ &= \sum_{N=M_1}^{\infty} X^{N-M_1} \sum_{\substack{\lambda \vdash N \\ \ell(\lambda) \leq 4}} \sum_{\substack{\beta' \text{ even} \\ \alpha' \text{ even}}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) (c_{\nu, \alpha}^\lambda - c_{\nu^+, \alpha}^\lambda) \\ &= \sum_{\substack{\lambda \supseteq \mu, \nu \\ \ell(\lambda) \leq 4}} X^{|\lambda|-M_1} \sum_{\substack{\beta' \text{ even} \\ \alpha' \text{ even}}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) (c_{\nu, \alpha}^\lambda - c_{\nu^+, \alpha}^\lambda), \end{aligned}$$

so that by (3.34),

$$\begin{aligned} \Psi^*(s, W_1^\circ, W_2^\circ, f_s^\circ) &= \sum_{\substack{\mu, \nu: M_2 \geq m_1 \\ \ell(\mu), \ell(\nu) \leq 2}} \chi_{\text{Sp}(4)}(\mu)(x_{\pi_1}) \chi_{\text{Sp}(4)}(\nu)(y_{\pi_2}) X^{M_1} P(\mu, \nu; X^2) \\ &= \sum_{\substack{\mu, \nu \\ \ell(\mu), \ell(\nu) \leq 2}} \chi_{\text{Sp}(4)}(\mu)(x_{\pi_1}) \chi_{\text{Sp}(4)}(\nu)(y_{\pi_2}) \\ &\quad \cdot \sum_{\substack{\lambda \supseteq \mu, \nu \\ \ell(\lambda) \leq 4}} X^{|\lambda|} \sum_{\substack{\beta' \text{ even} \\ \alpha' \text{ even}}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) (c_{\nu, \alpha}^\lambda - c_{\nu^+, \alpha}^\lambda) \\ &= \sum_{\lambda: \ell(\lambda) \leq 4} X^{|\lambda|} \left(\sum_{\mu: \ell(\mu) \leq 2} \chi_{\text{Sp}(4)}(\mu)(x_{\pi_1}) \sum_{\beta' \text{ even}} (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda) \right) \\ &\quad \cdot \left(\sum_{\nu: \ell(\nu) \leq 2} \chi_{\text{Sp}(4)}(\nu)(y_{\pi_2}) \sum_{\alpha' \text{ even}} (c_{\nu, \alpha}^\lambda - c_{\nu^+, \alpha}^\lambda) \right) \\ &= \sum_{\lambda: \ell(\lambda) \leq 4} \chi_{\text{GL}(4)}(\lambda)(x_{\pi_1}) \chi_{\text{GL}(4)}(\lambda)(y_{\pi_2}) q^{-s'|\lambda|} \quad (\text{by (3.21)}) \end{aligned}$$

$$= L(s', \pi_1 \times \pi_2, \rho \otimes \rho) \quad (\text{by (3.39)})$$

which is the desired result. $\square$

3.3 Proof of combinatorial lemmas

This section is devoted to proving Prop. 3.1.12, Prop. 3.1.13 and Prop. 3.2.3. Before giving the proofs, it will be convenient to cast the Littlewood-Richardson rule in a slightly different language.

Fix partitions λ, μ, β satisfying $\lambda \supseteq \mu$, $\lambda \supseteq \beta$ and $|\lambda| = |\mu| + |\beta|$. Call a tableau T of skew-shape λ/μ and weight β whose word is a lattice permutation, an *LR-tableau*. To such a tableau we may associate an integral matrix $A = (a_{ij}) = (a_{ij}(T))$, where a_{ij} is the number of symbols j appearing in row i of T . Because the word of T is a lattice permutation, the symbol j cannot appear until at least row j , starting from the top, so $A = A(T)$ is lower triangular. We can similarly translate all the conditions on an LR-tableau into conditions on such matrices:

Lemma 3.3.1. *For λ, μ, β as above, set $n = \ell(\lambda)$. Then $c_{\beta, \mu}^\lambda = c_{\mu, \beta}^\lambda$ is the number of $n \times n$ integral matrices $A = (a_{i,j})$ with nonnegative entries which are lower triangular and satisfy the following inequalities:*

- (i). $\sum_j a_{i,j} = \lambda_i - \beta_i$ and $\sum_i a_{i,j} = \mu_j$;
- (ii). $\sum_{i=1}^k a_{i,j} \leq \sum_{i=1}^{k-1} a_{i,j-1}$ for $k \geq 1, j \geq 2$ (lattice permutation);
- (iii). $\beta_i + \sum_{j=1}^k a_{i,j} \leq \beta_{i-1} + \sum_{j=1}^{k-1} a_{i-1,j}$ for $k \geq 1, i \geq 2$ (column-strictness). $\square$

Proof of Prop. 3.1.12. Let $A = (a_{ij})$ be a matrix as in the above lemma. Note that β' is even, so $\beta_1 = \beta_2$ and $\beta_3 = \beta_4$. The column-strict condition with $k = 1$ and $i = 2, 4$ respectively shows that $a_{21} = 0 = a_{41}$. So our matrix has the following form:

$$\begin{array}{cccc} \lambda_1 - \beta_2 & \left(\begin{array}{cccc} a_{11} & & & \\ 0 & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ 0 & a_{42} & a_{43} & a_{44} \end{array} \right) & & \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array}.$$

Then we get

$$\begin{aligned} a_{11} &= \lambda_1 - \beta_2, \\ a_{31} &= \mu_1 - a_{11} = \mu_1 - \lambda_1 + \beta_2 = L_1, \\ a_{22} &= \lambda_2 - \beta_2, \\ a_{44} &= \mu_4. \end{aligned}$$

The remaining four entries satisfy

$$\begin{aligned} a_{32} + a_{33} &= \lambda_3 - \beta_4 - a_{31} \\ a_{42} + a_{43} &= \lambda_4 - \beta_4 - \mu_4 = -L_4 \\ a_{32} + a_{42} &= \mu_2 - \lambda_2 + \beta_2 = L_2 \\ a_{33} + a_{43} &= \mu_3, \end{aligned}$$

and one can check that the last equation is a consequence of the first three.

Now set $a = a_{42} \geq 0$. Then

$$a_{43} = -L_4 - a \geq 0$$

$$a_{32} = L_2 - a \geq 0$$

$$a_{33} = \mu_3 - a_{43} = \mu_3 + L_4 + a \geq 0.$$

Now we write down all the inequalities forced on the a_{ij} in terms of a :

$$\left. \begin{array}{l} L_1 + \mu_2 - \mu_1 = L_2 + \lambda_2 - \lambda_1 \\ -L_3 + \lambda_4 - \lambda_3 = -L_4 + \mu_4 - \mu_3 \\ L_1 + L_2 + \beta_4 - \beta_2 \\ 0 \end{array} \right\} \leq a \leq \left\{ \begin{array}{l} L_1 \\ L_2 \\ -L_3 \\ -L_4 \\ L_1 + L_2 + \lambda_2 - \lambda_3 + \beta_4 - \beta_2 \\ = L_1 + L_3 + \mu_2 - \mu_3 \\ \mu_2 - \mu_3. \end{array} \right. \quad (3.42)$$

So if any $\lambda_i = \lambda_j$ or $\mu_i = \mu_j$ for $i \neq j$, then a and hence all the a_{ij} will be completely determined, so $c_{\mu,\beta}^\lambda \leq 1$. This proves part (i).

Now suppose $\mu_3 = \mu_4$. Then (3.42) gives $a = -L_4$, so applying the previous inequalities, eliminating redundancies and using that $\sum_i L_i = 0$ gives what we want.¹ $\square$

Before giving the next proof, it will be a helpful reference to rephrase Prop. 3.1.12 for the particular cases we need.

¹Specifically, the redundant inequalities are $a = -L_4 \leq \mu_2 - \mu_3$, $a = -L_4 \leq -L_3$, $-L_4 + \mu_4 - \mu_3 \leq a \leq -L_4$.

Corollary 3.3.2. *Assume all the hypotheses of Prop. 3.1.12.*

(i). *If $\mu_3 = \mu_4 = 0$, then $c_{\mu,\beta}^\lambda = 1$ if and only if the following hold:*

$$(a). \lambda_2 - \mu_2 + \lambda_4 \leq \beta_2 + \beta_4;$$

$$(b). \lambda_1 - \mu_1 + \lambda_4 \leq \beta_2 + \beta_4 \leq \lambda_1 - \mu_2 + \lambda_4;$$

$$(c). \lambda_3 \leq \beta_2.$$

(ii). *If $\mu_3 = \mu_4 = 1$, then $c_{\mu,\beta}^\lambda = 1$ if and only if the following hold:*

$$(a). \lambda_2 - \mu_2 + \lambda_4 - 1 \leq \beta_2 + \beta_4;$$

$$(b). \lambda_1 - \mu_1 + \lambda_4 - 1 \leq \beta_2 + \beta_4 \leq \lambda_1 - \mu_2 + \lambda_4 - 1;$$

$$(c). \lambda_3 \leq \beta_2 + 1;$$

$$(d). \lambda_2 - \beta_2 \geq 1 \text{ and } \lambda_4 - \beta_4 \geq 1. \quad \square$$

Remark 3.3.3. Consider the inequalities (a) and (b) above, under the same hypotheses but $\mu_3 = \mu_4$ arbitrary. Since

$$\beta_2 + \beta_4 = \beta_1 + \beta_3 = \frac{1}{2} |\beta| = \frac{1}{2} (|\lambda| - |\mu|),$$

the statements are equivalent, respectively, to

$$\lambda_2 + \lambda_4 + \delta \leq \lambda_1 + \lambda_3, \tag{3.43}$$

$$\lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3 + \delta \leq \lambda_1 + \lambda_4 + 2\delta, \tag{3.44}$$

independent of the value of $\mu_3 = \mu_4$.

Figure 3.4: The partitions λ , β and μ

Recall that in order to calculate both $B_{\mu, |\mu|+2N}$ and $D_{\mu, \nu, 2N+M_1}$ we need to know sums of the form $\sum (c_{\mu, \beta}^\lambda - c_{\mu^+, \beta}^\lambda)$, where the sum is over β' even and $\mu^+ = \mu + (0, 0, 1, 1)$. We know that the coefficient $c_{\mu, \beta}^\lambda$ is zero unless $\lambda \supseteq \mu, \beta$ and $|\lambda| = |\mu| + |\beta|$. Suppose the triple (λ, μ, β) satisfies the hypotheses of Prop. 3.1.12. Then we may think of λ as being built up from β by adding a_1 blocks to each of rows 1 and 2, a_2 blocks to each of rows 3 and 4, b_1 blocks to row 1 and b_2 blocks to row 3 (see Figure 3.4). Then (recall $\beta_1 = \beta_2$ and $\beta_3 = \beta_4$)

$$\lambda_1 = \beta_2 + a_1 + b_1$$

$$\lambda_2 = \beta_2 + a_1$$

$$\lambda_3 = \beta_4 + a_2 + b_2$$

$$\lambda_4 = \beta_4 + a_2.$$

So we may translate the conditions of Prop. 3.1.12(ii) on λ , μ and β into conditions

on a_1, a_2, b_1, b_2 . Fix λ and μ and let $N = \beta_2 + \beta_4 = \frac{1}{2}(|\lambda| - |\mu|)$ and $r = \beta_4$, so $\beta_2 = N - r$. Then the map

$$(\lambda, \mu, \beta) \mapsto (a_1, a_2, b_1, b_2, r; N)$$

sets up a bijection between triples (λ, μ, β) such that $c_{\mu, \beta}^\lambda \neq 0$ and tuples $(a_1, a_2, b_1, b_2, r; N)$ satisfying the corresponding conditions

$$2a_1 + b_1 + 2a_2 + b_2 = |\mu| = 2\gamma + \delta \quad (|\lambda| - |\beta| = |\mu|)$$

$$|b_1 - b_2| \leq \delta \leq b_1 + b_2 \quad (\text{by Remark 3.3.3})$$

$$b_1 + b_2 \equiv \delta \pmod{2}$$

$$a_1 + a_2 \leq \gamma$$

$$a_2 + b_2 \leq N - 2r \quad (\text{by Cor. 3.3.2(i)(c)})$$

$$a_1 + N - r \geq \max\{\mu_1 - b_1, \gamma\} \quad (\lambda \supseteq \mu)$$

$$a_1, a_2, b_1, b_2 \geq 0$$

$$0 \leq r \leq \left\lfloor \frac{N}{2} \right\rfloor,$$

where we have set $\gamma = \mu_2$ and $\delta = \mu_1 - \mu_2$, and $\left\lfloor \frac{N}{2} \right\rfloor$ denotes the greatest integer function of $N/2$. Note that the third and fourth equations are consequences of the first two. Let $\mathcal{C}(\lambda, \mu; N)$ be the set of all tuples $(a_1, a_2, b_1, b_2, r; N)$ satisfying the above conditions (so by Prop. 3.1.12, $c_{\mu, \beta}^\lambda = 1$), so that

$$\sharp \mathcal{C}(\lambda, \mu; N) = \sum_{\beta' \text{ even}} c_{\mu, \beta'}^\lambda.$$

Now consider the partially-defined map

$$\begin{cases} \mathcal{C}(\lambda, \mu; N) \rightarrow \mathcal{C}(\lambda, \mu^+; N-1) \\ (a_1, a_2, b_1, b_2, r; N) \mapsto (a_1 + 1, a_2, b_1, b_2, r; N-1). \end{cases} \quad (3.45)$$

The equations satisfied by a tuple $(a'_1, a'_2, b'_1, b'_2, r'; N') \in \mathcal{C}(\lambda, \mu^+; N-1)$ are

$$2a'_1 + b'_1 + 2a'_2 + b'_2 = |\mu^+| = 2\gamma + 2 + \delta \quad (|\lambda| - |\beta| = |\mu^+|)$$

$$|b'_1 - b'_2| \leq \delta \leq b'_1 + b'_2 \quad (\text{by Remark 3.3.3})$$

$$b'_1 + b'_2 \equiv \delta \pmod{2}$$

$$a'_1 + a'_2 \leq \gamma + 1$$

$$a'_2 + b'_2 \leq N - 2r' \quad (\text{by Cor. 3.3.2(ii)(c)})$$

$$a'_1 + N - 1 - r' \geq \max\{\mu_1 - b'_1, \gamma\} \quad (\lambda \supseteq \mu^+)$$

$$b'_1, b'_2 \geq 0$$

$$a'_1, a'_2 \geq 1 \quad (\text{by Cor. 3.3.2(ii)(d)})$$

$$0 \leq r' \leq \left\lfloor \frac{N-1}{2} \right\rfloor.$$

Comparing these with the equations for $\mathcal{C}(\lambda, \mu; N)$ shows that the map (3.45) is defined on tuples $\mathbf{A} = (a_1, a_2, b_1, b_2, r; N)$ with $a_2 \geq 1$ and $r \leq \left\lfloor \frac{N-1}{2} \right\rfloor$. Moreover, it

is easy to see that this map is surjective where it is defined. Thus we have

$$\begin{aligned}
\sum_{\beta' \text{ even}} (c_{\mu, \beta}^{\lambda} - c_{\mu^+, \beta}^{\lambda}) &= \sharp \mathcal{C}(\lambda, \mu; N) - \sharp \mathcal{C}(\lambda, \mu^+; N-1) \\
&= \sum_{\substack{\mathbf{A} \in \mathcal{C}(\lambda, \mu; N) \\ a_2=0, \ r \leq \lfloor \frac{N-1}{2} \rfloor}} 1 + \sum_{\substack{\mathbf{A} \in \mathcal{C}(\lambda, \mu; N) \\ N \text{ even}, \ r=N/2}} 1 \\
&= \sum_{\substack{\mathbf{A} \in \mathcal{C}(\lambda, \mu; N) \\ a_2=0, \ r \leq \lfloor \frac{N}{2} \rfloor}} 1,
\end{aligned} \tag{3.46}$$

where the last equality follows from the fact that for $\mathbf{A} \in \mathcal{C}(\lambda, \mu; N)$, N even and $r = N/2$ implies $a_2 = 0$. But there is at most one $\mathbf{A} \in \mathcal{C}(\lambda, \mu; N)$ with $a_2 = 0$: the equations for $\mathcal{C}(\lambda, \mu; N)$ give

$$\begin{aligned}
a_2 &= 0 \\
b_1 &= \lambda_1 - \lambda_2 \\
b_2 &= \lambda_3 - \lambda_4 \\
r &= \lambda_4 \\
a_1 &= \lambda_2 + \lambda_4 - N.
\end{aligned}$$

Hence we have shown the

Lemma 3.3.4. *Let λ, μ be partitions such that $\lambda \supseteq \mu$, $\ell(\lambda) \leq 4$ and $\ell(\mu) \leq 2$. Then*

$$\sum_{\beta' \text{ even}} (c_{\mu, \beta}^{\lambda} - c_{\mu^+, \beta}^{\lambda})$$

is either 0 or 1, and is 1 if and only if there is an $\mathbf{A} \in \mathcal{C}(\lambda, \mu; N)$ such that $a_2 = 0$.

Corollary 3.3.5. *Let μ have length at most two. Then if $\nu = \mu$,*

$$D_{\mu,\nu,2N+M_1} = B_{\mu,|\mu|+2N}.$$

Proof. For $\nu = \mu$, $M_1 = |\mu|$ and so

$$\begin{aligned} D_{\mu,\nu,2N+M_1} &= D_{\mu,\mu,2N+|\mu|} = \sum_{\substack{\lambda \vdash |\mu|+2N \\ \ell(\lambda) \leq 4}} \left(\sum_{\beta' \text{ even}} (c_{\mu,\beta}^\lambda - c_{\mu^+,\beta}^\lambda) \right)^2 \\ &= \sum_{\substack{\lambda \vdash |\mu|+2N \\ \ell(\lambda) \leq 4}} \sum_{\beta' \text{ even}} (c_{\mu,\beta}^\lambda - c_{\mu^+,\beta}^\lambda) = B_{\mu,|\mu|+2N}. \end{aligned}$$

□

Notice also that in the case $\mu = \nu$, the right-hand side of (3.41) simplifies to the right-hand side of (3.23), so that Prop. 3.1.13 follows from Prop. 3.2.3.

Proof of Prop. 3.2.3. The plan of proof is as follows. First, we compute the N -th coefficient of the right-hand side of (3.41). Second, by Lemma 3.3.4 determining $D_{\mu,\nu,2N+M_1}$ essentially reduces to counting the set of tuples $\mathbf{A} = (a_1, a_2, b_1, b_2, r; N) \in \mathcal{C}(\lambda, \mu; N)$ with $a_2 = 0$. We show the resulting sum simplifies to the aforementioned N -th coefficient.

For the remainder of the proof, set $\varepsilon' = \frac{1}{2}(m_1 - m_2)$, $\gamma = \frac{1}{2}(M_2 - m_2)$, $\delta = \frac{1}{2}(m_1 + m_2)$ and $\varepsilon = \frac{1}{2}(M_2 + m_2)$, so $\gamma + \delta = \varepsilon + \varepsilon'$. (Note that if $\mu = \nu$, this agrees with the previous definitions of γ and δ , and $\varepsilon' = 0$.) Also set $N_\mu = N + \frac{1}{2}(M_1 - |\mu|)$ and $N_\nu = N + \frac{1}{2}(M_1 - |\nu|)$, so that

$$2N_\mu + |\mu| = 2N + M_1 = 2N_\nu + |\nu|.$$

For a statement P , we will use the notation

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

As above, $\llbracket n \rrbracket$ denotes the greatest integer function of n . Finally, in accordance with standard convention, empty sums are interpreted as zero.

Now the right-hand side of (3.41) is

$$\frac{X^{\varepsilon'}(1 - X^{1+\gamma-\varepsilon'})(1 - X^{1+\delta-\varepsilon'})}{(1 - X)^3(1 - X^2)} = \frac{P(X)}{(1 - X)(1 - X^2)}, \quad (3.47)$$

where

$$P(X) = X^{\varepsilon'}(1 + X + \cdots + X^{\gamma-\varepsilon'})(1 + X + \cdots + X^{\delta-\varepsilon'}) = \sum_{k=0}^{\mu_1} d_k X^k \quad (3.48)$$

and with $k_0 = \min\{\gamma, \delta\}$,

$$d_k = \begin{cases} 1 + k - \varepsilon' & \text{if } \varepsilon' \leq k \leq k_0; \\ 1 + k_0 - \varepsilon' & \text{if } k_0 \leq k \leq \varepsilon + \varepsilon' - k_0; \\ 1 + \varepsilon - k & \text{if } \varepsilon + \varepsilon' - k_0 \leq k \leq \varepsilon; \\ 0 & \text{otherwise.} \end{cases} \quad (3.49)$$

The rational function $((1 - X)(1 - X^2))^{-1}$ is the generating function for the numbers $p_2(n)$, the number of partitions of n with length at most 2. So the N -th coefficient of (3.47) is

$$\sum_{k=0}^N p_2(N - k) d_k = \sum_{k=\varepsilon'}^{\min\{N, \varepsilon\}} p_2(N - k) d_k. \quad (3.50)$$

Figure 3.5: The partitions λ , α and ν

Now by Lemma 3.3.4, $D_{\mu, \nu, 2N+M_1}$ counts pairs of triples $\{(\lambda, \mu, \beta), (\lambda, \nu, \alpha)\}$ (all with the same μ and ν) for which (λ, μ, β) and (λ, ν, α) *simultaneously* have realizations as tuples $\mathbf{A} = (a_1, a_2, b_1, b_2, r; N_\mu) \in \mathcal{C}(\lambda, \mu; N_\mu)$ with $a_2 = 0$ and $\mathbf{C} = (c_1, c_2, d_1, d_2, t; N_\nu) \in \mathcal{C}(\lambda, \nu; N_\nu)$ with $c_2 = 0$, respectively. (See Figures 3.4 and 3.5.) This gives

$$\lambda_1 = \beta_2 + a_1 + b_1 = \alpha_2 + c_1 + d_1$$

$$\lambda_2 = \beta_2 + a_1 = \alpha_2 + c_1$$

$$\lambda_3 = \beta_4 + a_2 + b_2 = \alpha_4 + c_2 + d_2$$

$$\lambda_4 = \beta_4 + a_2 = \alpha_4 + c_2.$$

So immediately we see that if $a_2 = c_2 = 0$,

$$b_1 = d_1, b_2 = d_2$$

$$\beta_4 = r = \alpha_4 = t$$

and since $\beta_2 + \beta_4 = N_\mu$ and $\alpha_2 + \alpha_4 = N_\nu$,

$$a_1 + N_\mu - r = \lambda_2 = c_1 + N_\nu - t \implies a_1 + N_\mu = c_1 + N_\nu.$$

So our integers a_1, b_1, b_2, c_1, r must satisfy the following conditions:

$$2a_1 + b_1 + b_2 = |\mu|$$

$$2c_1 + b_1 + b_2 = |\nu|$$

$$|b_1 - b_2| \leq \delta_\mu, \delta_\nu \leq b_1 + b_2$$

$$b_1 + b_2 \equiv \delta_\mu \equiv \delta_\nu \pmod{2}$$

$$a_1 \leq \mu_2$$

$$c_1 \leq \nu_2$$

$$b_2 \leq \min\{N_\mu - 2r, N_\nu - 2r\}$$

$$a_1 + N_\mu - r \geq \max\{\mu_1 - b_1, \mu_2\}$$

$$c_1 + N_\nu - r \geq \max\{\nu_1 - b_1, \nu_2\}$$

$$a_1, b_1, b_2, c_1 \geq 0$$

$$0 \leq r \leq \left\lfloor \frac{N}{2} \right\rfloor.$$

Set $a = a_1 + N_\mu - N = c_1 + N_\nu - N$. Then writing these conditions in terms of a, b_1, b_2

gives

$$\begin{aligned}
2a + b_1 + b_2 &= M_1 \\
|b_1 - b_2| &\leq m_2 \leq m_1 \leq b_1 + b_2 \\
b_1 + b_2 &\equiv m_1 \equiv m_2 \pmod{2} \\
a &\leq \frac{1}{2}(M_1 - m_1) \\
b_2 &\leq N - 2r \\
a + N - r &\geq \max\{\mu_1 - b_1, \nu_1 - b_1, \mu_2, \nu_2\} \\
a &\geq \frac{1}{2}(M_1 - M_2) \\
b_1, b_2 &\geq 0 \\
0 \leq r &\leq \left\lfloor \frac{N}{2} \right\rfloor.
\end{aligned}$$

Since b_1 is determined by a and b_2 , we may count such tuples by summing over r, a, b_2 satisfying

- $0 \leq r \leq \left\lfloor \frac{N}{2} \right\rfloor$
- $\max\{\frac{1}{2}(M_1 - M_2), \mu_2 + N - r, \nu_2 + N - r\} \leq a \leq \frac{1}{2}(M_1 - m_1)$
- $0 \leq b_2 \leq N - 2r$.

One immediate consequence of the summation range for a is that

$$D_{\mu, \nu, 2N+M_1} = 0 \text{ if } M_2 < m_1. \quad (3.51)$$

We must also ensure the constraints on $b_1 = M_1 - 2a - b_2$ are met:

(i). $b_2 - m_2 \leq b_1 \leq b_2 + m_2$ if and only if

$$\frac{1}{2}(M_1 - m_2) - a \leq b_2 \leq \frac{1}{2}(M_1 + m_2) - a.$$

Note that the leftmost term is nonnegative, since $a \leq \frac{1}{2}(M_1 - m_1) \leq \frac{1}{2}(M_1 - m_2)$.

(ii). $b_1 \geq m_1 - b_2$ if and only if $a \leq \frac{1}{2}(M_1 - m_1)$, which already holds.

(iii). The previous constraints show $b_1 \geq m_1 - b_2 \geq m_2 - b_2$ and $b_1 \geq b_2 - m_2$, so already $b_1 \geq 0$.

(iv). $b_1 \geq \max\{\mu_1 - N + r - a, \nu_1 - N + r - a\}$ if and only if $b_2 \leq \min\{N + M_1 - \mu_1 - a - r, N + M_1 - \nu_1 - a - r\} = N + M_1 - a - r - \max\{\mu_1, \nu_1\}$.

So the b_2 -range should actually be

$$\frac{1}{2}(M_1 - m_2) - a \leq b_2 \leq \min\{N - 2r, \frac{1}{2}(M_1 + m_2) - a, N + M_1 - a - r - \max\{\mu_1, \nu_1\}\}.$$

But some of these constraints are redundant:

(i). $N - 2r \leq N + M_1 - a - r - \max\{\mu_1, \nu_1\}$ if and only if $a \leq M_1 + r - \max\{\mu_1, \nu_1\}$ (*).

Now we already have $a \leq \frac{1}{2}(M_1 - m_1)$, and $\frac{1}{2}(M_1 - m_1) \leq M_1 + r - \max\{\mu_1, \nu_1\}$ if and only if $\frac{1}{2}(M_1 + m_1) - \max\{\mu_1, \nu_1\} + r \geq 0$. But clearly

$$M_1 + m_1 \geq \max\{|\mu| + \delta_\mu, |\nu| + \delta_\nu\} \geq \max\{2\mu_1, 2\nu_1\},$$

and $r \geq 0$. So (*) always holds and thus the first statement does as well.

(ii). The b_2 -range is nonempty $\iff$ either $N - 2r \geq \frac{1}{2}(M_1 + m_2) - a$ or $N - 2r \geq \frac{1}{2}(M_1 - m_2) - a$. Since the second inequality is weaker in a , we see that the b_2 -range is nonempty if and only if $a \geq \frac{1}{2}(M_1 - m_2) - N + 2r$. Now

$$M_1 - m_2 \geq \max\{|\mu| - \delta_\mu, |\nu| - \delta_\nu\} = \max\{2\mu_2, 2\nu_2\},$$

so $\frac{1}{2}(M_1 - m_2) - N + 2r \geq \max\{\mu_2 - N + r, \nu_2 - N + r\}$. Thus the range of summation for a is

$$\max\{\frac{1}{2}(M_1 - M_2), \frac{1}{2}(M_1 - m_2) - N + 2r\} \leq a \leq \frac{1}{2}(M_1 - m_1).$$

(iii). Finally, the a -range is nonempty $\iff$ either $\frac{1}{2}(M_1 - m_2) - N + 2r \leq \frac{1}{2}(M_1 - M_2)$ or $\frac{1}{2}(M_1 - m_2) - N + 2r \leq \frac{1}{2}(M_1 - m_1)$. Since the second inequality is weaker in r , we conclude that the a -range is nonempty if and only if $r \leq \llbracket \frac{N - \frac{1}{2}(m_1 - m_2)}{2} \rrbracket = \llbracket \frac{N - \varepsilon'}{2} \rrbracket$. In particular,

$$D_{\mu, \nu, 2N + M_1} = 0 \text{ if } N < \varepsilon' = \frac{1}{2}(m_1 - m_2). \quad (3.52)$$

Hence the summation is over $r, a, b = b_2$ such that

- $0 \leq r \leq \llbracket \frac{N - \varepsilon'}{2} \rrbracket$
- $\max\{\frac{1}{2}(M_1 - M_2), \frac{1}{2}(M_1 - m_2) - N + 2r\} \leq a \leq \frac{1}{2}(M_1 - m_1)$
- $\frac{1}{2}(M_1 - m_2) - a \leq b \leq \min\{N - 2r, \frac{1}{2}(M_1 + m_2) - a\}$.

Finally, for later reference we give the various cutoffs in summation:

$$b: \frac{1}{2}(M_1 + m_2) - a \leq N - 2r \iff a \geq \frac{1}{2}(M_1 + m_2) - N + 2r.$$

$$a: \text{ (i). } \frac{1}{2}(M_1 - m_2) - N + 2r \leq \frac{1}{2}(M_1 - M_2) \text{ if and only if}$$

$$r \leq \llbracket \frac{N - \frac{1}{2}(M_2 - m_2)}{2} \rrbracket = \llbracket \frac{N - \gamma}{2} \rrbracket.$$

$$\text{(ii). } \frac{1}{2}(M_1 - M_2) \leq \frac{1}{2}(M_1 + m_2) - N + 2r \leq \frac{1}{2}(M_1 - m_1) \text{ if and only if}$$

$$\frac{N - \frac{1}{2}(M_2 + m_2)}{2} \leq r \leq \frac{N - \frac{1}{2}(m_1 + m_2)}{2} \text{ if and only if}$$

$$\frac{N - \varepsilon}{2} \leq r \leq \llbracket \frac{N - \delta}{2} \rrbracket$$

$$r: \left\lfloor \frac{N-\delta}{2} \right\rfloor \leq \left\lfloor \frac{N-\gamma}{2} \right\rfloor \iff \gamma \leq \delta.$$

So we must consider the cases $\gamma \leq \delta$ and $\gamma \geq \delta$ separately.

Case $\gamma \leq \delta$.

$$\begin{aligned}
D_{\mu, \nu, 2N+M_1} = & \\
& [N \geq \varepsilon] \sum_{r=0}^{\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} \left(1 + \left(\frac{1}{2}(M_1+m_2) - a \right) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \\
& + [N \geq \delta] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\delta}{2} \right\rfloor} \left\{ \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1+m_2)-N+2r} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right. \\
& \quad \left. + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} \left(1 + \left(\frac{1}{2}(M_1+m_2) - a \right) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right\} \\
& + [N \geq \gamma] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\delta}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\gamma}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \\
& + [N \geq \varepsilon'] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\delta}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\varepsilon'}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-m_2)-N+2r}^{\frac{1}{2}(M_1-m_1)} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= [N \geq \varepsilon] \sum_{r=0}^{\llbracket \frac{N-\varepsilon}{2} \rrbracket} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} (1+m_2) \\
&\quad + [N \geq \delta] \sum_{r=\max\{0, 1+\llbracket \frac{N-\varepsilon}{2} \rrbracket\}}^{\llbracket \frac{N-\delta}{2} \rrbracket} \left\{ \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1+m_2)-N+2r} (1+a+N-2r-\frac{1}{2}(M_1-m_2)) + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} (1+m_2) \right\} \\
&\quad + [N \geq \gamma] \sum_{r=\max\{0, 1+\llbracket \frac{N-\delta}{2} \rrbracket\}}^{\llbracket \frac{N-\gamma}{2} \rrbracket} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} (1+a+N-2r-\frac{1}{2}(M_1-m_2)) \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\llbracket \frac{N-\delta}{2} \rrbracket\}}^{\llbracket \frac{N-\varepsilon'}{2} \rrbracket} \sum_{a=\frac{1}{2}(M_1-m_2)-N+2r}^{\frac{1}{2}(M_1-m_1)} (1+a+N-2r-\frac{1}{2}(M_1-m_2))
\end{aligned}$$

$$\begin{aligned}
&= [N \geq \varepsilon] \sum_{r=0}^{\lfloor \frac{N-\varepsilon}{2} \rfloor} \sum_{a=0}^{\frac{1}{2}(M_2-m_1)} (1+m_2) \\
&\quad + [N \geq \delta] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon}{2} \rfloor\}}^{\lfloor \frac{N-\delta}{2} \rfloor} \left\{ \sum_{a=N-2r}^{m_2} (1+a) + \sum_{m_2+1}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+m_2) \right\} \\
&\quad + [N \geq \gamma] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\gamma}{2} \rfloor} \sum_{a=N-2r-\frac{1}{2}(M_2-m_2)}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+a) \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=0}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+a) \\
&= [N \geq \varepsilon] \sum_{r=0}^{\lfloor \frac{N-\varepsilon}{2} \rfloor} \sum_{a=0}^{\gamma-\varepsilon'} (1+\delta-\varepsilon') \\
&\quad + [N \geq \delta] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon}{2} \rfloor\}}^{\lfloor \frac{N-\delta}{2} \rfloor} \left\{ \sum_{a=N-2r-\gamma}^{\delta-\varepsilon'} (1+a) + \sum_{\delta-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\delta-\varepsilon') \right\} \\
&\quad + [N \geq \gamma] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\gamma}{2} \rfloor} \sum_{a=N-2r-\gamma}^{N-2r-\varepsilon'} (1+a) \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=0}^{N-2r-\varepsilon'} (1+a). \tag{3.53}
\end{aligned}$$

Case $\delta \leq \gamma$.

$$\begin{aligned}
D_{\mu, \nu, 2N+M_1} = & \\
& [N \geq \varepsilon] \sum_{r=0}^{\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} \left(1 + \left(\frac{1}{2}(M_1+m_2) - a \right) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \\
& + [N \geq \gamma] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\gamma}{2} \right\rfloor} \left\{ \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1+m_2)-N+2r} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right. \\
& \quad \left. + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} \left(1 + \left(\frac{1}{2}(M_1+m_2) - a \right) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right\} \\
& + [N \geq \delta] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\gamma}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\delta}{2} \right\rfloor} \left\{ \sum_{a=\frac{1}{2}(M_1-M_2)-N+2r}^{\frac{1}{2}(M_1+m_2)-N+2r} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right. \\
& \quad \left. + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} \left(1 + \left(\frac{1}{2}(M_1+m_2) - a \right) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right) \right\} \\
& + [N \geq \varepsilon'] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\delta}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\varepsilon'}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-M_2)-N+2r}^{\frac{1}{2}(M_1-m_1)} \left(1 + (N-2r) - \left(\frac{1}{2}(M_1-m_2) - a \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= [N \geq \varepsilon] \sum_{r=0}^{\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1-m_1)} (1+m_2) \\
&\quad + [N \geq \gamma] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\varepsilon}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\gamma}{2} \right\rfloor} \\
&\quad \left\{ \sum_{a=\frac{1}{2}(M_1-M_2)}^{\frac{1}{2}(M_1+m_2)-N+2r} \left(1+a+N-2r-\frac{1}{2}(M_1-m_2)\right) + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} (1+m_2) \right\} \\
&\quad + [N \geq \delta] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\gamma}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\delta}{2} \right\rfloor} \\
&\quad \left\{ \sum_{a=\frac{1}{2}(M_1-m_2)-N+2r}^{\frac{1}{2}(M_1+m_2)-N+2r} \left(1+a+N-2r-\frac{1}{2}(M_1-m_2)\right) + \sum_{a=\frac{1}{2}(M_1+m_2)-N+2r+1}^{\frac{1}{2}(M_1-m_1)} (1+m_2) \right\} \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\left\{0, 1+\left\lfloor \frac{N-\delta}{2} \right\rfloor\right\}}^{\left\lfloor \frac{N-\varepsilon'}{2} \right\rfloor} \sum_{a=\frac{1}{2}(M_1-m_2)-N+2r}^{\frac{1}{2}(M_1-m_1)} \left(1+a+N-2r-\frac{1}{2}(M_1-m_2)\right)
\end{aligned}$$

$$\begin{aligned}
&= [N \geq \varepsilon] \sum_{r=0}^{\lfloor \frac{N-\varepsilon}{2} \rfloor} \sum_{a=0}^{\frac{1}{2}(M_2-m_1)} (1+m_2) \\
&\quad + [N \geq \gamma] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon}{2} \rfloor\}}^{\lfloor \frac{N-\gamma}{2} \rfloor} \left\{ \sum_{a=N-2r}^{m_2} (1+a) + \sum_{m_2+1}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+m_2) \right\} \\
&\quad + [N \geq \delta] \sum_{r=\max\{0, 1+\lfloor \frac{N-\gamma}{2} \rfloor\}}^{\lfloor \frac{N-\delta}{2} \rfloor} \left\{ \sum_{a=0}^{m_2} (1+a) + \sum_{a=m_2+1}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+m_2) \right\} \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=0}^{N-2r-\frac{1}{2}(m_1-m_2)} (1+a) \\
&= [N \geq \varepsilon] \sum_{r=0}^{\lfloor \frac{N-\varepsilon}{2} \rfloor} \sum_{a=0}^{\gamma-\varepsilon'} (1+\delta-\varepsilon') \\
&\quad + [N \geq \gamma] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon}{2} \rfloor\}}^{\lfloor \frac{N-\gamma}{2} \rfloor} \left\{ \sum_{a=N-2r-\gamma}^{\delta-\varepsilon'} (1+a) + \sum_{\delta-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\delta-\varepsilon') \right\} \\
&\quad + [N \geq \delta] \sum_{r=\max\{0, 1+\lfloor \frac{N-\gamma}{2} \rfloor\}}^{\lfloor \frac{N-\delta}{2} \rfloor} \left\{ \sum_{a=0}^{\delta-\varepsilon'} (1+a) + \sum_{a=\delta-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\delta-\varepsilon') \right\} \\
&\quad + [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-\delta}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=0}^{N-2r-\varepsilon'} (1+a). \tag{3.54}
\end{aligned}$$

In order to simplify further, we consider the various terms of (3.53) and (3.54) with

the goal of writing everything in terms of $k_0 = \min\{\gamma, \delta\}$. (Recall $\gamma + \delta = \varepsilon + \varepsilon'$, so that $\{k_0, \varepsilon + \varepsilon' - k_0\} = \{\gamma, \delta\}$.)

Third term.

The inner sum of the third term of (3.53) is

$$\begin{aligned}
\sum_{a=N-2r-\gamma}^{N-2r-\varepsilon'} (1+a) &= \sum_{a=0}^{\gamma-\varepsilon'} (1+a+N-2r-\gamma) \\
&= \sum_{a=0}^{\gamma-\varepsilon'} (1+a) + (N-2r-\gamma)(\gamma-\varepsilon'+1) \\
&= \sum_{a=0}^{\gamma-\varepsilon'} (1+a) + \sum_{a=\gamma-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\gamma-\varepsilon'),
\end{aligned}$$

which is the third term inner sum of (3.54) with γ and δ interchanged. So the third term inner sum of both cases is

$$\sum_{a=0}^{k_0-\varepsilon'} (1+a) + \sum_{a=k_0-\varepsilon'+1}^{N-2r-\varepsilon'} (1+k_0-\varepsilon'). \tag{3.55}$$

Second term.

The inner sum of the second term of both (3.53) and (3.54) is

$$\begin{aligned}
& \sum_{a=N-2r-\gamma}^{\delta-\varepsilon'} (1+a) + \sum_{a=\delta-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\delta-\varepsilon') \\
&= \sum_{a=N-2r-\gamma+\varepsilon'}^{\delta} (1+a-\varepsilon') + \sum_{a=\delta+1}^{N-2r} (1+\delta-\varepsilon') \\
&= \sum_{a=N-2r-\gamma+\varepsilon'}^{N-2r} - \sum_{a=\delta+1}^{N-2r} ((1+a-\varepsilon') - (1+\delta-\varepsilon')) \\
&= \sum_{a=0}^{\gamma-\varepsilon'} (1+a+N-2r-\gamma) - \sum_{a=\delta+1}^{N-2r} (a-\delta) \\
&= \sum_{a=0}^{\gamma-\varepsilon'} (1+a) + (N-2r-\gamma)(\gamma-\varepsilon'+1) - \sum_{a=0}^{N-2r-\delta-1} (1+a) \\
&= \sum_{a=N-2r-\delta}^{\delta-\varepsilon'} (1+a) + \sum_{a=\gamma-\varepsilon'+1}^{N-2r-\varepsilon'} (1+\gamma-\varepsilon'),
\end{aligned}$$

which is symmetric in γ and δ , hence may be written

$$\sum_{a=N-2r-\varepsilon+k_0-\varepsilon'}^{k_0-\varepsilon'} (1+a) + \sum_{a=k_0-\varepsilon'+1}^{N-2r-\varepsilon'} (1+k_0-\varepsilon'). \quad (3.56)$$

First term.

It suffices to note that

$$\begin{aligned}
& \sum_{a=0}^{\gamma-\varepsilon'} (1+\delta-\varepsilon') = (1+\gamma-\varepsilon')(1+\delta-\varepsilon') \\
&= \sum_{a=0}^{\delta-\varepsilon'} (1+\gamma-\varepsilon') = \sum_{a=0}^{\varepsilon-k_0} (1+k_0-\varepsilon').
\end{aligned} \quad (3.57)$$

Now substituting (3.55),(3.56) and (3.57) into (3.53) (or (3.54)) yields

$$\begin{aligned}
D_{\mu,\nu,2N+M_1} &= [N \geq \varepsilon] \sum_{r=0}^{\lfloor \frac{N-\varepsilon}{2} \rfloor} \sum_{a=0}^{\varepsilon-k_0} (1+k_0-\varepsilon') \\
&+ [N \geq \varepsilon - k_0] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon+k_0-\varepsilon'}{2} \rfloor} \left\{ \sum_{a=N-2r-\varepsilon+k_0-\varepsilon'}^{k_0-\varepsilon'} (1+a) + \sum_{k_0-\varepsilon'+1}^{N-2r-\varepsilon'} (1+k_0-\varepsilon') \right\} \\
&+ [N \geq k_0] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon-k_0-\varepsilon'}{2} \rfloor\}}^{\lfloor \frac{N-k_0}{2} \rfloor} \left\{ \sum_{a=0}^{k_0-\varepsilon'} (1+a) + \sum_{a=k_0-\varepsilon'+1}^{N-2r-\varepsilon'} (1+k_0-\varepsilon') \right\} \\
&+ [N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-k_0}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=0}^{N-2r-\varepsilon'} (1+a).
\end{aligned} \tag{3.58}$$

Recall here that

$$d_a = \begin{cases} 1+a-\varepsilon' & \text{if } \varepsilon' \leq a \leq k_0; \\ 1+k_0-\varepsilon' & \text{if } k_0 \leq a \leq \varepsilon-k_0+\varepsilon'; \\ 1+\varepsilon-a & \text{if } \varepsilon-k_0+\varepsilon' \leq a \leq \varepsilon; \\ 0 & \text{otherwise,} \end{cases}$$

so the last two terms are clearly just

$$[N \geq \varepsilon'] \sum_{r=\max\{0, 1+\lfloor \frac{N-\varepsilon+k_0-\varepsilon'}{2} \rfloor\}}^{\lfloor \frac{N-\varepsilon'}{2} \rfloor} \sum_{a=\varepsilon'}^{N-2r} d_a.$$

(We are using the fact that, for $r \in \mathbb{Z}$, $1 + \lfloor \frac{N-\alpha}{2} \rfloor \leq r \leq \lfloor \frac{N-\beta}{2} \rfloor$ if and only if $\beta \leq N-2r \leq \alpha-1$.)

Now the first term inner sum is

$$\begin{aligned}
\sum_{a=0}^{\varepsilon-k_0} (1+k_0-\varepsilon') &= \sum_{a=\varepsilon'}^{k_0} (1+k_0-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon-k_0+\varepsilon'} (1+k_0-\varepsilon') \\
&= (1+k_0-\varepsilon')^2 + \sum_{a=k_0+1}^{\varepsilon-k_0+\varepsilon'} (1+k_0-\varepsilon') \\
&= \frac{1}{2}(1+k_0-\varepsilon')(2+k_0-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon-k_0+\varepsilon'} (1+k_0-\varepsilon') \\
&\quad + \frac{1}{2}(k_0-\varepsilon')(1+k_0-\varepsilon') \\
&= \sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon-k_0+\varepsilon'} (1+k_0-\varepsilon') + \sum_{a=0}^{k_0-\varepsilon'-1} (1+a) \\
&= \sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon-k_0+\varepsilon'} (1+k_0-\varepsilon') + \sum_{a=\varepsilon-k_0+\varepsilon'+1}^{\varepsilon} (1+\varepsilon-a) \\
&= \sum_{a=\varepsilon'}^{\varepsilon} d_a.
\end{aligned}$$

The second term inner sum is

$$\begin{aligned}
& \sum_{a=N-2r-\varepsilon+k_0-\varepsilon'}^{k_0-\varepsilon'} (1+a) + \sum_{a=k_0-\varepsilon'+1}^{N-2r-\varepsilon'} (1+k_0-\varepsilon') \\
&= \sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') - \sum_{a=\varepsilon'}^{N-2r-\varepsilon+k_0-1} (1+a-\varepsilon') \\
&\quad + \sum_{a=k_0+1}^{\varepsilon+\varepsilon'-k_0} (1+k_0-\varepsilon') + (N-2r-\varepsilon-\varepsilon'+k_0)(1+k_0-\varepsilon') \\
&= \sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon+\varepsilon'-k_0} (1+k_0-\varepsilon') \\
&\quad - \frac{1}{2}(N-2r-\varepsilon-\varepsilon'+k_0)(N-2r-\varepsilon-\varepsilon'+k_0+1) \\
&\quad + (N-2r-\varepsilon-\varepsilon'+k_0)(1+k_0-\varepsilon') \\
&= \sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon+\varepsilon'-k_0} (1+k_0-\varepsilon') \\
&\quad + \frac{1}{2}(N-2r-\varepsilon-\varepsilon'+k_0)(1+k_0-\varepsilon'-N+2r+\varepsilon).
\end{aligned} \tag{3.59}$$

Now since

$$\sum_{a=\alpha+1}^{\beta} a = \frac{1}{2}\beta(\beta+1) - \frac{1}{2}\alpha(\alpha+1) = \frac{1}{2}(\beta-\alpha)(\beta+\alpha+1),$$

we have

$$\begin{aligned}
\sum_{a=\varepsilon+\varepsilon'-k_0+1}^{N-2r} (1+\varepsilon-a) &= \sum_{\varepsilon-N+2r+1}^{k_0-\varepsilon'} a \\
&= \frac{1}{2}(k_0-\varepsilon'-\varepsilon+N-2r)(k_0-\varepsilon'+\varepsilon-N+2r+1),
\end{aligned}$$

which is the last term in (3.59). So the second term inner sum becomes

$$\sum_{a=\varepsilon'}^{k_0} (1+a-\varepsilon') + \sum_{a=k_0+1}^{\varepsilon+\varepsilon'-k_0} (1+k_0-\varepsilon') + \sum_{a=\varepsilon+\varepsilon'-k_0+1}^{N-2r} (1+\varepsilon-a) = \sum_{a=\varepsilon'}^{N-2r} d_a.$$

Combining all the above with (3.58) and dropping the initial $[N \geq \varepsilon']$ finally gives

$$\begin{aligned}
D_{\mu, \nu, 2N+M_1} &= \sum_{r=0}^{\min\{\varepsilon, N-2r\}} \left\lfloor \frac{N-\varepsilon'}{2} \right\rfloor \sum_{a=\varepsilon'} d_a \\
&= [N \geq \varepsilon] \sum_{r=0}^{\left\lfloor \frac{N-\varepsilon-1}{2} \right\rfloor} \sum_{a=\varepsilon'}^{\varepsilon} d_a + \sum_{r=\max\{0, \left\lfloor \frac{N-\varepsilon+1}{2} \right\rfloor\}}^{\left\lfloor \frac{N-\varepsilon'}{2} \right\rfloor} \sum_{a=\varepsilon'}^{N-2r} d_a \\
&= [N \geq \varepsilon] \left(1 + \left\lfloor \frac{N-\varepsilon-1}{2} \right\rfloor\right) \sum_{a=\varepsilon'}^{\varepsilon} d_a + \sum_{\varepsilon' \leq a \leq N-2r \leq \min\{\varepsilon, N\}} d_a \\
&= [N \geq \varepsilon] \left\lfloor \frac{N-\varepsilon+1}{2} \right\rfloor \sum_{a=\varepsilon'}^{\varepsilon} d_a + \sum_{a=\varepsilon'}^{\min\{\varepsilon, N\}} d_a \sum_{r: a \leq N-2r \leq \min\{\varepsilon, N\}} 1 \\
&= [N \geq \varepsilon] \left\lfloor \frac{N-\varepsilon+1}{2} \right\rfloor \sum_{a=\varepsilon'}^{\varepsilon} d_a + \sum_{a=\varepsilon'}^{\min\{\varepsilon, N\}} d_a \left(\left\lfloor \frac{N-a}{2} \right\rfloor - [N \geq \varepsilon] \left\lfloor \frac{N-\varepsilon+1}{2} \right\rfloor + 1 \right) \\
&= \sum_{a=\varepsilon'}^{\min\{\varepsilon, N\}} p_2(N-a) d_a,
\end{aligned}$$

which is the result. $\square$

CHAPTER 4

ANALYSIS OF LOCAL INTEGRALS

In order to complete our understanding of the twisted tensor L -function, we need more information about the local integrals appearing in (2.26). Specifically, we need to know that all the integrals are convergent in some region and nonvanishing. Moreover, in order to justify the absolute convergence of the Euler product, we need to establish a uniform line of convergence for the unramified integrals. This essentially depends on estimates on the unramified Whittaker functions.

Second, we would like to understand the location and order of the poles of our global L -function. In principle these should be the same as those of the global integral, but proving this is somewhat delicate. One must first show that the local integrals can be meromorphically continued, and then that the orders of poles of the local integrals are “no worse” than those of the corresponding local L -factor. We will say more about this in the closing remarks. As of now, these results are not included but are work in progress of the author.

Finally, we note that we will show the nonvanishing of the archimedean integrals for *smooth* data. Namely, for an archimedean place v , one can find a smooth section $f_{s,v}$ and a Whittaker function W_v such that the integral $\Psi_v(s, W_v, f_{s,v}) \neq 0$. However, for the theory of Eisenstein series (and in particular for the results of [24]) one needs

K -finite sections. So in order for our results here to be complete, we would need to show nonvanishing for K -finite data. For the moment, though, this disparity remains. By our arguments above, our local integral agrees with Jiang's local integral at all split places, so we will confine ourselves to inert and ramified places. (So if F is archimedean, then we assume $F = \mathbb{R}$ and $E = \mathbb{C}$.) We should note that most of this material is a close parallel to, and at times a near-identical repetition of, corresponding material in [24], Chapter 5. Also, many of the ideas here are ultimately borrowed from or inspired by work in [17, 18, 21, 20], and the author greatly benefited from reading parts of [39].

Let F be a local field of characteristic zero, and let E/F be a quadratic extension generated by an element θ whose square lies in F . Let π be an irreducible admissible generic unitary representation of $\mathrm{GSp}_4(E)$ with central character $\omega_\pi = 1$. So π has a Whittaker model, $\mathcal{W}(\pi, \psi)$, in the space $\mathcal{W}(\psi)$ of smooth complex-valued functions W on $\mathrm{GSp}_4(E)$ satisfying $W(n g) = \psi(n)W(g)$ for $n \in N_4(E)$.

If F is archimedean, we denote by $\mathcal{W}(\pi, \psi)_K$ the subspace of K_4 -finite vectors of $\mathcal{W}(\pi, \psi)$, where K_4 is an appropriate maximal compact subgroup of $\mathrm{GSp}_4(\mathbb{C})$ (to be defined below). Similarly denote the K_8 -finite vectors of $I_8(s) = \mathrm{Ind}_{P_8(F)}^{\mathrm{GSp}_8(F)}(\delta_{P_8}^{s/6})$ by $I_8(s)_K$, with K_8 a maximal compact subgroup of $\mathrm{GSp}_8(\mathbb{R})$.

4.1 Absolute Convergence

We now give some properties of Whittaker functions on GSp_4 which will be needed below.

Recall that the root system of GSp_4 has a basis of positive roots $\{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_4\}$ as in Lemma 2.2.1. In (3.7) we fixed a parametrization of the quotient $Z(F) \setminus T_0(F)$, which we recall here:

$$t = t'(a_1, a_2, b_1, b_2) \stackrel{\text{def}}{=} t(a_1(1 + a_2\theta), b_1(1 + b_2\theta), \lambda_{a,b}(1 - a_2\theta), 1 - b_2\theta), \quad (4.1a)$$

$$\lambda_{a,b} = \frac{b_1(1 - b_2^2\theta^2)}{a_1(1 - a_2^2\theta^2)} \in F^\times. \quad (4.1b)$$

Note that

$$t = t'(a_1, a_2, b_1, b_2) = z \cdot h(AB, A, B^{-1}, 1), \quad z \in Z(E),$$

where

$$A = b_1 \frac{1 + b_2\theta}{1 - b_2\theta} = \alpha_1(t), \quad (4.2a)$$

$$B = \frac{a_1(1 + a_2\theta)}{b_1(1 + b_2\theta)} = \alpha_2(t), \quad (4.2b)$$

and h is defined by (3.1). We note for later reference that

$$\delta_{B_4(E)}(t) = |A|^3 |B|^4. \quad (4.3)$$

Setting $a = 1 + a_2\theta$, $b = 1 + b_2\theta$ and $\mathbb{N}a = N_{E/F}a$, $\mathbb{N}b = N_{E/F}b$ as in Section 3.1 we see that

$$|A| = |b_1|, \quad (4.4a)$$

$$|B| = \left| \frac{a_1}{b_1} \right| \left| \frac{\mathbb{N}a}{\mathbb{N}b} \right|^{1/2}. \quad (4.4b)$$

Since any Whittaker function $W \in \mathcal{W}(\pi, \psi)$ satisfies $W(zg) = \omega_\pi(z)W(g) = W(g)$, such a W is determined by its restriction to the subgroup

$$T_4(E)' = \{h(ab, a, b^{-1}, 1) | a, b \in E^\times\}.$$

Proposition 4.1.1. *Let F be a local field and π an irreducible (admissible) generic representation of $\mathrm{GSp}_4(E)$ with Whittaker model $\mathcal{W}(\pi, \psi)$. There is a finite set $\mathbb{X} = \mathbb{X}(\pi)$ of finite functions on $T_4(E)'$, depending only on π , such that for $W \in \mathcal{W}(\pi, \psi)$ (K_4 -finite if F is archimedean) there are Schwartz functions $\phi_\xi \in \mathcal{S}(F^2)$ satisfying*

$$W(t) = \sum_{\xi \in \mathbb{X}(\pi)} \xi(\alpha_1(t), \alpha_2(t)) \phi_\xi(\alpha_1(t), \alpha_2(t)). \quad (4.5)$$

Proof. For F nonarchimedean, the argument is found in [17, 2.2], and its modification for GSp_4 is in [24] in the discussion preceding Prop. V.1.1.1. For F archimedean, see [24] and [20, 4.1-4.2]. $\square$

In the archimedean case, we also have the following variant:

Proposition 4.1.2. *Let F be an archimedean local field and π an irreducible (admissible) generic representation of $\mathrm{GSp}_4(E)$ with Whittaker model $\mathcal{W}(\pi, \psi)$. There is a finite set $\mathbb{X} = \mathbb{X}(\pi)$ of finite functions on $T_4(E)'$, depending only on π , such that for $W \in \mathcal{W}(\pi, \psi)_K$ there are Schwartz functions $\phi_\xi \in \mathcal{S}(F^2 \times K_4)$ satisfying*

$$W(tk) = \sum_{\xi \in \mathbb{X}(\pi)} \xi(\alpha_1(t), \alpha_2(t)) \phi_\xi((\alpha_1(t), \alpha_2(t)), k). \quad (4.6)$$

Proof. See [20], Prop. 4.3.2. $\square$

For convergence of some auxiliary integrals, we will majorize our Whittaker functions by a gauge, a notion first used in [17, 2.2] and refined in [21, p. 132] for GL_n . We correspondingly define a *gauge* on $\mathrm{GSp}_4(E)$ to be a function ξ which is left-invariant under $N_4(E)Z(E)$, right-invariant under K_4 and whose restriction to $T_4(E)'$ satisfies

$$\xi(t) = \sigma(t) \phi(\alpha_1(t), \alpha_2(t)),$$

where σ is a sum of positive quasicharacters of $T_4(E)$ and $\phi \in \mathcal{S}(F^2)$ is nonnegative. Since π is generic, there is a nonzero linear functional $\Lambda: V_\pi \rightarrow \mathbb{C}$ such that $\Lambda(\pi(n)v) = \psi(n)\Lambda(v)$ for $n \in N_4(E)$, $v \in V_\pi$. Set

$$W_v(g) = \Lambda(\pi(g)v).$$

Then W_v is an element of the space $\mathcal{W}(\psi)$ of smooth functions on $\mathrm{GSp}_4(E)$ satisfying

$$W(n g) = \psi(n)W(g), \quad n \in N_4(E), \quad W \in \mathcal{W}(\psi),$$

and the map $V_\pi \rightarrow \mathcal{W}(\psi)$ given by $v \mapsto W_v$ is an intertwining isomorphism from π to a subspace of $\mathcal{W}(\psi)$, which we denote by $\mathcal{W}(\pi, \psi)$. Following arguments in [17], [18, 8.3] and [21], we arrive at

Proposition 4.1.3 (cf. [21], Prop. 2.1). *Let F be a nonarchimedean (archimedean) local field and π an irreducible admissible generic representation of $\mathrm{GSp}_4(E)$. For any $f \in C_c^\infty(G)$ there is a gauge ξ and a seminorm p on V_π such that $|W_{\pi(f)v}(g)| \leq p(v)\xi(g)$, for all $v \in V_\pi$ (for all K_4 -finite $v \in V_\pi$).*

Proposition 4.1.4 ([15], Lem. 8.1). *For F a local field, any finite (continuous) function on $F^\times$ is a (finite) linear combination of functions of the form*

$$a \mapsto |a|^u \xi_1(a) (\log |a|)^m,$$

for $u \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$ and ξ_1 a unitary character of $F^\times$. □

We also need some basic information about a certain intertwining operator that appears in the course of evaluating the local integrals. Recall that in the unramified calculation we need to evaluate an integral of the form

$$\int_{F^3} f_s(\chi(x, y, w, u, v)) \psi_\circ(\alpha x + \beta y) \, dx \, dy \, dw,$$

with $\psi_\circ$ an additive character of F and $\chi(x, y, w, u, v)$ as in (3.3). We may write this as

$$\chi(x, y, w, u, v) = \gamma_0 n \bar{n}_1,$$

with

$$n = n'(x, y, w) = \begin{pmatrix} 1 & & & 0 & & \\ & 1 & -x & & 0 & -w \\ & & 1 & & 0 & -y \\ & & & 1 & -w & \\ & & & & 1 & \\ & & & & x & 1 \\ & & & & & 1 \end{pmatrix},$$

$$\bar{n}_1 = \bar{n}_1(u, v) = \begin{pmatrix} 1 & & & & & \\ -u & 1 & & & & \\ & & 1 & & & \\ & & & -v & 1 & \\ & & & & 1 & u \\ & & & & & 1 \\ & & & & & & 1 & v \\ & & & & & & & 1 \end{pmatrix}.$$

Let $N'_8(F) = \{n'(x, y, w) \mid x, y, w \in F\}$.

Lemma 4.1.5 ([24], Lem. V.1.2.1). *The integral*

$$M_s f_s(g) = \frac{\zeta_v(s+1)^2}{\zeta_v(s+2)\zeta_v(s+3)} \int_{N'_8(F)} f_s(\gamma_0 n g) \, dn$$

defines an intertwining operator from $I_8(s)$ to $I_{B,\beta}(s) = \text{Ind}_{B_8(F)}^{\text{GSp}_8(F)}(\beta_s)$, where for $t = h(t_1, \dots, t_8)$

$$\beta_s(t) = |t_1|^{s+3} |t_2|^2 |t_3|^{s+2} |t_4|^{-s} |\lambda|^{-\frac{s+7}{2}}.$$

M_s takes the unramified section $f_s^\circ \in I_8(s)$ to the unramified section $\tilde{f}_s^\circ \in I_{B,\beta}(s)$, where both are normalized by $f_s^\circ(1) = 1 = \tilde{f}_s^\circ(1)$.

Proof. All standard calculations. □

Lemma 4.1.6. *Let $f_s \in I_8(s)$ be a flat section and let f_s° and $\tilde{f}_s^\circ$ be the normalized unramified sections as in Lemma 4.1.5.*

(i). There is a constant C_f which depends only on $f_s|_K$ and such that

$$|f_s(g)| \leq C_f f_s^\circ(g), \quad \sigma = \operatorname{Re}(s). \quad (4.7)$$

(ii). For F nonarchimedean,

$$\tilde{f}_s^\circ(\bar{n}_1(u, v)) = \max\{1, |u|^{s+1}\}^{-1} \max\{1, |v|^{2s+2}\}^{-1}.$$

(iii). For $F = \mathbb{R}$,

$$\tilde{f}_s^\circ(\bar{n}_1(u, v)) = (1 + u^2)^{-\frac{1}{2}(s+1)} (1 + v^2)^{-(s+1)}.$$

Proof. Standard computations. □

4.1.1 Nonarchimedean integrals

Here we take F to be nonarchimedean.

Proposition 4.1.7. *There is a $\sigma_0 \in \mathbb{R}$ such that for any section f_s and any $W \in \mathcal{W}(\pi, \psi)$, the integral $\Psi(s, W, f_s)$ is absolutely convergent for $\operatorname{Re}(s) > \sigma_0$.*

Proof. As in Section 3.1 we have $G_0(F) = N_0(F)T_0(F)K_0$. For a subgroup $K \subseteq K_0$, let $\bar{K} = (K \cap Z(F)) \setminus K$. Our integral is

$$\begin{aligned} \Psi(s, W, f_s) &= \int_{Z(F)N^Q(F)X_2^{Q'}(F) \setminus G_0(F)} f_s(\gamma_0 g) W(g) dg \\ &= \int_{\bar{K}_0} \int_{Z(F) \setminus T_0(F)} \int_{X_2^{Q'}(F)N^Q(F) \setminus N_0(F)} f_s(\gamma_0 ntk) W(ntk) \delta_{B_0}^{-1}(t) dn dt dk. \end{aligned}$$

Recall that $N_0(F) = N^Q(F) \cdot N^{Q'}(F)$. Now choose an open compact subgroup $K' \subseteq K_0$ such that both W and f_s are right- K' -invariant, and write $K_0 = \cup_{i=1}^m k_i K'$.

Then our integral is a sum of integrals

$$\operatorname{vol}_{d\bar{k}}(K') \int_{Z(F) \setminus T_0(F)} \int_{X_2^{Q'}(F) \setminus N^{Q'}(F)} f_{s,i}(\gamma_0 nt) \psi(n) W_i(t) \delta_{B_0}^{-1}(t) dn dt,$$

where $W_i = \pi(k_i)W \in \mathcal{W}(\pi, \psi)$ and $f_{s,i} = \pi(k_i)f_s$, and $d\bar{k}$ is the quotient measure on $\overline{K}_0$ normalized appropriately. So it is enough to show convergence of the integral

$$\begin{aligned} & \int_{Z(F) \backslash T_0(F)} |W(t)\delta_{B_0}^{-1}(t)| \int_{X_2^{Q'}(F) \backslash N^{Q'}(F)} |f_s(\gamma_0 nt)\psi(n)| \, dn \, dt \\ & \leq C_f \int_{Z(F) \backslash T_0(F)} |W(t)\delta_{B_0}^{-1}(t)| \int_{X_2^{Q'}(F) \backslash N^{Q'}(F)} f_s^\circ(\gamma_0 nt) \, dn \, dt, \end{aligned}$$

where second line follows from (4.7) and the unitarity of ψ .

Now the inner integral is exactly $I(t, f_s^\circ, 1)$ in the notation of Section 3.1, which was calculated in Remark 3.1.8. With $t = t'(a_1, a_2, b_1, b_2)$ (see (4.1)), $a = 1 + a_2\theta$ and $b = 1 + b_2\theta$, (3.9) gives

$$I(t, f_s^\circ, 1) = |\theta|^{-6} |a_1|^{\sigma+5} |b_1|^{-\frac{\sigma+3}{2}} |\mathbb{N}a|^{\frac{\sigma+5}{2}} |\mathbb{N}b|^{-\frac{\sigma+5}{2}} \frac{\zeta_v(\sigma+1)^2}{\zeta_v(\sigma+2)\zeta_v(\sigma+3)},$$

which is defined and positive for $\sigma > -1$. Using (4.2) and (4.4), this becomes

$$I(t, f_s^\circ, 1) = |\theta|^{-6} |A|^{\frac{\sigma+7}{2}} |B|^{\sigma+5} \frac{\zeta_v(\sigma+1)^2}{\zeta_v(\sigma+2)\zeta_v(\sigma+3)}.$$

We also replace $W(t)$ by its expansion (4.5):

$$W(t) = \sum_{\xi \in \mathbb{X}(\pi)} \xi(A, B) \phi_\xi(A, B).$$

By Prop. 4.1.4, $|\xi(A, B)|$ is bounded above by a finite linear combination of functions of the form

$$|A|^{u_1} |B|^{u_2} |\log |A||^{m_1} |\log |B||^{m_2}, \quad u_i \in \mathbb{R}, m_i \in \mathbb{Z}_{\geq 0}.$$

The net effect is that $\Psi(s, W, f_s)$ is bounded above by a linear combination of integrals of the form

$$\int_{\{0 < |A|, |B| < \infty\}} |A|^{\frac{\sigma+1+2u_1}{2}} |B|^{\sigma+1+u_2} |\log |A||^{m_1} |\log |B||^{m_2} |\phi(A, B)| \, d^\times A \, d^\times B.$$

We have $|\log x| \leq Cx^\epsilon$ for $x \geq 1$ and $|\log x| \leq Cx^{-\epsilon}$ for $0 < x \leq 1$, for any $\epsilon > 0$ and some constant $C = C(\epsilon)$. Now since ϕ is compactly supported the integrand vanishes for $|A|$ and $|B|$ sufficiently large. So we may replace $|\log |A||^{m_1}$ and $|\log |B||^{m_2}$ by $|A|^{-\epsilon}$ and $|B|^{-\epsilon}$, respectively, for any $\epsilon > 0$, for the purpose of proving convergence. Then the integral will converge near 0 when σ is large enough, say $\sigma > \max\{-1 - 2u_1, -1 - u_2\}$. So our integral $\Psi(s, W, f_s)$ also converges for $\sigma = \operatorname{Re}(s)$ large enough, being bounded above by a finite combination of such integrals. Finally we note that by our analysis a lower bound σ_0 for the line of convergence depends only on the finite functions $\chi \in \mathbb{X}(\pi)$, hence only on π and not on the particular choice of W or f_s . $\square$

4.1.2 Archimedean integrals

Let $K_4 = \operatorname{Sp}_4(\mathbb{C}) \cap \operatorname{U}(4)$ denote the maximal compact subgroup of $\operatorname{GSp}_4(\mathbb{C})$ and $K_8 = \operatorname{Sp}(8) = \operatorname{Sp}_8(\mathbb{R}) \cap \operatorname{O}_8(\mathbb{R})$ the maximal compact subgroup of $\operatorname{GSp}_8(\mathbb{R})$. The maximal compact subgroup K_0 in G_0 is

$$K_0 = \operatorname{Sp}(4)^\pm \stackrel{\text{def}}{=} \operatorname{Sp}_4(\mathbb{C})^\pm \cap \operatorname{U}(4) \stackrel{\text{def}}{=} \{g \in \operatorname{GL}_4(\mathbb{C}) \mid gJ_4 {}^t g = \pm J_4, g {}^t \bar{g} = I_4\}. \quad (4.8)$$

For the decomposition $N_4(\mathbb{C}) = N^Q(\mathbb{R})N^{Q'}(\mathbb{R})$, we may assume $\theta^2 = -1$.

Proposition 4.1.8. *There is a $\sigma_0 \in \mathbb{R}$ such that for any smooth section f_s and any K_4 -finite Whittaker function $W \in \mathcal{W}(\pi, \psi)_K$, the integral $\Psi(s, W, f_s)$ is absolutely convergent for $\operatorname{Re}(s) > \sigma_0$.*

Proof. Using the Iwasawa decomposition just as in the nonarchimedean case, we have

$$\begin{aligned}\Psi(s, W, f_s) &= \int_{Z(F)N^Q(F)X_2^{Q'}(F)\backslash G_0(F)} f_s(\gamma_0 g)W(g) dg \\ &= \int_{\overline{K}_0} \int_{Z(F)\backslash T_0(F)} \int_{X_2^{Q'}(F)N^Q(F)\backslash N_0(F)} f_s(\gamma_0 ntk)W(ntk)\delta_{B_0}^{-1}(t) dn dt dk,\end{aligned}$$

where $\overline{K}_0 = (Z(F) \cap K_0) \backslash K_0$ as before. Given $W \in \mathcal{W}(\pi, \psi)_K$, we may write $\pi(k)W = \sum_{i=1}^n c_i(k)W_i$ for fixed $W_i \in \mathcal{W}(\pi, \psi)_K$ and continuous K_4 -finite functions c_i . From this our integral becomes a sum

$$\begin{aligned}\sum_{i=1}^n \int_{Z(F)\backslash T_0(F)} W_i(t)\delta_{B_0}^{-1}(t) \int_{X_2^{Q'}(F)N^Q(F)\backslash N_0(F)} \psi(n) \\ \cdot \left(\int_{\overline{K}_0} f_s(\gamma_0 ntk)c_i(k) dk \right) dn dt.\end{aligned}$$

The integral over $\overline{K}_0$ defines a new section $f_{s,i}$ by

$$f_{s,i}(g) = \int_{\overline{K}_0} f_s(gk)c_i(k) dk$$

which is K_8 -finite and also in $I_8(s)$. So it suffices to show the convergence of

$$\int_{Z(F)\backslash T_0(F)} |W(t)| \delta_{B_0}^{-1}(t) \left(\int_{X_2^{Q'}(F)N^Q(F)\backslash N_0(F)} f_\sigma^\circ(\gamma_0 nt) dn \right) dt, \quad \sigma = \text{Re}(s).$$

Now the inner integral is just $I(t, f_\sigma^\circ, 1)$ as in the nonarchimedean case. The proof of Lemma 3.1.2 carries over without change to the archimedean case; one only needs to check that $\gamma_0 \in K_8$, and this is clear. So this fact together with Lemma 4.1.6 and Lemma 4.1.5 gives

$$\begin{aligned}I(t, f_\sigma^\circ, 1) &= |a_1|^{\sigma+5} |b_1|^{-\frac{\sigma+3}{2}} (1+a_2^2)^{\frac{\sigma+5}{2}} (1+b_2^2)^{-\frac{\sigma+5}{2}} \frac{\zeta_v(\sigma+1)^2}{\zeta_v(\sigma+2)\zeta_v(\sigma+3)} \\ &= |A|^{\frac{\sigma+7}{2}} |B|^{\sigma+5} c_\sigma,\end{aligned}$$

where we have used $\mathbb{N}a = 1 + a_2^2$, $\mathbb{N}b = 1 + b_2^2$ and (4.2) and (4.4), with

$$c_\sigma = \frac{\zeta_v(\sigma + 1)^2}{\zeta_v(\sigma + 2)\zeta_v(\sigma + 3)},$$

and as in [43, 3.1],

$$\zeta_v(s) = \begin{cases} \pi^{-s/2} \Gamma(\frac{s}{2}), & \text{if } F_v = \mathbb{R}, \\ 2(2\pi)^{-s} \Gamma(s), & \text{if } F_v = \mathbb{C}. \end{cases}$$

Now using Prop. 4.1.1 we reduce to showing convergence of integrals of the form

$$\int_{\{0 < |A|, |B| < \infty\}} |\xi(A, B)| |\phi_\xi(A, B)| |A|^{-3} |B|^{-4} \cdot |A|^{\frac{\sigma+7}{2}} |B|^{\sigma+5} d^\times A d^\times B.$$

As in the nonarchimedean case, using Prop. 4.1.4 shows this integral is bounded above by a finite linear combination of integrals of the form

$$\int_{\{0 < |A|, |B| < \infty\}} |A|^{\frac{\sigma+1+2u_1}{2}} |B|^{\sigma+1+u_2} |\log |A||^{m_1} |\log |B||^{m_2} |\phi(A, B)| d^\times A d^\times B.$$

Now estimating $\log |A|$ and $\log |B|$ as before and noting that the rapid decay of the Schwartz functions ϕ makes the integrals converge away from zero, we get that the integrals are convergent for $\sigma = \text{Re}(s)$ large enough, say $\sigma > \max\{-1 - 2u_1, -1 - u_2\}$.

Then the convergence of $\Psi(s, W, f_s)$ independently of W and f_s follows as before. $\square$

4.2 Nonvanishing

Most of the results here are minor modifications of the analogous results in [24], Section V.2.

Denote, for $x, z, w \in F$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)$,

$$m(g) = \begin{pmatrix} e(g) & & & \\ & a & -b\theta & \\ & & 1 & \\ & -c\theta & & d \end{pmatrix} \in Q(F),$$

$$e(g) = ad - bc\theta^2 \in F^\times,$$

$$u(x, z, w) = \begin{pmatrix} 1 & x & w\theta & z\theta \\ & 1 & z\theta & \\ & & 1 & \\ & -x & & 1 \end{pmatrix} \in U^Q(F),$$

$$u'(x, z, w) = \begin{pmatrix} 1 & x(1+\theta) & w(1+\theta) & z \\ & 1 & z & \\ & & 1 & \\ & & -x(1+\theta) & 1 \end{pmatrix} \in U^{Q'}(F),$$

$$u''(y) = \begin{pmatrix} 1 & & & \\ & 1 & y(1-\theta) & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in N^{Q'}(F),$$

$$n(x, z, w) = \begin{pmatrix} 1 & x & w & z \\ & 1 & z & \\ & & 1 & \\ & -x & & 1 \end{pmatrix} \in P_4(E),$$

$$h(a, b, c, d) = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} \in T_4(F).$$

Note that for $g \in B_2(F)$, the standard Borel subgroup of $\mathrm{GL}_2(F)$, $e(g) = \det g$.

Let $C_c^\infty(F^n)$ be the space of smooth compactly supported functions on F^n ; recall for F nonarchimedean, smooth functions are just locally constant functions. It is easy to see that the linear map $\rho: C_c^\infty(U^{Q'}(F)) \rightarrow C_c^\infty(F^2)$ given by

$$\rho(\phi)(x, z) = \int_F \phi(u'(x, z, w)) \, dw \quad (4.9)$$

is well-defined and onto.

Let $\mathcal{S}(F^n)$ be the Schwarz space of smooth, rapidly decreasing functions on F^n , so for F nonarchimedean $\mathcal{S}(F^n) = C_c^\infty(F^n)$. The Fourier transform defined on $L^1(F^n)$ restricts to an isomorphism $\mathcal{S}(F^n) \rightarrow \mathcal{S}(F^n)$. In particular, for the case $n = 2$ we will need the Fourier transform $\Phi \mapsto \hat{\Phi}$ on $\mathcal{S}(F^2)$, given (with suitably normalized measures) by

$$\hat{\Phi}(u, v) = \int_{F^2} \Phi(x, z) \psi_\circ(\theta^2(-uz - vx)) \, dx \, dz, \quad (4.10)$$

$$\Phi(x, z) = \int_{F^2} \hat{\Phi}(u, v) \psi_\circ(\theta^2(uz + vx)) \, du \, dv, \quad (4.11)$$

which is an isomorphism $\mathcal{S}(F^2) \rightarrow \mathcal{S}(F^2)$.

To show the local integrals are nonzero, we will need a choice of section $f_s \in I_8(s) = \mathrm{Ind}_{P_8(F)}^{\mathrm{GSp}_8(F)}(\delta_{P_8}^{s/6})$. Recall that $G_0(F)$ has two orbits on $P_8(F) \backslash \mathrm{GSp}_8(F)$, the open orbit

$P_8(F)\gamma_0 G_0(F)$ and the closed, negligible orbit $P_8(F)G_0(F)$. For sections supported on the open orbit (see (2.10)), we have an isomorphism of representations

$$\{f \in I_8(s) \mid \text{supp } f \subseteq P_8\gamma_0 G_0\} \simeq \text{ind}_{QX_2^{Q'}}^{G_0}(\Xi_s), \quad (4.12)$$

where “ind” denotes non-normalized induction and $\Xi_s(x) = \delta_{P_8}(\gamma_0 x \gamma_0^{-1})^{\frac{s+3}{6}}$. The isomorphism is given by $f \mapsto f^*$, where $f^*(g) = f(\gamma_0 g)$. A simple computation shows that for $g \in \text{GL}_2(F)$, $u(x, y, z) \in U^Q(F)$ and $h \in Z(F)$,

$$\delta_{P_8}(\gamma_0 m(g) \gamma_0^{-1}) = e(g)^3,$$

so that

$$\Xi_s(m(g)u(x, z, w)) = |e(g)|^{\frac{s+3}{2}}.$$

4.2.1 Nonarchimedean case

Let v be a finite place of F which is either inert or ramified. As usual we will drop the subscript v .

Recall that the integral under consideration is

$$\Psi(s, W, f_s) = \int_{\mathcal{D}(F)} f_s(\gamma_0 g) W(g) dg.$$

The main result here is the following.

Proposition 4.2.1. *Let σ_0 be as in Prop. 4.1.7. Then the integral $\Psi(s, W, f_s)$ is not identically zero for $\text{Re}(s) > \sigma_0$. More precisely, there is a choice of Whittaker function W_0 and section $f_{s,0}$ such that $\Psi(s, W_0, f_{s,0}) = 1$, independent of s , for $\text{Re}(s) > \sigma_0$.*

This will be accomplished through several reductions.

Let P'_2 be the standard mirabolic subgroup of GL_2 with unipotent radical N_2 , so $\mathrm{GL}_2(F) = P'_2(F)Z(F)K_2$, with $K_2 = \mathrm{GL}_2(\mathcal{O})$. For a Whittaker function $\widetilde{W}$ and $g \in \mathrm{GL}_2(F)$, consider the integral

$$J(s, g, \widetilde{W}) = \int_{N_2(F) \backslash P'_2(F)} |\det p|^s \widetilde{W}(m(pg)) dp. \quad (4.13)$$

We note that $J(s, pg, \widetilde{W}) = J(s, g, \widetilde{W})$ for $p \in P'_2(F) \cap K_2$, and $J(s, g, \widetilde{W})$ is absolutely convergent for $\mathrm{Re}(s)$ large enough by Prop. 4.1.3.

Lemma 4.2.2. *The integral $J(s, g, \widetilde{W})$ defines a smooth function of g , and we may choose $\widetilde{W}$ so that $J(s, 1, \widetilde{W})$ is a nonzero constant independent of s , for $\mathrm{Re}(s)$ sufficiently large.*

Proof. The smoothness in g is clear from that of the Whittaker function $\widetilde{W}$. Choose $\widetilde{W}' \in \mathcal{W}(\pi, \psi)$ such that $\widetilde{W}'(1) = 1$, and set

$$\widetilde{W}(g) = \int_F \widetilde{W}'(gu''(y)) \hat{\Phi}'(y) dy, \quad (4.14)$$

with $\Phi' \in C_c^\infty(F)$ to be chosen below, and with Fourier transform given by

$$\hat{\Phi}(y) = \int_F \Phi(x) \psi_\circ(xy)^{-1} dx.$$

Then

$$\begin{aligned} J(s, 1, \widetilde{W}) &= \int_{N_2(F) \backslash P'_2(F)} |\det p|^s \widetilde{W}(m(p)) dp \\ &= \int_{F^\times} |a|^s \widetilde{W}(h(a, a, 1, 1)) d^\times a \\ &= \int_{F^\times} |a|^s \int_F \widetilde{W}'(h(a, a, 1, 1)u''(y)) \hat{\Phi}'(y) dy d^\times a \\ &= \int_{F^\times} |a|^s \widetilde{W}'(h(a, a, 1, 1)) \int_F \psi(u''(ay)) \hat{\Phi}'(y) dy d^\times a. \end{aligned}$$

Now since $\psi(u''(ay)) = \psi_\circ(\frac{1}{2}\text{Tr}(ay(1-\theta))) = \psi_\circ(ay)$, the inner integral is

$$\int_F \hat{\Phi}'(y) \psi_\circ(ay) dy = \Phi'(a),$$

so

$$J(s, 1, \widetilde{W}) = \int_{F^\times} |a|^s \widetilde{W}'(h(a, a, 1, 1)) \Phi'(a) d^\times a.$$

Now take $\Phi' \in C_c^\infty(F)$ nonnegative and with support containing 1 and small enough so that the last integrand is the positive constant $\widetilde{W}'(1) = 1$ on the support of Φ' . This choice of Φ' depends only on $\widetilde{W}'$, not on s , and we may scale Φ' (or $\widetilde{W}'$) so the integral is equal to 1, when s is in the domain of convergence. $\square$

Next, let Φ be a Schwarz-Bruhat function on F^2 and consider the integral

$$I_1(s, \widetilde{W}, \Phi) = \int_{N_2(F) \backslash \text{GL}_2(F)} |e(g)|^s \widetilde{W}(m(g)) \Phi((0, 1)g) dg.$$

This also converges for $\text{Re}(s) \gg 0$ by Prop. 4.1.3.

Lemma 4.2.3. *We may choose Φ and $\widetilde{W}$ so that $I_1(s, \widetilde{W}, \Phi)$ is a nonzero constant independent of s , for $\text{Re}(s)$ sufficiently large.*

Proof. Using the decomposition $N_2 \backslash \text{GL}_2 = (N_2 \backslash P'_2)Z_2K_2$, we may write the integral as

$$\begin{aligned} I_1(s, \widetilde{W}, \Phi) &= \int_{N_2(F) \backslash P'_2(F)} \int_{Z_2(F)} \int_{K_2} |e(pzk)|^s \delta_{B_2}^{-1}(pz) \\ &\quad \cdot \widetilde{W}(m(pzk)) \Phi((0, 1)pzk) dk dz dp \\ &= \int_{K_2} \int_{Z_2(F)} \Phi((0, 1)zk) |e(z)|^s \left(\int_{N_2(F) \backslash P'_2(F)} |e(p)|^{s-1} \widetilde{W}(m(pzk)) dp \right) dz dk \\ &= \int_{K_2} \int_{Z_2(F)} \Phi((0, 1)zk) |e(z)|^s J(s-1, zk, \widetilde{W}) dz dk. \end{aligned}$$

Now by Lemma 4.2.2, $J(s', g, \widetilde{W})$ is smooth in g and nonvanishing at $g = 1$, so there is an open compact subgroup $K'_2 \subseteq K_2$ so that $J|_{K'_2}$ is a constant, say 1. In fact, we may choose K'_2 of the form

$$K'_2 = \{k \in K_2 \mid k \equiv I_2 \pmod{\mathfrak{p}^\ell}\} = \left\{ \begin{pmatrix} 1 + \mathfrak{p}^\ell & \mathfrak{p}^\ell \\ \mathfrak{p}^\ell & 1 + \mathfrak{p}^\ell \end{pmatrix} \right\}, \quad \text{some } \ell \geq 0,$$

with $\mathfrak{p} = \varpi\mathcal{O}$ the maximal ideal in $\mathcal{O}$, so that $(0, 1)K'_2 \subseteq (\mathfrak{p}^\ell, 1 + \mathfrak{p}^\ell)$. Set $\Phi = \mathbf{1}_{(0, 1)K'_2}$. Note that for $z = \begin{pmatrix} a & \\ & a \end{pmatrix}$ and $k = (k_{ij}) \in K_2$, either k_{21} or k_{22} is a unit, so if $zk \in K'_2$ then $(0, 1)zk = (ak_{21}, ak_{22}) \in (\mathfrak{p}^\ell, 1 + \mathfrak{p}^\ell)$. If $k_{22} \in \mathcal{O}^\times$, then $ak_{22} \in 1 + \mathfrak{p}^\ell \subseteq \mathcal{O}^\times$ implies $a \in \mathcal{O}^\times$. If $k_{21} \in \mathcal{O}^\times$, then $ak_{21} \in \mathfrak{p}^\ell$ implies $a \in \mathfrak{p}^\ell$, but then $k_{22} \in \mathcal{O}$ forces $ak_{22} \in \mathfrak{p}^\ell$, a contradiction. Thus $a \in \mathcal{O}^\times$. Also, if $\Phi((0, 1)k) \neq 0$ for $k \in K_2$, then $k = pk'$ for $k' \in K'_2$ and $p = kk'^{-1} \in P'_2(F) \cap K_2$. By [27], Thm. 8.32 and its proof (noting that in this case both $S = P'_2(F) \cap K_2$ and $T = K'_2$ are compact), we can decompose the integral as

$$\begin{aligned} I_1(s, \widetilde{W}, \Phi) &= \int_{K_2} \Phi((0, 1)k) J(s - 1, k, \widetilde{W}) dk \\ &= \int_{K'_2} \int_{P'_2(F) \cap K_2} J(s - 1, pk, \widetilde{W}) dp dk' \\ &= \text{vol}_{dp}(P'_2(F) \cap K_2) \int_{K'_2} J(s - 1, k, \widetilde{W}) dk' \\ &= \text{vol}_{dp}(P'_2(F) \cap K_2) \text{vol}_{dk'}(K'_2) \neq 0. \end{aligned} \quad \square$$

Proof of Proposition 4.2.1. With $\widetilde{W}$ as in Lemma 4.2.3, define

$$W_0(g) = \int_{U^{Q'}(F)} \widetilde{W}(gu') \phi(u') du',$$

where $\phi \in C_c^\infty(U^{Q'}(F))$ is such that $\rho(\phi) = \hat{\Phi}$, with Φ as in Lemma 4.2.3. Then W_0 is

another Whittaker function which is right-invariant by some open compact subgroup K'_0 of K_0 .

Claim. We can choose $f^* \in \text{ind}_{QX_2^{Q'}}^{G_0}(\Xi_s)$ with $\text{supp } f^* \subseteq (QX_2^{Q'})(F) \cdot K'_0$, such that f^* restricted to K'_0 is a nonzero constant.

We will prove this below. Assuming it for now, we proceed to the integral, taking $f_{s,0}$ to correspond to f^* under (4.12):

$$\begin{aligned} \Psi(s, W_0, f_{s,0}) &= \int_{Z(F)N^Q(F)X_2^{Q'}(F) \backslash G_0(F)} f_{s,0}(\gamma_0 g) W_0(g) dg \\ &= \int_{Z(F)N^Q(F)X_2^{Q'}(F) \backslash Q(F)X_2^{Q'}(F)K'_0} f^*(g) W_0(g) dg \\ &= \int_{Z(F)N^Q(F) \backslash Q(F)} \int_{\tilde{K}'_0} f^*(qk') W_0(qk') dk' dq \\ &= \int_{Z(F)N^Q(F) \backslash Q(F)} \Xi_s(q) W_0(q) \int_{\tilde{K}'_0} f^*(k') dk' dq, \end{aligned}$$

where $\tilde{K}'_0 = (K'_0 \cap Z(F)N^Q(F)X_2^{Q'}(F)) \backslash K'_0$. So the inner integral is a constant $C = C_{f^*} \neq 0$, and since $Z(F)N^Q(F) \backslash Q(F) \simeq N_2(F) \backslash \text{GL}_2(F)$, this becomes

$$\begin{aligned} \Psi(s, W_0, f_{s,0}) &= C \int_{N_2(F) \backslash \text{GL}_2(F)} |e(g)|^{\frac{s+3}{2}} W_0(m(g)) dg \\ &= C \int_{N_2(F) \backslash \text{GL}_2(F)} |e(g)|^{s'} \int_{U^{Q'}(F)} \widetilde{W}(m(g)u') \phi(u') du' dg \\ &= C \int_{N_2(F) \backslash \text{GL}_2(F)} |e(g)|^{s'} \widetilde{W}(m(g)) \int_{U^{Q'}(F)} \phi(u') \psi(m(g)u'm(g)^{-1}) du dg \end{aligned}$$

with $s' = \frac{s+3}{2}$. Now for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $m(g)u'(x, z, w)m(g)^{-1} = n(x_1, z_1, w_1)$, where

$$x_1 = cz\theta + dx(1 + \theta), \quad (4.15)$$

$$z_1 = az + bx\theta(1 + \theta), \quad (4.16)$$

$$w_1 = e(g)w(1 + \theta), \quad (4.17)$$

so

$$\psi(m(g)u'(x, z, w)m(g)^{-1}) = \psi_\circ(\theta^2(cz + dx)). \quad (4.18)$$

Hence, by our choice of ϕ , the inner integral is

$$\begin{aligned} & \int_{F^2} \left(\int_F \phi(u'(x, z, w)) dw \right) \psi_\circ(\theta^2(cz + dx)) dx dz \\ &= \int_{F^2} \rho(\phi)(x, z) \psi_\circ(\theta^2(cz + dx)) dx dz \\ &= \int_{F^2} \hat{\Phi}(x, z) \psi_\circ(\theta^2(cz + dx)) dx dz \\ &= \Phi(c, d) = \Phi((0, 1)g). \quad (\text{since } (c, d) \neq (0, 0)) \end{aligned}$$

Putting all this together yields

$$\Psi(s, W_0, f_{s,0}) = C \int_{N_2(F) \backslash \text{GL}_2(F)} |e(g)|^{s'} \widetilde{W}(m(g)) \Phi((0, 1)g) dg.$$

By Lemma 4.2.3, this is a nonzero constant independent of s' , so we are done. $\square$

Now we prove the Claim. Let Y be the root subgroup of GSp_4 for the root $-2\epsilon_1$, so

that $Y \cdot B_4 = P_4$, and let $Y^Q(F) = Q(F) \cap Y(E)$. Then

$$\begin{aligned}
G_0(F) &= Y^Q(F)B_0(F)K_0 \\
&= Y^Q(F)N^Q(F)N^{Q'}(F)T_4(E)^\circ K_0 \\
&= Y^Q(F)N^Q(F)T_4(F)N^{Q'}(F)\tilde{T}K_0 \\
&= Q(F)N^{Q'}(F)\tilde{T}K_0 \\
&= (QX_2^{Q'})(F) \cdot \tilde{N}^{Q'}(F)\tilde{T}K_0,
\end{aligned} \tag{4.19}$$

where

$$\tilde{T} = \begin{cases} (\mathcal{O}_F^\times \setminus \mathcal{O}_E^\times)^2 & \text{if } E/F \text{ is unramified,} \\ \{t_{\epsilon_1, \epsilon_2} | \epsilon_1, \epsilon_2 \in \{0, 1\}\} \cdot (\mathcal{O}_F^\times \setminus \mathcal{O}_E^\times)^2 & \text{if } E/F \text{ is ramified} \end{cases} \simeq T_4(F) \setminus T_4(E)^\circ, \tag{4.20}$$

with

$$t_{\epsilon_1, \epsilon_2} = \begin{pmatrix} \theta^{\epsilon_1} & & & \\ & \theta^{\epsilon_2} & & \\ & & \theta^{-\epsilon_1} & \\ & & & \theta^{-\epsilon_2} \end{pmatrix} \tag{4.21}$$

and

$$\tilde{N}^{Q'}(F)(F) = \{u''(y)u'(x, z, 0) \mid x, y, z \in F\} \subseteq N^{Q'}(F). \tag{4.22}$$

Note that $\tilde{N}^{Q'}(F)$ is a subset (!) which clearly commutes with $X_2^{Q'}(F)$, satisfies $X_2^{Q'}(F) \cdot \tilde{N}^{Q'}(F) = N^{Q'}(F)$ and is mapped bijectively onto $X_2^{Q'}(F) \setminus N^{Q'}(F)$ under the quotient map.

Also, in decomposing $N_4(E) = N^Q(F) \cdot N^{Q'}(F)$, we may replace θ by $\varpi_F^* \theta$ and so assume $\theta \in \mathcal{O}_E^\times$ (in the unramified case) or $\theta \in \varpi_E \mathcal{O}_E^\times$ (in the ramified case). In particular, $\tilde{T}K_0$ is either K_0 or $(\mathbb{Z}/2\mathbb{Z})^2 \cdot K_0$.

To define f^* , we need to understand the intersection $(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)\tilde{T}K_0$. For this we have a

Lemma 4.2.4. *Let $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and let $t_{\epsilon_1, \epsilon_2}$ be as in (4.21), noting that $\epsilon_1 = 0 = \epsilon_2$ if E/F is unramified.*

(i). *The sets $\tilde{N}^{Q'}(F)t_{\epsilon_1, \epsilon_2}K_0$ are disjoint for distinct pairs (ϵ_1, ϵ_2) .*

(ii). *$(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)t_{\epsilon_1, \epsilon_2}K_0 = \emptyset$ unless $\epsilon_1 = 0$.*

(iii). *$(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)t_{0,0}K_0 \subseteq (QX_2^{Q'})(F) \cap K_0$.*

(iv). *$(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)t_{0,1}K_0 \subseteq (QX_2^{Q'})(F) \cap t_\theta K_0 t_\theta^{-1}$, for some $t_\theta \in T_4(E)$.*

Proof. Recall that $Q(F)$ has Levi decomposition

$$Q(F) = M^Q(F)U^Q(F) = Z(F)\widetilde{M}^Q(F)U^Q(F),$$

with $\widetilde{M}^Q$ as in (2.11), and similarly for $P_4(E)$. We will make use of the isomorphisms given by

$$\widetilde{M}^{P_4}(E) \simeq \mathrm{GL}_2(E), \quad \begin{pmatrix} e & & & \\ & a & b & \\ & & 1 & \\ & c & & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4.23a)$$

$$X_2^{P_4}(E) \setminus U^{P_4}(E) \simeq E^2, \quad \begin{pmatrix} 1 & x & w & s \\ & 1 & s & \\ & & 1 & \\ & -x & & 1 \end{pmatrix} \mapsto \begin{pmatrix} s \\ -x \end{pmatrix}, \quad (4.23b)$$

respectively. Note that the (left) action of $\widetilde{M}^{P_4}(E)$ on $U^{P_4}(E)$ by conjugation is transferred to left multiplication by $\mathrm{GL}_2(E)$ on E^2 under these isomorphisms.

(i). Let $t_1 = t_{\epsilon_1, \epsilon_2}$ and $t_2 = t_{\epsilon_3, \epsilon_4}$ and suppose $\widetilde{N}^{Q'}(F)t_1K_0 \cap \widetilde{N}^{Q'}(F)t_2K_0$ is nonempty.

Then there is an $n' \in N^{Q'}(F)$ such that $t_2^{-1}n't_1 = (t_2^{-1}n't_2)t_2^{-1}t_1 \in K_0$. But this implies $t_2^{-1}t_1 \in K_0$, which can only happen if $\epsilon_1 = \epsilon_3$ and $\epsilon_2 = \epsilon_4$, so $t_1 = t_2$.

(ii). Let $g = qv = q'tk \in (QX_2^{Q'})(F) \cap \widetilde{N}^{Q'}(F)t_{\epsilon_1, \epsilon_2}K_0$, with $t = t_{\epsilon_1, \epsilon_2}$. Then $k = t^{-1}q'^{-1}qv \in P_4(E)$, so we have Levi decompositions $k = z_k m_k n_k$, $q = zmn$ and $q' = m'n'$ where $m' = u''(y)$ and $n' = u'(x', s', 0)$. Then

$$zmnv = m'n'tz_k m_k n_k \quad (4.24)$$

implies equality of Levi factors, $zm = z_k m' t m_k$. Noting that $e'(m) = e'(m_k) = e'(m') = 1$, we get $e'(z) = e'(z_k)e'(t)$ and hence $\theta^{\epsilon_1} = e'(t)^{-1} = e'(z_k)e'(z)^{-1} \in F^\times \cdot \mathcal{O}_E^\times$, which means $\epsilon_1 = 0$.

(iii). Taking $t = t_{0,0} = 1$ in the previous argument, we see $e'(z) = e'(z_k) \in F^\times \cap \mathcal{O}_E^\times = \mathcal{O}_F^\times$, so $z = z_k \in Z(F) \cap K_0$. Using uniqueness of Levi decomposition in (4.24) we see that

$$m = m' m_k \quad (4.25)$$

$$nv = m_k^{-1} n' m_k n_k. \quad (4.26)$$

First let us show that $m, m' \in K_0$. By our choice of $m \in \widetilde{M}^Q(F)$ and $m' = u''(y) \in \widetilde{M}^{Q'}(F)$ and using (4.23), it is enough to show that

$$g = g' g_k \implies g, g' \in \mathrm{GL}_2(\mathcal{O}_E)$$

for

$$g = \begin{pmatrix} a & b\theta \\ c\theta & d \end{pmatrix} \in \mathrm{GL}_2(E), \quad (a, b, c, d \in F) \quad (4.27a)$$

$$g' = \begin{pmatrix} 1 & y(1 - \theta) \\ & 1 \end{pmatrix}, \quad (y \in F) \quad (4.27b)$$

$$g_k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_E). \quad (4.27c)$$

This is a matter of checking entry-by-entry, and noting that if $\alpha = \alpha_1 + \alpha_2\theta \in \mathcal{O}_E$, then both $\alpha_1, \alpha_2 \in \mathcal{O}_F$ since $\mathrm{ord}_E \theta = 1$ and $\mathrm{ord}_E a$ is even for any $a \in F$. Now since $m_k = m'^{-1}m$ by (4.25), we may rewrite (4.26) as

$$nv = m^{-1}m'n'm'^{-1}mn_k = m^{-1}n'_1mn_k, \quad n'_1 \in N^{\mathcal{Q}'}(F). \quad (4.28)$$

Since $m' \in K_0$, we have $n' \in K_0$ if and only if $n'_1 \in K_0$, so we may assume for our purpose that $n' = n'_1$. Since $n' \in \tilde{N}^{\mathcal{Q}'}(F)$, the X_2 -entry of $m^{-1}n'mn_k$ is in $\mathcal{O}_E$ and is equal to the X_2 -entry of v (which is in $F(1 + \theta) \subseteq E$) plus the X_2 -entry of n (which is in $F\theta \subseteq E$), which implies $v \in K_0$ and the X_2 -entry of n is in $\mathcal{O}_E$. So we must show $n = m^{-1}n'mn_k$ implies $n, n' \in K_0$. Using (4.23) with $n = u(x, s, w)$, $n' = u'(x', s', w')$ and $n_k = n(\xi, \sigma, \omega)$, this may be rewritten as

$$\begin{pmatrix} s\theta \\ -x \end{pmatrix} = \begin{pmatrix} a & b\theta \\ c\theta & d \end{pmatrix}^{-1} \begin{pmatrix} s' \\ -x'(1 + \theta) \end{pmatrix} + \begin{pmatrix} \sigma \\ -\xi \end{pmatrix}, \quad (4.29)$$

for $s, s', x, x' \in F$ and $\sigma, \xi \in \mathcal{O}_E$. Again, computing each entry, using rationality

and integrality shows that $x', s' \in \mathcal{O}_F$, so $n' \in K_0$ and hence $n \in K_0$. Thus $q \in K_0$.

(iv). The method here is identical to the previous step, except with the extra element $t_{0,1}$ included in the calculations. With g, g', g_k as in (4.27), the equation now becomes

$$g = g' \begin{pmatrix} \theta & \\ & \theta^{-1} \end{pmatrix} g_k,$$

which implies that g satisfies

$$\begin{aligned} a &\in \theta^2 \mathcal{O}_F, & b &\in \mathcal{O}_F^\times, \\ c &\in \theta^{-2} \mathcal{O}_F^\times, & d &\in \mathcal{O}_F, \end{aligned}$$

which we abbreviate by

$$g \in \left\{ \begin{pmatrix} \theta^2 \mathcal{O}_F & \theta \mathcal{O}_F^\times \\ \theta^{-1} \mathcal{O}_F^\times & \mathcal{O}_F \end{pmatrix} \right\}. \quad (4.30)$$

The unipotent equation (4.26) likewise becomes $nv = (tm_k)^{-1}n'(tm_k)n_k$, which leaves (4.28) as before. (Here we do *not* assume $n' = n'_1$.) Using (4.30) and proceeding as before we find that $v \in K_0$ and

$$x \in \mathcal{O}_F, \quad s \in \theta^{-2} \mathcal{O}_F, \quad (4.31)$$

with x and s as in (4.29). Thus we may (symbolically) write

$$q \in \left\{ \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F & \theta \mathcal{O}_F & \theta^{-1} \mathcal{O}_F \\ & \theta^2 \mathcal{O}_F & \theta \mathcal{O}_F & \theta \mathcal{O}_F^\times \\ & & \mathcal{O}_F^\times & \\ & \theta^{-1} \mathcal{O}_F^\times & \theta^{-2} \mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \right\} \subseteq Q(F) \cap t_\theta K_0 t_\theta^{-1},$$

where

$$t_\theta = h(\theta^{-1}, \theta, \theta^2, 1).$$

□

One immediate consequence of (iii) and (iv) is that

$$\Xi_s|_{(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)t_{0,\epsilon}K_0} \equiv 1,$$

since for g in either set, both $e(g)$ and $e'(g)$ are units.

Now we may define f^* as follows. Let K'_0 be the open compact subgroup of K_0 chosen above, and choose an open compact subgroup $K'_1 \subseteq K_0$ such that $t_\theta K'_1 t_\theta^{-1} \subseteq K'_0$. Then $(QX_2^{Q'})(F) \cap K'_0$ has finite index in $(QX_2^{Q'})(F) \cap K_0$; let $\{k_{i,0}\}_{i=1}^{r_0}$ represent the quotient and set $\Omega_0 = \cup_{i=1}^{r_0} k_{i,0} K'_0$, so Ω_0 is an open compact subset of K_0 . Similarly, let $\{k_{i,1}\}_{i=1}^{r_1}$ represent the (finite) quotient of $(QX_2^{Q'})(F) \cap t_\theta K_0 t_\theta^{-1}$ by $(QX_2^{Q'})(F) \cap t_\theta K'_1 t_\theta^{-1}$ and set $\Omega_1 = \cup_{i=1}^{r_1} k_{i,1} t_\theta K'_1 t_\theta^{-1}$. Define

$$f^*|_{\tilde{N}^{Q'}(F)t_{0,0}K_0} = \mathbf{1}_{\Omega_0},$$

$$f^*|_{\tilde{N}^{Q'}(F)t_{0,1}K_0} = \mathbf{1}_{\Omega_1}.$$

Then by (iii) and (iv) of Lemma 4.2.4,

$$f^*|_{(QX_2^{Q'})(F) \cap \tilde{N}^{Q'}(F)t_{0,\epsilon}K_0} \equiv 1$$

by choice of Ω_1, Ω_2 , and

$$\text{supp } f^* \subseteq (QX_2^{Q'})(F) \cdot K'_0 \cup (QX_2^{Q'})(F) \cdot t_\theta K'_1 t_\theta^{-1} \subseteq (QX_2^{Q'})(F) \cdot K'_0,$$

by choice of K'_1 . Finally we have $f^*|_{K'_0} \equiv 1$ by definition. This proves the claim.

4.2.2 Archimedean case

Since we have restricted our attention to nonsplit places, we have $F = \mathbb{R}$, $E = \mathbb{C}$, and we can take $\theta^2 = -1$. Also, it is easy to check that

$$\begin{aligned} N_4(\mathbb{C}) \cap K_0 &= 1, \\ T_4(\mathbb{C}) \cap K_0 &= T_4(\mathbb{C})^1, \end{aligned}$$

where $K_0 = \mathrm{Sp}(4)^\pm$ as in (4.8). Here $T_4(\mathbb{C})^1$ is the subgroup of $T_4(\mathbb{C})$ with entries in $\mathbb{C}^1$ (=complex numbers of modulus 1). Similarly we denote by $T_4(\mathbb{R})_+^\times$ the subgroup of $T_4(\mathbb{R})$ whose elements have positive real entries.

We have a similar group decomposition to (4.19), with the modification that since $T_4(\mathbb{C}) = T_4(\mathbb{R})_+^\times \cdot T_4(\mathbb{C})^1$, we have $T_4(\mathbb{C})^\circ K_0 = T_4(\mathbb{R})_+^\times \cdot K_0 = T_4(F)K_0$ and so

$$G_0(F) = (QX_2^{Q'})(F) \cdot \tilde{N}^{Q'}(F)K_0.$$

The isomorphism (4.12) also holds, and because $\tilde{N}^{Q'}(F) \cap K_0 = 1$, we can choose the section f^* to be given by

$$f^*(q'k) = \phi_1(q')\phi_2(k), \quad \text{for } q' \in \tilde{N}^{Q'}(F), k \in K_0, \quad (4.32)$$

where $\phi_1 \in C_c^\infty(\tilde{N}^{Q'}(F))$ and $\phi_2 \in C_c^\infty(K_0)$ will be determined below. Note that if f_s corresponds to f^* under (4.12), then f_s will not be K_8 -finite: a K_8 -finite function will be real analytic, hence cannot vanish on an open set.

Recall our integral

$$\Psi(s, W, f_s) = \int_{Z(F)X_2^{Q'}(F)N^Q(F)\backslash G_0(F)} f_s(\gamma_0 g)W(g) dg.$$

Proposition 4.2.5. *Let σ_0 be as in Prop. 4.1.8. Then the integral $\Psi(s, W, f_s)$ is not identically zero for $\operatorname{Re}(s) > \sigma_0$. More precisely, there is a choice of K_4 -finite Whittaker function W_0 and smooth section $f_{s,0}$ such that $\Psi(s, W_0, f_{s,0})$ is not zero, independent of s , for $\operatorname{Re}(s) > \sigma_0$.*

To show this we need some preliminary results, as before.

First consider the integral

$$J(s, g, \widetilde{W}) = \int_{N_2(F) \backslash P'_2(F)} |\det p|^s \widetilde{W}(m(pg)) dp;$$

this is exactly the integral (4.13) and it converges for $\operatorname{Re}(s) \gg 0$ by the same argument. Also note that $J(s, g, \widetilde{W})$ will be K_2 -finite when $\widetilde{W}$ is K_4 -finite.

Lemma 4.2.6. *We may choose $\widetilde{W}$ such that $J(s, 1, \widetilde{W}) > 0$ for all s with $\operatorname{Re}(s)$ sufficiently large.*

Proof. The choice of $\widetilde{W}$ and the formal manipulation of the integral proceed exactly as in Lemma 4.2.2, noting that $\widetilde{W}$ will be K_4 -finite if $\widetilde{W}'$ is. We only need to note that since $\Phi' \in C_c^\infty(F)$, its Fourier transform $\hat{\Phi}'$ is in $\mathcal{S}(F)$, hence is rapidly decreasing. So by the moderate growth of $\widetilde{W}'$, the integral (4.14) defining $\widetilde{W}$ is absolutely convergent. Unfolding the integral and choosing Φ' nonnegative with small support (and independent of s) as before, we note that though the integrand is no longer constant on the support of Φ' , the integral will still be positive for all s with $\operatorname{Re}(s)$ sufficiently large. This completes the argument. $\square$

Next we examine the integral given by

$$A(s, y, k; W, \Phi) = \int_{N_2(F) \backslash \operatorname{GL}_2(F)} |e(g)|^s W(m(g)u''(y)k)\Phi((0, 1)g) dg \quad (4.33)$$

for $y \in F$, $k \in K_0$, $\Phi \in C_c^\infty(F^2)$ and W a Whittaker function. This converges for $\operatorname{Re}(s) > 0$ by the gauge estimate, Prop. 4.1.3.

Lemma 4.2.7. *Given s with $\operatorname{Re}(s)$ large enough, there are W and Φ such that $A(s, 1, 1; W, \Phi) > 0$.*

Proof. The steps are roughly the same as in the proof of Lemma 4.2.3. Decomposing $\operatorname{GL}_2(F) = P'_2(F)Z(F)K_2$ with $K_2 = \operatorname{O}(2)$ and the measure $dg = \delta_{B_2}^{-1}(pz) dp dz dk$, we see that

$$\begin{aligned} A(s, 1, 1; W, \Phi) &= \int_{K_2} \int_{Z_2(F)} \int_{N_2(F) \setminus P'_2(F)} |\det p|^{s-1} |\det z|^s \\ &\quad \cdot W(m(pzk)) \Phi((0, 1)zk) dp dz dk \\ &= \int_{K_2} \int_{F^\times} |a|^{2s} J(s-1, ak, W) \Phi((0, 1)ak) d^\times a dk, \end{aligned}$$

where we have identified $z = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$ with the scalar $a \in F^\times$. Take W as in Lemma 4.2.6 so that $J(s-1, 1, W) = 1$, and let

$$\Phi(u, v) = \begin{cases} \Phi((0, 1)ak) \stackrel{\text{def}}{=} \phi_3(a) \overline{J(s-1, k, W)} & \text{if } (u, v) \neq (0, 0), \\ 0 & \text{if } (u, v) = (0, 0), \end{cases}$$

for $\phi_3 \in C_c^\infty(F^\times)$ to be chosen. Since for $(u, v) \neq (0, 0)$ we have

$$(u, v) = (0, 1)ak = (0, 1)ak_\alpha = (a \sin \alpha, a \cos \alpha)$$

and $u^2 + v^2 = a^2$, Φ is supported on an annulus (with positive inner radius) about the origin; it is K_2 -finite, hence smooth in k , since J is K_2 -finite. So $\Phi \in C_c^\infty(F^2)$.

The integral in question now becomes

$$A(s, 1, 1; W, \Phi) = \int_{F^\times} |a|^{2s} \phi_3(a) \int_{K_2} J(s-1, ak, W) \overline{J(s-1, k, W)} dk d^\times a.$$

If we denote the inner integral by $A_1(s, a)$, then $A_1(s, 1) > 0$. Thus we may take ϕ_3 to have support containing 1 and small enough that $A(s, 1, 1; W, \Phi) \neq 0$. $\square$

Proof of Prop. 4.2.5. Let f^* be as in (4.32) and let $f_{s,0}$ correspond to f^* under (4.12).

Then

$$\begin{aligned}\Psi(s, W, f_{s,0}) &= \int_{Z(F)X_2^{Q'}(F)N^Q(F)\backslash G_0(F)} f^*(g)W(g) dg \\ &= \int_{\{\pm I\}\backslash K_0} \int_{X_2^{Q'}(F)\backslash N^{Q'}(F)} \int_{Z(F)N^Q(F)\backslash Q(F)} f^*(qq'k)W(qq'k) dq dq' dk \\ &= \int_{\overline{K}_0} \phi_2(k) \int_{X_2^{Q'}(F)\backslash N^{Q'}(F)} \phi_1(q') \int_{N_2(F)\backslash \text{GL}_2(F)} |e(g)|^{s'} W(m(g)q'k) dg dq' dk,\end{aligned}$$

where $\overline{K}_0 = \{\pm I\}\backslash K_0$ and $s' = \frac{s+3}{2}$. Now let q' have Levi decomposition $q' = n'm' = u'(x, z)u''(y)$, for $n' = u'(x, z) = u'(x, z, 0) \in U^{Q'}(F)$ and $m' = u''(y) \in Q'(F)$. As varieties, $X_2^{Q'}(F) \backslash N^{Q'}(F) \simeq F^3$, so we may take ϕ_1 to be $\phi_1(q') = \phi_1(n'm') = \phi'_1(x, z)\phi''_1(y)$ for $\phi'_1 \in C_c^\infty(F^2)$ and $\phi''_1 \in C_c^\infty(F)$ to be chosen. The integral becomes

$$\begin{aligned}\Psi(s, W, f_{s,0}) &= \int_{\overline{K}_0} \phi_2(k) \int_{X_2^{Q'}(F)\backslash N^{Q'}(F)} \phi_1(n'm') \\ &\quad \cdot \int_{N_2(F)\backslash \text{GL}_2(F)} |e(g)|^{s'} W(m(g)n'm'k) dg d(n'm') dk \\ &= \int_{\overline{K}_0} \phi_2(k) \int_F \phi''_1(y) \int_{N_2(F)\backslash \text{GL}_2(F)} |e(g)|^{s'} W(m(g)u''(y)k) \\ &\quad \cdot \left(\int_{F^2} \phi'_1(x, z) \psi(\text{Ad}(m(g))u'(x, z)) dx dz \right) dg dy dk.\end{aligned}$$

(Note that although $d(n'm') = \delta_{P_4}^{-1}(m') dn' dm'$, $m' = u''(y)$ has unipotent image in $\text{GL}_2(E)$ under (4.23) and so $\delta_{P_4}(u''(y)) = 1$.) By (4.15) and (4.18), the integral over x and z is

$$\int_{F^2} \phi'_1(x, z) \psi_\circ(\theta^2(cz + dx)) dx dz, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which is the Fourier transform of ϕ'_1 . Hence we may choose ϕ'_1 so that the integral equals $\Phi(c, d)$ for our choice of Φ in Lemma 4.2.7, and for $(c, d) \neq (0, 0)$ we may write $\Phi(c, d) = \Phi((0, 1)g)$. This yields

$$\begin{aligned} \Psi(s, W, f_{s,0}) &= \int_{\overline{K}_0} \phi_2(k) \int_F \phi_1''(y) \int_{N_2(F) \backslash \mathrm{GL}_2(F)} |e(g)|^{s'} \\ &\quad \cdot W(m(g)u''(y)k) \Phi((0, 1)g) \, dg \, dy \, dk \\ &= \int_{\overline{K}_0} \phi_2(k) \int_F \phi_1''(y) A(s', y, k; W, \Phi) \, dy \, dk, \end{aligned}$$

with $A(s', y, k; W, \Phi)$ given by (4.33). By Lemma 4.2.7, there is a Whittaker function W_0 such that $A(s', 1, 1; W_0, \Phi) > 0$. So choose $\phi_1'' \geq 0$ and $\phi_2 \geq 0$ with support containing 1 and small enough that $A(s', y, k; W_0, \Phi) > 0$ on $\mathrm{supp} \phi_1'' \times \mathrm{supp} \phi_2$. Then the inner integral is positive for $k \in \mathrm{supp} \phi_2$, so $\Psi(s, W_0, f_{s,0}) > 0$ as well. $\square$

4.3 Final Remarks

Before concluding we would like to return to consideration of the global integral (2.5). After this, we outline our continuing work with this integral and highlight some possible future directions of research.

By [24, IV, Prop. 1.1.2], the Eisenstein series $E^*(g, s, f_s)$ has simple poles at $s = 3$ and $s = 2$ and a double pole at $s = 1$. Since the meromorphic continuation of the series propagates to that of the integral, studying poles of the integral $I^*(s, \varphi, f_s) \stackrel{\mathrm{def}}{=} d_G(s)I(s, \varphi, f_s)$ amounts to studying residues of this Eisenstein series. Jiang carries this out in chapter 4 of [24], and the result is as follows.

Proposition 4.3.1 (cf. [24], IV, Thm. 4.0.2). *The integral $I^*(s, \varphi, f_s)$ is holomorphic for all s except possibly at $s = 1$, where it may achieve a pole of order at most two.*

Proof. We only need to check holomorphy at $s = 3$ and $s = 2$, and Jiang's proof carries over with minimal modification. We only outline the argument. At $s = 3$, the residue of $E^*(g, s, f_s)$ is constant, so the residue is

$$\text{Res}_{s=3} I^*(s, \varphi, f_s) = c_3 \int_{Z(\mathbb{A})G_0(F)\backslash G_0(\mathbb{A})} \varphi(g) dg,$$

which vanishes since φ is a cusp form.

At $s = 2$, Jiang proves an identity of between the residue of $E^*(g, s, f_s)$ and the constant term of an Eisenstein series $E^{1,*}(g, s, M(f_s))$ on the (standard) parabolic subgroup P_8^1 of Sp_8 with Levi factor isomorphic to $\text{Sp}_6 \times \text{GL}_1$, where M is an intertwining operator. So the residue at $s = 2$ is

$$\text{Res}_{s=2} I^*(s, \varphi, f_s) = c_2 \int_{Z(\mathbb{A})G_0(F)\backslash G_0(\mathbb{A})} \varphi(g) E^{1,*}(g, s, M(f_s)) dg.$$

Here the integral vanishes after unfolding the Eisenstein series, because all the orbits of $G_0(F)$ on $P_8^1(F) \backslash \text{Sp}_8(F)$ are negligible (in the sense of [35]). $\square$

This leaves the question of behavior of the integral at $s = 1$. In order to motivate the discussion that follows, first recall that Jiang's global integral is

$$\tilde{I}^*(s, \varphi_1, \varphi_2, f_s) = \int_{Z(\mathbb{A})\tilde{G}_0(F)\backslash \tilde{G}_0(\mathbb{A})} \varphi_1(g_1)\varphi_2(g_2)E^*((g_1, g_2), s, f_s) d(g_1, g_2)$$

where φ_1 and φ_2 are cusp forms belonging to (irreducible, smooth, generic) cuspidal automorphic representations π_1 and π_2 of $\text{GSp}_4(\mathbb{A}_F)$, respectively. Jiang shows [24, IV, Thm. 3.2.1] another identity between the second-order residue of $E^*(g, s, f_s)$ at

$s = 1$ and the first-order residue of the Siegel-type Eisenstein series $E_8^{4,*}(g, s, M'(f_s))$ on Sp_8 at $s = \frac{1}{2}$. So his integral has residue at $s = 1$

$$\begin{aligned} \mathrm{Res}_{s=1} \tilde{I}^*(s, \varphi_1, \varphi_2, f_s) \\ = c_1 \cdot \mathrm{Res}_{s=\frac{1}{2}} \int_{Z(\mathbb{A})\tilde{G}_0(F)\backslash\tilde{G}_0(\mathbb{A})} \varphi_1(g_1)\varphi_2(g_2)E_8^{4,*}(g, s, M'(f_s)) \, d(g_1, g_2). \end{aligned}$$

But the integral on the right is the doubling integral of Piatetski-Shapiro and Rallis, which is nonzero if and only if π_1 and π_2 are contragredient [35, pp. 3–6]. Furthermore, by [31, Thm. 7.2.5], if the doubling integral has a pole at $s = \frac{1}{2}$, then $\pi_1 = \check{\pi}_2$ has a nonvanishing theta lift to $\mathrm{GO}_{2,2}$. This is Thm. 4.0.3, chapter IV, in [24].

Now we return to the integral $I^*(s, \varphi, f_s)$. Repeating the above analysis and using the same identities of Eisenstein series, we arrive at

$$\mathrm{Res}_{s=1} I^*(s, \varphi, f_s) = c'_1 \cdot \mathrm{Res}_{s=\frac{1}{2}} \int_{Z(\mathbb{A})G_0(F)\backslash G_0(\mathbb{A})} \varphi(g)E_8^{4,*}(g, s, f_s) \, dg.$$

The integral on the right is a kind of “twisted doubling” integral, and it can be shown that this integral is nonvanishing if and only if the period

$$\mathcal{P}(\varphi) = \int_{Z(\mathbb{A})\mathrm{GSp}_4(F)\backslash\mathrm{GSp}_4(\mathbb{A})} \varphi(g) \, dg$$

is nonzero for some $\varphi \in V_\pi$. By analogy with Jiang’s case, we pose the question: does the nonvanishing of this period indicate that π is a base change (functorial) lift from an automorphic representation π_0 on $\mathrm{GSp}_4(\mathbb{A}_F)$? Some evidence or at least heuristic in the affirmative direction is found in [16, 14]. Furthermore, π is a functorial lift from some π_0 , we may also ask if the existence of a pole of the “twisted doubling” integral is indicative of π_0 having a nonvanishing theta lift to a certain orthogonal group. The author intends to pursue these questions in later research.

Another set of questions the author is currently researching concerns the local integrals of Chapter 4. We have already mentioned that in the archimedean case, we need to show nonvanishing of the integrals for K -finite data, not just smooth data. This will likely be done using a continuity result and density argument similar to that in [7]. We also would like to show all the local integrals can be meromorphically continued to the complex plane. This is a standard argument in the nonarchimedean case (cf. [39]), and ultimately relies on asymptotic expansions for the Whittaker functions in the archimedean case (see [40]). It is useful because it allows us to relate poles of the global integral back to poles of the global L -function, which could then give a criterion for π begin a functorial lift in terms of poles of the twisted tensor L -function. However, more precise information is needed for this: we would need to know not only the continuation of the local integrals, but also that the ratio of local integral to local L -factor,

$$e_v(s, W_v, f_{s,v}) = \frac{\Psi_v^*(s, W_v, f_{s,v})}{L_v(s, \pi_v, \rho_{\text{TT}})},$$

is entire and that for any s , W_v and $f_{s,v}$ can be chosen to make the ratio nonzero. This is subtle and difficult, though conceivable since the L -factor is already known at archimedean places, and is now known at nonarchimedean places thanks to recent work of Gan and Takeda [9]. The global argument would then be based on the

equality

$$\begin{aligned}
I(s, \varphi, f_s) &= \prod_{v \in \Sigma_F} \Psi_v^*(s, W_v, f_{s,v}) \\
&= \left(\prod_{v \in S} \Psi_v^*(s, W_v, f_{s,v}) \right) \prod_{v \notin S} L_v(s, \pi_v, \rho_{\text{TT}}) \\
&= \left(\prod_{v \in S} \frac{\Psi_v^*(s, W_v, f_{s,v})}{L_v(s, \pi_v, \rho_{\text{TT}})} \right) \prod_{v \in \Sigma_F} L_v(s, \pi_v, \rho_{\text{TT}}) \\
&= \left(\prod_{v \in S} e_v(s, W_v, f_{s,v}) \right) L(s, \pi, \rho_{\text{TT}}).
\end{aligned}$$

Since there are W_v and $f_{s,v}$ for $v \in S$ such that the product of the factors $e_v(s, W_v, f_{s,v})$ is entire and nonzero, $I^*(s, \varphi, f_s)$ and $L(s, \pi, \rho_{\text{TT}})$ would have the same set of poles to the same orders. This would allow us to pose the above questions in terms of poles of $L(s, \pi, \rho_{\text{TT}})$, rather than those of the global integral.

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