INDUCED CHARACTERS WITH EQUAL DEGREE CONSTITUENTS

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by

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CHAPTER 1

INTRODUCTION

1.1 Background

A nonempty set G together with an associative binary operation (usually denoted by juxtaposition) is a *group* if the set G contains an identity element (usually denoted 1) and an inverse element for every element of G . The binary operation need not be commutative, but when it is we say G is *abelian*. In this dissertation all groups mentioned will be finite groups. If a subset H of G forms a group under the same operation as G , we say H is a *subgroup* of G and we denote it by $H \leq G$. Let S be a subset of G . We define the *index* of S in G to be $|G|/|S|$ ($= |G : S|$) and the subgroup generated by S to be the smallest subgroup of G that contains S , denoted $\langle S \rangle$. A subgroup H of G is said to be *normal* (denoted $H \trianglelefteq G$) if $H^g = g^{-1}Hg \subseteq H$ for all $g \in G$. Moreover, a subgroup H of G is said to be *subnormal*, denoted $H \triangleleft\triangleleft G$, if there exists a sequence of subgroups $H_1, H_2, \dots, H_n$ of G such that $H = H_1 \trianglelefteq H_2 \trianglelefteq \dots \trianglelefteq H_n = G$. Given a subgroup H of G there is a natural way to form two normal subgroups of G . The first is the called *core* of H in G , denoted $\text{core}_G(H)$, and is defined to be $\bigcap_{g \in G} H^g$. The core of H in G is the unique largest normal subgroup of G contained in H . The second is called the *normal closure* of H in G , denoted H^G . The normal closure is defined to be $\langle H^x : x \in G \rangle$, and it is the unique smallest normal subgroup of G that contains H . If H is normal in G , then $\text{core}_G(H) = H = H^G$.

If x and y are elements of G we define the *commutator* of x and y , denoted $[x, y]$, to be the element $x^{-1}y^{-1}xy$. Furthermore, we define the *derived subgroup* or the *commutator subgroup* of G , denoted G' , to be $\langle [x, y] : x, y \in G \rangle$. We define $G^{(0)} = G$ and when $G^{(n-1)}$ is defined we define $G^n = (G^{(n-1)})'$. The subgroup $G^{(n)}$ is normal in G for any n , and G is abelian if and only if $G^{(1)} = G' = 1$. Furthermore, we say G is solvable if $G^{(n)} = 1$ for some positive integer n .

Let G be a group and let Ω be a nonempty finite set. We say G *acts* on Ω if $\omega \cdot g$ is a uniquely

defined element of Ω , $\omega \cdot 1 = \omega$, and $\omega \cdot (gh) = (\omega \cdot g) \cdot h$ for all $g, h \in G$ and $\omega \in \Omega$. We define the *stabilizer* of ω in G to be the set of all elements of G that fix ω , denoted $G_\omega (= \{g \in G : \omega \cdot g = \omega\})$. Similarly, we define the orbit of ω to be the set $\{\omega \cdot g : g \in G\}$, denoted $\mathcal{O}^G(\omega)$ or simply $\mathcal{O}(\omega)$ when G is known. The Fundamental Counting Principal relates the sizes of these sets, that is, $|\mathcal{O}^G(\omega)| = |G : G_\omega|$. Arguably the most fundamental group action is the group acting on itself by conjugation. That is, $x \cdot g = x^g = g^{-1}xg$ for $x, g \in G$. If $x \in G$, then $G_x (= \{g \in G : x^g = x\})$ is called the *centralizer* of x in G and is denoted $\mathbf{C}_G(x)$. Furthermore, $\mathcal{O}(x) (= \{x^g : g \in G\})$ is called the *conjugacy class* of x in G and is denoted $cl_G(x)$. This action motivates the alternative notation for group actions $(\omega \cdot g = \omega^g)$. If G and H are groups and X is a function from G into H we say X is a *homomorphism* if $X(ab) = X(a)X(b)$ for all $a, b \in G$. The *kernel* of X is defined to be $\{g \in G : X(g) = 1\}$, denoted $\ker(X)$, and we note $\ker(X) \trianglelefteq G$. For a more in-depth discussion of groups please see [3].

A homomorphism X from a group G into the group of n by n invertible matrices over a field F is said to be a *F-representation* of G . We define the *character* afforded by the representation X to be the map χ from G into $\mathbb{C}$ defined by $\chi(g) = \text{trace}(X(g))$. The positive integer $\chi(1) = n$ is called the degree of χ . We say that a character χ of G is *irreducible* if χ cannot be written as a sum of other characters of G . We denote the set of irreducible characters of G by $\text{Irr}(G)$. The set $\text{Irr}(G)$ is a finite set and it encodes information about the group G . The set of irreducible character degrees of G is denoted by $\text{cd}(G)$. If θ and ϕ are characters of G , then the inner product of θ and ϕ is defined by $[\theta, \phi] = \frac{1}{|G|} \sum_{g \in G} \theta(g)\phi(g^{-1})$ and it is well-known that $[\theta, \phi]$ is a nonnegative integer. If χ is character of G and $\text{Irr}(G) = \{\psi_1, \psi_2, \dots, \psi_k\}$ we can decompose χ as a sum of irreducible characters of G , where $[\chi, \psi_i]$ is the coefficient of ψ_i in the decomposition. That is, $\chi = \sum_{i=1}^k [\chi, \psi_i] \psi_i$. Furthermore, we say ψ_i is an irreducible constituent of χ if $[\chi, \psi_i]$ is nonzero.

If X is the trivial homomorphism such that $X(g) = [1]$, the 1 by 1 identity matrix, and χ is the character afforded by X , then $\chi(g) = 1$ for all $g \in G$. This irreducible character χ is called the *principal character* of G and is denoted 1_G . We denote the set of nonprincipal irreducible characters of G by $\text{Irr}(G)^\#$. If χ is a character of G we define $\text{Irr}(\chi) = \{\psi \in \text{Irr}(G) : [\chi, \psi] \neq 0\}$,

which is the set of irreducible constituents of χ . If H is a subgroup of G and ϕ is a character of H , then ϕ^G , the *induced character* on G , is a character of G defined by $\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^\circ(xgx^{-1})$, where $\phi^\circ(h) = 0$ when h is not an element of H and $\phi^\circ(h) = \phi(h)$ when $h \in H$. If $\phi \in \text{Irr}(H)$ and ϕ^G is an irreducible character of G we say ϕ *induces irreducibly* to G . On the other hand, if χ is a character of G then the restriction of χ to H , denoted χ_H , is a character of H . If $\chi \in \text{Irr}(G)$ and χ_H is an irreducible character of H , then we say χ_H *extends* to χ . If $H \trianglelefteq G$ then G acts on the set $\text{Irr}(H)$ via $\lambda \cdot x = \lambda^x$, where $\lambda^x(h) = \lambda(xhx^{-1})$ for $\lambda \in \text{Irr}(H)$, $x \in G$, and $h \in H$. If $\lambda \in \text{Irr}(H)$ we define the *inertia subgroup* of λ in G to be the stabilizer of λ in G , denoted $I_G(\lambda)$ ($= \{x \in G : \lambda^x = \lambda\}$). Furthermore, the map $\psi \mapsto \psi^G$ is a bijection from the set $\text{Irr}(\lambda^{I_G(\lambda)})$ to the set $\text{Irr}(\lambda^G)$. For a more in-depth discussion of character theory please see [4].

1.2 Motivation

Dr. Stephen Gagola and Dr. Sezgin Sezer wrote a paper studying a character theoretic hypothesis (denoted $(*)$) [5]. We seek to extend their work. The following is a definition that we will be using frequently throughout this dissertation.

Definition 1.1. We say a subgroup H of a group G satisfies Hypothesis $(*)$ if all of the nonprincipal irreducible characters of H induce to G as a sum of irreducible characters all of equal degree. When this occurs we write (G, H) satisfies Hypothesis $(*)$.

If G is any group and H is the trivial subgroup, then (G, H) satisfies Hypothesis $(*)$ vacuously. Moreover, if $H = G$, then (G, H) satisfies Hypothesis $(*)$ trivially. Before we present some interesting examples we first state some of Gagola's and Sezer's preliminary results from [5].

Proposition 1.2. Suppose H is a subgroup of G such that (G, H) satisfies Hypothesis $(*)$. Then the following are true statements.

- (i) If $x \in G$, then (G, H^x) satisfies Hypothesis $(*)$.
- (ii) If $K \leq G$ and (G, K) satisfies Hypothesis $(*)$, then $(G, \langle H, K \rangle)$ satisfies Hypothesis $(*)$.
- (iii) The normal closure of H in G , H^G , satisfies Hypothesis $(*)$ in G .

(iv) If $N \trianglelefteq G$, then $(G/N, NH/N)$ satisfies Hypothesis (*).

The following result of Gagola and Sezer demonstrates that the only interesting examples of subgroups satisfying Hypothesis (*) occur as subgroups of G' .

Proposition 1.3. Suppose (G, H) satisfies Hypothesis (*). Then either $H < G'$ or else $G' \leq H$. In fact, if H is any subgroup of G that contains G' , then (G, H) satisfies Hypothesis (*).

The first interesting example that Gagola and Sezer provide is a Frobenius group. Let G be a Frobenius group with kernel K . Then all of the nonprincipal irreducible characters of K induce to G irreducibly. Thus (G, K) satisfies Hypothesis (*). The interesting case is when the complement of K in G is nonabelian. In this case $K < G'$. Furthermore, if $K' \leq H \leq K$, then (K, H) satisfies Hypothesis (*) and hence (G, H) satisfies Hypothesis (*). This provides examples of non-normal subgroups satisfying Hypothesis (*), although these examples are subnormal in G , as $H \trianglelefteq K \trianglelefteq G$.

Gagola and Sezer showed that if G is a nonabelian simple group and $1 < H < G$, then (G, H) does not satisfy Hypothesis (*). We investigate the situation where G is an almost simple group, that is, $S \leq G \leq \text{Aut}(S)$, where S is a nonabelian simple group.

Let G be a Frobenius group with $\text{SL}_2(5)$ as the Frobenius complement and once again let K be the Frobenius kernel. Then (G, K) satisfies Hypothesis (*) and G/K is nonsolvable. Here it is important to note that $K = K^G < G'$ and K is abelian. Gagola and Sezer proved that if (G, H) satisfies Hypothesis (*) with $H^G < G'$, then H is solvable [5]. This theorem opens a natural question, “what can be said if $H < H^G = G'$?” In this dissertation we answer this question by showing that G is solvable in this situation.

CHAPTER 2

Lemmas and Reductions

2.1 General Lemmas

Suppose (G, H) satisfies Hypothesis $(*)$ and let ϕ be a nonprincipal irreducible character of H that extends to a character of G . Then all irreducible constituents of ϕ^G have degree $\phi(1)$. This motivates the following lemma, which is our version of Lemma 12.17 of [4].

Lemma 2.1. Let ϕ be an irreducible character of $H \leq G$ and suppose all irreducible constituents of ϕ^G have degree $\phi(1)$. Then $(\phi^G)_H = |G : H|\phi$ and ϕ vanishes off of $C = \text{core}_G(H)$. Furthermore, if $V(\phi)$ is the vanishing-off subgroup of ϕ , then $V(\phi)$, $\ker(\phi)$, and $\mathbf{Z}(\phi)$ are all normal subgroups of G . Moreover, $\phi_C \in \text{Irr}(C)$ if and only if $H \trianglelefteq G$, in which case $G' \leq H$.

Proof. All irreducible constituents of ϕ^G are extensions of ϕ since they all have degree $\phi(1)$. If χ is any fixed irreducible constituent of ϕ^G , then $[\phi^G, \chi] = [\phi, \chi_H] = [\phi, \phi] = 1$, showing that each irreducible constituent of ϕ^G has multiplicity 1. Write $\phi^G = \sum_{i=1}^n \chi_i$, where each χ_i is an irreducible constituent of ϕ^G . Then

$$|G : H|\phi(1) = \phi^G(1) = \sum_{i=1}^n \chi_i(1) = \sum_{i=1}^n \phi(1) = n\phi(1),$$

and hence $|G : H| = n$. Observe that

$$(\phi^G)_H = \sum_{i=1}^{|G:H|} (\chi_i)_H = \sum_{i=1}^{|G:H|} \phi = |G : H|\phi.$$

Fix $h \in H$ such that $\phi(h) \neq 0$ and set $S = \{g \in G : ghg^{-1} \in H\}$ and compute

$$\begin{aligned}
|G : H|\phi(h) &= (\phi^G)_H(h) = \frac{1}{|H|} \sum_{g \in G} \phi^\circ(ghg^{-1}) \\
&= \frac{1}{|H|} \sum_{g \in S} \phi(ghg^{-1}) \\
&= \frac{1}{|H|} \sum_{g \in S} \chi_H(ghg^{-1}) \\
&= \frac{1}{|H|} \sum_{g \in S} \chi_H(h) \\
&= \frac{1}{|H|} \sum_{g \in S} \phi(h) \\
&= \frac{|S|}{|H|} \phi(h).
\end{aligned}$$

Thus $S = G$ and hence $h \in \text{core}_G(H)$. In particular, ϕ vanishes off of $\text{core}_G(H)$. Therefore, $V(\phi)$, $\ker(\phi)$, and $\mathbf{Z}(\phi)$ are all contained in $\text{core}_G(H)$. If x is a generator of $V(\phi)$, then $x^g \in H$ for all $g \in G$. Now $\phi(x^g) = \chi_H(x^g) = \chi_H(x) = \phi(x) \neq 0$ and hence $x^g \in V(\phi)$, proving $V(\phi)$ is normal in G . A similar argument shows $\ker(\phi)$ and $\mathbf{Z}(\phi)$ are normal in G . Now we make the following computation:

$$[\phi_C, \phi_C] = \frac{1}{|C|} \sum_{c \in C} \phi_C(c) \phi_C(c^{-1}) = |H : C| \frac{1}{|H|} \sum_{h \in H} \phi(h) \phi(h^{-1}) = |H : C| [\phi, \phi] = |H : C|.$$

Notice that this implies $\phi_C \in \text{Irr}(C)$ if and only if $H \trianglelefteq G$. If $H \trianglelefteq G$, then Gallagher's Theorem implies all irreducible characters of G/H are linear and hence G/H is abelian, proving $G' \leq H$. ■

Lemma 2.1 shows the importance of extending characters and motivates the following discussion and lemmas. Suppose N is a minimal normal subgroup of G and suppose N is nonsolvable. Then $N \leq G'$ and $N = \prod_{i=1}^n S_i$, where the S_i form a full G -conjugacy class of isomorphic subgroups and each S_i is a nonabelian simple group. Now G' acts on the collection of S_i by conjugation and hence partitions the collection of S_i into G' -orbits. Let Ω be the set of G' -orbits of the collection of S_i and let $\omega \in \Omega$. Then $T_\omega = \prod_{S_i \in \omega} S_i$ is a minimal normal subgroup of G' . Since $G' \trianglelefteq G$ and G acts on the collection of S_i transitively, we know G acts on the G' -orbits transitively and hence

$N = \prod_{\omega \in \Omega} T_\omega$. Write $M = \mathbf{N}_G(T_\omega)$ and observe M is independent of the choice of ω , as $G' \leq M$ implies $M \trianglelefteq G$.

Lemma 2.2. Let N be a nonsolvable minimal normal subgroup of G as described above. There exists a nonprincipal irreducible character ϕ of N such that ϕ extends to some $\chi \in \text{Irr}(G)$. Furthermore, χ may be chosen so that $\mathbf{C}_G(N) \leq \ker(\chi)$ and $\phi = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_n$, where $\alpha_i \in \text{Irr}(S_i)$ and the α_i are G -conjugate.

Proof. Write $\mathbf{C}_G(N) = C$ and observe that $C \trianglelefteq G$ as $N \trianglelefteq G$. Since N is a minimal normal subgroup of G and N is nonsolvable, we know $N \cap C = 1$, hence $NC/C \cong N$ and $G/C \lesssim \text{Aut}(N)$. Since $NC/C \cong N$ we may use Proposition 4.2 of [5] to find a nonprincipal irreducible character $\hat{\theta} \in \text{Irr}(NC/C)$ that extends to an irreducible character of $\text{Aut}(N)$. Hence $\hat{\theta}$ extends to some $\chi \in \text{Irr}(G/C)$. Now θ , the inflation of $\hat{\theta}$, is in $\text{Irr}(NC)$ and extends to χ with $C \leq \ker(\theta)$. Since NC is a direct product of N and C , and $C \leq \ker(\theta)$, we may write $\phi = \theta_N \in \text{Irr}(N)$ and observe that ϕ extends to χ with $C \leq \ker(\chi)$. The proof of Proposition 4.2 of [5] shows $\phi = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_n$, where $\alpha_i \in \text{Irr}(S_i)$ and the α_i are G -conjugate. ■

Lemma 2.3. Let N be a minimal normal subgroup of G as described above. Let ω be a G' -orbit of the collection of S_i . We may renumber the S_i so that $\omega = \{S_1, S_2, \dots, S_{|\omega|}\}$, and write $T_\omega = \prod_{i=1}^{|\omega|} S_i$ and $M = \mathbf{N}_G(T_\omega)$. Then there exists a nonprincipal irreducible character $\phi \in \text{Irr}(T_\omega)$ such that ϕ extends to some $\chi \in \text{Irr}(M)$. In fact, $\phi = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_{|\omega|}$, where $\alpha_j \in \text{Irr}(S_j)$ and the α_j are all G' -conjugate. Furthermore, χ may be chosen so that $\mathbf{C}_G(T_\omega) \leq \ker(\chi)$ and the inertia group $I_G(\chi) = M \trianglelefteq G$, hence $\chi^G \in \text{Irr}(G)$.

Proof. Lemma 2.2 applied to the group M shows us the existence of $\phi \in \text{Irr}(T_\omega)$ extending to $\chi \in \text{Irr}(M)$ and $\mathbf{C}_G(T_\omega) \leq \ker(\chi)$. It remains to show that $I_G(\chi) = M$. If $x \in I_G(\chi)$ and $g \in \ker(\chi_N)$, then $\chi_N(g^x) = \chi(g^x) = \chi(g) = \chi(1)$ and hence $g^x \in \ker(\chi_N)$. This shows $x \in \mathbf{N}_G(\ker(\chi_N))$. Since $\ker(\chi_N) = \prod_{i=|\omega|+1}^n S_i$ we have

$$I_G(\chi) \leq \mathbf{N}_G(\ker(\chi_N)) = \mathbf{N}_G\left(\prod_{i=|\omega|+1}^n S_i\right) = M \leq I_G(\chi).$$

Hence $I_G(\chi) = M$ and thus $\chi^G \in \text{Irr}(G)$. ■

Lemma 2.4. If S is a nonabelian simple group, then there exists a nonprincipal irreducible character ϕ of S that is extendible to $\text{Aut}(S)$, and $\theta \in \text{Irr}(S)$ such that $\theta(1) \nmid 2\phi(1)$.

Proof. We mimic the proof of Lemma 4.1 of [5] with slight modifications. If S is one of the 26 sporadic simple groups or if $S = {}^2F_4(2)'$, then select $\phi \in \text{Irr}(S)^\#$ of smallest degree such that ϕ is invariant in $\text{Aut}(S)$ and hence ϕ extends to $\text{Aut}(S)$ as $\text{Aut}(S)/S$ is cyclic. Select $\theta \in \text{Irr}(S)$ such that θ has largest degree. Then $\theta(1) > 2\phi(1)$ and hence $\theta(1) \nmid 2\phi(1)$. The Atlas [10] verifies this fact.

If S is a group of Lie type that is not ${}^2F_4(2)'$, then we may choose ϕ to be the Steinberg character, which is known to extend to $\text{Aut}(S)$. Since ϕ is the Steinberg character of S it has prime power degree for some prime p (in fact, p is the defining characteristic). Next we show that since S is nontrivial, there exists a nonprincipal irreducible character of S such that p does not divide its degree. Suppose for a contradiction that, $p \mid \chi(1)$ for every $\chi \in \text{Irr}(S)^\#$. Then $p \mid \sum \chi(1)^2 = |S| - 1$, where the sum runs over all of the nonprincipal irreducible characters of S . Since p divides $|S|$ it does not divide $|S| - 1$, a contradiction. Let θ be a nonprincipal irreducible character of S such that $p \nmid \theta(1)$. Since S is simple, $\theta(1) > 2$, and hence $\theta(1) \nmid 2\phi(1)$. In particular, since A_5 (viewed as either $\text{PSL}_2(4)$ or as $\text{PSL}_2(5)$) and A_6 are groups of Lie type, they are included in this argument.

If S is an alternating group A_n with $n \geq 7$, then we set ϕ to be the nonprincipal irreducible constituent of the permutation character of A_n in its natural action on n points. Observe, ϕ has degree $n - 1$ and extends to $\text{Aut}(S)$. If $n \geq 7$, the partition $\lambda = (2^2, 1^{n-4})$ is not self conjugate and hence $A_n = S$ has an irreducible character of degree

$$\frac{n!}{\prod(\text{Hook lengths of } \lambda)} = \frac{n!}{2(n-4)!(n-1)(n-2)} = \frac{n(n-3)}{2}$$

[2]. Select θ to be an irreducible character of degree $\frac{n(n-3)}{2}$ and observe $\theta(1) \nmid 2\phi(1)$ and in fact $\theta(1) = \frac{1}{2}n(n-3) > 2(n-1) = 2\phi(1)$. ■

Lemma 2.5. Let T be a direct product of isomorphic nonabelian simple groups. Then there exists a nonprincipal irreducible character α of T which is extendible to $\text{Aut}(T)$, and $\beta \in \text{Irr}(T)$ such that $\beta(1) \nmid 2\alpha(1)$.

Proof. Write $T = S \times S \times \cdots \times S = S^n$ for S a nonabelian simple group, and find irreducible characters ϕ and θ of S satisfying the conclusion of Lemma 2.4. Set $\alpha = \phi \times \phi \cdots \times \phi \in \text{Irr}(T)$ and $\beta = \theta \times \theta \times \cdots \times \theta \in \text{Irr}(T)$. The proof of Proposition 4.2 of [5] shows us that α extends to $\text{Aut}(T)$. Observe, $\beta(1) \nmid 2\alpha(1)$ since $\theta(1) \nmid 2\phi(1)$. ■

2.2 Reductions

Let G be a Frobenius group with kernel K and abelian complement. As previously mentioned, (G, K) satisfies Hypothesis (*). If $K' \leq H < K$, then (K, H) satisfies Hypothesis (*) and hence (G, H) satisfies Hypothesis (*). Here it is important to note that there are examples of H such that $H < H^G = G' = K$. Thus providing an example where the hypothesis of our main theorem is satisfied.

Let (G, H) satisfy Hypothesis (*) with $H < G'$ and hence $H^G \leq G'$. Gagola and Sezer showed that if $H^G < G'$, then H is solvable. We explore the residual situation when $H^G = G'$. In fact, in the following sections we show that if $H < G'$ and $H^G = G'$, then G is solvable.

If (G, H) satisfies Hypothesis (*) and $\text{core}_G(H)$ is nonsolvable, then $\text{core}_G(H)$ has a nonprincipal irreducible character that extends to some character χ of G . Then χ_H is a nonprincipal irreducible character of H that extends to χ and the following situation arises.

Lemma 2.6. Let (G, H) satisfy Hypothesis (*) and assume there exists a nonprincipal character $\phi \in \text{Irr}(H)$ that has an extension to a character of G . Then $(\phi^G)_H = |G : H|\phi$ and ϕ vanishes off of $C = \text{core}_G(H)$. Moreover, $\phi_C \in \text{Irr}(C)$ if and only if $H \trianglelefteq G$.

Proof. As (G, H) satisfies Hypothesis (*), every irreducible constituent of ϕ^G is an extension of ϕ and hence has degree $\phi(1)$. Thus by Lemma 2.1, we have $(\phi^G)_H = |G : H|\phi$ and ϕ vanishes off of $C = \text{core}_G(H)$. Finally, we apply Lemma 2.1 again to see that $\phi_C \in \text{Irr}(C)$ if and only if $H \trianglelefteq G$. ■

Proposition 2.7. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$. Then $\text{core}_G(H)$ is solvable.

Proof. If $H^G < G'$, then H is solvable by Theorem B of [5] and hence $\text{core}_G(H)$ is solvable. Thus we may assume $H^G = G'$. In particular, this implies H is not normal in G . Let $C = \text{core}_G(H)$ and suppose that C is nonsolvable. By Theorem 4.3 of [5], there exists a nonprincipal character $\lambda \in \text{Irr}(C)$ that has an extension to an irreducible character χ of G . Let $\phi = \chi_H$ and observe that ϕ is an extension of λ . Now ϕ extends to $\chi \in \text{Irr}(G)$ and $\phi_C = \lambda \in \text{Irr}(C)$. By Lemma 2.6, this implies $H \trianglelefteq G$, a contradiction. Hence C is solvable. ■

Let (G, H) satisfy Hypothesis $(*)$ with $1 < H < G'$. If $C = \text{core}_G(H)$, then $(G/C, H/C)$ satisfies Hypothesis $(*)$ with H/C core-free in G/C and $H/C < G'/C = (G/C)'$. If G/C is solvable, then G is also solvable, as $C \trianglelefteq G$ and C is solvable by Proposition 2.7. Hence we may assume H is core-free in G whenever convenient.

Let (G, H) satisfy Hypothesis $(*)$ with $1 < H < G'$ and $H^G = G'$. Assume G is nonsolvable and let N be minimal with the property that N is normal in G and N is nonsolvable. Since, $N \cap G' \leq N$ is also a nonsolvable normal subgroup of G , $N \cap G' = N$ and thus N is contained in G' . Theorem A of [5] shows that $G' < G$ and hence $N < G$. The following Lemmas describe how H interacts with nonsolvable normal subgroups of G contained in G' .

Lemma 2.8. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$ and $H^G = G'$. If N is any nonsolvable normal subgroup of G contained in G' , then $G' = NH$.

Proof. Since $\text{core}_G(H)$ is solvable by Proposition 2.7, we know $N \not\leq H$. Furthermore, there exists a nonprincipal irreducible character $\mu \in \text{Irr}(N)$ that has an extension to an irreducible character χ of G by Lemma 4.1 of [5]. Let $M = NH$ and observe that $\phi = \chi_M \in \text{Irr}(M)$ also extends to χ . If $H \leq \ker(\phi) \leq \ker(\chi) \trianglelefteq G$, then $N \leq G' = H^G \leq \ker(\chi)$ and hence $\mu = 1_N$, a contradiction. Thus $H \not\leq \ker(\phi)$. Now all irreducible constituents of ϕ^G lie above some fixed nonprincipal irreducible character of H and hence they all have degree $\phi(1)$. If $C = \text{core}_G(M)$, then we know $N \leq C \leq M$ and ϕ_C is an irreducible character of C . Hence $M \trianglelefteq G$ and $G' \leq M$ by Lemma 2.1. Furthermore, $G' = M$, as H and N are contained in G' . ■

Lemma 2.9. Let (G, H) satisfy Hypothesis $(*)$ with $\text{core}_G(H) = 1$ and $H > 1$. Let N be any normal subgroup of G such that $NH \trianglelefteq G$. Then $H/(H \cap N) \cong NH/N$ is solvable.

Proof. Suppose $H/(H \cap N)$ is nonsolvable. Then $NH/N \trianglelefteq G/N$ is nonsolvable, as $NH/N \cong H/(H \cap N)$, and NH/N has a nonprincipal irreducible character that is extendible to G/N . Let $\hat{\phi} \in \text{Irr}(NH/N)$ be one such character. Since $NH/N \cong H/(H \cap N)$, we can find $\hat{\lambda}$ in $\text{Irr}(H/(H \cap N))$ such that λ , the inflation of $\hat{\lambda}$, is an irreducible character of H and is extendible to $\phi \in \text{Irr}(NH)$. Hence λ is extendible to G and all irreducible constituents of λ^G have degree $\lambda(1)$. By Lemma 2.1, λ vanishes off of $\text{core}_G(H) = 1$. This is a contradiction as λ is an irreducible character of H . ■

Corollary 2.10. If (G, H) satisfies Hypothesis $(*)$ with $H < G'$, $\text{core}_G(H) = 1$, and $H^G = G'$, then G has at most one nonsolvable chief factor.

Proof. We may assume G is nonsolvable, for otherwise the result is trivial. Therefore select N minimal with the property that N is normal in G and N is nonsolvable. If N/K is a chief factor of G , then K is solvable. Now N is contained in G' , as $N \cap G' \trianglelefteq G$ is nonsolvable. Since $N \leq G'$, we have $NH = G'$ from Lemma 2.8 and G'/N is solvable from Lemma 2.9. Thus G/N is solvable. ■

Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $\text{core}_G(H) = 1$, and $H^G = G'$. Our main goal is to show that G is solvable under these assumptions. Let $S = O_\infty(G)$ be the solvable radical of G . Now $(G/S, SH/S)$ satisfies Hypothesis $(*)$ and so either $SH/S < (G/S)'$ or else $(G/S)' \leq SH/S$. If $(G/S)' \leq SH/S$, then $SH \trianglelefteq G$ and by Lemma 2.9 we have SH/S is solvable and hence G is solvable. If $SH/S < (G/S)'$ it is clear that $(SH/S)^{G/S} = (G/S)'$. This shows that for our purposes we may assume $SH/S < (G/S)'$ and $O_\infty(G) = 1$ whenever convenient.

Corollary 2.11. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, and $O_\infty(G) = 1$. Select N minimal with the property that N is a normal nonsolvable subgroup of G . Then N is a minimal normal subgroup of G and $\mathbf{C}_G(N) = 1$. In particular, $G \lesssim \text{Aut}(N)$.

Proof. If $K \trianglelefteq G$ with $K < N$, then K is solvable by the minimality of N and hence $K = 1$, as $O_\infty(G) = 1$. Since $N \trianglelefteq G$ we have $C = \mathbf{C}_G(N) \trianglelefteq G$ and hence $N \cap C \trianglelefteq G$. Since $N \cap C \leq N$ we

know $N \cap C$ is solvable and so $N \cap C \leq O_\infty(G) = 1$. Now $C \cong NC/N \leq G/N$ is solvable from Corollary 2.10 and hence $C = 1$. Observe that $G \cong \mathbf{N}_G(N)/\mathbf{C}_G(N) \lesssim \text{Aut}(N)$. $\blacksquare$

Proposition 2.12. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, and $O_\infty(G) = 1$. Then H is not normal in G' .

Proof. Suppose for a contradiction that $H \trianglelefteq G'$. Select $N \leq G$ minimal with the property that N is a normal nonsolvable subgroup of G . Then N is a minimal normal subgroup of G by Corollary 2.11 and $NH = G'$ by Lemma 2.8. We may write $N = \prod_{i=1}^n S_i$, where each S_i is a simple nonabelian normal subgroup of N , and the S_i constitute a full G -conjugacy class of subgroups of N . If $N \cap H = 1$, then $H \leq \mathbf{C}_G(N) = 1$, which is a contradiction since $H^G = G'$. Hence $L = N \cap H$ is a nontrivial normal subgroup of N (showing $n \geq 2$) and we may write $L = \prod_{i=1}^l S_i$ with $l < n$ and a possible renumbering of the S_i . If $K = \prod_{i=l+1}^n S_i$, then $K \trianglelefteq G'$ and $K \times L = N$. Now $K \cap H = K \cap N \cap H = K \cap L = 1$ and $KH = K L H = NH = G'$, which shows that $K \times H = G'$. Let $T = S_n$ and notice $T \trianglelefteq N$ and we show that $T \trianglelefteq G'$ next. If $h \in H$ and $l+1 \leq i \leq n$ then $S_i^h = S_j$ for some $l+1 \leq j \leq n$ and hence H normalizes K . Now $[K, H] \leq K \cap H = 1$ and hence H centralizes K and hence $T \trianglelefteq NH = G'$. Thus $M = \mathbf{N}_G(T)$ contains G' and hence is normal in G . Since all of the S_i are G -conjugate, we have $\mathbf{N}_G(S_i) = M$ for all i . Let $S = S_1$ so that $M = \mathbf{N}_G(S)$. Lemma 2.3 guarantees a nonprincipal irreducible character α of S that extends to $\psi \in \text{Irr}(M)$ with $S_2 \times \cdots \times S_n \leq \ker \psi$, $I_G(\psi) = M$, and $\psi^G \in \text{Irr}(G)$. Now $\psi_N = \alpha \times 1_{S_2} \times \cdots \times 1_T \in \text{Irr}(N)$ and ψ lies above ψ_H , a nonprincipal irreducible character of H .

Let $\beta \in \text{Irr}(T)$ be a G -conjugate of α so that β extends to $\chi \in \text{Irr}(M)$ with $\mathbf{C}_M(T) \leq \ker(\chi)$. Now $\psi\chi$ is an irreducible character of M by Theorem 6.16 of [4]. Since $H \leq \mathbf{C}_M(T) \leq \ker(\chi)$, we have $[(\psi\chi)_H, \psi_H] = \alpha(1)$ and hence $\psi\chi$ lies above ψ_H . Thus any irreducible constituent of $(\psi\chi)^G$ must have degree $\psi^G(1)$.

Let $I = \mathbf{N}_G(S \times T)$. Observe,

$$I_G(\psi\chi) \leq \mathbf{N}_G(\ker((\psi\chi)_N)) = \mathbf{N}_G\left(\prod_{i=2}^{n-1} S_i\right) = \mathbf{N}_G(S \times T) = I,$$

which shows $I_G(\psi\chi) \leq I$. Now I/M acts faithfully on the set $\{S, T\}$ by conjugation and hence $|I : M| \leq 2$.

If $M = I_G(\psi\chi)$, then $(\psi\chi)^G$ is an irreducible character of G that lies above ψ_H and thus

$$\alpha(1)^2|G : M| = \psi(1)\chi(1)|G : M| = (\psi\chi)^G(1) = \psi^G(1) = \alpha(1)|G : M|.$$

This implies $\alpha(1) = 1$, a contradiction since α is a nonprincipal irreducible character of the nonabelian simple group S . Thus $M < I_G(\psi\chi)$, hence $I_G(\psi\chi) = I$ and $|I : M| = 2$. Now $\psi\chi$ is I -invariant and hence $\psi\chi$ extends to $\eta \in \text{Irr}(I)$ and $\eta^G \in \text{Irr}(G)$ as I/M is cyclic. We see that both η^G and ψ^G lie above $\psi_H \in \text{Irr}(H)$ and hence they have equal degree. Thus $\alpha(1)|G : M| = \alpha(1)^2|G : I|$, so $|I : M| = \alpha(1)$. This shows $\alpha(1) = |I : M| = 2$, a contradiction since α is a nonprincipal irreducible character of a nonabelian simple group. $\blacksquare$

Throughout the rest of this section we keep the following notation fixed. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, and $O_\infty(G) = 1$. Select $N \leq G$ minimal with the property that N is a normal nonsolvable subgroup of G . Then N is a minimal normal subgroup of G by Corollary 2.11 and $NH = G'$ by Lemma 2.8. We may write $N = \prod_{i=1}^n S_i$, where each S_i is a simple nonabelian normal subgroup of N , and the S_i constitute a full G -conjugacy class of subgroups of N . As previously discussed, G' acts on the collection of S_i by conjugation and hence partitions the collection of S_i into G' -orbits. Let Ω be the set of G' -orbits of the collection of S_i and let $\omega \in \Omega$. Then $T_\omega = \prod_{S_i \in \omega} S_i$ is a minimal normal subgroup of G' . Since $G' \trianglelefteq G$ and G acts on the collection of S_i transitively, we know G acts on the G' -orbits transitively and we may write $N = \prod_{\omega \in \Omega} T_\omega$. In order to simplify notation, we number the collection of T_ω so that $N = \prod_{i=1}^m T_i$ and $m = |\Omega|$. Let $M = \mathbf{N}_G(T_1)$ and observe $M \trianglelefteq G$ as $G' \leq M$, and hence $M = \mathbf{N}_G(T_i)$ for $1 \leq i \leq m$.

Remark 2.13. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, and $O_\infty(G) = 1$. For the purpose of proving G is solvable we may assume $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$.

If $x \in G$, then $(G, \langle H, H^x \rangle)$ satisfies Hypothesis $(*)$ with $\langle H, H^x \rangle^G = G'$. Therefore, we replace H with $\langle H, H^x \rangle$ if $\langle H, H^x \rangle < G'$. Repeating this process yields $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$.

Proposition 2.14. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, $O_\infty(G) = 1$, and $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$. Then $H \trianglelefteq M$ for any $H \leq M < G'$ (i.e. $\mathbf{N}_{G'}(H)$ is a maximal subgroup of G'). Recall, if ω_i is a G' -orbit of the collection of S_j , then $\prod_{S \in \omega_i} S$ denotes T_i . Furthermore, $T_i H = G'$ for some T_i .

Proof. Suppose $H \leq M < G'$ and let $m \in M$. As $H^m \leq M$ we have $\langle H, H^m \rangle \leq M$. Thus $H^m = H$, for otherwise $M = G'$, and hence $H \trianglelefteq M$. Now $H \not\trianglelefteq G'$ by Proposition 2.12 and hence there exists $T_i \not\leq \mathbf{N}_{G'}(H)$, for otherwise $H \trianglelefteq G' = H \prod_i T_i$. Since $H \not\trianglelefteq T_i H$, we have $T_i H = G'$.

■

Proposition 2.14 guarantees the existence of some T_i such that $T_i H = G'$. With possible renumbering of the T_i we set $T = T_1$ so that $T H = G'$. Throughout this section we fix this notation so that $TH = G'$.

Proposition 2.15. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, $O_\infty(G) = 1$, and $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$. If $T_i \cap H > 1$ for any $i \geq 2$, then $T_i \leq H$.

Proof. Observe that $T_i \cap H \trianglelefteq H$ and $T \leq \mathbf{N}_G(T_i \cap H)$ as $T \leq \mathbf{C}_G(T_i)$. Hence $T_i \cap H \trianglelefteq TH = G'$ and hence $T_i \cap H = T_i$ as T_i is a minimal normal subgroup of G' . This proves $T_i \leq H$. ■

If the hypothesis of Proposition 2.15 is satisfied and $T_i \cap H > 1$ for some $i \geq 2$, then $T_i \leq H$ and hence a nonprincipal irreducible character of T_i extends to a character of $\mathbf{N}_G(T_i)$, say ψ . Furthermore, $I_G(\psi) = \mathbf{N}_G(T_i)$ and hence $\psi^G \in \text{Irr}(G)$. On the other hand, $\psi_H \in \text{Irr}(H)$ such that ψ_H extends to ψ and hence all irreducible constituents of $(\psi_H)^G$ have degree $\psi(1)|G : \mathbf{N}_G(T_i)| = m\psi(1)$.

Proposition 2.16. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, $O_\infty(G) = 1$, and $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$. Recall m is the number of G' -orbits of the

collection of S_i . Then either $m = 1$ and H acts transitively on the collection of S_i , or $m = 2$ and $T_i \cap H = 1$ and $T_i H = G'$ for $i = 1, 2$.

Proof. Recall $T = T_1$ and $TH = G'$. Since $TH = G'$ we know m is the number of H -orbits of the collection of S_i . If $m = 1$ then H acts transitively on the collection of S_i . Therefore, we assume $m \geq 2$ and suppose for a contradiction that $T_i \cap H > 1$ for some $i \geq 2$. Without loss of generality we may assume $i = m$ and set $V = T_m$. Proposition 2.15 guarantees $V \leq H$ as $V \cap H > 1$. Set $M = \mathbf{N}_G(V)$ and note $M = \mathbf{N}_G(T_i)$ for all i . Lemma 2.3 guarantees a nonprincipal character $\alpha \in \text{Irr}(V)$ such that α extends to $\psi \in \text{Irr}(M)$ with the property $T_j \leq \ker(\psi)$ for all $j < m$ and $\psi^G \in \text{Irr}(G)$. Now set $\phi = \psi_H$, a nonprincipal irreducible character of H . Since $T \leq \ker(\psi)$ we have $\phi_{T \cap H} = \alpha(1)1_{T \cap H}$ and hence $\alpha(1)(1_{T \cap H})^T = (\phi^{G'})_T$. Thus there exists $\beta \in \text{Irr}(T)^\#$ with the property $0 \neq [(\phi^{G'})_T, \beta]$. Hence $[\phi^M, \beta^M] \neq 0$. This allows us to find a common irreducible constituent χ of ϕ^M and β^M . There exists an irreducible constituent of χ_N of the form $\mu = \beta_1 \times \beta_2 \times \cdots \times \beta_m$, where $\beta_m = \alpha$, and $\beta_i \in \text{Irr}(T_i)$. All irreducible constituents of χ_N are M -conjugate to μ , thus β_1 is M -conjugate to β showing $\beta_1 \neq 1_T$. Set $\Omega = \{i \in \{1, 2, \dots, m\} : \beta_i = 1_{T_i}\}$ and $\Lambda = \{1, 2, \dots, m\} - \Omega$. Since all irreducible constituents of χ_N are M -conjugate to μ , we have $\gamma_i = 1_{T_i}$ if and only if $i \in \Omega$ for any $\gamma_1 \times \gamma_2 \times \cdots \times \gamma_m \in \text{Irr}(\chi_N)$ with $\gamma_i \in \text{Irr}(T_i)$. Observe $\prod_{i \in \Omega} T_i \leq \ker(\chi_N)$ and $m - |\Omega| = |\Lambda| \geq 2$. The only normal subgroups of M contained in N are products of the T_i 's. Thus $\ker(\chi_N) = \prod_{i \in \Omega} T_i$.

We know the inertia group of ψ in G is M and hence ψ^G is an irreducible character of G of degree $|G : M|\alpha(1)$ that lies above ϕ . Let $I = \mathbf{N}_G(\prod_{i \in \Lambda} T_i)$ and observe that $M \leq I_G(\chi) \leq I$ as

$$I_G(\chi) \leq \mathbf{N}_G(\ker(\chi_N)) = \mathbf{N}_G(\prod_{i \in \Omega} T_i) = \mathbf{N}_G(\prod_{i \in \Lambda} T_i) = I.$$

Now choose a subgroup K of G containing M as large as possible so that χ extends to a character, say $\eta \in \text{Irr}(K)$. Observe, $K \leq I_G(\chi) \leq I$. As $G' \leq M \leq K$ we know $K \trianglelefteq G$ and hence $K = I_G(\eta)$, as K was chosen maximal with the property that χ extends to a character of K . Now η^G is an

irreducible character of G of degree $\eta(1)|G : K|$. Observe

$$\eta(1)|G : K| \geq \eta(1)|G : I| = \chi(1)|G : I| \geq |G : I| \prod_{i \in \Lambda} \beta_i(1) \geq \alpha(1)|G : I| 3^{m-|\Omega|-1}.$$

Since η^G lies above ϕ , we have $\eta^G(1) = |G : M|\alpha(1)$. This implies $|I : M| \geq 3^{m-|\Omega|-1}$. Now G/M is abelian and acts faithfully and transitively the set $\{T_1, T_2, \dots, T_m\}$, hence I/M acts faithfully on $\{T_i : i \in \Lambda\}$ with $|I/M| \leq |\Lambda| = m - |\Omega|$. Thus $m - |\Omega| \geq 3^{m-|\Omega|-1}$, a contradiction since $m - |\Omega| \geq 2$. This proves $T_i \cap H = 1$ for $i \geq 2$. Furthermore, if $T \cap H > 1$, then $1 < |T \cap H| \leq |T_i \cap H|$ for any i , a contradiction. Since $TH = G'$ and $T_i \cap H = 1$, we have $T_i H = G'$ for any i .

Next we prove that H is core-free in G' . If L is a minimal normal subgroup of G' different than any T_i , then $L \cap T_i = 1$ for all i . Hence $L \leq \mathbf{C}_G(N) = 1$, a contradiction. Since $T_i \cap H = 1$ for all i , we have H is core-free in G' .

Now we prove that $m \leq 2$. Let $C = \mathbf{C}_{G'}(T)$ and note $C \trianglelefteq G'$, as $T \trianglelefteq G'$. Observe that $C \cap H \trianglelefteq H$ and $C \cap H$ centralizes T and hence $C \cap H \trianglelefteq TH = G'$. Thus $C \cap H = 1$, as H is core-free in G' . We know $\prod_{i \geq 1} T_i \leq C = C \cap T_2 H = T_2(C \cap H) = T_2$, which shows $C = T_2$ and thus $m = 2$. ■

The next proposition eliminates the case $m = 2$ in Proposition 2.16.

Proposition 2.17. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, $O_\infty(G) = 1$, and $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$. Let N be minimal with the property that N is a normal subgroup of G and N is nonsolvable. Then N is the unique minimal normal subgroup of G' and H acts transitively on the S_i .

Proof. Suppose for a contradiction that $m = 2$. Then $T_1 \cap H = T_2 \cap H = 1$, $T_1 H = G'$, and $T_2 H = G'$ by Proposition 2.16. Write $D = N \cap H$ and observe $DT_1 = N = DT_2$ and $D \cap T_1 = 1 = D \cap T_2$. This is exactly what it means for D to be a diagonal subgroup of N and hence D is of the form $D = \{g\theta(g) : g \in T\}$ where θ is an isomorphism from T_1 to T_2 . Keeping our previous notation $T_1 = T$, use Lemma 2.4 to select $\alpha, \beta \in \text{Irr}(T)$ such that α extends to ψ , an irreducible character of $M = \mathbf{N}_G(T)$ with $T_2 \leq \ker(\psi)$ and $\beta(1) \nmid 2\alpha(1)$. Now $\psi_N = \alpha \times 1_{T_2} \in \text{Irr}(N)$

and since D is a diagonal subgroup we have $\psi_D(g\theta(g)) = \psi_N(g\theta(g)) = \alpha(g) \times 1_{T_2}(\theta(g)) = \alpha(g)$. Thus $\psi_D \in \text{Irr}(D)$ as $D \cong T$. Thus ψ_D extends to ψ and $\phi = \psi_H \in \text{Irr}(H)^\#$ extends to ψ . Since the inertia group of ψ in G is M , we have ψ^G is an irreducible character of G of degree $|G : M|\alpha(1) = 2\alpha(1)$ that lies above ϕ .

We next show that if $\chi \in \text{Irr}(M|\phi^M)$, then $\chi(1) = \alpha(1)$ or $\chi(1) = 2\alpha(1)$. If $\eta \in \text{Irr}(\chi^G)$, then η lies above ϕ and so $\eta(1) = 2\alpha(1)$. Furthermore, either $\eta_M = \chi$ or $\eta_M = \chi_1 + \chi_2$, where χ_i are the G -conjugates of χ . The former case shows us that $\chi(1) = \eta(1) = 2\alpha(1)$. The latter case shows us that $\alpha(1) = \chi(1)$.

We know $(\phi^{G'})_T = (\phi_{(1)})^T = \alpha(1)(1_{(1)})^T = \alpha(1)\rho$, where ρ is the regular character of T , and hence $0 \neq \alpha(1)\gamma(1) = [(\phi^{G'})_T, \gamma]$ for every $\gamma \in \text{Irr}(T)$. Thus $0 \neq [(\phi^{G'})_T, \beta]$. Recall that $\beta(1) \nmid 2\alpha(1)$. Now $[\phi^M, \beta^M] \neq 0$, and this allows us to find a common irreducible constituent χ of ϕ^M and β^M . Thus $\chi(1) = \alpha(1)$ or $\chi(1) = 2\alpha(1)$. Since $N \trianglelefteq M$, we have $\beta(1) \mid \chi(1)$, and hence $\beta(1) \mid 2\alpha(1)$. But this is a contradiction and hence $m = 1$. Since m was the number of G' -orbits, G' acts transitively on the collection S_i . ■

Proposition 2.18. Let (G, H) satisfy Hypothesis $(*)$ with $H < G'$, $H^G = G'$, $O_\infty(G) = 1$, and $G' = \langle H, H^x \rangle$ for all $x \in G$ such that $H^x \neq H$. Let N be minimal with the property that N is a normal subgroup of G and N is nonsolvable. Then N is a nonabelian simple group, i.e., $n = 1$.

Proof. Suppose $n \geq 2$. We write $\mathbf{N}_G(S_1) = M$ and observe $M < G$. Since H acts transitively on the collection of S_i , we have $G = MH$. We see that M cannot be normal in G by Lemma 3.1 of [5]. Now $(1_M)^G$ is not a sum of linear characters, for otherwise $G' \leq \ker((1_M)^G) = \text{core}_G(M) \leq M$ and hence $M \trianglelefteq G$, a contradiction. Now let $\chi \in \text{Irr}((1_M)^G)$ such that $\chi(1) > 1$ and note $\chi(1) \leq n - 1$. Since $\chi(1) > 1$ we may select $\theta \in \text{Irr}(\chi_H)^\#$. Next, we observe that $N \not\leq \ker(\eta)$ for some $\eta \in \text{Irr}(\theta^G)$ for otherwise this would imply $N \leq \ker(\theta^G) \leq \text{core}_G(H) = 1$, a contradiction. Select $\eta \in \text{Irr}(\theta^G)$ such that $N \not\leq \ker(\eta)$. Observe, $\ker(\eta_N) = \ker(\eta) \cap N \trianglelefteq G$ is contained in N and hence is trivial as N is a minimal normal subgroup of G by Corollary 2.11. Since η_N is a faithful character of N there exist $\mu_i \in \text{Irr}(\eta_N)$ such that μ_i is nontrivial in the i^{th} coordinate for every $1 \leq i \leq n$. This shows that $\chi(1) = \eta(1) = \eta_N(1) \geq 3n$. On the other hand $\chi(1) \leq n - 1$, a contradiction. ■

CHAPTER 3

Almost Simple Groups

3.1 Useful Lemmas

If G is any group, we know $G/\mathbf{Z}(G) \cong \text{Inn}(G)$ and $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$. If S is a nonabelian simple group we have $S \cong \text{Inn}(S)$. Therefore S is naturally imbedded as a normal subgroup of $\text{Aut}(S)$.

Lemma 3.1. Let G be an almost simple group with $S \leq G \leq \text{Aut}(S)$ for some nonabelian simple group S . Let $m > 1$ be the smallest nontrivial character degree of S and let $k \geq 1$ be the largest character degree of G/S . Then $m > k$.

Proof. If K/S is any subgroup of $\text{Aut}(S)/S$ and $\theta \in \text{Irr}(K/S)$, then $\theta(1) \leq \chi(1)$ for some $\chi \in \text{Irr}(\text{Aut}(S)/S)$. Hence without loss of generality we may assume $G = \text{Aut}(S)$. If S is an alternating group or one of the 26 sporadic simple groups, then G/S is abelian and hence the result holds trivially. Therefore, we assume S is of Lie type with defining characteristic p over a field with q elements. The factor group G/S is a semi-direct product of cyclic groups of orders d, f , and g (in that order) with the exception of $S = D_4(q)$, whose group of graph automorphisms of order g is isomorphic to S_3 . Here d denotes the size of the group of diagonal automorphisms of S , f is the size of the group of field automorphisms of S ($q = p^f$), and g is the size of the group of graph automorphisms of S . If S is in one of the families of groups $A_1(q)$, ${}^2B_2(q)$ (f odd), $B_n(q)$ ($n \geq 3$), $C_n(q)$ ($n \geq 3$), ${}^3D_4(q)$, ${}^2G_2(q)$ (f odd), ${}^2F_4(q)$ (f odd), $E_7(q)$, or $E_8(q)$, then the factor group G/S is abelian and hence the result holds on these groups [10].

If G/S has an abelian subgroup of index l , then the character degrees of G/S are bounded above by l and hence $k \leq l$. If S is in one of the families of groups $B_2(q)$, $G_2(q)$, or $F_4(q)$, then the character degrees of G/S are at most 2 since G/S has an abelian subgroup of index 2. Thus the result holds on these groups. We know G/S has a cyclic subgroup of index fg and hence $k \leq fg$.

The following table summarizes our results. Lower bounds on m are obtained from [8].

Table 1: Bounds on character degrees

Conditions	S	Definition of f	lower bound on m	upper bound on k
$n \geq 2$	$A_n(q)$	$q = p^f$	$q^n - 1$	$2f$
$n \geq 2$	${}^2A_n(q)$	$q^2 = p^f$	$q(q^n - 1)/(q + 1)$ or $(q^n - 1)/(q + 1)$	f
$n \geq 4$	$D_n(q)$	$q = p^f$	$(q^{n-1} - 1)(q^{n-2} + 1)$ or $q^{n-2}(q^{n-1} - 1)$	$6f$
$n \geq 4$	${}^2D_n(q)$	$q^2 = p^f$	$(q^{n-1} + 1)(q^{n-2} - 1)$	f
	$E_6(q)$	$q = p^f$	$q^9(q^2 - 1)$	$2f$
	${}^2E_6(q)$	$q^2 = p^f$	$q^8(q^4 + 1)(q^3 - 1)$	f

If S is one of the families in Table 1, then $m > k$. ■

Definition 3.2. Let p and m be integers greater than 1. A Zsigmondy prime for (p, m) is a prime r such that $r \mid p^m - 1$, but $r \nmid p^i - 1$ for $1 \leq i \leq m - 1$.

It is well-known that a Zsigmondy prime exists for (p, m) except for two families of exceptions when $(p, m) = (2, 6)$ and when $m = 2$ and $p = 2^k - 1$ [12]. If r is a Zsigmondy prime for (p, m) , then m is the order of p in $\mathbb{Z}_r^\times$, the multiplicative group of $\mathbb{Z}_r$. Moreover, $m \mid |\mathbb{Z}_r^\times| = r - 1$ and hence $m < r$.

Lemma 3.3. Let G be an almost simple group with $S \leq G \leq \text{Aut}(S)$ for some nonabelian simple group S of Lie type of characteristic p . Then there exists a prime $r \neq p$ dividing $|S|$ such that $r \nmid |G : S|$.

Proof. It suffices to show the result for $G = \text{Aut}(S)$. If $p^k - 1$ is a factor of $|S|$ and r is a Zsigmondy prime for (p, k) , then $k < r$. If $|\text{Aut}(S) : S| \leq k$, then we have found our prime r that satisfies the lemma. Let $m(S)$ be the largest integer k such that $p^k - 1$ is a factor of $|S|$. Let S be a simple nonabelian group of Lie type from one of the families in Table 2 and let r be a Zsigmondy prime for $(p, m(S))$. Observe, $|\text{Aut}(S) : S|$ can be factored so that $m(S)$ is greater than or equal to each of the factors and hence r is greater than each of the factors. Thus, $r \nmid |\text{Aut}(S) : S|$ and we have found our prime satisfying the lemma, assuming a Zsigmondy prime exists for $(p, m(S))$. As

mentioned previously, there are two families of exceptions when a Zsigmondy prime does not exist. The first family only occurs once, when $p = 2$ and $m(S) = 6$. The second family potentially occurs infinitely many times when p is in the form $2^k - 1$ and $m(S) = 2$. When the case where $p = 2$ and $m(S) = 6$ arises we have computed $|\text{Aut}(S) : S|$ and listed it in Table 2. In this case we observe $|\text{Aut}(S) : S| = 2^a 3^b$ for some nonnegative integers a and b . Since S is a simple nonabelian group it must have at least three prime divisors. We can choose r to be the smallest prime divisor of $|S|$ that is not 2 or 3 and such r satisfies the lemma. The case where $m(S) = 2f = 2$ only occurs when S is a member of the family $A_1(q)$ and $f = 1$. In this case $|\text{Aut}(S) : S| = 2$ and we may choose r to be any prime divisor of $|S|$ that is not p or 2.

Table 2: Simple groups of Lie type: part 1

Conditions	S	Definition of f	$m(S)$	$ \text{Aut}(S) : S $	$ \text{Aut}(S) : S $ when $m(S) = 6$ and $p = 2$
$n \geq 2$	$A_1(q)$	$q = p^f$	$2f$	$f(2, q - 1)$	3
	$A_n(q)$	$q = p^f$	$f(n + 1)$	$2f(n + 1, q - 1)$	2 or 12
$n \geq 2$, odd	${}^2A_n(q)$	$q^2 = p^f$	$\frac{(n+1)f}{2}$	$f(n + 1, q + 1)$	2
	$B_2(q)$	$q = p^f$	$4f$	$2f$	
$n \geq 3$	$B_n(q)$	$q = p^f$	$2nf$	$f(2, q - 1)$	1
$n \geq 3$	$C_n(q)$	$q = p^f$	$2nf$	$f(2, q - 1)$	1
	$D_4(q)$	$q = p^f$	$6f$	$6f(2, q - 1)^2$	6
	${}^3D_4(q)$	$q^3 = p^f$	$2f$	f	3
$n > 4$, even	$D_n(q)$	$q = p^f$	$2f(n - 1)$	$2f(2, q - 1)^2$	
$n > 4$, odd	$D_n(q)$	$q = p^f$	$2f(n - 1)$	$2f(4, q^n - 1)$	
	$G_2(q)$	$q = p^f$	$6f$	f or $2f$	1 or 2
	$F_4(q)$	$q = p^f$	$12f$	$2f$ or f	
	$E_6(q)$	$q = p^f$	$12f$	$2f(3, q - 1)$	
	${}^2E_6(q)$	$q^2 = p^f$	$6f$	$f(3, q + 1)$	
	$E_7(q)$	$q = p^f$	$18f$	$f(2, q - 1)$	
	$E_8(q)$	$q = p^f$	$30f$	f	

If $p^k + 1$ is a factor of $|S|$ and r is a Zsigmondy prime for $(p, 2k)$, then $2k < r$, $r \mid p^{2k} - 1$, and $r \nmid p^i - 1$ for $1 \leq i \leq 2k - 1$. Since $r \mid p^{2k} - 1 = (p^k - 1)(p^k + 1)$ and $r \nmid p^k - 1$ we have $r \mid p^k + 1$ and hence $r \mid |S|$. If $|\text{Aut}(S) : S| \leq 2k$, then we have found our prime r satisfying the lemma. Let

$a(S)$ be the largest integer k such that $p^k + 1$ is a factor of $|S|$. Consider the following table.

Table 3: Simple groups of Lie type: part 2

Conditions	S	Definition of f	$2a(S)$	$ \text{Aut}(S) : S $	$ \text{Aut}(S) : S $ when $2a(S) = 6$ and $p = 2$
$n \geq 2$, even	${}^2A_n(q)$	$q^2 = p^f$	$f(n+1)$	$f(n+1, q+1)$	6
f odd	${}^2B_2(q)$	$q = 2^f$	$4f$	f	
$n \geq 4$,	${}^2D_n(q)$	$q^2 = p^f$	nf	$f(4, q^n + 1)$	
f odd	${}^2G_2(q)$	$q = 3^f$	$6f$	f	
f odd	${}^2F_4(q)$	$q = 2^f$	$12f$	f	

Let S be a simple nonabelian group of Lie type from one of the families in Table 3 and suppose r is a Zsigmondy prime for $(p, 2a(S))$. Observe, $|\text{Aut}(S) : S|$ can be factored so that $2a(S)$ is greater than or equal to each of the factors and hence r is greater than each of the factors. Thus, $r \nmid |\text{Aut}(S) : S|$ and we have found our prime satisfying the lemma. Of the two families of exceptions when Zsigmondy primes do not exist, we need only consider the case when $2a(S) = 6$ and $p = 2$, since $2a(S) \geq 3$. Fortunately, this only occurs when S is a member of the family ${}^2A_n(q)$. In this case $|\text{Aut}(S) : S| = 2 \cdot 3$ and $p = 2$. Once again, since S is a simple nonabelian group it must have at least three prime divisors. We can choose r to be the smallest prime divisor of $|S|$ that is not 2 or 3. ■

Lemma 3.4. Suppose S is a simple group of Lie type over a field of characteristic p and $S \neq {}^2F_4(2)'$. Then the Steinberg character is the unique nonprincipal irreducible character of p -power degree.

Proof. First, we consider Lemma 2.2 of [1], which states that if S is a simple group of Lie type over a field of characteristic p and $S \neq PSL_2(4), PSL_3(2), PSL_3(4), PSL_4(2), PSU_4(2), PSU_6(2), Sp_4(2)', Sp_6(2), \Omega_8^+(2), {}^2B_2(8), {}^2E_6(2), F_4(2), {}^2F_4(2)', G_2(4), PSL_2(9), PSU_4(3), \Omega_7(3)$, or $G_2(3)$, then the Steinberg character is the unique irreducible projective character of degree $p^r > 1$. In fact, more can be said if we restrict our view to ordinary complex characters. If $S = Sp_4(2)'$ (with $p = 2$), we may view S as $PSL_2(9)$ (with $p = 3$) since $Sp_4(2)' \cong PSL_2(9)$ and

observe that the Steinberg character of $PSL_2(9)$ is the unique nonprincipal irreducible character of 3-power degree. The character degrees of the remaining groups listed are known from either the Atlas or Frank Lübeck's website [9]. One can verify that the Steinberg character is the unique nonprincipal irreducible character of p -power degree, except for the group $S = {}^2F_4(2)'$ which has two characters of degree $|S|_2$. ■

3.2 Almost Simple Groups

Gagola and Sezer [5] showed that if G is a nonabelian simple group and H is a nontrivially proper subgroup of G , then (G, H) does not satisfy Hypothesis (*). In this section we investigate the case when (G, H) satisfies Hypothesis (*), with $H < H^G = G'$, N is a minimal normal subgroup of G that is simple and nonabelian, and $O_\infty(G) = 1$. Since $\mathbf{C}_G(N)$ is solvable by Corollary 2.10, we have $\mathbf{C}_G(N) = 1$ and hence $G \lesssim \text{Aut}(N)$. Therefore, G is almost simple. We will quickly reduce to the case where G/N is abelian and hence we will be in the case $H < H^G = G' = N < G \leq \text{Aut}(N)$. The proof of Lemma 3.5 is nearly identical to the proof of Lemma 3.2 of [5] and it is omitted.

Lemma 3.5. Assume (G, H) satisfies Hypothesis (*) where $H < H^G = G'$. Let $k_G(H)$ denote the number of conjugacy classes of G that intersect H nontrivially. Then $|\text{cd}(G)| \leq k_G(H) + 1$.

Lemma 3.6. Assume (G, H) satisfies Hypothesis (*) with $H < H^G = G'$, N is a minimal normal subgroup of G , N is a nonabelian simple group, and $O_\infty(G) = 1$. Then G/N is abelian and hence $H < H^G = G' = N < G \leq \text{Aut}(N)$.

Proof. Recall that $N \leq G'$, as N is a minimal normal subgroup of G and N is a simple nonabelian group. Since $G' < G$ by Theorem A of [5] we also have $N < G$. If $N = A_n$, an alternating group for $6 \neq n \geq 5$, or if N is one of the sporadic simple groups, then $|\text{Aut}(N)/N| \leq 2$ and hence $G = \text{Aut}(N)$ and $G' = N$. Assume N is of Lie type (note that $A_6 \cong \text{PSL}_2(9)$ is of Lie type). Suppose for a contradiction that G/N is nonabelian. Let $\chi \in \text{Irr}(G/N)$ with $\chi(1) > 1$. Then $\chi \in \text{Irr}(G)$ and $\chi_N = \chi(1)1_N$. Let $\phi \in \text{Irr}(\chi_H)^\#$, which exists since $\chi(1) > 1$ and $H^G = G'$. If $\psi \in \text{Irr}(\phi^G)$, then $\psi(1) = \chi(1)$ and hence $\psi_N = \psi(1)1_N$ by Lemma 3.1. Therefore, we have $N \leq \ker(\phi^G) \leq \text{core}_G(H) \leq O_\infty(G) = 1$, a contradiction. ■

Theorem 3.7. Let H be a nontrivial proper subgroup of $G' = H^G$ with $O_\infty(G) = 1$. Suppose N is a minimal normal subgroup of G such that N is a nonabelian simple group. Then (G, H) does not satisfy Hypothesis (*).

Proof. Assume (G, H) satisfies Hypothesis (*). We apply Lemma 3.6 and mimic the proof of Theorem 3.5 of [5]. If N is one of the 14 sporadic simple groups with trivial outer automorphism group, then G is simple and is eliminated by Theorem 3.5 of [5]. If p is a prime divisor of $|H|$, then G does not have a block of defect 0 for the prime p by Proposition 3.4 of [5]. On the other hand, p -blocks of G of defect 0 do exist for almost all primes p . For the remaining 12 sporadic groups, which do not have trivial outer automorphism group, let π denote the set of prime divisors p of $|G|$ for which G has no p -block of defect 0, and let $k_\pi(G)$ be the number of conjugacy classes of G consisting of π -elements. Now H must be a π -group and $k_G(H) \leq k_\pi(G)$. We apply Lemma 3.5, which yields $|\text{cd}(G)| \leq k_G(H) + 1 \leq k_\pi(G) + 1$. Hence the first 10 groups in Table 4 are eliminated since the data in the table contradicts the previous inequality.

Table 4: Comparing $|\text{cd}(G)|$ to $k_\pi(G)$

Group	π	$ \text{cd}(G) $	$k_\pi(G)$
$\text{Aut}(M_{12})$	$\{2\}$	12	8
$\text{Aut}(J_2)$	$\{2\}$	16	10
$\text{Aut}(HS)$	$\{2\}$	20	16
$\text{Aut}(McL)$	$\{2\}$	17	7
$\text{Aut}(He)$	$\{2\}$	25	14
$\text{Aut}(Fi_{22})$	$\{2\}$	51	27
$\text{Aut}(Fi'_{24})$	$\{2\}$	91	20
$\text{Aut}(HN)$	$\{2\}$	43	16
$\text{Aut}(O'N)$	$\{2\}$	19	11
$\text{Aut}(J_3)$	$\{2\}$	14	8
$\text{Aut}(M_{22})$	$\{2\}$	10	10
$\text{Aut}(Suz)$	$\{2, 3\}$	36	47

Now we eliminate $N = M_{22}$ with the following observation. Since (N, H) does not satisfy Hypothesis (*) from Theorem 3.5 of [5], there exist $\theta \in \text{Irr}(H)^\#$ and $\psi, \chi \in \text{Irr}(\theta^N)$ such that $\psi(1) < \chi(1)$. Since θ^G is a sum of characters of equal degree we must have ψ induces irreducibly

to G and χ extends to G with $\psi^G(1) = \chi(1)$. The only two irreducible characters of N that induce irreducibly to G are of degree 280. This eliminates M_{22} , since M_{22} does not have an irreducible character of degree $560 = \chi(1) = \psi^G(1)$. Now if $N = \text{Suz}$, there is only one irreducible character χ of $\text{Aut}(N)$ of degree 187110. Let θ be a nonprincipal irreducible constituent of χ_H . Since θ induces to a sum of irreducible characters of equal degree we must have $\theta^G = [\theta^G, \chi]\chi$. Hence χ vanishes on elements of order 5, since H is a π -group and $5 \notin \pi$. But χ takes the value 10 on the conjugacy class 5A, which contains elements of order 5. This eliminates the Suzuki group and hence all of the sporadic simple groups are eliminated.

Now we assume that N is one of the alternating groups A_n for $n \geq 7$. Since G is not simple we must have $G = S_n$, as $G \leq \text{Aut}(A_n) = S_n$ for $n \geq 7$. For $n \geq 7$, the symmetric group S_n has characters of p -defect zero for all primes $p \geq 5$ [7]. Applying Proposition 3.4 of [5] we see that H must be a $\{2, 3\}$ -group. Now select a 5-cycle g in S_n . Let χ be the nonprincipal irreducible constituent of the permutation character of S_n in its natural action on n points so that $\chi(1) = n - 1$ and $\chi(g) = n - 5 - 1 = n - 6 \neq 0$. Let $\sigma \in \text{Irr}(S_n)$ be the sign character of S_n and observe $\sigma\chi \in \text{Irr}(S_n)$. Now χ and $\sigma\chi$ are the unique irreducible characters of S_n of degree $n - 1$ [2]. Let $\theta \in \text{Irr}(\chi_H)^\#$. Then $\theta^G = e_1\chi + e_2\sigma\chi$, and $\sigma\chi(g) = 1(n - 6) \neq 0$. Since H is a $\{2, 3\}$ -group, θ^G vanishes on $\{2, 3\}'$ -elements. Thus $0 = \theta^G(g) = e_1\chi(g) + e_2\sigma\chi(g) = e_1(n - 6) + e_2(n - 6) \neq 0$, a contradiction. This eliminates $N = A_n$ for $n \geq 7$. The groups A_5 and A_6 are of Lie type and are eliminated in the following paragraph.

We assume N is a simple group of Lie type such that $N \neq {}^2F_4(2)'$. Since $G \leq \text{Aut}(N)$, we have $|G : N| \leq \text{dfg}$ and we apply Lemma 3.6 to conclude G/N is abelian and hence $H < N = G' = H^G$. Let $\psi \in \text{Irr}(N)$ be the Steinberg character of N which, by Lemma 3.4, is the unique nonprincipal irreducible character of N of p -power degree. Let $\chi \in \text{Irr}(G)$ be an extension of ψ to G and let $\phi \in \text{Irr}(\chi_H)^\#$, which exists as $\chi(1) > 1$. We will show that $\phi^N = m\psi$ for $m = [\phi^N, \psi]$. If $\eta \in \text{Irr}(\phi^N)$, then any irreducible constituent of η^G lies above ϕ and hence has degree $\chi(1)$. Since $N \triangleleft G$, we know $\eta(1)$ divides $\chi(1)$ and hence $\eta(1)$ must be a power of p . Thus $\eta = 1_N$ or $\eta = \psi$. The former is not possible since 1_G lies above 1_N , and hence $\eta = \psi$. This shows that $\phi^N = m\psi$.

Let r be a prime satisfying Lemma 3.3, that is, r divides the order of N , but r does not divide $|G : N|$. Let $g \in N$ be an element of order r . Then $\phi^N(g) = m\psi(g) \neq 0$, as g is a p -regular element. Hence g belongs to a conjugate of H and thus $r \mid |H|$. Now N has a character of r -defect zero, and hence G must have one as well, since $r \nmid |G : N|$. On the other hand, Proposition 3.4 of [5] says G has no defect zero characters with respect to r , a contradiction. Now, we assume $N = {}^2F_4(2)'$ and we observe that $|\text{Aut}(N) : N| = 2$. A Steinberg character has degree 2^{11} and induces irreducibly to a character of degree 2^{12} . Hence $\text{Aut}(N)$ has a character of t -defect zero for every prime t . We apply Proposition of 3.4 [5], which implies no prime divides $|H|$. This eliminates all almost simple groups of Lie type. ■

CHAPTER 4

Subnormality and Examples

Let $G = S_4$ and let H be a subgroup of G such that H is subnormal in G . Then (G, H) satisfies Hypothesis (*). Gagola and Sezer posed the question: if H is a subgroup of G and (G, H) satisfies Hypothesis (*) is H subnormal in G ? The following is a partial result.

Proposition 4.1. Let (G, H) satisfy Hypothesis (*). Suppose n can be chosen so that n is maximal with respect to $H \leq G^{(n)}$. If $(|G : G^{(n)}|, d) = 1$ for every $d \in \text{cd}(G^{(n)})$ with $d \leq |G : G^{(n)}|$, then $G^{(n+1)} \leq H \leq G^{(n)}$. In particular, $H \triangleleft\triangleleft G$.

Proof. If $G^{(n+1)} \leq H$ then we are done. Therefore, we assume $G^{(n+1)} \not\leq H$ and by maximality of n we know $H \not\leq G^{(n+1)}$. Now $G^{(n+1)}H/G^{(n+1)} \cong H/(H \cap G^{(n+1)})$ is a nontrivial abelian subgroup of $G^{(n)}/G^{(n+1)}$. Hence we can find λ , a nontrivial linear character of H , such that $H \cap G^{(n+1)} \leq \ker(\lambda)$ and λ extends to $\mu \in \text{Irr}(G^{(n)})$. If $\chi \in \text{Irr}(\mu^G)$, then $\chi(1) = \frac{\chi(1)}{\mu(1)} \mid |G : G^{(n)}|$. Observe, $\chi \in \text{Irr}(\lambda^G)$ and hence all irreducible constituents of λ^G have degree $\chi(1)$. If $\phi \in \text{Irr}(\lambda^{G^{(n)}})$, then $\phi(1)$ divides $|G : G^{(n)}|$ since $\chi(1)$ divides $|G : G^{(n)}|$. On the other hand, $\phi(1) \in \text{cd}(G^{(n)})$ with $\phi(1) \leq |G : G^{(n)}|$ and hence $(|G : G^{(n)}|, \phi(1)) = 1$. This forces $\phi(1) = 1$. Thus all irreducible constituents of $\lambda^{G^{(n)}}$ have degree 1 and $\phi_C \in \text{Irr}(C)$ where $C = \text{core}_{G^{(n)}}(H)$. Hence Lemma 2.1 applies and we conclude $G^{(n+1)} = (G^{(n)})' \leq H$, a contradiction. Therefore, $G^{(n+1)} \leq H$. ■

The next example shows that the conclusion of Proposition 4.1 does not always hold when the character degree hypothesis of Proposition 4.1 is not satisfied.

Example 4.2. Select primes r and s and a power of s , say q , such that $p = \frac{q^r - 1}{q - 1}$ is a prime. Let $V = \text{GF}(q^r)$ and let $\sigma \in \text{Gal}(\text{GF}(q^r)/\text{GF}(q))$ be the q power map. Let $\delta \in \text{GF}(q^r)$ be a primitive $(q^r - 1)^{\text{th}}$ root of unity and let $\epsilon = \delta^{q-1}$, a primitive p^{th} root of unity. Let

$$L = \langle a, b, c : a^p = b^p = c^p = [a, c] = [b, c] = 1 \text{ and } [b, a] = c \rangle.$$

Notice $\text{Gal}(\text{GF}(q^r)/\text{GF}(q))$ acts on L by $a^\sigma = a^q$, $b^\sigma = b$, and $c^\sigma = c^q$. If $v \in V$, then L acts on V by $v^a = v\epsilon$, $v^b = v$, and $v^c = v$. Now we define an action of $\langle \sigma \rangle \rtimes L$ on V by setting $v^{\sigma^i a^j b^k c^l} = v^{q^i \epsilon^j}$. Let $G = (\langle \sigma \rangle \rtimes L) \rtimes V$ and notice $|G| = p^3 q^r r$. Now the group $H = \langle c \rangle = \mathbf{Z}(L)$ is normal in G . We will show that (G, H) satisfies Hypothesis (*), but first we discuss some convenient notation.

Let μ_0 be the natural irreducible character of $\text{GF}(s)$ so that $\mu_0(1) = e^{\frac{2\pi i}{s}}$. The trace map from V onto $\text{GF}(s)$ is defined by $\text{trace}(v) = \sum v^\tau$ where τ runs over the elements of $\text{Gal}(\text{GF}(q^r)/\text{GF}(s))$. The irreducible characters of V are indexed by the elements of V in a natural way. Fix $w \in V$, we define $\lambda_w(v) = \mu_0(\text{trace}(wv))$. Observe $\text{Irr}(V) = \{\lambda_w : w \in V\}$. Let $\mu \in \text{Irr}(\langle b \rangle)$ such that $\mu(b) = e^{\frac{2\pi i}{p}}$, and $\eta \in \text{Irr}(H)$ such that $\eta(c) = e^{\frac{2\pi i}{p}}$.

Now $A = H \times \langle b \rangle \times V$ is an abelian group of order $p^2 q^r$ and hence η extends to $p q^r$ characters of A . All of the irreducible constituents of η^A are of the form $\eta \times \mu^m \times \lambda_w$ with $1 \leq m \leq p$. Direct computation shows us $\lambda_w^a = \lambda_{w\epsilon^{-1}}$, $\lambda_w^\sigma = \lambda_{w^q}$, $(\eta^n \times \mu^m)^a = \eta^n \times \mu^{m-n}$, and $(\eta^n \times \mu^m)^\sigma = \eta^{nq^{r-1}} \times \mu^m$. Next we show $I_G(\eta \times \mu^m \times \lambda_w) = A$. Any element of G can be written in the form $va^i b^j c^k \sigma^l$ where $v \in V$, $1 \leq i, j, k \leq p$, and $1 \leq l \leq r$. Let $g \in I_G(\eta \times \mu^m \times \lambda_w) \leq I_G(\eta \times \mu^m)$ and write $g = va^i b^j c^k \sigma^l$. Now we compute $\eta \times \mu^m = (\eta \times \mu^m)^g = \eta^{q^{l(r-1)}} \times \mu^{m-i}$, which implies $r \mid l$ and $p \mid i$, and so $g = vb^j c^k \in A$. Hence all characters of the form $\eta \times \mu^m \times \lambda_w$ induce irreducibly to G . Moreover, all irreducible constituents of η^G have degree pr . Since all of the nonprincipal irreducible characters of H are G -conjugate we have shown (G, H) satisfies Hypothesis (*).

Fix $K \leq HV$ such that $H \leq K$. We show that (G, K) satisfies Hypothesis (*). Let β be a nonprincipal irreducible character of K . If $H \not\leq \ker(\beta)$, then β^G is a sum of irreducible characters all of equal degree. Thus we assume $H \leq \ker(\beta)$ and we observe that β extends to irreducible characters of A of the form $1_H \times \mu^i \times \lambda_w$, where w is necessarily not 0. Now $(1_H \times \mu^i \times \lambda_w)^a = 1_H \times \mu^i \times \lambda_{w\epsilon^{-1}}$ shows that the LV -orbit containing $(1_H \times \mu^i \times \lambda_w)$ has size $p = |LV : A|$ and hence $(1_H \times \mu^i \times \lambda_w)^{LV} \in \text{Irr}(LV)$. As $0 \neq w \in V$, we may write $w = \delta^j$ for some $1 \leq j \leq q^r$ and $(1_H \times \mu^i \times \lambda_w)^\sigma = 1_H \times \mu^i \times \lambda_{w^q} = (1_H \times \mu^i \times \lambda_w)^{a^{-j}}$ since $\delta^{q-1} = \epsilon$. This shows the LV -orbit containing $(1_H \times \mu^i \times \lambda_w)$ is $\langle \sigma \rangle$ -invariant and hence $(1_H \times \mu^i \times \lambda_w)^{LV}$ is G -invariant. Since $|G : LV| = r$, a prime, $(1_H \times \mu^i \times \lambda_w)^{LV}$ extends to G . Thus all nonprincipal irreducible characters

of K with H in their kernel induce to G as a sum of irreducible characters all of degree p .

Next we compute G' and G'' . Any element of G can be written in the form $va^ib^jc^k\sigma^l$ where $v \in V$, $1 \leq i, j, k \leq p$, and $1 \leq l \leq r$. We define a homomorphism ϕ_1 from G onto $\langle b, \sigma \rangle$ by $\phi_1(va^ib^jc^k\sigma^l) = b^j\sigma^l$. As $\langle b, \sigma \rangle$ is abelian we have $G' \leq \ker(\phi_1)$. Furthermore, $\ker(\phi_1) = \{va^ic^j : v \in V \text{ and } 1 \leq i, j \leq p\}$. Similarly, define a homomorphism ϕ_2 from $\ker(\phi_1)$ into $\langle a, c \rangle$ by $\phi_2(va^ic^j) = a^ic^j$ with $\ker(\phi_2) = V$. Now $\langle a, c \rangle$ is abelian and hence $G'' \leq (\ker(\phi_1))' \leq \ker(\phi_2) = V$. Since the action of G on V is irreducible, V has no nontrivial proper G -invariant subgroups, and hence $G'' = V$ as clearly $G'' \neq 1$. Since G/H is nonabelian and (G, H) satisfies Hypothesis (*), we must have $H \leq G'$, and hence $HV \leq G' \leq \ker(\phi_1)$ and $|\ker(\phi_1) : HV| = p$. Now $HV \trianglelefteq G$ and G/HV is nonabelian, hence $HV < G'$. This forces $G' = \ker(\phi_1)$. Observe, H and K are contained in G' , yet H and K are not contained in G'' , nor is G'' contained in either H or K (except in the case where $K = HV$).

The next example shows that the hypothesis from Example 4.2, $\frac{q^r-1}{q-1}$ is a prime, is needed.

Example 4.3. Select a prime $r \geq 5$ and a prime q , where the order of q modulo r is either $r-1$ or $(r-1)/2$ and such that $\frac{q^r-1}{q-1}$ is composite. Then there exists a prime p dividing $\frac{q^r-1}{q-1}$ such that $p \leq q^{(r+1)/2} + 1$. Since r is a prime that is greater than or equal to 5 we know that $p \nmid q-1$. Therefore, p is a Zsigmondy prime divisor of $q^r - 1$ and we write $\frac{q^r-1}{q-1} = pt$ for some integer $t > 1$. Let $V = \text{GF}(q^r)$ and let $\sigma \in \text{Gal}(\text{GF}(q^r)/\text{GF}(q))$ be the q power map, hence $\langle \sigma \rangle$ acts on V . Let $\delta \in \text{GF}(q^r)$ be a primitive $(q^r - 1)^{\text{th}}$ root of unity and let $\epsilon = \delta^{t(q-1)}$, a primitive p^{th} root of unity. The cyclic group $C_p = \langle a \rangle$ of order p acts on V by $v^a = v \cdot \epsilon$. Now we define an action of $\langle \sigma \rangle \times \langle a \rangle$ on V by setting $v^{\sigma^i a^j} = v^{q^i} \epsilon^j$. Let $G = (\langle \sigma \rangle \times \langle a \rangle) \ltimes V$ and notice $|G| = pq^r r$.

Fix $1 \subsetneq K \subsetneq V$. We show that (G, K) does not satisfy Hypothesis (*). Assume (G, K) satisfies Hypothesis (*). If $\lambda \in \text{Irr}(K)^{\#}$ then λ extends to V . Since (G, K) satisfies Hypothesis (*) all of the irreducible constituents of λ^V lie in G -orbits with equal size. Fix $w \in V$ and define λ_w as in Example 4.2. Thus $\text{Irr}(V) = \{\lambda_w : w \in V\}$. Direct computation shows us $\lambda_w^a = \lambda_{w\epsilon^{-1}}$ and $\lambda_w^\sigma = \lambda_{w^q}$. There are exactly $q-1$ orbits of size p , hence there are exactly $p(q-1)$ irreducible

characters of V in orbits of size p . Thus $p(q-1) = |V : K|n + m$, where

$$n = |\{\lambda \in \text{Irr}(K)^\# : \lambda \text{ lies below a } G\text{-orbit of } \text{Irr}(V) \text{ of size } p\}|$$

and

$$m = |\{\lambda \in \text{Irr}((1_K)^V) : \lambda \text{ is in a } G\text{-orbit of } \text{Irr}(V) \text{ of size } p\}|.$$

Since $|V : K|$ is a power of q we must have $m > 0$. Thus $K \leq \ker(\lambda_w)$ with λ_w in a G -orbit of size p . Any G -orbit of size p contains exactly one character λ_α such that $\alpha \in \text{Fix}(\sigma)$. Hence we can find $g \in G$ such that $(\lambda_w)^g = \lambda_\alpha$ with $\alpha \in \text{Fix}(\sigma)$. Thus $K^g \leq \ker(\lambda_\alpha) = \ker(\text{trace})$ (the trace map is define in Example 4.2). Let $H = \langle (K^g)^{\sigma^i} : 1 \leq i \leq r \rangle$ and notice that (G, H) satisfies Hypothesis $(*)$ by Corollary 2.4 of [5] with H a σ -invariant subgroup of $\ker(\text{trace})$. Since the order of q modulo r is $r-1$ or $(r-1)/2$, we know that either $H = \ker(\text{trace})$ or $|H| = q^{(r-1)/2}$.

First, we assume $H = \ker(\text{trace})$. Observe, $\ker(\text{trace}) \leq \ker(\lambda_\alpha)$ if and only if $\alpha \in \text{Fix}(\sigma)$. If $(\lambda_w)_H \in \text{Irr}(H)$ then $\lambda_{w+x} = \lambda_w \lambda_x$ is an extension of $(\lambda_w)_H$ if and only if λ_x in $\text{Irr}(V/H) = \{\lambda_\alpha : \alpha \in \text{Fix}(\sigma)\}$. Furthermore, λ_w is in a G -orbit of size p if and only if $w = \alpha\epsilon^i$ for some i and some $\alpha \in \text{Fix}(\sigma)$. Define $P = \{\alpha\epsilon^i : 1 \leq i \leq p \text{ and } \alpha \in \text{Fix}(\sigma)\}$ and observe P is the union of all G -orbits of V of size p and $\{0\}$. Fix $\alpha \in \text{Fix}(\sigma)$ and $w \in P - \text{Fix}(\sigma)$, then λ_w and $\lambda_{w+\alpha}$ are both extensions of $(\lambda_w)_H \in \text{Irr}(H)^\#$. Hence $\lambda_{w+\alpha}$ lies in a G -orbit of $\text{Irr}(V)$ of size p . In particular, $w + \alpha \in P$ for $w \in P - \text{Fix}(\sigma)$ and $\alpha \in \text{Fix}(\sigma)$. Clearly P is closed under multiplication by ϵ . If $\alpha\epsilon^i, \beta\epsilon^j \in P$ with $i \geq j$ and $\alpha, \beta \in \text{Fix}(\sigma)$, then $\alpha\epsilon^i + \beta\epsilon^j = \epsilon^j(\alpha\epsilon^{i-j} + \beta) \in P$. This shows that P is a subgroup of V . Hence $|P|$ divides q^r , but on the other hand we also have $|P| = p(q-1) + 1$. Thus $p \mid q^i - 1$ for some $1 \leq i \leq r$, but this implies $i = r$ as p is a Zsigmondy prime divisor of $q^r - 1$. Hence $P = V$, a contradiction. This proves that $(G, H = \ker(\text{trace}))$ does not satisfy Hypothesis $(*)$.

Next we assume $|H| = q^{(r-1)/2}$. Then we write $p(q-1) = |V : H|n + m$, where

$$n = |\{\lambda \in \text{Irr}(H)^\# : \lambda \text{ lies below a } G\text{-orbit of } \text{Irr}(V) \text{ of size } p\}|$$

and

$$m = |\{\lambda \in \text{Irr}((1_H)^V) : \lambda \text{ is in a } G\text{-orbit of } \text{Irr}(V) \text{ of size } p\}|.$$

Since the action of G on V is irreducible we know $\ker(\lambda_1 + \lambda_\epsilon + \dots + \lambda_{\epsilon^{p-1}})$ is trivial and hence $H \not\leq \ker(\lambda_{\epsilon^i})$ for some $0 < i < p$. Now $\ker(\lambda_{\epsilon^i}) = \ker(\lambda_{\alpha\epsilon^i})$ for any $\alpha \in \text{Fix}(\sigma)$ and hence $H \not\leq \ker(\lambda_{\alpha\epsilon^i})$ for all $\alpha \in \text{Fix}(\sigma)$. If $\alpha, \beta \in \text{Fix}(\sigma)$ with $\alpha \neq \beta$, then $(\lambda_{\alpha\epsilon^i})_H \neq (\lambda_{\beta\epsilon^i})_H$ and thus $n \geq q - 1$. As $H \leq \ker(\text{trace}) = \ker(\lambda_\alpha)$ for every $\alpha \in \text{Fix}(\sigma)$, we know $m \geq q - 1$. If $m = q - 1$, then we can show P (as constructed in the previous paragraph) is a subgroup of V , a contradiction, hence $m > q - 1$. Thus $p(q - 1) = |V : H|n + m = q^{(r+1)/2}n + m > q^{(r+1)/2}(q - 1) + q - 1$ and hence $p > q^{(r+1)/2} + 1$, a contradiction. Thus (G, H) does not satisfy Hypothesis $(*)$ and so (G, K) does not satisfy Hypothesis $(*)$.

In the next example, G' and G'' form bottle necks. That is, if (G, H) satisfies Hypothesis $(*)$ then $G' \leq H \leq G$, or $G'' \leq H \leq G'$, or $H = 1$.

Example 4.4. Let p be an odd prime and let P be the extra special group of order p^{2n+1} with exponent p . Let q be a Zsigmondy prime divisor of $p^{2n} - 1$. Then there exists an automorphism σ of P such that σ has order q and σ fixes the elements of $\mathbf{Z}(P)$ [11]. Let $G = \langle \sigma \rangle \rtimes P$. Note that $G' = P$ and $G'' = P' = \mathbf{Z}(P)$ and there are exactly $p^{2n} = |P : P'|$ linear characters of P . Since q is a Zsigmondy prime divisor of $p^{2n} - 1$, we know G/P acts irreducibly on $P/\mathbf{Z}(P)$. Hence G/P acts irreducibly on $\text{Irr}(P/\mathbf{Z}(P)) = \{\lambda \in \text{Irr}(P) : \lambda(1) = 1\}$. If λ is a nonprincipal linear character of P , then $I_G(\lambda) = P$ and hence λ induces irreducibly to G . There are exactly $p - 1$ faithful irreducible characters of P , all of which have degree p^n [6]. If χ is a faithful irreducible character of P then $\chi_{\mathbf{Z}(P)} = p^n \eta$, where $\eta \in \text{Irr}(\mathbf{Z}(P))^\#$. Hence χ and η are fully ramified with respect to $P/\mathbf{Z}(P)$ and thus χ vanishes off of $\mathbf{Z}(P)$. This shows us that $\chi(g) = p^n \eta(g)$ if $g \in \mathbf{Z}(P)$ and $\chi(g) = 0$ otherwise. Since G/P acts trivially on $\mathbf{Z}(P)$ we know that G/P acts trivially on the faithful irreducible characters of P . Thus $I_G(\chi) = G$ and hence χ extends to a character of G , as G/P is cyclic.

Suppose $G'' \leq H \leq G'$, so (G', H) satisfies Hypothesis $(*)$. If $\mu \in \text{Irr}(H)^\#$, then $\mu^{G'}$ is either a

sum of linear characters or a sum of irreducible characters of degree p^n . In either case, μ^G is a sum of characters of equal degree. Thus (G, H) satisfies Hypothesis (*). On the other hand, suppose (G, H) satisfies Hypothesis (*) and $H < G'$. Since all irreducible characters of G' of equal degree induce to G as a sum of characters of equal degree we must have (G', H) satisfies Hypothesis (*). Hence either $G'' \leq H \leq G'$ or $H \leq G'' = \mathbf{Z}(P)$. Furthermore, if $G'' < H < G'$ then $H^G = G'$, providing us with another example where the hypothesis of Theorem 5.1 is satisfied.

We say a group G is $(*)$ -saturated if (G, H) satisfies Hypothesis (*) whenever $H < G'$. In other words, all subgroups that can satisfy Hypothesis (*) do satisfy Hypothesis (*). The following are examples of $(*)$ -saturated groups.

Example 4.5. Let G be a Frobenius group with abelian kernel K and an abelian complement. Since the complement is abelian, we know $K = G'$. Since K is the Frobenius kernel, all of the nonprincipal irreducible characters of $K = G'$ induce irreducibly to G . Let H be a subgroup of G' and $\lambda \in \text{Irr}(H)^\#$. Then λ extends to nonprincipal linear characters of G' that induce irreducibly to G . Hence all irreducible constituents of λ^G have degree $|G : G'| \lambda(1) = |G : G'|$. Thus (G, H) satisfies Hypothesis (*) and hence G is $(*)$ -saturated.

Example 4.6. Let G be any group with exactly two character degrees, say 1 and d , and let $H < G'$. We show that (G, H) satisfies Hypothesis (*). If $\lambda \in \text{Irr}(H)^\#$ and $\mu \in \text{Irr}(G)$ such that μ is linear, then $H \leq G' \leq \ker(\mu)$. Hence all of the irreducible constituents of λ^G must have degree d . Thus (G, H) satisfies Hypothesis (*) and hence G is $(*)$ -saturated.

Observe that Example 4.5 is a special case of Example 4.6, as any Frobenius group with abelian kernel and abelian complement has exactly two character degrees. The next example shows that there are groups that are $(*)$ -saturated with arbitrarily many character degrees.

Example 4.7. Let p be an odd prime, $C_{p^{n+1}} = \langle x \rangle$ be the cyclic group of order p^{n+1} under multiplication, and σ be the automorphism of $C_{p^{n+1}}$ that sends x to x^{p+1} . Set $G = C_{p^{n+1}} \rtimes \langle \sigma \rangle$. Let $C = \langle x \rangle$ be the image of $C_{p^{n+1}}$ in G so that $x^\sigma = x^{p+1}$. Direct computation shows $[x, \sigma] = x^p$ and $[x^i \sigma^j, x^k \sigma^l] = x^{pm} = (x^p)^m$ for some integer m . Thus $G' = \langle x^p \rangle$ and $|G'| = p^n$. Let λ be an

irreducible character of C such that $\lambda(x) = \epsilon$, where ϵ is a primitive p^{n+1} root of unity. Let $\mathcal{O}(k)$ be the orbit of $\lambda^k \in \text{Irr}(C)$ under the action of G/C on $\text{Irr}(C)$ and let k_p denote the p -part of k . The size of $\mathcal{O}(k)$ is p^n/k_p as the multiplicative order of $(p+1)$ modulo p^r is p^{r-1} for $r > 0$.

Next we compute the character degrees of G . Let $\lambda^k \in \text{Irr}(C)$ and write $k_p = p^l$. Then $p^{n-l} = |\mathcal{O}(k)| = |G : I_G(\lambda^k)|$. Furthermore, $I_G(\lambda^k)/C$ is cyclic, as G/C is cyclic. Thus λ^k extends to an irreducible character η of $I_G(\lambda^k)$, and $\eta^G \in \text{Irr}(G)$ by Theorem 6.11 of [4]. Observe that $|\mathcal{O}(k)| = p^{n-l} = \eta^G(1) \in \text{cd}(G)$. The orbit sizes for the action of G/C on $\text{Irr}(C)$ are $1, p, p^2, \dots, p^n$ and thus $\{1, p, p^2, \dots, p^n\} = \text{cd}(G)$, showing $|\text{cd}(G)| = n+1$.

Let H be a subgroup of C . We show that (G, H) satisfies Hypothesis (*). Let $\mu \in \text{Irr}(H)^\#$. If λ^i is an extension of μ , then $\ker(\mu) = \ker(\lambda^i)$ since the subgroups of C are linearly ordered under subset inclusion. If λ^i and λ^j are extensions of μ , then $\ker(\lambda^i) = \ker(\mu) = \ker(\lambda^j)$. As $\langle x^{p^{n+1}/i_p} \rangle = \ker(\lambda^i)$ and $\ker(\lambda^i) = \ker(\lambda^j)$, we have $i_p = j_p$. Moreover, $|\mathcal{O}(i)| = |\mathcal{O}(j)|$. This shows that all irreducible constituents of $(\lambda^i)^G$ and of $(\lambda^j)^G$ have degree $p^n/i_p = p^n/j_p$. Thus (G, H) satisfies Hypothesis (*) and hence G is (*)-saturated.

In the previous three examples the commutator subgroup has been abelian. In the next example the commutator subgroup is nonabelian.

Example 4.8. Let $G = \text{SL}_2(3)$. The irreducible characters of $G' \cong Q_8$, the quaternion group, are well known. There are exactly three nonprincipal linear characters of G' that are permuted by G/G' . If $\lambda \in \text{Irr}(G')^\#$ is linear, then $I_G(\lambda) = G'$ and hence $\lambda^G \in \text{Irr}(G)$. If ψ is the unique irreducible character of G' of degree 2, then $I_G(\psi) = G$ and hence ψ is extendible to G . If H is a nontrivial subgroup of $G' \cong Q_8$, then $G'' \leq H \leq G'$. Hence (G', H) satisfies Hypothesis (*). Let $\mu \in \text{Irr}(H)^\#$, then $\mu^{G'}$ is a sum of irreducible characters all of equal degree. Thus either $\mu^{G'} = \psi$ or else $\mu^{G'}$ is a sum of nonprincipal linear characters of G' . In either case, μ^G is a sum of characters all of equal degree. Thus (G, H) satisfies Hypothesis (*) and hence G is (*)-saturated.

CHAPTER 5

Conclusion

5.1 Main Result

Theorem 5.1. Let (G, H) satisfy property $(*)$ with $H < G'$ and $H^G = G'$. Then G is solvable.

Proof. Suppose G is nonsolvable. Let $S = O_\infty(G)$ be the solvable radical of G and observe $\text{core}_G(H) \leq S$, as $\text{core}_G(H)$ is solvable by Proposition 2.7. Now $(G/S, SH/S)$ satisfies property $(*)$ with $SH/S \leq SG'/S = (G/S)'$, H/S is core-free in G/S , and $(SH/S)^{G/S} = (G/S)'$. If $SH/S = (G/S)'$, then $SH \trianglelefteq G$ and hence SH/S is solvable by Lemma 2.9. This implies $SH = G'$ is solvable, and hence G is solvable. Without loss of generality we may assume $SH/S < (G/S)'$. Therefore, we may replace G with G/S and H with SH/S . Using Remark 2.13 we may also assume that $G' = \langle H, H^x \rangle$ for any $x \in G$ such that $H^x \neq H$.

Select $N \leq G$ minimal with the property that N is nonsolvable and normal in G . Then N is a minimal normal subgroup of G by Corollary 2.11, and N is a nonabelian simple group by Proposition 2.18. This contradicts Theorem 3.7. Hence G is solvable. ■

5.2 Closing Remarks

Let G be a group. Subgroups of G that satisfy Hypothesis $(*)$ in G “act” like subnormal subgroups of G . It is already known that if H and K are subnormal subgroups of G , then $H \cap K$, $\langle H, K \rangle$, and H^g are subnormal subgroups of G for any $g \in G$. Gagola and Sezer also showed that if (G, H) and (G, K) satisfy Hypothesis $(*)$, then $(G, \langle H, K \rangle)$ and (G, H^g) satisfy Hypothesis $(*)$ for any $g \in G$. This motivates the conjecture, “if (G, H) and (G, K) satisfy Hypothesis $(*)$, then $(G, H \cap K)$ satisfies Hypothesis $(*)$.” If this is true, then $(G, \text{core}_G(H))$ would also satisfy Hypothesis $(*)$. This shows that a good starting point might be to show that $(G, \text{core}_G(H))$ satisfies Hypothesis $(*)$.

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