**NONLINEAR UNCERTAINTY QUANTIFICATION, SENSITIVITY ANALYSIS,
AND UNCERTAINTY PROPAGATION OF A DYNAMIC ELECTRICAL
CIRCUIT**

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ABSTRACT

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The development of a statistically-based process for verification and validation of computational experiments is presented in this study. The process can be used to identify sources of uncertainty, quantify magnitudes of uncertainty, and propagate uncertainty through a model. Model form error was identified through prototype experiments with the system and subsystem, and methods for reducing model form error are presented. Existing validation metrics are applied to the system in this analysis, and a new statistical validation metric is introduced. The methodology for performing Uncertainty Analysis (UA), nonlinear Sensitivity Analysis (SA), and nonlinear Uncertainty Propagation (UP) is presented in this investigation as part of the validation process. The results of this portion of the methodology guided the development of the experimental design and evaluation.

Experimental validation experiments were developed for a simple electrical system in order to demonstrate the computational-to-experimental validation

process. The process is applicable to any system, but a simple example was chosen so that any interested person can follow the implementation. Despite the simplicity of the system selected, the analysis proved to be complicated and tedious, while also identifying many avenues for future work. A numerical example is presented along with the relevant data in sufficient detail to demonstrate how the analysis was performed. By applying this new process, the electrical system is studied from a statistical perspective, with an emphasis on uncertainty quantification and propagation. The sensitivity analysis discovered that the behavior of each component varied significantly, and several critical parameters were identified. By identifying and quantifying the uncertainty in each parameter, the quality of the computational model can be improved, and decisions can be made with quantifiable confidence.

Dedicated posthumously to Ace Doty.

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LIST OF SYMBOLS

$\frac{d\{E_{In}(t)\}}{dt}$	=	Rate of change of energy input to system
$\frac{d\{E_{Out}(t)\}}{dt}$	=	Rate of change of energy loss from system
$\frac{d\{E_{system}(t)\}}{dt}$	=	Rate of change of energy change within system
$\frac{d\{I_L(t)\}}{dt}$	=	Rate of change of inductor current
$\frac{d\{V_C(t)\}}{dt}$	=	Rate of change of capacitor voltage
$E_{In}(t)$	=	Dynamic energy input to system
$E_{Out}(t)$	=	Dynamic energy loss from system
$E_{system}(t)$	=	Dynamic energy change within system
R	=	Resistance
L	=	Inductance
C	=	Capacitance
$I(t)$	=	General dynamic circuit current
$I_{In}(t)$	=	Dynamic input current
I_{L_0}	=	Initial inductor current
$V_R(t)$	=	Dynamic resistor voltage
$V_L(t)$	=	Dynamic inductor voltage
$V_C(t)$	=	Dynamic capacitor voltage
$V_{In}(t)$	=	Dynamic input voltage, step change herein
$P_R(t)$	=	Dynamic resistor power
$P_L(t)$	=	Dynamic inductor power
$P_C(t)$	=	Dynamic capacitor power
$P_{In}(t)$	=	Dynamic input power, step change herein
$\dot{S}_{gen}$	=	Entropy generation rate
T_0	=	Reference temperature of dead state
$W_C(t)$	=	Dynamic capacitor power
$W_{In}(t)$	=	Dynamic input work, step change herein
$W_L(t)$	=	Dynamic inductor work
$W_R(t)$	=	Dynamic resistor work
$\dot{W}_{Actual}$	=	Actual work rate (power)
$\dot{W}_{In}$	=	Input work rate (power)
$\dot{W}_{System}$	=	Work rate in system (power)
$\dot{W}_R$	=	Work rate in resistor (power)
$\dot{W}_{Reversible}$	=	Reversible work rate (power)

CHAPTER 1

INTRODUCTION

1.1 Background

Characterization and quantification of uncertainty in both experiments and computer-based simulations is a growing area of interest for scientists and engineers. As the complexity and cost of engineered systems increases, so does the need for physics-based modeling and simulation (M&S). Practical applications of physical systems differ from most simulations in that applications are subject to uncertainty due to measurement and instrumentation, manufacturing processes, and system environment. On the other side, the M&S process has its own imperfections stemming from model inputs, numerical approximations, and model form error. These problems can be addressed from the process of verification and validation. While several guides document the *process* of verification and validation, few documents focus on the *application* of these methods. This investigation is part of a methodology under development to improve the effectiveness and efficiency of the validation process¹.

This thesis supports the need for developing a systems-level, physics-based approach for analysis, synthesis, and design optimization of aerospace systems. Particular focus is to use hardware, modeling, and simulation as part of the “Science of Integration” for efficient aircraft subsystem integration

1.2 The Need for Verification and Validation

With the emergence and availability of technology, it has become common in many fields to create computational models of physical systems. The intent of these models is not only to represent, but to also predict the behavior of such physical phenomena². When a model is used to predict an outcome, the result has little meaning without an indication of accuracy. Determining the accuracy is part of Uncertainty Quantification (*UQ*), while assessing this accuracy is part of the verification and validation process.

Verification and validation can mean many different things based on the context of the analysis. The American Society of Mechanical Engineers³ (*ASME*) provides a definition which is also applicable to this investigation. The definition reads, “*Verification*: The process of determining that a computational model accurately represents the underlying mathematical model and its solution”. Due to the presence of an analytical solution in this study, this process is rather straightforward. Similarly, the *ASME* also provides a definition for validation that is relevant to the process detailed herein. This definition reads, “*Validation*: The process of determining the degree to which a model is an accurate representation of the real world from the perspective of the intended uses of the model”. Simply put, verification is performed to assure that the model is correctly implemented, while validation is performed to assess the agreement between the model and the physical system that the model represents. The accuracy of the model is determined by considering both the verification and validation process.

Before beginning a process of verification and validation, it is important to determine if the model *can* be validated. What happens to a project if the goal of an

investigation is to determine the accuracy of a result beyond that of which the System Response Quantities (*SRQs*) can be calculated? The answer is that either the expectations must be adjusted, or the process was an inefficient use of time and money. This reason underscores the importance of understanding and stating the quality of the result required before beginning the analysis. Part of determining the required level of an outcome may involve enlisting the help of subject matter experts, studying the published literature, assessing the instrumentation available, and performing trial or scaled experiments when possible.

This thesis addresses the uncertainty quantification, sensitivity analysis, and uncertainty propagation of a dynamic electrical circuit. The analysis of a dynamic system with this level of scrutiny has not been performed before, and thus this document may provide useful insight for future practitioners pose the question: How good do you need the answer?

At the onset of this project, an exact answer could not be provided to the preceding question. However, one could infer some basic guidelines. For example, in this investigation the dynamic voltage of an electrical system subject to a voltage input was analyzed. It is reasonable then that, as a preliminary guideline, we would need to measure this voltage to within a fraction of the input signal. If the only available instrumentation could deliver a voltage reading within 1 V , then one would either need to purchase better equipment or not conduct the analysis. This is an obvious example, however it illustrates the point of determining how accurate a measurement can be provided *before* stating how accurate the answer needs to be to validate the system model.

1.3 Thesis Objectives

The motivation for this thesis stems from the need of a verification and validation process to analyze and assess the uncertainty in computational models of nonlinear dynamic systems. Previous work by J.H. Doty^{4,5} completed on the study of electrical systems explored the advantages of a slightly underdamped, nonequilibrium electrical system when compared to a traditional critically-damped system.

The specific goal of this work is develop a statistically based process for verification and validation of computational models and experiments. This is achieved by constructing and analyzing a computational model that accurately represents the physics of a second-order dynamic electrical system. The accuracy will be determined based on agreement between the model and experiment through validation metrics defined within this investigation after performing uncertainty quantification, sensitivity analysis, and uncertainty propagation for the dynamic electrical circuit.

CHAPTER 2

LITERATURE REVIEW

This chapter will examine a selection of papers that show the current knowledge and methodology implemented in similar types of analysis. Many publications exist that discuss the process of verification and validation through uncertainty quantification and uncertainty propagation. However, after searching the topics of uncertainty quantification, sensitivity analysis, and verification and validation, few publications were discovered that implemented these methodologies in a rigorous fashion with real world experiments. The first manuscript reviewed was the most influential study for the development of this thesis in terms of examination of a complex real-world problem.

2.1 Statistical Analysis and Assessment of Missile Performance Characteristics – (Doty, J.H., 2012)

This manuscript is an example of ongoing research that demonstrates the potential benefits of uncertainty analysis, nonlinear sensitivity analysis, and nonlinear uncertainty propagation as applied to experimental data of a generic missile “FG5”. An advanced statistical process is presented in sufficient detail such that an interested reader may fully understand the uncertainty analysis process.

The first objective of this reference was to develop a surrogate model using ANalysis Of Variance (ANOVA)⁶. The process began by postulating a class of models

for response surfaces using the input as the angle of attack (*AoA*) and for the output System Response Quantities (*SRQs*) as rolling moment coefficient (*CLA*); side force coefficient (*CYA*); and yawing moment coefficient (*CNA*). A Taylor-series approach was selected for its advantages in: quantification of accuracy and precision, presence of analytical derivatives, and support of statistically-based methodologies such as ANOVA. The author then applied the ANOVA-based techniques to transform the input variables to non-dimensional scaled values and developed regression equations for the inputs mentioned above.

Next, the author described the uncertainty and sensitivity analysis to be performed and defined aleatory and epistemic uncertainty in the context of the investigation. Probability density functions (*PDFs*) were postulated for the *AoA* with respect to the coded variables. Partial derivatives were then used to represent the sensitivities of the response, *AoA*, to variations in the input(s), *CLA*, *CYA*, *CNA*. These partial derivatives are also used in the standard formulation of uncertainty propagation, not to be confused with the uncertainty-modified form of uncertainty propagation implemented later in this investigation via randomized sampling techniques and Monte Carlo analysis. The author presented solutions and discussion of uncertainty analysis, sensitivity analysis, and uncertainty propagation with no uncertainty in the experimental data or surrogate models. The process was described as a “quasi-deterministic” in that it is not strictly deterministic. One data set that represents the input distribution is chosen and randomized sampling is not used, so it is therefore termed pseudo-deterministic.

Figure 1: Conceptual illustration of forward uncertainty propagation for quasi-analytical process defined by J.H. Doty [1].

The following section begins with the first activity in determining the quasi-analytical solution, which is to invert the probability density function (PDF). A step-by-step guide to this inversion process is detailed which starts with designating a number "n" of discrete values for the coded input variable ξ . Next, the discrete values for the input variable are calculated for a linear distribution on a closed interval $\xi \subseteq [-1,1]$ to get equally-spaced values for the coded input variable $\xi = \xi_1, \dots, \xi_n$. Then the coded variable is un-coded to get the input variable as a function of the coded variable for each discrete value. The PDF developed previously is evaluated as $PDF(\xi_i) = \frac{15}{16}(1 - \xi_i^2)^2$. Finally, the value of the cumulative probability density function (CDF) is determined from

$[-1, \xi_i]$, such that $CDF(\xi_i) = \left[\frac{\xi(3\xi^4 - 10\xi^2 + 15)}{16} \right]_{-1}^{\xi_i}$. A table of the coded input variable,

PDF, and CDF values is then presented. Using the numerical values generated in the table, the CDF is inverted and mapped to the input variable. Another detailed process is presented to complete this activity, with an alternate solution of graphically inverting the CDF shown in Figure 2.

Figure 2: Illustration of CDF inversion process for input variable of AOA. Specified probability $p_k=0.16$ intersects with CDF and mapped to the angle of attack (AoA) to yield $AoA(p_k)=11.187$ deg. J.H. Doty [1].

Following the inversion process, the author presents the quasi-analytical solution to the input and output uncertainty analysis, the sensitivity analysis, and the uncertainty propagation. The results are summarized as statistical metrics which will be used to gauge how well the sampling is performed.

The main objective of Professor Doty's work was to illustrate the importance of a statistically-based sampling process in nonlinear uncertainty propagation and nonlinear sensitivity analysis. He stresses the importance of statistical interpretation of the data, rather than simply generating the data itself. A Monte Carlo technique was presented in detail with a focus on quantifying the effectiveness of such a sampling methodology with reference to the quasi-analytical solutions previously developed. Professor Doty states that one of the key issues with any sampling methodology is that random sampling is an important concept that is not currently implemented in practice.

The Monte Carlo simulation results are presented in terms of uncertainty analysis of the input, uncertainty analysis of the output, the sensitivity of the output responses to the uncertainty in the input, and the uncertainty propagation from input-to-output. The results of the Monte Carlo simulations are compared to the results from the quasi-analytical solution using statistical metrics. Professor Doty achieved his goals to perform a nonlinear sensitivity analysis and uncertainty propagation, as well as compare the differences in two different sampling methodologies. Substantial evidence was presented to support the influence of sampling methodology on the interpretation of results.

2.2 Guide for Verification and Validation in Computational Solid Mechanics (ASME 2006)

A guide for verification and validation of computational models in solid mechanics is presented in detail. This publication provides a clear and concise explanation of the necessary steps in verification and validation of a model and simulation with regard to a physical experiment.

The ASME presented an approach to modeling complex systems that is similar to the lumped parameter model used in electrical circuit theory. A hierarchical representation was described to discretize real-world physical systems. The methodology shows how a system is comprised of smaller assemblies, subassemblies, and components.

Figure 3: Discretization of physical systems defined by ASME [3]. Top-down method on left vs. bottom-up method on right. Source: Guide for Verification and Validation in Computational Solid Mechanics, ASME.

The process illustrated in Figure 3 can be used to develop a model of a complex system, however an inverse process is recommended by the ASME for verification and validation. The procedure advocated is a bottom-up approach beginning with the lowest tier in the hierarchy. The authors also state that it is important to assess whether the bottom-up approach appropriately considered nonlinear effects at all levels of the process. For systems with complex nonlinear interactions, this approach must be validated at each level of the hierarchy. Thus it is tedious to implement properly, and

performing validation at the component level may not be useful in a complex model due to the complex interactions at the sub-system and full system level.

The authors continue the process with development of a conceptual model that identifies the physical processes, which are anticipated to have significant effects on the system's response. Following this procedure is the development of the mathematical model which takes into account the physical phenomena determined in the conceptual model. The final result of this modeling process is to arrive at the computational model, which is a numerical implementation of the mathematical model. The computational model will be used to predict the system response quantity similar to the experiment. The computational model can be improved through model updates such as parametric model calibration when experimental data is available.

The ASME presented sensitivity analysis as a method to identify important phenomena when intuition and experience fail to explain results from an experiment or simulation. After the physics can be explained, it is necessary to follow the sensitivity analysis with uncertainty quantification. The authors discuss two types of uncertainty, aleatory and epistemic, and methods for propagating the uncertainty once it is properly quantified and categorized. The approaches for uncertainty propagation discussed include probabilistic analysis using sampling-based methods such as Monte Carlo and Latin Hypercube.

The next area of discussion presented is model verification. Verification is the process of assessing the model's accuracy with regard to the mathematical formulations upon which it is based. Code verification and calculation verification are the two principal activities in the verification process. The computational model must be checked

to ensure the solutions are accurate with respect to the true solutions as determined by the physics based differential equations. Calculation verification is the practice of estimating numerical errors due to factors such as discretization of continuous functions and round-off error. The authors go on to describe methods for both code verification and calculation verification with examples such as a comparison of simulation results to analytical solutions and the implementation of multiple solution methods.

Finally, the authors offer insight into the validation process. The ultimate goal of validation is to evaluate the predictive capability of a computational model. This is achieved by comparing the results from a simulation with the outcomes from an experiment. This process determines the agreement between the model and the physical system that the model represents. There are three activities which must be completed prior to validating any model: (1) clearly define the model's intended use, (2) complete both code verification and calculation verification, and (3) identify and quantify uncertainties in both the experiment and model. After these necessary procedures are completed, the practitioner can begin with validation experiments. The ASME authors continue with descriptions of experiment design, sources of error in experiments, and uncertainty quantification in experiments. The final step in a validation process is the comparison of simulation results to experimental results by a means of accuracy assessment using a validation metric. A validation metric is a statistical or mathematical evaluation of the difference between the two outcomes with the best agreement possible, resulting in a value of zero for an L1 norm, or no deviation between the computational model and physical experiment.

The authors of this reference provided an in-depth process for verification and validation in the field of computational solid mechanics. The process described was generally formulated and applicable to many engineering fields where computational models are constructed with the intent of predicting physical phenomena. This work is essential for communicating the idea and process of verification and validation.

2.3 The critique of modeling and sensitivity analysis in the scientific discourse. An overview of good practices. Transatlantic Uncertainty Colloquium (Saltelli 2006)

Each scientific community addresses modeling and simulation from their separate respective viewpoint. The author discusses the lack of a common methodology for encoding physical systems into computational models. This manuscript is described by the author as an easy to read manual for sensitivity analysis. The first step in the development of a common process of sensitivity analysis is to define general terminology.

The first definition presented by the author is that of sensitivity analysis, which is described as “*the study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input.*” Uncertainty analysis is the next important term to define, which focuses on quantifying uncertainty in the model output. This leads to the explanation of what exactly is meant by the term ‘model’.

There is reason to believe that any system, natural or manmade, is governed by physical or mathematical rules. Typically, we wish to describe or discover such relationships in order to use this knowledge to our advantage as we see fit. A model is a

representation of any type of system or process, formulated using differential equations or similar mathematical equations. With any model there is the issue of uncertainty.

Uncertainty in modeling and simulation is a topic which has been debated at length. The topic of model validation is influenced by uncertainty, in that validation can be viewed as the results of a model to be ‘proven true’, and uncertainty affects the accuracy, or ‘truth’, of a model. The author comments here that most modelers would agree that a model cannot be validated, at least in this context of the word. Instead it is more correct to say that a model has received extensive corroboration through a series of tests which evaluate the model’s ability to explain or predict the physical system which it represents. Prior to evaluating a model’s predictive ability, it is recommended to perform a sensitivity analysis.

Sensitivity analysis can serve many different purposes which are useful in modeling any system. It can find technical errors, identify critical regions, establish priorities for research, and simplify models. Several methods of sensitivity analysis are discussed including derivatives, scatterplots, sigma-normalized derivatives, and Monte Carlo and linear regression.

The author presents several conclusions which are summarized here. Models should never be considered as truth, and always subjected to sensitivity analysis. Transparency is an asset when communicating scientific results. The assumptions of a model can be assessed and evaluated when implementing a sensitivity analysis.

CHAPTER 3

MODEL DESCRIPTION AND TECHNICAL BACKGROUND

3.1 Electronic Circuit Design

The use of Maxwell's Equations in conjunction with the Lorentz force law form the basis of electric circuit theory^{7,8}. To connect physics and electrical engineering, there is a layer of abstraction on top of Maxwell's Equations called lumped circuit analysis⁹. By using abstractions one can increase the complexity of engineering systems, layer by layer while maintain a tractable simplicity in the analysis. These layers are simple building blocks that allow engineers a way to dissect intricate systems. Matter is discretized, or lumped, to deal with individual elements instead of complete systems.

To illustrate this methodology of the lumped circuit abstraction, we consider a simple light bulb connected to a battery with cables. An engineer may be interested in determining the amount of voltage required to adequately light the bulb. Analyzing the physical properties of the bulb, battery, and cables would be a lengthy and complicated process. Instead we ignore the internal properties of the bulb and instead model it as a resistor. Thus, Ohm's Law can be used to model the relationship between the light bulb and battery with respect to the net current flowing through the circuit as

$$R = \frac{V}{I}. \tag{1}$$

By discretizing matter in this sense we can simplify many complex systems to simple elements in circuit theory.

In this investigation, a simple electrical circuit is examined to illustrate the concepts of sensitivity analysis and uncertainty propagation. Although many models were considered, the electrical system was ultimately chosen for its ease of use and quick construction. This electrical system has the advantages of being portable, modular, and producing repeatable results while still exhibiting the behavior of a second-order system.

This electrical system consists of a resistor (R), inductor (L), and capacitor (C), and is subject to a DC voltage input applied via power supply. A schematic of this circuit is illustrated in Figure 4.

Figure 4: Sketch of simplified series RLC circuit. Input is DC step change in voltage (V_{In}) while output is capacitor voltage (V_{Out}).

This investigation studies a second-order electrical system, which introduces nonlinear effects due to the parasitic interactions of the inductor and capacitor. The methodology implemented here can be extended to any other dynamic system, as second-order systems can be described by the same ordinary differential equation (ODE)

$$a_2\ddot{y} + a_1\dot{y} + a_0y = F(t) \quad (2)$$

where a_0 , a_1 , and a_2 are parameters used to describe the system, $F(t)$ is an arbitrary input, and $\ddot{y} = d^2y/dt^2$.

3.2 Electrical Circuit Theory

From circuit theory^{10,11} we know to apply Kirchoff's voltage law (*KVL*) to the circuit shown in Figure 4, which gives

$$V_{In}(t) = V_R(t) + V_L(t) + V_C(t) \quad (3)$$

where the subscripts R, L , and C refer to the resistor, inductor, and capacitor, respectively.

Equation 3 is a statement of conservation of electrical energy for the circuit; the sum of the voltage through each component is equal to the voltage applied to the circuit. Making substitutions for each component results in

$$V_{In}(t) = I(t)R + L \frac{dI(t)}{dt} + V_C(t) \quad (4)$$

The current $I(t)$ is related to the capacitor voltage $V_C(t)$ by the expression

$$I(t) = C \frac{dV_C(t)}{dt} \quad (5)$$

Taking the derivative of this expression gives

$$\frac{dI(t)}{dt} = C \frac{d^2V_C(t)}{dt^2} \quad (6)$$

Substituting Equation 5 and Equation 6 into Equation 4

$$\frac{d^2V_C(t)}{dt^2} + \frac{R}{L} \frac{dV_C(t)}{dt} + \frac{1}{LC} V_C(t) = \frac{1}{LC} V_{In}(t) \quad (7)$$

Since this differential equation cannot be solved by separation of variables, we instead find the total solution to be a combination of the homogenous solution, $V_{CH}(t)$, and the particular solution, $V_{CP}(t)$.

$$V_C(t) = V_{CH}(t) + V_{CP}(t) \quad (8)$$

The solution is expected to be of the form

$$V_C(t) = Ae^{st} \quad (9)$$

where A is a coefficient and s is a frequency. Substituting Equation 9, into equation 7 yields

$$As^2e^{st} + A\frac{R}{L}se^{st} + A\frac{1}{LC}e^{st} = 0 \quad (10)$$

$$A\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)e^{st} = 0 \quad (11)$$

which leads to

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad (12)$$

Equation 12 is the characteristic equation for the series RLC circuit. This characteristic equation can be simplified as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \quad (13)$$

The amount of damping in the circuit is defined by α ; known as the *neper frequency*, or *attenuation*, and for the series (RLC) circuit is given by:

$$\alpha \equiv \frac{R}{2L} \quad (14)$$

The undamped angular resonance frequency (ω_0), is given by

$$\omega_0 \equiv \frac{1}{\sqrt{LC}}. \quad (15)$$

The damping factor (ζ) is the ratio of the neper frequency (α) to the undamped angular resonance frequency (ω_0), and for the series (RLC) circuit is:

$$\zeta = \frac{\alpha}{\omega_0} = \frac{R}{2} \sqrt{\frac{C}{L}} \quad (16)$$

Further, the critical damping factor is defined for the system that returns to equilibrium as quickly as possible after a disturbance, *without oscillating*, and is determined by setting the damping factor (ζ) to unity. For given values of inductance and capacitance, the value of the resistance at critical damping, called the critical resistance (R_{crit}), may be determined from equation 16 as

$$\zeta_{crit} = 1 = \frac{R_{crit}}{2} \sqrt{\frac{C}{L}} \rightarrow R_{crit} = 2(\zeta_{crit} = 1) \sqrt{\frac{L}{C}} = 2 \sqrt{\frac{L}{C}} \quad (17)$$

The time constant τ :

$$\tau = \frac{1}{\alpha} \quad (18)$$

The Quality Factor Q

$$Q = \frac{\omega_0}{2\alpha} \quad (19)$$

Substituting the definitions of α and ω_0 for the series RLC circuit, the Quality Factor becomes

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} \quad (20)$$

Equation 13 is a quadratic with two roots

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad (21)$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad (22)$$

We can now rewrite the form of the homogenous solution in Equation 8 as a superposition of two exponential functions

$$V_{CH} = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (23)$$

The particular solution, $V_{CP}(t)$, is determined for the voltage step input:

$$V_{In}(t) = V_O u(t) \quad (24)$$

Equation 24 is then substituted into Equation 11

$$\frac{d^2 V_C(t)}{dt^2} + \frac{R}{L} \frac{dV_C(t)}{dt} + \frac{1}{LC} V_C(t) = \frac{1}{LC} V_{out} \quad (25)$$

for $t > 0$. Note that the particular solution need not satisfy the initial conditions. So an acceptable solution is:

$$V_{CP}(t) = V_{In}(t) \quad (26)$$

Recalling that the total solution is the sum of the homogeneous solution and particular solution, as shown in Equation 8. The total solution becomes

$$V_C(t) = V_{In}(t) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (27)$$

Rewriting equation 4 and recalling that current is the same through each component for a series circuit

$$I_L(t) = I_C(t) = C \frac{dV_C(t)}{dt} \quad (28)$$

$$\frac{dV_C(t)}{dt} = \frac{I_L(t)}{C} \quad (29)$$

The inductor voltage can be written as

$$V_L(t) = L \frac{dI_L(t)}{dt} \quad (30)$$

$$\frac{dI_L(t)}{dt} = \frac{1}{L} \{V_L(t)\} \quad (31)$$

By Kirchhoff's Voltage Law¹² the result is

$$V_L(t) = V_{In}(t) - V_C(t) - V_R(t). \quad (32)$$

Using Ohm's law

$$V_R(t) = I_R(t) R \quad (33)$$

and noting that $I_R(t) = I_L(t)$:

$$V_L(t) = V_{In}(t) - V_C(t) - I_L(t) R \quad (34)$$

$$\frac{dI_L(t)}{dt} = \frac{1}{L} \{V_{In}(t) - V_C(t) - I_L(t) R\} \quad (35)$$

The relative values of α and ω_0 , determine the dynamic behavior of the circuit as:

$$\alpha < \omega_0 \rightarrow \text{under - damped dynamics}$$

$$\alpha = \omega_0 \rightarrow \text{critically - damped dynamics}$$

$$\alpha > \omega_0 \rightarrow \text{over - damped dynamics}$$

The dynamic behavior of the series RLC circuit is drastically different for each of the three cases.

3.2.1 Underdamped Dynamics

The case of under-damped dynamics is characterized by a damping ratio less than unity

$$\zeta < 1$$

Or we can compare the natural frequency of the system to the neper frequency as

$$\alpha < \omega_0$$

Which we can show in terms of the resistance (R), inductance (L), and capacitance (C) by substituting Equations 14 and 15 to get

$$R/2 < \sqrt{L/C}$$

With $\alpha < \omega_0$, the quantity under the radical in Equations 29 and 30 is negative, so s_1 and s_2 are two complex conjugate roots.

Next, we define the damped angular resonance frequency as

$$\omega_d \equiv \sqrt{\omega_0^2 - \alpha^2} \quad (36)$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad (37)$$

so that Equations 21 and 22 become

$$s_1 = -\alpha + j\omega_d \quad (38)$$

$$s_2 = -\alpha - j\omega_d \quad (39)$$

We can use the Euler relation

$$e^{j\omega_d t} = \cos(\omega_d t) + j\sin(\omega_d t) \quad (40)$$

to rewrite the total solution, Equation 27, becomes

$$V_C(t) = V_{In}(t) + A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t) \quad (41)$$

Similarly, the inductor current becomes

$$I_L(t) = (\omega_d C A_2 - \alpha C A_1) e^{-\alpha t} \cos(\omega_d t) - (\omega_d C A_1 + \alpha C A_2) e^{-\alpha t} \sin(\omega_d t) \quad (42)$$

The coefficients A_1 and A_2 can be determined from the initial conditions:

$$V_C(t = 0) = V_{C_0} ; I_L(t = 0) = I_{L_0} \quad (43)$$

Substituting these initial conditions into Equations 41 and 42

$$V_C(0) = V_{In}(t) + A_1 \quad (44)$$

$$\frac{dV_C(0)}{dt} = A_2\omega_d - A_1\alpha = 0 \quad (45)$$

$$I_L(0) = \omega_d C A_2 - \alpha C A_1 \quad (46)$$

Solving Equations 44 and 45 while substituting Equation 46 yields

$$A_1 = V_C(0) - V_{In}(t) \quad (47)$$

$$A_2 = \frac{I_L(0)/C + \alpha V_C(0) - \alpha V_{In}(t)}{\omega_d} \quad (48)$$

Applying the initial conditions

$$V_{C_0} = 0 ; I_{L_0} = 0 \quad (49)$$

Equations 47 and 48 become

$$A_1 = -V_{In}(t) \quad (50)$$

$$A_2 = \frac{-\alpha}{\omega_d} V_{In}(t) \quad (51)$$

Substituting Equations 50 and 51 into yields Equations 41 and 42 yields

$$V_C(t) = V_{In}(t) \left(1 - \frac{\omega_0}{\omega_d} e^{-\alpha t} \cos \left(\omega_d t - \tan^{-1} \left(\frac{\alpha}{\omega_d} \right) \right) \right) \quad (52)$$

And

$$I_L(t) = V_{In}(t) \left(\frac{1}{\omega_d L} \right) e^{-\alpha t} \sin(\omega_d t) \quad (53)$$

These equations can be expressed more conveniently as

$$\frac{dV_C(0)}{dt} = A_2 \omega_d - A_1 \alpha = 0 \quad (54)$$

$$A = \begin{bmatrix} 1 & 0 \\ -\alpha & \omega_d \end{bmatrix} \quad (55)$$

$$b = \begin{bmatrix} I_L(0) \\ \frac{dI_L(0)}{dt} \end{bmatrix} \quad (56)$$

$$b = \begin{bmatrix} V_C(0) - V_{In}(0) \\ \frac{dV_C(0)}{dt} \end{bmatrix} \quad (57)$$

$$I_C(t) = e^{-\alpha t} (A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) \quad (58)$$

$$V_C(t) = V_{In} + e^{-\alpha t} (A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) \quad (59)$$

3.2.2 Critically-damped Dynamics

The case of critically-damped dynamics is characterized by a damping ratio of unity

$$\zeta = 1.$$

Or we can compare the natural frequency of the system to the neper frequency as

$$\alpha = \omega_0.$$

With this relation established, it follows from Equations 21 and 22 that

$$s_1 = s_2 = -\alpha$$

and the characteristic equation, Equation 13, has a repeated root. Such that

$$e^{s_1 t} = e^{s_2 t} = e^{-\alpha t}.$$

Where the total solution takes the form of

$$V_C(t) = V_{In}(t) + (A_1 + A_2 t)e^{s_1 t} \quad (60)$$

$$I_L(t) = (A_1 + A_2 t)e^{s_1 t}. \quad (61)$$

We apply the initial conditions as

$$V_C(0) = V_{In}(t) + A_1 = 0 \quad (62)$$

$$\frac{dV_C(0)}{dt} = A_1 s_1 + A_2 = 0 \quad (63)$$

$$I_L(0) = A_1 = 0 \quad (64)$$

$$\frac{dI_L(0)}{dt} = A_1 s_1 + A_2 = 0 \quad (65)$$

Which can be shown in matrix notation as

$$A = \begin{bmatrix} 1 & 0 \\ s_1 & 1 \end{bmatrix} \quad (66)$$

$$b = \begin{bmatrix} I_L(0) \\ \frac{dI_L(0)}{dt} \end{bmatrix} \quad (67)$$

$$b = \begin{bmatrix} V_C(0) - V_{In}(0) \\ \frac{dV_C(0)}{dt} \end{bmatrix} \quad (68)$$

It follows that the coefficients of the A matrix are

$$A_1 = -V_{In}(t) \quad (69)$$

$$A_2 = s_1 V_{In}(t) \quad (70)$$

Which results in the final expressions for voltage and current of

$$V_C(t) = V_{In}(t) - (1 + \alpha t)V_{In}(t)e^{-\alpha t} \quad (71)$$

$$I_L(t) = s_1 V_{In}(t)te^{-\alpha t} \quad (72)$$

$$V_R(t) = I_R(t) R \quad (73)$$

$$V_L(t) = V_{In}(t) - V_R(t) - V_C(t) \quad (74)$$

3.2.3 Over-damped Dynamics

The case of critically-damped dynamics is characterized by a damping ratio greater than unity

$$\zeta > 1$$

Or we can compare the natural frequency of the system to the neper frequency as

$$\alpha > \omega_0$$

Which we can show in terms of the resistance (R), inductance (L), and capacitance (C) by substituting Equations 14 and 15 to get

$$R/2 > \sqrt{L/C}$$

Where the total solution takes the form of

$$V_C(t) = V_{In}(t) + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (75)$$

$$I_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (76)$$

Then we can apply the initial conditions to evaluate the constants as

$$V_C(0) = V_{In}(t) + A_1 + A_2 = 0 \quad (77)$$

$$\frac{dV_C(0)}{dt} = A_1 s_1 + A_2 s_2 = 0 \quad (78)$$

$$I_L(0) = A_1 + A_2 = 0 \quad (79)$$

$$\frac{dI_L(0)}{dt} = A_1 s_1 + A_2 s_2 = 0 \quad (80)$$

Which can be shown in matrix notation as

$$A = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \quad (81)$$

$$b = \begin{bmatrix} I_L(0) \\ \frac{dI_L(0)}{dt} \end{bmatrix} \quad (82)$$

$$b = \begin{bmatrix} V_C(0) - V_{In}(0) \\ \frac{dV_C(0)}{dt} \end{bmatrix} \quad (83)$$

We can solve for the coefficients of the matrix which results in

$$A_1 = \frac{\left(\frac{dV_C(0)}{dt} - V_C(0)\right)s_2}{s_1 - s_2} \quad (84)$$

$$A_2 = \frac{-\left(\frac{dV_C(0)}{dt} - V_C(0)\right)s_1}{s_1 - s_2} \quad (85)$$

$$\begin{aligned} V_C(t) = V_{In}(t) + & \left(\frac{\left(\frac{dV_C(0)}{dt} - V_C(0)\right)s_2}{s_1 - s_2} \right) e^{s_1 t} \\ & + \left(\frac{-\left(\frac{dV_C(0)}{dt} - V_C(0)\right)s_1}{s_1 - s_2} \right) e^{s_2 t} \end{aligned} \quad (86)$$

$$I_L(t) = \left(\frac{\left(\frac{dI_L(0)}{dt} - I_L(0)\right)s_2}{s_1 - s_2} \right) e^{s_1 t} + \left(\frac{-\left(\frac{dI_L(0)}{dt} - I_L(0)\right)s_1}{s_1 - s_2} \right) e^{s_2 t} \quad (87)$$

$$V_R(t) = I_R(t) R \quad (88)$$

$$V_L(t) = V_{In}(t) - V_R(t) - V_C(t) \quad (89)$$

With the voltage expression defined for all cases, we can now derive relationships which characterize the performance of our electrical system. We start by deriving the expressions for the electrical power of the resistor (P_R), inductor (P_L), and capacitor (P_C) as

$$P_R(t) = V_R(t)I(t) = [I(t)R] I(t) = I(t)^2 R \quad (90)$$

$$P_L(t) = V_L(t) I(t) = \left[L \frac{d\{I(t)\}}{dt} \right] I_L(t) = L I(t) \frac{d\{I(t)\}}{dt} \quad (91)$$

$$P_C(t) = V_C(t) I(t) = V_C(t) \left[C \frac{dV_C(t)}{dt} \right] = V_C(t) C \frac{dV_C(t)}{dt} \quad (92)$$

3.3 Second-Order Systems

The focus of this investigation is the verification and validation *process* for a second-order system, which can be implemented to *any* system. To illustrate this point we make several analogies to other common systems. The table below shows these relationships by defining the categorical variables: resistance, inductance, compliance, effort variable, and flow variable.

Table 1: Equivalent systems

System Type	Flow Variable	Effort Variable	Compliance	Inductance	Resistance
Mechanical	Displacement (x)	Force (F)	Spring (k)	Mass (m)	Damper (c)
Electrical	Current (i)	Voltage (V)	Capacitance (C)	Inductance (L)	Resistance (R)
Thermal	Heat flow rate (Q)	Temperature change (ΔT)	Surface (S)	-	Thermal resistance (R)
Fluid	Mass/volume flow rate ($\dot{m}/Q$)	Pressure/height (P/h)	Hydraulic Accumulator (D)	Mass (m)	Pipe resistance (R)

Perhaps the most common example of a second-order system for engineers is the mass-spring-damper system (MSD). The components of a MSD system are similar to those in the RLC system. We can think of the resistance as a damper, the inductance as mass, and the capacitance as a spring.

3.4 Experimental Procedure Overview

As part of the uncertainty analysis, it was necessary to determine the uncertainty in each of the input parameters: resistance (R), capacitance (C), and inductance (L), as well as the uncertainty in the input signal (V_{in}). The details of the experimental procedure are presented in Appendix A, however a brief discussion is included here. A batch of 30 resistors, inductors, and capacitors were ordered from Digi-Key Corporation with guidance¹³ from an expert in electrical engineering. The components were selected at random and labeled (1 – 30) prior to measurement. The manufacturer of each component typically reports a nominal value along with an associated tolerance. This information can be useful in selecting components, but for the experiment performed in this study, there is room for improvement in quantifying the variance in a sample of these components.

The model was originally developed to simulate a step response. The step response is one of the most common cases studied for dynamic systems, as it provides information on the stability of the system as it moves from one state to another. However, producing a repeatable step response with the equipment available in the University of Dayton laboratory proved to be a difficult task.

Several options were available in the lab to supply the necessary voltage to the circuit. The first method attempted was to use a Agilent 33220A Function / Arbitrary Waveform Generator to produce a square wave with a magnitude 5 V. The result was an inconsistent signal that routinely overshoot the target voltage. The second method attempted was to implement a MOSFET in line with an Agilent 3631A DC Power Supply. The MOSFET did not provide the desired step response and also presented

challenges in modeling the additional physics associated with adding a transistor to the circuit. The last option considered was to wire the DC power supply to the circuit directly. The power supply produced a repeatable voltage signal that did not overshoot the set point and did not introduce additional physics. Instead of focusing on the *source* of the voltage input, this investigation emphasizes the process of uncertainty quantification and uncertainty propagation in the voltage input.

CHAPTER 4

SOLUTION METHODOLOGY

4.1 Uncertainty Background

Aleatory uncertainty, also known as *random* uncertainty, is an irreducible uncertainty due to the nature of the variation exhibited in physical systems and measurements. Examples of aleatory uncertainty are variations caused by manufacturing processes, such as fluctuations in mass, length, or geometry. This type of uncertainty can never be eliminated. Aleatory uncertainty is typically represented using a probability distribution. The two most common ways to characterize the irreducible uncertainty are a probability density function (PDF) and a cumulative density function (CDF)¹⁴. *Epistemic* uncertainty, also known as *systematic* uncertainty, is uncertainty attributed to a lack of knowledge about the physics of the system or model, the inputs to the model, or any quantity that cannot be accurately measured or represented. Epistemic uncertainty can be reduced but never completely eliminated. An example of epistemic uncertainty is model form uncertainty. Model form uncertainty can be introduced via modeling assumptions, statistical sampling methods, and numerical errors. If experimental data for a system is available, it can be used to reduce the errors and uncertainty introduced via improper modeling of physics or parasitic effects that were unknown when the model was developed.

4.2 Uncertainty Sources

It is necessary to identify all known sources of uncertainty prior to performing an uncertainty analysis. This process can be tedious and difficult to perform properly, but it is an essential part of the uncertainty analysis. In this study, there is uncertainty in both the experiment and simulation, so it is important to analyze and quantify each separately.

The sources of error and uncertainty in the experiment and data acquisition include manufacturing uncertainty, instrumentation uncertainty, calibration error, and quantization error related to the resolution of the data acquisition device, errors due to offset, gain, random noise, temporal variation, spatial variation, and environmental effects. In addition, there is uncertainty in each component used in the experiment. This uncertainty stems from manufacturing methods, material variation, and random error.

For the computational model, primary uncertainty sources are model inputs, numerical approximation, and the form of the model. For this model, the inputs of resistance, inductance, capacitance, and even the voltage input can be measured and represented using a probability density function. This aleatory uncertainty introduced via manufacturing processes and random variation uncertainty cannot be eliminated, but it can be quantified to describe the distribution.

4.3 Uncertainty Characterization

Now that the sources of uncertainty have been identified, the focus is to accurately quantify and represent all known sources of uncertainty present in this experiment and simulation.

4.3.1 Input Uncertainty Characterization

We first begin with characterizing the uncertainty of the input variables. For this investigation, the factors input to the experiment and model are the resistance (R), inductance (L), capacitance (C), and voltage input (V_{In}). For the hardware components R , L , and C , the uncertainty types include manufacturing, instrumentation, and random measurement. For the voltage input, the uncertainty stems from the randomness associated with applying a dynamic voltage signal. There is some variation in the path that the signal takes each time the machine is cycled. By measuring a sufficient number of cycles, the uncertainty in the voltage input can be quantified.

4.3.1.1 Resistor Uncertainty

The resistors used in this experiment were manufactured by Stackpole Electronics, Inc. and have a nominal value of $4.7\ \Omega \pm 5\%$, as stated on the manufacturer's data sheet¹⁵. As described in the experimental procedure, the actual value of a batch of 30 resistors was measured using a digital multimeter. By measuring the resistance value of each component, we can quantify the manufacturing uncertainty. Using the process described in the experimental procedure, the distribution of 30 resistors is represented in Figure 5 below.

Figure 5: Histogram of 30 resistors measured with digital multimeter

Examining Figure 5 it can be seen that the measured values of resistors are different than the nominal value indicated by the manufacturer. All of the resistors still fall within the specified tolerance of 5%, with the greatest difference from the nominal value at 3.6%. The red line represents a normal distribution fit, which is used as a first approximation to describe the component data. We will use the range of measured resistance, u_{range} , in computing the total uncertainty in the resistor.

As with any measurement, there is an uncertainty involved in this process. The first source of uncertainty to consider was the instrument itself. The uncertainty due to instrument resolution, u_0 , is defined by Figliola and Beasley¹⁶ as

$$u_0 = \pm \frac{1}{2} \text{ resolution} \quad (95\%) \quad (93)$$

This means that we assume one out of 20 measured values (95%) would have a value exceeding the interval defined by u_0 . For the Agilent 34405A measuring in the range of (1 – 100 Ω) the uncertainty due to instrument resolution of

$$u_0 = \pm 0.0005 \quad (94)$$

In addition, the instrument manufacturer has also published an accuracy in resistance measurement for this multimeter in a data sheet¹⁷. The accuracy of a resistance measurement is given as (*% of reading + % of range*) with the values of

$$u_{0,manf.} = \pm (0.0060\% + 0.0008\%) \quad (95)$$

Using the mean value of resistance measured of 4.567 Ω , the equation above becomes

$$u_{0,manf.} = \pm (0.00027\Omega + 0.00004\Omega) = \pm 0.31 \text{ m}\Omega \quad (96)$$

The final error for quantifying the uncertainty in the resistor value is the effect of random variation, which is covered through repeated measurement. Five resistors were selected at random and measured ten times each using a method of quasi-replication. True replication would mean turning off the multimeter, allowing it cool down completely, restarting the multimeter, waiting 30 minutes for warm-up, and *then* making a second measurement. Since this was not practical for the experimenter, the following process was implemented: measure resistance, remove leads from resistor, wait five minutes, make second measurement of resistance. Each resistor produced similar results in this experiment, so the results for one resistor's random error, $u_{rand.}$, were determined to be

$$u_{rand} = \pm 0.0160 \Omega \quad (97)$$

Each measurement error will combine with the errors introduced via manufacturing and instrumentation. Each individual error is referred to as an *elemental error*. The estimate of uncertainty in the resistor, u_R , due to elemental errors can be computed using the root-sum-squares (*RSS*) method:

$$u_R = \pm \sqrt{u_{range}^2 + u_0^2 + u_{0,manf.}^2 + u_{rand.}^2} \quad (98)$$

Substituting in the corresponding values results in the estimate of uncertainty in the resistor of

$$u_R = \pm \sqrt{0.07199^2 + 0.0005^2 + 0.00031^2 + 0.0160^2} \quad (99)$$

$$u_R = \pm 0.07375 \, \Omega \, (95\%) \quad (100)$$

We can now represent the resistors as the mean value, R' , with a measured tolerance as

$$R' = 4.567 \, \Omega \pm 1.615\% \quad (101)$$

4.3.1.2 Capacitor Uncertainty

The capacitors used in this experiment were manufactured by Panasonic Electronic Components and have a nominal value of $100 \, \mu F \pm 20\%$, as stated on the manufacturer's data sheet¹⁸. As described in the experimental procedure, the actual value of a batch of 30 capacitors was measured using a digital multimeter. By measuring the capacitance value of each component, we can quantify the manufacturing uncertainty. Using the process described in the experimental procedure, the distribution of 30 capacitors is represented in the figure below.

Figure 6: Histogram of 30 capacitors measured with digital multimeter

The remaining uncertainty values for the capacitors were calculated in a similar fashion to the resistors, with differences only in the numerical values for each type of uncertainty. For the Agilent 34405A measuring in the range of (1 – 100 μF) the uncertainty due to instrument resolution of

$$u_0 = \pm 0.05 \mu F \quad (102)$$

In addition, the instrument manufacturer has also published an indication of accuracy in resistance measurement for this multimeter in a data sheet¹⁹. The accuracy of a capacitance measurement is given as (*% of reading + % of range*) with the values of

$$u_{0,manf.} = \pm (0.02\% + 0.001\%) \quad (103)$$

Using the mean value of resistance measured as 96.8 μF , the equation above becomes

$$u_{0,manf.} = \pm (0.02 \mu F + 0.001 \mu F) = \pm 0.021 \mu F \quad (104)$$

Similar to the resistors, there was a random error associated with each capacitor measurement. This random error, $u_{rand.}$, was estimated by measuring five capacitors each ten times. This produced a range of measurements for the random error of

$$u_{rand} = \pm 0.9 \mu F \quad (105)$$

The estimate of uncertainty in the capacitor, u_C , due to elemental errors can be computed using the root-sum-squares (*RSS*) method:

$$u_C = \pm \sqrt{u_{range}^2 + u_0^2 + u_{0,manf.}^2 + u_{rand.}^2} \quad (106)$$

Substituting the corresponding values the result is

$$u_C = \pm 4.00 \mu F \quad (95\%) \quad (107)$$

We can state this is as the measured mean value for the capacitors, C' , with a measured tolerance as

$$C' = 96.8 \mu F \pm 4.13\% \quad (108)$$

4.3.1.3 Inductor Uncertainty

The inductors in this experiment were manufactured by Bourns Inc. and have a nominal value of $1.000 \text{ mH} \pm 20\%$, as stated on the manufacturer's data sheet²⁰. Due to the equipment available, a direct measurement of the inductance was not performed. Instead, a series resistor-inductor (*RL*) circuit was constructed and a voltage was applied to this circuit in the form of a sine wave via a function generator. The voltage was measured entering and exiting the inductor and used to determine the measured value of each inductor.

The theory behind this measurement was to compare the impedance of the inductor to that of a known resistor. The relationship between inductance and frequency is given by the impedance of the inductor, Z_L as

$$Z_L = j\omega L \quad (109)$$

where j is the symbol for the imaginary unit and ω is the natural frequency. Now we introduce the concept of a voltage divider, which can be used to reduce a voltage source to any arbitrary fraction of the voltage input. The voltage divider is created by connecting two electrical impedances in series and attaching a ground.

Figure 7: Voltage divider schematic. Taken from the website: http://en.wikipedia.org/wiki/Voltage_divider

Since this investigation includes an experiment and not just a simulation, it is important to model the inductor as a *real* inductor instead of an *ideal* inductor. One factor that differs between the two is that a real inductor will have resistance due to the length of wire used in the construction. The resistance of each inductor was measured and is quantified later in this section.

Now we can apply Ohm's Law to Figure 7 to determine the relationship between the input and output voltage. Ohm's Law is simply $V = IR$, so for the voltage input we have

$$V_{in} = I(Z_1 + Z_2) \quad (110)$$

Similarly, for the voltage output we have

$$V_{out} = I \cdot Z_2 \quad (111)$$

If we solve Equation 110 for the current, I , and substitute the result into Equation 111 we arrive at the equation for the voltage divider as

$$V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in} \quad (112)$$

We then replace Z_1 with the impedance of the inductor to be measured and replace Z_2 with the impedance of a resistor plus the resistive impedance of the inductor. This results in

$$V_{out} = \frac{(R + r)}{j\omega L + (R + r)} V_{in} \quad (113)$$

where R is the resistance of the resistor in series and r is the DC resistance of the inductor. Since the impedance of the inductor is imaginary we look at the absolute value of this equation, or the magnitude squared.

$$\left| \frac{V_{out}}{V_{in}} \right| = \left| \frac{(R + r)}{j\omega L + (R + r)} \right| = \frac{\sqrt{(R + r)^2}}{\sqrt{(j\omega L)^2 + (R + r)^2}} \quad (114)$$

If we square both sides of this equation and substitute the relationship for the natural frequency as, $\omega = 2\pi f$, the equation for the measured inductance is

$$L = \frac{1}{2\pi f} \sqrt{\left((R + r) \frac{V_{in}}{V_{out}} \right)^2 - (R + r)^2} \quad (115)$$

The resistor used in this RL circuit had a nominal value of 100Ω , which measured 99.1Ω . The sinusoidal frequency of the applied voltage was tuned such that the ratio of $\frac{V_{out}}{V_{in}}$ was approximately one half. The resulting frequency was 27 KHz , and the actual values of $\frac{V_{in}}{V_{out}}$ were measured with an oscilloscope. After measuring the DC resistance of

each inductor, the *RL* circuit was tested for the batch of 30 inductors and the results are shown in the histogram below.

Figure 8: Histogram of 30 inductors

The inductors were subject to the same types of uncertainty as both the resistors and capacitors: manufacturing uncertainty, instrument uncertainty, and random uncertainty. However, there are additional factors that contribute to the total uncertainty in the inductance, due to the additional experiment performed. These factors include the measurement of the $100\ \Omega$ resistor used in the *RL* circuit, the measurement of the DC resistance of the inductor, the frequency of the voltage applied to the circuit, and the measurement of the voltage in and out of the inductor element. The uncertainty in the measurement and calculation can be propagated through a numerical method known as sequential perturbation approach²¹. This method should yield identical results as a first-order Taylor series approach using partial derivatives which is not practical for the number of variables in this calculation. This method uses finite differences to approximate the derivatives and is easily implemented by perturbing the value of each

variable with respect to its original fixed value. The magnitude of perturbation is determined by the practitioner but it is typically within 5% of the nominal value.

We can implement this sequential perturbation method using the measured value of input voltage on the oscilloscope, V_{In} , as an example. First we determine the nominal value for a given inductor; for convenience assume that the inductor was measured to be equal to the mean value of $L = 1.133$ mH. Next we reevaluate Equation 115 for a slightly perturbed value of V_{In} . In this case we have insight about the magnitude for this perturbation from the manufacturer of the Agilent 3000 Series Oscilloscope²². For a voltage reading using 5 mV/div the DC gain accuracy is stated as $\pm 4\%$, so we perturb the variable V_{In} by $\pm 4\%$ and calculate the result of Equation 115. Then we calculate the difference of the positive and negative perturbation with reference to the original value, V_{In0} , as

$$\delta V_{In}^{+} = V_{In}^{+} - V_{In0} \quad (116)$$

$$\delta V_{In}^{-} = V_{In}^{-} - V_{In0} \quad (117)$$

And finally we can evaluate the approximation of uncertainty contribution from this variable as

$$\delta V_{In} = \frac{\delta V_{In}^{+} - \delta V_{In}^{-}}{2} \approx \theta_{V_{In}} u_{V_{In}} \quad (118)$$

This process was repeated for each variable present in the calculation of the inductance and combined with the uncertainty due to manufacturing error to give the estimate for total uncertainty as

$$u_L = \pm \sqrt{(\theta_f u_f)^2 + (\theta_R u_R)^2 + (\theta_r u_r)^2 + (\theta_{V_{In}} u_{V_{In}})^2 + (\theta_{V_{Out}} u_{V_{Out}})^2 + u_{manf.}} \quad (119)$$

Substituting the corresponding values the result is

$$u_L = \pm 0.08941 \text{ mH} \quad (95\%) \quad (120)$$

We can state this is as the measured mean value for the capacitors, L' , with a measured tolerance as

$$L' = 1.133 \text{ mH} \pm 7.891\% \quad (121)$$

As mentioned previously in the calculation of inductance, a real inductor will have a measureable DC resistance. Each inductor was measured using the same process as the resistors and the results are shown in Figure 9 below.

Figure 9: Histogram of DC resistance of 30 inductors

The manufacturer specifies²³ a maximum DC resistance of $R_{inductor} = 0.555 \Omega$.

Looking at Figure 9, we can see that all inductors measured fall within this maximum resistance specification. The justification behind measuring the inductor resistance was that the specified maximum DC resistance is over 10% of the nominal value of

resistance, so it is likely to have a noticeable effect on the response. Since this investigation is the study of a *series RLC* circuit, we can simply add these values to the total system resistance.

4.3.1.4 Summary of Measured Hardware Components R, L, C

Recall from the experimental procedure that prior to measuring the components they were labeled (1 – 30) for easy identification. The results of the measurements described in this chapter are presented in Table 2.

Table 2: Measured values of R, L, C

Component set #	R (Ohm)	L (mH)	C (μ F)
1	4.569	1.121	95.5
2	4.573	1.121	96.2
3	4.571	1.133	96.3
4	4.558	1.133	96.6
5	4.572	1.156	96.6
6	4.573	1.159	96.6
7	4.589	1.121	96.8
8	4.575	1.133	95.8
9	4.567	1.159	96.8
10	4.585	1.133	96.6
11	4.567	1.133	96.7
12	4.585	1.146	97.1
13	4.581	1.133	97.7
14	4.564	1.109	95.8
15	4.568	1.109	97.6
16	4.522	1.133	97.6
17	4.571	1.172	97.6
18	4.543	1.109	97.1
19	4.531	1.121	98.3
20	4.573	1.133	96.1
21	4.556	1.133	97.8
22	4.570	1.133	97.6
23	4.594	1.133	96.6
24	4.561	1.146	95.0
25	4.563	1.120	98.0
26	4.564	1.109	98.0
27	4.576	1.133	96.7
28	4.564	1.133	94.4
29	4.556	1.133	97.7
30	4.569	1.133	96.0

Each of the 30 measured values is summarized in Table 2. For the resistors, we see that all of the values are below the nominal value of $4.7\ \Omega$ and have four significant figures, which corresponds to the resolution of the multimeter. The inductors all measured above the nominal value of $1.000\ \text{mH}$ and have four significant figures, which corresponds to the resolution on the instrumentation. The capacitors all measured below the nominal value of $100\ \mu\text{F}$ and have three significant figures, which represents the resolution of the instrumentation.

4.3.1.5 Voltage Input Uncertainty

The final input to the model was the dynamic voltage input signal, $V_{In}(t)$. The input was recorded for 90 trials and the results are presented below.

Figure 10: Data recorded for 90 trials of voltage input

Looking at Figure 10, we see that the input is comprised of two signals that form a piecewise function. The first section is linear and the second section is a decaying exponential, which is due to the physics of the DC power supply itself. The principle is to create the fastest rise time while maintaining zero overshoot. This is achieved by

implementing two circuits, one underdamped and one critically damped. The power supply uses the underdamped circuit as long as possible, and then switches to the critically damped circuit around when the signal has reached approximately 90% of the target voltage. The voltage input can also be represented in terms of the standard deviation as a function of time, and this relationship is shown below in Figure 11.

Figure 11: Standard deviation of dynamic voltage input

In Figure 11 we see the variation in the voltage input signal as a function of time. This plot is the standard deviation of Figure 10 calculated for each instance in time. The greatest variation occurs in the initial “turn-on” phase of the response, which is due to the physics of the DC power supply previously described. The instrument is attempting to turn on as fast as possible, and thus a high variation from one response to the next is to be expected in this region. If we compare Figure 10 to Figure 11, we see that when the input signal switches to its critically damped circuit at $t = 1.1 \text{ ms}$ the standard deviation between responses drops significantly. In this second phase of the response the rate of

change of the voltage with respect to time also decreases significantly. Logically, it follows that this region will have a low standard deviation, as the system is now responding relatively slowly compared to the first phase of response where the voltage was changing rapidly with respect to time.

4.4 Data Acquisition Uncertainty

Another source of uncertainty in this investigation is due to the error introduced from measuring dynamic values. The process of converting a continuous physical quantity is achieved through analog-to-digital conversion. The error introduced by this process is known as a quantization error and is directly related to the resolution of the data acquisition device. In addition to the quantization error, we have errors introduced via system noise, offset, gain, and nonlinearity²⁴.

The device used to capture data in this experiment was a Measurement Computing USB-1208FS analog and digital I/O device with 12-bit analog inputs. The quantization error, or resolution, is determined from the following relationship between " n " number of bits and the range of the voltage measurement, V_{range} . Since this experiment is subject to a 6V input, the range selected on the device was $\pm 10\text{ V}$.

$$Resolution = \frac{V_{range}}{2^n} \quad (122)$$

Substituting the appropriate values the result for the quantization error, or resolution, is given as

$$Resolution = \frac{20V}{2^{12}} = \pm 4.9\text{ mV} \quad (123)$$

Next we can address the errors introduced due to random noise in the system. We can determine this error from the number of bits in our data acquisition device as

$$\text{Random Noise} = \frac{1}{2^{n+1}} = \frac{1}{2^{13}} = \pm 0.12 \text{ mV} \quad (124)$$

The next type of error examined is the error introduced due to offset error in the device.

The manufacturer of the data acquisition (DAQ) device has provided²⁵ a visual representation of this type of error which is shown below.

Figure 12: Offset error in USB-1208FS. Source:
www.mccdaq.com/PDFs/Manuals/usb-1208fs.pdf

Offset error is constant across the input range and affects all voltages equally by shifting the response up or down along the input voltage axis. The manufacturer states that the typical offset error for the $\pm 10 \text{ V}$ range is $\pm 9.77 \text{ mV}$.

The user's manual for this DAQ device plainly states that the error induced due to nonlinearity is small and not significant as an error source with respect to offset and gain. Thus, the final type of error in data acquisition is the error due to gain.

Gain error is caused by imperfections in the amplifier and affects the signal by modifying the slope of the input. The manufacturer has again provided a visual representation of this type of error and is shown below.

Figure 13: Gain error in USB-1208FS. Source:
www.mccdaq.com/PDFs/Manuals/usb-1208fs.pdf

Gain error is typically given as a percentage of full-scale, and can be converted to a voltage by multiplying the full-scale voltage, 10 V, by the input error of 2% as

$$\text{Gain Error} = 10 \text{ V} \cdot \pm 0.002 = \pm 20 \text{ mV} \quad (125)$$

With all types of error estimated in the data acquisition process, we can combine each type of error with the root-sum-squares (RSS) method as

$$\text{DAQ Error} = \pm \sqrt{4.9^2 + 0.12^2 + 9.77^2 + 20^2} = \pm 23 \text{ mV} \quad (126)$$

For each data point acquired in the experiment, this is the error in the voltage measurement.

4.5 Numerical Approximation Uncertainty

Numerical approximation is a type of error present in nearly all computational models. This error stems from the use of differential equations to model physical systems. One of the reasons this system was selected is that despite being based on differential equations, an analytical solution is readily available in many standard electric circuit textbooks. The analytical solution was implemented in the model of this system and is verified through comparison of a response simulated using MATLAB's ODE23,

ODE45, and a state space representation. The results of this comparison are presented in the Chapter 5 section on verification.

Another source of numerical approximation is due to a digital computer's ability to represent numbers to a finite precision. The default numeric type for MATLAB is the double-precision data type. The model and simulation were evaluated on a 64-bit platform which means that 52 bits were used to represent the mantissa, 11 bits were used for the exponent, and 1 bit was used for the sign²⁶. The mantissa is the part of a floating-point number consisting of its significant digits, so the model implemented in this investigation can represent 52 significant digits. Considering the magnitude of uncertainties presented in the previous sections, this source of error is insignificant in the model and simulation. This conclusion is corroborated in the verification process presented in later sections.

4.6 Validation Metrics

The process of validation in the context of this study involves assessing the agreement between the experimentally measured results and the results produced by the model and simulation. To validate a model or process, we need to have a means of evaluating the accuracy based on a statistical metric or benchmark. Deciding what to use for this validation benchmark is the responsibility of the practitioner²⁷. One of the validation metrics suggested by experts in the field of verification and validation is the Area Validation Metric (Oberkampf and Roy 2010). This metric is defined to be the area between the CDF from the simulation and the empirical distribution function (EDF) from the experiment.

The validation metric introduced in this study is used to compare the time and voltage series recorded in the experiment with the time and voltage series generated by the model and simulation. This validation metric, the Absolute Area Deviation Metric (AAD), provides a measure of the disagreement between the experiment and simulation. Mathematically this metric is formulated as

$$\text{Abs. Area Deviation} = \sum_{i=1}^n |V_{C, \text{Experiment}, i} - V_{C, \text{Simulation}, i}| dt \quad (127)$$

where n is the total number of data points in the response. The AAD was developed based on the principals used to formulate the L_1 norm.

Both the Area Validation Metric (AVM) and the Absolute Area Deviation (AAD) serve as terms of evaluating the difference between two results, however they are not statistically significant. One example of a statistical validation metric is the Anderson-Darling test.

The Anderson-Darling k-sample test is used to determine if two samples likely come from the same distribution such that one or the other may be used interchangeably. The null hypothesis of the Anderson-Darling k-sample test is that the sampled populations come from the same distribution, without specifying the type of that common distribution F .

$$H_0 : F_1 = \dots = F_k \quad (128)$$

The k-sample Anderson-Darling test statistic is defined as

$$A_{kN}^2 = \sum_{i=1}^k n_i \int_{B_N} \frac{\{F_{n_i}(x) - H_N(x)\}^2}{H_N(x)\{1 - H_N(x)\}} dH_N(x) \quad (129)$$

Which is then implemented computationally as

$$A_{kN}^2 = \frac{1}{N} \sum_{i=1}^k \frac{1}{n_i} \sum_{j=1}^{N-1} \frac{(NM_{ij} - jn_i)^2}{j(N-j)} \quad (130)$$

The Mathworks File Exchange provides an add-on²⁸ for the Statistics Toolbox which was used in this study to perform the Anderson-Darling test.

4.7 Model Form Uncertainty

As part of the initial validation process the form of the model was evaluated with respect to the experiment. Trial runs of the experiment were performed to assess the performance of the available instrumentation and data acquisition.

The model was designed to simulate the step response of a dynamic electrical system. Through initial experimentation it was discovered that generating a repeatable step input was not realistic. In Figure 14 we see the intended step response from the simulation in the dark green data set compared to the data recorded in the experiment shown in blue.

Figure 14: Results from experiment compared to simulation with a perfect step input

To improve the accuracy of the model, the next step was to subject the simulation to the same voltage input as the experiment. An average of 90 trials of the voltage input was recorded and used to replace the step input in the simulation shown in Figure 15.

Figure 15: Simulation updated with experimental input

Looking at Figure 15 we can see that the simulation shown in green is still lacking physics present in the experiment. The simulation exceeds the magnitude of the voltage input while the experiment exhibits a highly damped response. This was an indication that the experiment was experiencing more resistance than originally included in the model. In addition, the slope of the experiment versus the model also diverge after approximately 0.25 ms . The difference in slope helped to identify additional lacking physics due to parasitic inductance and capacitance.

Figure 16: Simulation updated with additional information on experimental physics

Figure 16 illustrates the final revision of the model taking into account all of the parasitic effects from the inductor, capacitor, power supply, breadboard, and test leads. Such parasitic effects are unavoidable in the use of any electrical conductor. To help determine the parasitic inductance, capacitance and resistance in the system, several variations of subsystems were tried experimentally. One of the most beneficial experiments was the trial of a "*LC*" circuit, with the parenthesis indicating that a purely inductive-capacitive circuit is merely an idealized model and not a practical implementation.

4.7.1 Updated Parameters

The model inputs were updated to factor in the new information regarding parasitic resistance, inductance, and capacitance. The results are summarized in Table 3 below.

Table 3: Updated model inputs

Component set #	R (Ohm)	L (mH)	C (μF)
1	8.002	1.233	106
2	8.007	1.123	107
3	8.002	1.247	107
4	7.995	1.247	107
5	8.006	1.275	107
6	8.010	1.275	107
7	8.018	1.233	107
8	8.006	1.245	106
9	7.999	1.275	107
10	8.017	1.247	107
11	7.994	1.246	107
12	8.011	1.261	108
13	8.009	1.246	108
14	7.987	1.219	106
15	7.996	1.220	108
16	7.948	1.246	108
17	8.001	1.289	108
18	7.963	1.219	108
19	7.953	1.233	109
20	8.001	1.246	107
21	7.987	1.247	109
22	8.006	1.247	108
23	8.028	1.247	107
24	7.997	1.261	105
25	7.995	1.233	109
26	7.993	1.220	109
27	8.005	1.246	107
28	7.991	1.246	105
29	7.988	1.247	108
30	8.009	1.247	107

Table 3 is a summary for each of the 30 components after considering the parasitic and coupling effects. Comparing this table to Table 2, it can be seen that each value has increased. The resistance increased the most on a percentage basis, which is due to using such a low resistance to begin with.

4.8 Sensitivity Analysis

Sensitivity analysis is the study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input.²⁹ Sensitivity analysis and uncertainty analysis are frequently performed

in tandem; once the factors that contribute to output variability are determined via sensitivity analysis, they need to be quantified using uncertainty analysis. By properly implementing and assessing these methods, the goal is to increase the understanding of the relationships between input variables and output System Response Quantities (*SRQs*).

For any number of " n " inputs ($x_1, x_2, \dots, x_n$) which generate any number of " k " outputs, we are able to evaluate a function f or other result recorded as a System Response Quantity (*SRQ*). This relationship is explained mathematically as

$$SRQ = f_k(x_i); i = 1, \dots, n \text{ inputs.} \quad (131)$$

Considering a general nonlinear combination of the variables ' x_i ', Goodman³⁰ employs a Taylor-series-based approach for the functional approximation to give

$$f_k(x_i) \approx f_k^0 + \sum_{i=1}^n \left(\frac{\partial f_k}{\partial x_i} \right). \quad (132)$$

where $\frac{\partial f_k(x_i)}{\partial x_i}$ is the partial derivative of f_k with respect to variable x_i . The localized sensitivity analysis methodology estimates the sensitivity (Sen_k^i) of each response ($f_k(x_i)$) to each input variable (x_i) and is represented as using its respective partial derivative. Mathematically, such sensitivity may be approximated as

$$Sen_k^i \approx \frac{\partial f_k}{\partial x_i}. \quad (133)$$

As provided by Saltelli³¹, localized sensitivity analysis based upon derivatives has advantages and disadvantages. The advantage of the simplicity of the localized sensitivity analysis is typically offset by the fact that the localized derivatives are typically evaluated in a linear fashion. As such, they are generally not applicable to

nonlinear problems³². The modified form of the sensitivity coefficient for uncertain input data from sampled distributions and propagated uncertainty to the output as recommended by the intergovernmental panel for climate change (IPCC) is³³

$$\left\{ Sen_{SRQ_k}^{x_i} \right\}_{\sigma_{SRQ_k}} \approx \frac{\sigma_{x_i}}{\sigma_{SRQ_k}} \frac{\partial(SRQ_k)}{\partial x_i} \quad (134)$$

where the quotient, $\left(\frac{\sigma_{x_i}}{\sigma_{SRQ_k}} \right)$, is introduced to normalize the sensitivity coefficient by the ratio of input to output uncertainty.

Partial derivatives may be used to represent the sensitivities of the response to variations in the input. To do this, we need to substitute the following relationships into the expression for dynamic capacitor voltage

$$\alpha \equiv \frac{R}{2L} \quad ; \quad \omega_d \equiv \omega_0 \sqrt{1 - \zeta^2} \quad ; \quad \omega_0 \equiv \frac{1}{\sqrt{LC}} \quad ; \quad \zeta \equiv \frac{\alpha}{\omega_0} \quad ; \quad A_1 = -V_{In}(N) \quad ; \quad A_2 = \frac{-\alpha}{\omega_d} V_{In}(N);$$

which results in the new form for the capacitor voltage as a function of time

$$V_C(t) = 1 + e^{-\frac{R}{2L}t} \left[-V_{In}(t) * \cos \left(\frac{t \sqrt{1 - \frac{CR^2}{4L}}}{\sqrt{LC}} \right) - \frac{R\sqrt{LC}}{2L \sqrt{1 - \frac{CR^2}{4L}}} * V_{In}(t) * \sin \left(\frac{t \sqrt{1 - \frac{CR^2}{4L}}}{\sqrt{LC}} \right) \right] \quad (135)$$

It follows that we can determine the sensitivity of the dynamic capacitor voltage to the resistance $\left(\frac{\partial(V_c(t))}{\partial(R)}\right)$ is obtained by analytically differentiating Equation 135 with respect to resistance. Performing this operation yields

$$\begin{aligned} \frac{\partial(V_c(t))}{\partial(R)} = & (t(V_{in}(t) * \cos((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) + (RV_{in}(t) * \sin((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(2L \\ & * (1 - (CR)/(2L))^{1/2}))/ (2L * e^{(Rt)/(2L)}) - ((V_{in}(t) \\ & * \sin((t * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(2L \\ & * (1 - (CR)/(2L))^{1/2}) + (CRV_{in} * \sin((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(8L^2 \\ & * (1 - (CR)/(2L))^{3/2}) - (CR^2 * V_{in}(t) * t * \cos((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}))/ (8L^2 \\ & * (1 - (CR^2)/(4L))^{1/2} \\ & * (1 - (C * R)/(2 * L))^{1/2}) + (CRV_{in} * t * \sin((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}))/ (4L \\ & * (1 - (CR^2)/(4L))^{1/2} * (CL)^{1/2}))/ e^{(Rt)/(2L)} \end{aligned} \quad (136)$$

We similarly determine the sensitivity of the dynamic capacitor voltage with respect to the inductance $\left(\frac{\partial(V_c(t))}{\partial(L)}\right)$ by analytically differentiating Equation 135 with respect to inductance to yield

$$\begin{aligned} \frac{\partial(V_c(t))}{\partial(L)} = & ((R * V_{in} * \sin((t * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) \\ & * (CL)^{1/2})/(2L^2 * (1 - (CR)/(2L))^{1/2} - V_{in} * \sin((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * ((Ct \\ & * (1 - (CR^2)/(4L))^{1/2})/(2 * (CL)^{3/2}) - (CR^2t)/(8L^2 \\ & * (1 - (CR^2)/(4L))^{1/2} * (CL)^{1/2})) + (CR^2 * V_{in} \\ & * \sin((t * (1 - (CR^2)/(4L))^{1/2})/(C * L)^{1/2}) \\ & * (CL)^{1/2})/(8L^3 * (1 - (CR)/(2L))^{3/2}) + (RV_{in} * \cos((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * ((Ct \\ & * (1 - (CR^2)/(4L))^{1/2})/(2(CL)^{3/2}) - (CR^2t)/(8L^2 \\ & * (1 - (CR^2)/(4 * L))^{1/2} * (CL)^{1/2})) * (CL)^{1/2})/(2L \\ & * (1 - (CR)/(2L))^{1/2}) - (CR * V_{in} * \sin((t \\ & * (1 - (CR^2)/(4 * L))^{1/2})/(CL)^{1/2}))/(4L * (CL)^{1/2} \\ & * (1 - (CR)/(2L))^{1/2}))/e^{(Rt)/(2L)} - (Rt * (V_{in} * \cos((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) + (R * V_{in} * \sin((t \\ & * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(2L \\ & * (1 - (C * R)/(2L))^{1/2}))/ (2 * L^2 * e^{(Rt)/(2L)})) \end{aligned} \quad (137)$$

Next, we determine the sensitivity of the dynamic capacitor voltage with respect to the capacitance $\left(\frac{\partial(V_c(t))}{\partial(C)}\right)$ by analytically differentiating Equation 135 with respect to capacitance to yield

$$\begin{aligned}
\frac{\partial(V_c(t))}{\partial(C)} = & -(V_{in} * \sin((t * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * ((Lt \\
& * (1 - (CR^2)/(4 * L))^{1/2})/(2 * (CL)^{3/2}) + (R^2 t)/(8L \\
& * (1 - (CR^2)/(4 * L))^{1/2} * (CL)^{1/2})) + (R * V_{in} * \sin((t \\
& * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}))/ (4 * (CL)^{1/2} \\
& * (1 - (CR)/(2L))^{1/2}) + (R^2 * V_{in} * \sin((t \\
& * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(8L^2 \\
& * (1 - (CR)/(2L))^{3/2}) - (R * V_{in} * \cos((t \\
& * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * ((Lt \\
& * (1 - (CR^2)/(4L))^{1/2})/(2 * (CL)^{3/2}) + (R^2 t)/(8L \\
& * (1 - (CR^2)/(4L))^{1/2} * (CL)^{1/2})) * (CL)^{1/2})/(2L \\
& * (1 - (CR)/(2L))^{1/2}))/e^{(Rt)/(2L)}
\end{aligned} \tag{138}$$

Next, we determine the sensitivity of the dynamic capacitor voltage with respect to the voltage input $\left(\frac{\partial(V_c(t))}{\partial(V_{in})}\right)$ by analytically differentiating Equation 135 with respect to capacitance to yield

$$\begin{aligned}
\frac{\partial(V_c(t))}{\partial(V_{in})} = & 1 - (\cos((t * (1 - (CR^2)/(4L))^{1/2})/(CL)^{1/2}) + (R \sin((t(1 - \\
& (CR^2)/(4L))^{1/2})/(CL)^{1/2}) * (CL)^{1/2})/(2L(1 - (CR)/(2L))^{1/2}))/e^{(Rt)/(2L)}
\end{aligned} \tag{139}$$

According to the modified sensitivities recommended by the IPCC, the sensitivities calculated via partial differentiation above are multiplied by the normalizing ratio, which is the standard deviation of the input divided by the standard deviation of the output.

$$\{Sen_{V_c(t)}^R\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_R}{\sigma_{V_c(t)}}\right) \frac{\partial(V_c(t))}{\partial(R)} \tag{140}$$

$$\{Sen_{V_c(t)}^L\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_L}{\sigma_{V_c(t)}}\right) \frac{\partial(V_c(t))}{\partial(L)} \tag{141}$$

$$\{Sen_{V_c(t)}^C\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_C}{\sigma_{V_c(t)}}\right) \frac{\partial(V_c(t))}{\partial(C)} \tag{142}$$

$$\left\{ Sen_{V_c(t)}^{V_{In}(t)} \right\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_{V_{In}(t)}}{\sigma_{V_c(t)}} \right) \frac{\partial(V_c(t))}{\partial(V_{In}(t))} \quad (143)$$

The actual partial derivatives as determined previously can be substituted into the expressions above. The values for the standard deviation from the various types of uncertainty sources were calculated in the previous chapter. The only remaining standard deviation is that of the dynamic capacitor voltage.

Recall that the standard deviation for both the dynamic voltage input, $V_{In}(t)$, and the dynamic capacitor voltage, $V_c(t)$, change with time. The standard deviation for the voltage input was presented in Figure 11, similarly, now we present the standard deviation for the dynamic capacitor voltage below.

Figure 17: Standard deviation for 30 trials of capacitor voltage response

The implications of Figure 17 are discussed in the results section. For now, we just want to show how this value changes with time and thus quantify all of the terms present in the equations for the Sensitivity Analysis of this dynamic electrical system. The results of the methodology are shown and discussed in the final chapter of this thesis.

4.9 Uncertainty Propagation

With the sources of uncertainty identified and an estimate for the uncertainties we quantified, we now examine how those uncertainties are propagated through the model or process. Uncertainty Propagation is the process of determining the effect of the input uncertainty on the System Response Quantity (SRQ) or system output. We can show mathematically how the uncertainty is propagated as

$$\sigma_{x_{input} \rightarrow SRQ_k}^2 = \left(Sen_{x_{input}}^{SRQ_k} \right)^2 \left(\sigma_{x_{input}}^2 \right). \quad (144)$$

Here we see that it is actually the *variance* which is propagated, or the square of the uncertainty. For the inputs of resistance (R), inductance (L), capacitance (C), and voltage input ($V_{In}(t)$) and the SRQ as dynamic capacitor voltage ($V_C(t)$), we may write Equation 144, respectively for each input, as

$$\sigma_{R \rightarrow V_C(t)}^2 = \left(Sen_R^{V_C(t)} \right)^2 (\sigma_R^2) \quad (145)$$

$$\sigma_{L \rightarrow V_C(t)}^2 = \left(Sen_L^{V_C(t)} \right)^2 (\sigma_L^2) \quad (146)$$

$$\sigma_{C \rightarrow V_C(t)}^2 = \left(Sen_C^{V_C(t)} \right)^2 (\sigma_C^2) \quad (147)$$

$$\sigma_{V_{In}(t) \rightarrow V_C(t)}^2 = \left(Sen_{V_{In}(t)}^{V_C(t)} \right)^2 (\sigma_{V_{In}(t)}^2) \quad (148)$$

Now we can substitute the sensitivity calculated in the previous section into Equations 145-148 which results in

$$\sigma_{R \rightarrow V_C(t)}^2 = \left(\frac{\partial(V_C(t))}{\partial(R)} \right) \sigma_R^2 \quad (149)$$

$$\sigma_{L \rightarrow V_C(t)}^2 = \left(\frac{\partial(V_C(t))}{\partial(L)} \right) \sigma_L^2 \quad (150)$$

$$\sigma_{C \rightarrow V_C(t)}^2 = \left(\frac{\partial(V_C(t))}{\partial(C)} \right) \sigma_C^2 \quad (151)$$

$$\sigma_{V_{In}(t) \rightarrow V_C(t)}^2 = \left(\frac{\partial(V_C(t))}{\partial(V_{In}(t))} \right) \sigma_{V_{In}(t)}^2 \quad (152)$$

The proposed formulation of the uncertainty propagation using the nonlinear weighted sensitivity estimate ala Equation 133 for the SRQ of $V_C(t)$ follows that

$$\sigma_{R \rightarrow V_C(t)}^2 = \left[\left(\frac{\sigma_R}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(R)} \right) \right]^2 \sigma_R^2 \quad (153)$$

$$\sigma_{L \rightarrow V_C(t)}^2 = \left[\left(\frac{\sigma_L}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(L)} \right) \right]^2 \sigma_L^2 \quad (154)$$

$$\sigma_{C \rightarrow V_C(t)}^2 = \left[\left(\frac{\sigma_C}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(C)} \right) \right]^2 \sigma_C^2 \quad (155)$$

$$\sigma_{V_{In}(t) \rightarrow V_C(t)}^2 = \left[\left(\frac{\sigma_{V_{In}(t)}}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(V_{In}(t))} \right) \right]^2 \sigma_{V_{In}(t)}^2 \quad (156)$$

With the uncertainties quantified we are now in a position to calculate the sensitivities and propagate the uncertainties.

4.10 Statistical Representation

We can also represent the input variables as probability density functions (PDFs).

A symmetric beta distribution is used to represent this data in the form of

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad (157)$$

where $B(\alpha, \beta)$ is the beta function evaluated as

$$B(\alpha, \beta) = \frac{(\alpha-1)! (\beta-1)!}{(\alpha+\beta-1)!} \quad (158)$$

We can apply the beta distribution to the resistance (R) by introducing the coded variable ξ to represent the distribution of resistance that have been transformed to the interval $[0,1]$. The result is the PDF for the resistance in units of ohms is

$$PDF(\xi) = \frac{1}{26.05} \xi^{0.0546-1} (1 - \xi)^{0.1252-1}; \text{ with } \xi = \frac{(R - 7.949)}{0.08} \quad (159)$$

Which we can then visualize in Figure 18 where the coded resistance is discretized for plotting purposes with $n = 10^3$ data points on the interval $[0,1]$.

Figure 18: Illustration of the probability density function PDF for the coded resistance (ξ).

In order to prove that the PDF is, indeed, a *density* function it is required that the sum of all possible points in the PDF equals to unity. Mathematically for $\xi \subseteq [0,1]$ we write this as

$$\int_0^1 PDF(\xi) d\xi \equiv 1. \quad (160)$$

Substituting the $PDF(\xi)$ from Equation 173 into Equation 174 gives

$$\int_0^1 PDF(\xi) d\xi \equiv \int_0^1 \frac{1}{26.05} \xi^{-.9454} (1 - \xi)^{-.8748} d\xi = 1 \quad (161)$$

We can now represent the distribution of resistance as a cumulative distribution function (CDF) as shown in Figure 19.

Figure 19: Illustration of the cumulative density function CDF for the coded resistance (ξ).

Similarly for the inductance PDF we use the coded variable ψ and units of henrys to result in

$$PDF(\psi) = \frac{1}{25.50} \psi^{0.0531-1} (1 - \psi)^{0.1436-1}; \text{ with } \psi = \frac{(L - 1.219 * 10^{-3})}{6.268 * 10^{-5}} \quad (162)$$

And lastly for the capacitance PDF we use the coded variable φ and units of farad to result in

$$PDF(\varphi) = \frac{1}{26.24} \varphi^{0.0544-1} (1 - \varphi)^{0.1233-1}; \text{ with } \varphi = \frac{(C - 105 * 10^{-6})}{4.33 * 10^{-6}} \quad (163)$$

The PDF and CDF for capacitance and inductance are not shown graphically but are similar in shape to that of the resistance which should be apparent when comparing Equation 159, Equation 162, and Equation 163.

CHAPTER 5

RESULTS AND DISCUSSION

5.1 Results of Capacitor Voltage Response

The main result of this investigation is the comparison of the simulated capacitor voltage response versus the experimentally-recorded capacitor voltage response.

Figure 20: Comparison of simulation versus experiment for dynamic capacitor voltage

The simulation of dynamic capacitor voltage is compared to the experimentally-recorded dynamic capacitor voltage in Figure 20. Visually, the simulation appears to predict the experiment reasonably well. This is corroborated with the low value for the Absolute Area Deviation. Upon closer examination, it appears that the simulation may still be

lacking physics that are present in the physical experiment, as indicated by the “bump” in the simulation response at $t = 1.12ms$ which is not present in the experimental response. This is an indication that the experiment may exhibit either more damping (resistance), or a greater inductance (mass) such that the response is unaffected by the bifurcation in the input signal.

5.1.1 Subsystem Response

Figure 21: RC circuit response: $R = 7.998 \text{ Ohm}$, $C = 105 \mu\text{F}$

As part of the pre-validation exercises we examined the experimental response of a RC circuit with respect to the simulated response illustrated in Figure 21. By eliminating the inductor from the circuit we can get a better idea of the parasitic elements present in only the resistor and capacitor. The Absolute Area Deviation is an indication of the difference between the experimental results and simulation results. With a low magnitude for the AAD of $2.53 * 10^{-4}$ and a visual comparison of the results in Figure 21 is it obvious that the model is a good estimation of the experimental response.

Figure 22: LC circuit response: $R=2.998 \text{ Ohm}$, $L = 1.233 \text{ mH}$, $C = 107 \text{ }\mu\text{F}$

Another subsystem tested was the *LC* circuit. A true *LC* circuit only exists in theory due to the presence of resistance in any wire as well as any parasitic elements present in the system. This experiment was valuable in determining the system resistance in the circuit. Although the slope of the response begins to diverge around $t = 0.5 \text{ ms}$, the magnitude and period of oscillation is similar between the two responses. We can refer to the Absolute Area Deviation again as a measure of the agreement between the two responses, which has a low magnitude of $4.21 * 10^{-4}$. As an example of a poor agreement between responses we present Figure 23 below.

Figure 23: LC circuit with poor agreement between responses: $R = 6.000$ Ohms, $L = 1.233$ mH, $C = 107$ μ F

Figure 23 illustrates a poor agreement between the experimental response and the simulated response for a LC circuit. With an Absolute Area Deviation of $1.28 * 10^{-3}$, Figure 23 is an example of how the AAD varies with difference in responses. In this case the AAD is an order of magnitude greater than both Figure 21 and Figure 22. This should put the AAD metric in perspective with regards to quantifying differences in response.

5.1.2 System Response

Figure 24: Comparison of experiment vs. simulation for 30 trials of capacitor voltage response

Now we can compare the experimental results to the results generated by the simulation.

Figure 24 illustrates the response using the numbered component sets (1 – 30) of resistors, capacitors, and inductors for both the experiment shown in red and the simulation shown in blue.

The component sets that were randomly numbered (1 – 30) were again randomized and tested experimentally. Table 4 illustrates the randomization of each component set, such that run #1 was not tested with resistor #1, inductor #1, and capacitor #1 as it was tested previously in Figure 24.

Table 4: Randomized component sets

R	L	C
1	11	16
2	12	24
3	17	30
4	18	8
5	27	28
6	8	1
7	14	3
8	5	27
9	29	20
10	21	6
11	26	18
12	13	22
13	1	4
14	7	19
15	2	29
16	25	11
17	19	7
18	20	17
19	4	14
20	30	26
21	23	10
22	6	12
23	15	13
24	3	9
25	28	21
26	22	15
27	16	23
28	9	25
29	10	2
30	24	5

Figure 25: Comparison of experiment vs. simulation for 30 trials of capacitor voltage response for randomized component sets

If we compare Figure 25 to Figure 24 we see a large difference in the spread of the responses. Also in Figure 25 we notice that there are two outliers in the experiment that were not present previously. This is due to the component values, specifically the capacitance of two capacitors, changing significantly between the two experiments. The experiments were run approximately three months apart, and during that time the capacitance of two capacitors decreased by approximately $20\ \mu F$ and $10\ \mu F$, respectively. In addition, the variance was greater between experiments for successive trials of the voltage input. These two factors explain the wider spread of responses in this experiment.

Figure 26: Standard deviation of Figure 25

We can compare the uncertainty estimated by the simulation with the experimental results by plotting the standard deviation of Figure 25 as shown in Figure 26. We see that the simulation has less variation than the experiment which indicates the model may still be lacking physics present in the experiment. This is still a good qualitative comparison which illustrates the correct trend in uncertainty propagation.

5.2 Results of Sensitivity Analysis

The results of the sensitivity analysis performed in Chapter 4.4 are now examined in detail. The equations used to generate these results are presented again here as

$$\{Sen_{V_c(t)}^R\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_R}{\sigma_{V_c(t)}} \right) \frac{\partial(V_c(t))}{\partial(R)} \quad (164)$$

$$\{Sen_{V_c(t)}^L\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_L}{\sigma_{V_c(t)}} \right) \frac{\partial(V_c(t))}{\partial(L)} \quad (165)$$

$$\{Sen_{V_c(t)}^C\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_C}{\sigma_{V_c(t)}} \right) \frac{\partial(V_c(t))}{\partial(C)} \quad (166)$$

$$\{Sen_{V_c(t)}^{V_{In}(t)}\}_{\sigma_{V_c(t)}} = \left(\frac{\sigma_{V_{In}(t)}}{\sigma_{V_c(t)}} \right) \frac{\partial(V_c(t))}{\partial(V_{In}(t))} \quad (167)$$

In addition to the dynamic nature of the system response, we previously showed that the standard deviation for both the input and the output are dynamic also values. This is a critical point to realize in this investigation, as it makes this process a *dynamic* Sensitivity Analysis. The electrical system is dynamic itself, and we now have highlighted the fact that the sensitivity and hence the uncertainty are also changing with respect to time.

5.2.1 Sensitivity Analysis of Resistance

Figure 27: Illustration of sensitivity for resistance as a function of time

The sensitivity of the dynamic capacitor voltage with respect to resistance ($Sen_{V_c(t)}^R$) is presented in Figure 27. Note that the sensitivity response has two inflection points, one

at approximately $t = 0.012\text{ ms}$, and another at $t = 1.1\text{ ms}$.

Figure 28: Experimental log of voltage in and out of resistor element

In Figure 28 we see that the voltage out of the resistor (blue line) has a distinct nonlinear “bend” in the initial phase of the response. Resistance changes with temperature as defined by a material property known as the temperature coefficient of resistance³⁴. For the carbon film resistor used in this experiment, the resistance actually decreases with increasing temperature. It was hypothesized that the first inflection point and nonlinear “bending” effects were due to thermal lag; however, when comparing the response in Figure 30 to the response of an RC circuit illustrated in Figure 21 we can see that there is no such bend in the response. This does not necessarily mean that thermal lag is not present.

Looking at the manufacturer’s data sheet for the resistor [17] we see that the temperature coefficient is listed as $\pm 400\text{ppm}/^{\circ}\text{C}$, which is equivalent to $0.0188\text{ }\Omega/^{\circ}\text{C}$ for a nominal 4.7Ω resistor. Without a thermocouple, an accurate temperature reading is

not possible due to the rapid response present in this system. Bourns Designer's Guide³⁵ does however provide a method to calculate the self-heating of a resistor based on the power derating curve³⁶ for the resistor in this circuit. Evaluating this formula results in $85^{\circ}\text{C}/0.5\text{ watt}$. The power for this system is presented in Figure 29.

Figure 29: Power for RLC circuit

In Figure 29 we see that the power of this circuit is 0.43W which is near the max power rating for the resistor, although it is for a very short duration. It follows that the temperature increase to self-heating is approximately 73°C for the resistor. If we also consider the ambient temperature in the laboratory of 23°C in addition to the self-heat generated through heat dissipation we arrive at a value for the resistance due to temperature change of $1.80\ \Omega$. This change in resistance is significant and would produce visually different results.

After the first inflection point in Figure 27, the rate of change of $(Sen_{V_c(t)}^R)$ increases to a maximum value at the second inflection point. Looking again at Figure 27, we see that the second inflection point corresponds to the switch between the two circuits

inside the DC power supply at $t = 1.1 \text{ ms}$. The behavior of this switch was discussed at the end of Chapter 4. The important point to note is that after this switch occurs, the sensitivity decreases as the system settles to its final value.

5.2.2 Sensitivity Analysis of Inductance

Relatively high values of sensitivity as well as rates of change of the sensitivity of the dynamic capacitor voltage with respect to the inductance ($Sen_{V_c(t)}^L$) are illustrated in Figure 30.

Figure 30: Illustration of sensitivity for inductance as a function of time

Comparing this system to a mass-spring-damper system, the mass is analogous to the inductance in this electrical circuit. The sensitivity magnitude increases to a maximum value at $t = 1.1 \text{ ms}$, and then begins to decrease, just as we saw with the sensitivity analysis of the resistance. The inductance is analogous to the mass in the mass-spring-damper system, so we expect that as the DC power supply switches from an underdamped circuit to a circuit with greater damping that we would observe a change in

the sensitivity of the inductance. As the damping is increased through the application of additional resistance, the sensitivity due to inductance also begins to decrease as the system approaches equilibrium. Globally, the dynamic capacitor voltage is most sensitive to the inductance.

5.2.3 Sensitivity Analysis of Capacitance

The sensitivity of the dynamic capacitor voltage with respect to capacitance ($Sen_{V_c(t)}^C$) is presented in Figure 31.

Figure 31: Illustration of sensitivity for capacitance as a function of time

Figure 31 shows that relatively low values of the capacitor voltage sensitivity with respect to the capacitance, as well as a low rate of change of the sensitivity as time increases are present in this system. The inflection point for the sensitivity due to capacitance occurs at approximately $t = 2.6 \text{ ms}$, or about 1.5 ms after the major inflection point in both the sensitivity due to resistance and inductance. Another point to note is that the sensitivity of the system with respect to capacitance has only a small

discontinuity induced by the bifurcation in the input signal at $t = 1.1 \text{ ms}$, and this sensitivity response does not change its sign at this point. Globally, the dynamic capacitor voltage is least sensitive to the capacitance.

5.2.4 Sensitivity Analysis of Voltage Input

Figure 32: Illustration of sensitivity for voltage input as a function of time

The sensitivity of the dynamic capacitor voltage with respect to the dynamic voltage input ($Sen_{V_c(t)}^{V_{In}(t)}$) is presented in Figure 32. Here we see the greatest sensitivity corresponds to the initial phase of the response. This phenomenon was explained in the input uncertainty quantification in Chapter 4. Studying Figure 11, we see the largest standard deviation corresponding to the initial power on of the DC power supply.

5.2.5 Summary of Sensitivity Analysis

Based on the visual results presented in the previous section, we may determine that the relative decreasing order of sensitivity of output to input voltage is $Sen_{V_c(t)}^L >$

$Sen_{V_C(t)}^{V_{In}(t)} > Sen_{V_C(t)}^R > Sen_{V_C(t)}^C$. In words, the sensitivity of the inductance (L) is greater than the sensitivity of the voltage input ($V_{In}(t)$), and also greater than the sensitivity of the resistance (R), all three of which are greater than the sensitivity of the capacitance (C).

5.3 Results of Uncertainty Propagation

The results of the sensitivity analysis performed in Chapter 4 are now examined in detail. Recall the equations developed in that chapter

$$\sigma_{R \rightarrow V_C(t)}^2 = [Sen_{V_C(t)}^R]^2 \sigma_R^2 \quad (168)$$

$$\sigma_{L \rightarrow V_C(t)}^2 = [Sen_{V_C(t)}^L]^2 \sigma_L^2 \quad (169)$$

$$\sigma_{C \rightarrow V_C(t)}^2 = [Sen_{V_C(t)}^C]^2 \sigma_C^2 \quad (170)$$

$$\sigma_{V_{In}(t) \rightarrow V_C(t)}^2 = [Sen_{V_C(t)}^{V_{In}(t)}]^2 \sigma_{V_{In}(t)}^2 \quad (171)$$

The uncertainty that is propagated is represented by the standard deviation, which is the positive square root of the propagated variance. Performing this process for Equation 168 above yields

$$\sigma_{R \rightarrow V_C(t)} = \sqrt{[Sen_{V_C(t)}^R]^2 \sigma_R^2} = + \left| \left(\frac{\sigma_R}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(R)} \right) \right| (\sigma_R) \quad (172)$$

Note that the absolute value on the right-most part of Equation 186 preserves the positive-definite nature of the positive square root. Due to the length of the partial derivative for the dynamic capacitor voltage it is not shown here, but can be verified in Equation 136. The solution will take the form of

$$\sigma_{R \rightarrow V_c(t)} \left(\frac{\sigma_R^2}{\sigma_{V_c(t)}} \right) \left| \left(\frac{\partial(V_c(t))}{\partial(R)} \right) \right| \quad (173)$$

Recall that Equation 136 is the partial derivative of dynamic capacitor voltage with respect to resistance, $\left(\frac{\partial(V_c(t))}{\partial(R)} \right)$, which is a highly nonlinear and dynamic function of time, resistance, inductance, capacitance, and the voltage input. In Equation 187, note that the uncertainty includes this nonlinear and dynamic partial derivative term, but it is also scaled in magnitude by the input variance (σ_R^2) as well as output uncertainty ($\sigma_{V_c(t)}$).

Similarly, we can repeat this process for propagating the uncertainty from the inductance to the output $V_c(t)$. The uncertainty for the System Response Quantity (SRQ) as the dynamic capacitor voltage $V_c(t)$ may be determined from the positive square root of the variance propagation from Equation 169 as

$$\sigma_{L \rightarrow V_c(t)} = \sqrt{[Sen_{V_c(t)}^L]^2 \sigma_L^2} = + \left| \left(\frac{\sigma_L}{\sigma_{V_c(t)}} \right) \left(\frac{\partial(V_c(t))}{\partial(L)} \right) \right| (\sigma_L) \quad (174)$$

Recall from Equation 137 that the partial derivative of dynamic capacitor voltage with respect to inductance is a lengthy nonlinear and time-varying equation; we can see that the propagated uncertainty will take the form of

$$\sigma_{L \rightarrow V_c(t)} \left(\frac{\sigma_L^2}{\sigma_{V_c(t)}} \right) \left| \left(\frac{\partial(V_c(t))}{\partial(L)} \right) \right| \quad (175)$$

Again note that the uncertainty that is propagated from the input to the output $V_c(t)$ is a nonlinear *function* of time but scaled with respect to the input variance and output uncertainty.

The uncertainty for the System Response Quantity (SRQ) as the dynamic capacitor voltage $V_C(t)$ may be determined from the positive square root of the variance propagation from Equation 170 as

$$\sigma_{C \rightarrow V_C(t)} = \sqrt{[Sen_{V_C(t)}^C]^2 \sigma_C^2} = + \left| \left(\frac{\sigma_C}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(C)} \right) \right| (\sigma_C) \quad (176)$$

The partial derivative of dynamic capacitor voltage with respect to capacitance, $\left(\frac{\partial(V_C(t))}{\partial(C)} \right)$, was presented in Equation 138 as a nonlinear and dynamic function. Due to the length of Equation 138 it is not included here as it would only convolute the process illustrated, but the solution takes the form of

$$\sigma_{C \rightarrow V_C(t)} \left(\frac{\sigma_C^2}{\sigma_{V_C(t)}} \right) \left| \left(\frac{\partial(V_C(t))}{\partial(C)} \right) \right| \quad (177)$$

Note that the uncertainty that is propagated from the input to the output $V_C(t)$ is a nonlinear *function* of time but scaled with respect to the input variance and output uncertainty.

$$\sigma_{V_{In}(t) \rightarrow V_C(t)} = \sqrt{[Sen_{V_C(t)}^{V_{In}(t)}]^2 \sigma_{V_{In}}^2} = + \left| \left(\frac{\sigma_{V_{In}}}{\sigma_{V_C(t)}} \right) \left(\frac{\partial(V_C(t))}{\partial(V_{In})} \right) \right| (\sigma_{V_{In}}) \quad (178)$$

The partial derivative of dynamic capacitor voltage with respect to the dynamic voltage input is the shortest partial derivative calculated, we can show the substitution of Equation 139 here as

$$\begin{aligned} \sigma_{V_{In}(t) \rightarrow V_C(t)} = & \left(\frac{\sigma_{V_{In}(t)}}{\sigma_{V_C(t)}} \right)^2 \left| (1 - (\cos((t * (1 - (CR^2)/(4L))^{1/2}/(CL)^{1/2}) \right. \right. \\ & + (R \sin((t(1 - (CR^2)/(4L))^{1/2}/(CL)^{1/2}) \\ & * (CL)^{1/2})/(2L(1 - (CR)/(2L))^{1/2}))/e^{(Rt)/(2L)} \left. \right| \end{aligned} \quad (179)$$

As with the other results for uncertainty propagation, we see that the uncertainty that is propagated from the input to the output $V_C(t)$ is a nonlinear *function* of time but scaled with respect to the input variance and output uncertainty. Equation 179 is unique in that both the input and the output are dynamic functions. A more powerful presentation of the uncertainty that is propagated from the inputs through the process to the System Response Quantity (*SRQ*) is obtained by plotting the dynamic uncertainty propagation functions.

5.3.1 Uncertainty Analysis of Resistance

Figure 33: Uncertainty propagation for resistance

The uncertainty that is propagated to the dynamic capacitor voltage through the process from the resistance ($\sigma_{R \rightarrow V_C(t)}$) is illustrated in Figure 33. This propagated uncertainty for $V_C(t)$ is a direct manifestation of the sensitivity $Sen_{V_C(t)}^R$ as shown in Figure 30. Because the sensitivity of capacitor voltage with respect to resistance changes the sign of its slope over the range of time considered, the propagated uncertainty also exhibits this behavior.

We see that the maximum uncertainty propagated occurs at $t = 1.1 \text{ ms}$, at the same point of maximum sensitivity for this input variable. The propagated uncertainty is smallest as the system reaches its equilibrium state at maximum voltage.

5.3.2 Uncertainty Analysis of Inductance

Figure 34: Uncertainty propagated due to inductance

The uncertainty that is propagated to the dynamic capacitor voltage through the process by the inductance ($\sigma_{L \rightarrow V_C(t)}$) is presented in Figure 34. The propagated uncertainty for $V_C(t)$ is directly related to the sensitivity $Sen_{V_C(t)}^L$ as illustrated in Figure 34. Since the sensitivity of the inductance changes its slope from negative to positive at $t = 1.1 \text{ ms}$, the propagated uncertainty also changes its slope, except now from positive to negative. The positive-definite nature of the uncertainty propagated is a result of the absolute value in the equation for uncertainty propagation.

5.3.3 Uncertainty Analysis of Capacitance

Figure 35: Uncertainty propagated due to capacitance

The uncertainty that is propagated to the dynamic capacitor voltage through the process by the capacitance ($\sigma_{C \rightarrow V_C(t)}$) is presented in Figure 35. The maximum uncertainty propagated due to the capacitor occurs around 2.6 ms. Globally, the capacitance is the smallest contributor to the total uncertainty propagated.

5.3.4 Uncertainty Analysis of Voltage Input

Figure 36: Uncertainty propagated due to voltage input

The uncertainty that is propagated to the dynamic capacitor voltage through the process by the dynamic voltage input ($\sigma_{V_{In}(t) \rightarrow V_C(t)}$) is presented in Figure 36. The uncertainty propagated due to the voltage input is directly related to the sensitivity which was presented in the previous section. As expected, the greatest value of uncertainty propagated occurs at the beginning of the response, when the power supply is attempting to turn on as fast as possible. The uncertainty decreases as the system reaches its equilibrium state near 4 *ms*.

5.3.5 Total Uncertainty Propagated

Figure 37: Total uncertainty propagated with respect to time

The total uncertainty propagated to the output from the inputs of resistance, inductance, capacitance, and the voltage input is illustrated in Figure 37. We see that the maximum uncertainty propagated occurs at $t = 1.1 \text{ ms}$, or the end of the linear phase of the input signal, which is near the point of the maximum current in the circuit as shown in Figure 38.

Figure 38: Experimental current $I(t)$

Again using the analogy of the mass-spring-damper (*MSD*) system, the velocity of the *MSD* system is comparable to the current in this electrical circuit. It is logical then that we would expect the maximum uncertainty of the system to occur at the point of maximum velocity. We should also be aware that the time constant of this system also falls within this region of maximum uncertainty propagated.

Figure 39: Comparison of calculated uncertainty results with experimental results

We can now compare the uncertainty propagated from the theoretical analysis to the actual uncertainty propagated as recorded experimentally. The experimental uncertainty is simply the standard deviation of the dynamic plots in Figure 25. Looking at Figure 39 we see that the uncertainty propagated by the computational model is initially greater than the experimentally recorded uncertainty propagation. This is due to the model weighting the uncertainty in the input at a greater value than what is actually propagated in the experiment. The uncertainty in the initial turn-on of the DC power supply does not affect the output as much as anticipated from the simulation results. The simulation results agree with the experimental results very well in the time range of $0.2\text{ms} - 0.9\text{ms}$, and after that point the simulation overpredicts the uncertainty with respect to the experiment.

5.4 Final Results

Figure 40: Dynamic capacitor voltage with error bars

Figure 40 illustrates the average of 30 trials of the dynamic capacitor voltage with error bars applied to each data point in time. The error bars correspond to a 95% confidence interval using the calculated values of uncertainty shown with blue data points in Figure 39. The maximum uncertainty in the response occurs at data point #28, or $t = 1.12 \text{ ms}$, with a value of $V = 3.1119 \text{ V} \pm 0.20232 \text{ V}$.

5.5 Verification

Verification is the process used to evaluate the accuracy and credibility of a model or simulation. It is essentially a check to ensure that the model is correctly implemented and properly represents the mathematical equations upon which it is based. Verification is the first step in providing confidence in the results generated by a model or simulation. There are two critical operations that cover verification: verification of the code and calculation verification³⁷. Code verification is the process of reviewing the code to ensure that no errors were made in programming. Calculation verification ensures that no

programming errors exist in the computational simulation and that the numerical algorithms implemented accurately represent the differential functions on which they are based. Calculation verification is the process of estimating numerical errors due to round-off and discretization. In addition to hand calculations, the simulation was implemented using four different methodologies to ensure agreement with the physics based ODE. The results from the analytical solution are plotted simultaneously in Figure 41 with the results using MATLAB's ODE23 solver, ODE45, and a state space representation.

Figure 41: Results for a DC step change of 100 Volts to a system initially at equilibrium with zero initial conditions ($R = 2 \text{ Ohm}$, $L = 1 \text{ mH}$, $C = 10 \text{ }\mu\text{F}$; under-damped). a) dynamic current response, b),c),d): dynamic voltage responses for the resistor, capacitor and inductor, respectively.

The dynamic current response is presented in Figure 41a, while the dynamic voltage responses for the resistor, capacitor, and inductor are illustrated in Figure 41b, c, and d respectively. In all cases, the analytical solution and three numerical solutions are visually indistinguishable. The largest difference in any numerical comparison was less than 0.2% with a mean absolute deviation of 0.05% for all cases studied.

Figure 42: Corroboration for Uncertainty Propagation. Comparison of three methods: sequential perturbation (blue), partial derivative (black), modified (IPCC) partial derivative (red). Experimental uncertainty shown in green.

Figure 42 is one of the most meaningful plots in this investigation. This figure is similar to Figure 39, with the addition of two other methods for uncertainty propagation. The first additional method is the sequential perturbation method³⁸ shown in the black data points, which has the greatest values for uncertainty propagation of the three methods. The second additional method shown is the first-order Taylor series approximation using partial derivatives³⁹, which is represented by the blue data points. We see that the perturbation method and the partial derivative method both follow the same trend, with a discontinuity at the bifurcation point in the voltage input. Both of these methods also

show a large deviation from the experimental uncertainty after $t = 1.12 \text{ ms}$. We can see that all three methods capture the same general trend of uncertainty, with the modified partial derivative method resulting in the best representation of the experimentally propagated uncertainty.

Another part of verification is to ensure that the model does not introduce bias to the result. Ideally, the results of a simulation or experiment will contain no obvious patterns. One method implemented to check for bias in the model was to construct a normal probability plot. The normality plot should form an approximate straight line, and any departures from the line indicate a deviation from normality.

Figure 43: Normal probability plot for one point in time of voltage input

Figure 43 illustrates the normal probability plot for 30 trials of the experimentally recorded input voltage, evaluated at the same point in time. Examining Figure 43 we see

that there is one data point which deviates from the straight line of normality, near 1.55V. The rest of the data shows no bias and agrees with the assumption of a structureless input voltage source.

Figure 44: Normal probability plot for one point in time of experimentally recorded capacitor voltage

Figure 44 is an illustration of the normality plot for 30 trials of experimentally recorded capacitor voltage, evaluated at the same point in time. This plot was generated using the same combination of resistor, capacitor, and inductor, repeated 30 times. Although we see some grouping in the plot, the experiment shows no major deviations from normality. This is a good check that our instrumentation and measurements do not bias the result one way or another.

Figure 45: Normal probability plot for one point in time of simulated capacitor voltage

Figure 45 is an illustration of the normality plot for 30 trials of simulated capacitor voltage, evaluated at the same point in time. This plot was generated using the same combination of resistor, capacitor, and inductor, repeated 30 times. The simulation was subject to the same voltage input as the experimentally recorded voltage shown in Figure 44. The purpose of this graph is to check for bias introduced by the simulation. There are no major deviations from normality, and we can see from Figure 45 that the model predicts approximately the same number of points both above and below the mean value.

5.6 Validation

The validation process assesses the accuracy of the simulation with respect to the physical experiment it represents. Depending on the purpose of the exercise and the stringency of the validation metric developed by the practitioner, the model may be deemed as validated, even when other validation metrics dispute the agreement between simulation and experiment. For example, a validation metric may simply have the form: estimated error of the model $\pm$ an interval that represents experimental uncertainty with 90% confidence⁴⁰.

In this study we use a statistical validation metric; the Anderson-Darling k-sample test was implemented to determine if the data from the experiment and the data from the simulation could be drawn from an identical population. This test is useful because it determines if the two sets of data can be combined, i.e. they represent the same population. There are several different ways we can implement the Anderson-Darling k-sample test. The first test is to compare the dynamic response generated in the simulation to the dynamic response recorded in the experiment.

Figure 46: Comparison of dynamic capacitor voltage, simulation versus experiment

The voltage vectors from the experiment and simulation use the same time increment of $\Delta t = 40 \mu s$. The total number of data points in each response is $n = 133$. Figure 46 illustrates that the simulation predicts the simulation reasonably well, which is corroborated by the low value of the AAD. However, if we apply the Anderson-Darling k-sample test, the result is that the samples were drawn from different populations. This is indicated by an Anderson-Darling rank statistic of 7.1553 with an associated probability of 0.0001355.

If we only consider the response from $n = 1$ to $n = 100$ data points by removing 33 data points at the trailing end of voltage vectors in Figure 46, or essentially ending the response at $t = 4 ms$, the Anderson-Darling k-sample test reveals that the populations are identical. This is indicated by an Anderson-Darling rank statistic of 1.6258 with an associated probability of 0.1485.

We can also use the Anderson-Darling test to compare one data point in time for both the experiment and the simulation. A single combination of resistor, capacitor, and inductor was tested experimentally for 30 trials. The voltage input used in these trials was recorded and fed into the computational simulation, where 30 similar results were generated. The results are summarized in the table below.

Table 5: Comparison of a single data point for 30 trials

	Experiment Voltage (V)	Simulation Voltage (V)
Mean	0.2188	0.2413
Std. Dev.	0.02718	0.01614

Applying the Anderson-Darling test to these two samples of data resulted in an Anderson-Darling rank statistic of 7.5189 with an associated probability of 0.0000566. This indicates that the samples were drawn from different populations, so we cannot say that the simulation represents the experiment with statistical confidence.

Based on the results of the Anderson-Darling tests, it should be no surprise to the reader that this model cannot be statistically validated. We can say that the model agrees favorably with the experiment, but from a statistical standpoint we cannot say that the model is an accurate representation of the real world experiment.

CHAPTER 6

CONCLUSION

6.1 Summary

This investigation studied the influence of uncertainty in inputs upon the uncertainty in the output, sensitivity of the output to the inputs, and uncertainty propagation of the inputs through the process to the output. The focus of this study was to provide an example of a computational simulation of a real-world experiment that could easily be understood by a wide variety of engineers and scientists, while providing the technical rigor required in such an analysis. This investigation illustrates the difficulty in the proper verification and validation of any computational model. A statistically validated model should be the goal for any computational simulation of a physical experiment, yet it is rarely seen in practice.

This work began by creating a computational model of a series RLC circuit. The model was implemented using four different solution methods which allowed for corroboration of results. Ultimately, only the analytical solution was used in this investigation as it allowed for the computation of partial derivatives. The experiment was designed based on the results of the simulation. The intention of this thesis was to study the benefits of a slightly underdamped system. However, after initial experimentation it was obvious that the results from the model were not easily reproduced in the experiment.

The reduction of model form error was a major part of this analysis. In this investigation, the results of a model based on known theory were initially presented. However, there was a noticeable difference between the results generated from this simulation when compared to the results recorded from the model-based experiment. The model form error was reduced by using the experimental input in the model and quantifying the parasitic effects through experiments with various subsystems.

Identifying uncertainty and quantifying uncertainty in each component was an important part of this analysis. There were many sources of uncertainty present in each factor, and it is important to consider each one when developing any computational model or experiment. A key point to consider in any uncertainty analysis is to understand how accurate an answer needs to be, and if that level of accuracy can be achieved.

6.2 Conclusions

This investigation details the implementation of uncertainty quantification, sensitivity analysis and uncertainty propagation for a dynamic and nonlinear system. A process was developed and presented for verification and validation of such time-varying systems. Although the model developed for this study could not be validated from a statistical standpoint, it is important to note the stringency of the validation metric. In section 5.7 the example was presented where only the transient response was considered instead of the full response (100 data points vs. 133 data points) which resulted in passing criteria for the validation metric. The conclusions that are noteworthy are listed below:

1. Uncertainty quantification, sensitivity analysis, and uncertainty propagation were performed for a dynamic system. The uncertainty of the system examined in this investigation had a strong dependence on time, and these

findings can be useful when studying other dynamic systems where the path of the transient response is important.

2. Uncertainty propagation was performed using an uncertainty-weighted technique, which when compared with several accepted methods of uncertainty propagation to the experimental results proved to better represent the measured uncertainty in the system.
3. Model form uncertainty was shown to be very large when considering only an analytical solution modeled with known physics. This is an important discovery with respect to the way we currently model and simulate complex systems. It is critical to consider installation and parasitic effects when developing any model, otherwise the model may misrepresent a system.
4. Statistical validation of a dynamic model is a necessary task in establishing confidence in the results produced computationally. The results of successful validation for individual models will have a compounding effect, and over time the modeling and simulation community will be better able to predict physical phenomena without actually building any physical systems.

6.3 Recommendations for Future Work

While performing this analysis, it became obvious to the author that there were many avenues available to examine and explore further. For example, a study could be performed on the voltage input with regard to the bifurcation to prevent overshoot.

Previous work⁴¹ completed suggests the benefits of using a split-circuit system to deliver the fastest rise time while still maintaining zero overshoot. The DC power supply used in

this investigation appears to exhibit this type of temporally-bifurcated behavior, and it may be of interest to reverse-engineer such a circuit.

To improve the predictive ability of this simulation, the total uncertainty propagated can be reduced. From the results of the sensitivity analysis and uncertainty quantification we discovered the two main sources of uncertainty in this experiment were the inductance and the voltage input. In addition to sourcing a more precise inductor, a more precise measurement of the inductance could be achieved by using a LCR meter or constructing a Maxwell bridge. A more repeatable voltage source could also be implemented to reduce the input uncertainty.

An advanced application of the uncertainty analysis conducted in this study may involve a more complex system or experiment where the path of a power signal is tangible. One example would be an electromechanical actuator which controls a flight surface. A scale model could be built and studied with similar principles described in this thesis to analyze the uncertainty in such a system. The benefits of studying electromechanical systems like this could lead to improved efficiency and performance of flight control systems.

The uncertainty-weighted technique for sensitivity analysis and uncertainty propagation should be implemented for additional cases with analytical solutions, and compared to indirect approaches like the sequential perturbation analysis. Then, when we encounter “black box” cases where a direct solution is not available, we can better estimate the uncertainty introduced by such techniques.

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APPENDIX A

EXPERIMENTAL PROCEDURE

Materials List

- 30 resistors: Digikey Part Number: S4.7HCT-ND
- 30 capacitors: Digikey Part Number: P5123-ND
- 30 inductors: Digikey Part Number: M8275-ND
- One 100 Ω resistor
- Breadboard
- Label maker
- Multimeter test leads w/ alligator clips
- Power supply test leads
- Agilent 34405A Multimeter
- Agilent 3631A DC Power Supply
- Agilent 33220A Signal Generator
- Agilent DSO3062A Oscilloscope
- Data Acquisition Device: USB-1208FS (Measurement Computing)
- Laptop

Experimental Procedure – Measuring Components: Resistors

1. Power on the multimeter and allow to warm up for 30 minutes.
2. Connect the multimeter leads with alligator clips to the HI and LO terminals on the multimeter.
3. Number each component (1-30) using the label maker.
4. Select the appropriate range of measurement (1-100 Ohms).
5. Attach the alligator clips to each end of the resistor.
6. Set the resistor flat on the lab table.
7. Allow three seconds to elapse.
8. Record reading.
9. If reading is still varying, record the greater value

Experimental Procedure – Measuring Components: Capacitors

1. Power on the multimeter and allow to warm up for 30 minutes.
2. Connect the multimeter leads with alligator clips to the HI and LO terminals on the multimeter.
3. Number each component (1-30) using the label maker.
4. Select the appropriate range of measurement (1-100 Microfarads).
5. Attach the alligator clips to each end of the capacitor.
6. Set the resistor flat on the lab table.
7. Allow three seconds to elapse.
8. Record reading.

9. If reading is still varying, record the greater value.

Experimental Procedure – Measuring Components: Inductors

1. Measure the true value of the $100\ \Omega$ resistor, following the procedure above
2. Power on the signal generator and allow to warm up for 30 minutes.
3. Power on the oscilloscope and allow to warm up for 30 minutes.
4. Connect the negative terminal (black) from the 6V section of the power supply to the grounded terminal (green) on the power supply.
5. Connect the negative terminal on the power supply to the breadboard to create a row of grounded contact points on the breadboard.
6. Connect the positive terminal to a row on the breadboard.
7. Insert the inductor to be measured into the breadboard with one end connected to the powered row of terminals.
8. Insert the $100\ \Omega$ resistor in series with the inductor and connect one end to ground.
9. Connect one oscilloscope probe to the voltage input side of the inductor.
10. Connect another oscilloscope probe to the opposite end of the inductor.
11. Apply a 2V signal at a frequency of 27 kHz in the form of a sine wave using the signal generator.
12. Record the actual voltage in and voltage out of the inductor (should be near a $\frac{1}{2}$ ratio).
13. Use these recording to calculate the inductance value.

Experimental Procedure – Recording Transient Response

1. Power on DC power supply and allow to warm up for 30 minutes.
2. Power on laptop and connect DAQ device.
3. Calibrate DAQ device according to manufacturer's guidelines.
4. Connect the negative terminal (black) from the 6V section of the power supply to the grounded terminal (green) on the power supply.
5. Connect the negative terminal on the power supply to the breadboard to create a row of grounded contact points on the breadboard.
6. Connect the positive terminal to a row on the breadboard.
7. Insert resistor into breadboard – connect one end to the row of powered terminals.
8. Insert inductor into breadboard.
9. Insert capacitor into breadboard – connect one end to the row of grounded terminals.
10. Connect DAQ channel 0 HI to the row of powered terminals – the same row the power supply input and resistor are connected.
11. Connect the DAQ channel 0 LOW and GND to the grounded row of terminals on the breadboard. This will allow you to record the input signal.
12. Connect DAQ channel 1 HI at row where the inductor and capacitor meet.
13. Connect the DAQ channel 1 LOW and GND to the grounded row of terminals on the breadboard. This will allow you to record the output signal.
14. Set power supply to 6V and ensure it is in standby mode
15. Begin recording data via the software on your laptop
16. Cycle the power supply from standby to ON

17. Verify recorded data by plotting in Excel or MATLAB

The assembled circuit should look similar to the figure below.

Figure 47: Assembled RLC circuit