A LOW-ORDER NONLINEAR STATE-SPACE MODEL
FOR DELTA WING LEADING EDGE VORTICES

A thesis presented to
the faculty of
the Fritz J. and Dolores H. Russ
College of Engineering and Technology
of Ohio University

In partial fulfillment
of the requirements for the degree
Master of Science

Bo Liao
March 2006

This thesis entitled
A LOW-ORDER NONLINEAR STATE-SPACE MODEL
FOR DELTA WING LEADING EDGE VORTICES

by
BO LIAO

has been approved for
the Department of Electrical Engineering and Computer Science
and the Russ College of Engineering and Technology by

J. Jim Zhu
Professor of Electrical Engineering and Computer Science

Dennis Irwin
Dean, Russ College of Engineering and Technology

LIAO, BO. M.S. March 2006. Electrical Engineering

A Low-Order Nonlinear State-Space Model For Delta Wing Leading Edge Vortices

(106 pp.)

Director of Thesis: J. Jim Zhu

The delta wing has been studied extensively since the mid-1940s. One unique characteristic of the delta wing is its highly swept, sharp leading edges that cause the flow to separate at the leading edge and roll up into tight spiral vortices on the upper surface of the wing, which contribute to significant lift increase. At high angles of attack, vortex breakdown occurs which creates a region of highly turbulent flow aft of the burst point and consequently a reduction in lift. The interaction between the vortices and the air frame can induce aero-elastic oscillations such as wing rocking.

In different flying motions the vortices above the delta wing change position and strength in the cross flow plane,, and the vortex breakdown location also changes along the wing longitudinal axis. These changes have a great influence on the flying motion. Therefore understanding the flow mechanism and developing closed-loop and active control of the flow become very important in delta wing research.

This thesis provides a unique way for low-order flow modeling suitable for feedback flow control design. It combines two theoretical models based on canonical flow assumptions for flow field over a delta wing, a wing rock model and a vortex breakdown model, to develop into a set of standard non-linear state equations describing the dynamics of the flow field as well as the aircraft roll motion through a series of transformations and simplifications. The non-linear state equation model allows a variety

of flow control designs to be directly and explicitly applied to the system. Since the two theoretical models have been validate with wind tunnel experiments, and the new model presented here has been validated with the theoretical models, control designs based on this new model should be directly applicable on the experimental setup to be tested for improved performance of a delta wing.

Approved:

Jim Zhu

Professor of Electrical Engineering and Computer Science

Acknowledgments

First and most importantly, thanks to my advisor, Dr. Jim Zhu, who always scientifically, arduously and patiently guided me through the research. Not only in scientific research, but also in many other aspects I learned many valuable things that will always help in my life and work.

Thanks to other professors who always helped in the research whenever needed, especially thanks to Dr. Douglas Lawrence who always combines intelligence with humor.

Thanks to my classmates with whom I have worked: M. Wu, Y. Liu and D. Zhu. They always promptly provided me a hand when I needed.

Thanks to Dr. Andrew Arena and Dr. James Myatt for their candid help in explaining the detailed theory in their works which form the theoretical basis for the current model.

Thanks to Dayton Area Graduate Studies Institute (DAGSI) and Ohio University EECS Department for their financial support on the research.

Thanks to my father and mother who always supported me so that I could finish my research work efficiently.

Thanks to all of you, I appreciate it very much.

Table of Contents

	Page
Abstract	3
Acknowledgments	5
List of Figures	8
Chapter 1. Introduction to Delta Wing Modeling	10
1.1 Background.....	10
1.2 A survey on flow modeling methods.....	12
1.3 Low-order modeling method by flowfield simplification.....	19
1.4 Problem Statement.....	20
1.5 Objective, Scope and Basic Methodology of Current Model.....	21
1.6 Experimental setup for control design.....	22
1.7 Contents of the thesis.....	24
Chapter 2. Theory and Methodology	25
2.1 Overview.....	25
2.2 Some fundamental theories.....	29
2.3 Kutta condition.....	35
2.4 Zero-Force condition.....	36
2.5 Nondimensionalization of Kutta condition equations.....	41
2.6 Rolling moment calculation.....	42
2.7 The relationship between all the equations.....	48
Chapter 3. Transformation from Complex Quantities to Real Quantities	50
3.1 Modeling goal.....	50
3.2 Kutta condition equations.....	52
3.3 Zero-force condition equations.....	56
3.4 Flow potential.....	60
3.5 Flow potential rate.....	64
3.6 Total roll moment.....	69
Chapter 4. Solving for State Equations and Simplification	72
4.1 Solve for state equations.....	72
4.2 The system output.....	75
4.3 Simplification of the state equations.....	76
Chapter 5. Model Validation --Results and Analysis....	87
5.1 SIMULINK diagram.....	88
5.2 Simulation result.....	91
Chapter 6. Summary and Conclusions	98

	7
Appendix: Nomenclature	100
References	105

List of Figures

Figure	Page
1.1 CFD image of Delta Wing, 1995, NASA/Ames Research Center	15
1.2 TECPLOT tips-using TECPLOT's CFD analyzer add-on	16
1.3 Vortex breakdown on a delta wing using Adaptive Grid Refinement method	16
1.4 Experimental delta wing setup	23
2.1 The complex potential for static flow model	27
2.2 The complex potential for the unsteady-wing model	28
2.3 Local surface velocity distribution due to rolling motion.....	29
2.4 Joukowski Transformation between two planes.....	30
2.5 The Joukowski's conformal transformation between the circle plane and the physical plane.....	31
2.6 Zero-force condition	36
5.1 The SIMULINK diagram for the dynamic state system.....	88
5.2 The wing rigid-body dynamic sub diagram	89
5.3 The flow dynamic sub diagram	90
5.4.1 Vortex breakdown location movement in a harmonic motion of delta wing (the current model)	92
5.4.2 Vortex breakdown location movement in a harmonic motion of delta wing (Myatt model)	92
5.5 Roll angle vs. time for the wing rock phenomenon (the current model). Initial roll angle: -0.5° , angle of attack: 20°	93

5.6 Roll angle vs. time for the wing rock phenomenon (Arena model). Initial roll angle: -0.5°, angle of attack: 20°	93
5.7 Roll angle vs. time for the wing rock phenomenon (the current model). Initial roll angle 85°, angle of attack 20°	94
5.8 Roll moment vs. time for the wing rock phenomenon (Arena model). Initial roll angle of 85°, angle of attack: 20°	94
5.9 The limit cycle of roll angle and roll angle rate, initial roll angle -0.5°	95
5.10 The limit cycle of roll angle and roll angle rate, initial roll angle 85°	95
5.11 Roll angle vs. time with vortex breakdown. Initial roll angle: -0.5°, angle of attack: 20°, sweep angle: 80°	96

Chapter 1

Introduction to Delta Wing Modeling

In this Chapter, after a brief survey on delta wing research, some other research methods on delta wing besides the thesis research method are described. Then the research method in this thesis is emphasized, together with a concise introduction on experimental setup. Finally the contents of the thesis are summarized.

1.1 Background

The delta-wing has been studied extensively since the mid-1940s. The research on delta-wing is important in that many aircraft utilize high sweep-angle to achieve high efficiency when cruising at supersonic or hypersonic velocity, and they are expected to fly at large angles of attack with a effective attitude control and robust stability.

One unique characteristic of the delta wing is its swept, sharp leading edges that cause the flow to separate at the leading edge and roll up into tight spiral vortices on the upper surface of the wing. The flow of these vortices has much higher velocity than the free stream velocity, therefore a strong suction on the upper surface of the wing occurs, and the lift increases.

The leading edge vortices not only result in lift increase, but also lead to unacceptable wing motion such as self induced oscillations like wing rock from an operational and safety point of view. For example, at a higher angle of attack, highly swept angle delta wings are susceptible to wing rock which is a sustained self-induced periodic oscillation in roll. Wing rock is a rolling oscillation due to vortex asymmetries

interacting with an aircraft motion and loss of damping at high angles of attack. This dynamic condition can result in departure from controlled flight. Two particularly important cases are the highly swept delta wing and the slender body forebody-delta wing. To avoid wing rock, angle of attack limits must be established. However, by doing so, the maneuvering envelope of these aircraft is degraded. More serious problems resulted from this kind of self induced oscillation include degradation of aiming accuracy and loss of control.

At high angles of attack, vortex breakdown occurs over the wing resulting in a deceleration of the flow along the axis of vortex, a decrease in the circumferential velocity, and a region of highly turbulent flow aft of the burst point. The suction pressure in the region aft of the vortex breakdown is reduced and results in a reduction in lift. And with zero sideslip the vortex breakdown point moves towards the apex of wing when the angle of attack increases, and vice versa. In some cases, they moved cross the trailing edge in a discontinuous manner.

When the wing is in different flying motion, the vortices above the delta wing change as well as strength in the cross flow plane, and the vortex breakdown locations also change along the wing longitudinal axis. They have such a great influence on the flying motion that understanding the flow mechanism and developing effective flow control techniques have been emphasized in delta wing researches.

To maintain the stability and maneuverability of an airplane at high angles of attack, the understandings of the structure and the behavior of vortex bursting is fundamental. Without this basic knowledge, active control of vortex breakdown will be extremely difficult, if not impossible. Active flow control is important in that the control of vortices

has the potential to increase lift and rolling moments on demand, which will enhance aircraft maneuverability, agility, and controllability.

This thesis develops a unique way for low-order modeling of the flow mechanism that is suitable for flow control design. It is unique in that it combines two low-order theoretical models of the delta wing, a wing rock model of Arena [1] and a vortex breakdown model of Myatt [2], to derive a set of non-linear state equations describing the flow as well as aircraft rolling dynamics. This model allows a variety of flow control designs to be explicitly implemented on the delta wing.

In the next section, some existing flow field modeling for delta wings, such as CFD and Modeling Scheduling method are surveyed.

1.2 A Survey on flow modeling methods

1.2.1 CFD method

Computational Fluid Dynamics, or simply CFD, is the numerical solution of the equations describing fluid and gas motion. CFD is used for basic understanding of fluid dynamics. CFD is concerned with obtaining a numerical solution to fluid flow problems by using digital computers. The advent of high-speed and large-memory computers has enabled CFD to obtain solutions to many flow problems including those that are compressible or incompressible, laminar or turbulent, chemically reacting or non-reacting.

Computer modeling makes it possible to study cases which are virtually impossible to measure in practice. Unlike the practical measurements, the CFD solution often yields

all flow field quantities of interest. The drawbacks of CFD are the long time required to create a high quality, detailed grid and uncertainties related to turbulence modeling.

The equations governing the fluid flow problem are the Navier-Stokes (conservation of momentum), the continuity (conservation of mass), and the energy equations. These equations form a system of coupled non-linear partial differential equations (PDEs). Because of the non-linear terms in these PDEs, analytical methods can yield very few solutions. In general, closed form analytical solutions are possible only if these PDEs can be made linear, because of two reasons. One is that non-linear terms naturally drop out and the other is that nonlinear terms are small compared to other terms so that they can be neglected. If the non-linearities in the governing PDEs cannot be neglected, which is the situation for most engineering flows problems, and then numerical methods are needed to obtain solutions.

CFD is the art of replacing the differential equation governing the Fluid Flow with a set of iterative algebraic equations (the process is called discretization) which in turn can be solved with the aid of a digital computer to get an *approximate* solution. The well known discretization methods used in CFD are Finite Element Method (FEM), Finite Difference Method (FDM), Finite Volume Method (FVM), and Boundary Element Method(BEM).

Solving a particular problem generally involves first discretizing the physical domain that the flow occurs in, such as the interior of a turbine engine or the radiator system of a car. This discretization is straightforward for very simple geometries such as rectangles or circles, but is a difficult problem in CFD for more complicated objects. This leads to problems in human-computer interfaces (HCI) tools, as well as fundamental

problems in graph theory since the resulting discretization gives a mesh that is best dealt with as a graph.

On the discretized mesh the Navier-Stokes equations take the form of a large system of nonlinear equations; going from the continuum to the discrete set of equations requires a combination of both physics and numerical analysis. At each node in the mesh, some variables are associated: the pressure, the three velocity components, density, temperature, etc. Furthermore, capturing physically important phenomena such as turbulence requires extremely fine meshes in parts of the physical domain. The system of nonlinear equations is typically solved by a Newton-type numerical method, which in turn requires solving a large, sparse system of equations on each step.

Once the solution is found, visualization and graphics techniques are needed in analyzing, validating, and presenting the solution. Those techniques are useful for more than just viewing the computed flow field. Visualization can help with understanding the nature of the problem, the interaction of algorithms with the computer architecture, the performance analysis of the code. Therefore CFD needs graph theory and algorithms. Computational complexity, numerical analysis, parallel programming, graphics and visualization are all needed here.

In 1995, in a finite-difference setting, sixth-order compact schemes were employed at the Air Force Research Laboratory (AFRL) to understand the physics of complex vortical flows. The high-order method proved crucial for understanding the nature and origin of helical substructures found in the shear layer above a delta wing. Fig. 1.1 shows the CFD image of delta wing from NASA.

Figure 1.1 CFD image of Delta Wing, 1995, NASA/Ames Research Center

Caroline Weishaepf (1999) simulated the occurring unsteady vortical flow to analyze the dynamic stability of a delta wing. Computational Fluid Dynamics is used to investigate and predict wing rock motions. Usually in order to reduce the computational costs the Euler equations together with the conical flow assumption are used as the governing equations. But some physical mechanisms associated with wing rock cannot be modeled using the Euler equations. Therefore three-dimensional unsteady Navier-Stokes calculations for a delta wing are investigated for the intensive analysis of wing rock.

CFD has matured significantly in the computation of flows around large structures, such as an airplane at designed conditions. However, it is still difficult to deal with complex flows at off-design conditions where flow separates and vortices characterize the flow features. These vortical flows are often encountered in various important engineering problems, including the leading edge vortices over a delta-wing.

J.A. Ekaterinaris and L.B. Schiff (2003) used Tecplot's CFD Analyzer 3.0 Add-on to calculate the total vorticity magnitude and the trajectory of particles with mass. They extracted vortex cores and used Tecplot to view the vortical flow around the primary vortex of a delta wing at 40° of angle of attack as shown in Fig.1.2.

Figure 1.2 TECPLOT tips-using TECPLOT's CFD analyzer add-on

M. Murayama (2003) applied the Adaptive Grid Refinement Method for vortical flows to study on vortex breakdown on a delta wing. Navier-Stokes Equations are solved on an unstructured hybrid grid. Mach number is 0.3, and Re is 1.0×10^6 , with high angle of attack. As shown in Fig. 1.3, the thin line shows instantaneous vortex center lines, the irregular regions stand for reversed flow regions.

Figure 1.3 Vortex breakdown on a delta wing using Adaptive Grid Refinement method

CFD models for vortices over a delta wing are typically of very high order, which are not suitable for feedback flow control design. Typical approach for dealing with this problem is order reduction, such as the Proper Orthogonal Decomposition (POD) method.

The POD method is a powerful method for system identification aiming at obtaining low-dimensional approximate descriptions for multidimensional systems. The Proper Orthogonal Decomposition provides a basis for the modal decomposition of a system of functions, usually with data obtained from experiments, measurements or numerical simulations. The basis functions retrieved are called proper orthogonal modes. It provides an efficient way of capturing the dominant components of a multidimensional system and representing it to the desired precision by using the relevant set of modes, thus reducing the order of the system. There are different interpretations for the POD. For example, among them the principal components analysis (PCA) is a technique that can be used to simplify a dataset; more formally it is a linear transformation that chooses a new coordinate system for the data set such that the greatest variance by any projection of the data set comes to lie on the first axis (then called the first principal component), the second greatest variance on the second axis, and so on. At the present time, the POD method is effective only for steady flow field. Additional research for unsteady flow field is still needed [15] [16].

Instead of obtaining a flow field model by order reduction of CFD models, low-order models can be obtained either by simplifying assumptions on a highly organized flow field such as the leading edge vortices on a delta wing, or using a “black-box” approach from experimental data. Both methods will be reviewed next.

1.2.2 Model Scheduling

Dr. Douglas Laurence and M. Wu (2002) have applied the Model Scheduling method to a delta wing [4] for closed-loop control of vortices. Jet actuators are amounted on the

top surface of the delta wing. If the jet blowing of air is taken as the input of the system, and the air pressure on the surface of the delta wing is taken as the output of the system, these variables are governed by a nonlinear dynamic system. Methods for nonlinear control system design are typically model-based. Usually first principle modeling yields parametric models that require identification and tuning of a relatively small set of parameters. Complex nonlinear phenomena often do not lend themselves to this type of modeling, necessitating a black-box approach commonly referred to as system identification. Techniques for linear system identification are more widely available than for the nonlinear case, however, linear models can only predict the behavior of a nonlinear system in a vicinity of an equilibrium point, or a nominal operating condition. In order to exploit the maturity of linear system identification techniques while overcoming the inherent limitation of linear models, an approach for nonlinear system identification motivated by gain scheduled control system design is applied which is referred to as model scheduling. The basic idea is to identify a set of linear models about a collection of equilibriums from which a nonlinear model is constructed which instantaneously schedules the linear models based on the input signals and/or other internal variables.

It is important to keep in mind that while all model implementations satisfying the linear requirement should behave similarly when operated near an equilibrium point, their behavior can be dissimilar in a non-local operation.

Advantages of this approach: Obtaining experimental data from wind tunnel tests is considered common practice in aerodynamic modeling and this approach takes advantage of the wealth of experience and practical knowledge in this area. Linear system

identification is a much more mature research area as compared to nonlinear system identification and a variety of techniques have been devised to successfully solve complex, real-world problems. Gain scheduling is a commonly practiced technique for nonlinear control system design, especially in flight control applications. As an added benefit, it would seem that nonlinear models possessing such a structure would lend themselves to subsequent gain scheduled control system design. Finally, it is shown that the nonlinear models generated by this approach can be formally related to nonlinear indicial response (NIR) Theory [4] and so the numerous advantages of the NIR approach carry over to this technique.

1.3 Low-order modeling method by flowfield simplification

In 1994, A. Arena investigated the experimental and the computational methodologies of slender body delta wings undergoing wing rock [1]. Though his research is focused on the wing rock characteristics of a flat plate delta wing, it provides a computational model based on the discrete vortex potential flow theory. The key simplifying assumption for the modeling method is the locally conical flow field assumption, which was validated by the experimental study. This computational model investigates the dominant features of the flow field and captures a limit cycle behavior which is the wing rock motion. The vortices positions and strengths are solved by coupling the flow field equations to the rigid body equation of the rolling motion of a delta wing. This computational model has captured the major qualitative characteristics of the wing motion and the flow field behavior. Also the theory can account for variations in sweep angle and angle of attack.

In 1997, J. Myatt extends Arena's delta-wing flow field model by developing a computational model on the effects of vortex breakdown over the planform by generalizing from empirical results [1]. The model provides the equations to approximate the pressure aft of the vortex breakdown. Experimental data proves that it is important to account for the movement of the vortex breakdown location when studying on delta wing vortices. Both the Arena and Myatt modeling approaches constitutes the theoretical basis for the new modeling method developed in the dissertation.

1.4 Problem Statement

In summary, each method has its unique characteristics. Computational Fluid Dynamics methods (CFD) apply PDE which usually yield high order equations, therefore it needs order reduction such as POD method. Model Scheduling method derives a low order model, and all the parameter values come from experimental test, therefore the model is a black box model which does not offer detailed flow mechanisms. Arena/Myatt Conical flow model is a First-Principle theoretical model. With suitable simplification assumptions on the flow, the model has spatial solutions, and it delves into the flow mechanism, but it is not suitable for control design.

The current model provides a unique way of modeling by converting the spatial flow field solutions such as Arena and Myatt models into low-order ordinary differential state equations. It preserves all the flow dynamics analysis of the conical flow model, while it is expressed in a set of standard low-order nonlinear state equations. Therefore it is suitable for control design.

In addition, each modeling method has its pros and cons for feedback flow control system design. Complex models have higher precision depicting flow dynamics, but usually have high order equations that need order reduction by methods such as POD, which will result in some precision loss. While simpler models such as the current model has low order equations, but it is based on those assumptions that simplified the model, which also leads to some precision loss. However, the resulting model can be used for controller design without or with little further simplification.

1.5 Objective, Scope and Basic Methodology of Current Model

In general, most control methods are model based, and most flow models are PDE or High-order ODE. Therefore in combination of both, the objective of the thesis research is to develop a model with a set of standard low-order nonlinear state space equations for the delta wing with leading edge vortices so as to facilitate nonlinear feedback control methods.

The scope of the thesis research is to develop the nonlinear model, not including applying control designs. It does not include validating Arena and Myatt models either, because the current model relies on the same assumptions and flow theories as by Arena and Myatt models. The difference is that it develops through a series of mathematical transformations. Therefore the current model has the same validity as Arena and Myatt models does. And also since Arena and Myatt models have been validated by experiments described in their dissertations [1][2], the research on current model is mainly focused on accomplishing the standard nonlinear state equations and in the meantime keeping accurate flow dynamics agreement with Arena and Myatt models.

The basic methods in deriving the current model is through a series of mathematical transformations including: transforming flow equations from complex quantities to real quantities, solving the flow equations and differentiating the state variables to obtain the state equations and simplifying them to get standard nonlinear low-order space state equations finally.

This model provides a unique way on delta wing research in that the new dynamic model not only sheds some light on the air flow and the rigid body motion, but also provides a set of standard nonlinear state equations. Therefore the significance of the new models is in that a variety of nonlinear control methods can be readily applied to the state equation model so that, by controlling the vortices on the wing surface, the delta-wing aircraft maneuver and stability can be significantly improved.

1.6 Experimental setup for control design

As Fig.1.4 shows an experimental delta wing model built by University of Cincinnati that was used in Scheduling Design model [4], which has jet actuators and air pressure sensors that are mounted on the top surface of the delta wing. Jet actuators can eject air in controlled time period and with controlled strength, its ejecting direction can be adjusted too. They work as the control input to affect vortex speed, vortex strength rate, as well as subsequent vortex breakdown locations. Air pressure sensors are mounted on the surface to measure air pressure on the top surface of the delta wing. They work as the system feedback and part of the system output, so that the controller can adjust the jet actuators in response to the measured air pressure changes.

Fig. 1.4 Experimental delta wing setup

In the new flowfield modeling method developed in the thesis, a set of nonlinear state equations is derived in which vortex strength, vortex breakdown location and other independent variables are chosen as system states. In this model, the control input is assumed to be able to affect vortex strength rate, vortex breakdown locations and other parameters. Surface air pressure together with roll angle and roll rate are the output or feedback of this system. Therefore on the whole, this model deals with the same input and output as the experimental model does. Since the two Arena and Myatt models have been validated with the experimental result, and the new method has been validated with Arena and Myatt models, this set of derived nonlinear state equations are also validated with experimental results. Control designs applied to the model will be implemented and tested on the experimental model as well.

1.7 Contents of the thesis

The basic flow theory and analysis methods are described in Chapter 2. Since all the dynamic equations in the original theories are expressed in complex quantities, Chapter 3 mainly deals with transforming the complex-valued equations into real-valued. Then the

differential equations are decoupled and solved step by step until finally every state equation is derived and simplified as is described in Chapter 4. Chapter 5 compares the computation results of the new model with that of Arena's and Myatt's models for verification of the new model. Finally Chapter 6 summarizes and concludes the thesis.

In this Chapter, we have identified the need for low-order models for vortex flow over a delta-wing, which are suitable for flow control design. We have also surveyed related research, and offered an overview of the research presented in this thesis. We will review in Chapter 2 the fundamental theories that form the basis of the new computational model.

Chapter 2

Theory and Methodology

In this Chapter, the methodology and supporting fundamental theories for the delta wing vortex modeling are presented. After an overview, the fundamental theories are briefed, and then two important conditions including Kutta condition and Zero-Force condition are reviewed. After it, the two condition equations get simplified by nondimensionalizing. After the flow dynamics, the delta wing rigid body dynamics is examined. Finally, the relationship of all the equations is established.

2.1 Overview

In this chapter, the theories applied in the Arena and Myatt theoretical models are reviewed. The Arena model is the wing rock model that can simulate the static-wing and the unsteady-wing situations. The static-wing situation means the wing's roll angle remains unchanged. The unsteady-wing situation refers to the wing rock motion. The static-wing model first calculates the vortex position and vortex strength while the wing is motionless. Then the unsteady-wing model is used to calculate the motion of the wing when it is released to roll freely.

These two models involve some important assumptions so that the application of a variety of equations becomes valid. The assumptions include:

- 1) The flow is inviscid; the only effect of viscosity is to fix separation at the leading edge, and the flow field can be represented by velocity potential. This allows the Navier-Stokes equation be replaced by Laplace equation.

2) The delta wing flow is conical. The slender body theory applies to the delta wing, therefore 3D flow field can be replaced with quasi-2D cross-flow field.

3) All vortices are concentrated into two leading edge vortices, and each vortex is connected to the wing leading edge by a feeding sheet.

4) Boundary conditions include: no fluid passes the wing, and fluid sustain no force. Therefore Zero-force condition can be applied. The velocity must be finite at the leading edge. Therefore the Kutta condition can be applied.

5) The wing surface pressure is induced by flow velocity on the wing surface. Therefore the unsteady Bernoulli equation is valid.

6) The airframe is rigid, and it has one degree of freedom (DOF). Therefore the one DOF equation of rolling motion can be applied.

7) The motion always starts from a static equilibrium. This doesn't limit the model in any way because the integration can be started at any time and allowed to proceed to the desired starting condition. And the varying motion of delta wing is slow in a nondimensional sense. Therefore mathematical calculation of the vortex breakdown location can be performed,

Since the flow potential equations are expressed by complex quantities, the complex velocity can be obtained by differentiating the complex potential with respect to ζ , which is the complex coordinate in the circle plane of the cross flow plane of the delta wing.

When the wing is static, the cross flow field can be decomposed into two flow potentials, one for the free stream and the other for the two vortices each on one leading edge of the delta wing with opposite rotating directions. Fig. 1 shows the components of the air flow when the wing is static.

Figure 2.1 The complex potential for static flow model

Runge-Kutta method is applied in the static-wing model. The Runge-Kutta algorithm lets us solve a differential equation numerically. This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

Suppose there are only two variables x, y and two differential equations

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

We let x_n be the value of x at time t_n , and similarly for y_n . Then the formulas for the Runge-Kutta algorithm are

$$x_{n+1} = x_n + (k_1 + 2k_2 + 2k_3 + k_4)h/6$$

$$y_{n+1} = y_n + (j_1 + 2j_2 + 2j_3 + j_4)h/6$$

where

$$k_1 = f(t_n, x_n, y_n)$$

$$j_1 = g(t_n, x_n, y_n)$$

$$k_2 = f(t_n + h/2, x_n + hk_1/2, y_n + hj_1/2) \quad j_2 = g(t_n + h/2, x_n + hk_1/2, y_n + hj_1/2)$$

$$k_3 = f(t_n + h/2, x_n + hk_2/2, y_n + hj_2/2) \quad j_3 = g(t_n + h/2, x_n + hk_2/2, y_n + hj_2/2)$$

$$k_4 = f(t_n + h, x_n + hk_3, y_n + hj_3) \quad j_4 = g(t_n + h, x_n + hk_3, y_n + hj_3)$$

Given starting values x_0, y_0 we can plug them into the formula to find x_1 and y_1 .

Then we can plug in x_1, y_1 to find x_2, y_2 and so on.

When the delta wing is static, the air flow over the delta wing is steady. But when the wing is rocking around its longitudinal axis during wing rock and the delta wing is assumed as a thin flat board, the rolling motion of the board results in a third flow potential. It is the unsteady flow induced by the surface of the wing due to wing rock.

Fig. 2 shows the flow potentials in the unsteady-wing model.

Figure 2.2 The complex potential for the unsteady-wing model

When considering the unsteady flow, every point on the wing surface during wing rolling is taken as a source-sink doublet. Therefore the flow potential at any point in the

cross flow plane is the summation of the flow potential at this point produced by each point on the surface of the rocking wing. The local surface velocity distribution of the delta wing due to the rolling motion as the delta wing is rolling with a clockwise angular velocity as shown in Fig. 2.3.

Figure 2.3 Local surface velocity distribution due to rolling motion

2.2 Some fundamental theories

2.2.1 Joukowski transformation, circle plane, and physical plane

The Joukowski transformation is the transformation between two planes, the physical plane and the circle plane. The physical plane is the plane which contains the coordinates describing the airfoil in the real physical world. This conformal mapping transformation transforms the much more complicated calculations of flow velocity and pressure around airfoils in physical plane into another set of calculations in circle plane where the calculations become relatively easier. It is expressed as $\sigma = \zeta + \frac{a^2}{\zeta}$, where ζ is the complex coordinates for any point in the circle plane, and σ is the corresponding complex coordinates for this point in the physical plane. The circle plane is the plane whose coordinates are the Joukowski transformation of the coordinates in the physical plane.

Figure 2.4 Joukowski Transformation between two planes

2.2.2 Circle theorem

This lays the foundation for formulating equations for complex flow potential and flow velocity for all the three flows. According to the Circle Theorem (Milne-Thomson, 1973), for the irrotational two-dimensional flow of incompressible inviscid fluid in the z -plane, under the condition that there is no rigid boundaries, let $f(z)$ be the complex potential of the flow, where the singularities of $f(z)$ are all at a distance greater than the distance, a , from the origin. If a circular cylinder with its cross-section the circle C , $|z| = a$, is introduced into the field of flow, the complex potential becomes as:

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right), \text{ where } \bar{f} \text{ stands for the conjugate function.}$$

2.2.3 Conformal mapping and tangential velocity

The conformal mapping, $Y=Y(f)$, refers to the mapping between a vector, $Y=x+iz$, in the physical plane, and another vector, $f=X+iZ$, in the circle plane. Let F stands for a complex potential function, and W stands for a complex velocity function, and assume that $F(f)$ is the complex potential, and $W(f)$ is the complex velocity in the circle plane, then the results in the physical plane are:

$$F(Y) = F[Y(f)]$$

$$W(Y) = \frac{dF}{dY} = \frac{dF}{df} \frac{df}{dY} = W(f) \frac{1}{dY/df} \quad (2.1)$$

According to Joukowski transformation (Bisplinghoff, p.254), the circle $r = \frac{1}{2}b$ in the circle plane is transformed into the line or ‘slit’

$$-b \leq x \leq b, \quad z = 0,$$

in the physical plane by means of

$$x + iz = (X + iZ) + \frac{b^2}{4(X + iZ)} \quad (2.2)$$

Applying the polar coordinates r and θ , related to X and Z ,

$$X = r \cos \theta, \quad Z = r \sin \theta,$$

$$X + iZ = r(\cos \theta + i \sin \theta) = re^{i\theta} \quad (2.3)$$

To find the corresponding points on the circle plane and on the slit, we substitute Eq. (2.3) into Eq. (2.2), and set $r = \frac{1}{2}b$. Equating real parts and imaginary parts, we get

$$x = b \cos \theta, \quad z = 0.$$

Figure 2.5 The Joukowski's conformal transformation between the circle plane and the physical plane

Let $W(Y) = u' - iw$ and $W(f) = q_x - iq_z$, from Eq. (2.1), flow velocities in the two planes are obtained, containing the derivative of the conformal transformation function.

$$u' - iw = \frac{q_x - iq_z}{[d(x + iz)/d(X + iZ)]} \quad (2.4)$$

where q_x and q_z are the Cartesian perturbation velocity components in the X-Z plane.

Substitute $(X + iZ)$ into Eq. (2.2),

$$\left[\frac{d(x + iz)}{d(X + iZ)} \right]_{r=b/2} = \left[1 - \frac{b^2}{4(X + iZ)^2} \right]_{r=b/2} = 1 - \frac{b^2}{b^2 e^{i2\theta}} = 2(\sin \theta) e^{i\theta} \quad (2.5)$$

Substitute Eq. (2.5) into Eq. (2.4), and take the absolute magnitudes of both sides of Eq. (2.4), we have

$$|u' - iw| = \sqrt{u'^2 + w^2} = \frac{\sqrt{q_x^2 + q_z^2}}{|2 \sin \theta|} = \frac{\sqrt{q_\theta^2 + q_r^2}}{|2 \sin \theta|} \quad (2.6)$$

where q_r and q_θ are the radial and tangential velocity components in the X-Z plane, the circle plane.

Eq. (2.6) may also be used to relate the magnitudes of the components separately. Since conformal transformation preserves the angle at which two lines meet, in this case, the angle at which the local velocity vector in the circle plane is the same as the angle at which the corresponding velocity vector meets the slit in the physical plane. Therefore the radial and tangential components are in the same proportions to u and w , respectively, in two planes:

$$|u'| = \frac{|q_\theta|}{|2 \sin \theta|}$$

$$|w| = \frac{|q_r|}{|2 \sin \theta|}$$

2.2.4 Uniform stream and singular solutions

Let $Y=x+iz$ be any point in the cross flow physical plane, the complex potential for the flow of a uniform stream of velocity U_∞ in the x direction is :

$$F = \Phi + i\Psi = U_\infty(x + iz) = U_\infty Y$$

Where Φ and Ψ are the velocity potential and the stream functions, respectively. Now consider the stream to be at an angle α to the x axis and repeat the process. The complex potential becomes

$$F = U_\infty(x \cos \alpha + z \sin \alpha) + iU_\infty(-x \sin \alpha + z \cos \alpha) = U_\infty(\cos \alpha - i \sin \alpha)(x + iz) = U_\infty Y e^{-i\alpha}$$

This illustrates the general result that the complex potential for one flow field can be made to represent the same flow field rotated counterclockwise by α if Y is replaced by $Y e^{-i\alpha}$.

2.2.5 Velocity potential components and velocity components:

The flow field components for unsteady-wing model include: the free stream, the vortices above the wing, and the source-sink sheet due to wing rolling. The source-sink velocity is also called the unsteady velocity. After a series of derivation [1], the free stream velocity potential is:

$$F_{fs} = U_\infty \sin \alpha \left[\frac{\zeta}{e^{i(\pi/2+\phi)}} - \frac{a^2}{\zeta} e^{i(\pi/2+\phi)} \right]$$

The velocity potential due to the vortices is:

$$F_{vortex} = \frac{i\Gamma_l}{2\pi} \left[\ln(\zeta - \zeta_l) - \ln(\zeta - a^2/\bar{\zeta}_l) \right] - \frac{i\Gamma_r}{2\pi} \left[\ln(\zeta - \zeta_r) - \ln(\zeta - a^2/\bar{\zeta}_r) \right]$$

The velocity potential due to the source-sink sheet is:

$$F_{s/s} = \frac{-4a^2\dot{\phi}}{\pi} \int_0^{2\pi} \sin \theta \cos \theta \ln(\zeta - ae^{i\theta}) d\theta$$

Since the flow field model consists of three flow components:

$$F_c = F_{fs} + F_{vortex} + F_{s/s}$$

the complex potential for the cross-flow in the circle plane is:

$$F_c(\zeta) = U_\infty \sin \alpha \left[\frac{\zeta}{e^{i(\pi/2+\phi)}} - \frac{a^2}{\zeta} e^{i(\pi/2+\phi)} \right] + \frac{i\Gamma_l}{2\pi} \left[\ln(\zeta - \zeta_l) - \ln(\zeta - a^2/\bar{\zeta}_l) \right] \\ - \frac{i\Gamma_r}{2\pi} \left[\ln(\zeta - \zeta_r) - \ln(\zeta - a^2/\bar{\zeta}_r) \right] + \frac{-4a^2\dot{\phi}}{\pi} \int_0^{2\pi} \sin \theta \cos \theta \ln(\zeta - ae^{i\theta}) d\theta$$

The complex velocity in the circle plane $W_c(\zeta) = (v_c - iw_c)$ can be derived by taking the

derivative of the complex potential $W_c(\zeta) = \frac{dF}{d\zeta}$,

$$W_c(\zeta) = U_\infty \sin \alpha \left[\frac{1}{e^{i(\pi/2+\phi)}} - \frac{a^2}{\zeta^2} e^{i(\pi/2+\phi)} \right] + \frac{i\Gamma_l}{2\pi} \left[\frac{1}{\zeta - \zeta_l} - \frac{1}{\zeta - a^2/\bar{\zeta}_l} \right] \\ - \frac{i\Gamma_r}{2\pi} \left[\frac{1}{\zeta - \zeta_r} - \frac{1}{\zeta - a^2/\bar{\zeta}_r} \right] + \frac{-4a^2\dot{\phi}}{\pi} \int_0^{2\pi} \frac{\sin \theta \cos \theta}{\zeta - ae^{i\theta}} d\theta \quad (2.7)$$

2.3 Kutta condition

By Joukowski transformation, $\sigma = \zeta + \frac{a^2}{\zeta}$, we get

$$\frac{d\zeta}{d\sigma} = 1 / \frac{d\sigma}{d\zeta} = \frac{\zeta^2}{\zeta^2 - a^2}$$

Therefore the complex velocity is

$$W(\sigma) = \frac{dF}{d\sigma} = \frac{dF}{d\zeta} \frac{d\zeta}{d\sigma} = W_c(\zeta) / \frac{d\sigma}{d\zeta} = W_c(\zeta) \frac{\zeta^2}{\zeta^2 - a^2}$$

By the above equation, the velocity at $\zeta = (\pm a, 0)$ will be infinite. To avoid this, the velocity there should be set to zero in the circle plane so that the velocity at the leading edge in the physical plane would be finite and continuous. Substitute $\zeta = (\pm a, 0)$ into Eq.(2.7), and the Kutta condition for the left edge $\zeta = (-a, 0)$ becomes as:

$$\begin{aligned} W_c(-a, 0) = \text{Im} \left\{ U_\infty \sin \alpha \left[e^{-i(\pi/2 + \phi)} - e^{i(\pi/2 + \phi)} \right] + \frac{i\Gamma_l}{2\pi} \left[\frac{-1}{a + \zeta_l} + \frac{1}{a + a^2/\bar{\zeta}_l} \right] \right. \\ \left. + \frac{i\Gamma_r}{2\pi} \left[\frac{1}{a + \zeta_r} - \frac{1}{a + a^2/\bar{\zeta}_r} \right] \right\} + 2a\dot{\phi} = 0 \end{aligned} \quad (2.8)$$

Kutta condition for the right edge $\zeta = (a, 0)$ becomes as:

$$W_c(a, 0) = \text{Im} \left\{ U_\infty \sin \alpha \left[e^{-i(\pi/2 + \phi)} - e^{i(\pi/2 + \phi)} \right] + \frac{i\Gamma_l}{2\pi} \left[\frac{1}{a - \zeta_l} - \frac{1}{a - a^2/\bar{\zeta}_l} \right] \right\}$$

$$-\frac{i\Gamma_r}{2\pi} \left[\frac{1}{a - \zeta_r} - \frac{1}{a - a^2/\bar{\zeta}_r} \right] \} - 2a\dot{\phi} = 0 \quad (2.9)$$

2.4 Zero-Force condition

Figure 2.6 Zero-force condition

2.4.1 Zero force in combination of the two forces, F_{sl} and F_{vl}

Denote by F_{sl} the force on the left vortex feeding sheet, and by F_{vl} the force on the left vortex. The sum of the forces, F_{sl} and F_{vl} , is required to be zero in the flow. By making use of the unsteady Bernoulli equation and small-disturbance assumptions, and integrating from the leading edge to the vortex, the force (chordwise) on the left vortex feeding sheet is:

$$F_{sl} = -i\rho \left[U_{\infty} \cos \alpha \frac{\partial \Gamma_l}{\partial x} + \frac{\partial \Gamma_l}{\partial t} \right] (\sigma_l - \sigma_o)$$

and the force on the vortex itself is:

$$F_{vl} = i\rho \Gamma_l v_{rel} = i\rho \Gamma_l (v_l^* - \frac{d\sigma_l}{dt})$$

where v_{rel} is the velocity of the free stream relative to the vortex.

Since $F_{sl} + F_{vl} = 0$ is required,

$$-i\rho\left[U_{\infty}\cos\alpha\frac{\partial\Gamma_l}{\partial x}+\frac{\partial\Gamma_l}{\partial t}\right](\sigma_l-\sigma_o)+i\rho\Gamma_l(v_l^*-\frac{d\sigma_l}{dt})=0 \quad (2.10)$$

2.4.2 Components of the velocity at the vortex, v_l^*

The velocity at the vortex can be decomposed into two parts, the axial free stream across the vortex and the velocity due to the complex potential in the cross-flow plane:

$$v_l^* = -U_{\infty}\cos\alpha\frac{d\sigma_l}{dx} + \overline{\tilde{W}(\sigma_l)}$$

The complex conjugate of $\tilde{W}(\sigma)$ is used because the complex velocity found from the complex potential is the conjugate of the actual flow velocity. Substitute into Eq. (2.10) and rearrange the equation, the velocity of left vortex with respect to an inertial frame in the physical plane becomes as:

$$\frac{d\sigma_l}{dt} = -\left[U_{\infty}\cos\alpha\frac{\partial\Gamma_l}{\partial x}+\frac{\partial\Gamma_l}{\partial t}\right]\frac{(\sigma_l-\sigma_o)}{\Gamma_l} - U_{\infty}\cos\alpha\frac{d\sigma_l}{dx} + \overline{\tilde{W}(\sigma_l)} \quad (2.11)$$

2.4.3 Transformation of the complex velocity from the inertial coordinate system to the body fixed coordinate system

The equation (2.11) is expressed in the inertial frame, which needs to be expressed in the body fixed frame of the wing, because other velocity potential and complex velocity are expressed in body fixed coordinate system.

$$\frac{\delta\sigma_l}{\delta t} = \frac{d\sigma_l}{dt} - \bar{\Omega}X\sigma = \frac{d\sigma}{dt} + i\phi\sigma_l \quad (2.12)$$

where δ indicates the derivative with respect to the body fixed coordinates rotating with an angular velocity $\bar{\Omega}$. Rearrange and substitute Eq. (2.12) into Eq. (2.11), the velocity of the left vortex with respect to the body fixed coordinate system becomes as:

$$\frac{\delta\sigma_l}{\delta t} = - \left[U_\infty \cos\alpha \frac{\partial\Gamma_l}{\partial x} + \frac{\partial\Gamma_l}{\partial t} \right] \frac{(\sigma_l - \sigma_o)}{\Gamma_l} - U_\infty \cos\alpha \frac{d\sigma_l}{dx} + \overline{\widetilde{W}(\sigma_l)} + i\dot{\phi}\sigma_l \quad (2.13)$$

2.4.4 Conical flow assumption

In this delta wing flow field model, the vortex flow is nearly conical. Thus, the conical flow assumption is valid, where the vortex positions and strengths are assumed to be linear functions of chordwise position x :

$$\frac{d\sigma}{dx} = \frac{\sigma_v}{2a} \tan\epsilon \quad (2.14)$$

$$\frac{\partial\Gamma}{\partial x} = \frac{\Gamma_v}{2a} \tan\epsilon \quad (2.15)$$

where $2a$ is the wing semi-span, and ϵ is the apex half angle. Substituting Eq. (2.14) and (2.15) into (2.13) and rearrange them, the velocity of the left vortex becomes as:

$$\frac{\delta\sigma_l}{\delta t} = - \frac{U_\infty \cos\alpha \tan\epsilon}{a} (\sigma_l + a) - \frac{\dot{\Gamma}_l}{\Gamma_l} (\sigma_l + 2a) + \overline{\widetilde{W}(\sigma_l)} + i\dot{\phi}\sigma_l \quad (2.16)$$

2.4.5 Transformation from the physical plane to the circle plane

Finally the equation will be transformed to the circle plane where the complex potential and the other equations for Kutta condition are calculated. By the chain rule,

$$\frac{\delta\zeta_l}{\delta t} = \frac{d\zeta_l}{d\sigma} \frac{\delta\sigma_l}{\delta t} = \frac{\zeta_l^2}{\zeta_l^2 - a^2} \frac{\delta\sigma_l}{\delta t}$$

Substituting into Eq. (2.16), the velocity of the left vortex becomes as:

$$\begin{aligned} \frac{\delta \zeta_l}{\delta t} = & \frac{\zeta_l^2}{\zeta_l^2 - a^2} \left[-\frac{U_\infty \cos \alpha \tan \varepsilon}{a} \left(\zeta_l + \frac{a^2}{\zeta_l} + a \right) \right. \\ & \left. - \frac{\dot{\Gamma}_l}{\Gamma_l} \left(\zeta_l + \frac{a^2}{\zeta_l} + a \right) + \overline{\widetilde{W}(\sigma_l)} \right] + i\dot{\phi} \frac{\zeta_l(\zeta_l^2 + a^2)}{\zeta_l^2 - a^2} \end{aligned} \quad (2.17)$$

Because the complex velocity at the vortex center is singular, transferring the complex velocity to the circle plane must be done with caution. As in Pappas and Kunen [2], the velocity at the vortex center in the physical plane can be transformed to the circle plane as:

$$\widetilde{W}(\sigma_l) = W_c(\zeta_l) \frac{\zeta_l^2}{\zeta_l^2 - a^2} - \frac{i\Gamma_l}{2\pi} \frac{\zeta_l a^2}{(\zeta_l^2 - a^2)^2}$$

where $\widetilde{W}(\sigma_l)$ is the velocity at the left vortex in the physical plane, and $W(\zeta_l)$ is the complex velocity in the circle plane, where the contribution of the left vortex is removed in $W(\zeta_l)$. Substitute into Eq. (2.17), the velocity of the left vortex in the circle plane becomes as:

$$\begin{aligned} \frac{\delta \zeta_l}{\delta t} = & \frac{\zeta_l^2}{\zeta_l^2 - a^2} \left[-\frac{U_\infty \cos \alpha \tan \varepsilon}{a} \left(\zeta_l + \frac{a^2}{\zeta_l} + a \right) - \frac{\dot{\Gamma}_l}{\Gamma_l} \left(\zeta_l + \frac{a^2}{\zeta_l} + 2a \right) \right. \\ & \left. + W_c(\zeta_l) \frac{\zeta_l^2}{\zeta_l^2 - a^2} - \frac{i\Gamma_l}{2\pi} \frac{\zeta_l a^2}{(\zeta_l^2 - a^2)^2} \right] + i\dot{\phi} \frac{\zeta_l(\zeta_l^2 + a^2)}{\zeta_l^2 - a^2} \end{aligned} \quad (2.18)$$

By similar procedure, the velocity of the right vortex in the circle plane can be derived as:

$$\begin{aligned} \frac{\delta \zeta_r}{\delta t} = & \frac{\zeta_r^2}{\zeta_r^2 - a^2} \left[-\frac{U_\infty \cos \alpha \tan \varepsilon}{a} \left(\zeta_r + \frac{a^2}{\zeta_r} - a \right) - \frac{\dot{\Gamma}_r}{\Gamma_r} \left(\zeta_r + \frac{a^2}{\zeta_r} - 2a \right) \right. \\ & \left. + \overline{W_c(\zeta_r) \frac{\zeta_r^2}{\zeta_r^2 - a^2} + \frac{i\Gamma_r}{2\pi} \frac{\zeta_r a^2}{(\zeta_r^2 - a^2)^2}} \right] + i\dot{\phi} \frac{\zeta_r(\zeta_r^2 + a^2)}{\zeta_r^2 - a^2} \end{aligned} \quad (2.19)$$

2.4.6 Nondimensionalizing variables

Finally, for simpler expression and computation, the Eq. (2.18) and (2.19) need to be nondimensionalized. The nondimensional variables are defined as:

$$\begin{aligned}\zeta^* &= \zeta / a \\ \gamma &= \frac{\Gamma}{2\pi a U_\infty \sin \alpha} \\ t^* &= \frac{t}{c_r / U_\infty} \\ \dot{\phi}^* &= \frac{d\phi}{dt^*} = \dot{\phi} \frac{c_r}{U_\infty}\end{aligned}$$

Substituting the variables into Eq. (2.17) and (2.18), the left vortex velocity becomes:

$$\begin{aligned}\frac{\delta \zeta_l}{\delta t} &= \frac{\zeta_l^{*2}}{\zeta_l^{*2} - 1} \left[-\frac{2 \cos \alpha}{x/c_r} \left(\zeta_l^* + \frac{1}{\zeta_l^*} + 1 \right) - \frac{\dot{\gamma}_l}{\gamma_l} \left(\zeta_l^* + \frac{1}{\zeta_l^*} + 2 \right) \right. \\ &\quad \left. + \left[\frac{W_c(\zeta_l^*)}{U_\infty} \frac{\zeta_l^{*2}}{\zeta_l^{*2} - 1} - \frac{i \gamma_l \sin \alpha \zeta_l^*}{(\zeta_l^{*2} - 1)^2} \right] \frac{2}{\frac{x}{c_r} \tan \varepsilon} \right] + i \dot{\phi}^* \frac{\zeta_l^* (\zeta_l^{*2} + 1)}{\zeta_l^{*2} - 1}\end{aligned}\quad (2.20)$$

where $W_c(\zeta)$ is similar to Eq. (2.7) except that one term is removed in the equation,

$$\begin{aligned}\frac{W_c(\zeta_l)}{U_\infty} &= \sin \alpha \left[\frac{1}{e^{i(\pi/2+\phi)}} - \frac{1}{\zeta_l^2} e^{i(\pi/2+\phi)} \right] + i \sin \alpha \left[-\frac{\gamma_l}{\zeta_l - 1/\bar{\zeta}_l} - \frac{\gamma_r}{\zeta_l - \zeta_r} + \frac{\gamma_r}{\zeta_l - a^2/\bar{\zeta}_r} \right] \\ &\quad - \frac{2}{\pi} \frac{x \dot{\phi}}{c_r} \tan \varepsilon \int_0^{2\pi} \frac{\sin \theta \cos \theta}{\zeta_l - a e^{i\theta}} d\theta\end{aligned}\quad (2.21)$$

Similarly, the velocity of right vortex becomes as:

$$\begin{aligned}
\frac{\delta \zeta_r}{\delta t} = & \frac{\zeta_r^{*2}}{\zeta_r^{*2} - 1} \left[-\frac{2 \cos \alpha}{x/c_r} (\zeta_r^* + \frac{1}{\zeta_r^*} - 1) - \frac{\dot{\gamma}_r}{\gamma_r} (\zeta_r^* + \frac{1}{\zeta_r^*} - 2) \right. \\
& \left. + \overline{\left[\frac{W_c(\zeta_r^*)}{U_\infty} \frac{\zeta_r^{*2}}{\zeta_r^{*2} - 1} + \frac{i \gamma_r \sin \alpha \zeta_r^*}{(\zeta_r^{*2} - 1)^2} \right] \frac{2}{x/c_r \tan \varepsilon}} \right] + i \dot{\phi}^* \frac{\zeta_r^* (\zeta_r^{*2} + 1)}{\zeta_r^{*2} - 1}
\end{aligned} \tag{2.22}$$

where $W_c(\zeta)$ is similar to Eq. (2.7),

$$\begin{aligned}
\frac{W_c(\zeta_r)}{U_\infty} = & \sin \alpha \left[\frac{1}{e^{i(\pi/2+\phi)}} - \frac{1}{\zeta_r^2} e^{i(\pi/2+\phi)} \right] + i \sin \alpha \left[\frac{\gamma_l}{\zeta_r - \zeta_l} - \frac{\gamma_l}{\zeta_r - 1/\bar{\zeta}_l} + \frac{\gamma_r}{\zeta_r - 1/\bar{\zeta}_r} \right] \\
& - \frac{2}{\pi} \frac{x \dot{\phi}}{c_r} \tan \varepsilon \int_0^{2\pi} \frac{\sin \theta \cos \theta}{\zeta_r - a e^{i\theta}} d\theta
\end{aligned} \tag{2.23}$$

2.5 Nondimensionalization of Kutta condition equations

Kutta condition equations also need to be nondimensionalized. After that, the equations of Kutta condition become ('Im' stands for imaginary part):

Kutta condition for the left edge $\zeta = (-a, 0)$:

$$\begin{aligned}
& \text{Im} \left\{ \left[e^{-i(\pi/2+\phi)} - e^{i(\pi/2+\phi)} \right] + \text{Im} \left[\frac{-i}{1 + \zeta_l^*} + \frac{i}{1 + 1/\bar{\zeta}_l^*} \right] \gamma_l \right. \\
& \left. + \text{Im} \left[\frac{i}{1 + \zeta_r^*} - \frac{i}{1 + 1/\bar{\zeta}_r^*} \right] \gamma_r \right\} + \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* = 0
\end{aligned} \tag{2.24}$$

Kutta condition for the right edge $\zeta = (a, 0)$:

$$\begin{aligned}
& \text{Im}\left\{ \left[e^{-i(\pi/2+\phi)} - e^{i(\pi/2+\phi)} \right] + \text{Im}\left[\frac{i}{1-\zeta_l^*} - \frac{i}{1-\frac{1}{\zeta_l^*}} \right] \gamma_l \right. \\
& \left. - \text{Im}\left[\frac{i}{1-\zeta_r^*} - \frac{i}{1-\frac{1}{\zeta_r^*}} \right] \gamma_r \right\} - \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* = 0
\end{aligned} \tag{2.25}$$

2.6 Rolling moment calculation

2.6.1 The pressure before the vortex breakdown

The unsteady Bernoulli equation (Katz and Plotkin, 1991) is

$$C_p = 1 - \frac{U^2 + 2 \frac{\partial \Phi}{\partial t}}{U_\infty^2}$$

where $U = (U_\infty \cos \alpha + \Phi_x, \Phi_y, \Phi_z)$

The induced axial flow potential on the surface is

$$\Phi_x = \left[(\phi - \phi_\infty) - \frac{\partial \Phi}{\partial (y/s)} \frac{y}{s} \right] \tan \varepsilon$$

The tangential potential is applied to calculate the velocity in the cross-flow plane, and

then it is transformed from the circle plane to the physical plane,

$$\Phi_x^2 + \Phi_y^2 = \left[\frac{q_\theta}{2 \sin \theta} \right]^2$$

Finally, the final equation for the air pressure on the surface of the wing is:

$$C_p = \sin^2 \alpha - \frac{2 \cos \alpha \Phi_x}{U_\infty} - \frac{\Phi_x^2 + \left[\frac{q_\theta}{2 \sin \theta} \right]^2}{U_\infty^2} - 8 \left[\frac{x}{c_r} \dot{\phi}^* \tan \varepsilon \cos \theta \right]^2 - \frac{x}{c_r} \frac{\tan \varepsilon}{U_\infty} \frac{\partial \Phi}{\partial t^*} \tag{2.26}$$

where, the tangential velocity can be expressed as:

$$q_\theta = U_\infty x_{cr} \tan(\varepsilon) \dot{\phi} (1 - 2 \sin^2 \theta) - \text{Re} al(\overline{W}_l) \sin \theta + \text{Im}(\overline{W}_l) \cos \theta \quad (2.27)$$

where W_l is

$$W_l = U_\infty \sin(\alpha) \left[(e^{-i\rho} - \frac{e^{i\rho}}{(e^{i\theta})^2}) + i \left(\frac{\gamma_l}{e^{i\theta} - \zeta_l} - \frac{\gamma_l}{e^{i\theta} - \frac{1}{\bar{\zeta}_l}} - \frac{\gamma_r}{e^{i\theta} - \zeta_r} + \frac{\gamma_r}{e^{i\theta} - \frac{1}{\bar{\zeta}_r}} \right) \right] \quad (2.28)$$

where

$$\rho = \frac{\pi}{2} + \phi \quad (2.29)$$

and Φ_x can be expressed as:

$$\Phi_x = \left[(\tilde{\phi} - \tilde{\phi}_\infty) - \frac{\partial \Phi}{\partial (y/s)} \frac{y}{s} \right] \tan \varepsilon \quad (2.30)$$

and

$$\tilde{\phi} - \tilde{\phi}_\infty = \int_\infty^r W_p dr' + \int_{\pi/2}^\theta W_p d\theta' \quad (2.31)$$

where W_p is

$$W_p = U_\infty \sin \alpha \left[(e^{-i\rho} - \frac{e^{i\rho}}{\zeta^2}) + i \sin \alpha \left(\frac{\gamma_l}{\zeta - \zeta_l} - \frac{\gamma_l}{\zeta - \frac{1}{\bar{\zeta}_l}} - \frac{\gamma_r}{\zeta - \zeta_r} + \frac{\gamma_r}{\zeta - \frac{1}{\bar{\zeta}_r}} \right) \right] - U_\infty x_{cr} \tan(\varepsilon) \\ - U_\infty x_{cr} \tan(\varepsilon) \frac{1}{2} \int_0^{2\pi} \frac{2\dot{\phi} \cos \theta \sin \theta}{\pi} \left(\frac{1}{\zeta - e^{i\theta}} \right) d\theta \quad (2.32)$$

and

$$\Phi = \tilde{\phi} - \tilde{\phi}_{\infty} = \int_{\infty}^r W_p dr' + \int_{\pi/2}^{\theta} W_p d\theta' \quad (2.33)$$

$\frac{\partial \Phi}{\partial t}$ is the derivative of it with respect to the normalized time, t . Substitute into Eq.

(2.30), we get

$$\Phi_x = \left[(\tilde{\phi} - \tilde{\phi}_{\infty}) + (q_{\theta} \frac{\cos \theta}{\sin \theta}) - (2U_{\infty} \sin \alpha \cos \theta \sin \phi) \right] \frac{\tan \varepsilon}{2} \quad (2.34)$$

2.6.2 The air pressure aft of vortex breakdown

The air pressure at all stations aft of vortex breakdown is estimated from an empirical relation developed by Hanff and Huang (1991), it is expressed as:

$$C_p = C_{p0} \frac{\sin(\alpha_{\phi}) \cos(\alpha_{\phi})}{\sin \alpha \cos \alpha} \quad (2.35)$$

where α is the body-axis inclination and α_{ϕ} is the angle of attack at an roll angel ϕ expressed as:

$$\alpha_{\phi}(\phi) = \tan^{-1}(\tan \alpha \cos \phi)$$

C_{p0} is from the experimental data for the pressure behind vortex burst points, and its value changes with different sweep angles and different body-axis inclinations [2]. Finally this is the single air pressure expression for the entire area aft of breakdown, greatly simplified the actual pressure in the region.

2.6.3 The location of the vortex breakdown

The vortex breakdown location along the longitudinal axis is modeled as [2]:

$$\frac{dx_{vblag}(t)}{dt} = \frac{[x_{vbs}(\phi(t)) - x_{vbs}(\phi(0))] - [x_{vblag}(t) - x_{vblag}(0)]}{\beta} \quad (2.36)$$

where β is the measured time constant for the lag of the vertical static vortex position and is found from the experimental data. It has the value of $30.89t$, $28.43t$ and $18.38t$ respectively for regions of $-4^\circ < \phi < 5^\circ$, $5^\circ < \phi < 8.5^\circ$ and $8.5^\circ < \phi < 11.3^\circ$ [2], where t is the nondimensional time t' . Let $x_{vblag}(t)$ be the distance by which the vortex breakdown point lags its static position. Suppose the initial condition is given by,

$$x_{vblag}(0) = x_{vbs}(\phi(0)) \quad (2.37)$$

then the dynamic equation for vortex breakdown, Eq. (2.36), becomes:

$$\dot{x}_l(t) = -\frac{1}{\beta}x_l(t) + \frac{1}{\beta}x_s(t) \quad (2.38)$$

As we know

$$x_{vb} = x_{vbs} + x_{vblag}$$

therefore,

$$\dot{x}_{vb} = \dot{x}_{vbs} + \dot{x}_{vblag} \quad (2.39)$$

A linear fit to the static data is used for $x_{vbs}(\phi)$. The discontinuous behavior of the vortex

burst point as it jumps across the trailing edge is ignored. $\frac{d(x_{vbs}(\phi(\tau)))}{d\phi(\tau)}$ is from the

experimental data, and its value changes correspondingly when sweep angle or body-axis inclination changes.

Set

$$x_{vbs}(\phi) = k_1 + k_2\phi, \quad (2.40)$$

with the initial condition $x_{vbs}(\phi(0)) = k_1$, $k_1 = 0.65$, $k_2 = 0.05$ for the left wing and $k_2 = -0.05$ for the right wing, substitute Eq. (2.38) and the derivative of Eq. (2.40) into (2.39), we have

$$\dot{x}_{vb} = \frac{-x_{vb} + 2k_2\phi + k_2\beta\dot{\phi} + 2k_1}{\beta} \quad (2.41)$$

and the initial condition can be derived from Eq. (2.37) and (2.40):

$$x_{vb}(0) = x_{vbs}(\phi(0)) + x_{vblag}(0) = 2x_{vbs}(\phi(0))$$

2.6.4 The equation of rolling motion

According to the aero dynamics and rigid-body dynamics, the equation of motion of the delta wing is:

$$\ddot{\phi}(t) = \frac{qSb}{I_{xx}} C_{l_int}(t) \quad (2.42)$$

where, q is the dynamic pressure,

$$q = \frac{1}{2} \rho U_{\infty}^2$$

and, the roll velocity is the initial velocity plus the integration of acceleration by time:

$$\dot{\phi}_{t2} = \dot{\phi}_{t1} + \int_{t1}^{t2} \ddot{\phi}(t) dt$$

The moment coefficient at any cross flow section, C_l , before vortex breakdown is a function of $\frac{x}{c_r}$, which is the axis location of cross-section. To get the total moment of the delta wing, C_l needs to be integrated along the body-axis till the vortex break points. The integration of the difference of the pressure between the upper and lower surface over the

distance from the longitudinal body-axis along the span, will give the rolling moment per unit length at any chord wise section:

$$C_l = \frac{1}{Sb} \int_{-2a}^{2a} (C_{pu} - C_{pl}) y dy$$

In terms of the circle plane, it becomes as:

$$C_l = \frac{4a^2}{Sb} \int_0^{2\pi} C_p \sin \theta \cos \theta d\theta$$

or

$$C_l = \frac{x^2}{2c_r^3} \int_0^{2\pi} C_p \sin \theta \cos \theta d\theta \quad (2.43)$$

Without vortex breakdown, for conical flow the rolling moment for the complete wing can be found from integrating the pressure at one chordwise station. For flow fields with vortex breakdown, the sectional rolling moments must be integrated in the chordwise direction. It is accomplished by

$$C_{l_int} = \int_0^{c_r} C_l(x) dx$$

Where C_l is calculated by Eq. (2.43). Coupling with the rigid body equation of motion Eq. (2.42) and after nondimensionalizing, it becomes as:

$$\ddot{\phi}(t) = \frac{\rho c_r^5 \tan^2 \epsilon}{I_{xx}} C_{l_int}(t)$$

With vortex breakdown, the total roll moment is a summation of roll moment from the area before and aft of the vortex breakdown point on both the left and the right wing. The wing is longitudinally divided into two parts by the vortex breakdown point for both the left and the right wing. Therefore we have

$$\begin{aligned}
C_{l_int} &= \int_0^{x_{lag_right}} C_{l_right_bef}(x)dx + \int_{x_{lag_right}}^{c_r} C_{l_right_aft}(x)dx \\
&+ \int_0^{x_{lag_left}} C_{l_left_bef}(x)dx + \int_{x_{lag_left}}^{c_r} C_{l_left_aft}(x)dx \\
&+ \int_{\pi}^{2\pi} C_{l_bot}(x)dx
\end{aligned} \tag{2.44}$$

Where by substituting Eq. (2.43) into it, we have

$$\begin{aligned}
C_{l_int} &= \frac{x_{lag_right}^3}{6c_r^3} \int_0^{\frac{\pi}{2}} C_p \sin \theta \cos \theta d\theta + \frac{c_r^3 - x_{lag_right}^3}{6c_r^3} \int_0^{\frac{\pi}{2}} C_{p_vb} \sin \theta \cos \theta d\theta \\
&+ \frac{x_{lag_left}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} C_p \sin \theta \cos \theta d\theta + \frac{c_r^3 - x_{lag_left}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} C_{p_vb} \sin \theta \cos \theta d\theta \\
&+ \frac{1}{6} \int_0^{\pi} C_{pl} \sin \theta \cos \theta d\theta
\end{aligned} \tag{2.45}$$

2.7 The relationship between all the equations

The roll angle, the roll velocity, and the roll acceleration have the following relationship:

$$\begin{aligned}
\dot{\phi} &= \frac{d}{dt} \phi \\
\ddot{\phi} &= \frac{d}{dt} \dot{\phi}
\end{aligned}$$

The equation of motion is given by Eq. (2.42), and C_{l_int} is calculated by Eq. (2.44), in which C_l is calculated by Eq. (2.43). The pressure coefficient C_p before and aft of the vortex breakdown is calculated by Eq. (2.26) and Eq. (2.35) respectively. The vortex

position, and vortex strength are solved by Zero-force condition and Kutta condition equations expressed by Eq. (2.20), (2.22), (2.46) and Eq. (2.47) respectively. The vortex velocity Eq. (2.20) and (2.22) (Zero-force condition equation) need terms involving γ_l , γ_r , $\dot{\gamma}_l$ and $\dot{\gamma}_r$. They come from the Kutta condition equations and their derivatives.

Eq. (2.24) of Kutta condition can be expressed as the followings after simplification.

The left side Kutta condition Eq. (2.8) becomes

$$\begin{aligned} & \text{Im} \left[\frac{-i}{1 + \zeta_l^*} + \frac{i}{1 + \frac{1}{\bar{\zeta}_l^*}} \right] \gamma_l + \text{Im} \left[\frac{i}{1 + \zeta_r^*} - \frac{i}{1 + \frac{1}{\bar{\zeta}_r^*}} \right] \gamma_r \\ &= 2 \sin(\phi + \frac{\pi}{2}) - \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* \end{aligned} \quad (2.46)$$

and similarly, the right side Kutta condition Eq. (2.9) becomes as:

$$\begin{aligned} & \text{Im} \left[\frac{i}{1 - \zeta_l^*} - \frac{i}{1 - \frac{1}{\bar{\zeta}_l^*}} \right] \gamma_l - \text{Im} \left[\frac{i}{1 - \zeta_r^*} - \frac{i}{1 - \frac{1}{\bar{\zeta}_r^*}} \right] \gamma_r \\ &= 2 \sin(\phi + \frac{\pi}{2}) + \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* \end{aligned} \quad (2.47)$$

These two equations together can be solved for γ_l and γ_r , by taking derivative of them, we will get their derivatives $\dot{\gamma}_l$ and $\dot{\gamma}_r$.

After familiarization with all the theories and methodologies for the delta-wing vortex flow model, it is ready to solve for the state equations. But since most equations now are expressed in complex quantities, they first need to be converted to real quantity equations as described in Chapter 3.

Chapter 3

Transformation from Complex Quantities to Real Quantities

In order to derive a state equation for the delta-wing vortex flow field model, each pivotal equations described in Chapter 2 have to be first transformed to real-valued equations. Therefore one by one each equation is converted. They include: the Kutta condition equations, Zero-force condition equations, flow potential, flow potential rate and total roll moment.

3.1 Modeling goal

After all the equations are prepared as described in Chapter 2, they will be transformed into a new form of equations. The modeling goal of transforming the equations is to derive a series of nonlinear ordinary differential equations that involves all the dynamic states:

$$\dot{\vec{x}} = F(\vec{x}) + G(\vec{x})\vec{u}$$

$$\vec{y} = H(\vec{x})$$

where

$\vec{x}$, the states vector

$\vec{y}$, the output vector

$\vec{u}$, the control input vector

In this delta wing model, there are 10 variables:

$x_1 = \phi$, the roll angle

$x_2 = \dot{\phi}$, the roll velocity

$x_3 = y_l$, the left vortex position in y direction

$x_4 = z_l$, the left vortex position in z direction

$x_5 = y_r$, the right vortex position in y direction

$x_6 = z_r$, the right vortex position in z direction

$x_7 = \gamma_l$, the left vortex strength

$x_8 = \gamma_r$, the right vortex strength

with vortex breakdown, we have the following two states:

$x_9 = x_{cr_vb_lag_right}$, the right wing vortex breakdown location

$x_{10} = x_{cr_vb_lag_left}$, the left wing vortex breakdown location

System output includes:

$y_1 = x_1 = \phi$, the roll angle

$y_2 = x_2 = \dot{\phi}$, the roll velocity

$y_{k+2} = C_p(\theta_k, x_{cr})$, the pressure coefficient at (θ_k, x_{cr}) ,

$$(0 < \theta_k < \pi), k = 1, 2, \dots, 38$$

The output variables $y_3, \dots, y_{38}$ represents wing surface pressure sensor measurements. Assuming the flow control effectors, such as a micro jet, can affect the vortices position movement, vortices strength rate and the vortex breakdown location movement, the control inputs are modeled by:

$u_1 = \dot{y}_{lc}$, the control effect on the left vortex velocity in y direction

$u_2 = \dot{z}_{lc}$, the control effect on the left vortex velocity in z direction

$u_3 = \dot{y}_{rc}$, the control effect on the right vortex velocity in y direction

$u_4 = \dot{z}_{rc}$, the control effect on the right vortex velocity in z direction

$u_5 = \dot{\gamma}_{lc}$, the control effect on the left vortex strength rate

$u_6 = \dot{\gamma}_{rc}$, the control effect on the right vortex strength rate

$u_7 = \dot{x}_{left_vb_ctr}$, the control effect on the left vortex breakdown location movement

$u_8 = \dot{x}_{right_vb_ctr}$, the control effect on the right vortex breakdown location movement

To solve for state equations, all the present equations first must be expressions in real valued quantities. Therefore at first all the equations expressed in complex quantities will be transformed to equations in real quantities. Since in many places, computation of some variables is repeated many times, intermediate variables are used to simplify the expressions. They will significantly reduce the computation time.

3. 2 Kutta condition equations

The left and the right side Kutta condition are described by Eq. (2.46) and (2.47) respectively. The vortex strength rates, $\dot{\gamma}_l$ and $\dot{\gamma}_r$, can be derived by first solving for γ_l and γ_r by solving the Kutta condition equations, and then taking derivative of them. Kutta condition equations can be expressed in the following form:

$$\begin{aligned} a_{11}\gamma_l + a_{12}\gamma_r &= b_1 \\ a_{21}\gamma_l + a_{22}\gamma_r &= b_2 \end{aligned}$$

where

$$a_{11} = \text{Im} \left[\frac{-i}{1 + \zeta_l^*} + \frac{i}{1 + 1/\bar{\zeta}_l^*} \right] \quad (3.1)$$

$$a_{12} = \text{Im} \left[\frac{i}{1 + \zeta_r^*} - \frac{i}{1 + 1/\bar{\zeta}_r^*} \right] \quad (3.2)$$

$$\begin{aligned} b_1 &= -\text{Im} \left[e^{-i(\pi/2+\phi)} - e^{i(\pi/2+\phi)} \right] - \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* \\ &= 2 \sin(\phi + \frac{\pi}{2}) - \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi} \end{aligned} \quad (3.3)$$

$$a_{21} = \text{Im} \left[\frac{i}{1 - \zeta_l^*} - \frac{i}{1 - 1/\bar{\zeta}_l^*} \right] \quad (3.4)$$

$$a_{22} = -\text{Im} \left[\frac{i}{1 - \zeta_r^*} - \frac{i}{1 - 1/\bar{\zeta}_r^*} \right] \quad (3.5)$$

$$\begin{aligned} b_2 &= -\text{Im} \left[e^{-i(\pi/2+\phi)} - e^{i(\pi/2+\phi)} \right] + \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}^* \\ &= 2 \sin(\phi + \frac{\pi}{2}) + \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi} \end{aligned} \quad (3.6)$$

then, the vortex strength can be derived as

$$\gamma_l = (a_{22}b_1 - a_{12}b_2) / \det \quad (3.7)$$

$$\gamma_r = (a_{11}b_2 - a_{21}b_1) / \det \quad (3.8)$$

where

$$\det = a_{11}a_{22} - a_{12}a_{21} \quad (3.9)$$

$\dot{\gamma}_l$ and $\dot{\gamma}_r$, the vortex strength rates, are brought forth by taking the derivative of γ_l and γ_r , the vortex strength. First both γ_l and γ_r need to be expressed in real quantities. To break down a complex number to a real and an imaginary part, we set

$$\zeta_l^* = a + ib \quad \text{and} \quad \zeta_r^* = c + id \quad (3.10)$$

then

$$\dot{\zeta}_l^* = \dot{a} + i\dot{b} \quad \text{and} \quad \dot{\zeta}_r^* = \dot{c} + i\dot{d} \quad (3.11)$$

Substitute Eq. (3.11) into Eq. (3.1) through (3.7), we get

$$\begin{aligned} a_{22}b_1 - a_{12}b_2 &= \frac{4(c^2 + d^2)^2 - 1}{[(1-c)^2 + d^2][(1+c)^2 + d^2]} \sin(\phi + \frac{\pi}{2}) \\ &\quad - \frac{4c(c^2 + d^2 - 1)}{[(1-c)^2 + d^2][(1+c)^2 + d^2]} \left(\frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi} \right) \end{aligned}$$

and substitute into Eq. (3.9)

$$\det = \frac{4(a^2 + b^2 - 1)(c^2 + d^2 - 1)(ca^2 - ac^2 - a - ad^2 + c + cb^2)}{[(1+a)^2 + b^2][(1-a)^2 + b^2](1+c)^2 + d^2][(1-c)^2 + d^2]}$$

Therefore, substitute the above equation into Eq. (3.7), we get the right vortex strength

$$\begin{aligned} \gamma_l &= \frac{[(1-a)^2 + b^2][(1+a)^2 + b^2]}{(a^2 + b^2 - 1)(ca^2 - ac^2 - a - ad^2 + c + cb^2)} \\ &\quad \times [(c^2 + d^2 + 1)\sin(\phi + \frac{\pi}{2}) - c \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}] \end{aligned} \quad (3.12)$$

Similarly, for the right vortex strength, we have

$$\begin{aligned} a_{11}b_2 - a_{21}b_1 &= \frac{a^2 + b^2 - 1}{(1+a)^2 + b^2} [2\sin(\phi + \frac{\pi}{2}) + \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}] \\ &\quad + \frac{a^2 + b^2 - 1}{(1-a)^2 + b^2} [2\sin(\phi + \frac{\pi}{2}) - \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}] \end{aligned}$$

therefore, substitute into Eq. (3.8)

$$\gamma_r = \frac{[(1-c)^2 + d^2][(1+c)^2 + d^2]}{(c^2 + d^2 - 1)(ca^2 - ac^2 - a - ad^2 + c + cb^2)} \times [(a^2 + b^2 + 1)\sin(\phi + \frac{\pi}{2}) - a \frac{x}{c_r} \frac{\tan \varepsilon}{\sin \alpha} \dot{\phi}] \quad (3.13)$$

Since Eq. (3.12) have variables including a, b, c, d , taking derivative of Eq. (3.12) with respect to time will bring forth the derivatives of these variables, i.e. $\dot{a}, \dot{b}, \dot{c}$ and $\dot{d}$.

Therefore, it can be expressed as:

$$\dot{\gamma}_l = (f_1 \dot{a} + f_2 \dot{b} + f_3 \dot{c} + f_4 \dot{d} + f_5) / m_2 \quad (3.14)$$

where

$$\begin{aligned} m_2 &= (a^2 + b^2 - 1)^2 (ca^2 - ac^2 - a - ad^2 + c + cb^2)^2 \\ f_1 &= \{4an_{11} n_{11} n_{41} - n_6 n_5 [(2a)n_{41} + n_{11}(2ca - c^2 - 1 - d^2)]\} [n_{31} \sin(\phi + \frac{\pi}{2}) - cc_3 \dot{\phi}] \\ f_2 &= (2b)\{2n_{12} n_{11} n_{41} - n_6 n_5 [n_{41} + n_{11}(c)]\} [n_{31} \sin(\phi + \frac{\pi}{2}) - cc_3 \dot{\phi}] \\ f_3 &= n_6 n_5 n_{11} \{[(2c)\sin(\phi + \frac{\pi}{2}) - c_3 \dot{\phi}] n_{41} - [n_{31} \sin(\phi + \frac{\pi}{2}) - cc_3 \dot{\phi}] (a^2 - 2ac + 1 + b^2)\} \\ f_4 &= 2dn_6 n_5 n_{11} \{[\sin(\phi + \frac{\pi}{2})] n_{41} - [n_{31} \sin(\phi + \frac{\pi}{2}) - cc_3 \dot{\phi}] (-a)\} \\ f_5 &= k_1 + k_2 \ddot{\phi} \end{aligned} \quad (3.15)$$

$$k_1 = n_6 n_5 [n_{31} \cos(\phi + \frac{\pi}{2}) \dot{\phi}] n_{11} n_{41}$$

$$k_2 = n_6 n_5 [-cc_3] n_{11} n_{41}$$

Similarly, take derivative of the right vortex in Eq. (3.13), we get

$$\dot{\gamma}_r = (f_6 \dot{a} + f_7 \dot{b} + f_8 \dot{c} + f_9 \dot{d} + f_{10}) / m_4 \quad (3.16)$$

where,

$$m_4 = (c^2 + d^2 - 1)^2 (ca^2 - ac^2 - a - ad^2 + c + cb^2)^2$$

$$f_6 = n_{26} n_{25} n_{32} \{ [(2a) \sin(\phi + \frac{\pi}{2}) - c_3 \dot{\phi}] n_{41} - [n_{12} \sin(\phi + \frac{\pi}{2}) - ac_3 \dot{\phi}] (2ac - c^2 - 1 - d^2) \}$$

$$f_7 = 2bn_{26} n_{25} n_{32} \{ \sin(\phi + \frac{\pi}{2}) n_{41} - [n_{12} \sin(\phi + \frac{\pi}{2}) - ac_3 \dot{\phi}] c \}$$

$$f_8 = \{ 4cn_{32} n_{32} n_{41} - n_{26} n_{25} [(2c)n_{41} + n_{32}(a^2 - 2ac + 1 + b^2)] \} [n_{12} \sin(\phi + \frac{\pi}{2}) - ac_3 \dot{\phi}]$$

$$f_9 = (2d) \{ 2n_{31} n_{32} n_{41} - n_{26} n_{25} [n_{41} + n_{32}(-a)] \} [n_{12} \sin(\phi + \frac{\pi}{2}) - ac_3 \dot{\phi}]$$

$$f_{10} = k_3 + k_4 \ddot{\phi} \quad (3.17)$$

$$k_3 = n_{26} n_{25} [n_{12} \cos(\phi + \frac{\pi}{2}) \dot{\phi}] n_{32} n_{41}$$

$$k_4 = n_{26} n_{25} (-ac_3) n_{32} n_{41}$$

3.3 Zero-force condition equations

The left and the right vortex core velocity, $\frac{\delta \zeta_l}{\delta t}$ and $\frac{\delta \zeta_r}{\delta t}$, are expressed by Eq. (2.20)

and (2.21) respectively. For the left vortex velocity, substitute Eq. (3.10) into Eq. (2.21),

and set

$$\frac{W_c(\zeta_l)}{U_\infty} = m_{g1} + in_{g1}$$

then we have

$$\begin{aligned}
m_{g1} = & c_9 \left[-\sin \phi - \frac{n_2(-\sin \phi) + n_1 \cos \phi}{n_3^2} \right] + c_9 \left\{ \frac{-\gamma_l b}{n_{11}} \right. \\
& \left. + \gamma_r \left[\frac{-(b-d)}{n_{42}} + \frac{(bn_{23}-d)}{n_3 n_{23} - 2n_{43} + 1} \right] \right\} - c_5 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (-n_{t9})}{n_{t1}} d\theta
\end{aligned} \quad (3.18)$$

and

$$\begin{aligned}
n_{g1} = & c_9 \left[-\cos \phi - \frac{n_2(\cos \phi) + n_1 \sin \phi}{n_3^2} \right] + c_9 \left[\frac{-\gamma_l a}{n_{11}} + \gamma_r \left[\frac{-(a-c)}{n_{42}} + \frac{(an_{23}-c)}{n_3 n_{23} - 2n_{43} + 1} \right] \right\} \\
& - c_5 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (-n_{t11})}{n_{t1}} d\theta
\end{aligned} \quad (3.19)$$

Set

$$\begin{aligned}
& -\frac{2 \cos \alpha}{x/c_r} (\zeta_l^* + \frac{1}{\zeta_l^*} + 1) - \frac{\dot{\gamma}_l}{\gamma_l} (\zeta_l^* + \frac{1}{\zeta_l^*} + 2) + \left[\frac{W_c(\zeta_l^*)}{U_\infty} \frac{\zeta_l^{*2}}{\zeta_l^{*2} - 1} - \frac{i \gamma_l \sin \alpha \zeta_l^*}{(\zeta_l^{*2} - 1)^2} \right] \frac{2}{\frac{x}{c_r} \tan \varepsilon} \\
& = m_2 + i n_2
\end{aligned} \quad (3.20)$$

Substitute Eq. (3.10) into the right side of it, we have

$$\begin{aligned}
m_2 = & -\frac{2 \cos \alpha}{x/c_r} \left(a + \frac{a}{a^2 + b^2} + 1 \right) - \frac{\dot{\gamma}_l}{\gamma_l} \left(a + \frac{a}{a^2 + b^2} + 2 \right) \\
& + \left\{ \frac{m_{g1} [(a^2 - b^2)(a^2 - b^2 - 1) + (2ab)^2] + n_{g1}(2ab)}{(a^2 - b^2 - 1)^2 + (2ab)^2} \right. \\
& \left. - \frac{(\gamma_l \sin \alpha) b (3a^4 + 2a^2 b^2 - b^4 - 2a^2 - 2b^2 - 1)}{[(a+1)^2 + b^2]^2 [(a-1)^2 + b^2]^2} \right\} \frac{2}{\frac{x}{c_r} \tan \varepsilon} \\
n_2 = & -\frac{2 \cos \alpha}{x/c_r} \left(b - \frac{b}{a^2 + b^2} \right) - \frac{\dot{\gamma}_l}{\gamma_l} \left(b - \frac{b}{a^2 + b^2} \right)
\end{aligned} \quad (3.21)$$

$$\begin{aligned}
& - \left\{ \frac{n_{g1}[(a^2 - b^2)(a^2 - b^2 - 1) + (2ab)^2] - m_{g1}(2ab)}{(a^2 - b^2 - 1)^2 + (2ab)^2} \right. \\
& \left. - \frac{(\gamma_l \sin \alpha)a(a^4 - 2a^2b^2 - 3b^4 - 2a^2 - 2b^2 + 1)}{[(a+1)^2 + b^2]^2[(a-1)^2 + b^2]^2} \right\} \frac{2}{\frac{x}{c_r} \tan \varepsilon}
\end{aligned} \quad (3.22)$$

Therefore, substitute Eq. (3.20) into the vortex core velocity Eq. (2.20), we have

$$\begin{aligned}
\frac{\delta \zeta_l}{\delta t} &= \frac{m_2[(a^2 - b^2)(a^2 - b^2 - 1) + (2ab)^2] + n_2(2ab)}{(a^2 - b^2 - 1)^2 + (2ab)^2} \\
& - (b\dot{\phi}^*) \frac{(a^4 + 2a^2b^2 + b^4 - 4a^2 - 1)}{[(a+1)^2 + b^2][(a-1)^2 + b^2]} \\
& + i \left\{ \frac{n_2[(a^2 - b^2)(a^2 - b^2 - 1) + (2ab)^2] - m_2(2ab)}{(a^2 - b^2 - 1)^2 + (2ab)^2} \right. \\
& \left. + (a\dot{\phi}^*) \frac{(a^4 + 2a^2b^2 + b^4 + 4b^2 - 1)}{[(a+1)^2 + b^2][(a-1)^2 + b^2]} \right\}
\end{aligned} \quad (3.23)$$

since $\frac{\delta \zeta_l}{\delta t} = \dot{a} + i\dot{b}$, substitute Eq. (3.21) and (3.22) into Eq. (3.23), we get $\dot{a}$ and $\dot{b}$.

Further more, because both $\dot{a}$ and $\dot{b}$ have $\dot{\gamma}_l$ in their equations, $\dot{a}$ and $\dot{b}$ can be broken into two terms as the following:

$$\dot{a} = g_1 + g_2 \dot{\gamma}_l \quad (3.24)$$

and

$$\dot{b} = g_3 + g_4 \dot{\gamma}_l \quad (3.25)$$

where,

$$g_1 = \frac{1}{n_{52}} \{-c_1[+n_{51} n_{54} + n_1 n_{53}] + c_2 \left[\frac{m_{g1}(n_{51}^2 + n_1^2)}{n_{52}} \right]$$

$$\begin{aligned}
& + (\gamma_l \sin \alpha) \frac{an_1 n_7 - bn_8 n_{51}}{n_5^2 n_6^2} \} - (b\dot{\phi}) \frac{n_9}{n_5 n_6} \\
g_2 &= -\frac{1}{\gamma_l} \frac{1}{n_{52}} \{n_{51} (n_{54} + 1) + n_1 n_{53}\} \\
g_3 &= \frac{1}{n_{52}} \{-c_1 [n_{51} n_{53} - n_1 n_{54}] + c_2 \left[\frac{-n_{g1} (n_{51}^2 + n_1^2)}{n_{52}} \right. \\
& \left. + (\gamma_l \sin \alpha) \frac{an_{51} n_7 + bn_8 n_1}{n_5^2 n_6^2} \} \} + (a\dot{\phi}) \frac{n_{10}}{n_5 n_6}
\end{aligned}$$

and

$$g_4 = -\frac{1}{\gamma_l} \frac{1}{n_{52}} \{n_{51} n_{53} - n_1 (n_{54} + 1)\}$$

Similarly, for the right vortex velocity $\frac{\delta \zeta_r}{\delta t}$ in Eq.(3.10), following the same procedure

as for the left vortex equation to transform Eq. (2.22) and (2.23), finally we have

$$\dot{c} = g_5 + g_6 \dot{\gamma}_r \quad (3.26)$$

and

$$\dot{d} = g_7 + g_8 \dot{\gamma}_r \quad (3.27)$$

where

$$\begin{aligned}
g_5 &= \frac{1}{n_{62}} \{-c_1 [n_{61} n_{64} + n_{21} n_{63}] + c_2 \left[\frac{m_{g2} (n_{61}^2 + n_{21}^2)}{n_{62}} \right. \\
& \left. - (\gamma_r \sin \alpha) \left[\frac{cn_{21} n_{27} - dn_{28} n_{61}}{n_{25}^2 n_{26}^2} \right] \} - (d\dot{\phi}) \frac{n_{29}}{n_{25} n_{26}} \\
g_6 &= -\frac{1}{\gamma_r} \frac{1}{n_{62}} \{n_{61} (n_{64} - 1) + n_{21} n_{63}\}
\end{aligned}$$

$$g_7 = \frac{1}{n_{62}} \{ -c_1 [n_{61} n_{63} - n_{21} n_{64}] + c_2 \left[\frac{-n_{g2}(n_{61}^2 + n_{21}^2)}{n_{62}} \right. \\ \left. - (\gamma_r \sin \alpha) \frac{cn_{27}n_{61} + dn_{28}n_{21}}{n_{25}^2 n_{26}^2} \right] \} + (c\dot{\phi}) \frac{n_{30}}{n_{25}n_{26}}$$

and

$$g_8 = -\frac{1}{\gamma_r} \frac{1}{n_{62}} [n_{61} n_{63} - n_{21} (n_{64} - 1)]$$

where,

$$m_{g2} = c_9 \left[-\sin \phi - \frac{n_{22}(-\sin \phi) + n_{21} \cos \phi}{n_{23}^2} \right] - c_9 \left[\frac{-\gamma_r d}{n_{32}} + \gamma_l \left[\frac{-(d-b)}{n_{42}} \right. \right. \\ \left. \left. + \frac{(dn_3 - b)}{n_3 n_{23} - 2n_{43} + 1} \right] \right] - c_5 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (-n_{t10})}{n_{t4}} d\theta$$

and

$$n_{g2} = c_9 \left[-\cos \phi - \frac{n_{22}(\cos \phi) + n_{21} \sin \phi}{n_{23}^2} \right] - c_9 \left[\frac{-\gamma_r c}{n_{31} - 2} + \gamma_l \left[\frac{-(c-a)}{n_{42}} \right. \right. \\ \left. \left. + \frac{(cn_3 - a)}{n_3 n_{23} - 2n_{43} + 1} \right] \right] - c_5 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (-n_{t12})}{n_{t4}} d\theta$$

3.4 Flow potential

The flow potential at any desired point on the wing surface is an integration of velocity from a point which is far away from the wing to the point on the wing surface.

The flow potential Φ is a function of a, b, c, d, γ_l , and γ_r . Since it is not independent from these six states, it may not be taken as a system state. For the wing top surface, it is expressed as:

$$\Phi = - \int_r^{10r} \text{Im}[\overline{W(\zeta)}] \cdot dr + \int_{\pi/2}^{\theta_k} q_\theta \cdot d\theta, \quad 0 < \theta_k < \pi \quad (3.28a)$$

Similarly for bottom surface, it is

$$\Phi = - \int_{-10r}^0 \text{Im}[\overline{W(\zeta)}] \cdot dr + \int_\pi^{\theta_k} q_\theta \cdot d\theta, \quad \pi < \theta_k < 2\pi \quad (3.28b)$$

3.4.1 Tangential velocity potential, q_θ

The integration of tangential velocity on the surface of the wing starts from the middle point of the top wing surface to any desired point on the wing surface. In the circle plane, the starting point is at 90° on the circle. Since the tangential velocity q_θ is expressed by Eq. (2.27),

$$W(\zeta) = U_\infty \sin \alpha \left\{ \frac{1}{e^{i(\pi/2+\phi)}} - \frac{1}{\zeta^2} e^{i(\pi/2+\phi)} \right\} + i \left[\frac{\gamma_l}{\zeta - \zeta_l} - \frac{\gamma_l}{\zeta - 1/\bar{\zeta}_l} - \frac{\gamma_r}{\zeta - \zeta_r} + \frac{\gamma_r}{\zeta - 1/\bar{\zeta}_r} \right] \quad (3.29)$$

Set

$$\zeta = \cos \theta + i \sin \theta$$

and substitute into Eq. (3.29), and set

$$W(\zeta) = m + in$$

we have

$$q_\theta = U_\infty xcr \tan \epsilon \dot{\phi} (1 - 2 \sin^2 \theta) - m \sin \theta - n \cos \theta \quad (3.30)$$

where

$$\begin{aligned}
m = U_{\infty} \sin \alpha \{ & -\sin \phi - \sin(2\theta - \phi) + \frac{\gamma_l(\sin \theta - b)}{(\cos \theta - a)^2 + (\sin \theta - b)^2} \\
& - \frac{\gamma_l(a^2 + b^2)[(a^2 + b^2)\sin \theta - b]}{[(a^2 + b^2)\cos \theta - a]^2 + [(a^2 + b^2)\sin \theta - b]^2} - \frac{\gamma_r(\sin \theta - d)}{(\cos \theta - c)^2 + (\sin \theta - d)^2} \\
& + \frac{\gamma_r(c^2 + d^2)[(c^2 + d^2)\sin \theta - d]}{[(c^2 + d^2)\cos \theta - c]^2 + [(c^2 + d^2)\sin \theta - d]^2} \} \quad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
n = U_{\infty} \sin \alpha \{ & -\cos \phi - \cos(2\theta - \phi) + \frac{\gamma_l(\cos \theta - a)}{(\cos \theta - a)^2 + (\sin \theta - b)^2} \\
& - \frac{\gamma_l(a^2 + b^2)[(a^2 + b^2)\cos \theta - a]}{[(a^2 + b^2)\cos \theta - a]^2 + [(a^2 + b^2)\sin \theta - b]^2} - \frac{\gamma_r(\cos \theta - c)}{(\cos \theta - c)^2 + (\sin \theta - d)^2} \\
& + \frac{\gamma_r(c^2 + d^2)[(c^2 + d^2)\cos \theta - c]}{[(c^2 + d^2)\cos \theta - c]^2 + [(c^2 + d^2)\sin \theta - d]^2} \} \quad (3.32)
\end{aligned}$$

therefore

$$\begin{aligned}
m \sin \theta + n \cos \theta = c_4 \Big[& -2 \cos(\phi - \theta) + \gamma_l \left(\frac{1 - n_{t13}}{n_{t1}} - \frac{n_3(n_3 - n_{t13})}{n_{t2}} \right) \\
& - \gamma_r \left(\frac{1 - n_{t14}}{n_{t4}} - \frac{n_{23}(n_{23} - n_{t14})}{n_{t5}} \right) \Big]
\end{aligned}$$

Substitute into Eq.(3.30), we get the tangential velocity

$$\begin{aligned}
q_{\theta} = c_8 \dot{\phi} (1 - 2 \sin^2 \theta) - c_4 \Big[& -2 \cos(\phi - \theta) + \gamma_l \left(\frac{1 - n_{t13}}{n_{t1}} - \frac{n_3(n_3 - n_{t13})}{n_{t2}} \right) \\
& - \gamma_r \left(\frac{1 - n_{t14}}{n_{t4}} - \frac{n_{23}(n_{23} - n_{t14})}{n_{t5}} \right) \Big] \quad (3.33)
\end{aligned}$$

3.4.2 Vertical velocity integration:

Part of the potential for the top surface is the integration of the vertical velocity from a point which is 10 times of the wing span above the wing to the middle point on the wing surface. It is calculated in the circle plane, and is expressed by the first term of Eq. (3.28).

$$\begin{aligned}
 W(\zeta) = & U_{\infty} \sin \alpha \left\{ \left[\frac{1}{e^{i(\pi/2+\phi)}} - \frac{1}{\zeta^2} e^{i(\pi/2+\phi)} \right] \right. \\
 & + i \left[\frac{\gamma_l}{\zeta - \zeta_l} - \frac{\gamma_l}{\zeta - 1/\bar{\zeta}_l} - \frac{\gamma_r}{\zeta - \zeta_r} + \frac{\gamma_r}{\zeta - 1/\bar{\zeta}_r} \right] \Bigg\} \\
 & - U_{\infty} \frac{2}{\pi} \frac{x\dot{\phi}}{c_r} \tan \varepsilon \int_0^{2\pi} \frac{\sin \theta \cos \theta}{\zeta_l - a e^{i\theta}} d\theta
 \end{aligned} \tag{3.34}$$

Set

$$\zeta = 0 + ir \tag{3.35}$$

and

$$\overline{W(\zeta)} = m_3 - in_3$$

therefore

$$\int_r^{10r} \text{Im}[\overline{W(\zeta)}] \cdot dr = \int_r^{10r} (-n_3) dr \tag{3.36}$$

Substitute Eq. (3.35) into Eq. (3.34),

$$\begin{aligned}
 n_3 = & c_4 \left[\cos \phi \left(-1 + \frac{1}{r^2} \right) - a \gamma_l \left(\frac{1}{n_{r3}} - \frac{n_3}{n_{r2}} \right) + c \gamma_r \left(\frac{1}{n_{r6}} - \frac{n_{23}}{n_{r5}} \right) \right] \\
 & - c_6 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (r - \sin \theta)}{(-\cos \theta)^2 + (r - \sin \theta)^2} d\theta
 \end{aligned} \tag{3.37}$$

The potential for the bottom surface is similar to the one for the top surface except that the starting point is 10 times the wing span below the wing. If we set

$$\zeta = -1 - ir \quad (3.38)$$

and

$$\overline{W(\zeta)} = m_4 - in_4$$

therefore

$$\int_r^{10r} \text{Im}[\overline{W(\zeta)}] \cdot dr = \int_r^{10r} (-n_4) dr \quad (3.39)$$

Substitute Eq. (3.38) into Eq. (3.34), after simplification we have

$$\begin{aligned} n_4 = c_4 \left\{ -\cos \phi - \frac{(2r) \sin \phi + (1-r^2) \cos \phi}{(1+r^2)^2} + \gamma_l \left[\frac{-n_{13}}{n_{r7}} + \frac{n_3 n_{12}}{n_{r8}} \right] \right. \\ \left. + \gamma_r \left[\frac{n_{33}}{n_{r9}} + \frac{-n_{23} n_{32}}{n_{r10}} \right] \right\} - c_6 \dot{\phi} \int_0^{2\pi} \frac{\sin \theta \cos \theta (r + \sin \theta)}{(1 + \cos \theta)^2 + (r + \sin \theta)^2} d\theta \end{aligned} \quad (3.40)$$

3.5 Flow potential rate

The flow potential rate, $\frac{\partial \Phi}{\partial t}$, is part of the calculation of the pressure coefficient C_p ,

and it can be derived by taking derivative of the Φ expressed by Eq. (3.28). Since n_3 , n_4

and q_θ all have $\dot{\phi}$, the derivative of Φ will have $\dot{\phi}$ in it.

For the top surface,

$$\frac{\partial \Phi}{\partial t} = \int_r^{10r} \dot{n}_3 \cdot dr + \int_{\pi/2}^{\theta_k} \dot{q}_\theta \cdot d\theta \quad (\pi < \theta_k < 2\pi) \quad (3.41)$$

Eq. (3.37) contains $a, b, c, d, \gamma_l, \gamma_r$ and ϕ as state variables, the derivative of Eq.

(3.37) will contain $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{\gamma}_l, \dot{\gamma}_r$ and $\ddot{\phi}$ consequently, therefore n_3 can be set as

$$\dot{n}_3 = v_1 \dot{a} + v_2 \dot{b} + v_3 \dot{c} + v_4 \dot{d} + v_5 \dot{\gamma}_l + v_6 \dot{\gamma}_r + v_7 \ddot{\phi} + v_8 \quad (3.42)$$

By taking derivative of Eq. (3.37), and after simplification, we have

$$\begin{aligned} v_1 &= c_4 \gamma_l \left\{ \frac{-n_{r3} + [(2a^2)]}{n_{r3}^2} - \frac{(-3a^2 - b^2)}{n_{r2}} - \frac{n_3 [2a^2 (1 + 2n_{r1}r)]}{n_{r2}^2} \right\} \\ v_2 &= c_4 \gamma_l \left\{ \frac{-[(a)][2(r-b)]}{n_{r3}^2} - \frac{[(-n_1)]}{n_{r2}} - \frac{n_3 (a) \{2n_{r1} [(2b)r - 1]\}}{n_{r2}^2} \right\} \\ v_3 &= c_4 \gamma_r \left\{ -\frac{-n_{r6} + 2c^2}{n_{r6}^2} + \frac{[(-3c^2 - d^2)]}{n_{r5}} + \frac{[n_{23} (2c^2)][1 + 2(n_{r4})r]}{n_{r5}^2} \right\} \\ v_4 &= c_4 \gamma_r \left\{ -\frac{[(-c)][2(r-d)]}{n_{r6}^2} - \frac{n_{21}}{n_{r5}} + \frac{[n_{23}]c \{2(n_{r4})[2dr - c]\}}{n_{r5}^2} \right\} \\ v_5 &= c_4 \left[(-a) \left(\frac{1}{n_{r3}} - \frac{n_3}{n_{r2}} \right) \right] \\ v_6 &= c_4 \left[(-c) \left(-\frac{1}{n_{r6}} + \frac{n_{23}}{n_{r5}} \right) \right] \\ v_7 &= -c_6 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r - \sin \theta)}{(-\cos \theta)^2 + (r - \sin \theta)^2} d\theta \\ v_8 &= c_4 [(\sin \phi) \dot{\phi} (1 - \frac{1}{r^2})] \end{aligned}$$

For the bottom surface,

$$\frac{\partial \Phi}{\partial t} = \int_{-10r}^0 \dot{n}_4 \cdot dr + \int_{\pi}^{\theta_k} \dot{q}_{\theta} \cdot d\theta \quad (\pi < \theta_k < 2\pi) \quad (3.43)$$

Similarly for $\dot{n}_4$, we can set

$$\dot{n}_4 = v_{11}\dot{a} + v_{12}\dot{b} + v_{13}\dot{c} + v_{14}\dot{d} + v_{15}\dot{\gamma}_l + v_{16}\dot{\gamma}_r + v_{17}\ddot{\phi} + v_{18}$$

By taking derivative of Eq. (3.40), and after simplification we have

$$v_{11} = c_4 \gamma_l \left\{ -\frac{n_{r7} - 2n_{13}^2}{n_{r7}^2} + \frac{4n_3 a + 2a^2 + n_3}{n_{r8}} - \frac{2n_3 n_{12} \{ [n_{12}(2a+1) + [n_3 r + b][(2a)r] \}}{n_{r8}^2} \right\}$$

$$v_{12} = c_4 \gamma_l \left\{ -\frac{n_{13}[2(r+b)]}{n_{r7}^2} + \frac{2b(2n_3 + a)}{n_{r8}} - \frac{2n_3 n_{12} \{ [2n_{12}b + [n_3 r + b][(2b)r + 1] \}}{n_{r8}^2} \right\}$$

$$v_{13} = c_4 \gamma_r \left\{ \frac{n_{r9} - 2n_{33}^2}{n_{r9}^2} - \frac{4n_{23}c + 4c^2 + n_{23}}{n_{r10}} + \frac{2n_{22}n_{33} \{ n_{32}(2c+1) + [n_{23}r + d][(2c)r] \}}{n_{r10}^2} \right\}$$

$$v_{14} = c_4 \gamma_r \left\{ \frac{-n_{33}[2(r+d)]}{n_{r9}^2} - \frac{(2n_{23} + c)(2d)}{n_{r10}} + \frac{2n_{23}n_{32} \{ 2n_{32}d + [n_{23}r + d][(2d)r + 1] \}}{n_{r10}^2} \right\}$$

$$v_{15} = c_4 \left\{ -\frac{n_{13}}{n_{r7}} + \frac{n_3 n_{12}}{n_{r8}} \right\}$$

$$v_{16} = c_4 \left\{ \frac{n_{33}}{n_{r9}} - \frac{n_{23}n_{32}}{n_{r10}} \right\}$$

$$v_{17} = -c_6 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r + \sin \theta)}{(1 + \cos \theta)^2 + (r + \sin \theta)^2} d\theta$$

$$v_{18} = c_4 \dot{\phi} \left\{ \sin \phi - \frac{(2r) \cos \phi - (1 - r^2) \sin \phi}{(1 + r^2)^2} \right\}$$

and similarly, to get the tangential acceleration q_θ , take derivative of Eq. (3.33) and set

$$\dot{q}_\theta = v_{41}\dot{a} + v_{42}\dot{b} + v_{43}\dot{c} + v_{44}\dot{d} + v_{45}\dot{\gamma}_l + v_{46}\dot{\gamma}_r + v_{47}\ddot{\phi} + v_{48}$$

then we have

$$v_{41} = -c_4 \gamma_l \left\{ \frac{-n_{t1} \cos \theta + 2(1 - n_{t13})n_{t9}}{n_{t1}^2} - \frac{[4an_3 - n_1 \sin \theta - (3a^2 + b^2) \cos \theta]}{n_{t2}} \right\}$$

$$\begin{aligned}
& + \frac{2n_3(n_3 - n_{t13}) \{n_7[(2a)\cos\theta - 1] + n_{t3}[(2a)\sin\theta]\}}{n_{t2}^2} \} \\
v_{42} = & -c_4 \gamma_l \left\{ \frac{-n_{t1} \sin\theta + 2n_{t11}(1 - n_{t13})}{n_{t1}^2} - \frac{[4bn_3 - (a^2 + 3b^2)\sin\theta - n_1 \cos\theta]}{n_{t2}} \right. \\
& + \frac{2n_3(n_3 - n_{t13}) \{n_7[(2b)\cos\theta] + n_{t3}[(2b)\sin\theta - 1]\}}{n_{t2}^2} \} \\
v_{43} = & -c_4 \gamma_r \left\{ -\frac{-n_{t4} \cos\theta + 2n_{t10}(1 - n_{t14})}{n_{t4}^2} + \frac{[4cn_{23} - (3c^2 + d^2)\cos\theta - n_{21} \sin\theta]}{n_{t5}} \right. \\
& - \frac{2n_{23}(n_{23} - n_{t14}) \{n_8[(2c)\cos\theta - 1] + n_{t6}[(2c)\sin\theta]\}}{n_{t5}^2} \} \\
v_{44} = & -c_4 \gamma_r \left\{ -\frac{-n_{t4} \sin\theta + 2n_{t12}(1 - n_{t14})}{n_{t4}^2} + \frac{[4dn_{23} - (c^2 + 3d^2)\sin\theta - n_{21} \cos\theta]}{n_{t5}} \right. \\
& - \frac{2n_{23}(n_{23} - n_{t14}) \{[n_{23} \cos\theta - c][(2d)\cos\theta] + n_{t6}[(2d)\sin\theta - 1]\}}{n_{t5}^2} \} \\
v_{45} = & -c_4 \left\{ \frac{1 - n_{t13}}{n_{t1}} - \frac{n_3(n_3 - n_{t13})}{n_{t2}} \right\} \\
v_{46} = & -c_4 \left\{ -\frac{1 - n_{t14}}{n_{t4}} + \frac{n_{23}(n_{23} - n_{t14})}{n_{t5}} \right\} \\
v_{48} = & -2c_4 \dot{\phi} \sin(\phi - \theta)
\end{aligned}$$

Therefore by Eq. (3.41), the flow potential rate for the top surface is:

$$\begin{aligned}
\frac{\partial \Phi_\theta}{\partial t} = & \int_r^{10r} (v_1 \dot{a} + v_2 \dot{b} + v_3 \dot{c} + v_4 \dot{d} + v_5 \dot{\gamma}_l + v_6 \dot{\gamma}_r + v_7 \ddot{\phi} + v_8) \cdot dr \\
& + \int_{\pi/2}^{\theta} (v_{41} \dot{a} + v_{42} \dot{b} + v_{43} \dot{c} + v_{44} \dot{d} + v_{45} \dot{\gamma}_l + v_{46} \dot{\gamma}_r + v_{47} \ddot{\phi} + v_{48}) \cdot d\theta
\end{aligned}$$

We can set

$$\frac{\partial \Phi_\theta}{\partial t} = v_{51}\dot{a} + v_{52}\dot{b} + v_{53}\dot{c} + v_{54}\dot{d} + v_{55}\dot{\gamma}_l + v_{56}\dot{\gamma}_r + v_{57}\ddot{\phi} + v_{58} \quad (3.44)$$

similarly for the bottom surface by Eq. (3.43)

$$\begin{aligned} \frac{\partial \Phi_\theta}{\partial t} = & \int_{-10r}^0 (v_{11}\dot{a} + v_{12}\dot{b} + v_{13}\dot{c} + v_{14}\dot{d} + v_{15}\dot{\gamma}_l + v_{16}\dot{\gamma}_r + v_{17}\ddot{\phi} + v_{18}) \cdot dr \\ & + \int_{\pi}^{\theta_k} (v_{41}\dot{a} + v_{42}\dot{b} + v_{43}\dot{c} + v_{44}\dot{d} + v_{45}\dot{\gamma}_l + v_{46}\dot{\gamma}_r + v_{47}\ddot{\phi} + v_{48}) \cdot d\theta \end{aligned}$$

We can set

$$\frac{\partial \Phi_\theta}{\partial t} = v_{91}\dot{a} + v_{92}\dot{b} + v_{93}\dot{c} + v_{94}\dot{d} + v_{95}\dot{\gamma}_l + v_{96}\dot{\gamma}_r + v_{97}\ddot{\phi} + v_{98} \quad (3.45)$$

$C_{P\theta}$ is calculated by Eq. (2.26), therefore we set

$$C_{P\theta} = q_{5\theta} + q_6 \frac{\partial \Phi_\theta}{\partial t} \quad (3.46)$$

where

$$q_{5\theta} = \sin^2 \alpha - \frac{2 \cos \alpha \Phi_x}{U_\infty} - \frac{\Phi_x^2 + [q_\theta / 2 \sin \theta]^2}{U_\infty^2} - 8 \left[\frac{x}{c_r} \dot{\phi} \tan \varepsilon \cos \theta \right]^2$$

$$q_6 = c_7$$

where according to Eq.(2.30)

$$\Phi_x = \left[\Phi + (q_\theta \frac{\cos \theta}{\sin \theta}) - (2U_\infty \sin \alpha \cos \theta \sin \phi) \right] \frac{\tan \varepsilon}{2}$$

where Φ is calculated by Eq. (3.28) and (3.36) for the top surface, and by Eq. (3.28) and (3.40) for the bottom surface.

Substitute Eq. (3.28) and (3.36) into it, for the top surface we get:

$$\Phi_x = \left[\left(\int_r^{10r} n_3 \cdot dr + \int_{\pi/2}^{\theta} q_\theta \cdot d\theta \right) + \left(q_\theta \frac{\cos \theta}{\sin \theta} \right) - (2U_\infty \sin \alpha \cos \theta \sin \phi) \right] \frac{\tan \varepsilon}{2} \quad (3.47)$$

and substitute Eq. (3.28) and (3.40) into it, for the bottom surface we get:

$$\Phi_x = \left[\left(\int_{-10r}^0 n_4 \cdot dr + \int_{\pi}^{\theta} q_\theta \cdot d\theta \right) + \left(q_\theta \frac{\cos \theta}{\sin \theta} \right) - (2U_\infty \sin \alpha \cos \theta \sin \phi) \right] \frac{\tan \varepsilon}{2} \quad (3.48)$$

3.6 Total roll moment

The calculation of roll moment at any cross flow plane is described in Chapter 2.

Substitute Eq. (3.46) into Eq. (2.43), we have

$$\begin{aligned} m_{ctop} = \frac{1}{2} \frac{x_{cr}^2}{c_r} \int_0^\pi & [q_{8\theta} (v_{51}\dot{a} + v_{52}\dot{b} + v_{53}\dot{c} + v_{54}\dot{d} + v_{55}\dot{\gamma}_l + v_{56}\dot{\gamma}_r) + (q_{8\theta}v_{58} + q_{7\theta}) \\ & + \ddot{\phi}(v_{57}q_{8\theta})] d\theta \end{aligned} \quad (3.49)$$

We can set

$$m_{ctop} = v_{61}\dot{a} + v_{62}\dot{b} + v_{63}\dot{c} + v_{64}\dot{d} + v_{65}\dot{\gamma}_l + v_{66}\dot{\gamma}_r + v_{67}\ddot{\phi} + v_{68}$$

where

$$q_{7\theta} = q_{5\theta} \sin \theta \cos \theta$$

$$q_{8\theta} = \left(-\frac{x}{c_r} \frac{\tan \varepsilon}{U_\infty} \right) \sin \theta \cos \theta$$

Similarly, we can set

$$m_{cbot} = v_{71}\dot{a} + v_{72}\dot{b} + v_{73}\dot{c} + v_{74}\dot{d} + v_{75}\dot{\gamma}_l + v_{76}\dot{\gamma}_r + v_{77}\ddot{\phi} + v_{78} \quad (3.50)$$

To calculate the total roll moment on the delta wing, Eq. (3.49) and (3.50) are substituted into Eq. (2.45). C_{p_vb} is calculated by Eq. (2.35), and it remains the same value at every point after the vortex breakdown point, therefore we have

$$\frac{c_r^3 - x_{lag_right}^3}{6c_r^3} \int_0^{\frac{\pi}{2}} C_{p_vb} \sin \theta \cos \theta d\theta = -\frac{1}{2} \frac{c_r^3 - x_{lag_right}^3}{6c_r^3} C_{p_vb}$$

$$\frac{c_r^3 - x_{lag_left}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} C_{p_vb} \sin \theta \cos \theta d\theta = \frac{1}{2} \frac{c_r^3 - x_{lag_left}^3}{6c_r^3} C_{p_vb}$$

Therefore substitute into Eq. (2.45), we have

$$\begin{aligned} C_{l_int} = & \frac{x_{lag_right}^3}{6c_r^3} \int_0^{\frac{\pi}{2}} [q_{8\theta} (v_{51}\dot{a} + v_{52}\dot{b} + v_{53}\dot{c} + v_{54}\dot{d} + v_{55}\dot{\gamma}_l + v_{56}\dot{\gamma}_r) + (q_{8\theta}v_{58} + q_{7\theta}) \\ & + \ddot{\phi}(v_{57}q_{8\theta})] d\theta + \frac{x_{lag_left}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} [q_{8\theta} (v_{51}\dot{a} + v_{52}\dot{b} + v_{53}\dot{c} + v_{54}\dot{d} + v_{55}\dot{\gamma}_l + v_{56}\dot{\gamma}_r) \\ & + (q_{8\theta}v_{58} + q_{7\theta}) + \ddot{\phi}(v_{57}q_{8\theta})] d\theta + \frac{1}{6} \int_0^{\pi} [q_{8\theta} (v_{91}\dot{a} + v_{92}\dot{b} \\ & + v_{93}\dot{c} + v_{94}\dot{d} + v_{95}\dot{\gamma}_l + v_{96}\dot{\gamma}_r) + (q_{8\theta}v_{98} + q_{7\theta}) + \ddot{\phi}(v_{97}q_{8\theta})] d\theta \\ & + \frac{1}{2} \frac{x_{lag_right}^3 - x_{lag_left}^3}{6c_r^3} C_{p_vb} \end{aligned}$$

If we set

$$C_{l_int} = w_{31}\dot{a} + w_{32}\dot{b} + w_{33}\dot{c} + w_{34}\dot{d} + w_{35}\dot{\gamma}_l + w_{36}\dot{\gamma}_r + w_{37}\ddot{\phi} + w_{38} \quad (3.51)$$

then for $k=1, \dots, 7$:

$$\begin{aligned} w_{3k} = & \frac{x_{lag_right}^3}{6c_r^3} \int_0^{\frac{\pi}{2}} q_{8\theta} \left[\int_r^{10r} v_k \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} -(v_{2k} \sin \theta + v_{3k} \cos \theta) \cdot d\theta \right] d\sigma \\ & + \frac{x_{lag_left}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} q_{8\theta} \left[\int_r^{10r} v_k \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} -(v_{2k} \sin \theta + v_{3k} \cos \theta) \cdot d\theta \right] d\sigma \\ & + \frac{1}{6} \int_0^{\pi} q_{8\theta} \left[\int_{-10r}^0 v_{1k} \cdot dr + \int_{\pi}^{\sigma} -(v_{2k} \sin \theta + v_{3k} \cos \theta) \cdot d\theta \right] d\sigma \end{aligned}$$

and

$$\begin{aligned}
w_{38} = & \frac{x_{lag_right}^3}{6c_r^3} \left[\int_0^{\frac{\pi}{2}} q_{8\theta} \left(\int_r^{10r} v_8 \cdot dr + \int_{\frac{\pi/2}{2}}^{\frac{\theta k}{2}} v_{48} \cdot d\theta \right) d\theta + \int_0^{\frac{\pi}{2}} q_{7\theta} d\theta \right] \\
& + \frac{x_{lag_left}^3}{6c_r^3} \left[\int_{\frac{\pi}{2}}^{\pi} q_{8\theta} \left(\int_r^{10r} v_8 \cdot dr + \int_{\frac{\pi/2}{2}}^{\frac{\theta k}{2}} v_{48} \cdot d\theta \right) d\theta + \int_{\frac{\pi}{2}}^{\pi} q_{7\theta} d\theta \right] \\
& + \frac{1}{6} \left[\int_0^{\pi} q_{8\theta} \left(\int_{-10r}^0 v_{18} \cdot dr + \int_{\pi}^{\frac{\theta k}{2}} v_{48} \cdot d\theta \right) d\theta + \int_0^{\pi} q_{7\theta} d\theta \right] \\
& + \frac{1}{2} \frac{x_{lag_right}^3 - x_{lag_left}^3}{6c_r^3} C_{p_vb}
\end{aligned}$$

Therefore substitute Eq. (3.51) into Eq. (2.42),

$$\ddot{\phi} = \frac{\rho c_r^5 \tan^2 \varepsilon}{I_{xx}} (w_{31}\dot{a} + w_{32}\dot{b} + w_{33}\dot{c} + w_{34}\dot{d} + w_{35}\dot{\gamma}_l + w_{36}\dot{\gamma}_r + w_{37}\ddot{\phi} + w_{38})$$

and solve for $\ddot{\phi}(t)$

$$\ddot{\phi} = \frac{(w_{31}\dot{a} + w_{32}\dot{b} + w_{33}\dot{c} + w_{34}\dot{d} + w_{35}\dot{\gamma}_l + w_{36}\dot{\gamma}_r + w_{38})}{C_1 - w_{37}}$$

therefore we can set

$$\ddot{\phi} = h_1\dot{a} + h_2\dot{b} + h_3\dot{c} + h_4\dot{d} + h_5\dot{\gamma}_l + h_6\dot{\gamma}_r + h_7 \quad (3.52)$$

where

$$C_1 = \frac{I_{xx}}{\rho c_r^5 \tan^2 \varepsilon}$$

Now all the state variables are clearly defined, and all the equations are transformed into real valued calculations. They are ready for solving for the state equations systematically.

Chapter 4

Solving for State Equations and Simplification

As described in the goal of the new model in Chapter 3, 10 state variables are identified, The flow equations and rigid body equations together will be used to derive 10 state equations. The system inputs are assumed to affect the vortex strength rate in a linear fashion. The system outputs which are wing surface air pressure measurements need to be solved as well. Finally, all the state equations and output equations constituting the model are simplified.

4.1 Solve for state equations

Now all equations governing the flow field and rigid body state variables have been transformed into equations in real quantities. However, since the system state derivatives are entangled with each other in the equations, which need to be solved for the state derivatives to get the system state equations. Note that the equations can be taken as a series of linear equations in respect to the state derivatives. The vortex core velocity in y and z direction have been solved in Chapter 3, they are

$$\dot{a} = g_1 + g_2 \dot{\gamma}_l \quad (3.24)$$

$$\dot{b} = g_3 + g_4 \dot{\gamma}_l \quad (3.25)$$

$$\dot{c} = g_5 + g_6 \dot{\gamma}_r \quad (3.26)$$

$$\dot{d} = g_7 + g_8 \dot{\gamma}_r \quad (3.27)$$

Substitute Eq. (3.15) and (3.52) into Eq. (3.14), we have

$$\begin{aligned}
m_2 \dot{\gamma}_l &= (f_1 + k_2 h_1) \dot{a} + (f_2 + k_2 h_2) \dot{b} + (f_3 + k_2 h_3) \dot{c} + (f_4 + k_2 h_4) \dot{d} \\
&\quad + (k_2 h_5) \dot{\gamma}_l + (k_2 h_6) \dot{\gamma}_r + (k_1 + k_2 h_7)
\end{aligned} \tag{4.1}$$

Similarly, substitute Eq. (3.17) and (3.52) into Eq. (3.16), we have

$$\begin{aligned}
m_4 \dot{\gamma}_r &= (f_6 + k_4 h_1) \dot{a} + (f_7 + k_4 h_2) \dot{b} + (f_8 + k_4 h_3) \dot{c} + (f_9 + k_4 h_4) \dot{d} \\
&\quad + (k_4 h_5) \dot{\gamma}_l + (k_4 h_6) \dot{\gamma}_r + (k_3 + k_4 h_7)
\end{aligned} \tag{4.2}$$

To solve for $\dot{\gamma}_l$ and $\dot{\gamma}_r$, Eq. (3.24) through (3.27) are substituted into Eq. (4.1) and (4.2),

therefore we have:

$$\begin{aligned}
m_2 \dot{\gamma}_l &= [(f_1 + k_2 h_1) g_1 + (f_2 + k_2 h_2) g_3 + (f_3 + k_2 h_3) g_5 + (f_4 + k_2 h_4) g_7 + (k_1 + k_2 h_7)] \\
&\quad + [(f_1 + k_2 h_1) g_2 + (f_2 + k_2 h_2) g_4 + k_2 h_5] \dot{\gamma}_l \\
&\quad + [(f_3 + k_2 h_3) g_6 + (f_4 + k_2 h_4) g_8 + k_2 h_6] \dot{\gamma}_r
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
m_4 \dot{\gamma}_r &= [(f_6 + k_4 h_1) g_1 + (f_7 + k_4 h_2) g_3 + (f_8 + k_4 h_3) g_5 + (f_9 + k_4 h_4) g_7 + (k_3 + k_4 h_7)] \\
&\quad + [(f_6 + k_4 h_1) g_2 + (f_7 + k_4 h_2) g_4 + k_4 h_5] \dot{\gamma}_l \\
&\quad + [(f_8 + k_4 h_3) g_6 + (f_9 + k_4 h_4) g_8 + k_4 h_6] \dot{\gamma}_r
\end{aligned} \tag{4.4}$$

Set

$$p_1 = [(f_1 + k_2 h_1) g_1 + (f_2 + k_2 h_2) g_3 + (f_3 + k_2 h_3) g_5 + (f_4 + k_2 h_4) g_7 + (k_1 + k_2 h_7)] \tag{4.5}$$

$$p_2 = (f_1 + k_2 h_1) g_2 + (f_2 + k_2 h_2) g_4 + k_2 h_5 \tag{4.6}$$

$$p_3 = (f_3 + k_2 h_3) g_6 + (f_4 + k_2 h_4) g_8 + k_2 h_6 \tag{4.7}$$

$$p_4 = [(f_6 + k_4 h_1) g_1 + (f_7 + k_4 h_2) g_3 + (f_8 + k_4 h_3) g_5 + (f_9 + k_4 h_4) g_7 + (k_3 + k_4 h_7)] \tag{4.8}$$

$$p_5 = (f_6 + k_4 h_1) g_2 + (f_7 + k_4 h_2) g_4 + k_4 h_5 \tag{4.9}$$

$$p_6 = (f_8 + k_4 h_3)g_6 + (f_9 + k_4 h_4)g_8 + k_4 h_6 \quad (4.10)$$

$$p_7 = \frac{p_1(m_4 - p_6) + p_3 p_4}{(m_4 - p_6)(m_2 - p_2) - p_3 p_5} \quad (4.11)$$

$$p_8 = \frac{p_4(m_2 - p_2) + p_1 p_5}{(m_4 - p_6)(m_2 - p_2) - p_3 p_5} \quad (4.12)$$

Thus Eq. (4.3) and (4.4) become:

$$m_2 \dot{\gamma}_l = p_1 + p_2 \dot{\gamma}_l + p_3 \dot{\gamma}_r$$

$$m_4 \dot{\gamma}_r = p_4 + p_5 \dot{\gamma}_l + p_6 \dot{\gamma}_r$$

Solve these equations, we get

$$\dot{\gamma}_l = p_7 \quad (4.5)$$

$$\dot{\gamma}_r = p_8 \quad (4.6)$$

Substitute into Eq. (3.24) through (3.27)

$$\dot{a} = g_1 + g_2 p_7 \quad (4.7)$$

$$\dot{b} = g_3 + g_4 p_7 \quad (4.8)$$

$$\dot{c} = g_5 + g_6 p_8 \quad (4.9)$$

$$\dot{d} = g_7 + g_8 p_8 \quad (4.10)$$

and substitute Eq. (4.5) through (4.10) into Eq. (3.52), we get

$$\ddot{\phi} = (h_1 g_1 + h_2 g_3 + h_3 g_5 + h_4 g_7 + h_7) + (h_1 g_2 + h_2 g_4 + h_5) p_7 + (h_3 g_6 + h_4 g_8 + h_6) p_8$$

Together with the equation of the vortex breakdown location Eq. (2.40), all the state equations have been derived according to the modeling goal described at the beginning of Chapter 3.

4.2 The system output

As derived in Chapter 3, the system has the output as:

$$y_1 = x_1 = \phi, \text{ the roll angle}$$

$$y_2 = x_2 = \dot{\phi}, \text{ the roll velocity}$$

The rest output variables are the pressure coefficients $C_p(\theta_k)$ that are described by Eq. (3.46). It involves terms including $\frac{\partial \Phi_\theta}{\partial t}$, Φ_x and q_θ . The flow potential Φ_x is expressed by Eq. (3.47) and (3.48) for the top surface and the bottom surface respectively, and the tangential velocity q_θ are calculated by Eq. (3.33).

Now to get the flow potential rate $\frac{\partial \Phi_\theta}{\partial t}$, for the top surface, substitute Eq. (4.5)

through (4.10) into Eq. (3.44), we have

$$\begin{aligned} \frac{\partial \Phi_\theta}{\partial t} = & (v_{51} + v_{57}h_1)(g_1 + g_2p_7) + (v_{52} + v_{57}h_2)(g_3 + g_4p_7) + (v_{53} + v_{57}h_3)(g_5 + g_6p_8) \\ & + (v_{54} + v_{57}h_4)(g_7 + g_8p_8) + (v_{55} + v_{57}h_5)(p_7) + (v_{56} + v_{57}h_6)(p_8) + (v_{58} + v_{57}h_7) \end{aligned}$$

similarly, for the bottom surface, we have

$$\begin{aligned} \frac{\partial \Phi_\theta}{\partial t} = & (v_{91} + v_{97}h_1)(g_1 + g_2p_7) + (v_{92} + v_{97}h_2)(g_3 + g_4p_7) + (v_{93} + v_{97}h_3)(g_5 + g_6p_8) \\ & + (v_{94} + v_{97}h_4)(g_7 + g_8p_8) + (v_{95} + v_{97}h_5)(p_7) + (v_{96} + v_{97}h_6)(p_8) + (v_{98} + v_{97}h_7) \end{aligned}$$

4.3 Simplification of the state equations

4.3.1 State equations

Even though all the state equations have been derived, there are many repeated calculations. Simplifying these equations will not only simplify the expression of the system, but also will simplify the control design, and computation/simulation time. Sorting out and merging common terms and applying intermediate variables and functions, finally we have the simplified state equations together with the control input as:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (h_1 g_1 + h_2 g_3 + h_3 g_5 + h_4 g_7 + h_7) + (h_1 g_2 + h_2 g_4 + h_5) p_7 + (h_3 g_6 + h_4 g_8 + h_6) p_8$$

$$\dot{x}_3 = g_1 + g_2 p_7 + u_1$$

$$\dot{x}_4 = g_3 + g_4 p_7 + u_2$$

$$\dot{x}_5 = g_5 + g_6 p_8 + u_3$$

$$\dot{x}_6 = g_7 + g_8 p_8 + u_4$$

$$\dot{x}_7 = p_7 + u_5$$

$$\dot{x}_8 = p_8 + u_6$$

$$\dot{x}_9 = \frac{K_{s1}}{\beta} x_1 - \frac{1}{\beta} x_9 + \frac{K_{s2}}{\beta} + u_7$$

$$\dot{x}_{10} = -\frac{K_{s1}}{\beta} x_1 - \frac{1}{\beta} x_{10} + \frac{K_{s2}}{\beta} + u_8$$

p_1 through p_8 are expressed by Eq. (4.5) through (4.10). And

$$m_2 = n_{11}^2 (x_3, x_4) n_{41}^2$$

$$m_4 = n_{11}^2(x_5, x_6)n_{41}^2$$

Each state equation is expressed by some of or all the states. The more states involved in a state equation, the more complicated the system is for control design. These 10 state equations are the functions of the states:

$$\dot{x}_1 = F_1(x_2)$$

$$\dot{x}_{2,\dots,8} = F_{2,\dots,8}(x_1, x_2, \dots, x_{10})$$

$$\dot{x}_9 = F_9(x_1, x_9)$$

$$\dot{x}_{10} = F_{10}(x_1, x_{10})$$

The equations show that the system is not a sparse system of equations. The control design calculation will be very complicated. Therefore further simplification of the equations before designing control is obviously important.

4.3.2 The output of the system

The output of the system includes:

$$y_1 = x_1 = \phi, \text{ the roll angle}$$

$$y_2 = x_2 = \dot{\phi}, \text{ the roll velocity}$$

$$y_{k+2} = C_p(\theta_k, x_{cr}), \quad 0 < \theta_k < \pi$$

where

$$C_p(\theta) = q_{5\theta} + c_7 \frac{\partial \Phi_\theta}{\partial t}$$

where

$$q_{5\theta} = c_9^2 - c_{12}\Phi_x - \frac{\Phi_x^2 + [q_\theta / 2 \sin \theta]^2}{U_\infty^2} - 8 \left[x_2 \frac{2 \cos \theta}{c_2} \right]^2$$

for the top surface,

$$\begin{aligned} \frac{\partial \Phi_\theta}{\partial t} &= (v_{51} + v_{57}h_1)(g_1 + g_2p_7) + (v_{52} + v_{57}h_2)(g_3 + g_4p_7) + (v_{53} + v_{57}h_3)(g_5 + g_6p_8) \\ &+ (v_{54} + v_{57}h_4)(g_7 + g_8p_8) + (v_{55} + v_{57}h_5)(p_7) + (v_{56} + v_{57}h_6)(p_8) + (v_{58} + v_{57}h_7) \end{aligned}$$

$$\Phi_x = \left[\left(\int_r^{10r} n_{\Phi 3} \cdot dr + \int_{\pi/2}^{\theta k} q_\theta \cdot d\theta \right) + \left(q_\theta \frac{\cos \theta}{\sin \theta} \right) - (2c_4 \cos \theta n_{\phi 2}) \right] c_{11}$$

$$\begin{aligned} n_{\Phi 3} &= c_4 \left[n_{\phi 3} \left(-1 + \frac{1}{r^2} \right) - x_3 x_7 \left(\frac{1}{n_{r3}(x_3, x_4, r)} - \frac{n_3(x_3, x_4)}{n_{r2}(x_3, x_4, r)} \right) + x_5 x_8 \left(\frac{1}{n_{r3}(x_5, x_6, r)} \right. \right. \\ &\quad \left. \left. - \frac{n_3(x_5, x_6)}{n_{r2}(x_5, x_6, r)} \right) \right] - c_6 x_2 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r - \sin \theta)}{(-\cos \theta)^2 + (r - \sin \theta)^2} d\theta \end{aligned}$$

and for the bottom surface,

$$\begin{aligned} \frac{\partial \Phi_\theta}{\partial t} &= (v_{91} + v_{97}h_1)(g_1 + g_2p_7) + (v_{92} + v_{97}h_2)(g_3 + g_4p_7) + (v_{93} + v_{97}h_3)(g_5 + g_6p_8) \\ &+ (v_{94} + v_{97}h_4)(g_7 + g_8p_8) + (v_{95} + v_{97}h_5)(p_7) + (v_{96} + v_{97}h_6)(p_8) + (v_{98} + v_{97}h_7) \end{aligned}$$

$$\Phi_x = \left[\left(\int_{-10r}^0 n_{\Phi 4} \cdot dr + \int_{\pi}^{\theta k} q_\theta \cdot d\theta \right) + \left(q_\theta \frac{\cos \theta}{\sin \theta} \right) - (2c_4 \cos \theta n_{\phi 2}) \right] c_{11}$$

$$\begin{aligned} n_{\Phi 4} &= c_4 \left\{ -n_{\phi 3} - \frac{(2r)n_{\phi 2} + (1-r^2)n_{\phi 3}}{(1+r^2)^2} + x_7 \left[\frac{-n_{13}(x_3, x_4)}{n_{r7}(x_3, x_4, r)} + \frac{n_3(x_3, x_4)n_{12}(x_3, x_4)}{n_{r8}(x_3, x_4, r)} \right] \right. \\ &\quad \left. + x_8 \left[\frac{n_{13}(x_5, x_6)}{n_{r7}(x_5, x_6, r)} + \frac{-n_3(x_5, x_6)n_{12}(x_5, x_6)}{n_{r8}(x_5, x_6, r)} \right] \right\} - c_6 x_2 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r + \sin \theta)}{(1 + \cos \theta)^2 + (r + \sin \theta)^2} d\theta \end{aligned}$$

and for all the surfaces, the tangential velocity becomes as:

$$q_\theta = c_8 x_2 (1 - 2 \sin \theta^2) - c_4 \{-2 \cos(\phi - \theta) + x_7 \left[\frac{1 - n_{13}(x_3, x_4, \theta)}{n_{11}(x_3, x_4, \theta)} \right]$$

$$\begin{aligned}
& - \frac{n_3(x_3, x_4)[n_3(x_3, x_4) - n_{t13}(x_3, x_4, \theta)]}{n_{t2}(x_3, x_4, \theta)}] - x_8 \left[\frac{1 - n_{t13}(x_5, x_6, \theta)}{n_{t1}(x_5, x_6, \theta)} \right. \\
& \left. - \frac{n_3(x_5, x_6)[n_3(x_5, x_6) - n_{t13}(x_5, x_6, \theta)]}{n_{t2}(x_5, x_6, \theta)} \right] \}
\end{aligned}$$

4.3.3 Intermediate functions

After simplification, f_i and k_i functions become:

$$\begin{aligned}
f_1 &= \{4x_3 n_{11}^2(x_3, x_4) n_{41} - n_6(x_3, x_4) n_5(x_3, x_4) [2x_3 n_{41} \\
&+ n_{11}(x_3, x_4)(2x_5 x_3 - x_5^2 - 1 - x_6^2)]\} [n_{14}(x_5, x_6) n_{\phi 1} - x_5 c_3 x_2] \\
f_2 &= 2x_4 [2n_{14}(x_3, x_4) n_{11}(x_3, x_4) n_{41} - n_6(x_3, x_4) n_5(x_3, x_4) (n_{41} + n_{11}(x_3, x_4) x_5)] \\
&\times [n_{14}(x_5, x_6) n_{\phi 1} - x_5 c_3 x_2] \\
f_3 &= n_6(x_3, x_4) n_5(x_3, x_4) n_{11}(x_3, x_4) [(2x_5 n_{\phi 1} - c_3 x_2) n_{41} \\
&- [n_{14}(x_5, x_6) n_{\phi 1} - x_5 c_3 x_2] (x_3^2 - 2x_3 x_5 + 1 + x_4^2)] \\
f_4 &= 2x_6 n_6(x_3, x_4) n_5(x_3, x_4) n_{11}(x_3, x_4) [n_{\phi 1} n_{41} - [n_{14}(x_5, x_6) n_{\phi 1} - x_5 c_3 x_2] (-x_3)] \\
f_6 &= n_6(x_5, x_6) n_5(x_5, x_6) n_{11}(x_5, x_6) [(2x_3 n_{\phi 1} - c_3 x_2) n_{41} \\
&- (n_{14}(x_3, x_4) n_{\phi 1} - x_3 c_3 x_2) (2x_3 c - x_5^2 - 1 - x_6^2)] \\
f_7 &= 2x_4 n_6(x_5, x_6) n_5(x_5, x_6) n_{11}(x_5, x_6) [n_{\phi 1} n_{41} - (n_{14}(x_3, x_4) n_{\phi 1} - x_3 c_3 x_2) x_5] \\
f_8 &= \{4x_5 n_{11}^2(x_5, x_6) n_{41} - n_6(x_5, x_6) n_5(x_5, x_6) [2x_5 n_{41} \\
&+ n_{11}(x_5, x_6)(x_3^2 - 2x_3 x_5 + 1 + x_4^2)]\} [n_{14}(x_3, x_4) n_{\phi 1} - x_3 c_3 x_2] \\
f_9 &= 2x_6 [2n_{14}(x_5, x_6) n_{11}(x_5, x_6) n_{41} - n_6(x_5, x_6) n_5(x_5, x_6) (n_{41} - n_{11}(x_5, x_6) x_3)]
\end{aligned}$$

$$\times [n_{14}(x_3, x_4)n_{\phi 1} - x_3 c_3 x_2]$$

$$k_1 = n_6(x_3, x_4)n_5(x_3, x_4) [n_{14}(x_5, x_6) \cos(x_1 + \frac{\pi}{2})x_2] n_{11}(x_3, x_4)n_{41}$$

$$k_2 = n_6(x_3, x_4)n_5(x_3, x_4) (-x_3 c_3) n_{11}(x_3, x_4)n_{41}$$

$$k_3 = n_6(x_5, x_6)n_5(x_5, x_6) [n_{14}(x_3, x_4) \cos(x_1 + \frac{\pi}{2})x_2] n_{11}(x_5, x_6)n_{41}$$

$$k_4 = n_6(x_5, x_6)n_5(x_5, x_6) (-x_3 c_3) n_{11}(x_5, x_6)n_{41}$$

And after simplification, g_i functions become:

$$\begin{aligned} g_1 = & \frac{1}{n_{52}(x_3, x_4)} \{-c_1[n_{51}(x_3, x_4) n_{54} + n_1(x_3, x_4) n_{53}(x_3, x_4)] \\ & + c_2 \left[\frac{m_{g1}[n_{51}^2(x_3, x_4) + n_1^2(x_3, x_4)]}{n_{52}(x_3, x_4)} \right. \\ & \left. + (x_7 \sin \alpha) \frac{x_3 n_1(x_3, x_4) n_7(x_3, x_4) - x_4 n_8(x_3, x_4) n_{51}(x_3, x_4)}{n_5^2(x_3, x_4) n_6^2(x_3, x_4)} \right] \} \\ & - x_4 x_2 \frac{n_9(x_3, x_4)}{n_5(x_3, x_4) n_6(x_3, x_4)} \end{aligned}$$

$$g_2 = -\frac{1}{x_7} \frac{1}{n_{52}(x_3, x_4)} [n_{51}(x_3, x_4) (n_{54} + 1) + n_1(x_3, x_4) n_{53}(x_3, x_4)]$$

$$\begin{aligned} g_3 = & \frac{1}{n_{52}} \{-c_1[n_{51}(x_3, x_4) n_{53}(x_3, x_4) \\ & - n_1(x_3, x_4) n_{54}] + c_2 \left[\frac{-n_{g1}(n_{51}^2(x_3, x_4) + n_1^2(x_3, x_4))}{n_{52}(x_3, x_4)} \right. \\ & \left. + (x_7 \sin \alpha) \frac{x_3 n_{51}(x_3, x_4) n_7(x_3, x_4) + x_4 n_8(x_3, x_4) n_1(x_3, x_4)}{n_5^2(x_3, x_4) n_6^2(x_3, x_4)} \right] \} \\ & + x_3 x_2 \frac{n_{10}(x_3, x_4)}{n_5(x_3, x_4) n_6(x_3, x_4)} \end{aligned}$$

$$g_4 = -\frac{1}{x_7} \frac{1}{n_{52}(x_3, x_4)} [n_{51}(x_3, x_4) n_{53}(x_3, x_4) - n_1(x_3, x_4) (n_{54} + 1)]$$

where if we set

$$\begin{aligned} f_{n_0}(\xi_1, \xi_2, \xi_8, \xi_9) &= -c_5 \xi_8 \int_0^{2\pi} \frac{\sin \theta \cos \theta \xi_9}{(\xi_1 - \cos \theta)^2 + (\xi_2 - \sin \theta)^2} d\theta \\ f_{n_1}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_{71}, \xi_{72}, n_{y_1}, f_{n_0}(\xi_1, \xi_2, \xi_8, \xi_9)) \\ &= c_9 [-\xi_{71} - \frac{(\xi_1^2 - \xi_2^2)(-\xi_{71}) + (2\xi_1 \xi_2) \xi_{72}}{n_3^2}] + c_9 n_{y_1} \left\{ \frac{-\xi_5 \xi_2}{\xi_1^2 + \xi_2^2 - 1} \right. \\ &\quad \left. - \xi_6 \left[\frac{(\xi_2 - \xi_4)}{(\xi_1 - \xi_3)^2 + (\xi_2 - \xi_4)^2} - \frac{\xi_2 (\xi_3^2 + \xi_4^2) - \xi_4}{(\xi_1^2 + \xi_2^2)(\xi_3^2 + \xi_4^2) - 2(\xi_1 \xi_3 + \xi_2 \xi_4) + 1} \right] \right\} \\ &\quad + f_{n_0}(\xi_1, \xi_2, \xi_8, \xi_9) \end{aligned}$$

then

$$\begin{aligned} m_{g_1} &= f_{n_1}(x_3, x_4, x_5, x_6, x_7, x_8, \sin x_1, \cos x_1, 1, f_{n_0}(x_3, x_4, x_2, (x_3 - \cos \theta))) \\ n_{g_1} &= f_{n_1}(x_4, x_3, x_6, x_5, x_7, x_8, \cos x_1, \sin x_1, 1, f_{n_0}(x_1, x_2, x_2, (x_4 - \sin \theta))) \end{aligned}$$

and

$$\begin{aligned} g_5 &= \frac{1}{n_{52}(x_5, x_6)} \{-c_1 [n_{51}(x_5, x_6) n_{64} \\ &\quad + n_1(x_5, x_6) n_{53}(x_5, x_6)] + c_2 \left[\frac{m_{g_2} [n_{51}^2(x_5, x_6) + n_1^2(x_5, x_6)]}{n_{52}(x_5, x_6)} \right. \\ &\quad \left. - (x_8 \sin \alpha) \left(\frac{x_5 n_1(x_5, x_6) n_7(x_5, x_6) - x_6 n_8(x_5, x_6) n_{51}(x_5, x_6)}{n_5^2(x_5, x_6) n_6^2(x_5, x_6)} \right) \right\} \\ &\quad - x_6 x_2 \frac{n_9(x_5, x_6)}{n_5(x_5, x_6) n_6(x_5, x_6)} \\ g_6 &= -\frac{1}{x_8} \frac{1}{n_{52}(x_5, x_6)} [n_{51}(x_5, x_6) (n_{64} - 1) + n_1(x_5, x_6) n_{53}(x_5, x_6)] \end{aligned}$$

$$\begin{aligned}
g_7 = & \frac{1}{n_{52}(x_5, x_6)} \{ -c_1 (n_{51}(x_5, x_6) n_{53}(x_5, x_6) \\
& - n_1(x_5, x_6) n_{64}) + c_2 \left[\frac{-n_{g2} [n_{51}^2(x_5, x_6) + n_1^2(x_5, x_6)]}{n_{52}(x_5, x_6)} \right. \\
& \left. - (x_8 \sin \alpha) \frac{x_5 n_7(x_5, x_6) n_{51}(x_5, x_6) + x_6 n_8(x_5, x_6) n_1(x_5, x_6)}{n_5^2(x_5, x_6) n_6^2(x_5, x_6)} \right] \} \\
& + x_5 x_2 \frac{n_{10}(x_5, x_6)}{n_5(x_5, x_6) n_6(x_5, x_6)}
\end{aligned}$$

$$g_8 = -\frac{1}{x_8} \frac{1}{n_{52}(x_5, x_6)} [n_{51}(x_5, x_6) n_{53}(x_5, x_6) - n_1(x_5, x_6) (n_{64} - 1)]$$

where

$$m_{g1} = f_{n1}(x_5, x_6, x_3, x_4, x_8, x_7, \sin x_1, \cos x_1, -1, f_{n0}(x_5, x_6, x_2, (x_5 - \cos \theta)))$$

$$n_{g1} = f_{n1}(x_6, x_5, x_4, x_3, x_8, x_7, \cos x_1, \sin x_1, -1, f_{n0}(x_5, x_6, x_2, (x_6 - \sin \theta)))$$

and after simplification, h_k functions become:

$$h_k = \frac{w_{3k}}{c_{10} - w_{37}}, \quad k = 1, \dots, 6$$

$$h_7 = \frac{w_{38}}{c_{10} - w_{37}}$$

where for $k=1, \dots, 7$:

$$\begin{aligned}
w_{3k} = & \frac{x_9^3}{6c_r^3} \int_0^{\frac{\pi}{2}} q_{8\theta} \left[\int_r^{10r} v_k \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} v_{4k} \cdot d\theta \right] d\sigma \\
& + \frac{x_{10}^3}{6c_r^3} \int_{\frac{\pi}{2}}^{\pi} q_{8\theta} \left[\int_r^{10r} v_k \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} v_{4k} \cdot d\theta \right] d\sigma \\
& + \frac{1}{6} \int_0^{\pi} q_{8\theta} \left[\int_{-10r}^0 v_{1k} \cdot dr + \int_{\pi}^{\sigma} v_{4k} \cdot d\theta \right] d\sigma
\end{aligned}$$

and

$$\begin{aligned}
w_{38} = & \frac{x_9^3}{6c_r^3} \left\{ \int_0^{\frac{\pi}{2}} q_{8\theta} \left[\int_r^{10r} v_8 \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} v_{48} \cdot d\theta \right] d\sigma + \int_0^{\frac{\pi}{2}} q_{7\theta} d\theta \right\} \\
& + \frac{x_{10}^3}{6c_r^3} \left\{ \int_{\frac{\pi}{2}}^{\pi} q_{8\theta} \left[\int_r^{10r} v_8 \cdot dr + \int_{\frac{\pi}{2}}^{\sigma} v_{48} \cdot d\theta \right] d\sigma + \int_{\frac{\pi}{2}}^{\pi} q_{7\theta} d\theta \right\} \\
& + \frac{1}{6} \left\{ \int_0^{\pi} q_{8\theta} \left[\int_{-10r}^0 v_{18} \cdot dr + \int_{\pi}^{\sigma} v_{48} \cdot d\theta \right] d\sigma + \int_0^{\pi} q_{7\theta} d\theta \right\} \\
& + \frac{1}{2} \frac{x_9^3 - x_{10}^3}{6c_r^3} C_{p-vb}
\end{aligned}$$

where

$$q_{7\theta} = (q_{5\theta}) \sin \theta \cos \theta$$

$$q_{8\theta} = c_7 \sin \theta \cos \theta$$

$$C_{p-vb} = C_{p0} \frac{\sin(\alpha_\phi) \cos(\alpha_\phi)}{\sin \alpha \cos \alpha}, \quad C_{p0} = -1.1$$

$$\alpha_\phi(\phi) = \tan^{-1}(\tan \alpha n_{\phi 3})$$

and

$$\begin{aligned}
v_1 = & c_4 x_7 \left\{ \frac{-n_{r3}(x_3, x_4, r) + 2x_3^2}{n_{r3}^2(x_3, x_4, r)} - \frac{-3x_3^2 - x_4^2}{n_{r2}(x_3, x_4, r)} - \frac{n_3(x_3, x_4)[2x_3^2(1 + 2n_{r1}(x_3, x_4, r)r)]}{n_{r2}^2(x_3, x_4, r)} \right\} \\
v_2 = & c_4 x_7 \left\{ \frac{-x_3[2(r - x_4)]}{n_{r3}^2(x_3, x_4, r)} - \frac{-n_1(x_3, x_4)}{n_{r2}(x_3, x_4, r)} - \frac{n_3(x_3, x_4)x_3[2n_{r1}(x_3, x_4, r)(2x_4r - 1)]}{n_{r2}^2(x_3, x_4, r)} \right\} \\
v_3 = & c_4 x_8 \left\{ -\frac{-n_{r3}(x_5, x_6, r) + 2x_5^2}{n_{r3}^2(x_5, x_6, r)} + \frac{-3x_5^2 - x_6^2}{n_{r2}(x_5, x_6, r)} \right. \\
& \left. + \frac{n_3(x_5, x_6)(2x_5^2)[1 + 2n_{r1}(x_5, x_6, r)r]}{n_{r2}^2(x_5, x_6, r)} \right\}
\end{aligned}$$

$$v_4 = c_4 x_8 \left\{ -\frac{x_5[2(r-x_6)]}{n_{r3}^2(x_5, x_6, r)} - \frac{n_1(x_5, x_6)}{n_{r2}(x_5, x_6, r)} + \frac{n_3(x_5, x_6)x_5[2n_{r1}(x_5, x_6, r)(2x_6r-1)]}{n_{r2}^2(x_5, x_6, r)} \right\}$$

$$v_5 = c_4 \left[-x_3 \left(\frac{1}{n_{r3}(x_3, x_4, r)} - \frac{n_3(x_3, x_4)}{n_{r2}(x_3, x_4, r)} \right) \right]$$

$$v_6 = c_4 \left[-x_5 \left(-\frac{1}{n_{r3}(x_5, x_6, r)} + \frac{n_3(x_5, x_6)}{n_{r2}(x_5, x_6, r)} \right) \right]$$

$$v_7 = -c_6 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r - \sin \theta)}{(-\cos \theta)^2 + (r - \sin \theta)^2} d\theta$$

$$v_8 = c_4 [n_{\phi 2} x_2 (1 - \frac{1}{r^2})]$$

and

$$v_{11} = c_4 x_7 \left\{ -\frac{n_{r7}(x_3, x_4, r) - 2n_{13}^2(x_3, x_4)}{n_{r7}^2(x_3, x_4, r)} + \frac{4n_3(x_3, x_4)x_3 + 2x_3^2 + n_3(x_3, x_4)}{n_{r8}(x_3, x_4, r)} \right. \\ \left. - \frac{2n_3(x_3, x_4)n_{12}(x_3, x_4)[n_{12}(x_3, x_4)(2x_3+1) + (n_3(x_3, x_4)r + x_4)(2x_3r)]}{n_{r8}^2(x_3, x_4, r)} \right\}$$

$$v_{12} = c_4 x_7 \left\{ -\frac{-n_{13}(x_3, x_4)[2(r+x_4)]}{n_{r7}^2(x_3, x_4, r)} + \frac{2x_4(2n_3(x_3, x_4) + x_3)}{n_{r8}(x_3, x_4, r)} \right. \\ \left. - \frac{2n_3(x_3, x_4)n_{12}(x_3, x_4)[2n_{12}(x_3, x_4)x_4 + (n_3(x_3, x_4)r + x_4)(2x_4r+1)]}{n_{r8}^2(x_3, x_4, r)} \right\}$$

$$v_{13} = c_4 x_8 \left\{ \frac{n_{r7}(x_5, x_6, r) - 2n_{13}^2(x_5, x_6)}{n_{r7}^2(x_5, x_6, r)} - \frac{4n_3(x_5, x_6)x_5 + 4x_5^2 + n_3(x_5, x_6)}{n_{r8}(x_5, x_6, r)} \right. \\ \left. + \frac{2n_3(x_5, x_6)n_{12}(x_5, x_6)[n_{12}(x_5, x_6)(2x_5+1) + (n_{23}r + x_6)(2x_5r)]}{n_{r8}^2(x_5, x_6, r)} \right\}$$

$$v_{14} = c_4 x_8 \left\{ \frac{-n_{13}(x_5, x_6)[2(r+x_6)]}{n_{r7}^2(x_5, x_6, r)} - \frac{2x_6(2n_{23} + x_5)}{n_{r8}(x_5, x_6, r)} \right\}$$

$$+ \frac{2n_3(x_5, x_6)n_{12}[(x_5, x_6)2n_{12}(x_5, x_6)x_6 + (n_{23}r + x_6)(2x_6r + 1)]}{n_{r8}^2(x_5, x_6, r)}\}$$

$$v_{15} = c_4 \left\{ -\frac{n_{13}(x_3, x_4)}{n_{r7}(x_3, x_4, r)} + \frac{n_3(x_3, x_4)n_{12}(x_3, x_4)}{n_{r8}(x_3, x_4, r)} \right\}$$

$$v_{16} = c_4 \left\{ \frac{n_{13}(x_5, x_6)}{n_{r7}(x_5, x_6, r)} - \frac{n_3(x_5, x_6)n_{12}(x_5, x_6)}{n_{r8}(x_5, x_6, r)} \right\}$$

$$v_{17} = -c_6 \int_0^{2\pi} \frac{\sin \theta \cos \theta (r + \sin \theta)}{(1 + \cos \theta)^2 + (r + \sin \theta)^2} d\theta$$

$$v_{18} = c_4 x_2 \left\{ n_{\phi 2} - \frac{2rn_{\phi 3} - (1 - r^2)n_{\phi 2}}{(1 + r^2)^2} \right\}$$

Now let

$$\begin{aligned} & f_{n2}(y_1, y_2, y_3, y_{4s}, y_{4c}, n_{y1}) \\ &= -n_{y1} c_4 \xi_3 \left\{ \frac{-[(\xi_{4c} - \xi_1)^2 + (\xi_{4s} - \xi_2)^2] \xi_{4c} + 2[(1 - (\xi_2 \xi_{4s} + \xi_1 \xi_{4c}))](\xi_{4c} - \xi_1)}{\{(\xi_{4c} - \xi_1)^2 + (\xi_{4s} - \xi_2)^2\}^2} \right. \\ & \quad \frac{[4\xi_1(\xi_1^2 + \xi_2^2) - 2\xi_1 \xi_2 \xi_{4s} - (3\xi_1^2 + \xi_2^2)\xi_{4c}]}{[(\xi_1^2 + \xi_2^2)\xi_{4c} - \xi_1]^2 + [(\xi_1^2 + \xi_2^2)\xi_{4s} - \xi_2]^2} \\ & \quad \left. + \frac{2(\xi_1^2 + \xi_2^2)[(\xi_1^2 + \xi_2^2) - (\xi_2 \xi_{4s} + \xi_1 \xi_{4c})] \{[(\xi_1^2 + \xi_2^2)\xi_{4c} - \xi_1][2\xi_1 \xi_{4c} - 1] + [(\xi_1^2 + \xi_2^2)\xi_{4s} - \xi_2][2\xi_1 \xi_{4s}]\}}{[(\xi_1^2 + \xi_2^2)\xi_{4c} - \xi_1]^2 + [(\xi_1^2 + \xi_2^2)\xi_{4s} - \xi_2]^2} \right\} \end{aligned}$$

then

$$v_{41} = f_{n2}(x_3, x_4, x_7, \sin \theta, \cos \theta, 1)$$

$$v_{42} = f_{n2}(x_4, x_3, x_7, \cos \theta, \sin \theta, 1)$$

$$v_{43} = f_{n2}(x_5, x_6, x_8, \sin \theta, \cos \theta, -1)$$

$$v_{44} = f_{n2}(x_6, x_5, x_8, \cos \theta, \sin \theta, -1)$$

$$v_{45} = -c_4 \left\{ \frac{1 - n_{t13}(x_3, x_4, \theta)}{n_{t1}(x_3, x_4, \theta)} - \frac{n_3(x_3, x_4)[n_3(x_3, x_4) - n_{t13}(x_3, x_4, \theta)]}{n_{t2}(x_3, x_4, \theta)} \right\}$$

$$v_{46} = -c_4 \left\{ -\frac{1 - n_{t13}(x_5, x_6, \theta)}{n_{t1}(x_5, x_6, \theta)} + \frac{n_3(x_5, x_6)[n_3(x_5, x_6) - n_{t13}(x_5, x_6, \theta)]}{n_{t2}(x_5, x_6, \theta)} \right\}$$

$$v_{47} = c_8 (1 - 2 \sin^2 \theta)$$

$$v_{48} = -2c_4 x_2 \sin(x_1 - \theta)$$

The equations are greatly simplified by applying the intermediate variables and functions. After simplification, many equations are found to have the same terms as some other equations do except their signs are opposite of each other. This will greatly help control design. The simplified equations will greatly reduce the working load for control designs.

Now that the system can be completely expressed by all the state equations and output equations derived and simplified in this Chapter, it is ready to validate the new model by comparing the computation results to that of the original Arena and Myatt models.

Chapter 5

Model Validation- Results and Analysis

After we have derived the state equations, the system can be implemented in Simulink to simulate the dynamic wing rocking. In this Chapter, the logic and structure of the model implemented in Simulink are described first. Then the simulation results of the new model are compared with that of the original Arena and Myatt models to verify the new model.

The validation of the current model can be accomplished by comparing the simulation results from the current model with that from Arena and Myatt models. Because the current model evolves from Arena and Myatt models applying the same assumptions, theories and flow dynamic equations, the only difference is that from the same basis the current model is developed through a series of mathematical transformation without adding any additional theories or equations. Therefore if Arena and Myatt models are correct, the current model is correct. And Arena and Myatt models have been proven valid by experiment results having good agreement with their mathematical model simulation results in delta wing flow dynamics [1][2]. Therefore based on these facts and reasoning, the current model will be validated by comparing the simulation results of the current model with that of Arena and Myatt models.

5.1 SIMULINK diagram

The new model is implemented by SIMULINK. Fig. 5.1 shows the main structure of the model. It consists of two blocks: the Flow Dynamics block that represents the flow field dynamics and the Wing Dynamics block which represents the rigid-body wing motion equations. On the left side of the Flow Dynamics block are the parameters and inputs that can be adjusted including the angle of attack α and the sweep-angle λ . The control input is the jet or other flow control effector for both the left and right side of the wing. The control input is assumed to affect the vortex strength and the vortex breakdown location. On the right side of the Wing Dynamics block is the output including the roll angle ϕ , the roll rate $\dot{\phi}$ and the surface air pressure C_p . The roll angle and roll angle rate are not only the system output, but also function as an internal feedback driving the Flow Dynamics block.

1. Flow dynamics and Wing Rigid body dynamics are 2 subblocks consisting the system
2. gamal, gamar, and a, b, c, d , all 6 states, are using integrators , their state equations are inside one subfunction block, with an integrator following it (05/10/04)

Figure 5.1 The SIMULINK diagram for the dynamic state system

In the Wing rigid-body Dynamics block as shown in Fig. 5.2, the roll acceleration comes out as the result of the equation of motion of the rigid-body wing. By integration over time, the roll angle rate and the roll angle are derived.

Figure 5.2 The wing rigid-body dynamic sub diagram

In the Flow Dynamics block, as shown in Fig. 5.3, the vortex breakdown location and the air pressure before and aft of the vortex breakdown points are calculated individually in separate blocks. The calculation for the total roll moment of the wing requires all these results together. All the state equations except the vortex breakdown location equations are implemented in one function block. After the integration of the state variables over time, the vortex strength and positions function as an internal feedback to the state equation block as well.

5.2 Simulation result

Arena model mainly focuses on wing rock and Myatt's related research is on vortex breakdown location, therefore validating the current model is implemented by focusing on the comparison between this model and Arena and Myatt models on wing rock and vortex breakdown location movement. And since of all the state variables in the system, the roll angle and roll rate are among the most important states in flying motion, their variations during the flying motion is focused on in the comparison.

Fig. 5.4.1 shows the vortex breakdown location movement along the wing longitudinal axis in a harmonic roll motion. Fig. 5.4.1.A shows the location of the starboard wing vortex. When at the zero roll angle and the wing is still, the vortex breakdown location is at 0.65 of the wing root chord. After the wing starts to roll, the movement of the vortex breakdown point stabilizes on the outer curve as shown in Fig. 5.4.1.A. If the delta wing rolls in the positive direction, the vortex breakdown point moves in the direction which is from the trailing edge toward the wing apex. The vortex breakdown point moves in the opposite direction when the wing rolls towards negative roll angle. Fig. 5.4.1.B shows for the left wing vortex which is similar to the right vortex, except its movement direction is opposite of Fig. 5.4.1.A.

Figures 5.4.1 result shows exactly the same as that of Figures 5.4.2 that are derived by Matlab programming based on Myatt model. Both use the fixed-point-step calculation adopting 0.01 as the step the current model proves correct depicting vortex breakdown location movement along the root chord axis with respect to Myatt model.

A) Right vortex

B) Left vortex

Figure 5.4.1 Vortex breakdown location movement in a harmonic motion of delta wing (current model). The roll angle of 33° and frequency of 0.02 Hz (x: roll angle; y: vortex breakdown location)

A) Right vortex

B) Left vortex

Figure 5.4.2 Vortex breakdown location movement in a harmonic motion of delta wing (Myatt model). The roll angle of 33° and frequency of 0.02 Hz ((x: roll angle; y: vortex breakdown location)

Fig. 5.5 from the SIMULINK simulation of the new model and Fig. 5.6 from the original model program show that without vortex breakdown, if the wing initially is released from a very small roll angle such as -0.5° , the wing will roll back and forth to gradually larger roll angles until it reaches a stabilized oscillation. The two pictures are very similar to each other except that the SIMULINK model has about 5% faster rolling frequency rpm and about 5% smaller oscillation amplitude of Arena model result. This

may be attributed to that SIMULINK uses variable time step numerical integration while the original program uses fixed time step.

Figure 5.5 Roll angle vs. time for the wing rock phenomenon (the current model). Initial roll angle: -0.5° , angle of attack: 20°

Figure 5.6 Roll angle vs. time for the wing rock phenomenon (Arena model). Initial roll angle: -0.5° , angle of attack: 20°

Figs. 5.7 form the new model and Fig. 5.8 from the original model both show that without vortex breakdown, after the wing is released from a large roll angle such as 85° , the wing will also reach a stabilized oscillation with the same roll amplitude and frequency as in the previous case except that the stabilizing time is much shorter than in the previous case. That is because in this situation, the initial vortex strength and position are much closer to the stabilized oscillation condition than in the previous case where the initial roll angle is very small.

Figure 5.7 Roll angle vs. time for the wing rock phenomenon (the current model). Initial roll angle 85° , angle of attack 20°

Figure 5.8 Roll moment vs. time for the wing rock phenomenon (Arena model). Initial roll angle of 85° , angle of attack: 20°

Figs. 5.9 and 5.10 for the initial roll angle of -0.5° and 85° respectively show the current and the Arena model show that the limit cycle consisting of the roll angle as 'x' axis and the roll rate as 'y' axis. Both of them start from a point, i.e. (85, 0), that is far from the limit cycle, and gradually merge onto the stable limit cycle as time goes by.

A) The current model

B) The Arena model

Figure 5.9 The limit cycle of roll angle and roll angle rate, initial roll angle -0.5°

A) The current model

B) The Arena model

Figure 5.10 The limit cycle of roll angle and roll angle rate, initial roll angle 85°

Fig. 5.11 shows that with vortex breakdown if the wing is released from a small roll angle such as -0.5° , it will roll down to larger negative roll angles while oscillating with amplitude that is gradually getting smaller until finally the wing is stabilized at -8.6° as

time goes by. This is the effect of both the vortex impacts on the area before the vortex breakdown point and the surface air pressure aft of the vortex breakdown point. The vortex breakdown point moves along the wing longitudinal axis as the wing rolls, the vortex breakdown point decides the border between the vortex area and the aft-of-vortex breakdown area, and the roll moment comes from the impacts of both of them. Therefore the vortex breakdown effect has a great influence on the wing rolling.

Figure 5.11 Roll angle vs. time with vortex breakdown. Initial roll angle: -0.5° , angle of attack: 20° , sweep angle: 80°

Through all the comparison and analysis, the current model proves to have correctly depicted the flow dynamics around delta wing, so that this model has the same simulation results in every aspect as that from Arena and Myatt models. The calculation results are accurate compared to that of Arena and Myatt models, though there are small differences such as the current model has 5% lower stable roll angle and 5% higher roll rate. These discrepancies come from the fact that the current model in Simulink uses automatic

adjusted sampling-time step, while Arena and Myatt models in Matlab program use fixed time step in mathematical simulation.

On the whole, since the current model have proved being correct depicting the flow dynamics on the delta wing in accordance with Arena and Myatt models, and it now consists a set of standard low-order non-linear state-space equations on which control methods can be relatively more easily applied, the goal and objectives described at the end of Chapter 2 has been accomplished.

Chapter 6

Summary and Conclusions

In this thesis, a unique low-order, nonlinear state equation model has been developed. In this model, there are 10 independent variables and they are defined as the 10 states of this nonlinear system. This model evolved from two theoretical models on air flow mechanics above delta wing including the wing rock model of Arena [1] and the vortex breakdown model of Myatt [2], through a series of transformations, differentiations, and simplifications. Simulation results of the new model have been shown to have good agreement with that of the original models.

This model is important in that it not only provides an analytical insight in the delta wing flow field characteristics, but also facilitates closed-loop flow control system design and simulation. Because this new model has the same input and output definition as the experimental delta wing model does, flow controllers designed and verified with this model can be applied to an experimental test article or an aircraft. In addition, since applying jet actuators can implement the system control input including vortices position movement, vortices strength rate and the vortex breakdown location movement, attitude control can be accomplished without flaps. Therefore controlling the experimental model to improve flying performance can be achieved by applying a variety of control methods to the computational model. Thus the computational model and the experimental model complement each other.

In future studies, the new model may be further simplified by omitting some insignificant components in each equation, which will greatly simplify the applied control

mechanism calculations that involve 10 states. And among the system outputs, the pressure sensor's relative positions along the root cord in the new computational model should be adjusted to have the same relative position of the sensor in the experimental model, so that outputs from both models can be directly compared and verified.

A variety of nonlinear control methods can be applied to this model such as Trajectory Linearization, Gain Scheduling and Sliding Mode Control methods. Since the system control inputs affect the vortex strength, y and z vortex position, and vortex breakdown location, in the future actuators on the top surface of delta wing can efficiently implement control input on vortex parameters by being adjusted to appropriate angle and eject appropriate amount of air as a function of time. But before that some preparation work needs to be done such as the movement of vortex position and breakdown location during wing rock need to be tracked, and the control actuator effectiveness on vortex strength and breakdown location needs to be established either theoretically or experimentally.

Appendix

Nomenclature

A. General symbols

c_r	= root chord
C_p	= pressure coefficient
c_l	= roll moment coefficient in one cross flow section
c_{l_int}	= total roll moment coefficient
b	= trailing edge span
I_{xx}	= roll moment of inertia
q_θ	= tangential velocity on the surface of the circle
S	= wing area
U_∞	= free-stream velocity
W	= complex velocity in the circle plane
x	= axial distance from wing apex in physical plane
x_{vb}	= vortex breakdown location
x_{vbs}	= static vortex breakdown location
x_{vblag}	= lag of vortex breakdown location
t'	= nondimensional time, $tU_\infty/(b/2)$
ϕ	= roll angle
$\dot{\phi}$	= roll angle velocity
$\ddot{\phi}$	= roll angle acceleration
α	= angle of attack
λ	= sweep angle
σ	= complex coordinate in the physical plane
ζ	= complex coordinate in the circle plane
ζ_l	= complex coordinate of left vortex
ζ_r	= complex coordinate of right vortex
γ	= vortex strength or circulation in conical coordinates
γ_l	= left vortex strength
γ_r	= right vortex strength
ε	= apex half-angle, rad
θ	= angular coordinate in the circle plane
Γ	= vortex strength or circulation
Φ	= total velocity potential

B. Intermediate variables

$$c_1 = \frac{2 \cos \alpha}{\frac{x}{c_r}}$$

$$c_2 = \frac{2}{\frac{x}{c_r} \tan \varepsilon}$$

$$c_3 = \frac{x \tan \varepsilon}{c_r \sin \alpha}$$

$$c_4 = U_\infty \sin \alpha$$

$$c_5 = \frac{2x \tan \varepsilon}{c_r \pi}$$

$$c_6 = U_\infty \frac{2x \tan \varepsilon}{c_r \pi}$$

$$c_7 = -\frac{x \tan \varepsilon}{c_r U_\infty}$$

$$c_8 = U_\infty x_{cr} \tan \varepsilon$$

$$c_9 = \sin \alpha$$

$$c_{10} = \frac{I_{xx}}{\rho c_r^5 \tan^2 \varepsilon}$$

$$c_{11} = \frac{\tan \varepsilon}{2}$$

$$n_1 = 2ab$$

$$n_2 = a^2 - b^2$$

$$n_3 = a^2 + b^2$$

$$n_4 = a^2 - b^2 - 1 = n_2 - 1$$

$$n_5 = (a+1)^2 + b^2$$

$$n_6 = (a-1)^2 + b^2$$

$$n_7 = a^4 - 2a^2b^2 - 3b^4 - 2a^2 - 2b^2 + 1$$

$$n_8 = 3a^4 + 2a^2b^2 - b^4 - 2a^2 - 2b^2 - 1$$

$$n_9 = a^4 + 2a^2b^2 + b^4 - 4a^2 - 1 = n_3^2 - 4a^2 - 1$$

$$n_{10} = a^4 + 2a^2b^2 + b^4 + 4b^2 - 1 = n_3^2 + 4b^2 - 1$$

$$n_{11} = a^2 + b^2 - 1 = n_3 - 1$$

$$n_{12} = a^2 + b^2 + a = n_3 + a$$

$$n_{21} = 2cd$$

$$n_{22} = c^2 - d^2$$

$$\begin{aligned}
n_{23} &= c^2 + d^2 \\
n_{24} &= c^2 - d^2 - 1 = n_{22} - 1 \\
n_{25} &= (c+1)^2 + d^2 \\
n_{26} &= (c-1)^2 + d^2 \\
n_{27} &= c^4 - 2c^2d^2 - 3d^4 - 2c^2 - 2d^2 + 1 \\
n_{28} &= 3c^4 + 2c^2d^2 - d^4 - 2c^2 - 2d^2 - 1 \\
n_{29} &= c^4 + 2c^2d^2 + d^4 - 4c^2 - 1 = n_{23}^2 - 4c^2 - 1 \\
n_{30} &= c^4 + 2c^2d^2 + d^4 + 4d^2 - 1 = n_{23}^2 + 4d^2 - 1 \\
n_{31} &= c^2 + d^2 + 1 = n_{23} + 1 \\
n_{32} &= c^2 + d^2 + c = n_{23} + c \\
n_{41} &= (ca^2 - ac^2 - a - ad^2 + c + cb^2) \\
n_{51} &= n_2n_4 + n_1^2 \\
n_{52} &= n_4^2 + n_1^2 \\
n_{53} &= b - \frac{b}{n_3} \\
n_{54} &= a + \frac{a}{n_3} + 1 \\
n_{61} &= n_{22}n_{24} + n_{21}^2 \\
n_{62} &= n_{24}^2 + n_{21}^2 \\
n_{63} &= d - \frac{d}{n_{23}} \\
n_{64} &= c + \frac{c}{n_{23}} - 1 \\
n_{r1} &= (a^2 + b^2)r - b = n_3r - b \\
n_{r2} &= a^2 + n_{r1}^2 \\
n_{r3} &= a^2 + (r - b)^2 \\
n_{r4} &= (c^2 + d^2)r - d = n_{23}r - d \\
n_{r5} &= c^2 + n_{r4}^2 \\
n_{r6} &= c^2 + (r - d)^2 \\
n_{r7} &= (1 + a)^2 + (r + b)^2 \\
n_{r8} &= (a^2 + b^2 + a)^2 + [(a^2 + b^2)r + b]^2 \\
n_{r9} &= (1 + c)^2 + (r + d)^2 \\
n_{r10} &= (c^2 + d^2 + c)^2 + [(c^2 + d^2)r + d]^2 \\
n_{t1} &= (\cos \theta - a)^2 + (\sin \theta - b)^2
\end{aligned}$$

$$n_{t2} = [(a^2 + b^2)\cos\theta - a]^2 + [(a^2 + b^2)\sin\theta - b]^2$$

$$n_{t3} = (a^2 + b^2)\sin\theta - b$$

$$n_{t7} = (a^2 + b^2)\cos\theta - a$$

$$n_{t9} = \cos\theta - a$$

$$n_{t11} = \sin\theta - b$$

$$n_{t4} = (\cos\theta - c)^2 + (\sin\theta - d)^2$$

$$n_{t5} = [(c^2 + d^2)\cos\theta - c]^2 + [(c^2 + d^2)\sin\theta - d]^2$$

$$n_{t6} = (c^2 + d^2)\sin\theta - d$$

$$n_{t8} = (c^2 + d^2)\cos\theta - c$$

$$n_{t10} = \cos\theta - c$$

$$n_{t12} = \sin\theta - d$$

C. Intermediate functions

$$n_1(\xi_1, \xi_2) = 2\xi_1\xi_2$$

$$n_2(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2$$

$$n_3(\xi_1, \xi_2) = \xi_1^2 + \xi_2^2$$

$$n_4(\xi_1, \xi_2) = n_2(\xi_1, \xi_2) - 1$$

$$n_5(\xi_1, \xi_2) = (\xi_1 + 1)^2 + \xi_2^2$$

$$n_6(\xi_1, \xi_2) = (\xi_1 - 1)^2 + \xi_2^2$$

$$n_7(\xi_1, \xi_2) = \xi_1^4 - 2\xi_1^2\xi_2^2 - 3\xi_2^4 - 2\xi_1^2 - 2\xi_2^2 + 1$$

$$n_8(\xi_1, \xi_2) = 3\xi_1^4 + 2\xi_1^2\xi_2^2 - \xi_2^4 - 2\xi_1^2 - 2\xi_2^2 - 1$$

$$n_9(\xi_1, \xi_2) = n_3^2(\xi_1, \xi_2) - 4\xi_1^2 - 1$$

$$n_{10}(\xi_1, \xi_2) = n_3^2(\xi_1, \xi_2) + 4\xi_2^2 - 1$$

$$n_{11}(\xi_1, \xi_2) = n_3(\xi_1, \xi_2) - 1$$

$$n_{12}(\xi_1, \xi_2) = n_3(\xi_1, \xi_2) + \xi_1$$

$$n_{13}(\xi_1, \xi_2) = 1 + \xi_1$$

$$n_{14}(\xi_1, \xi_2) = n_3(\xi_1, \xi_2) + 1$$

$$n_{41} = (ca^2 - ac^2 - a - ad^2 + c + cb^2)$$

$$n_{42} = (a - c)^2 + (b - d)^2$$

$$n_{43} = ac + bd$$

$$n_{51}(\xi_1, \xi_2) = n_2(\xi_1, \xi_2)n_4(\xi_1, \xi_2) + n_1^2(\xi_1, \xi_2)$$

$$n_{52}(\xi_1, \xi_2) = n_4^2(\xi_1, \xi_2) + n_1^2(\xi_1, \xi_2)$$

$$n_{53}(\xi_1, \xi_2) = \xi_2 - \frac{\xi_2}{n_3(\xi_1, \xi_2)}$$

$$n_{54} = x_3 + \frac{x_3}{n_3(x_3, x_4)} + 1$$

$$n_{64} = x_5 + \frac{x_5}{n_3(x_5, x_6)} - 1$$

$$n_{r1}(\xi_1, \xi_2, r) = n_3(\xi_1, \xi_2)r - \xi_2$$

$$n_{r2}(\xi_1, \xi_2, r) = \xi_1^2 + n_{r1}^2(\xi_1, \xi_2, r)$$

$$n_{r3}(\xi_1, \xi_2, r) = \xi_1^2 + (r - \xi_2)^2$$

$$n_{r7}(\xi_1, \xi_2, r) = (1 + \xi_1)^2 + (r + \xi_2)^2$$

$$n_{r8}(\xi_1, \xi_2, r) = [n_3(\xi_1, \xi_2) + \xi_1]^2 + [n_3(\xi_1, \xi_2)r + \xi_2]^2$$

$$n_{t1}(\xi_1, \xi_2, \theta) = (\cos \theta - \xi_1)^2 + (\sin \theta - \xi_2)^2$$

$$n_{t2}(\xi_1, \xi_2, \theta) = [n_3(\xi_1, \xi_2) \cos \theta - \xi_1]^2 + [n_3(\xi_1, \xi_2) \sin \theta - \xi_2]^2$$

$$n_{t3}(\xi_1, \xi_2, \theta) = n_3(\xi_1, \xi_2) \sin \theta - \xi_2$$

$$n_{t7}(\xi_1, \xi_2, \theta) = n_3(\xi_1, \xi_2) \cos \theta - \xi_1$$

$$n_{t9}(\xi_1, \xi_2, \theta) = \cos \theta - \xi_1$$

$$n_{t11}(\xi_1, \xi_2, \theta) = \sin \theta - \xi_2$$

$$n_{t13}(\xi_1, \xi_2, \theta) = \xi_2 \sin \theta + \xi_1 \cos \theta$$

$$n_{\phi 1} = \sin(x_1 + \frac{\pi}{2})$$

$$n_{\phi 2} = \sin(x_1)$$

$$n_{\phi 3} = \cos(x_1)$$

References

1. Arena, Andrew Salvatore, Jr ., “An Experimental and Computational Investigation Of Slender Wings Undergoing Wing Rock”, 1994, Louisiana State University
2. Myatt, James Harold, “Modeling the Rolling Moment on the 65-Degree Delta Wing For Rolling Motion s at High Angle of Attack”, 1997, Stanford
3. J. Jim Zhu, , “Nonlinear Tracking and Decoupling by Trajectory Linearization”, 1998, EECE department, Louisiana State University
4. D. Lawrence, M. Wu “ Model Scheduling design on delta wing”, 2002, Ohio University
5. Hassan K. Khalil, “Nonlinear systems”, 2002, Prentice Hall, NJ, US
6. Lawrence, D. A. and Rugh, W. J., “Gain scheduling dynamic linear compensators for a nonlinear plant,” 1995, *Automatica*, 31(3), 381 – 390.
7. Kurt Arbenz, and Alfred Wohlhauser, “Advanced Mathematics for Practicing Engineers”, (P.93-95), 1986, Artech house, Mass, US
8. Munson, Young, Okiishi, “Fundamentals of Fluid Mechanics”, 1990, Wiley, New York, US
9. Anderson, John David, “Fundamentals of Aerodynamics”, 2001, McGraw-Hill, Boston, US
10. I.G.Currie, “Fundamental Mechanics of Fluids”, McGraw-Hill, New York, US
11. Gad-el-Hak, “Flow Control (passive, active, and reactive flow management)”, 2000, Cambridge University Press, New York
12. Joseph Katz, and Allen Plotkin, “Low-Speed Aerodynamics From Wing Theory to Panel Methods”, 1991, McGraw-Hill, Inc., NY, US
13. Bisplinghoff, Raymond L., Holt Ashley, and Robert L. Halfman, “Aeroelasticity”, 1955, Addison-Wesley Publishing Co., Inc, Cambridge, MA, pp251-261
14. Milne-Thomson, L. M., “Theoretical Aerodynamics”, 1973, 4th ed., Dover Publications, Inc., NY, US

15. Deman Tang, Denis Kholodar, "System Identification and Pod Method Applied to Unsteady Aerodynamics", 2001, Duke University, US
16. T. Heijn, R. Markovinović, Delft U. of Technology; J.-D. Jansen, Delft U. of Technology and Shell Intl. E&P, "Generation of Low-Order Reservoir Models Using System-Theoretical Concepts", SPE Journal, Volume 9, Number 2, June, (p.202-218)