Modules over Infinite Dimensional Algebras

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ABSTRACT

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Let A be an infinite dimensional K - algebra, where K is a field and let B be a basis for A . In this Dissertation, we explore a property of the basis B that guarantees that K^B (the direct product of copies indexed by B of the field K) can be made into an A -module in a natural way. We call bases satisfying that property "amenable," and we show that not all amenable bases yield isomorphic A -modules. Then we consider a relation (which we name congeniality) that guarantees that two different bases yield isomorphic A -module structures on K^B . We present several examples in the familiar setting of the algebra $K[x]$ of polynomials with coefficients in K and the Laurent Polynomial Algebra $K[x, x^{-1}]$. Finally, we introduce some results regarding these notions in the context of Leavitt Path Algebras.

To My Parents, Prof. AlJawharah Bubshait and Dr. Mohammed Al-Essa

To my Brothers Eng.Omar, Anas and Mouathe

To my Lovely Little sister Sara

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1 INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Let A be an algebra over a field K and let $\mathcal{B}$ be a basis. Then $A = \bigoplus_{b \in \mathcal{B}} Kb = K^{(\mathcal{B})}$ is the direct sum of copies of the field K indexed by $\mathcal{B}$. This is because the elements of A are finite linear combinations of elements of the base. One can think of those linear combinations as being full, possibly infinite, linear combinations of elements of $\mathcal{B}$ with the restriction that only finitely many coefficients are allowed to be non-zero. Thusly, associated with every algebra A there is another K -vector space, the product $P = \prod_{b \in \mathcal{B}} Kb = K^{\mathcal{B}}$ of copies of K (namely, the set of all infinite linear combinations of elements of $\mathcal{B}$, without any restriction on coefficients). A natural question is whether P has a module structure over the algebra A that is naturally induced by the knowledge of the products of elements from the basis. The answer to the question is that while all the various bases produce versions of P that are isomorphic to one another as vector spaces, some versions will be amenable to be made into A -modules and some others will not. For that reason, we introduce here the notion of an amenable basis basically being one for which the leading question has a positive answer. We use the notation $P = \mathcal{M}_{\mathcal{B}}$ to refer to the module structure on P for an amenable basis $\mathcal{B}$.

A follow up question is whether amenable bases yield isomorphic module structures. One approach to attain a sufficient condition is through the notion of congeniality of bases. A basis $\mathcal{B}$ is *congenial* to a basis $\mathcal{C}$ if for every element $c \in \mathcal{C}$ there exists only finitely many elements $b \in \mathcal{B}$, when represented in terms of $\mathcal{C}$ use a non-zero coefficient in their representation of c in terms of $\mathcal{B}$. It turns out that the congeniality relation is not an equivalence relation; in fact, it is not symmetric. We provide examples of bases $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{B}$ is congenial to $\mathcal{C}$ but $\mathcal{C}$ is not congenial to $\mathcal{B}$. On the other hand, we show that congeniality is transitive. Moreover, we consider the relation between two bases $\mathcal{B}$ and

C being mutually congenial; that is, $\mathcal{B}$ is congenial to C and C is congenial to $\mathcal{B}$. We then show that mutually congenial bases have naturally isomorphic associated modules.

While mutual congeniality of two amenable bases $\mathcal{B}$ and C suffices to conclude that $\mathcal{M}_{\mathcal{B}}$ and $\mathcal{M}_C$ are naturally isomorphic, one can also conclude that if $\mathcal{B}$ is properly congenial to C ($\mathcal{B}$ is congenial to C but C is not congenial to $\mathcal{B}$) then there exists a natural epimorphism from $\mathcal{M}_{\mathcal{B}}$ onto $\mathcal{M}_C$. In the absence of mutual congeniality we cannot guarantee that the associated modules are isomorphic but to this moment we have not been able to find an argument to guarantee that they are not.

We explore various natural questions that arise in the context of proper congeniality. It seems that these questions will have different answers depending on the specific algebra in which they are considered. Our investigation was limited to the algebra of polynomials with coefficients in a field. Some consideration was given to semigroup and group algebras as well as to Leavitt Path Algebras but we have barely scratched the surface of what seem to be very deep problems.

1.2 Preliminaries

In this section we provide some basic modern and linear algebra definitions that are going to be used throughout this dissertation. These stated in standard references include [7], [9] and [15]. Then we extend the linear algebra ideas to countably infinite dimensional spaces as presented in [12] and [13].

In this dissertation rings are associative and have 1_R and modules are always unital.

Definition 1.2.1. *By a ring we mean a nonempty set R together with two binary operations (usually denoted as addition $(+)$ and multiplication) such that*

1. $(R, +)$ is an abelian group;
2. For all $a, b, c \in R$, $(ab)c = a(bc)$ (associative multiplication);

3. For all $a, b, c \in R$, $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ (left and right distributive laws)

4. For all $a \in R$, $1_R a = a 1_R$

Definition 1.2.2. Let R be a ring. A (left) R -module is an additive abelian group M together with a function $R \times M \rightarrow M$ (the image of (r, m) being denoted by rm) such that for all $r, s \in R$ and $m, n \in M$

1. $r(m+n) = rm + rn$.

2. $(r+s)m = rm + sm$

3. $r(sm) = (rs)m$

For 1, the identity in R ,

4. $1_R m = m$ for all $m \in M$

If R is a division ring, then a unitary R -module is called a (left) vector space.

Definition 1.2.3. Let K be a commutative ring with identity. We say that K -algebra A is a ring A such that

1. $(A, +)$ is a unitary K -module;

2. For all $k \in K$ and $a, b \in A$ $k(ab) = (ka)b = a(kb)$

A K -algebra A which is division ring is called a division K -algebra.

Next, we introduce the notion of direct sums and direct products, in the following definitions as they appear in [15].

Definition 1.2.4. Let $\{V_1, V_2, \dots, V_n\}$ be vector spaces over a field K . The direct sum of $\{V_1, \dots, V_n\}$, denoted by

$$V = V_1 * V_2 * \cdots * V_n$$

is the vector space whose elements are ordered n -tuples:

$$V = \{(v_1, \cdots, v_n) : v_i \in V_i, i = 1, \cdots, n\}$$

For all $k \in K$, with vector space operations (Addition is defined componentwise and multiplication by scalars is also defined componentwise) given by the formulas

$$(u_1, \cdots, u_n) + (v_1, \cdots, v_n) = (u_1 + v_1, \cdots, u_n + v_n)$$

and

$$k(v_1, \cdots, v_n) = (kv_1, \cdots, kv_n)$$

In particular, The vector space K^n is the direct sum of n copies of K , that is

$$K^n = K * \cdots * K$$

where there are n summands on the right-hand side.

We can generalize this construction to any family of vector spaces by the following:

An ordered n -tuple $(v_1, \cdots, v_n)$ can be defined as a function $\phi : \{1, \cdots, n\} \rightarrow \bigcup V_i$ from the index set to the union of the spaces such that $\phi(i) \in V_i$.

Definition 1.2.5. Let $\mathcal{V} = \{V_i : i \in I\}$ be any family of vector spaces over K . The direct product of $\mathcal{V}$ is the vector space

$$\prod_{i \in I} V_i = \{\phi : I \rightarrow \bigcup_{i \in I} V_i : \phi(i) \in V_i\}$$

thought of as a subspace of the vector space of all functions from index I to $\bigcup V_i$.

Now we define a support of a function and it would be useful to restrict the set of functions to those with finite support.

Definition 1.2.6. Let $\mathcal{V} = \{V_i : i \in I\}$ be a family of vector spaces over K . The set

$$\text{supp}(\phi) = \{i \in I : \phi(i) \neq 0\}$$

is the support of a function $\phi : I \rightarrow \bigcup V_i$.

Thus, a function ϕ has finite support if $\phi(i) = 0$ for all but finitely many of $i \in I$. The direct sum of the family $\mathcal{V}$ is the vector space

$$\bigoplus_{i \in I} V_i = \{\phi : I \rightarrow \bigcup_{i \in I} V_i, \phi \text{ has finite support}\}$$

thought of as a subspace of the vector space of all functions from I to $\bigcup V_i$.

In particular, let $V_i = V$ for all $i \in I$ and let V^I be denote the set of all functions from I to V and $V^{(I)}$ denote the set of all functions in V^I that have finite support, then

$$\prod_{i \in I} V = V^I \text{ and } \bigoplus_{i \in I} V = V^{(I)}$$

Note that the direct product and the direct sum are the same for a finite family of vector spaces.

Definition 1.2.7. A vector space V is a finite dimensional if it has a finite basis. All other vector spaces are infinite dimensional.

Definition 1.2.8. Let V and W be vector spaces (over K). We call a function $T : V \rightarrow W$ a linear transformation from V to W if, for all $v_1, v_2 \in V$ and $k \in K$, we have

1. $T(v_1 + v_2) = T(v_1) + T(v_2)$ and
2. $T(kv_1) = kT(v_2)$

If T is surjective and injective we called T an isomorphism linear transformation. For any vector space V over K , we define the identity transformation $I_V : V \rightarrow V$ by $I_V(v) = v$ for all $v \in V$. It is clear that the identity transformation is an isomorphism.

Definition 1.2.9. A linear transformation $T : V \rightarrow W$ that is bijective (that is, injective and surjective) is called an isomorphism linear transformation.

Definition 1.2.10. Let $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite dimensional vector space V . For $v \in V$, let $k_1, k_2, \dots, k_n$ be the unique scalars such that

$$\sum_{i=1}^n k_i u_i$$

we define the coordinate vector of v relative to $\mathcal{B}$, denoted $[v]_{\mathcal{B}}$ by

$$[v]_{\mathcal{B}} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

The correspondence $x \rightarrow [v]_{\mathcal{B}}$ provides us an isomorphism from V to K^n .

Suppose V and W are finite-dimensional vector spaces with ordered bases $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathcal{C} = \{w_1, w_2, \dots, w_m\}$, respectively and let $T : V \rightarrow W$ be a linear transformation. Then for each $j, 1 \leq j \leq n$, there exist unique scalars $k_{ij} \in K, 1 \leq i \leq m$, such that

$$T(v_j) = \sum_{i=1}^m k_{ij} w_i \text{ for } 1 \leq j \leq n.$$

Definition 1.2.11. Using the notation above, we call the $m \times n$ matrix A defined by $A_{ij} = k_{ij}$ the matrix representation of T relative to the ordered bases $(\mathcal{B}, \mathcal{C})$ and write $A = [T]_{\mathcal{B}}^{\mathcal{C}}$. If $V = W$ and $\mathcal{B} = \mathcal{C}$, then we simply write $[T]_{\mathcal{B}}$.

Notice that the j th column of A is simply $[T(v_j)]_{\mathcal{C}}$. Matrix representations of linear transformations serve to yield representations of outputs via matrix multiplication from representations of inputs due to the identity. Thus we have the theorem

Theorem 1.2.1. Let V and W be finite dimensional vector space having ordered bases $\mathcal{B}$ and $\mathcal{C}$, respectively, and let $T : V \rightarrow W$ be linear. Then for each $v \in V$, we have $[T(v)]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} \cdot [v]_{\mathcal{B}}$

Definition 1.2.12. Let $\mathcal{B}$ and $\mathcal{C}$ be two ordered bases for an finite-dimensional vector space V , the matrix $Q = [I_V]_{\mathcal{B}}^{\mathcal{C}}$ is called a change of coordinate matrix. Q changes $\mathcal{B}$ -coordinates into $\mathcal{C}$ -coordinates. Namely, $[v]_{\mathcal{C}} = Q \cdot [v]_{\mathcal{B}}$

Definition 1.2.13. Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. If $\text{Ker}(T)$ and $\text{Im}(T)$ are finite-dimensional, then we define the nullity of T , denoted $\text{nullity}(T)$, and the rank of T , denoted $\text{rank}(T)$, to be the dimensions of $\text{Ker}(T)$ and $\text{Im}(T)$, respectively.

Theorem 1.2.2. *Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. If V is finite-dimensional, then*

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

We conclude this section with an adaption of the previous linear algebra results to the countably infinite dimensional case as seen in [13] and [12].

1.3 Matrix Representations of Linear Mappings on $K^{(\omega)}$

We present the notion Matrix Representations of Linear Mappings on $K^{(\omega)}$ as it appears in [13]. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ (endowed its standard well ordering) is identified with the ordinal notation ω . Let K be a field and $K^{(\omega)}$ be the vector space of sequences in K of only finitely many nonzero terms. The canonical basis of $K^{(\omega)}$ is denoted by $E = \{e_i\}_{i \in \omega}$, where $e_i = (\delta_{ij})_{j \in \omega}$ and δ_{ij} is the Kronecker delta function. A row-finite $\omega \times \omega$ matrix A is a sequence of vectors in $K^{(\omega)}$ indexed by ω and denoted by $A = (A_r)_{r \in \omega} = (a_{ri})_{(r,i) \in \omega \times \omega}$. Also, let $I = (e_i)_{i \in \omega}$ be the identity $\omega \times \omega$ matrix. A row $A_r \in K^{(\omega)}$ of A is written as:

$$A_r = (a_{r0}, a_{r1}, \dots, a_{r\rho_r}, 0, 0 \dots) = \sum_{i=0}^{\rho_r} a_{ri} e_i$$

The index ρ_r is called rightmost index of A_r (rightmost is the nonzero right leading entry of the row) and it is denoted by $l(A_r)$. We have $l(e_i) = i$ for $i \in \omega$, thus $l(e_0) = 0$. Since $l(0)$ does not exist in ω , the convention $l(0) = -1$ is adopted. The rightmost nonzero coefficient $a_{r\rho_r}$ of A_r will be also called right leading coefficient of A_r .

We denote $\text{Hom}_K(K^{(\omega)})$ the space of linear mappings from $K^{(\omega)}$ to $K^{(\omega)}$ over K . Let $T \in \text{Hom}_K(K^{(\omega)})$. The null space and the range of T is denoted by $\text{Ker}(T)$ and $\text{Im}(T)$ respectively. Let $\mathcal{B} = \{b_i\}_{i \in \omega}$ and $\mathcal{C} = \{c_j\}_{j \in \omega}$, be bases of the domain and codomain of T respectively. The matrix representation of T , relative to $(\mathcal{B}, \mathcal{C})$, is a row-finite matrix $A = (a_{ij})_{(i,j) \in \omega \times \omega}$. The entries of A are determined by $T(b_i) = \sum a_{ij} c_j$ and we shall referred to it as $A = [T]_{\mathcal{C}}^{\mathcal{B}}$.

On the other hand, if $A \in \mathbb{RFM}_\omega(K)$, (The set of all matrices where each row has only finitely many nonzero entries) there is a unique linear extension of $b_i \rightarrow \sum a_{ij}c_j$ to $\text{Hom}_K(K^{(\omega)})$, call it U , such that $A = [U]_C^{\mathcal{B}}$. A row $A_i = (a_{ij})_{j \in \omega}$ of A represents the element $U(b_i)$, relative to C . If $\mathcal{B} = C$ then the matrix representation of U , relative to $\mathcal{B}$ is denoted by $[U]_{\mathcal{B}}$.

The matrix multiplication of two matrices in $\mathbb{RFM}_\omega(K)$ is defined in complete analogy with finite matrices, and results in a matrix in $\mathbb{RFM}_\omega(K)$. Moreover, If $\alpha \in K^{(\omega)}$, the mapping

$$\begin{aligned} T : K^{(\omega)} &\rightarrow K^{(\omega)} \\ \alpha &\rightarrow T(\alpha) = \alpha \cdot A \end{aligned}$$

Will be called the induced linear mapping by A . In this setting, A is treated as a right operator and $A = [T]_E$, since

$$T(e_k) = e_k \cdot A = (a_{k0}, a_{k1}, \dots, a_{k\rho_k}, 0, 0 \dots) = \sum_{i=0}^{\rho_k} a_{ki}e_i$$

Now we introduce the notion of change of bases and left association as it appears in [13] A is said to be non-singular or invertible If a left and a right inverse of $A \in \mathbb{RFM}_\omega(K)$ exist in $\mathbb{RFM}_\omega(K)$, then they necessarily coincide defining the two-sided inverse of A . Therefore, the existence of a left (resp. right) inverse implies that the induced linear mapping is surjective (respectively injective), whence the following Proposition:

Proposition 1.3.1. [13] *Let $T \in \text{Hom}_K(K^{(\omega)})$. Then T is an isomorphism if and only if $A = [T]_E^E$, is non-singular.*

If $\mathcal{G} \subset K^{(\omega)}$ spans a subspace $\mathcal{Z}$ of $K^{(\omega)}$, we write $\text{span}(\mathcal{G}) = \mathcal{Z}$. The null space and the row space of $A \in \mathbb{RFM}_\omega(K)$ are defined by $N(A) = \text{Ker}(T) = \{\alpha \in K^{(\omega)} : \alpha \cdot A = 0\}$ and $R(A) = \text{Im}(T) = \text{span}(A_r)_{r \in \omega}$. The dimensions of these spaces are called nullity and rank of A , respectively, writing $\text{nul}(A) = \dim(N(A))$, $\text{rank}(A) = \dim(R(A))$. Let $\mathcal{D} = (d_r)_{r \in \omega}$ be basis of $K^{(\omega)}$, then $T \in \text{Hom}_K(K^{(\omega)})$ and $A = [T]_{\mathcal{D}}^{\mathcal{C}}$, $B = [T]_{\mathcal{D}}^{\mathcal{C}}$. Let also $\psi : \mathcal{B} \rightarrow \mathcal{D}$ be defined by $d_r = \psi(b_r)$ and $\psi(b_r) = \sum_i q_{ri}b_i$. As the unique linear extension of the bijection

ψ , say $\varphi : K^{(\omega)} \rightarrow K^{(\omega)}$, is an isomorphism, the matrix $Q = (q_{ri})_{(r,i) \in \omega \times \omega} = [\varphi]_{\mathcal{B}}^{\mathcal{B}}$ is non-singular, called the matrix of passage from the basis $\mathcal{B}$ to the basis $\mathcal{D}$. The matrix Q is just the matrix with respect to the basis $\mathcal{B}$ of the isomorphism φ . It also can be interpreted as the matrix of the identity mapping with respect to the basis $\mathcal{D}$ to the basis $\mathcal{B}$ (in that order). This passage of bases causes a change of the matrix representation of T from A to B , for a fixed basis $\mathcal{C}$ of the codomain space, described by

$$Q \cdot A = B \quad *$$

Let $Q \in \mathbb{RFM}_{\omega}(K)$ be non-singular such that $*$ holds. Then A, B is said to be left associates or row-equivalent and we write $A \sim B$. Formally $\sim$ is an equivalence relation on $\mathbb{RFM}_{\omega}(K)$, which generalizes the the well known notion of "row-equivalence" for finite matrices. A change of the domain basis gives rise to row equivalent matrix representations of a linear mapping and vice versa.

Definition 1.3.1. $A, B \in \mathbb{RFM}_{\omega}(K)$ are said to be:

1. *Left associates if there exists a nonsingular matrix $Q \in \mathbb{RFM}_{\omega}(K)$ such that $Q \cdot A = B$.*
2. *Row equivalent if $R(A) = R(B)$ and $\text{nul}(A) = \text{nul}(B)$.*

2 RELATED TOPICS

2.1 Formal Calculus and Vertex Operator Algebra

Formal calculus is an essential topic in vertex operator algebra theory. The notion of amenability in this dissertation and the notion of algebraic summability in formal calculus seem to be similar at a first glance. In order to understand each topic, we shall provide a quick primer to vertex operator algebra and some basic elements of formal calculus. For more information, the reader should consult for [10].

Let V be vector space. We denote by $V[[x, x^{-1}]]$ the vector space of (doubly infinite) formal Laurent series in x with coefficient in V . The vector spaces W in vertex operator algebra are often in the form $W = \text{End } V$, the algebra of endomorphisms of a vector space V . Now we state the definition of vertex operator algebra which is taken directly from [10].

Definition 2.1.1. *A vertex algebra consists of a vector space V equipped, first, with a linear map (the vertex operator map) $V \otimes V \rightarrow V[[x, x^{-1}]]$, or equivalently, a linear map*

$$Y(\cdot, x) : V \rightarrow (\text{End } V)[[x, x^{-1}]]$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

Notice that we are using the notation v_n for the element of $\text{End } V$ which is the coefficient of x^{-n-1} in the formal series $Y(v, x)$; (in particular, the notation v_n does not refer to any sort of component of the vector v .) We call $Y(v, x)$ the vertex operator associated with v . We also have a distinguished element 1 of V (the vacuum vector). The following conditions are assumed for $u, v \in V$: First, the truncation condition:

$$u_n v = 0 \text{ for sufficiently large } n,$$

that is,

$$Y(u, x)v \in V((x));$$

Next, the following vacuum property: $Y(1, x) = 1$ (1 on the right being the identity operator); also, the creation property (whereby $Y(v, x)$ "creates" the vector v):

$$Y(v, x)1 \in V[[x]] \text{ and } \lim_{x \rightarrow 0} Y(v, x)1 = v$$

(that is, $Y(v, x)1$ involves only nonnegative integral powers of x and the constant term is v); and finally, and most importantly, the Jacobi identity:

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(u, x_1) = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_0^2}\right) Y(Y(u, x_0)v, x_2). \quad (1)$$

It is strait forward - and of course necessary- to check that when each expression in (1) is applied to any element of V , the coefficient of each monomial in the three formal variables is finite sum of vectors. On the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in an obvious way to $V[[x_0, x_0^{-1}]]$; this makes clear the meaning of the "iterate" $Y(Y(u, x_0)v, x_2)$ of the two vertex operators $Y(v, x_2)$ and $Y(u, x_0)$.

In formal calculus, everything depends upon formal Laurent series of the type $\sum_{n \in \mathbb{Z}} a_n x^n$ and analogous formal series in several variables. The aim of both definitions is that for the product of elements to be well behaved. However, the summability is needed not only for formal product of finitely many formal series but also for formal sums of infinitely many formal series so they can be well defined. In our case the amenability is needed only to get a well-defined multiplication of finite series, which is an element from the algebra, times an infinite formal series, which is an element from the constructed module. Now, we define the summability property of formal calculus.

Definition 2.1.2. *Let V be a vector space and $(f_i)_{i \in I}$ be family in $\text{End } V$ (I an index set). we say that $(f_i)_{i \in I}$ is summable if for every $v \in V$, $f_i v = 0$ for all but a finite number of $i \in I$. in this case we write the corresponding "sum" operator as*

$$\begin{aligned} \sum_{i \in I} f_i : V &\rightarrow V \\ v &\rightarrow \sum_{i \in I} f_i v \end{aligned}$$

The idea of the notion of algebraic summability is to be used for the product of finite formal series and the sum of infinite formal series so they can be well defined. Therefore, a finite formal product and an infinite formal sum of formal series is defined if and only if

the coefficient of every monomial in the formal product or sum in the formal expression is summable as we can see in the next definitions.

Definition 2.1.3. *We say that $\sum_{i \in I} F_i(x)$ exists if for every $n \in \mathbb{Z}$, the family $(f_i)_{i \in I}$ is summable. We then set*

$$\sum F_i(x) = \sum_{n \in \mathbb{Z}} \left(\sum_{i \in I} f_i(n) \right) x^n$$

Let $(F_i(x))_{i=1}^r$ be finite family in $\text{End } V[[x, x^{-1}]]$, with

$$F_i(x) = \sum_{n \in \mathbb{Z}} f_i(n) x^n$$

We say the product $F_1(x) \cdots F_r(x)$ exists if for every $n \in \mathbb{Z}$, the family

$$(f_1(n_1) \cdots f_r(n_r))_{n_1 + \cdots + n_r = n}^{n_1 \cdots n_r}$$

is summable. We then set

$$F_1(x) \cdots F_r(x) = \sum_{n \in \mathbb{Z}} \left(\sum_{n_1 + \cdots + n_r = n} f_1(n_1) \cdots f_r(n_r) \right) x^n \in (\text{End } V)[[x, x^{-1}]]$$

On the other hand, the idea of the notion of amenability is that the formal product of finite series times infinite series on a module that is produced by a basis will be understood to exist if and only if the coefficient of any monomial in the formal product is finite, that is that basis is amenable as we can see through this dissertation. For more details we refer the reader to [10].

2.2 Semigroup Rings and Group Rings

Here we present the notion of semigroup ring and group rings as it appears in [11]. Let G be a multiplicative group (finite or infinite) and R be a ring. We construct an R -module, with elements of G as a basis, and then use the operations in both G and R to define a ring structure on it. To do so, let RG to be the set of all formal linear combinations of the form

$$\alpha = \sum_{g \in G} a_g g$$

with $a_g \in R$ and $a_g = 0$ for almost everywhere, that is, α has only finitely many of the coefficients of a_g is nonzero. whenever convenient, we may also write α in the form

$$\alpha = \sum_{g \in G} a_g g$$

Given an element $\alpha = \sum_{g \in G} a_g g$ we define the *support* of α to be the subset of elements in G that they really accrue in the representation of α , that is:

$$\text{supp}(\alpha) = \{g \in G : a_g \neq 0\}$$

Notice that it follows from our definition that given two elements, $\alpha = \sum_g \in G a_g g$ and $\beta = \sum_{g \in G} b_g g \in RG$, we have that $\alpha = \beta$ if and only if $a_g = b_g$, $\forall g \in G$.

We define the addition of two elements in RG componentwise:

$$(\sum_{g \in G} a_g g) + (\sum_{g \in G} b_g g) = \sum_g \in G (a_g + b_g) g$$

Also, given two elements $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$ in RG we define the multiplication by

$$\alpha \cdot \beta = \sum_{g, h \in G} a_g b_h g h$$

Reordering the terms in the formula above we can write the product by

$$\alpha \cdot \beta = \sum_{c \in G} c_u u$$

where

$$c_u = \sum_{gh=u} a_g b_h$$

It is easy to verify that that, with the operations above, RG is a ring, which has a unity; namely, the element $1 = \sum_{g \in G} u_g g$ where the coefficient corresponding to the unite element of the group is equal to 1 and $u_g = 0$ for every other element $g \in G$

We can also define a product of elements in RG by elements $\lambda \in R$ as

$$\lambda(\sum_{g \in G} a_g g) = \sum_{g \in G} \lambda(a_g g)$$

Again it is easy to verify that RG is an R -module. Actually, if R is commutative, it follows that RG is an algebra over R .

We can define an embedding $i : G \rightarrow RG$ by assigning to each element $h \in G$ the element $i(x) = \sum_{g \in G} a_g g$, where $a_x = 1$ and $a_g = 0$ if $g \neq h$, we may thus regard G as a subset of RG . With this identification in mind, we can say that G is a basis of RG over R . As in immediate consequence we see that, if R is commutative, then the rank of agree

module over R is well-define. Thus if G is finite, we can state that the $\text{rank}(RG)$ over R is precisely $|G|$.

2.3 Hamel Bases

In this section we explain the notion of *Hamel Basis* and compare it with other bases. We begin with the definition of Hamel bases, which are the bases that we are familiar with in linear algebra as it appears in [8].

Definition 2.3.1. (*Hamel Basis*). Let V be a vector space. A sequence of vectors $\{x_i\}_{i \in I}$ is a *Hamel basis* for V if

- the finite linear span of $\{x_i\}_{i \in I}$ is V , i.e., $\text{span}\{x_i\} = V$, and
- $\{x_i\}$ is finitely linear independent.

Note that the index set I of Hamel basis need not to be countable. Equivalent formulations of **Definition 2.3.1** are that $\{x_i\}_{i \in I}$ is a Hamel basis for V if and only if every nonzero vector $x \in V$ can be written as $x = \sum_{k=1}^N c_k x_{i_k}$ for some unique choice indices $i_1, \dots, i_N \in I$ and unique nonzero scalars $c_1, \dots, c_N$, or that every $x \in V$ can be written uniquely as $x = \sum_{i \in I} a_i(x) x_i$ for some unique choice of scalars $a_i(x)$

In finite-dimensional linear algebra, a Hamel basis is usually just called a "basis". Thus, the definition of a Hamel basis, as given in **Definition 2.3.1**, is made entirely in terms of finite linear combinations of vectors. Every vector space has a Hamel basis. The proof uses the Axiom of Choice in the form of Zorn's Lemma. In addition, all Hamel bases for a given vector space have the same cardinality, which we call the Hamel dimension of the space. It is highly convenient to use Hamel bases for finite-dimensional vector spaces and vector spaces with countably infinite dimension. For example, the set $\mathcal{B} = \{x^i : i \geq 0\}$ is a Hamel basis for the vector space of polynomials.

However, For infinite dimensional vector spaces, we only know that their Hamel bases exist because of the Axiom of Choice. Even if we know that every vector space has a Hamel basis, these bases are difficult to describe since there is no practical way in general to construct such bases. The obvious example is $\mathbb{R}$ as a vector space over $\mathbb{Q}$ where the Hamel basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$ is uncountable.

Similarly, from the perspective of real analysis, a Hamel basis for normed spaces, for example infinite-dimensional Banach space, is necessarily uncountable "Baire Category Theorem." Hence, new notions of infinite bases were introduced that are more appropriate for infinite-dimensional normed spaces. This new notion of bases allow the use of "infinite linear combinations" instead of using finite linear combinations to which Hamel bases are restricted, provided that the infinite sum of vectors are compatible. Thus, seeing that the notion of convergence in normed spaces is applicable, infinite series can be constructed in order to make sense of an infinite sum of vectors in different types of normed spaces. This leads to the notion of Schauder Bases.

3 THE CENTRAL NOTION

3.1 Introduction

Let K be a field and R be an infinite dimensional K -algebra with basis $\mathcal{B}$. We consider the feasibility to construct an R -module $\mathcal{M}$ from the basis $\mathcal{B}$. Let $\mathcal{M} = \{\phi : \mathcal{B} \rightarrow K\}$ the set of all functions from $\mathcal{B}$ to K . Then $\mathcal{M}$ is a K -vector space where, for all $b \in \mathcal{B}$ and $k \in K$, we define $\phi + \psi \rightarrow \phi(b) + \psi(b)$ and $k\phi : b \rightarrow k[\phi(b)]$. We may write

$$\phi = (\phi(b))_{b \in \mathcal{B}} = \sum_{b \in \mathcal{B}} \phi(b)b$$

whenever convenient, we may also write ϕ in the form

$$\phi = \sum_{b \in \mathcal{B}} \phi_b b.$$

Remark. We would like to define a product $b \in \mathcal{B}$ of elements of R with elements from $\mathcal{M}$. For $b, c \in \mathcal{B}$ and $k \in K$, define

$$[(kb)\phi](c) = \sum_{d \in {}_b\mathcal{B}_c} k(bd)_c \phi(d)$$

where ${}_b\mathcal{B}_c = \{d \in \mathcal{B} : (bd)_c \neq 0\}$.

We construct modules with respect to some bases under some conditions. We start with the basic basis, which is closed under product, and the cancelation property is valid. A more general basis could have one more condition; the product of two elements of that basis might be zero. We defined the most general basis we considered as the "amenable basis" as in **Lemma 1.4.1**.

We begin with the following useful Definitions.

Definition 3.1.1. Let $\mathcal{B}$ be a basis of R we say that $\mathcal{B}$ is amenable if for every $b, c \in \mathcal{B}$ there exist only finitely many $d \in \mathcal{B}$ such that $(bd)_c \neq 0$, In other words the set ${}_b\mathcal{B}_c = \{d \in \mathcal{B} : (bd)_c \neq 0\}$ is finite.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of R we say that $\mathcal{B}$ is amenable to $\mathcal{C}$ if for every $b \in \mathcal{B}$ and $c \in \mathcal{C}$ there exist only finitely many $d \in \mathcal{C}$ such that $(bd)_c \neq 0$, In other words the set ${}_b\mathcal{C}_c = \{d \in \mathcal{C} : (bd)_c \neq 0\}$ is finite.

Let $\mathcal{B}$ be amenable basis of R and for $r \in R$ let L_r be a linear transformation from R to R defined by $L_r = rb$ for all $b \in \mathcal{B}$. Then for all $r \in R$ the matrix $[L_r]_{\mathcal{B}}$ is row-finite.

Definition 3.1.2. For a field K and basis $\mathcal{B}$, $K^{(\mathcal{B})}$ will denote the direct sum of copies of K and the symbol $K^{\mathcal{B}}$ the direct product of K .

3.2 Modules Constructed from Amenable Bases

The condition that $\mathcal{B}$ is amenable guarantees that we can make the set $K^{\mathcal{B}}$ into a module (Which we call $\mathcal{M}$) as in Lemma.1. For instant, If $\mathcal{B}$ is closed under products i.e. For every b_1 and b_2 in $\mathcal{B}$ then $b_1b_2 \in \mathcal{B}$, and for any b_1, b_2 and b_3 in $\mathcal{B}$, if $b_1b_2 = b_1b_3 \Rightarrow b_2 = b_3$, then ${}_R\mathcal{M}$ is a module. In Particular, the bases in group algebras are closed under products. More generally, for every b_1 and b_2 in $\mathcal{B}$ either $b_1b_2 \in \mathcal{B}$ or $b_1b_2 = 0$ and for any b_1, b_2 and b_3 in $\mathcal{B}$, if $b_1b_2 = b_1b_3 \Rightarrow b_2 = b_3$, then ${}_R\mathcal{M}$ is a module. In Particular, the bases in Leavitt Path Algebras as we will explain in more details in section.3.

Lemma 3.2.1. Let $\mathcal{B}$ be a basis of R and if $\mathcal{B}$ is amenable then ${}_R\mathcal{M}$ is a module.

Proof. We want to show that ${}_R\mathcal{M}$ is a module. So we show that the product rm ($r \in R, m \in M$) is well-defined.

$$[(\sum_{i=1}^n k_i b_i)\phi](c) = \sum_{i=1}^n (\sum_{d \in b_i \mathcal{B}_c} k_i (b_i d)_c \phi(d)) \text{ this is a well defined.}$$

- $$\begin{aligned} \bullet [(\sum_{i=1}^n k_i b_i \cdot (\phi + \psi))](c) &= \sum_{i=1}^n (\sum_{d \in b_i \mathcal{B}_c} k_i (b_i d)_c (\phi + \psi)(d)) \\ &= \sum_{i=1}^n (\sum_{d \in b_i \mathcal{B}_c} k_i (b_i d)_c \phi(d)) + \sum_{i=1}^n (\sum_{d \in b_i \mathcal{B}_c} k_i (b_i d)_c \psi(d)) \\ &= [(\sum_{i=1}^n k_i b_i)\phi](c) + [(\sum_{i=1}^n k_i b_i)\psi](c) \end{aligned}$$
- $$\begin{aligned} \bullet [(\sum_{i=1}^n k_i b_i + \sum_{j=1}^m k'_j b_j)\phi](c) &= [(\sum_{i=1}^n k_i b_i)\phi](c) + [(\sum_{j=1}^m k'_j b_j)\phi](c) \\ &= \sum_{i=1}^n (\sum_{d \in b_i \mathcal{B}_c} k_i (b_i d)_c \phi(d)) + \sum_{j=1}^m (\sum_{d \in b_j \mathcal{B}_c} k'_j (b_j d)_c \phi(d)) \\ &= [\sum_{i=1}^n k_i b_i \phi](c) + [\sum_{j=1}^m k'_j b_j \phi](c) \end{aligned}$$

$$\bullet \left[\left(\sum_{i=1}^n k_i b_i \cdot \sum_{j=1}^m k'_j b_j \right) \phi \right](c) = \left[\sum_{i=1}^n k_i b_i \left(\left(\sum_{j=1}^m k'_j b_j \right) \phi \right) \right](c) \quad \square$$

Now in the theorem below we show that the following statements are equivalent: A basis $\mathcal{B}$ is amenable to C , all bases are amenable to C and C is amenable to itself. As a result, we will simply say that C is amenable if one of the previous statements is satisfied.

Theorem 3.2.2. *Let C and $\mathcal{B}$ be bases. Then the Following are equivalent:*

i) *There exists a basis $\mathcal{B}$ such that $\mathcal{B}$ is amenable to C .*

ii) *For every basis $\mathcal{B}$, $\mathcal{B}$ is amenable to C .*

iii) *C is amenable to itself.*

Proof. ii) $\Rightarrow$ i) clear

iii) $\Rightarrow$ ii) Suppose that C is amenable to itself, means that for fixed $c_1, c_2 \in C$ the set ${}_{c_1}C_{c_2} = \{v \in C : (c_1 v)_{c_2} \neq 0\}$ is finite. We claim that for fixed $b \in \mathcal{B}$ and $c \in C$ the set ${}_b C_c = \{v \in C : (bv)_c \neq 0\}$ is finite. Since C is a basis, we can write b as a linear combination of C as $b = \sum_{i=1}^n (b)_{c_i} c_i$ where $c_i \in C$, $1 \leq i \leq n$.

Since C is amenable to itself then for $1 \leq i \leq n$ the sets ${}_{c_i} C_c = \{v_{c_i} \in C : (c_i v_{c_i})_c \neq 0\}$ are finite. Let $\mathfrak{U} = \cup_{i=1}^n {}_{c_i} C_c$. We want to show that ${}_b C_c \subseteq \mathfrak{U}$.

Now suppose $v \notin \mathfrak{U}$ such that $(bv)_c \neq 0$.

$$\Rightarrow bv = \sum_{i=1}^n (b)_{c_i} (c_i v)$$

$$\Rightarrow (bv)_c = \sum_{i=1}^n (b)_{c_i} (c_i v)_c$$

Since $(bv)_c \neq 0$, there exist c_i such that $(c_i v)_c \neq 0$ which is a contradiction. Therefore if $v \in {}_b C_c$ then $v \in \mathfrak{U}$, but $\mathfrak{U}$ is finite then the set ${}_b C_c$ is finite as claimed.

i) $\Rightarrow$ iii) similar to iii) $\Rightarrow$ ii) $\square$

Not all bases of an algebra are amenable as we can see in the following examples.

Example 3.2.1. *Consider the polynomial ring $\mathbb{F}[x]$ over a field $\mathbb{F}$ and let $C = \{1\} \cup \{x^i + x^{i+1} : i \geq 0\}$ and let $\mathcal{D} = \{\sum_{i=0}^n x^i : n \geq 0\}$. It is easy to see that C and $\mathcal{D}$ are bases of*

$\mathbb{F}[x]$ and C is amenable, however, $\mathcal{D}$ is not amenable. To show that $\mathcal{D}$ is not amenable, fix $d_1 = 1 + x$ and $d_2 = 1 \in \mathcal{D}$, then the set $\{v \in \mathcal{D} : ((1+x)v)_{1_{\mathcal{D}}} \neq 0\}$ is infinite, since $(1+x)(\sum_{i=0}^n x^i) = (1)\sum_{i=0}^{n+1} x^i + (1)\sum_{i=0}^n x^i + (-1)1_{\mathcal{D}}$.

Take $x \in R$ and let T_x be the linear transformation from $\mathbb{F}^{(C)}$ to $\mathbb{F}^{(C)}$ defined by left multiplication by x . Clearly the representation $[T_x]_C$ is row finite. Since $T_x(1) = (x+1) - 1$ and $T_x(x^i + x^{i+1}) = x^{i+1} + x^{i+2}$ then we have

$$[T_x]_C = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

On the other hand, take $(x+1) \in R$ and let $T_{(x+1)}$ be the linear transformation from $\mathbb{F}^{(\mathcal{D})}$ to $\mathbb{F}^{(\mathcal{D})}$ defined by left multiplication by $(x+1)$. Then the matrix representation $[T_{(x+1)}]_{\mathcal{D}}$ is not row finite. Since

$T_{(x+1)}(1) = (x+1)$ and $T_{(x+1)}(\sum_{i=0}^n x^i) = (1)\sum_{i=0}^{n+1} x^i + (1)\sum_{i=0}^n x^i + (-1)1_{\mathcal{D}}$ Then we have

$$[T_{(x+1)}]_{\mathcal{D}} = \begin{pmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 3.2.2. Consider the Laurent polynomial ring $\mathbb{F}[x, x^{-1}]$ over a field $\mathbb{F}$ and let $C = \{1\} \cup \{x^i + x^{i+1} : i \in \mathbb{Z}\}$, $\mathcal{D} = \{1\} \cup \{x^i + (-1)^j x^{-i}, i \in \mathbb{Z}/\{0\}, j \in \{0, 1\}\}$ and $\mathcal{E} = \{1\} \cup \{1 + x^i, i \in \mathbb{Z}/\{0\}\}$. It is easy to see that C , $\mathcal{D}$ and $\mathcal{E}$ are bases of $\mathbb{F}[x, x^{-1}]$. In addition, C and $\mathcal{D}$ are amenable bases, however, $\mathcal{E}$ is not amenable. To show that $\mathcal{E}$ is

not amenable, take $x \in \mathbb{F}[x, x^{-1}]$ and $e = 1 \in \mathcal{E}$, then the set $\{v \in \mathcal{E} : (xv)_{1_{\mathcal{E}}} \neq 0\}$ is infinite, since $x \cdot (1) = (x + 1) - 1$ and $x \cdot (1 + x^i) = (x + 1) + (x^{i+1} + 1) + (-2)1_{\mathcal{E}}$ for every $i \in \mathbb{Z}^+$.

Take $x \in R$ and let T_x be the linear transformation from $\mathbb{F}^{(\mathcal{E})}$ to $\mathbb{F}^{(\mathcal{E})}$ defined by left multiplication by x . Then the representation matrix $[T_x]_{\mathcal{E}}$ is not row finite. Since

$T_x(1) = (x+1) - 1$ and $T_x(1+x^i) = (x+1) + (x^{i+1} + 1) + (-2)1_{\mathcal{E}}$ for every $i \in \mathbb{Z}^+$. then we have

[illegible]

4 ISOMORPHIC MODULES

4.1 Introduction

Our goal of this dissertation is to investigate how different amenable bases relate to one another and whether the various amenable bases yield isomorphic module structures. Let R be an infinite dimensional K -algebra and let $\mathcal{B}$ and $\mathcal{C}$ be two bases of R . From these two bases we can produce two R -Modules $\mathcal{M}_{\mathcal{B}}$, $\mathcal{M}_{\mathcal{C}}$ with respect $\mathcal{B}$, $\mathcal{C}$ respectively. Define next what we mean by that these two modules are *naturally* isomorphic.

Definition 4.1.1. *Let R be an infinite dimensional K -algebra and let $\mathcal{B}$ and $\mathcal{C}$ be two bases of R . From these two bases we can produce two R -Modules $\mathcal{M}_{\mathcal{B}}$, $\mathcal{M}_{\mathcal{C}}$ depends on $\mathcal{B}$, $\mathcal{C}$ respectively. A homomorphism $\phi : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_{\mathcal{C}}$, said to be ϕ is natural if and only if $\phi|_A = 1_A$. If ϕ is an isomorphism then it is a natural isomorphism.*

These conditions on the bases leads to the definition of a *Congeniality* relation between bases $\mathcal{B}$ and $\mathcal{C}$.

Definition 4.1.2. *We say that a basis $\mathcal{B}$ is congenial to a basis $\mathcal{C}$ if every element $c \in \mathcal{C}$ there exists only finitely many elements $b \in \mathcal{B}$, when represented in terms of $\mathcal{C}$ use a non-zero coefficient for c . We mean that $\forall c \in \mathcal{C} \Rightarrow |\mathcal{B}_c| = |\{b \in \mathcal{B} : (b)_c \neq 0\}| < \infty$. We say that $\mathcal{B}$ and $\mathcal{C}$ are mutually congenial if $\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is congenial to $\mathcal{B}$.*

Let $\mathcal{B} := (b_i)_{i \in I}$ and $\mathcal{C} := (c_{i'})_{i' \in I}$ be two bases of R , and $\mathcal{B}$ is congenial to $\mathcal{C}$ we may write $b_i = \sum \theta_{ii'} c_{i'}$. Then the matrix $[I_R]_{\mathcal{B}}^{\mathcal{C}} = (\theta_{ii'})_{(i,i')}$ is a row-column finite matrix. Also $[I_R]_{\mathcal{B}}^{\mathcal{C}}$ is invertible with inverse $([I_R]_{\mathcal{B}}^{\mathcal{C}})^{-1} = [I_R]_{\mathcal{C}}^{\mathcal{B}}$ column finite. Hence $([I_R]_{\mathcal{B}}^{\mathcal{C}})^{-1}$ may be not row-finite as in **Example 4.1.1**

Lemma 4.1.1. *$\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is congenial to $\mathcal{B}$ if and only if $[I_R]_{\mathcal{B}}^{\mathcal{C}}$ is an invertible row-column finite with inverse $[I_R]_{\mathcal{C}}^{\mathcal{B}}$ row-column finite.*

$[I_R]_C^{\mathcal{B}}$ is row-column finite.

However, C is not congenial to $\mathcal{B}$ then we have

$$[I_R]_{\mathcal{B}}^C = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & \cdots \\ 0 & 1 & -1 & 1 & -1 & \cdots \\ 0 & 0 & 1 & -1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$[I_R]_{\mathcal{B}}^C$ is column finite but not row finite.

Notice: *Congeniality* relation is not symmetric. For example, Let $\mathcal{B} = \{x^i : i \geq 0\}$ and $C = \{1\} \cup \{x^i + x^{i+1} : i \geq 0\}$. $\mathcal{B}$ is congenial to C but C is not congenial to $\mathcal{B}$. However, the transitivity property is satisfied as we prove in the Theorem below. Before that, we prove useful Lemma.

Alternatively, *mutually congeniality* is an equivalent relation. Since it is reflexive, symmetric and transitive.

Lemma 4.1.2. Let $\mathcal{B}, C$ and $\mathcal{D}$ be bases of R . $\forall d \in \mathcal{D}$ we have that $\mathcal{B}_d \subseteq \bigcup_{c \in C_d} \mathcal{B}_c$.

Proof. Take $b \in \mathcal{B}_d$ and suppose $b \notin \bigcup_{c \in C_d} \mathcal{B}_c \Rightarrow b \in \bigcap_{c \in C_d} \mathcal{B}_c^c = \{x \in \mathcal{B} : (x)_c = 0, \forall c \in C_d\}$. Since $\mathcal{D}$ is a basis, we write b as a linear combination of $\mathcal{D}$ as $b = \sum_{z \in \mathcal{D}} (b)_z z$. Let $Z = \text{supp}(b) = \{z_1, \dots, z_{t-1}, z_t\}$, where $z_t = d$, we can see that $(b)_d \neq 0$. Also we write $z_i, 1 \leq i \leq t$, as a linear combinations of C as $z_i = \sum_{y \in C} (z_i)_y y$.

$$\begin{aligned} b &= \sum_{i=1}^t (b)_{z_i} z_i \\ &= \sum_{i=1}^t (b)_{z_i} [\sum_{y \in C} (z_i)_y y] \\ &= \sum_{i=1}^t [\sum_{y \in C} (b)_{z_i} (z_i)_y y] \\ &= \sum_{y \in C} [\sum_{i=1}^t (b)_{z_i} (z_i)_y] y, \end{aligned}$$

$$\text{Thus } (b)_y = \sum_{i=1}^t (b)_{z_i} (z_i)_y$$

So if $y \in C_d$ then $\sum_{i=1}^t (b)_{z_i} (z_i)_y = (b)_y = 0$, since $b \notin \mathcal{B}_c$ for all $c \in C_d$. Thus $y \notin C_d$ then we have $(y)_d = 0$. Write y as linear combination of $\mathcal{D}$, $y = \sum_{z \in \mathcal{D}} (y)_z z$.

$$b = \sum_{y \notin C_d} (b)_y y = \sum_{y \notin C_d} (b)_y \sum_{z \in \mathcal{D}} (y)_z z = \sum_{z \in \mathcal{D}} [\sum_{y \notin C_d} (b)_y (y)_z] z$$

Therefore we have $0 \neq (b)_d = \sum_{y \notin C_d} (b)_y (y)_d$ but $(y)_d = 0$ which is a contradiction. $\square$

Theorem 4.1.3. *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be bases of R . If $\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is congenial to $\mathcal{D}$ then $\mathcal{B}$ is congenial to $\mathcal{D}$.*

Proof. Since $\mathcal{B}$ is congenial to $\mathcal{C}$ then, $\forall c \in \mathcal{C}$ the set $\mathcal{B}_c$ is finite. Also since $\mathcal{C}$ is congenial to $\mathcal{D}$ then, $\forall d \in \mathcal{D}$ the set $\mathcal{C}_d$ is finite. For fixed $d \in \mathcal{D}$, since each $\mathcal{B}_c$ and $\mathcal{C}_d$ are finite then the union of all sets $\mathcal{B}_c$ for all $c \in \mathcal{C}_d$ is finite. From Lemma.4, $\forall d \in \mathcal{D}$ we have that $\mathcal{B}_d \subseteq \bigcup_{c \in \mathcal{C}_d} \mathcal{B}_c$ which prove that the set $\mathcal{B}_d$ is finite. Therefore $\mathcal{B}$ is congenial to $\mathcal{D}$. $\square$

4.2 Isomorphic Modules Based on Different Bases

The following theorem shows that if we have two bases of algebra, if one of these bases is amenable then the other basis is amenable under the condition that they are mutually congenial.

Lemma 4.2.1. *For every $r \in R$ and $c_0 \in \mathcal{C}$, let $\mathfrak{U} = \bigcup_{b \in B_{c_0}} rB_b$ then we have that $rC_{c_0} \subseteq \bigcup_{y \in \mathfrak{U}} C_y$.*

Proof. Suppose $x \in rC_{c_0}$ and $x \notin \bigcup_{y \in \mathfrak{U}} C_y$ we show that this leads to a contradiction. So, if $x \in rC_{c_0}$ we get $(rx)_{c_0} \neq 0$. In addition, if $x \notin \bigcup_{y \in \mathfrak{U}} C_y$ we get $x \in \bigcap_{y \in \mathfrak{U}} C_y^c = \{v \in \mathcal{C} : (v)_y = 0, \forall y \in \mathfrak{U}\}$. First since B is a basis, we write x as a linear combination of elements from B , $x = \sum_{y \in B} (x)_y y$. Then multiplying on the right by r we get $rx = \sum_{y \in B} (x)_y (ry)$. Now we write ry as a linear combination of elements from B , we get

$$\begin{aligned} rx &= \sum_{b \in B} \sum_{y \in B} (x)_y (ry)_b b \\ \Rightarrow (rx)_{c_0} &= \sum_{b \in B} \sum_{y \in B} (x)_y (ry)_b (b)_{c_0}. \end{aligned}$$

Now $(rx)_{c_0} \neq 0$ then $\exists y', b' \in B$ such that $(x)_{y'} \neq 0$, $(ry')_{b'} \neq 0$ and $(b')_{c_0} \neq 0$. Here if $(b')_{c_0} \neq 0$ then $b' \in B_{c_0}$ and if $(ry')_{b'} \neq 0$ then $y \in \mathfrak{U}$, but $(x)_y = 0$ for every $y \in \mathfrak{U}$ which is a contradiction. $\square$

Theorem 4.2.2. *Given two bases $\mathcal{B}$ and $\mathcal{C}$. If $\mathcal{B}$ is amenable and $\mathcal{B}$ is congenial to $\mathcal{C}$ and $\mathcal{C}$ is congenial to $\mathcal{B}$ then $\mathcal{C}$ is amenable.*

Proof. Fix $r \in R$ and $c_0 \in C$, it is sufficient to show that the set ${}_rC_{c_0}$ contained in a finite set. From the previous Lemma we can see that ${}_rC_{c_0} \subseteq \bigcup_{y \in \mathfrak{U}} C_y$. Now, Since $\mathcal{B}$ is congenial to $\mathcal{C}$, we have $|\mathcal{B}_{c_0}| < \infty$. Also, since $\mathcal{B}$ is amenable, for $r \in R$ and $\forall b \in \mathcal{B}_{c_0}$ we have $|{}_r\mathcal{B}_b| < \infty$. Let $\mathfrak{U} = \bigcup_{b \in \mathcal{B}_{c_0}} \mathcal{B}_b$ we can see that the set $\mathfrak{U}$ is finite. Now Since $\mathcal{C}$ is congenial to $\mathcal{B}$, $\forall y \in \mathfrak{U}$ we have $|C_y| < \infty$. Therefore the set $\bigcup_{y \in \mathfrak{U}} C_y$ is finite. $\square$

In the previous theorem, we had that if we have two mutually congenial bases and one of them is amenable basis, then so is the other one. The examples below show that if we just have congeniality in either direction, are cannot predict neither amenability nor its absence for the one.

Example 4.2.1. *Let $\mathcal{B} = \{x^i : i \geq 0\}$ and $\mathcal{E} = \{1\} \cup \{x^i + (-1)^{i+1}x^{i+1} : i \geq 0\}$ be two bases of $\mathbb{F}[x]$. Clearly $\mathcal{B}$ is amenable and $\mathcal{E}$ is congenial to $\mathcal{B}$ but yet $\mathcal{E}$ is not amenable. Thus in order to have $\mathcal{E}$ to be amenable we must have condition that $\mathcal{B}$ is congenial to $\mathcal{E}$.*

Example 4.2.2. *Let $\mathcal{B} = \{x^i : i \geq 0\}$ and $\mathcal{D} = \{\sum_{i=0}^n x^i : n \geq 0\}$ be two bases of $\mathbb{F}[x]$. Clearly $\mathcal{B}$ is amenable and $\mathcal{B}$ is congenial to $\mathcal{D}$ but yet $\mathcal{D}$ is not amenable. Thus in order to have $\mathcal{D}$ to be amenable we must have condition that $\mathcal{D}$ is congenial to $\mathcal{B}$.*

Remark: we deal with infinite linear systems $A \cdot \alpha = \beta$ where A is row-finite matrix as it appears in [12]. Let A be matrix and let H be a matrix constructed by the infinite Gauss-Jordan elimination algorithm. Let also Q be the nonsingular matrix representing the composite of elementary row operators transforming A to H i.e. $Q \cdot A = H$. Multiplying both sides of $A \cdot \alpha = \beta$ by Q we have: $A \cdot \alpha = \beta \Rightarrow H \cdot \alpha = Q \cdot \beta$ As Q is nonsingular the converse statement also holds by multiplying both sides by Q^{-1} of $H \cdot \alpha = Q \cdot \beta$ thus: $A \cdot \alpha = \beta \Leftrightarrow H \cdot y = Q \cdot \beta$. Meaning that any solution of $A \cdot \alpha = \beta$ is a solution of $H \cdot y = Q \cdot \beta$ and vice versa. For more details see [12].

Theorem 4.2.3. *If $\mathcal{B}$ is congenial to C then there exists an onto linear map $\phi : K^{\mathcal{B}} \rightarrow K^C$. If, In addition, $\mathcal{B}$ and C are amenable bases then ϕ is an R -linear i.e. $\phi : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_C$ is epimorphism.*

Proof. The claim is to show that ϕ is onto. Suppose that $\mathcal{B}$ is congenial to C , that means $\forall c \in C \rightarrow |\{b \in \mathcal{B} : (b)_c \neq 0\}| < \infty$. Suppose that $\phi : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_C$ is a map given by $\sum_{b \in \mathcal{B}} \alpha_b b \rightarrow \sum_{c \in C} \beta_c c$. We start with an arbitrary element, say $\sum_{c \in C} \beta_c c \in \mathcal{M}_C$ and write $\beta_c = \sum \alpha_b (b)_c$. We show that ϕ is homomorphism.

We show the map is onto by showing that the system $A\alpha = \beta$ has a solution where α is the representative column of the solution $(\alpha_1, \alpha_2, \dots, \alpha_{n_1}, \dots)$. So the goal is to find the α_b 's.

Let $A = [I_V]_{\mathcal{B}}^C$ be the change of coordinate row-column finite matrix that changes C -coordinates to $\mathcal{B}$ -coordinates.

Represent the system in the matrix form as

$$A \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n_1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n_1} \\ \vdots \end{pmatrix}$$

Reorder the columns in A so we get,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n_1} & 0 & \cdots & \cdots & \cdots \\ a_{21} & \cdots & \cdots & \cdots & a_{2n_2} & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n_1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n_1} \\ \vdots \end{pmatrix}$$

Case 1. Without loss of generality we assume that $n_i < n_j$ if $i < j$. Multiplying the first row of the matrix A by the α column, we get $a_{11}\alpha_1 + \cdots + a_{1n_1}\alpha_{n_1} = \beta_1$. Choose $\alpha_i = 0$ for $1 < i < n - 1$ thus $\alpha_{n_1} = a_{1n_1}^{-1}\beta_1$. Now take $\alpha_i = 0$ for $n_1 < i < n_2$. Then we have $a_{2n_1}\alpha_{n_1} + a_{2n_2}\alpha_{n_2} = \beta_2$ therefore $\alpha_{n_2} = a_{2n_1}^{-1}(\beta_2 - a_{2n_1}\alpha_{n_1})$. By continuing in this process

we can find α which satisfies the system. So we proved the lemma when the inequality is strictly less $n_i < n_j$.

Case 2. If $n_i = n_{i+1} = \dots = n_{i+t_1}$ for some i . Let t_1 to be the largest.

First let $i = 1$ then we have

$$\left(\begin{array}{ccc|cc} a_{11} & \cdots & a_{1n_1} & 0 & \cdots \\ a_{21} & \cdots & a_{2n_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{t_1 1} & \cdots & a_{t_1 n_1} & 0 & \cdots \\ \hline & B_2 & & A_2 & \end{array} \right) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n_1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n_1} \\ \vdots \end{pmatrix}$$

Now let

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1n_1} \\ a_{21} & \cdots & a_{2n_1} \\ \vdots & \vdots & \vdots \\ a_{t_1 1} & \cdots & a_{t_1 n_1} \end{pmatrix}$$

Now we want to show that the map $\psi = l_{A_1} : F^{n_1} \rightarrow F^{t_1}$ (left multiplication by A_1) is onto. Since the rows of A_1 are linearly independent then $t_1 \leq n_1$ then the nullity of ψ is equal to $n_1 - t_1$. We know that $\text{nullity}(\psi) + \text{rank}(\psi) = \dim(F^{n_1})$ this gives us $(n_1 - t_1) + \text{rank}(\psi) = n_1$, therefore $\text{rank}(\psi)$ is equal to t_1 . Since ψ is onto we can find $(\alpha_1, \alpha_2, \dots, \alpha_{n_1})$. So Let $\Gamma_1 = (\alpha_1, \alpha_2, \dots, \alpha_{n_1})$ and $\Omega_2 = (\beta_1, \beta_2, \dots, \beta_{n_1})$. Then $A_1 \Gamma_1 = \Omega_1 \Rightarrow \Gamma_1 = A_1^{-1} \Omega_1$.

Next let $\Gamma_2 = (\alpha_{n_1+1}, \dots, \alpha_{n_2})$ and $\Omega_2 = (\beta_{t_1+1}, \dots, \beta_{t_1+t_2})$.

Then we have

$$\begin{pmatrix} A_1 & O & \cdots & \cdots \\ B_2 & A_2 & O & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} \Gamma_1 \\ \alpha_{n_1+1} \\ \vdots \\ \alpha_{n_2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{t_1} \\ \beta_{t_1+1} \\ \vdots \\ \beta_{t_1+t_2} \\ \vdots \end{pmatrix}$$

Therefore $B_2\Gamma_1 + A_2\Gamma_2 = \Omega_2 \Rightarrow A_2\Gamma_2 = \Omega_2 - B_2\Gamma_1$. To find Γ_2 we have to show that the map of left multiplication by A_2 is onto. Suppose that the map is not onto then

$$\begin{pmatrix} A_1 & O & \cdots & \cdots \\ B_2 & A_2 & O & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} A_1 & O & \cdots & \cdots \\ B_{21} & A_2^* & O & \cdots \\ B_{22} & O & O & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} A_1 & O & \cdots & \cdots \\ B_{22} & O & O & \cdots \\ B_{21} & A_2^* & O & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Which is a contradiction since we chose t_1 to be large. Hence the map of left multiplication by A_2 is onto and $\Gamma_2 = A_2^{-1}\Omega_2 - A_2^{-1}B_2\Gamma_1$. Continue in this process we can find a solution α for the system. $\square$

Theorem 4.2.4. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are two amenable bases of R . if $\mathcal{B}$ and $\mathcal{C}$ are mutually congenial then $\mathcal{M}_{\mathcal{B}} \cong \mathcal{M}_{\mathcal{C}}$ naturally.*

Proof. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are mutually congenial and induced an R – linear maps $\Phi_{\mathcal{BC}} : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_{\mathcal{C}}$ and $\Phi_{\mathcal{CB}} : \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{B}}$ we show that these two maps are mutually inverse. First we show that $\Phi_{\mathcal{BC}}$ is a R -homomorphism.

Let $\mu : \mathcal{B} \rightarrow K$ and $\sigma : \mathcal{C} \rightarrow K$ where $\mu \in \mathcal{M}_{\mathcal{B}}$ and $\sigma \in \mathcal{M}_{\mathcal{C}}$. $\Phi_{\mathcal{BC}}$ is defined by $\Phi_{\mathcal{BC}}(\mu)(c) = \sum_{b \in \mathcal{B}} (b)_c \mu(b)$.

Now we have,

$$\begin{aligned} \bullet \quad \Phi_{\mathcal{BC}}(\mu_1 + \mu_2)(c) &= \sum_{b \in \mathcal{B}} (b)_c (\mu_1 + \mu_2)(b) \\ &= \sum_{b \in \mathcal{B}} (b)_c \mu_1(b) + \sum_{b \in \mathcal{B}} (b)_c \mu_2(b) \end{aligned}$$

$$= \Phi_{\mathcal{BC}}(\mu_1)(c) + \Phi_{\mathcal{BC}}(\mu_2)(c).$$

$$\begin{aligned} \bullet \quad \Phi_{\mathcal{BC}}(r \cdot \mu)(c) &= \sum_{b \in \mathcal{B}} (b)_c (r\mu)(b) \\ &= r \cdot \sum_{b \in \mathcal{B}} (b)_c \mu(b) \\ &= r \cdot \Phi_{\mathcal{BC}}(\mu)(c). \end{aligned}$$

Therefore $\Phi_{\mathcal{BC}}$ is an R -homomorphism, similarly $\Phi_{\mathcal{CB}}$ is homomorphism.

Now we want show that $\Phi_{\mathcal{BC}}$ is isomorphism, we claim that $\Phi_{\mathcal{CB}} \circ \Phi_{\mathcal{BC}} = 1$.

$$\begin{aligned} \Phi_{\mathcal{CB}} \circ \Phi_{\mathcal{BC}}(\mu)(b) &= \Phi_{\mathcal{CB}}(\Phi_{\mathcal{BC}}(\mu))(b) \\ &= \sum_{c \in \mathcal{C}} (c)_b \Phi_{\mathcal{BC}}(\mu)(c) \\ &= \sum_{c \in \mathcal{C}} (c)_b \sum_{a \in \mathcal{B}} (a)_c \mu(a) \\ &= \sum_{c \in \mathcal{C}} \sum_{a \in \mathcal{B}} (c)_b (a)_c \mu(a) \\ &= \sum_{a \in \mathcal{B}} \left(\sum_{c \in \mathcal{C}} (a)_c (c)_b \right) \mu(a) \\ &= \sum_{a \in \mathcal{B}} (a)_b \mu(a) \quad , \text{ Since } \sum_{c \in \mathcal{C}} (a)_c (c)_b = (a)_b = \mu(b) \\ \text{Since } \Rightarrow (a)_b &= \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \end{aligned}$$

Similarly $\Phi_{\mathcal{BC}} \circ \Phi_{\mathcal{CB}} = 1 \Rightarrow \Phi_{\mathcal{BC}}$ is isomorphism. Therefore $\Phi_{\mathcal{BC}}, \Phi_{\mathcal{CB}}$ are mutually inverse. $\square$

As an application of our main result we have these nice examples.

Example 4.2.3. Let $R = \mathbb{F}[x]$ and $\mathcal{B} = \{x^i : i \geq 0\}$ be the standard basis of R .

$$\forall n \geq 0 \text{ and let } \mathcal{G} = \begin{cases} x^i & \text{if } i = 2n \\ x^i + x^{i+1} & \text{if } i = 2n + 1 \end{cases}$$

Clearly $\mathcal{B}$ and $\mathcal{G}$ are mutually congenial then by **Theorem 4.2.3** $\mathcal{M}_{\mathcal{B}}$ and $\mathcal{M}_{\mathcal{G}}$ are naturally isomorphic.

Example 4.2.4. Let $\mathcal{B} = \{x^i : i \geq 0\}$ and let $\mathcal{C} = \{1\} \cup \{x^i + x^{i+1} : i \geq 0\}$ be bases of $\mathbb{F}[x]$. These two bases are amenable we can see that $\mathcal{C}$ is congenial to $\mathcal{B}$ and induced a map $\phi : \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{B}}$. Since $\mathcal{B}$ is not congenial to $\mathcal{C}$, ϕ is not one to one.

Let $P_0 = 1$

$$P_i = x^{i-1}(1+x), i \geq 0$$

and $\phi(\sum_{i=0}^n (-1)^i P_i) = (-1)^n x^n$. We show that $\phi(\sum_{i=0}^{\infty} (-1)^i P_i) = 0$. Let $a = \sum_{i=0}^{\infty} (-1)^i P_i$.

Then we have,

$$a = \sum_{i=0}^n (-1)^i P_i + (-1)^n x^n (a - 1)$$

$$\Rightarrow \phi(a) = \phi(\sum_{i=0}^n (-1)^i P_i) + (-1)^n x^n (\phi(a) - 1)$$

$$\Rightarrow \phi(a) = (-1)^n x^n + (-1)^n x^n (\phi(a) - 1) = (-1)^n x^n + (-1)^n x^n \phi(a) - (-1)^n x^n = (-1)^n x^n \phi(a)$$

$$\Rightarrow \phi(a)(1 - (-1)^n x^n) = 0$$

$$\Rightarrow \phi(a) = 0$$

Therefore ϕ is not one to one since $\phi(0) = 0$.

Definition 4.2.1. Let $\mathcal{B}$ and $\mathcal{C}$ be bases. Let $\mathcal{B}$ be congenial to the $\mathcal{C}$ then the matrix representation relative to $(\mathcal{B}, \mathcal{C})$, $A = [I_R]_{\mathcal{B}}^{\mathcal{C}}$ is called a congeniality matrix. Furthermore, if $\mathcal{C}$ not congenial to $\mathcal{B}$, then the matrix representation relative to $(\mathcal{B}, \mathcal{C})$, $[I_R]_{\mathcal{B}}^{\mathcal{C}} = A$ is called a proper congeniality matrix.

Definition 4.2.2. A Module $\mathcal{M}$ over infinite dimensional algebra R is called a basic module if there exists a basis $\mathcal{B}$ such that $\mathcal{M} = \mathcal{M}_{\mathcal{B}}$.

Definition 4.2.3. Let R be an infinite dimensional algebra and $\mathcal{B} = \mathcal{B}_0$ be an amenable basis and let A any proper congeniality matrix. Then we can produce a series

$$\cdots \xrightarrow{\quad} \mathcal{B}_{-2} \xrightarrow{\quad} \mathcal{B}_{-1} \xrightarrow{\quad} \mathcal{B}_0 \xrightarrow{\quad} \mathcal{B}_1 \xrightarrow{\quad} \mathcal{B}_2 \xrightarrow{\quad} \cdots$$

Where for every i , $\mathcal{B}_i$ is congenial to $\mathcal{B}_{i+1}$ and $A = [I_R]_{\mathcal{B}_i}^{\mathcal{B}_{i+1}}$. We call that series an A -series.

Now, If there exists $n \in \mathbb{Z}^+$ such that $j \geq n$, $\mathcal{B}_j$ is not amenable. A smallest such n yield $\mathcal{B}_n$ is the last amenable basis in the A -series. We say that $\mathcal{M}_{\mathcal{B}_n}$ is an A -simple basic module.

Example 4.2.5. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let $\mathcal{B}_0 = \{x^i : i \geq 0\}$ we can produced a series from A and $\mathcal{B}_0$

$$\cdots \xrightarrow{\quad} \mathcal{B}_{-2} \xrightarrow{\quad} \mathcal{B}_{-1} \xrightarrow{\quad} \mathcal{B}_0 \xrightarrow{\quad} \mathcal{B}_1 \xrightarrow{\quad} \cdots$$

Where $\mathcal{B}_{-1} = \{1\} \cup \{x^i + x^{i+1} : i \geq 0\}$, $\mathcal{B}_{-2} = \{1, x + 1\} \cup \{x^i + 2x^{i+1} + x^{i+2} : i \geq 0\}$ and $\mathcal{B}_1 = \{\sum_{i=0}^n (-1)^{n-i} x^i : n \geq 0\}$. The bases $\mathcal{B}_{-2}, \mathcal{B}_{-1}$ and $\mathcal{B}_0$ are amenable, however $\mathcal{B}_1$ is not amenable. In fact $\mathcal{B}_n$ is not amenable for any $n \geq 1$ and $\mathcal{M}_{\mathcal{B}_0}$ is an A -simple basic module.

For the algebra of the polynomials, there could be a series that non of the bases are amenable. Furthermore, it turns out that there is a series that is if one basis is amenable in one stage doesn't imply that it remain amenable basis in the next stage. Also it doesn't imply that the basis before is amenable. As we see in the following examples.

Example 4.2.6. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let $\mathcal{B}_0 = \{x^i : i \geq 0\}$ we can produced a series from A and $\mathcal{B}_0$

$$\cdots \xrightarrow{\quad} \mathcal{B}_{-2} \xrightarrow{\quad} \mathcal{B}_{-1} \xrightarrow{\quad} \mathcal{B}_0 \xrightarrow{\quad} \mathcal{B}_1 \xrightarrow{\quad} \cdots$$

Where $\mathcal{B}_{-1} = \{\sum_{i=0}^n (-1)^i x^i : n \geq 0\}$, $\mathcal{B}_{-2} = \{1\} \cup \{x^i + x^{i+2} : i \geq 0\}$ and $\mathcal{B}_1 = \{1, 1 - x, -1 + x + x^2, -1 + x + x^2 - x^3, 1 - x - x^2 + x^3 + x^4, 1 - x - x^2 + x^3 + x^4 - x^5, \cdots\}$.

The bases $\mathcal{B}_{-2}, \mathcal{B}_0$ are amenable, however $\mathcal{B}_{-1}$ and $\mathcal{B}_1$ are not amenable.

In the example below, we can produce an A-series where all bases are not amenable.

Example 4.2.7. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let $\mathcal{B}_0 = \{1 + x^i : i \geq 0\}$ we can produced a series from A and B_0

$$\cdots \longrightarrow \mathcal{B}_{-2} \longrightarrow \mathcal{B}_{-1} \longrightarrow \mathcal{B}_0 \longrightarrow \mathcal{B}_1 \longrightarrow \cdots$$

Where $\mathcal{B}_{-1} = \{1, 2 + x\} \cup \{2 + x^i + x^{i+1} : i \geq 1\}$, $\mathcal{B}_{-2} = \{1, 3 + x\} \cup \{4 + 2x^i + x^{i+1} : i \geq 1\}$, and $\mathcal{B}_1 = \{1, x, 1 - x + x^2, x - x^2 + x^3, 1 - x + x^2 - x^3 + x^4, x - x^2 + x^3 - x^4 + x^5, \dots\}$. The bases $\mathcal{B}_{-2}, \mathcal{B}_{-1}, \mathcal{B}_0$ and $\mathcal{B}_1$ all are not amenable.

5 LEAVITT PATH ALGEBRA

Now we introduce the notion of Leavitt Path Algebra as it appears in many papers, including [2], [14] and [1].

Definition 5.0.4. *Let K be a field and E be a row-finite graph i.e. every vertex emits only finitely many edges. The path K -algebra over E is defined as the free K -algebra $K[E^0 \cup E^1]$ with the relations:*

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
2. $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$.

This algebra is denoted by $A_K(E)$.

Definition 5.0.5. *Given a graph E we define the extended graph of E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as*

$$r'|_{E^1} = r, s'|_{E^1} = s, r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

The elements of E^1 are called (real) edges, while for $e \in E^1$ we call e^* a ghost edge. The set $e^* \in E^1$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

Definition 5.0.6. *For a field K and a row-finite graph E , the Leavitt path K -algebra $L_K(E)$ is defined as the K -algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $E^1 \cup E^1^*$, which satisfy the following relations:*

$$(V) \quad vv' = \delta_{v,v'} v \text{ for all } v, v' \in E^0,$$

$$(E1) \quad s(e)e = er(e) = e \text{ for all } e \in E^1$$

$$(E2) \quad r(e)e^* = e^*s(e) = e^* \text{ for all } e \in E^1$$

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$

(CK2) $v = \sum_{e \in E^1: s(e)=v} ee^*$ for every $v \in E^0$ that emits edges.

The conditions **CK1** and **CK2** are called the Cuntz-Krieger relations. In particular condition **CK2** is the Cuntz-Krieger relation at v_i . If v_i is a sink, we do not have a **CK2** relation at v_i . Note that the condition of row-finiteness is needed in order to define the equation **CK2**.

Definition 5.0.7. A vertex v in E^0 is a bifurcation (or there is a bifurcation at v) if $s^{-1}(v)$ has at least two elements.

The following are the well-known examples of Leavitt Path Algebras.

Example 5.0.8. Matrix algebras $M_n(K)$: Consider the graph E defined by $E^0 = v_1, \dots, v_n, E^1 = e_1, \dots, e_{n-1}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \dots, n-1$. Then $L_K(E) \cong M_n(K)$, via the map $v_i \rightarrow e_{(i,i)}$, $e_i \rightarrow ee_{(i,i+1)}$, and $e_i^* \rightarrow ee_{(i+1,i)}$ (where $e_{(i,j)}$ denotes the standard (i, j) -matrix unit in $M_n(K)$).

Example 5.0.9. Laurent polynomial algebras $K[x, x-1]$: Consider the "one vertex, one loop" graph E defined by $E^0 = \{*\}$, $E^1 = \{x\}$. Then clearly $L_K(E) \cong K[x, x-1]$.

Example 5.0.10. Consider the rose with n petals graph E defined by $E^0 = \{*\}$, $E_1 = y_1, \dots, y_n$ (where $n \geq 2$). Then $L_K(E) \cong L_K(1, n)$.

Lemma 5.0.5. [2] Every monomial in $L(E)$ is of the following form.

1. $k_i v_i$ with $k_i \in K$ and $v_i \in E^0$, or
2. $ke_{i_1} \cdots e_{i_\sigma} e_{j_1}^* \cdots e_{j_\tau}^*$ where $k \in K$; $\sigma, \tau \geq 0$, $\sigma + \tau > 0$, $e \in E^1$ and $e^* \in (E_1)^*$ for $0 \leq s \leq \sigma$, $0 \leq t \leq \tau$.

In this section we introduce a family of bases for Leavitt Path Algebras which we will call standard and show that all standard bases are amenable.

Now we define a basis of Leavitt Path Algebra $L_k(E)$ over directed graphs as it appears in [4] .

For an arbitrary vertex v which is not a sink, choose an edge $\phi(v)$ such that $s(\phi(v)) = v$. We will refer to this edge as special. In other words, we fix a function $\phi : V \rightarrow E^1$ such that $s(\phi(v)) = v$ for an arbitrary $v \in V$.

Definition 5.0.8. The union of the following sets form a basis of the Leavitt path algebra $L_K(E)$ denoted by $\mathcal{B}_\phi$.

- $\{v : v \in V\}$,
- $\{p : p \text{ is a path in } E\} \cup \{p^* : p \text{ is a path in } E\}$

- $\{pq^* : p = e_1 \dots e_n, q = f_1 \dots f_m, e_i, f_j \in E, r(e_n) = r(f_m) \text{ and } pq^* \text{ is reduced, with the condition that the last edges } e_n \text{ and } f_m \text{ are either distinct or equal, but not special}\}.$

Let α, β, γ and δ be paths in $L_k(E)$, then

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} \alpha\gamma'\delta^* & \text{if } \gamma = \beta\gamma' \\ \alpha\beta'^*\delta^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise} \end{cases}$$

Now we represent the product with respect to the basis $\mathcal{B}_\phi$.

$$(\alpha\beta^*)(\gamma\delta^*) = \begin{cases} \alpha\delta^* & \text{if } \gamma = \beta \text{ \& } \text{end}(\alpha) \text{ or } \text{end}(\delta) \notin \mathcal{F} \\ & \text{or if } \text{end}(\alpha), \text{end}(\delta) \in \mathcal{F} \text{ \& } s(\text{end}(\alpha)) \neq s(\text{end}(\delta)) \\ \sum \alpha'\delta'^* & \text{if } \text{end}(\alpha) = \text{end}(\delta) \in \mathcal{F} \end{cases}$$

For all α', β', γ' and δ' in $L_k(E)$.

The following notions and terminology were introduced in [6].

Definition 5.0.9. For a path $\alpha = \alpha_1 \dots \alpha_k$ and for $1 \leq l \leq k$, denote by $(\alpha)_{\tau \leq l} = \alpha_1 \dots \alpha_l$ and $(\alpha)_{\tau > l} = \alpha_{l+1} \dots \alpha_k$ the truncation at l .

Definition 5.0.10. Two paths α and β are tail-equivalent if there exists n and m such that $(\alpha)_{\tau > n} = (\beta)_{\tau > m}$.

In the **Theorem 1.6.5.** below we show that $\mathcal{B}$ is amenable. Before that we prove some useful Lemmas.

Lemma 5.0.6. Let $L_k(E)$ be a Leavitt path algebra and $\mathcal{B}$ be its standard basis induced by $\phi : V \rightarrow E^1$. Let α, δ, φ and ε be paths in $L_K(E)$ with $\alpha = \alpha_1 \dots \alpha_k$, $\beta = \beta_1 \dots \beta_l$, $\varepsilon = \varepsilon_1 \dots \varepsilon_m$ and $\varphi = \varphi_1 \dots \varphi_p$. For every $\alpha\beta^*, \varepsilon\varphi^* \in \mathcal{B}$, if $\alpha = \varepsilon$ and β and φ are tail-equivalent then we have $|\{\gamma\delta^* \in \mathcal{B} : (\alpha\beta^*\gamma\delta^*)_{\varepsilon\varphi^*} \neq 0\}| = n + 1$, where n is the number of the edges in the common tail of the paths φ and β .

Proof. We prove by induction on n that there are $n + 1$ ways of $\gamma\delta^*$ such that $(\alpha\beta^*\gamma\delta^*)_{\varepsilon\varphi^*} \neq 0$.

- We prove the statement when $n = 1$. If $n = 1$ then $\beta_l = \varphi_p$. We have two ways such that $(\alpha\beta^*\gamma\delta^*)_{\alpha\varphi^*} \neq 0$:

$$1. (\alpha\beta^*(\beta\varphi^*))_{\alpha\varphi^*} = (\alpha\varphi^*)_{\alpha\varphi^*} \neq 0.$$

$$2. (\alpha\beta^*(\beta_{\tau \leq l-1}\varphi_{\tau \leq p-1}^*))_{\alpha\varphi^*} = (\alpha\beta_l^*\varphi_{\tau \leq p-1}^*)_{\alpha\varphi^*} = (\alpha\varphi_p^*\varphi_{\tau \leq p-1}^*)_{\alpha\varphi^*} = (\alpha\varphi^*)_{\alpha\varphi^*} \neq 0.$$

When $n = 2$ then $\beta_{l-1}\beta_l = \varphi_{p-1}\varphi_p$.

We have 3 ways such that

$(\alpha\beta^*\gamma\delta^*)_{\alpha\varphi^*} \neq 0$:

$$1. (\alpha\beta^*(\beta\varphi^*))_{\alpha\varphi^*} = (\alpha\varphi^*)_{\alpha\mu^*} \neq 0.$$

$$2. (\alpha\beta^*(\beta_{\tau \leq l-1}\varphi_{\tau \leq p-1}^*))_{\alpha\varphi^*} = (\alpha\beta_l^*\varphi_{\tau \leq p-1}^*)_{\alpha\varphi^*} = (\alpha\varphi^*)_{\alpha\varphi^*} \neq 0.$$

$$3. (\alpha\beta^*(\beta_{\tau \leq l-2}\varphi_{\tau \leq p-2}^*))_{\alpha\mu^*} = (\alpha\beta_l^*\beta_{l-1}^*\varphi_{\tau \leq p-2}^*)_{\alpha\varphi^*} = (\alpha\varphi^*)_{\alpha\varphi^*} \neq 0.$$

- Suppose the argument is true for n .
- Now, we show that the argument is true for $n+1$. When we have $n+1$ then $\beta_{\tau > l-(n+1)} = \varphi_{\tau > p-(n+1)}$. Since $\beta_{l-n-1}\beta_{\tau > l-(n)} = \varphi_{p-n-1}\varphi_{\tau > p-n}$ then we have $\beta_{\tau > l-(n)} = \varphi_{\tau > p-n}$, by our assumption there are $n+1$ ways such that $(\alpha\beta^*\gamma\delta^*)_{\alpha\varphi^*} \neq 0$. Also we have

$$\begin{aligned} & (\alpha\beta^*(\beta_{\tau \leq l-n-1}\varphi_{\tau \leq l-n-1}^*))_{\alpha\varphi^*} \\ &= (\alpha\beta_{\tau > l-n-1}^*\varphi_{\tau \leq p-n-1}^*)_{\alpha\mu^*} \\ &= (\alpha\varphi^*)_{\alpha\varphi^*} \neq 0. \end{aligned}$$

Thus we have $n+2$ ways. Therefore by induction the lemma holds. $\square$

Lemma 5.0.7. For every $\alpha\beta^*, \varepsilon\varphi^* \in \mathcal{B}$, let ε be initial subpath of α but the last term i.e. $\varepsilon = \alpha_{\tau < i}e_i$ where $1 \leq t \leq n$. If α_i is special where $t \leq i \leq n$ and $\varphi = \varphi'e_i$, then we have one way such that $(\alpha\beta^*\gamma\delta^*)_{\alpha\varphi^*} \neq 0$.

$$\begin{aligned}
\text{Proof. } (\alpha\beta^*(\beta\alpha_n^*\alpha_{n-1}^*\dots\alpha_t^*\varphi'^*))_{\varepsilon\mu^*} &= (\alpha_{\tau<n}(\alpha_n\alpha_n^*)\alpha_{n-1}^*\dots\alpha_t^*\varphi V^*)_{\varepsilon\mu^*} \\
&= (\alpha_{\tau<n-1}(\alpha_{n-1}\alpha_{n-1}^*)\alpha_{n-2}\dots\alpha_t^*\varphi^*)_{\varepsilon\varphi^*} = (\alpha_{\tau<t}(\alpha_t\alpha_t^*)\varphi^*)_{\varepsilon\mu^*} \\
&= (\alpha_{\tau<t}(v - \sum_{s(e_j)=v} e_j e_j^*)\varphi^*)_{\varepsilon\varphi^*} = (\alpha_{\tau<t}(e_t e_t^*)\varphi^*)_{\varepsilon\varphi^*} \neq 0. \square
\end{aligned}$$

Lemma 5.0.8. For every $\alpha\beta^*, \varepsilon\varphi^* \in \mathcal{B}$, If ε is initial subpath of α i.e. $\varepsilon = \alpha_{\tau<t}$ where $1 \leq t \leq n$ and α_i are special where $t \leq i \leq n$ then we have one way such that $(\alpha\beta^*\gamma\delta^*)_{\varepsilon\varphi^*}$.

$$\begin{aligned}
\text{Proof. } (\alpha\beta^*(\beta\alpha_n^*\alpha_{n-1}^*\dots\alpha_t^*\phi^*))_{\varepsilon\phi^*} &= (\alpha_{\tau<n}(\alpha_n\alpha_n^*)\alpha_{n-1}^*\dots\alpha_t^*\phi^*)_{\varepsilon\phi^*} \\
&= (\alpha_{\tau<n-1}(\alpha_{n-1}\alpha_{n-1}^*)\alpha_{n-2}\dots\alpha_t^*\phi^*)_{\varepsilon\phi^*} = (\alpha_{\tau<t}(\alpha_t\alpha_t^*)\phi^*)_{\varepsilon\phi^*} \\
&= (\alpha_{\tau<t}(v - \sum_{s(e_j)=v} e_j e_j^*)\phi^*)_{\varepsilon\phi^*} = (\alpha_{\tau<t}\phi^*)_{\varepsilon\phi^*} \neq 0. \square
\end{aligned}$$

Theorem 5.0.9. Let $\mathcal{B}$ be a basis of Leavitt path algebra $L_k(E)$ over directed graphs that has bifurcations as describe above. Then $\mathcal{B}$ is amenable.

Proof. There are two cases $|\varepsilon| > |\alpha|$ and $|\varepsilon| \leq |\alpha|$ to be considered.

Case 1. $|\varepsilon| > |\alpha|$

Since $|\varepsilon| > |\alpha|$ we need to extend the path α . From the definition above we have $(\alpha\beta^*)(\gamma\delta^*) \neq 0 \Leftrightarrow \gamma = \beta\gamma_1$. Therefore $(\alpha\beta^*)(\gamma\delta^*) = (\alpha\beta^*)(\beta\gamma_1\delta^*) = \alpha\gamma_1\delta^*$.

Now, fix $\alpha\beta^*$ and $\varepsilon\varphi^*$ since $|\varepsilon| > |\alpha|$ then α is initial subpath of ε i.e. if $\alpha = \alpha_1\dots\alpha_n$ and $\varepsilon = \varepsilon_1\dots\varepsilon_m$ then $\varepsilon = \alpha\varepsilon_{\tau>n}$. Therefore the only way to get $\varepsilon\varphi^*$ is to multiply from the right $\alpha\beta^*$ by $\beta\varepsilon_{\tau>n}\varphi^*$.

Case 2. $|\varepsilon| \leq |\alpha|$

Since $|\varepsilon| \leq |\alpha|$ we have two sub-cases:

Case 2.1. When $\alpha = \varepsilon$.

If β and φ are not tail-equivalent then we have only one way so that $(\alpha\beta^*\gamma\delta^*)_{\varepsilon\varphi^*} \neq 0$. Which is $(\alpha\beta^*\beta\varphi^*)_{\varepsilon\varphi^*} = (\alpha\varphi^*)_{\varepsilon\phi^*} \neq 0$

Now if β and φ are tail-equivalent then see **Lemma 5.0.6.**

Case 2.2. When $\alpha \neq \varepsilon$.

Lemma 5.0.7 and **Lemma 5.0.8**. $\square$

Example 5.0.11. Let $R = L_K(E)$ be Leavitt path algebra over graph E . Let $\mathcal{B}_e$ and $\mathcal{B}_f$ be two bases of R where $\mathcal{B}_e = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ee^*\beta'^*\}$ and $\mathcal{B}_f = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ff^*\beta'^*\}$. We have that $\mathcal{B}_e$ and $\mathcal{B}_f$ are mutually congenial then by **Theorem 4.2.3** $\mathcal{M}_{\mathcal{B}_e}, \mathcal{M}_{\mathcal{B}_f}$ are naturally isomorphic.

Example 5.0.12. Let $R = L_K(F)$ be Leavitt path algebra over graph F . Let $\mathcal{B}_e$ and $\mathcal{B}_f$ be two bases of R where $\mathcal{B}_e = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ee^*\beta'^*\}$ and $\mathcal{B}_f = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ff^*\beta'^*\}$. We have that $\mathcal{B}_f$ is congenial to $\mathcal{B}_e$ but $\mathcal{B}_e$ is not congenial to $\mathcal{B}_f$. Therefore, the R – linear map $\psi : \mathcal{M}_{\mathcal{B}_e} \rightarrow \mathcal{M}_{\mathcal{B}_f}$ is an epimorphism.

Example 5.0.13. Let $R = L_K(G)$ be Leavitt path algebra over graph G . Let $\mathcal{B}_e$ and $\mathcal{B}_f$ be two bases of R where $\mathcal{B}_e = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ee^*\beta'^*\}$ and $\mathcal{B}_f = \{\alpha\beta^* : \alpha\beta^* \neq \alpha'ff^*\beta'^*\}$. We

have that $\mathcal{B}_f$ is not congenial to $\mathcal{B}_e$ and $\mathcal{B}_e$ is not congenial to $\mathcal{B}_f$. Therefore, the R – linear map $\xi : \mathcal{M}_{\mathcal{B}_e} \rightarrow \mathcal{M}_{\mathcal{B}_f}$ is not a bijection map.

$$G \quad f \bigcirc \bullet^v \bigcirc^{\leftarrow} e$$

6 FUTURE WORK

In this section, we list some open problems related to our topic.

1. Let $\mathcal{B}$, C be amenable bases of R . If $\mathcal{B}$ is congenial to C and induced a map $\phi : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_C$ such that ϕ is one to one and onto then C is congenial to $\mathcal{B}$.
2. If $M_B \cong M_C$ then there exists $\phi : M_B \rightarrow M_C$ which is natural.
3. Suppose that $\mathcal{B}$ is amenable and $\mathcal{B}$ is congenial to C . Let $\pi : \mathcal{M}_{\mathcal{B}} \rightarrow \mathcal{M}_C$ be a map. C is amenable if and only if $Ker(\pi)$ is an R -submodule of $M_{\mathcal{B}}$.
4. Does there exist an amenable basis $\mathcal{B}$ such that $\mathcal{B}$ is A -simple for all proper congeniality basis A ? If so, $\mathcal{B}$ is called simple and $\mathcal{M}_{\mathcal{B}}$ is called a simple basic module. Another way to say this is, if $\mathcal{M}_{\mathcal{B}}$ is a module and for every $\mathcal{B}$, $\mathcal{B}$ is congenial to C , if C is amenable then $\mathcal{B}$ and C are mutually congenial”.
5. If B is amenable then $Ker\phi$ is submodule of $\mathcal{M}_{\mathcal{B}}$ if and only if C is amenable.

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