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**Constrained identification for adaptive control: Application to
biomedical systems**

Timmons, William Donald, Ph.D.

Case Western Reserve University, 1992

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CONSTRAINED IDENTIFICATION FOR ADAPTIVE CONTROL:
APPLICATION TO BIOMEDICAL SYSTEMS

by

WILLIAM DONALD TIMMONS

Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

Thesis Advisors: P.G. Katona and H.J. Chizeck

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May 1992

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GRADUATE STUDIES

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William D. Tennison

CONSTRAINED IDENTIFICATION FOR ADAPTIVE CONTROL:
APPLICATION TO BIOMEDICAL SYSTEMS

Abstract

by

WILLIAM DONALD TIMMONS

For ill defined, time varying, or non-linear systems, adaptive controllers offer an attractive alternative to classical design techniques. These controllers require minimal information. Ironically, the information that is available is often discarded. As a result, control may become unstable. If the available information could be used instead of discarded, the instability might be avoided.

We therefore developed a real-time algorithm that imposes linear equality and inequality constraints on a time series model of the process. Thus commonly available information, such as open loop stability, settling time, and steady state gain, can be incorporated into the

control. When the information is imposed according to our guidelines, control errors due to mismodeling can be significantly attenuated (in one instance, we reduced the mean squared output error by more than two orders of magnitude).

We demonstrate our algorithm in two practical biomedical applications. In the first, we consider second order linear compartmental models (a popular model for pharmacodynamical systems). We develop novel constraints for these models. Then, as an example, we control plasma and tissue concentrations of methotrexate, an antimetabolite used in the treatment of certain neoplastic diseases. Our results show that the constraints improve both controller performance and patient safety.

In the second biomedical application, we lower mean arterial pressure with a vasodilator. This system is non-linear and time-varying; hence it can be difficult to control. We simulate this system with a series of models that are progressively more complex. For each model, we develop suitable constraints which we then impose during control. As the models become more complex, we show that the constraints become more important for safety and improved control.

While pursuing this work, we discovered a simple modification that significantly improves the efficiency and accuracy of positive semi-definite complementary linear programming (a technique for solving quadratic programs). We prove its validity and modify the pivot selection rules to implement least distance programming.

DEDICATION

To my loving wife, Debbie,
who earned this degree no less than I,
who stayed with me through thick and thin,
and who poured herself out that I might survive.

To my delightful daughter, Bethany,
who taught me to see the fascination in every day life.

To my parents,
who taught me to persevere.

And to my cat, Beau,
who kept me company through many a late night.

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Of course, I also wish to acknowledge sources of financial support. Early feasibility studies were supported by a contract from Eli Lilly and Company. During this period I developed much of my geometrical understanding of quadratic programming. This approach is reflected strongly in Chapter 3 (Mathematical Techniques). Under NSF grant ECS-84-00765, and later, BCS-8908713, I discovered our enhanced complementary

pivoting algorithm (target rejection), began to formulate the set of application guidelines for constrained identification, and started to analyze second order compartmental models. And, for most of this last year, my family's living expenses were supported by generous loans from my parents.

On a more personal note, I want to thank my past and present lab partners, Dr. Ranu Jung and Mr. Ping Xue, for providing a pleasant and friendly work place. Furthermore, I wish to acknowledge the tremendous help Mr. Xue provided. His knowledge of the cardiovascular system and his enhancements of the PK/PD CV model have been a great help to me.

I owe a debt of gratitude to Mr. Scott D. Heavner. His diligence and skill as a work study student were exemplary. I hope all my students will be as bright and capable as he. And not to be forgotten, I wish to thank Ms. Meredith Albreicht. Her contributions helped in my understanding of this work.

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CHAPTER 1 :

INTRODUCTION

Adaptive controllers, either implicitly or explicitly, consist of two parts: (1) a system identifier that models the controlled system in real time, and (2) a controller that, based on the identified model, chooses the next input to the system. For systems that are ill defined, slowly time varying, or non-linear, such controllers offer a ray of hope that near optimal control may be obtained. This prospect has been particularly attractive to the biomedical field, since many physiological systems meet these conditions.

Quite naturally, there has arisen an aura of glamour around these devices. It is often thought that an adaptive controller can be thrown at a problem and the hard work of a detailed system analysis avoided. However, adaptive control is not a panacea for all situations. In practice, the lessons and rules from classical control theory and system analysis must form the basis for safe and effective adaptive control.

Adaptive controllers have problems that can lead to

momentary instability, freezing of the inputs, or otherwise poor control. We have traced many of these problems to poorly identified or inadequate models [11]. If these devices are ever to be used in a clinical setting, such as for the automated infusion of therapeutics (see for example, [1]-[15]), safety dictates that these problems be remedied. Hence, among other concerns, it becomes important that the adaptive controller keep an accurate model of the process.

The objective for this work was to tackle the modeling issue. In pursuing this goal, we developed an algorithm that keeps the model reasonable by using a priori information about the controlled process. The information must be in the form of linear equality and inequality constraints on the identified model parameters, so we have included a section in which we translate commonly available knowledge into this form. Finally, and of paramount importance, we have examined the affect this information has on closed loop control. We have identified certain guidelines and rules that, if followed, can significantly improve transient control performance without loosing the long term convergence properties of standard adaptive controllers. Indeed, in one example the mean squared control error is reduced by more than two orders of magnitude!

1.1 Overview And Intent

Many times, due to momentarily inaccurate modeling, an adaptive controller in simulation or during real time application inappropriately turns off the inputs, saturates them at the maximum, or causes them to oscillate [11]. The reason that the model can become inaccurate is that black box models, typically used in adaptive controllers, are non-specific. They can be used to identify anything from a physiological process to a concrete mixer. Their utility is that any algorithm developed for one application can be used across a wide spectrum of other applications. However, their utility is also their weakness. They may mistakenly model the input/output characteristics of a concrete mixer, when instead the intent was to model a physiological process. Noise and nonlinearities are typically the culprit, and, unfortunately, cannot be avoided.

Despite these observations, most research concentrates on control algorithms rather than identification. The typical goal is to increase stability or improve performance, even though improvement of one is typically at the expense of the other. Improved system identification, on the other hand,

increases both stability and performance.

Clearly, any controller's performance, whether it is adaptive or not, depends on the model parameters used to design the control law. A seeming exception is the self-tuning regulator, which can have biased model parameters and still be optimal. Nevertheless, it too obeys this principle. Its parameters must converge to specific, well defined values. Anything else results in degraded performance. Also, since optimal control is guaranteed only as time goes to infinity, the self-tuning property is meaningless for systems that never reach steady state. Many physiological systems fall into this category.

Thus, our supposition is that improved modeling will improve control, and that modeling can be significantly improved by using additional information besides the standard input and output data during identification. That is, by using known information about a system, we can rule out anomalous models and force the identifier to produce reasonable ones only. However, our improved identifier must be able to use this information optimally and in real time.

To make this solution tenable, we restrict ourselves to knowledge that can be expressed or approximated by

linear equality and inequality constraints on the model parameters. This includes information such as open loop stability, steady state gain, settling time, certain pole and zero locations, baseline drift, and ranges for the parameters. Techniques from modeling and systems analysis can be used to obtain this information. Tools from continuous and discrete-time signal processing can be used to convert it to linear form.

With this restriction to linear constraints, standard quadratic and numerical programming can solve the constrained identification problem. Combined with adaptive control techniques for speed, and taking into account numerical considerations for robustness, we arrive at an algorithm suitable for real-time application. In developing this final product, we will have thus combined theory from four fields: Optimization (quadratic and numerical programming), Adaptive Control, Modeling and Systems Analysis, and Continuous and Discrete-Time Signal Processing.

If our hypotheses are correct, the possibilities become exciting. A real time identifier that can optimally impose an arbitrary set of mixed linear equality and inequality constraints could be used across a wide range of applications, each application having its

own set of constraints based on the known information. The identifier would produce not a general purpose black box model, but a more specific "grey" box model. The specificity, or shade of grey, would be determined by the number and nature of the constraints. The resulting controller would be a grey box adaptive controller (GBAC).

However, it still remains to be seen whether constrained identification actually results in improved controller performance or not. If it does, other questions arise. For example, which constraints are most helpful? Which are least? Is the added complexity and computational burden of optimal identification worth the effort? What can go wrong with this algorithm? And finally, in what other ways can we use it?

Simulations are useful in answering these questions. We will be able to demonstrate that, if certain guidelines are followed, transient performance can be significantly improved without losing the normal long term convergence properties of adaptive controllers. We will show that any information, if true, is generally useful, and if false, almost always harmful. Optimal identification will be shown to be important, and other uses for the constraining algorithm will be discussed.

After these issues are solved, we will then tackle specific biomedical applications, including the control of plasma and tissue concentration of methotrexate (an antimetabolite used in the treatment of certain neoplastic diseases), and the lowering and control of mean arterial pressure using a vasodilator.

1.2 Dissertation Structure and Outline

Earlier we stated that our objective was to tackle the modeling issue by developing an identifier that uses a priori information in the form of linear constraints on the model parameters to keep the model reasonable. This goal produces three tasks. First, we must develop a real-time identifier that optimally imposes linear constraints. Second, we must show that useful knowledge can be represented as linear constraints. And finally, we must demonstrate the utility of grey box adaptive control.

This dissertation will consist of three papers (Chapters 4-6) preceded by a literature review (Chapter 2) and a detailed development of our mathematical techniques (Chapter 3). Since the papers are self-contained, they will duplicate some material. Following

the papers, a final chapter unifies the dissertation (Chapter 7).

The first section of the literature review (Chapter 2) examines previously published theories on constrained identification used with adaptive control, while the second section reviews applications of control theory to biomedical systems. The first few sections on mathematical techniques (Chapter 3) introduce ARMAX models, linear constraints, and quadratic programming. However, the majority of Chapter 3 describes an enhanced version of positive semi-definite complementary linear programming (CLP), a numerical algorithm that solves quadratic programs.

The first paper (Chapter 4) focuses on our tasks of developing the basic algorithm, expressing a priori information in the form of linear constraints, and extracting guidelines that should be followed when using GBAC's. The resulting constraints will impose information such as open loop stability, settling time, certain pole and zero locations, and static sensitivity (steady state gain).

The second paper (Chapter 5) will concentrate on the application of GBAC's to pharmacological systems that can

be approximated by linear compartmental models. We will develop novel constraints on the poles and zeros for these systems and apply them to the control of blood plasma and tissue concentrations of methotrexate.

In the third paper (Chapter 6) we apply GBAC's to lower and control mean arterial blood pressure using a vasodilator. This system is inherently non-linear and time-varying. First, however, we tackle linear time-varying approximations to the canine and human cardiovascular systems. Finally, we develop novel constraints based on an analysis of a more complex pharmacokinetic/pharmacodynamic model of the human cardiovascular system. This paper brings us back full circle, since it was our early attempts to control blood pressure that motivated us to pursue this work (see [10], [11], [13], and [14]).

Finally, in Chapter 7, we summarize our results and conclusions, discuss future research possibilities, and list additional potential applications.

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CHAPTER 2 :

LITERATURE REVIEW

2.1 Adaptive Control and Constrained Identification

Except for specifying an upper limit on model order, or perhaps a delay, adaptive control theory typically ignores a priori information. Surveys and literature reviews mention that it ought to be used, but then fail to mention appropriate ways to use it (see, for example, [61], [57], or [74]). There are many reasons for avoiding this issue. For example, many consider the use of a priori information antithetic to the whole idea of general purpose adaptive control (e.g., see [84]). Others conclude that in many practical cases, restriction of the parameters to a constraint region "... is not feasible nor even necessary, e.g., in output error system identification and adaptive control, momentary unstable parameter settings can induce very rapid learning..." [48].

We do not agree for several reasons. First, in a clinical setting, momentary instability can produce

adverse sequelae. Second, while it is true that momentary instability can induce rapid learning, once this learning is finished the rate of adaptation would be greatly reduced. The parameters would thus be slow to track new changes in the system as they occur. Finally, constraints offer the chance for more sophisticated modeling. For example, alternate model structures could be tested and compared on-line. This capability will be useful in Chapter 6 (Paper 3), where the linearization of a non-linear cardiovascular model results in various structures at different operating points.

Once past these stumbling blocks, another barrier to the use of constraints is that the algorithms needed to impose them involve complex theory and can pose a computational burden. To bypass these shortcomings, some investigators have developed controllers such as the Multiple Model Adaptive Controller, where the controller chooses from a set of pre-defined models to base its control ([54], [89], and [33]). For these systems, the benefit is that the worst case is limited to selection of the worst model. This strength is also its weakness: the best case is limited to selection of the best model. If the best model is not too good, control may not be good either. And since the parameters cannot be adjusted, the model almost surely will not be optimal.

Other investigators put heuristic rules on the allowable inputs, such as limiting the maximum amount of change or the maximum deviation from the mean [38]. These techniques are often necessary for safety: they limit worst case oscillations. However, they also limit best case optimal control, especially in those instances when sudden changes in the input are really needed. Finally, they do not solve the problem of freezing and saturations due to modeling error. This approach is not a solution, but instead should be used along with grey box identification.

Some investigators suggest avoiding the complexity of constraining algorithms by enforcing only simple bounds (maximums and minimums) on the model parameters [32]. However, besides limiting the amount of a priori information that can be encoded, simple bounds only reduce the complexity when one knows which bounds will be imposed on the identification *before they are invoked*. And the only instances when one can know ahead of time which bounds will be invoked is the trivial case of bounds on only one parameter or when a non-optimal method is used.

Early ad hoc attempts to control non-minimum phase systems are commonly cited as practical examples of

bounding one parameter. A priori information, if it was even considered, was secondary to the goal of stabilizing control. It was well known that minimum variance control is unstable for these systems. A constraint was therefore constructed to impose incorrect information, so that the minimum variance control could be "fooled" into remaining stable. This constraint typically inflated the first non-zero term of the impulse response, so that only one parameter needed bounding [3], [18], [19], [51]. Generalized predictive control has since obviated the need for this fix [10]-[11], [81].

The recent genre of literature concerned with global asymptotic stability imposes simple bounds also (e.g., see [66], [58], [84], [36], [68], [46], [63], [52], [28], and [30]). Constraints are constructed based on the amount of modeling error that can be tolerated and still maintain stable control. This approach is loosely based on a priori information, since, to infer modeling error, one must set up a nominal parameter region. The stability proofs often require bounding of all the parameters, typically in the form of individual parameter bounds (minimum and maximum for each parameter) or a hypersphere about a nominal parameter vector (minimum and maximum for the norm of the parameter error vector).

Since all parameters are bounded, one must now choose between the complexity of optimally projectors and simple non-optimal projectors.¹ Since long term (asymptotic) stability is of primary importance for these studies, the most frequent choice has been simple sub-optimal projectors.

We have discovered, however, that short term control performance may suffer when using a sub-optimal projector, and that one should therefore use optimal projectors exclusively. Algorithm complexity thus cannot be avoided. Therefore, there is no reason to limit a priori information to simple constraints. One might as well exploit all the available information and construct complicated constraints as needed.

While algorithm complexity cannot be reduced with inequality constraints, it can be reduced when all the known information can be put into the form of equality constraints. Knowledge of a pole or zero location falls into this category. Two methods can implement this type

¹ Unless, of course, only one parameter is estimated. The only practical example of such a system (to our knowledge) is the closed-circuit anesthesia delivery system in [79]. However, the authors failed to realize that their estimator (a recursive least squares algorithm) reduced to a first order filter! Hence, even this example is spurious.

of information. One keeps the overall structure and form of the model, and constrains the parameters using the equality constraints [12], [9], [31], [24], [13], [14]. Chia, Chow, and Chizeck (1991) were able to enforce monotonicity of a cubic polynomial from a Hammerstein muscle model using equality constraints. They reported over a three-fold improvement in predictive capabilities of the model in open loop. The other, in the genre of generalized least squares (as in [32] or [31]), filters or factors out the known parts of the system so that only the unknown parts need to be identified [12], [5], [49], [90], [85].

The first technique is preferable when the constraints are time varying or when alternate model structures are compared, as in the statistical analysis of a null hypothesis versus an alternate. The second technique would require that a separate model be identified for each possible constraint or hypothesis. The second technique is preferable when the model structure is known and the constraints are not changing, since the number of estimated parameters is reduced by one for every equality constraint. This reduction can result in significant computational savings.

For most systems, the ability to specify equality constraints requires more information than is known. For example, exact knowledge of a system's poles and zeros (typically needed for forming equality constraints) is usually not available. Instead, usually we can only specify regions in which the poles and zeros lie, or a range for, say, static gain, or maybe a rise time or settling time. This type of information translates to inequality constraints, which again brings us back to the need for more complex algorithms.

For our approach, we shall certainly not ignore the important stability results, but shall additionally use more complex linear constraints to incorporate as much information as possible. Furthermore, we shall use optimal projection and supervisory control. Thus, not only will we be able to guarantee global asymptotic stability (based on the above results), but we will also be able to enhance performance and patient safety.

2.2 Biomedical Control Applications

Nowhere does control have a greater potential to improve people's lives than it does in patient care. Intravenous drips are often gauged by sight, quantitative

strategies are almost non-existent, and time consuming tasks are often performed by exhausted staff. Compared to industry, current medical practice for drug management is a stone-age art.

At the same time, nowhere does control have a greater potential for harm than it does in patient care. Control failure can lead to drug overdose, underdose, and possibly death. Hence, safety dictates that we use smart and safe devices for these tasks (humans would be ideal, except they cost too much). With the advent of smart adaptive controllers, expert systems, and fault tolerant systems, the hospital environment is ripe for the application of modern control systems. For the sake of humanity then, there is a great need and urgency for the development of suitable algorithms to control physiological variables.

From another point of view, physiological systems are often non-linear and time-varying. As such, biomedical control applications often push current theories and technologies to the limit, and often result in new and creative control strategies. For the sake of the advancement of control theory then, there is a great need and urgency for the development of suitable algorithms to control physiological variables.

Thus, the mean arterial blood pressure (MAP) system has been a popular test bed for a variety of controllers. Koivo (1981) was one of the first to adaptively control MAP in the anesthetized dog [47]. Slate and Sheppard (1982) were fast on his heels, using an adaptive PID with gain scheduling [69]. Arnsparger (1983) and Stern (1985) further explored adaptive control of MAP [1], [76]. Stern compared performance of the self-tuning regulator with that of a highly trained anesthesiologist, and found that they were quite similar.

Kaufman, Roy, and Xu (1984) used a model reference adaptive controller to control MAP (MRAC) [41]. MRACs are similar to the controllers used here, except that the objective of the control is to get the output of the system to follow the trajectory of a secondary model's output. Hence, they are not optimal in the sense of trying to minimize a cost function. Furthermore, the advent of receding horizon controllers similar to the one used in this dissertation obviate the need for MRACs.

An interesting outgrowth of MRACs is the Multiple Model Adaptive Controller (MMAC). MMACs pre-specify a bank of system models. The object during control is to choose the model or combination of models that best fit the data. The advantages and disadvantages of this type

of controller have been discussed earlier. He, et al. (1985) applied an MMAC to the regulation of MAP [33], while Yu, et al. (1987) applied it to the regulation of arterial oxygen concentration [89].

Neat, Kaufman, and Roy (1989) combined MRAC, MMAC, fuzzy control and expert systems to control MAP in a time-varying model. This approach was designed to handle uncertainties in the patient responses by using an expert system to match the dynamic structure of the plant with the best control scheme [59].

Voss, et al. (1987) used the MAP system as a test bed for a new type of predictive controller, CAMAC [82]. The same year, using this algorithm, they were the first to demonstrate feasibility of automated concomitant drug therapy by simultaneously controlling MAP and cardiac output in anesthetized dogs [83].

Timmons, et al. (1991) developed an approach to identify a drifting baseline during control. They demonstrated improvements over other identification techniques during the control of MAP in anesthetized dogs [78].

Anesthesia delivery systems were one of the first attempted applications of control in the medical device industry, and are still a popular testbed for control devices (see for example, [67] and [87]). Recently, Vishnoi, et al. (1991) used a first order filter to implement an adaptive controller during closed loop control of closed-circuit anesthesia [79]. Adaptive controllers are also being applied to the control of ventilation gasses [26], muscle relaxation [37], and diabetes [25], [23], [60], and to the control of left ventricular assist devices [55], to name a few. Indeed, an entire issue of the *IEEE Transactions on Biomedical Engineering* has been devoted to just closed loop drug delivery, with most of the controllers employing some kind of adaptive scheme [35]. There is also interest in using adaptive systems for patient controlled analgesia [34]. A summary of other attempts to automate control of biomedical variables may be found in [39].

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CHAPTER 3 :

MATHEMATICAL TECHNIQUES

Before we proceed with the development of the constraining algorithm in Paper 1 (Chapter 4), we need an introduction to ARMAX models, linear constraints, and quadratic programming. In the following sections, we shall define terms, set up a structure for linear constraints, and review some canonical forms of convex quadratic programs. In the process, we will explore equality and inequality constraints and explain why they cannot be treated the same way. Since we will be using the method of Lagrange multipliers to solve our QP (quadratic program), we will include an interpretation of the Lagrange variables. This interpretation in turn will allow us to use a least distance approach to motivate our real-time programming strategy.

We have included extensive sections on positive semi-definite complementary linear programming (a method for solving QP's), where we present a novel enhancement that can significantly reduce the number of iterations required for a solution (hence improving both speed and

accuracy). The enhancement is simple to add to existing code. It merely involves column rejection during selection of a simplex pivot.

In many ways, the sections on CLP (positive semi-definite complementary linear programming) are a separate work with an elegance and beauty all their own. We discovered the improvement while on our journey towards real-time constrained identification for adaptive control. And, like so many things in life, the serendipitous discoveries are often the best. The new algorithm has wide reaching applications, as it applies to any positive semi-definite QP.¹

The reader should be warned that the sections on CLP are steep reading. Fortunately, they are not required for an understanding of the quadratic programming applications in the remainder of the dissertation, and therefore may be skipped by those more interested in the application results. Nevertheless, at some point, the mathematics are needed if the constraining algorithm of Chapter 4 (Paper 1) is to be implemented. Thus, at some point, these sections are required reading.

¹ Indeed, it may even solve negative definite QP's, although we have not proven this aspect yet.

3.1 ARMAX Models

ARMAX models (autoregressive moving average models with exogenous inputs) are a class of linear time series models. They can be used to model any discrete time linear system, and many continuous time linear systems (hence they are often referred to as "black box" models). These models are also useful for non-linear systems that can be reasonably approximated by a linear system at its operating points. Also, if the parameters are allowed to change with time, ARMAX models can capture the behavior of some time-varying systems. Furthermore, these models directly convert to the form of a transfer function by a simple change of variables, so that it is straight forward to analyze the poles and zeros of an arbitrary ARMAX model.

The typical form is

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})w(t) \quad (3.1)$$

where the variables $y(t)$, $u(t)$, and $w(t)$ are the output, the input, and a zero-mean white noise, respectively. A , B , and C are polynomials in the backwards time shift operator, q^{-1} :

$$\begin{aligned}
A(q^{-1}) &\triangleq 1 - a_1 q^{-1} - a_2 q^{-2} - \dots - a_{n_1} q^{-n_1} \\
B(q^{-1}) &\triangleq b_0 + b_1 q^{-1} + b_2 q^{-2} + \dots + b_{n_2} q^{-n_2} \\
C(q^{-1}) &\triangleq 1 + c_1 q^{-1} + c_2 q^{-2} + \dots + c_{n_3} q^{-n_3}
\end{aligned} \quad (3.2)$$

where n_1 , n_2 , and n_3 represent the order of the polynomials. By substituting z^{-1} for q^{-1} , we obtain its transfer function:

$$A(z^{-1})Y(z) = B(z^{-1})U(z) + C(z^{-1})W(z) \quad (3.3)$$

where the variables $Y(z)$, $U(z)$, and $W(z)$ are the Z-transforms of the output, input, and noise.

An input delay in (3.1) may be represented by setting the appropriate number of initial B terms to zero. Alternately, B may be rewritten as a modified B polynomial multiplied by q^{-d} (where d indicates the number of initial zero terms in the original B polynomial). Also, an offset term may be added so that the output has a non-zero steady state value for zero inputs.

Assuming a unit delay, eq. (3.1) may be written in long hand as

$$\begin{aligned}
y(t) = & a_1 y(t-1) + a_2 y(t-2) + \dots + a_{n_1} y(t-n_1) \\
& + b_1 u(t-1) + b_2 u(t-2) + \dots + b_{n_2} u(t-n_2) \\
& + e(t)
\end{aligned} \quad (3.4)$$

where $e(t)$ combines $C(q^{-1})w(t)$ into one term. From this representation, it is readily seen that these models may be expressed in the vector form:

$$y(t) = \phi^T(t-1) \theta + e(t) \quad (3.5)$$

where the vector θ corresponds to the ARMAX parameters (the a_i 's and b_i 's) and is of length $p=n_1+n_2$, and $\phi(t-1)$ is an appropriate data regression vector:

$$\begin{aligned} \theta &\triangleq [a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}]^T \\ \phi(t-1) &\triangleq [y(t-1), \dots, y(t-n_1), u(t-1), \dots, u(t-n_2)]^T \end{aligned} \quad (3.6)$$

It is this form that is often used for identification in adaptive control.

3.2 Linear Constraints

If we have linear equality or inequality constraints on any of the ARMAX model parameters (or on any combination of the parameters) in (3.1), we can also put them into a form similar to (3.5):

$$\begin{aligned} m_i^T \theta &= k_i, & i &= 1, \dots, n_4 \\ \ell_i^T \theta &\leq c_i, & i &= 1, \dots, n_5 \end{aligned} \quad (3.7)$$

Here, m_i and ℓ_i are p length vectors and k_i and c_i are scalars. It is this form that we shall use for our identifier.

For ease of notation, we may rewrite (3.7) as

$$\begin{aligned} M \theta &= K \\ L \theta &\leq C \end{aligned} \quad (3.8)$$

where K and C are n_4 and n_5 length vectors made up of the k_i and c_i respectively, and M and L are $n_4 \times p$ and $n_5 \times p$ matrices defined as

$$\begin{aligned} M &= [m_1 \mid m_2 \mid \dots \mid m_{n_3}]^T \\ L &= [\ell_1 \mid \ell_2 \mid \dots \mid \ell_{n_4}]^T \end{aligned} \quad (3.9)$$

As an example of putting constraints into our standard form, consider the following hypothetical constraint on the parameters of the model

$$y(t) = \alpha y(t-1) + \beta u(t-1):$$

$$-4 \alpha + 3 \beta > 10$$

To be used with our algorithm, it should be re-written

$$4 \alpha - 3 \beta \leq -(10 + \delta)$$

where δ is a small positive value used to convert the "<" to a " $\leq$ ". For θ defined as $[\alpha \quad \beta]^T$, we should set ℓ to $[4 \quad -3]^T$ and c to $-(10+\delta)$.

At this point, it is important to understand the difference between equality and inequality constraints. Unless otherwise specified, we shall assume all constraints are linear in the following discussion.

In parameter space, each parameter can be represented by a different axis. For the case of one parameter, the space is one dimensional. An equality constraint in this space would exactly specify the value of this parameter (say, $\theta=1$), while an inequality constraint would specify a ray (say, $\theta>0$). Assuming no constraints are redundant, another equality constraint (say, $\theta=2$) would conflict with our first equality constraint (hence the constraints are inconsistent). On the other hand, many inequality constraints could be specified and not conflict with our first inequality constraint (say, $\theta>4$, $\theta>5$, and $\theta<10$). Of course, an inequality constraint could conflict (say, $\theta<0$).

For three parameters, the space is three dimensional. Again, assuming no redundancies or degeneracies, one equality constraint limits the

parameter estimates to a plane (Figure 3.1a); two limit them to a line (Figure 3.1b); and three to a point (Figure 3.1c). In contrast, one inequality constraint limits the parameter estimates to an infinite volume bordered on one side by a plane. Two can result in an infinite volume bordered by one plane as for one constraint, and, assuming no redundancies or degeneracies, in an infinite volume bordered by two planes. Three result in all the possibilities for two constraints, and also an infinite volume bordered by three constraints (Figure 3.2a). Finally, four result in all the possibilities for three constraints, and also a finite volume in the shape of a tetrahedron (Figure 3.2b). More constraints can result in all the previous possibilities, or they can further refine the shape of the volume. For example, in the limit, an ellipsoid can be formed with an infinite number of linear inequality constraints.

In general, for any p -length parameter vector θ , we can specify only p or fewer equality constraints. Similarly, if the equality constraints span a subspace of the parameter space, then the number of constraints must not exceed the dimensionality of the subspace. It is important that we do not over-determine the solution and potentially produce a set of inconsistent constraints.

The number of linear inequality constraints, on the other hand, can be infinite. Still, we must be careful that the constraints are consistent and non-empty (e.g., $\theta_i = 4$ and $\theta_i \geq 5$, for which there is no possible solution). Because an infinite number of inequality constraints can be used, we can approximate some non-linear inequality constraints (such as an ellipse) by a set of linear inequality constraints.

Finally, in a convex space, a line segment formed by any two points in the space must also be in the space (Figure 3.3a). Any non-empty space formed by linear constraints must be convex. A concave space, on the other hand, is any space that is not convex (Figure 3.3b). Usually, for quadratic programming, concave spaces are replaced by their convex hulls. A convex hull is the smallest convex space that encloses a concave space (Figure 3.3c). For our application, the hull will be made up of linear constraints.

3.3 Quadratic Programming

Quadratic programs (QP's) minimize (or maximize) a quadratic objective function subject to linear constraints [7]. A general canonical form (Gg) is

$$\min_{\theta} f_1(\theta) \triangleq \frac{1}{2} \theta^T G \theta + g^T \theta \quad (3.10)$$

subject to

$$\begin{aligned} M\theta &= K \\ L\theta &\leq C \end{aligned} \quad (3.11)$$

where G is a symmetric weighting matrix, g is a weighting vector, and the constraints are linear.

Another canonical form is that of weighted projection (WP):

$$\min_{\theta} f_2(\theta) \triangleq \frac{1}{2} (\theta - \theta_0)^T G (\theta - \theta_0) \quad (3.12)$$

subject to the constraints in (3.11), where θ_0 maps to g by relation $g = -G^T \theta_0$. The cost functionals f_1 and f_2 are equal except for a constant, so the solution is the same.

Another canonical form, one that we shall use later to motivate our constraint selection rules, is the least distance formulation (LD):

$$\min_{\theta} f_3(\bar{\theta}) \triangleq \frac{1}{2} (\bar{\theta} - \bar{\theta}_0)^T (\bar{\theta} - \bar{\theta}_0) \quad (3.13)$$

subject to

$$\begin{aligned} \overline{M\theta} &= K \\ \overline{L\theta} &\leq C \end{aligned} \quad (3.14)$$

LD form may be mapped to and from the Gg form and the WP form by the relations $\bar{\theta} = G^{\frac{1}{2}} \theta$, $M = \bar{M} G^{\frac{1}{2}}$, and $L = \bar{L} G^{\frac{1}{2}}$, where $G = (G^{\frac{1}{2}})^T G^{\frac{1}{2}}$ ($G^{\frac{1}{2}}$ is the matrix square root of G). For positive definite G , the mappings are unique, and the cost function f_3 is equal to the cost function f_2 .

The solution of the LD form is obtained by an orthogonal projection in $\bar{\theta}$ space. That is, the LD cost function will be a minimum for that point in the constraint space that is the least distance (in a Euclidean sense) from the point $\bar{\theta}_0$ (hence LD for least distance). Figure 3.4 illustrates solution of an LD problem. In this figure, the constraint space is represented by the set S , the solution by Z^* . Concentric circles centered at Z_0 indicate contours of equal cost for the functional f_3 (hence f_3 is said to be a circular cost function). The point Z^* is the point in S closest to Z_0 . The ray from Z_0 to Z^* (the projection of Z_0 onto the surface of S) is orthogonal to the tangent of S at Z^* (hence the ray is an orthogonal projection). Note that the gradient of the cost function at Z^* is also orthogonal to the tangent.

The solution of a Gg or WP problem is illustrated in Figure 3.5a. The concentric ellipses centered at X_0 are contours of equal cost for the functionals f_1 and f_2 . The point in the constraint set S that minimizes these functions is X^* . This solution is not the same as would be obtained by using orthogonal projection (indicated by the ray). The optimal solution is instead obtained by a weighted projection. Note that the gradient of the cost function is orthogonal to the tangent of S at X^* . A necessary condition for a minimum is that the cost function gradient be orthogonal to the constraint surface at the solution.

To convert this problem to LD form, one could stretch the X space along the minor axis of the cost function ellipse, or shrink it along the major axis (indicated by the arrows). The transformed space (now Z space) is illustrated in Figure 3.5b. The constraint space S is distorted, but now the cost function is circular. The projection from Z_0 to the point in S with the minimum cost (Z^*) is orthogonal. Note that the orthogonal projection in X space is not orthogonal in Z space.

The previous two examples illustrate the elliptical nature of the cost function for positive definite G in

two dimensional space. Figure 3.6 illustrates the case of positive definite G in three dimensional space (the constraints are represented by the box). Again, the functional is elliptical. In general, positive definite G results in an elliptical cost function regardless of dimensionality of the space. Furthermore, notice that, provided the constraint space is non-empty, problems with positive definite G have a unique solution.

For positive semi-definite G , however, the cost function is not generally elliptical, and the solution may not be unique. A three dimensional example of a cost function with positive semi-definite G (specifically with only one eigenvalue equal to zero) is illustrated in Figure 3.7a. Equal cost contours appear as concentric cylinders. Depending on the relative orientation of the cylinders to the constraint space, it is easily seen that the solution may or may not be unique. Figure 3.7b illustrates an example when the problem has a non-unique solution. The box represents the constraint space, and the solid heavy line indicates the loci of solutions. Figure 3.7c illustrates an example when the problem has a unique solution (indicated by the dot).

In summary, if G is positive semi-definite and the space defined by the constraints is non-empty, then the

solution is a global minimum (although it is not necessarily unique). Note that linear programs fall into this category (G equal to zero). If G is positive definite, then the solution is unique. These results are well known (see for example, [7] or [14]). As we will show in the first paper (Chapter 4), for our purposes G is positive definite, so that we will always have a unique solution.

So far we have illustrated inequality constraints. Obviously no projection is needed if the center of the cost function is already in the allowable space. In contrast, equality constraints almost always require projection, since the allowable space is made up of surface, or boundary, points only (cf. Figure 3.1a).

In Figure 3.8a, we illustrate a two dimensional problem with one equality constraint. Note that the shape of the cost function affects the location of the solution. However, in Figure 3.8b, where two equality constraints are specified, the solution is independent of the cost function. The number of the equality constraints (or more precisely, their rank) thus affects the degrees of freedom of the solution. In general, each equality constraint decreases the degrees of freedom by one.

Finally, compare Figure 3.8b with Figure 3.8c, for which the equality constraints have been converted to inequality constraints (and additionally, two more inequality constraints have been added). The solutions are identical. These two figures illustrate a basic approach to quadratic programming. Given a set of inequality constraints, if we can determine which ones are binding at the solution, then the problem can be reduced to an equivalent equality constraint problem. With this approach, the inequality constraint problem becomes a combinatorial problem -- that of determining the binding constraints.

Of the several methods available for solving general quadratic problems of type Gg, WP, and LD, most can be thought of as combinatorial methods. Since it cannot be known a priori which constraints will be binding at the solution, these solvers distinguish themselves based on their constraint selection rules. Since we have positive definite G , we can use a specialized, efficient method. However, before we present our algorithm, we must first introduce the concept of Lagrange multipliers, and then of constraint error.

The negative of the cost function gradient and the negative of the constraint gradients can be regarded as

force fields over the parameter space. For a point to be stationary in such a field, the forces at that point must sum to zero. Hence, a basic theorem of quadratic programming is that at the solution (a stationary point), a unique linear combination of the constraint gradients exists that will exactly match the gradient of the objective function [14]. The Lagrange multipliers are the coefficients of the constraint gradients that result in the unique linear combination. Using the force field analogy, the Lagrange multipliers modulate the strength of the force fields due to the constraints (or, alternately, they tell how strongly the solution is pulled toward a given constraint).

An example is the resting of a ball on a floor in the Earth's gravitational field. The force of the floor on the ball in the upward direction must be equal and opposite to the force of gravity downward. If the plane of the floor were described by a unit normal vector, the Lagrange multiplier would be the weight of the ball.

If the Lagrange multipliers tell how strongly the solution is pulled toward a given constraint, then the constraint errors tell how far the solution is away from a given constraint. For the constraints in (3.7), we can

define constraint error as $(k_i - m_i^T \theta)$ for the equality constraints and $(c_i - l_i^T \theta)$ for the inequality constraints.

For the inequality constraints, a negative error means the solution lies on the forbidden side of the constraint boundary, outside the acceptable space (which in turn means that the solution is not a correct solution). A positive error indicates it lies on the allowable side of the constraint boundary, inside the acceptable space. For equality constraints, any error means the solution does not satisfy the constraint condition (and is therefore not a correct solution). For the equality and inequality constraints, a zero error means the solution lies on the constraint boundary (the only acceptable solution for an equality constraint).

The Lagrange multipliers and the constraint errors have a complementary relationship with each other that can be exploited. At the optimal solution, either the constraint error or the corresponding Lagrange multiplier must be zero. Furthermore, for the inequality constraints, the Lagrange multipliers and the constraint errors must be non-negative at the solution. This condition means that either the solution lies on the acceptable side of the constraint boundary and does not

need to be pulled in the direction of the constraint (thus the constraint error is positive and the Lagrange multiplier is zero), or the solution is pulled in the direction of the constraint boundary just enough for it to lie on the boundary (thus the constraint error is zero and the Lagrange multiplier is positive). For equality constraints, the Lagrange multiplier can be positive or negative, since the constraint error is forced to zero no matter whether the solution approaches from one side or the other.

Combined with the gradient force balance, these conditions (the complementary relationship and the non-negativity conditions) are formally known as the Kuhn-Tucker conditions [14]. These conditions are the first order necessary conditions for a stationary point (a relative minimum or maximum). As discussed earlier, in the case of positive definite G , the stationary point is a unique global minimum [14]. Thus we now have necessary and sufficient conditions that our solution must satisfy for the case of positive definite G .

Before we delve into the mathematics of solving QPs, we introduce one more concept -- that of least distance programming. Returning to our force field analogy, constraint error corresponds to the scaled Euclidean

distance to a constraint. If the constraints are in unit normal form, then the constraint error is the Euclidean distance. Any constraint may be put into unit normal form by scaling the constraint by the magnitude of its normal vector [6]. Hence, given a point θ^1 and the constraint $\ell^T \theta \leq k$, the signed Euclidean distance from θ^1 to the constraint is

$$\frac{\ell^T \theta^1 - k}{\sqrt{\ell^T \ell}} \quad (3.15)$$

Theorem 3.1. Given a circular cost function (LD canonical form) and a set of constraints without redundancies or degeneracies, the violated constraint with the greatest Euclidean distance from the cost function centroid will be binding at the solution.

Proof: Proof is omitted. See Figure 3.9 for an illustration.

Hence, the LD canonical structure allows us to immediately eliminate half of the constraint combinations from consideration. Furthermore, once the problem is orthogonally projected onto the constraint surface, the

problem can be reformulated as an LD problem within the null space of the projection. The null space is the space perpendicular to the projection ray (hence, the null space is the constraint surface). Again, assuming no degeneracies or redundancies, we can immediately eliminate half the remaining constraint combinations from consideration. In this manner, we can proceed until the solution is found.

Unfortunately, the assumption that there are no redundancies or degeneracies is not practical. Even if the original, full dimensional problem contains no redundant or degenerate constraints, subsequent null spaces might. The possibility of a redundancy is illustrated in Figure 3.10. In the same way, a degeneracy could have occurred if the redundant constraint passed through the corner. Nevertheless, least distance programming moves us in the right direction. Later, we will show that this method of constraint selection is equivalent to choosing the constraint that increases the cost function the most.

3.4 CLP Equivalents of QP Canonical Forms

Positive semi-definite complementary linear programming (CLP) is a simplex-like algorithm. It solves linear programs that are subject to complementary and non-negativity conditions. The CLP problem is usually described as follows. Given a real n -vector y and a real positive definite or positive semi-definite (but not necessarily symmetric) $n \times n$ matrix Q , find vectors x and z satisfying the conditions

$$\begin{aligned} x &= Qz + y \\ x &\geq 0, \quad z \geq 0 \\ x^T z &= 0 \end{aligned} \tag{3.16}$$

Gg, WP, and LD canonical quadratic programs can be put into this form. Given form Gg (eqs. (3.10) and (3.11)), augment the objective function to form the Lagrangian:

$$\begin{aligned} \mathcal{L}(\theta, \mu, \lambda) &= \frac{1}{2} \theta^T G \theta + g^T \theta \\ &\quad - \mu^T (K - M\theta) - \lambda^T [(C - L\theta) - v] \end{aligned} \tag{3.17}$$

where μ and λ are the Lagrange multipliers associated with, respectively, the equality and inequality constraints; the quantities $(K - M\theta)$ and $(C - L\theta)$ are the errors for the equality and inequality constraints; and v

is a non-negative slack variable to convert the inequality constraints into equality constraints:

$$L\theta + v = C \quad (3.18)$$

To enforce the complementary conditions discussed earlier, we require that

$$\mu^T(K - M\theta) = 0 \quad (3.19)$$

$$\lambda^T(C - L\theta) = \lambda^T v = \lambda^T[(C - L\theta) - v] = 0 \quad (3.20)$$

If the complementary conditions are met, then the augmented terms in the Lagrangian (3.17) are zero and therefore do not directly contribute to the cost. The non-negativity conditions on the Lagrange multipliers and the slack variables associated with the inequality constraints are

$$\lambda \geq 0 \quad (3.21)$$

$$v \geq 0 \quad (3.22)$$

We now solve for a stationary point of the Lagrangian (the triplet $(\theta^*, \mu^*, \lambda^*)$) by setting the partial of (3.17), with respect to θ , μ , and λ , to zero:

$$\frac{\partial \mathcal{L}}{\partial \theta} = G\theta + g + M^T\mu + L^T\lambda = 0 \quad (3.23)$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = M\theta - K = 0 \quad (3.24)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = L\theta + v - C = 0 \quad (3.25)$$

Equation (3.23) represents the gradient force balance previously discussed. $G\theta + g$ is the gradient of the Gg cost function (3.10), and $M^T\mu$ and $L^T\lambda$ are the gradients for the equality and inequality constraints (3.11), all of which must sum to zero. Equations (3.24) and (3.25) impose the original constraints. Equations (3.19) - (3.25) are the Kuhn-Tucker conditions for Gg [7]. Hence, a stationary point of the Lagrangian also solves Gg .

Proceeding further, using (3.23) we may eliminate θ from (3.24) and (3.25):

$$K + MG^{-1}g + MG^{-1}M^T\mu + MG^{-1}L^T\lambda = 0 \quad (3.26)$$

$$C + LG^{-1}g + LG^{-1}M^T\mu + LG^{-1}L^T\lambda = v \quad (3.27)$$

We may eliminate μ also. However, it is more robust to include μ in the upcoming simplex tableau so that matrix factorizations can be used. Therefore, analogous to v in (3.27) for the inequality constraints, we will create an artificial zero vector, w , as the right hand side of (3.26):

$$K + MG^{-1}g + MG^{-1}M^T\mu + MG^{-1}L^T\lambda = w \quad (3.28)$$

We can now modify the CLP to handle equality constraints. Define

$$z \triangleq \begin{bmatrix} z_e \\ z_i \end{bmatrix} = \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \quad (3.29)$$

and

$$x \triangleq \begin{bmatrix} x_e \\ x_i \end{bmatrix} = \begin{bmatrix} w \\ v \end{bmatrix} \quad (3.30)$$

and initialize y and Q to

$$y = \begin{bmatrix} K + M G^{-1} g \\ C + L G^{-1} g \end{bmatrix} \quad (3.31)$$

$$Q = \begin{bmatrix} M G^{-1} M^T & M G^{-1} L^T \\ L G^{-1} M^T & L G^{-1} L^T \end{bmatrix} \quad (3.32)$$

The CLP becomes

$$\begin{bmatrix} x_e \\ x_i \end{bmatrix} = Q \begin{bmatrix} z_e \\ z_i \end{bmatrix} + y \quad (3.33)$$

$$x_e = 0, \quad x_i \geq 0, \quad z_i \geq 0$$

$$x_e^T z_e + x_i^T z_i = 0$$

(note the modified conditions). This CLP will yield v , w , μ^* , and λ^* . Given μ^* and λ^* , we can calculate θ^* from (3.23).

WP canonical form yields a CLP with z and x defined as in (3.29) and (3.30), with y initialized to the constraint errors at the cost function centroid (the unconstrained solution):

$$y = \begin{bmatrix} K - M\theta_0 \\ C - L\theta_0 \end{bmatrix} \quad (3.34)$$

But this variable is identical to (3.31), since $g = -G^T \theta_0$, hence the two CLPs are identical.

LD canonical form yields a CLP with the z and x defined as in (3.29) and (3.30), but with y and Q initialized to

$$y = \begin{bmatrix} K - \overline{M}\theta_0 \\ C - \overline{L}\theta_0 \end{bmatrix} \quad (3.35)$$

$$Q = \begin{bmatrix} \overline{M}\overline{M}^T & \overline{M}\overline{L}^T \\ \overline{L}\overline{M}^T & \overline{L}\overline{L}^T \end{bmatrix} \quad (3.36)$$

However, since $\overline{\theta} = G^{\frac{1}{2}} \theta$, $M = \overline{M}G^{\frac{1}{2}}$, $L = \overline{L}G^{\frac{1}{2}}$, and $G = (G^{\frac{1}{2}})^T G^{\frac{1}{2}}$, this y and Q are also equal to (3.31) and (3.32). Thus CLP is invariant under these different mappings. This result is useful, as we will be able to interpret the CLP from an LD perspective.

The projection costs (f_1 and f_3) may be written as a bilinear function of the Lagrange multipliers, the slack variables, and the initial constraint errors:

$$\begin{aligned}
 f(\theta) &= \frac{1}{2} z^T (x - y) \\
 &= \frac{1}{2} [\mu^T \quad \lambda^T] \left(\begin{bmatrix} w \\ v \end{bmatrix} - \begin{bmatrix} K - M \theta_0 \\ C - L \theta_0 \end{bmatrix} \right)
 \end{aligned} \tag{3.37}$$

When the complementary conditions are satisfied (such as at the solution), the cost function becomes linear:

$$\begin{aligned}
 f(\theta^*) &= -\frac{1}{2} z^T y \\
 &= -\frac{1}{2} [\mu^T (K - M \theta_0) + \lambda^T (C - L \theta_0)]
 \end{aligned} \tag{3.38}$$

The proof of (3.37) is as follows. From (3.33), we have

$$x = Qz + y \tag{3.39}$$

Substituting into (3.37), we obtain

$$f(\theta) = \frac{1}{2} z^T Qz \tag{3.40}$$

If we define $R \triangleq [M^T \quad L^T]^T$, then $Q = RG^{-1}R^T$ and (3.40) becomes

$$f(\theta) = \frac{1}{2} z^T RG^{-1}R^T z \tag{3.41}$$

From (3.23) we know that

$$R^T z = -G(\theta - \theta_0) \tag{3.42}$$

Substituting this identity into (3.41), we obtain (3.12):

$$\begin{aligned}
 f(\theta) &= \frac{1}{2} [(\theta - \theta_0)^T G^T] G^{-1} [G(\theta - \theta_0)] \\
 &= \frac{1}{2} (\theta - \theta_0)^T G (\theta - \theta_0)
 \end{aligned}
 \tag{3.43}$$

Q.E.D.

As we have shown, the addition of equality constraints to the CLP structure is straightforward. The modifications to the algorithm needed to solve the mixed constraint problem is equally simple. However, the additional book-keeping required for a mixed constraint problem will distract from more important theoretical concerns. For clarity, we shall therefore assume our quadratic program is subject to inequality constraints only. After we develop our algorithm, we will indicate where modifications should be made to incorporate equality constraints.²

Without equality constraints, the CLP reduces to (3.16):

$$\begin{aligned}
 x &= Qz + y \\
 x &\geq 0, \quad z \geq 0 \\
 x^T z &= 0
 \end{aligned}
 \tag{3.44}$$

with

² This approach may be justified from another point of view also. Although numerically inefficient, equality constraints can be represented as pairs of inequality constraints (e.g., $\theta=5$ is equivalent to $\theta \leq 5$ and $\theta \geq 5$).

$$\begin{aligned}
 z &\triangleq \lambda \\
 x &\triangleq v \\
 y &= C - L\theta_0 \\
 Q &= LG^{-1}L^T
 \end{aligned}
 \tag{3.45}$$

The cost function also reduces to

$$f = \frac{1}{2} z^T (x - y) = \frac{1}{2} \lambda^T [v - (C - L\theta_0)] \tag{3.46}$$

which, when the complementary conditions are satisfied, reduces to the linear cost function

$$f = -\frac{1}{2} z^T y = -\frac{1}{2} \lambda^T (C - L\theta_0) \tag{3.47}$$

3.5 The CLP Tableau: Definitions and Structure

The goal of CLP is to permute the equations $x = Qz + y$ using simplex pivots until all the complementary and non-negativity conditions are met.³ Basic and non-basic variables are defined as for the simplex method. Thus, a starting tableau, with x basic and z nonbasic, is $Ix - Qz = y$. When the basic and non-basic variables are complementary, we can augment the tableau to include the relative cost coefficients from

³ We shall assume some familiarity with the simplex method since it is extensively treated in most introductory textbooks on optimization (see for example [14], [7], or [16]).

(3.47) (their signs are flipped since we have placed all variables on the left side of the equations):

x_1	...	x_n	z_1	...	z_n	
1			$-Q_{11}$	...	$-Q_{1n}$	y_1
			$\vdots$	$\vdots$	$\vdots$	$\vdots$
		1	$-Q_{n1}$	...	$-Q_{nn}$	y_n
0	...	0	y_1	...	y_n	0

After a series of permutations, the elements of x and z will no longer be exclusively basic or non-basic. Therefore, let $\bar{x}$ and $\bar{z}$ respectively define the basic and non-basic variables. Obviously, for the starting tableau, $\bar{x}=x$ ($=y$) and $\bar{z}=z$ ($=0$). In general, by rearranging the columns, the permuted tableau can be written as

$\bar{x}_1$	...	$\bar{x}_n$	$\bar{z}_1$	...	$\bar{z}_n$	
1			$-\bar{Q}_{11}$	...	$-\bar{Q}_{1n}$	$\bar{y}_1$
			$\vdots$	$\vdots$	$\vdots$	$\vdots$
		1	$-\bar{Q}_{n1}$	...	$-\bar{Q}_{nn}$	$\bar{y}_n$
0	...	0	$\pm\bar{y}_1$	...	$\pm\bar{y}_n$	F

where $\bar{Q}$ defines the permuted basis, and $\bar{y}$ is the current value of $\bar{x}$. For positive definite Q , $\bar{Q}$ will also be positive definite [3]. Likewise, for positive semi-

definite Q , $\bar{Q}$ will also be positive semi-definite [10]. For our quadratic problem, Q is symmetric and at least positive semi-definite (cf. (3.45)). The sign of the relative cost coefficients depends on the column variable, as will be explained later.

If the complementary conditions are satisfied ($z_i x_i = 0$ for all i), then the tableau is complementary, and the vectors $\bar{x}$ and $\bar{z}$ constitute a complementary solution. For a complementary solution, the value of the cost function is $\frac{1}{2}F$ (where F is the lower right entry in the tableau). If one of the $x_i z_i \neq 0$, then the tableau is almost complementary, and the vectors $\bar{x}$ and $\bar{z}$ constitute an almost complementary solution. For an almost complementary solution, the value of the cost function is $\frac{1}{2}(F + \bar{x}^T \bar{z})$. From the Kuhn-Tucker conditions, we know that a complementary solution is optimal if its basic variables are non-negative.

For a complementary tableau, the relative cost coefficients will have signs depending on the column variables. We can infer the dependance by studying a series of complementary pivots to the starting tableau, where a complementary pivot exchanges a z_i for an x_i or visa versa. For convenience, we shall perform the

following operations on the tableau in condensed form.
 The condensed form is $x = Qz + y$ (the columns of x are
 tacit). The starting tableau is:

	z_1	...	z_n	
x_1	Q_{11}	...	Q_{1n}	y_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	Q_{n1}	...	Q_{nn}	y_n
F	$-y_1$	...	$-y_n$	0

Pivoting on Q_{11} :

	z_1	z_2	...	z_n	
z_1	$\frac{1}{Q_{11}}$	$-\frac{Q_{12}}{Q_{11}}$	...	$\frac{Q_{1n}}{Q_{11}}$	$\frac{y_1}{Q_{11}}$
x_2	$\frac{Q_{12}}{Q_{11}}$	$Q_{22} - \frac{Q_{12}^2}{Q_{11}}$	...	$Q_{2n} - Q_{12} \frac{Q_{1n}}{Q_{11}}$	$y_2 - Q_{12} \frac{y_1}{Q_{11}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$\frac{Q_{1n}}{Q_{11}}$	$Q_{2n} - Q_{1n} \frac{Q_{12}}{Q_{11}}$	...	$Q_{nn} - \frac{Q_{1n}^2}{Q_{11}}$	$y_n - Q_{1n} \frac{y_1}{Q_{11}}$
F	$-\frac{y_1}{Q_{11}}$	$-y_2 + y_1 \frac{Q_{12}}{Q_{11}}$	...	$-y_n + y_1 \frac{Q_{1n}}{Q_{11}}$	$\frac{y_1^2}{Q_{11}}$

Re-notating,

	x_1	x_2	...	x_n	
x_1	$\bar{Q}_{11}$	$\bar{Q}_{12}$	...	$\bar{Q}_{1n}$	$\bar{y}_1$
x_2	$-\bar{Q}_{12}$	$\bar{Q}_{22}$	...	$\bar{Q}_{2n}$	$\bar{y}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$-\bar{Q}_{1n}$	$\bar{Q}_{2n}$	...	$\bar{Q}_{nn}$	$\bar{y}_n$
F	$\bar{y}_1$	$-\bar{y}_2$	...	$-\bar{y}_n$	$\bar{F}$

Notice the skew symmetry indicated by the dashed lines.

Pivoting on $\bar{Q}_{22}$, the tableau becomes:

	x_1	x_2	z_3	...	z_n	
z_1	$\bar{Q}_{11} + \frac{\bar{Q}_{12}^2}{Q_{22}}$	$\frac{\bar{Q}_{12}}{Q_{22}}$	$\bar{Q}_{13} - \frac{\bar{Q}_{12}}{Q_{22}}$	...	$\frac{\bar{Q}_{1n}}{Q_{22}}$	$\frac{\bar{y}_2}{\bar{y}_1 - \bar{Q}_{12}} \frac{1}{Q_{22}}$
z_2	$\frac{\bar{Q}_{12}}{Q_{22}}$	$\frac{1}{Q_{22}}$	$-\frac{\bar{Q}_{23}}{Q_{22}}$	...	$-\frac{\bar{Q}_{2n}}{Q_{22}}$	$-\frac{\bar{y}_2}{Q_{22}}$
z_3	$-\bar{Q}_{13} + \frac{\bar{Q}_{12}}{Q_{22}}$	$\frac{\bar{Q}_{23}}{Q_{22}}$	$\bar{Q}_{33} - \frac{\bar{Q}_{23}^2}{Q_{22}}$	...	$\frac{\bar{Q}_{3n}}{Q_{22}}$	$\frac{\bar{y}_2}{\bar{y}_3 - \bar{Q}_{23}} \frac{1}{Q_{22}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_n	$-\bar{Q}_{1n} + \frac{\bar{Q}_{12}}{Q_{22}}$	$\frac{\bar{Q}_{2n}}{Q_{22}}$	$\bar{Q}_{3n} - \frac{\bar{Q}_{2n}}{Q_{22}}$	...	$\frac{\bar{Q}_{2n}^2}{Q_{22}}$	$\frac{\bar{y}_2}{\bar{y}_n - \bar{Q}_{2n}} \frac{1}{Q_{22}}$
F	$\frac{\bar{Q}_{12}}{\bar{y}_1 - \bar{y}_2} \frac{1}{Q_{22}}$	$-\frac{\bar{y}_2}{Q_{22}}$	$-\frac{\bar{Q}_{23}}{\bar{y}_3 + \bar{y}_2} \frac{1}{Q_{22}}$	...	$-\frac{\bar{Q}_{2n}}{\bar{y}_n + \bar{y}_2} \frac{1}{Q_{22}}$	$\frac{\bar{y}_2^2}{\bar{F} + \bar{y}_2} \frac{1}{Q_{22}}$

Re-notating to $\tilde{Q}$ for clarity:

	x_1	x_2	x_3	...	x_n	
x_1	$\tilde{Q}_{11}$	$\tilde{Q}_{12}$	$\tilde{Q}_{13}$	...	$\tilde{Q}_{1n}$	$\tilde{y}_1$
x_2	$\tilde{Q}_{21}$	$\tilde{Q}_{22}$	$\tilde{Q}_{23}$	...	$\tilde{Q}_{2n}$	$\tilde{y}_2$
x_3	$-\tilde{Q}_{13}$	$-\tilde{Q}_{23}$	$\tilde{Q}_{33}$	...	$\tilde{Q}_{3n}$	$\tilde{y}_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$-\tilde{Q}_{1n}$	$-\tilde{Q}_{2n}$	$\tilde{Q}_{3n}$	...	$\tilde{Q}_{nn}$	$\tilde{y}_n$
F	$\tilde{y}_1$	$\tilde{y}_2$	$-\tilde{y}_3$	...	$-\tilde{y}_n$	$\tilde{F}$

Again, notice the skew symmetry indicated by the dashed lines.

The patterns emerging from this exercise can be summarized in the following two theorems.

Theorem 3.2: For a complementary tableau,

1. If column i of the tableau is basic, then $r_i = 0$.
2. If column i of the tableau is non-basic, then two possibilities arise:
 - a. If $\bar{z}_i$ is an element of z (i.e., a Lagrange multiplier), then $r_i = -\bar{y}_i$.
 - b. If $\bar{z}_i$ is an element of x (i.e., a slack variable), then $r_i = \bar{y}_i$.

Theorem 3.3: For a complementary tableau, the tableau can be partitioned as

$$\bar{Q} = \begin{bmatrix} A & C^T \\ -C & B \end{bmatrix} \quad (3.48)$$

where A is comprised of the rows of the tableau corresponding to the (basic) elements of z and the columns corresponding to the (non-basic) elements of x ; B is comprised of the rows of the tableau corresponding to the (basic) elements of x , and the columns corresponding to the (non-basic) elements of z . Both A and B are symmetric positive semi-definite matrices.⁴

Proof: Both proofs depend on the tableau structure. Since the tableau is invariant under pivot transforms, it does not matter whether the tableau becomes complementary after one pivot, or after many pivots. Furthermore, the pattern will progress as additional complementary pairs are exchanged in the basis. Progression can be forwards (the z_i in, the x_i out) or backwards (the x_i in, the z_i out).⁵ Also, if the progression does not proceed in

⁴ The same structure occurs for G symmetric positive semi-definite (see [3] or [10]).

⁵ Backwards progression can be demonstrated by reversing the order of the tableaux in our example. That
(continued...)

order, the variables can be re-defined so that the progression does proceed in order (hence there is no loss of generality in assuming the progression is in order),
Q.E.D.

These theorems will be useful for our pivot selection rules. In addition, Theorems 3.2 and 3.3 allow us to efficiently store the tableau in computer memory. An $(n+1) \times (n+1)$ condensed tableau will require only $(\frac{1}{2}n+1) \times (n+1)$ floating point storage locations. Keller (1973) and van de Panne and Whinston (1969) also point out this fact [10], [22].

Before we present the pivot selection rules, we need four more definitions. During the tableau permutations, once a variable becomes positive, it is never allowed to become negative. This restriction may cause the formation of an almost complementary tableau. Therefore, define a major cycle as the pivots needed to restore (or maintain) the complementary condition. Each major cycle commences with the selection of a variable to leave the basis. The major cycle ends when the variable is finally

⁵(...continued)
is, from the final tableau, a pivot on $\tilde{Q}_{22}$ will result in the second tableau; a subsequent pivot on $\tilde{Q}_{11}$ will result in the starting tableau.

pivoted out. This variable is defined as the target variable. For each pivot within the major cycle, there will be a variable that enters the basis (the entering variable) and one that exits (the exiting or blocking variable). Keep in mind that, within a major cycle, the entering variable is not necessarily the complement of the target variable, and the exiting variable is not necessarily the target variable.

We are now ready to present the pivot selection rules.

3.6 CLP Pivot Selection Rules

Given a starting tableau, our CLP proceeds as follows:

Step 1. Select a target variable to leave the basis from the set of negative basic variables (only basic variables in x can be negative). To maximize the expected cost for any major cycle, choose that $\bar{x}_i$ out of the available variables for which $\bar{y}_i^2 / \bar{Q}_{ii}$ is a maximum. Define its complement as the entering variable. If no basic variables are negative, the solution is optimal. If all negative variables have been rejected as targets in any one major cycle (cf. Step 2), then there is no non-negative complementary solution.

Step 2. Increase the entering variable until it is blocked by either (a) the target variable increasing to zero, or (b) another basic variable decreasing to

zero. If the tableau is complementary and there is no blocking variable, then reject the target, return to Step 1 and choose another. If the tableau is almost complementary and there is no blocking variable, then there is no non-negative complementary solution. In case of ties, the target variable takes priority [21]. For degeneracies not involving the target variable, Charnes' method can be used (see, for example, [14]).

Step 3. Exchange the entering variable for the blocking variable (a standard simplex pivot). The entering variable is now basic, the blocking variable non-basic.

Step 4. If the tableau is now complementary, return to Step 1 with all negative variables allowed as targets (a major cycle has been completed). Otherwise select the complement of the blocking variable as the next entering variable and return to Step 2.

Italics have been added in Steps 1 and 2 to indicate our modifications. The italics in Step 1 implement least distance programming, while the italics in Step 2 implement target rejection. The modification in Step 2 is a novel technique, and should prove to be a major improvement to CLP. We shall therefore treat it first.

In the original algorithm, negative bounding is used in Step 2 when no blocking variable exists [4]. For each negatively bounded variable, at least two pivots are required -- one to bound the variable, and later one to unbound it. Neither pivot makes progress towards the solution. Our enhancement (target rejection) saves computational time (at least two pivots' worth per

rejection), improves the numerical accuracy of the solution (since each pivot contributes to roundoff errors), and reduces the number of variables that need testing in Step 1. The following theorem validates target rejection.

Theorem 3.4: If a non-negative basic feasible solution exists for a CLP derived for a quadratic program as above with positive definite G , then, given a complementary tableau for which no block occurs for some target variable in Step 2, another allowable target variable exists whose complement will be blocked.

Proof: We will prove by contradiction. Assume a non-negative feasible solution exists. Furthermore, for some positive semi-definite complementary tableau, assume that there is no target variable with a block in Step 2. To analyze one of these variables and its complement (say, $\bar{x}_j$ and $\bar{z}_j$), consider the following complementary tableau (with only the column and row for this pair filled in)

	$\bar{x}_1$	...	$\bar{x}_s$	...	$\bar{x}_{n_1+n_2}$	
$\bar{x}_1$			$\bar{Q}_{1,s}$			$\bar{y}_1$
$\vdots$			$\vdots$			$\vdots$
$\bar{x}_{n_1}$			$\bar{Q}_{n_1,s}$			$\bar{y}_{n_1}$
$\bar{x}_{n_1+1}$			$\bar{Q}_{n_1+1,s}$			$\bar{y}_{n_1+1}$
$\vdots$			$\vdots$			$\vdots$
$\bar{x}_s$	$\bar{Q}_{s,1}$	...	$\bar{Q}_{s,s}$	...	$\bar{Q}_{s,n_1+n_2}$	$\bar{y}_s$
$\vdots$			$\vdots$			$\vdots$
$\bar{x}_{n_1+n_2}$			$\bar{Q}_{n_1+n_2,s}$			$\bar{y}_{n_1+n_2}$
F			$\bar{F}_s$			$\bar{F}$

We have partitioned the tableau such that $\bar{y}_j \geq 0$ for $j \in [1, \dots, n_1]$, and $\bar{y}_j < 0$ for $j \in [n_1+1, \dots, n_1+n_2]$. Only negative basic variables are potential target variables (i.e., the $\bar{x}_j$ for $j \in [n_1+1, \dots, n_1+n_2]$). In order for the complements of these variables to be without blocks, the following two conditions must exist for $j \in [n_1+1, \dots, n_1+n_2]$:

1. $\bar{Q}_{jj} = 0$.
2. $\bar{Q}_{ij} \geq 0$ for $i \in [1, \dots, n_1]$.

Furthermore, since only slack variables (the elements of x) can ever be negative, the tableau can be partitioned as in Theorem 3.3 to obtain the B submatrix. Recall that this matrix is symmetric positive semi-definite. Define $\hat{B}$ as the submatrix of B formed by dropping the rows and columns of B corresponding to positive elements of x and their complements. This matrix must also be symmetric positive semi-definite. Note that $\hat{B}$ is the matrix formed by the rows in the bottom partition of the above tableau and their complementary columns. Hence, $\hat{B}_{jj} = 0$ for all j .

Now, consider any two basic variables from $\hat{B}$, and form the 2×2 submatrix

	$\bar{x}_i$	$\bar{x}_j$
$\bar{x}_i$	0	$\hat{B}_{ij}$
$\bar{x}_j$	$\hat{B}_{ji}$	0

This submatrix must be symmetric and positive semi-definite, since it is a principle minor of $\hat{B}$. To be symmetric, $\hat{B}_{ij} = \hat{B}_{ji}$. However, to be positive semi-definite, $\hat{B}_{ij} = -\hat{B}_{ji}$. The only possibility, then, is for $\hat{B}_{ij} = \hat{B}_{ji} = 0$. Mapping this condition back to the

tableau, all $\bar{Q}_{ij} = 0$ for $i \in [n_1+1, \dots, n_1+n_2]$ and $j \in [n_1+1, \dots, n_1+n_2]$.

Now consider the 2x2 combinations of the variables $\bar{x}_i$ and $\bar{x}_j$ ($i \neq j$), with $\bar{x}_i$ positive and $\bar{x}_j$ negative. Forming the 2x2 submatrix out of $\bar{Q}$ for these variables,

	$\bar{x}_i$	$\bar{x}_j$
$\bar{x}_i$	$\bar{Q}_{ii}$	$\bar{Q}_{ij}$
$\bar{x}_j$	$\bar{Q}_{ji}$	0

This submatrix must be positive semi-definite, since it is a principle minor of $\bar{Q}$. Therefore, $\bar{Q}_{ij} = -\bar{Q}_{ji}$. Additionally, $\bar{Q}_{ij} \geq 0$ since $\bar{y}_i \geq 0$ and $\bar{x}_j$ has no block.

Thus, negative basic variables in our tableau can be written:

$$\begin{aligned} \bar{x}_j &= \bar{Q}_{j1} \bar{z}_1 + \dots + \bar{Q}_{jn_1} \bar{z}_{n_1} \\ &\quad + 0 \bar{z}_{n_1+1} + \dots + 0 \bar{z}_{n_1+n_2} \\ &\quad + \bar{y}_j \leq 0 \end{aligned}$$

($j \in [n_1+1, \dots, n_1+n_2]$). $\bar{Q}_{ji} \leq 0$, hence $\bar{x}_j$ cannot be made positive. This contradicts our original assumptions, Q.E.D.

Incidentally, we may expand this theorem to include the more general case of dual quadratic programs with

positive semi-definite weighting. Consider the primal QP:

$$\begin{aligned} \min_{\theta, \zeta} f_4(\theta, \zeta) &= \frac{1}{2} (\theta^T D \theta + \zeta^T E \zeta) + c^T \theta \\ &\text{subject to} \\ -A\theta + E\zeta &\geq b \\ \theta &\geq 0 \end{aligned} \quad (3.49)$$

and its dual:

$$\begin{aligned} \max_{\theta, \zeta} f_5(\theta, \zeta) &= -\frac{1}{2} (\theta^T D \theta - \zeta^T E \zeta) + b^T \zeta \\ &\text{subject to} \\ -D\theta - A^T \zeta &\leq c \\ \zeta &\geq 0 \end{aligned} \quad (3.50)$$

where D and E are symmetric positive semi-definite. This pair can be converted to CLP form with

$$x = \begin{bmatrix} w \\ v \end{bmatrix}, \quad z = \begin{bmatrix} \theta \\ \zeta \end{bmatrix}, \quad y = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad Q = \begin{bmatrix} D & A^T \\ -A & E \end{bmatrix} \quad (3.51)$$

(see [3]).

Theorem 3.5: If a non-negative basic feasible solution exists for the CLP in (3.51), then, given a complementary tableau for which no block occurs for some target variable, another allowable target variable exists whose complement will be blocked.

Proof: The proof is in the Appendix to this chapter.

Another change from the original algorithm is our use of least distance programming to choose the target variable in Step 1. The original algorithm chose any negative variable [4]. Thus, long solution paths are just as likely as short solution paths. In real-time, long solution paths are not desirable. To make long paths unlikely, LD (least distance) programming chooses the target variable that will produce the largest potential increase in the cost function. The rule is: choose negative basic variable i for which $\bar{y}_i^2 / \bar{Q}_{ii}$ is maximum (Step 1). This rule implements the least distance programming approach discussed earlier, since $\bar{y}_i^2 / \bar{Q}_{ii}$ is the square of the Euclidean distance from the unconstrained optimum within the current null space to the i^{th} constraint within the null space (the null space is given by the matrix $\bar{Q}$).

In some variants of CLP, some Lagrange multipliers may start out negative, and hence will be potential targets (e.g., see [3], [10], and [26]). A negative multiplier means the cost function can be decreased. A complementary pivot on such a variable (say, basic variable i) would decrease the cost function by $\bar{y}_i^2 / \bar{Q}_{ii}$.

For these variant CLPs, the LD approach would be to pick as target that variable (slack or Lagrange multiplier) whose complementary pivot would change the cost function the most. Thus, the Step 1 selection rule would stay the same, except that negative Lagrange multipliers would be included in the test.

The equivalence of the target selection rule (Step 1) with least distance programming can be proven using the following approach.

Since the CLP tableau is invariant under the various canonical quadratic programs, we can, without loss of generality, consider this slightly modified LD canonical form:

$$\min_{\theta} \theta^T \theta \quad \text{subject to} \quad L\theta \leq C. \quad (3.52)$$

For the corresponding CLP starting tableau, $Q=LL^T$ and $y=C-L\theta_0$. Let us define this tableau as T_0 . The square of the Euclidean distance (eq. (3.15)) from the free estimate to any constraint is given by:

$$\left(\frac{e_i^T \theta - k_i}{\sqrt{e_i^T e_i}} \right)^2 = \frac{y_i^2}{Q_{ii}} \quad (3.53)$$

Since all distances are positive, their ordering is maintained if we square them. Hence, from any starting tableau, our rule (Step 1) implements LD programming since it selects the constraint that is furthest away from the unrestricted optimum.

Projection to the i^{th} constraint is identical to a simplex pivot on the i^{th} diagonal of tableau T_0 . Assume we do this, and thus obtain tableau T_1 . We will notate this sequence by $\{T_0:T_1\}$. Now, let us go back, and instead (and equivalently) treat the i^{th} constraint as an equality constraint and eliminate it from the original QP. We thus obtain a new QP with one less variable. Furthermore, we can now set up a new CLP starting tableau (T_0'). Interestingly, if we remove the row and column of T_1 corresponding to the i^{th} constraint, we obtain T_0' . The selection rule will thus generate the same target variable whether we use tableau T_1 or T_0' . Since, for any starting tableau, our selection rule implements LD programming, we can conclude that it implements it for T_1 . We have thus linked two successive tableau sequences ($\{T_1 \rightarrow T_0'\}$). Subsequent tableaus can be similarly linked. E.g.,

$$\begin{array}{ccccccc}
\{ T0 : T1 : T2 : T3 : \dots \} & & & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & & & \\
\{ T0' : T1' : T2' : \dots \} & & & & & & \\
\downarrow & \downarrow & \downarrow & & & & \\
\{ T0'' : T1'' : \dots \} & & & & & & \\
\downarrow & \downarrow & & & & & \\
& \downarrow & \downarrow & & & & \\
& \downarrow & \downarrow & & & &
\end{array} \tag{3.54}$$

The links establish the equivalence between our selection rules and LD programming.

In summary, each complementary pivot (a) provides us with the optimal reduced estimate within the current null space, (b) projects the remaining constraints into this null space, and (c) updates the cost function.

If some distance is infinite (e.g., when a $\bar{Q}_{ij} = 0$), then the constraint cannot be reached from within the current null space. This condition indicates that one of the constraints in the current basis is redundant. Fortunately, with our formulation of the CLP tableau (that is, with G positive definite), an increase in the Lagrange multiplier corresponding to the constraint with infinite distance will result in the redundant constraint's Lagrange multiplier being pivoted out. In other types of CLP problems (say, for G in (3.10) positive semi-definite), it may happen that no block

occurs, in which case we reject the target variable, and select another (cf. Step 2 and Theorems 3.4 and 3.5).

For target selection, the least distance approach requires additional computations compared to the original algorithm. However, as the number of constraints increases, the potential benefits of the least distance approach increase, since unproductive pivots are avoided. And if fewer pivots are used, the solution will be more accurate since there will be less chance for roundoff errors to accumulate.

Consider, for example, an $n \times (n+1)$ condensed tableau of the form $x = Qz + y$ (we can eliminate the relative costs row since it can be obtained from the last column). Assume there are $m \leq n$ negative basic variables (that is, m violated constraints). The number of floating point operations (FLOPs) can be tabulated as follows (ignoring additions and subtractions).

1. The formula for a simplex pivot [16]:

a. Save the pivot column and the pivot element.	0
b. Replace each row (except pivot row) by the linear combination of that row and pivot row that make the pivot column entry zero.	$n^2 + n$

c. Divide pivot row by the negative of the pivot.	n
d. Replace pivot by its reciprocal.	1
e. Replace pivot column by its saved values divided by the saved pivot element.	n-1

Total for pivot	n^2+3n
-----------------	----------

2. The formula to find a block.	n-m+1
---------------------------------	-------

3. The formula to find a target variable:

Original algorithm [4].	0
-------------------------	---

LD programming.	2m
-----------------	----

4. Total FLOPs/pivot:

Original algorithm.	$n^2+4n-m+1$
---------------------	--------------

LD programming.	$n^2+4n+m+1$
-----------------	--------------

5. Extra FLOPs/pivot for LD programming.	2m
--	----

Thus there are a total of $2m$ ($\leq 2n$) additional FLOPs for each iteration when using LD programming. For our applications, there are typically up to 20 constraints. Since the majority of the constraints impose upper and lower bounds on various quantities, the upper bound for m will be approximately $\frac{1}{2}n$. The number of negative variables decreases by at least one per major cycle [3]. For the sake of argument, let us assume that 10 pivots

are required for the original algorithm to solve the problem and that the average number of negative variables is 5. Furthermore, let us assume that only 1 pivot is saved by LD programming. With $n = 20$ and $m_{avg} = 5$, the original algorithm will require 4760 FLOPs, while the LD algorithm will require 4374 FLOPs. This represents a savings of 386 flops. If, per chance, no iterations are saved, then 100 extra FLOPs will be required.

Usually, several iterations will be saved. Considering that every wrong Lagrange multiplier pivoted into the basis requires two pivots for correction (one to pivot out the Lagrange multiplier and one to pivot in its corresponding slack variable), at least two pivots will be saved for every wrong pivot avoided. Thus the potential benefits of LD programming can be large. Nevertheless, for time limited applications, the most attractive aspect of LD programming is that it avoids long solution paths.

3.7 CLP Examples

Several examples of CLP should help to drive home the methodology of the algorithm and illustrate the

potential savings. The first example we will present is from [4] and [26]:

$$\begin{array}{ll}
 \text{Max} & -\theta_1 + \theta_2 \\
 \theta_1, \theta_2 & \\
 \text{subject to} & \\
 \theta_1 + \theta_2 & \leq 1 \\
 \theta_1 - \theta_2 & \leq -1 \\
 \theta_1 & \geq 0 \\
 \theta_2 & \geq 0
 \end{array}$$

This is a linear program and does not fall into the quadratic programming structure we have developed. Nevertheless, it can be put into CLP form with the starting tableau ($x = Qz + y$):

T0	z_1	z_2	z_3	z_4		
x_1	0	0	-1	1	:	-1
x_2	0	0	-1	-1	:	1
x_3	1	1	0	0	:	1
x_4	-1	1	0	0	:	-1

In this example, $z_3 = \theta_1$ and $z_4 = \theta_2$. For our algorithm, select x_1 as the target variable. z_1 has no block, so choose x_4 as the target variable instead. x_2 blocks z_4 , so pivot on $Q_{2,4}$. The tableau becomes

T1	z_1	z_2	z_3	x_2		
x_1	0	0	-2	-1	$\vdots$	0
z_4	0	0	-1	-1	$\vdots$	1
x_3	1	1	0	0	$\vdots$	1
x_4	-1	1	0	0	$\vdots$	-1

The tableau is almost complementary. Therefore increase z_2 , the complement of x_1 . The target variable x_4 blocks:

T2	z_1	x_4	z_3	x_2		
x_1	0	0	-2	-1	$\vdots$	0
z_4	0	0	-1	-1	$\vdots$	1
x_3	2	1	0	0	$\vdots$	2
z_2	1	1	0	0	$\vdots$	1

The tableau is complementary, the solution is non-negative. Hence, the optimal solution is $x = [0 \ 0 \ 2 \ 0]^T$ and $z = [0 \ 1 \ 0 \ 1]^T$ ($\theta^* = [0 \ 1]^T$) in two pivots.

We will now solve using the original algorithm. The starting tableau is

$T0$	z_1	z_2	z_3	z_4		
x_1	0	0	-1	1	$\vdots$	-1
x_2	0	0	-1	-1	$\vdots$	1
x_3	1	1	0	0	$\vdots$	1
x_4	-1	1	0	0	$\vdots$	-1

For the original algorithm, select x_1 as the target variable. z_1 has no block, so impose a negative bound on x_4 (say, -10). Therefore, define the variable $w_4 = x_4 + 10$. The tableau becomes

$\overline{T0}$	z_1	z_2	z_3	z_4		
x_1	0	0	-1	1	$\vdots$	-1
x_2	0	0	-1	-1	$\vdots$	1
x_3	1	1	0	0	$\vdots$	1
w_4	-1	1	0	0	$\vdots$	9

and w_4 blocks. Therefore, pivot on $Q_{4,1}$:

$\overline{T1}$	w_4	z_2	z_3	z_4		
x_1	0	0	-1	1	$\vdots$	-1
x_2	0	0	-1	-1	$\vdots$	1
x_3	-1	2	0	0	$\vdots$	10
z_1	-1	1	0	0	$\vdots$	9

The tableau is almost complementary, so increase z_4 (w_4 's complement). z_4 will be blocked by the target variable, x_1 , restoring the complementary conditions:

$\overline{T2}$	w_4	z_2	z_3	x_1		
z_4	0	0	1	1	:	1
x_2	0	0	-2	-1	:	0
x_3	-1	2	0	0	:	10
z_1	-1	1	0	0	:	9

The only remaining negative variable is x_1 , so substitute back in for w_4 and increase it (it becomes the target variable). z_1 blocks:

$T3$	z_1	z_2	z_3	x_1		
z_4	0	0	1	1	:	1
x_2	0	0	-2	-1	:	0
x_3	1	1	0	0	:	1
x_4	-1	1	0	0	:	-1

The tableau is almost complementary, so increase x_1 (the complement of z_1). x_2 immediately blocks:

T4	z_1	z_2	z_3	x_2		
z_1	0	0	-1	-1	:	1
x_1	0	0	-2	-1	:	0
x_3	1	1	0	0	:	1
x_4	-1	1	0	0	:	-1

The tableau is almost complementary, so increase z_2 (the complement of x_2). The target variable x_4 blocks:

T5	z_1	x_4	z_3	x_2		
z_1	0	0	-1	-1	:	1
x_1	0	0	-2	-1	:	0
x_3	2	1	0	0	:	2
z_2	1	1	0	0	:	1

The solution is optimal, and agrees with our previous solution. However, five pivots were required compared to two for our modified algorithm. Of course, if the tableau were ordered so that the first encountered negative target would be x_4 , only two pivots would have been required. Hence, for this example, target rejection saves on average $1\frac{1}{2}$ pivots. More importantly, it saved us from proceeding down a long path.

For the next example, consider the quadratic program (QP1):

$$\min_{\theta_1, \theta_2} f_1(\theta_1, \theta_2) = (\theta_1 + 6)^2 + (\theta_2 + 1)^2$$

subject to

$$3\theta_1 + \theta_2 \geq 3$$

$$\theta_1 \geq 0$$

$$\theta_2 \geq 0$$

$$\frac{1}{3}\theta_1 + \theta_2 \geq 1$$

QP1 is in LD canonical form with $\theta = [\theta_1, \theta_2]^T$,

$\theta_0 = [-6, -1]^T$, and

$$L = \begin{bmatrix} -3 & -1 \\ -1 & 0 \\ 0 & -1 \\ -\frac{1}{3} & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Figure 3.11 illustrates this problem. Upon inspection, the solution should be $\theta_1 = 0.6$ and $\theta_2 = 1.2$. Forming the CLP, we have the starting tableau

TO	z_1	z_2	z_3	z_4	
x_1	10	3	1	2	$\vdots$ -22
x_2	3	1	0	1/3	$\vdots$ -6
x_3	1	0	1	1	$\vdots$ -1
x_4	2	1/3	1	10/9	$\vdots$ -4

Keep in mind that each z_i corresponds to the Lagrange multiplier for the i^{th} constraint (as ordered above), and that each x_i corresponds to the constraint error for the i^{th} constraint. Each column variable has value 0, while each row variable has the value specified in the last column. Given these variables, we can calculate the corresponding value of θ for any tableau. We will thus be able to chart our progress in θ -space as we proceed with the CLP.

We will solve the CLP using our algorithm first. Each tableau will be mapped to Figure 3.11.

Step 1 chooses x_1 as the best target. z_1 is blocked by x_1 , so pivot on Q_{11} ($=10$):

T1	x_1	x_2	x_3	x_4	
z_1	1/10	-3/10	-1/10	-1/5	11/5
x_2	3/10	1/10	-3/10	-4/15	3/5
x_3	1/10	-3/10	9/10	4/5	6/5
x_4	1/5	-4/15	4/5	32/45	2/5

The tableau is optimal. The solution to QP1 is $\theta^1 = \theta_0 - L^1 z = [.6, 1.2]^1$.⁶ Each tableau corresponds to an intermediate solution of θ .

Comparing with the original method, the choice of the pivot depends on the ordering of the constraints. By the luck of the draw, the original method would have used only one pivot also. However, one cannot expect the constraints to be serendipitously ordered. Instead, there are 19 equally likely solution paths:

(a) Six paths have 7 pivots (for example, add constraint 3 → add constraint 2 → add constraint 4 → drop constraint 3 → add constraint 1 → drop constraint 2 → drop constraint 4).

(b) Nine paths have 5 pivots (for example, add constraint 2 → add constraint 4 → add constraint 1 → drop constraint 2 → drop constraint 4).

(c) Three paths have 3 pivots (for example, add constraint 4 → add constraint 1 → drop constraint 4).

(d) One path has 1 pivot (for example, add constraint 4).

Our method will always choose the shortest path (d), whereas the original algorithm could choose any. Thus,

⁶ Note that the two slack variables associated with the non-negativity of θ , (x_1 and x_2) are equal to θ_1 and θ_2 . In general, a non-negativity constraint on an element in θ allows one to read its value straight from the tableau. This characteristic can often be used advantageously.

the expected path length for the original version is 5.1 pivots. The expected savings using our algorithm is then 4.1 pivots. More importantly for real-time applications, our algorithm avoids long paths.

In the next example, we demonstrate how our algorithm can choose the longest path when a redundant constraint is present. Consider the quadratic program (QP2):

$$\begin{aligned} \min_{\theta_1, \theta_2} f_1(\theta_1, \theta_2) &= (\theta_1 + 2)^2 + (\theta_2 + 10)^2 \\ \text{subject to} \\ \theta_2 &\geq 0 \\ \theta_1 &\geq 0 \\ -3\theta_1 + \theta_2 &\geq 3 \end{aligned}$$

QP2 is also in LD canonical form with $\theta = [\theta_1, \theta_2]^T$, $\theta_0 = [-2, -10]^T$, and

$$L = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 3 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

Figure 3.12 illustrates this problem. Upon inspection, the solution should be $\theta_1 = 0$ and $\theta_2 = 3$. Forming the CLP, we have the starting tableau

T0	z_1	z_2	z_3	
x_1	1	0	1	$\vdots$ -10
x_2	0	1	-3	$\vdots$ -2
x_3	1	-3	10	$\vdots$ -7

Notice that $x_2 = \theta_1$ and $x_1 = \theta_2$ because of the non-negativity constraints. For this tableau, the redundant constraint (the first constraint) just happens to have the largest distance (10). Hence the first pivot choice is Q_{11} ($=-1$):

T1	x_1	z_2	z_3	
z_1	1	0	-1	$\vdots$ 10
x_2	0	1	-3	$\vdots$ -2
x_3	1	-3	9	$\vdots$ 3

x_2 is the only remaining target, hence increase z_2 . x_3 blocks:

T2	x_1	x_3	z_3	
z_1	1	0	-1	$\vdots$ 10
x_2	1/3	-1/3	0	$\vdots$ -1
z_2	1/3	-1/3	3	$\vdots$ 1

The tableau is almost complementary, so increase z_3 (the complement of x_3). z_1 blocks:

T3	x_1	x_3	z_1		
z_3	1	0	-1	$\vdots$	10
x_2	1/3	-1/3	0	$\vdots$	-1
z_2	10/3	-1/3	-3	$\vdots$	31

The tableau is almost complementary, so increase x_1 (the complement of z_1). x_2 blocks:

T4	x_2	x_3	z_1		
z_3	3	1	-1	$\vdots$	13
x_1	3	1	0	$\vdots$	3
z_2	10	3	-3	$\vdots$	41

The tableau is optimal, $\theta^* = [x_2, x_1]^T = [0, 3]^T$. Four pivots were required. As for the previous example, we have traced the solution path onto Figure 3.12.

Comparing with the original method, if the constraints are ordered as above, the same number of pivots would be required. However, one cannot expect this ordering. Instead, there are 5 equally likely solution paths:

(a) Three paths have 4 pivots (for example, add constraint 1 → add constraint 2 → add constraint 3 → drop constraint 1).

(b) Two paths have 2 pivots (for example, add constraint 2 → add constraint 3).

Our method will always choose the longest path for this problem, whereas the original algorithm could choose any. Thus, the expected path length for the original version is 3.2 pivots. The expected additional cost for using our algorithm is then 0.8 pivots.

Hence, LD programming does not guarantee savings. Nevertheless, expected savings typically increase as the problem size becomes bigger, or if redundant constraints are eliminated. More importantly for real-time applications, long paths are generally avoided. However, avoidance of long paths cannot be guaranteed. Therefore, it is important to limit the number of iterations. Fortunately, the longest solution path using LD programming will never be longer than the longest solution path using the original algorithm. Furthermore, each intermediate complementary solution in LD programming is closer to the optimal solution than the previous intermediate complementary solution is (hence it is closer than the unconstrained solution is) so that an

intermediate complementary solution can be used if necessary.

3.8 Equality Constraints

Finally, we address the issue of equality constraints. They may be put into the tableau along with the inequality constraints as indicated earlier. Since there is no restriction on the sign of their Lagrange multipliers, their multipliers can be pivoted in (using complementary pivots) one after the other and ignored during subsequent searches for a target variable. Furthermore, because the equality constraints are not eliminated from the problem, numerically robust techniques (such as LU decomposition) can be applied to the tableau (see for example [8]). If a zero pivot element is encountered while making the Lagrange multipliers basic, then the equality constraints are over-determined. If the x variable corresponding to the zero pivot element has zero value, then the equality constraints are consistent. In this case, the x and z variables corresponding to the zero pivot element can be eliminated from the tableau. On the other hand, if the x variable is not zero, then the equality constraints are not consistent, and there is no solution.

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Appendix: Proof of Theorem 3.5

Theorem 3.5: If a non-negative basic feasible solution exists for the CLP in (3.51), then, given a complementary tableau for which no block occurs for some target

variable, another allowable target variable exists whose complement will be blocked.

Proof: We will prove by contradiction. Assume a non-negative feasible solution exists. Furthermore, for some complementary tableau, assume that there is no target variable with a block. The partitioning theorem (Theorem 3.3) still applies (see, for example, [10]). Therefore, we may partition any of the resulting complementary tableaus as follows:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} \bar{D}_1 & \bar{D}_2^T & \bar{A}_1^T & \bar{A}_3^T \\ \bar{D}_2 & \bar{D}_3 & \bar{A}_2^T & \bar{A}_4^T \\ -\bar{A}_1 & -\bar{A}_2 & \bar{E}_1 & \bar{E}_2^T \\ -\bar{A}_3 & -\bar{A}_4 & \bar{E}_2 & \bar{E}_3 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{bmatrix} + \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \end{bmatrix} \quad (3.55)$$

with $\bar{x}_1$ and $\bar{x}_3$ non-negative, and $\bar{x}_2$ and $\bar{x}_4$ negative [10].

Rearranging,

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_3 \\ \bar{x}_2 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} \bar{D}_1 & \bar{A}_1^T & \bar{D}_2^T & \bar{A}_3^T \\ -\bar{A}_1 & \bar{E}_1 & -\bar{A}_2 & \bar{E}_2^T \\ \bar{D}_2 & \bar{A}_2^T & \bar{D}_3 & \bar{A}_4^T \\ -\bar{A}_3 & \bar{E}_2 & -\bar{A}_4 & \bar{E}_3 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_3 \\ \bar{z}_2 \\ \bar{z}_4 \end{bmatrix} + \begin{bmatrix} \bar{y}_1 \\ \bar{y}_3 \\ \bar{y}_2 \\ \bar{y}_4 \end{bmatrix} \quad (3.56)$$

Now define

$$\bar{Q} = \left[\begin{array}{cc|cc} \bar{D}_1 & \bar{A}_1^T & | & \bar{D}_2^T & \bar{A}_3^T \\ -\bar{A}_1 & \bar{E}_1 & | & -\bar{A}_2 & \bar{E}_2^T \\ - & - & - & - & - \\ \bar{D}_2 & \bar{A}_2^T & | & \bar{D}_3 & \bar{A}_4^T \\ -\bar{A}_3 & \bar{E}_2 & | & -\bar{A}_4 & \bar{E}_3 \end{array} \right] = \left[\begin{array}{cc} \bar{Q}_1 & \bar{Q}_2^T \\ -\bar{Q}_2 & \bar{Q}_3 \end{array} \right] \quad (3.57)$$

Using the same reasoning as for Theorem 3.4, for no blocks, $\bar{D}_3 = \bar{E}_3 = 0$ (since they are symmetric). Also, $\bar{Q}_2$ must have all non-negative elements. But non-negative $\bar{Q}_2$ means that $\bar{D}_2 = -\bar{D}_2$, hence $\bar{D}_2 = 0$. Similarly, $\bar{E}_2 = 0$. Therefore, $\bar{z}_1$ and $\bar{z}_3$ can only decrease $\bar{x}_2$ and $\bar{x}_4$ (who are already negative), and $\bar{z}_2$ and $\bar{z}_4$ can only increase (or leave unchanged) $\bar{x}_1$ and $\bar{x}_3$ (who are already positive). Hence, we can drop the first two partition rows and columns of (3.56) from consideration. That is, any non-negative solution to the CLP must result from pivots of $\bar{z}_2$ and $\bar{z}_4$ with $\bar{x}_2$ and $\bar{x}_4$. Hence, we need only test for a non-negative complementary solution to

$$\begin{bmatrix} \bar{x}_2 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & \bar{A}_4^T \\ -\bar{A}_4 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_2 \\ \bar{z}_4 \end{bmatrix} + \begin{bmatrix} \bar{y}_2 \\ \bar{y}_4 \end{bmatrix} \quad (3.58)$$

However, this CLP is equivalent to a dual pair of linear programs [3]. We can write the primal linear program as:

$$\begin{aligned}
\min_{\bar{\theta}} f_6(\bar{\theta}) &= \bar{y}_2^T \bar{\theta} \\
&\text{subject to} \\
\bar{A}_1 \bar{\theta} &\geq -\bar{y}_1 \\
\bar{\theta} &\geq 0
\end{aligned} \tag{3.59}$$

with the dual linear program:

$$\begin{aligned}
\max_{\bar{\zeta}} f_7(\bar{\zeta}) &= -\bar{\zeta}^T \bar{y}_1 \\
&\text{subject to} \\
\bar{\zeta}^T \bar{A}_1 &\leq \bar{y}_2 \\
\bar{\zeta} &\geq 0
\end{aligned} \tag{3.60}$$

For the systems to be consistent, the duality theorem of linear programming states that $\min f_6 = \max f_7$ (see, for example, [14]). The optimal solution must then satisfy

$$-\bar{\zeta}^T \bar{y}_1 = \bar{y}_2^T \bar{\theta} \tag{3.61}$$

But $\bar{y}_2$ and $\bar{y}_1$ are negative, hence no feasible non-negative solution exists. This completes the proof.

Figure 3.1: (a) One equality constraint in three dimensions is a plane. (b) Two non-degenerate, equality constraints in three dimensions intersect in a line. (c) Three non-degenerate, equality constraints in three dimensions intersect in a point.

(a)

(b)

Figure 3.2: (a) Three non-degenerate, non-redundant inequality constraints in three dimensions produce a volume bordered by a cone or a triangular prism. - (b) Four non-degenerate, non-redundant inequality constraints in three dimensions produce a tetrahedron.

(a)

(b)

(c)

Figure 3.3: (a) Examples of convex spaces. (b) Examples of concave spaces. (c) Example of the convex hull of a concave space.

Figure 3.4: Example of a quadratic program in least distance (LD) canonical form. S is the constraint space. The cost function is indicated by contours with minimum cost at z_0 , the unconstrained estimate. z^* is the optimal solution. Note that the projection ray is orthogonal to the constraint surface.

Figure 3.5: Example of a quadratic program (a) in Gg or WP canonical form; (b) translated to LD canonical form.

Figure 3.6: Example of a three dimensional quadratic program with positive definite G . There is a unique solution.

Figure 3.7: Examples of three dimensional quadratic programs with positive semi-definite G . (a) Contours of a possible cost function. (b) An example of a non-unique solution with the cost function in (a). (c) An example of a unique solution with the cost function in (a).

Figure 3.8: Examples of two dimensional quadratic programs with positive definite G . (a) An example with one equality constraint. (b) An example with two equality constraints. (c) An example with four inequality constraints. Note that the solution in (c) is equivalent to the solution in (b).

Figure 3.9: Least distance (LD) programming. For quadratic programs in LD canonical form with no constraint redundancies, the violated constraint that is furthest from the unconstrained solution will be binding at the constrained solution.

Figure 3.10: A three dimensional example for which a system of constraints without redundancies can result in redundancies in some null space. (a) The three dimensional constraint space, frontal perspective. (b) The three dimensional constraint space, side view. (c) The null space corresponding to the plane of the bottom surface. Notice the redundancy.

Figure 3.11: QP1. θ_0 is the unconstrained solution, θ^* is the constrained solution. The dashed line indicates the solution path for our enhanced CLP. Tableau T0 maps to θ_0 ; tableau T1 maps to θ^* .

Figure 3.12: QP2. θ_0 is the unconstrained solution, θ^* is the constrained solution. The hollow circles are intermediate solutions. For our enhanced CLP, the dashed line indicates the solution path from θ_0 to the first intermediate solution. The second and third intermediate solutions are identical. The fourth solution is optimal. Tableau T0 maps to θ_0 ; tableau T1 maps to the first hollow circle on the solution path; tableaux T2 and T3 map to the second hollow circle; tableau T4 maps to θ^* .

CHAPTER 4:

(PAPER 1)

PARAMETER CONSTRAINED ADAPTIVE CONTROL¹

Abstract

An identifier for adaptive control is presented that optimally imposes mixed linear equality and inequality constraints on a time series model. Linear constraints for ARMAX model parameters are derived from commonly available information, and application guidelines are demonstrated. It is shown that optimally constrained identification for adaptive control can dramatically improve transient controller performance, especially when combined with techniques that selectively adjust adaptation gain.

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4.1 Introduction

In this paper we show that, for certain adaptive controllers, the use of a priori information in the form of linear constraints on the model parameters results in significant improvements in transient performance and eliminates temporary instabilities. Our motivation for this work stems from the application of adaptive control to closed loop drug delivery (such as anesthesia delivery, blood pressure regulation, and blood sugar control for diabetics [67],[68],[65],[20],[60]), where temporary instability and poor transient responses are unacceptable.

The information required for application of standard control design methods is often unavailable. In such situations, adaptive control based on a time series model (ARMAX, CARIMA, etc.) may be a useful alternative since it requires very little information about the system. Ironically, the system knowledge that is available is often discarded. As a result, adaptive controllers need open loop probing before initiating control; after initiation, they are susceptible to gain and offset disturbances, temporary bursting, and other problems due to lack of persistent excitation.

Instead of discarding the available information, we propose to use it to enhance system identification. Specifically, if the knowledge can be expressed or approximated by linear constraints on the parameters of an ARMAX model, then the identified parameters can be optimally constrained in real-time to obey the known conditions [62],[61].

Two previous approaches to constrained identification for adaptive control exist. In one, stability proofs are the motivation. For example, it has been shown that if the parameters of the ARMAX model stay within a hypersphere of given radius and center (so that the ℓ^2 norm of the parameter error is bounded), then under certain conditions global asymptotic stability is ensured [54] [49] [53] [69] [31] [56] [35]. Other investigations in this genre prove the same result for parameters staying within a hypercube (so that the ℓ^1 norm is bounded) [51] [52] [44] [24] [33] [34]. One of the earliest methods ensures stability using bounds only on the controller's gain matrix [2], [16] [17] [18] [43]. While these constraints may be useful, there is typically much more information that can be used.

The second approach to constrained identification solves the much simpler problem of imposing exact

knowledge. Given exact information regarding one or more of the ARMAX model parameters, the model can be reduced either by factoring out the known parts (if not directly, then by filtering), or by constraining it with equality constraints [10] [7] [3] [4] [28] [23] [12] [13] [41] [71] [70]. Again, while these techniques may be useful, there is typically much more information that can be used.

Steady state gain, settling time, approximate pole and zero locations are often known. At the least, a range or a sign is known. Classifications such as "open loop stable" or "well damped" also convey information. To utilize this information, an algorithm must be capable of imposing both equality and inequality constraints.

In the following section we derive such an algorithm. To make the solution tenable in real-time, we consider linear constraints only. In the subsequent section we show how commonly known information may be represented as linear constraints. In the last section we demonstrate some important guidelines that should be followed when using constrained adaptive control.

4.2 Problem Formulation

Consider the ARMAX model (autoregressive moving average model with exogenous inputs):

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})w(t) \quad (4.1)$$

The variables $y(t)$, $u(t)$, and $w(t)$ are the output, the input, and a zero-mean white noise, respectively. A , B , and C are polynomials in the backwards time shift operator, q^{-1} :

$$\begin{aligned} A(q^{-1}) &\triangleq 1 - a_1q^{-1} - a_2q^{-2} - \dots - a_{n_1}q^{-n_1} \\ B(q^{-1}) &\triangleq b_0 + b_1q^{-1} + b_2q^{-2} + \dots + b_{n_2}q^{-n_2} \\ C(q^{-1}) &\triangleq 1 + c_1q^{-1} + c_2q^{-2} + \dots + c_{n_3}q^{-n_3} \end{aligned} \quad (4.2)$$

where n_1 , n_2 , and n_3 represent the order of the polynomials.

Letting $e(t)$ combine $C(q^{-1})w(t)$ into one term, (4.1) may be written as

$$y(t) = \phi^T(t-1) \theta + e(t) \quad (4.3)$$

where the vectors θ and $\phi(t-1)$ are defined by

$$\begin{aligned} \theta &\triangleq [a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}]^T \\ \phi(t-1) &\triangleq [y(t-1), \dots, y(t-n_1), u(t-1), \dots, u(t-n_2)]^T \end{aligned} \quad (4.4)$$

For θ unknown, the usual recursive least squares (RLS) estimate is

$$\begin{aligned} \theta_f(t) = & \theta_f(t-1) + \\ & \frac{P(t-2)\phi(t-1) \cdot [y(t) - \phi(t-1)^T \theta_f(t-1)]}{\alpha(t-1) + \phi(t-1)^T P(t-2) \phi(t-1)} \\ & (4.5) \\ P(t-1) = & \frac{1}{\alpha(t-1)} \cdot \left[P(t-2) - \right. \\ & \left. \frac{P(t-2)\phi(t-1)\phi(t-1)^T P(t-2)}{\alpha(t-1) + \phi(t-1)^T P(t-2) \phi(t-1)} \right] \end{aligned}$$

where $\alpha(t)$ is a scalar forgetting factor, $P(t-1)$ is a matrix proportional to the inverted data covariance matrix, and the subscript f signifies that θ is the free, or unconstrained solution. The RLS estimate minimizes the sum of the squared prediction errors:

$$J = \frac{1}{2} [Y(t) - X(t-1)\hat{\theta}]^T W(t) [Y(t) - X(t-1)\hat{\theta}] \quad (4.6)$$

where $\hat{\theta}$ is the parameters to be estimated, $W(t)$ is a diagonal matrix defined by the $\alpha(t)$, and

$$\begin{aligned} Y(t) \triangleq & [y(1), y(2), \dots, y(t)]^T \\ X(t-1) \triangleq & [\phi(0) | \phi(1) | \dots | \phi(t-1)]^T \end{aligned} \quad (4.7)$$

Note that the equivalent non-recursive solution is

$$\theta_f(t) = [X^T(t-1)W(t)X(t-1)]^{-1} X^T(t-1)W(t)Y(t) \quad (4.8)$$

and that $P(t-1) \triangleq [X(t-1)^T W(t) X(t-1)]^{-1}$. These variables arise in an intermediate step of the constraining algorithm, where we can substitute in their recursive forms.

Now consider the minimization of (4.6) subject to the linear equality and inequality constraints

$$\begin{aligned} M \theta &= K \\ L \theta &\leq C \end{aligned} \quad (4.9)$$

where, for the p -length vector θ , n_1 ($\leq p$) equality constraints, and n_2 inequality constraints, M and L are $n_1 \times p$ and $n_2 \times p$ matrices and K and C are n_1 and n_2 length vectors. Standard Lagrangian techniques can be used to solve this convex quadratic program [21]. Define the vectors of Lagrange multipliers μ and λ associated with the equality and inequality constraints respectively. The Lagrangian, a no-cost extension to (4.6) at the solution, may be constructed using the constraints (4.9) and the multipliers μ and λ :

$$\begin{aligned} \mathcal{Q}(\hat{\theta}, \mu, \lambda) = & \\ & \frac{1}{2} [Y(t) - X(t-1)\hat{\theta}]^T W(t) [Y(t) - X(t-1)\hat{\theta}] \\ & + \mu^T (M\hat{\theta} - K) + \lambda^T (L\hat{\theta} - C) \end{aligned} \quad (4.10)$$

We can now solve for (4.6) subject to the constraints in (4.9) by minimizing (4.10) subject to

$$M \hat{\theta} - K = 0 \quad (4.11)$$

$$L \hat{\theta} - C \leq 0 \quad (4.12)$$

$$\lambda \geq 0 \quad (4.13)$$

$$\lambda^T (L \hat{\theta} - C) = 0 \quad (4.14)$$

Taking the partial of (4.10) with respect to $\hat{\theta}$, and setting to zero, we get

$$-X^T W(Y - X\hat{\theta}) + M^T \mu + L^T \lambda = 0 \quad (4.15)$$

(dropping the time subscripts for convenience).

Equations (4.11)-(4.15) are the first order Kuhn-Tucker conditions, the necessary and sufficient conditions for a unique global minimum given positive definite $X^T W X$, or equivalently, positive definite P [46].

Using eq. (4.15) we may solve for $\hat{\theta}$ (which we denote as θ_c since it is the constrained solution):

$$\theta_c = \theta_f - P M^T \mu - P L^T \lambda \quad (4.16)$$

where P and θ_f have been defined above. Eq. (4.16) is in the form of the free estimates plus a correction due to the constraints. Furthermore, the correction terms are the projection of θ_f onto the constraint surface (4.9) weighted by the P -matrix. That is, the same solution would have been obtained if we had minimized the cost function

$$J_{proj} = [\hat{\theta} - \theta_f]^T P^{-1} [\hat{\theta} - \theta_f] \quad (4.17)$$

subject to the constraints in (4.9).

Equation (4.16) is the key to real time identification: θ_f and P can be determined by RLS. All that remains is to determine the Lagrange multipliers μ and λ .

At the solution, a complementary relationship exists between the Lagrange multipliers and the constraint errors $(M\hat{\theta} - K)$ and $(L\hat{\theta} - C)$. For each constraint, either its error or its corresponding Lagrange multiplier must be zero [21]. This complementary nature leads quite naturally to the solution of (4.16) using specialized quadratic programming techniques. We suggest positive semi-definite complementary linear programming (CLP) (a quadratic programming technique that is similar to the simplex algorithm for linear programming) [11] (see also [64] and [26]). We describe an enhanced version of CLP in Chapter 3.

If there are no inequality constraints, the solution reduces to the closed form equality constrained solution with

$$\mu = (MPM^T)^{-1} (M\theta_f - R) \quad (4.18)$$

$$(\lambda = 0).$$

4.3 ARMAX Constraints

In this section we develop linear constraints for ARMAX models based on commonly available design knowledge. We begin with information that is generally available for any system.

Individual Parameters: Information regarding individual parameter values in the ARMAX model is often known.

Individual bounds may be imposed by inequality constraints:

$$\theta_{i_{\min}} \leq \theta_i \leq \theta_{i_{\max}} \quad i \in [1, \dots, p] \quad (4.19)$$

where θ_i is the i^{th} element in θ , and p is the number of parameters in θ . Similarly, precise knowledge may be imposed by equality constraints.

Steady State Gain For Open Loop Stable Systems: An incorrect estimate of the steady state gain (G_{ss}) can be harmful during control. In the worst case, a wrong sign

may result in instability. Fortunately, information regarding the correct sign and range of G_{ss} is usually available.

G_{ss} is defined as the expected steady state output (change from baseline) resulting from a unit steady state input, and may be calculated from the ARMAX representation (4.1) as follows:

$$G_{ss} = \frac{\sum_i b_i}{1 - \sum_i a_i} \quad (4.20)$$

For $G_{ss_{\min}} \leq G_{ss} \leq G_{ss_{\max}}$, we obviously obtain two inequality constraints. However, to put the constraints in linear form, we must know the sign of the denominator of any possible $\hat{G}_{ss}$ (where $\hat{}$ indicates an estimated quantity).

If the model is constrained to be open loop stable (see below), then the denominator will be positive. In this case, the following constraints impose minimum and maximum bounds on $\hat{G}_{ss}$:

$$\begin{aligned} \sum \hat{b}_i + G_{ss_{\min}} \sum \hat{a}_i &\geq G_{ss_{\min}} \\ \sum \hat{b}_i + G_{ss_{\max}} \sum \hat{a}_i &\leq G_{ss_{\max}} \end{aligned} \quad (4.21)$$

If steady state gain is precisely known, the resulting equality constraint is independent of the sign of the denominator in (4.20).

Steady State Points: An SSP (steady state point) is a known steady state input/output pair associated with the model. Usually the background level (the steady state unforced output) is known. Sometimes another point, such as a steady state equilibrium point for a chemical reaction, is also known. For many systems, these points drift, or appear to drift due to linearization of a non-linear system. Nevertheless, a range can often be specified. In the following, we assume linearity.

If a time-invariant SSP is known, then it may be forced into the model by translating the output and input so that the model represents deviations from the point. Given the SSP (U_1, Y_1) , define the translation

$$\begin{aligned}\bar{y}(t) &\triangleq y(t) - Y_1 \\ \bar{u}(t) &\triangleq u(t) - U_1\end{aligned}\tag{4.22}$$

The parameters θ can now be estimated using the translated data signals $\bar{y}(t)$ and $\bar{u}(t)$.

Given a second SSP, (U_2, Y_2) , we may also impose a steady state gain:

$$Y_2 - Y_1 = (Y_2 - Y_1) \sum a_i + (U_2 - U_1) \sum b_i. \quad (4.23)$$

A different approach must be used to enforce a range instead of a precise point. Augment the ARMAX model (4.1) with an offset term D :

$$\begin{aligned} y(t) = & a_1 y(t-1) + a_2 y(t-1) + \dots + a_{n_1} y(t-n_1) \\ & + b_1 u(t-1) + b_2 u(t-2) + \dots + b_{n_2} u(t-n_2) \\ & + e(t) + D \end{aligned} \quad (4.24)$$

Now, D may be estimated by including it in the parameter vector θ and adding a corresponding "1" in the regressor vector ϕ . This technique is known as "the one-in-the-data-vector method" [9]. To form linear constraints, we must know the sign of $1 - \sum \hat{a}_i$. If the model is constrained to be open loop stable, then the sign will be positive. In this case, the SSP range, $(U_1, Y_{\min})$ to $(U_1, Y_{\max})$, may be imposed by

$$\begin{aligned} (1 - \sum a_i) Y_{\min} &\geq (\sum b_i) U_1 + D \\ (1 - \sum a_i) Y_{\max} &\leq (\sum b_i) U_1 + D. \end{aligned} \quad (4.25)$$

Another popular approach estimates the parameters using incremental outputs and inputs, $y_{\Delta}(t)$ and $u_{\Delta}(t)$, defined as

$$\begin{aligned} y_{\Delta}(t) &\triangleq y(t) - y(t-d) \\ u_{\Delta}(t) &\triangleq u(t) - u(t-d) \end{aligned} \quad (4.26)$$

where d is the system deadtime. By using this method, offsets are implicitly removed. While a known SSP or its range cannot be imposed on the identification, two SSP's can still be imposed as a steady state gain (eq. (4.23)).

A fourth approach high pass or band pass filters the outputs and inputs so that offsets are again implicitly removed [63]:

$$A(q^{-1})y_{hp}(t) = B(q^{-1})u_{hp}(t) + e_{hp}(t) \quad (4.27)$$

where the subscript hp indicates high pass or band pass filtering. The a 's and b 's can be estimated using this equation. The offset can then be re-introduced based on the estimated parameters and low pass filtered outputs and inputs:

$$\hat{A}(q^{-1})y_{lp}(t) = \hat{B}(q^{-1})u_{lp}(t) + e_{lp}(t) + \hat{D} \quad (4.28)$$

where the subscript lp indicates low pass filtering. In general, the high pass and low pass filters are chosen so that $e_{hp}(t) \approx e(t)$ and $e_{lp}(t) \approx 0$.

As with the one-in-the-data-vector method, a range on an SSP, $(U_1, Y_{\min})$ to $(U_1, Y_{\max})$, may be imposed if the sign of $1 - \sum \hat{a}_i$ is known. As before, if the model is constrained to be open loop stable, then the sign will be positive. In this case, we arrive at equation (4.25). Using (4.28) to eliminate D , we obtain the following two inequality constraints:

$$\begin{aligned} \hat{A}(q^{-1})(y_{lp}(t) - Y_{\min}) &\leq \hat{B}(q^{-1})(u_{lp}(t) - U_1) \\ \hat{A}(q^{-1})(y_{lp}(t) - Y_{\max}) &\geq \hat{B}(q^{-1})(u_{lp}(t) - U_1) \end{aligned} \quad (4.29)$$

Since these constraints depend on recent data, noise may invalidate the constraints. Hence, one must use care when imposing them.

In either this method or the one-in-the-data-vector method, multiple ranges for SSP's may be imposed as long as they do not conflict with each other.

Open Loop Stability: Open loop stability is almost always known ahead of time, and offers extremely useful information for an adaptive controller. It is well known that bounded input bounded output (BIBO) stable systems have all their poles within the unit circle in the Z -plane. The Jury Stability Test can be used to test the poles (e.g., see [38, pp. 277-280]). For up to second

order systems, the necessary and sufficient conditions for BIBO stability are linear:

$$\begin{aligned} a_1 + a_2 &< 1 \\ -a_1 + a_2 &< 1 \\ -a_2 &< 1. \end{aligned} \tag{4.30}$$

The constraint region is illustrated in Figure 4.1.

Open Loop Settling Time: Settling time (τ_s) is a commonly known design parameter. It is the time required for a step response to settle to within a band of its final value (typically, $\pm 1\%$). First and second order systems with an open loop settling time less than or equal to τ_s have discrete time poles within a circle of radius r (< 1) centered at the origin in the Z-plane, where $r \approx \exp(-4.6T/\tau_s)$ and T is the sampling interval [22, pp. 100-103].

Consider two poles with magnitude less than r as in Figure 4.2a. Now let $\bar{p}_1 = p_1/r$; $\bar{p}_2 = p_2/r$. In this new coordinate system, the original circle of radius r corresponds to the unit circle (Figure 4.2b). Applying the Jury Criteria in the transformed space we obtain the constraints in Figure 4.2c:

$$\begin{aligned}
\bar{a}_1 + \bar{a}_2 &\leq 1 \\
-\bar{a}_1 + \bar{a}_2 &\leq 1 \\
-\bar{a}_2 &\leq 1.
\end{aligned}
\tag{4.31}$$

By our definition of the ARMAX polynomial,

$$\begin{aligned}
a_1 &= p_1 + p_2 \\
a_2 &= -p_1 p_2.
\end{aligned}
\tag{4.32}$$

Likewise,

$$\begin{aligned}
\bar{a}_1 &= \bar{p}_1 + \bar{p}_2 = (p_1 + p_2) / r = a_1 / r \\
\bar{a}_2 &= -\bar{p}_1 \bar{p}_2 = -p_1 p_2 / r = a_2 / r.
\end{aligned}
\tag{4.33}$$

Hence, we obtain the following constraints in our original parameter space:

$$\begin{aligned}
ra_1 + a_2 &\leq r^2 \\
-ra_1 + a_2 &\leq r^2 \\
-a_2 &\leq r^2.
\end{aligned}
\tag{4.34}$$

Figure 4.2d illustrates these constraints. Note that they form a closed space within the stability space; hence, stability constraints are not needed if these are imposed.

Open Loop Zeros For Minimum Phase Systems: Minimum phase systems are commonly encountered. The zeroes of these systems must lie within the unit circle in the Z-plane. The B polynomial of a second order ARMAX model with a unit delay is given by

$$\begin{aligned}
 B(q^{-1}) &= b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3} \\
 &= b_1 q^{-1} \left(1 + \frac{b_2}{b_1} q^{-1} + \frac{b_3}{b_1} q^{-2} \right)
 \end{aligned} \tag{4.35}$$

Re-writing the constraints derived for settling time with $a_1 \rightarrow -b_2/b_1$ and $a_2 \rightarrow -b_3/b_1$, we obtain:

$$\begin{aligned}
 -r \frac{b_2}{b_1} - \frac{b_3}{b_1} &\leq r^2 \\
 r \frac{b_2}{b_1} - \frac{b_3}{b_1} &\leq r^2 \\
 \frac{b_3}{b_1} &\leq r^2,
 \end{aligned} \tag{4.36}$$

where r (<1) specifies the maximum radius of the zeros. To put these constraints in linear form, we must know the sign of b_1 . Since the sum of the polynomial coefficients in the brackets in (4.35) is positive (cf. the constraints in (4.36)), b_1 must have the same sign as G_{ss} if the model is constrained to be open loop stable. Hence, assuming stability, all that is needed is the sign of G_{ss} .

Stable Noise Polynomials: Often, it is assumed that $C(q^{-1})$ has its roots on or inside the unit circle (see for example [29]). For second order noise polynomials, we can re-write the constraints derived for settling time with $a_1 \rightarrow -c_1$ and $a_2 \rightarrow -c_2$:

$$\begin{aligned}
 -rc_1 - c_2 &\leq r^2 \\
 rc_1 - c_2 &\leq r^2 \\
 c_2 &\leq r^2,
 \end{aligned}
 \tag{4.37}$$

where r (<1) is the maximum radius of the roots of the noise polynomial.

Constraint Combinations: A judicious selection of constraints on individual parameters may allow more sophisticated use of the previous constraints. For example, minimum and maximum open loop settling time may be specified if the open loop discrete time poles are positive and real. This condition is typically met by continuous time systems with real poles [37]. For second order discrete time systems, positive real poles imply that

$$\begin{aligned}
 a_1 &\geq 0 \\
 a_2 &\leq 0 \\
 a_1^2 + 4a_2 &\geq 0
 \end{aligned}
 \tag{4.38}$$

These constraints restrict the parameters to the fourth quadrant, above a concave parabola (Figure 4.3a). Given these conditions, we can superimpose constraints for a maximum and minimum settling time (Figure 4.3b). By taking the linear convex hull of the feasible space, we

arrive at constraints that specify an approximate minimum and maximum settling time for the system (Figure 4.3c):

$$\begin{aligned} -r_{\max}^2 a_1 + (r_{\min} - 2r_{\max}) a_2 &\leq -r_{\max}^2 r_{\min} \\ r_{\max} a_1 + a_2 &\leq r_{\max}^2 \\ a_2 &\leq 0, \end{aligned} \quad (4.39)$$

where $r_{\min}$ and $r_{\max}$ are the radii associated with the minimum and maximum settling time.

4.4 Application Guidelines

The use of constraints for improved modeling and prediction leads to improved control if certain guidelines are followed. In this section, we demonstrate these guidelines with simulation examples, comparing control with and without constraints.

In the simulations, no constraints are invoked up to a certain time. At this time, the constraints are imposed and the plant's steady state gain is doubled as a challenge to the controller. This way, the constrained and unconstrained controllers enter the challenge in the same condition.

The simulated discrete time plant (an ARMAX model) has a pole at 0.7, a non-minimum phase zero at -2, a delay of 2 steps, and a steady state gain of 10:

$$y(t) = 0.7y(t-1) + u(t-2) + 2u(t-3) + e(t) \quad (4.40)$$

Except for the delay, the system is identical to the benchmark model of [8], [63], and [66]. After the challenge, the steady state gain becomes 20 (all else remains the same):

$$y(t) = 0.7y(t-1) + 2u(t-2) + 4u(t-3) + e(t) \quad (4.41)$$

Each simulation is repeated for three levels of zero mean white gaussian noise, defined as no noise ($e(t) \equiv 0$), small noise ($\sigma_e^2=1$), and large noise ($\sigma_e^2=5$).

In the protocol, a small step input (0.1 units for no noise, 0.3 units for small noise, and 1.0 unit for large noise) is administered for open loop identification. The loop is closed after 5 steps ($t=5$) with a setpoint of 50. Undisturbed control continues for 100 steps. At this point ($t=105$), the challenge begins (unknown to the controller), and control continues for another 100 steps.

For all simulations, control is implemented by a receding horizon predictive controller as in [66], with a fixed prediction horizon of 4 steps. For the control calculation at each step, this algorithm assumes that all future inputs will be equal to the current input.

The constraint sets used in the simulations are summarized in Table 4.I. For each simulation, the mean squared control error following the plant change ($t = 105$ to 205) is recorded in Table 4.II. The simulations have been plotted in Figures 4.4 (no noise), 4.5 (small noise), and 4.6 (large noise). We will refer to these simulations to illustrate the following guidelines.

The constraints should be tight enough to be invoked. If we want a performance improvement using constraints, the bounds must be tight enough that they will be invoked at some point. We have used two sets of constraints, "loose" and "tight", to demonstrate this point. Each set enforces minimum and maximum parameter values. In addition, since the tight set enforces open loop stability (the bounds force the pole to lie within the unit circle), we could also include steady state gain and a zero range in this set (see Table 4.I). The loose constraints are so broad that they are never invoked;

hence, there is no improvement compared to unconstrained control. The tight constraints, on the other hand, are invoked. For the no noise simulations, mean squared control error (MSE) for the tight constraint set (Figure 4.4b) is more than 26 times better than it is for the loose set (Figure 4.4a). As the noise level increases, the unconstrained (loose) controller improves, while the constrained (tight) controller slightly worsens (cf. Figures 4.5a,b and 4.6a,b). Nevertheless, the tight controller always performs significantly better than the loose controller (see Table 4.II).

The constraint set should enforce global asymptotic stability. The tight constraint set enforces global asymptotic stability. In this example, we eliminate some of the constraints to show what can happen without the full set.

For minimum variance control, it is common to bound the sign of b_1 or fix it to some known range of values to enforce global asymptotic stability [2] [43]. For receding horizon control, however, b_1 is no longer the only important parameter. This observation is well known (e.g., see [16] [17] [18] [43]), although easily overlooked. In this example, therefore, we only enforce

the sign of b_1 and a range for the zero (see Table 4.I). As shown by Figure 4.4c (the no noise case), these constraints do not eliminate the instability when the plant changes, although they do decrease the input and output excursions compared to the unconstrained case. However, for the small noise case, the MSE with these constraints is significantly worse than it is for no constraints (see Table 4.II and compare Figure 4.5 a and c).

Untrue constraints degrade performance. The set of tight constraints were modified to impose too small an upper limit on steady state gain. Although temporary instability is eliminated at the plant change, a constant steady state error remains. The MSE in Table 4.II is misleading in this regard. If we ran the simulation long enough, the MSE would instead be larger here than it would be for any of the previous examples. The sum of the squared errors will continue to increase over time (see Figures 4.4d, 4.5d, and 4.6d).

Use optimal projection. Orthogonal projection is guaranteed to reduce parameter bias (if the constraints are true), while optimal identification is not [7].

Thus, one might conjecture that identification and control would be more robust if orthogonal projection were used (in all of the previous simulations, optimal estimates were used). Orthogonal projection is obtained by replacing the P -matrix in (4.17) by the identity matrix.

We tested this hypothesis on our benchmark model using the tight set of constraints. While the control resulting from the orthogonal projector eliminates the instability, a constant steady state error remains. As in the example with the untrue constraint, the sum of the squared errors increases linearly with time (see Figures 4.4e, 4.5e, and 4.6e). Moreover, this steady state error is larger at all noise levels than it was with the untrue constraint (cf. Figures 4.4d, 4.5d, and 4.6d), so it will accumulate faster.

Re-identify after changes. Data from the new plant must compete with data from the old for estimating the parameters. Therefore old data should be discarded when a change has been detected. The data can be discarded by either resetting the P -matrix (blanking all past data) or increasing the speed of adaptation by lowering the estimator's exponential forgetting factor (rapidly

discounting past data). In this next example (Figures 4.4f, 4.5f, and 4.6f), we discount the past data by reducing the forgetting factor to 0.5 for five steps, then increasing it to 0.75 for eight steps, and finally returning it to its original value of 0.98.

Re-identification after the plant change when using the loose set of constraints does not eliminate instability -- in fact, when small noise is present, the input and output excursions are nearly twice as large as they were when re-identification was not used (compare Figure 4.5f with 4.5a). However, unlike the earlier example, the control error goes to zero immediately after the instability.

When using the tight set of constraints, control remains stable and the control error rapidly goes to zero (see Figures 4.4g, 4.5g, and 4.6g). For the no noise case, the MSE is improved more than a factor of 100 compared to the original control example (Figure 4.4a)! Furthermore, with the tight constraint set, noise level only slightly affects the MSE (see Table 4.II).

4.5 Discussion

Many consider the use of a priori information antithetic to the spirit of adaptive control (see for example [25]). Indeed, some investigators have concluded that the use of projectors is not necessary because "... momentary unstable parameter settings can induce very rapid learning" [36]. While this statement may well be true, as a practical matter momentary instability can be undesirable.

One problem with allowing momentary instability is that the adaptation gain becomes small, leaving the adaptive controller unable to compensate for later changes. Another problem is that instability may be dangerous in some situations (e.g., medical systems in human patients, where instability may be life-threatening). If rapid learning is desired, other techniques are available to increase the adaptation gain without sacrificing stability (e.g., varying the forgetting factor).

The lack of acceptance of constrained identification may stem partially from the use of suboptimal projectors. In particular, many of the " σ -modifiers" [50] [31] [33] only bound the parameters in the mean. As we have shown,

the orthogonal (suboptimal) projector can lead to poor steady state control performance. It is easy to demonstrate impaired transient control performance also.

This result is not obvious, as Chia et al. (1991) has proven that the orthogonal projector should produce less parameter bias than the optimal projector when large noise is present and when the inputs are uncorrelated with the noise [7]. However, this theorem does not apply to our results for two reasons. First, adaptive control depends on accurate prediction rather than on accurate parameter estimation (e.g., consider the self-tuning regulator of [2] that converges to the optimal predictor despite parameter bias). Second, in closed loop, the input is generally correlated with the noise, violating the assumptions of the theorem. Thus, at least for the types of noises and systems studied here, the optimal predictor is the better choice for adaptive control.

The "balloon" effect could also explain some of the reluctance to accept constrained identification. If one end of a balloon is squeezed, the other end will expand. Similarly, if only one part of a model is constrained, modeling error may be exaggerated in another. If control happens to be sensitive to errors in the unconstrained part, then control may be worse than if the constraints

had not been used! The simulations in which only the sign of b_1 and a range on the zero was constrained clearly illustrate this effect. The problem may be solved by using a different controller or by adding constraints reflecting the stability conditions, as in the examples using the tight constraint set. In general, it has been our experience that if constraints are developed for one part of a model, then similar constraints should be developed for the rest.

As we have shown by lowering the forgetting factor, when the appropriate constraints are in place, the adaptation gain can be increased without sacrificing stability. However, rapid learning depends on both the adaptation gain and the excitation level. Since it is possible to lose excitation for tightly constrained systems, it may be necessary to temporarily add an exciting signal to the control. This lack of excitation should not be viewed as a handicap, but instead as an advantage: potentially large excitation (such as temporary instability) can now be replaced by a carefully controlled temporary excitation (such as the decreasing dither in [55]). It should be noted that start-up control is a special case of re-identification. Hence, these techniques could be used to improve start-up as well.

Besides directly improving the robustness of adaptive control, constraints can be used in other ways too. For example, while rapid learning offers the potential for dramatic improvements, the controller (or some supervisory expert system) must determine when the learning is needed. The constraint set may be an excellent tool for this purpose. Indeed, if no noise is present, and the constraints are true, then they should never be invoked. In our experience, noise and non-linearities usually cause some of the constraints to be violated. The projection costs (4.17) are typically minor in these instances. However, when a significant event occurs (such as a change in the plant), the constraint costs often rise dramatically. Hence, by monitoring the constraint costs, system changes can be detected.

Finally, we re-emphasize a caution also stated in [10]. As demonstrated by the simulations with an untrue constraint, control performance can be poor, the control possibly even unstable. Thus, it is important to carefully scrutinize a priori assumptions about a plant, especially if the plant is non-linear or time-varying. Consider, for example, the constraints in (4.29) developed for a known range on an steady state point. It is common to know a range for the steady state unforced

output (defined as the background), since it is an easily grasped, intuitive quantity. However, in linearizing a non-linear model about an operating point, it is more appropriate to think of background as a slack variable (an extrapolated steady state intercept -- see Figure 4.7). The actual range could thus be much larger than anticipated.

4.6 Conclusions

We have shown that optimally constrained identification for adaptive control can dramatically improve controller performance, especially when combined with re-identification. The identification problem has been formulated as a quadratic program that can be solved in real-time. Linear constraints have been derived from commonly available information, and application guidelines have been demonstrated.

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Figure 4.1: BIBO Stability conditions (hatched region) for second order systems (the roots of the polynomial $A(q^{-1}) = 1 - a_1 q^{-1} - a_2 q^{-2}$ will be within the unit disc).

Figure 4.2: Settling time conditions for second order systems: (a) poles restricted to circle of radius r (<1); (b) transformed system poles are now restricted to the unit disc; (c) stability conditions for the transformed system; (d) inverse-transformed stability conditions result in settling time conditions.

Figure 4.3: Example of combined conditions for a second order system: (a) conditions for positive real poles; (b) conditions for minimum and maximum settling time superimposed on (a); (c) a closer look at the linear convex hull of (b).

Figure 4.4: Example closed loop simulations (no noise): (a) loose constraint set; (b) tight constraint set; (c) incomplete constraint set; (d) tight constraint set with a too small maximum steady state gain; (e) tight constraint set with orthogonal projection; (f) loose constraint set with re-identification; (g) tight constraint set with re-identification.

Figure 4.5: Example closed loop simulations (small noise): (a) loose constraint set; (b) tight constraint set; (c) incomplete constraint set; (d) tight constraint set with a too small maximum steady state gain; (e) tight constraint set with orthogonal projection; (f) loose constraint set with re-identification; (g) tight constraint set with re-identification.

Figure 4.6: Example closed loop simulations (large noise): (a) loose constraint set; (b) tight constraint set; (c) incomplete constraint set; (d) tight constraint set with a too small maximum steady state gain; (e) tight constraint set with orthogonal projection; (f) loose constraint set with re-identification; (g) tight constraint set with re-identification.

Figure 4.7: Non-linear background vs. linear background.

Table 4.I: Constraints used in simulations.

Constraint Set	G_{ss}		z_0		a_1		b_1		b_2	
	Min	Max	Min	Max	Min	Max	Min	Max	Min	Max
Actual Parameter Values	10	20	-2.0	-2.0	.700	.700	1.0	2.0	2.0	4.0
Loose										
Tight [†]	5	30	-2.5	-1.5	.596	.762	0.5	5.0	0.5	5.0
Incomplete			-2.5	-1.5			0.0			
Untrue [†]	5	10	-2.5	-1.5	.596	.762	0.5	5.0	0.5	5.0

[†] Minimum and maximum a_1 was chosen to achieve a minimum and maximum settling time of 8.91 and 16.91 steps (13 steps from the plant's settling time of 12.91 steps).

Table 4.II: Simulation results: mean squared control error
 (t = 105 to 205).

CONDITIONS	No Noise	Small Noise	Large Noise
Loose	4746.92	3491.04	386.70
Tight	179.24	191.18	210.68
Incomplete	2519.12	3552.27	312.26
Untrue	611.49	624.02	643.59
Sub-Optimal Tight	2226.18	2080.77	1736.14
Loose + Re-Ident.	3086.46	7116.13	210.39
Tight + Re-Ident.	44.76	47.81	56.13

CHAPTER 5:

(PAPER 2)

ADAPTIVE CONTROL OF SECOND ORDER LINEAR COMPARTMENTAL SYSTEMS USING CONSTRAINED IDENTIFICATION¹

Abstract

Second order linear compartmental models are popular for modeling physiological systems, but these models are not always convenient for on-line identification and adaptive control. ARMAX models, on the other hand, are convenient for on-line identification and adaptive control, but they are not physiologically based. Hence, adaptive control is potentially unsafe when ARMAX models are used as the system model. A solution to this dilemma is to restrict the identified ARMAX model to a subclass of compartmental-like models, thus avoiding potentially

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unsafe conditions and maintaining the convenience of ARMAX models.

In this paper, we delineate the ARMAX subspace corresponding to second order compartmental systems by converting a class of compartmental models to ARMAX form. We show that by constraining the ARMAX parameters to this subspace during adaptive control, we significantly improve the accuracy and consistency of the estimated model parameters. These improvements, besides increasing safety, can also dramatically improve the control.

5.1 Introduction

In this paper, we consider discrete time adaptive identification and control of systems that can be approximated by a class of second order linear compartmental models. We combine the structural constraints of these models with the ease of identification and control of linear time-series models (ARMAX, CARIMA, etc.) to obtain improved accuracy and consistency of the identification and improved control. When applied to physiological systems, the resulting control is safer than conventional adaptive control utilizing time-series models alone, and more practical

than adaptive control utilizing compartmental models alone.

When natural physiological regulating mechanisms go awry, it is often desirable to apply exogenous controls to bring the systems back to normal operating conditions. Examples of such interventions include the injection of insulin to regulate glucose levels in diabetics, the infusion of triglycerides in cases of acute hypertension, and the induction and maintenance of anesthesia (e.g., see [3], [17] and [20]). Other times, it is desirable to achieve a plasma concentration of a therapeutic agent during treatment of certain ailments (e.g., see [8]). In most instances, the therapeutic agents are toxic, yet ineffective below certain plasma concentrations. It is therefore critical to maintain the lowest possible plasma levels while still achieving the desired effect. Optimal control designs offer this possibility. In addition, adaptive control offers the ability to control these systems when they are uncertain or time-varying, since these controllers tune themselves as they control.

Compartmental modeling is a popular approach for capturing the behavior of pharmacodynamical systems (e.g., see [2], [9], or [4]). The compartments and variables typically have physiological meanings, and

often model the mechanisms underlying the system behavior. However, these models are not always practical for real-time analysis and control. For example, Furler et al. (1985) found that just the calculation of the optimal dosage regimen for diabetic patients based on a bilinear model took 35 minutes on an IBM PC [3]. While this speed may be adequate for control of blood sugar, it is not suitable for faster systems. Nevertheless, compartmental models have been used for the control of various pharmacodynamical systems, including control of cardiac arrhythmias using lidocaine [10] or disopyramide [21], as well as for digitalis therapy [10] and regulation of plasma concentrations of alfentanil [12].

Though compartmental models are physiologically based, black box time-series models (e.g., ARMAX or CARIMA models) are more practical for control. As such, they have been widely used for real-time system identification and control of physiological systems (e.g., see [11]). However, they are susceptible to inaccuracies due to noise, non-linearities and unmodeled dynamics. Thus, controllers designed from these models are potentially unsafe.

Since the input/output properties of linear compartmental models are a subset of the more general

time-series model, restriction of the time-series model parameters to this subspace may eliminate the potential safety problems associated with black box models. In Chapter 4 (Paper 1), we developed an algorithm that optimally constrains the parameters of a time-series model in real-time. By delineating the time-series subspace corresponding to compartmental models, we can use this algorithm to identify and control compartmental systems.

In this paper, we therefore convert a class of linear second order compartmental models to ARMAX form, and use the resulting constraints for identification and control. We then illustrate our constrained identifier and controller by adaptively controlling the plasma concentration of methotrexate (a potent anti-cancer drug) in a human model. For this example, we show that the consistency and accuracy of the parameter estimates are significantly improved.

5.2 Compartmental-Like Second Order Linear Models

Figure 5.1 illustrates a second order linear compartmental model. The input, $u(t)$, is a drug infused into one of the compartments, while the output, $y(t)$, is

the drug concentration in one of the compartments. Each compartment may correspond to a physiologically meaningful space, such as the blood plasma, an organ, or a group of organs in which it is assumed the drug concentration is uniform. The rate of loss of drug from a compartment is the sum of all flows out of that compartment into other compartments or the environment. It is assumed the rate of loss is linearly proportional to drug concentration in that compartment. The rate of gain of the drug in a compartment is the sum of all the flows entering from other compartments and the environment. The arrows indicate the direction and paths of flow, and the k_{ij} are the proportionality constants for the flow from compartment j to i , where compartment 0 is the environment. All k_{ij} must be non-negative for physical meaning. We define such models as strictly compartmental.

Two compartmental structures, which we shall define as Type I and Type II, are of practical interest for drug delivery. In Type I systems, the drug is input to the same compartment from which the output is sampled. In Type II systems, the drug is input to one compartment and the output sampled from the other. In a third possible system, Type III, either the drug is input to one compartment and the output is a combined sample from both

compartments, or (its dual) the drug is input to both compartments, and the output is sampled from one compartment. Without loss of generality, we shall assume that the drug is input to one compartment, namely, the first compartment ($b_1 \geq 0$, $b_2 = 0$), and that the output is sampled from one compartment (Type I and II) or both compartments (Type III). We shall not consider systems in which the drug is injected into both compartments and the output is a combined sample from both compartments (Type IV). Note that some Type IV models can be converted to Type I, II, or III by a change of variables.

Type I and II are mutually exclusive subsets of Type III. Therefore, we will solve for Type III, then set the appropriate k_{ij} to zero for our analysis of Types I and II. We assume for each type that all the $k_{ij} > 0$ ($i \neq 0$), and that at least one of the $k_{0j} > 0$.

If we define the state variables q_i (≥ 0) as the concentration of the drug in Compartment i , then the differential equations for Type III compartmental systems are

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} -(k_{21} + k_{01}) & k_{12} \\ k_{21} & -(k_{12} + k_{02}) \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u(t) \quad (5.1)$$

$$y(t) = [c_1 \ c_2] [q_1(t) \ q_2(t)]^T.$$

Let

$$A \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -(k_{21} + k_{01}) & k_{12} \\ k_{21} & -(k_{12} + k_{02}) \end{bmatrix}. \quad (5.2)$$

From the compartmental conditions, the $a_{ii} < 0$ and $a_{ij} > 0$ ($i \neq j$). This relationship will be the basis for our subsequent analysis and control. Thus, our analysis will apply equally well to systems for which the environment may supply flow into the compartments. That is, for $a_{jj} < 0$, some of the k_{0j} may be negative as long as $k_{0j} > -\sum_i k_{ij}$ ($i \neq 0, i \neq j$). We have defined this wider class

of models as compartmental-like. Often, these non-strictly compartmental models can be converted to strict form by a change of variables.

The transfer function is

$$\frac{Y(s)}{U(s)} = b_1 c_1 \frac{s - a_{22} + \frac{c_2}{c_1} a_{21}}{(s - a_{11})(s - a_{22}) - a_{12} a_{21}} \quad (5.3)$$

where s is the Laplace operator. The poles for all three types will be identical, while the zero and the gain will differ. We now derive continuous time conditions for these models. Results are summarized in Table 5.I.

Theorem 5.1: The transfer function poles of model Types I, II, and III will be real, distinct, and negative. Thus, the systems will all be bounded input bounded output (BIBO) stable.

Proof: BIBO stability and realness of second order linear compartmental systems is well known (see, for example, [4]). For distinctness, consider the roots of the characteristic equation $(s-a_{11})(s-a_{22}) - a_{12}a_{21} = 0$. The poles are given by

$$p_1, p_2 = \frac{1}{2}(a_{11} + a_{22}) \pm \frac{1}{2}\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \quad (5.4)$$

The radical will always be positive since $a_{12}a_{21} > 0$. Thus the poles will be distinct, Q.E.D.

For the following theorems, let the poles be ordered so that $p_2 < p_1 < 0$.

Theorem 5.2: For Type I systems, $p_2 < s_0 < p_1$, where s_0 is the transfer function zero.

Proof: For Type I systems ($c_2=0$) $s_0 = a_{22}$ (cf. (5.3)). We must show that the poles bracket this value. Hence, the poles must extend from their midpoint, $\frac{1}{2}(a_{11}+a_{22})$, by more than $\frac{1}{2}|a_{11}-a_{22}|$. This condition follows directly from (5.4), and completes the proof.

This theorem requires no knowledge about the magnitude or ordering of a_{11} and a_{22} . If their ordering is known, further refinement on the position of the zero is possible.

Theorem 5.3: The zero's upper or lower limit for Type I systems may be further restricted:

- 5.3.1: If $a_{22} \geq a_{11}$ then $s_0 \geq \frac{1}{2}(p_1+p_2)$.
- 5.3.2: If $a_{11} \geq a_{22}$ then $s_0 \leq \frac{1}{2}(p_1+p_2)$.
- 5.3.3: If $a_{11} = a_{22}$ then $s_0 = \frac{1}{2}(p_1+p_2)$.

Proof: The poles will be symmetrically centered about $\frac{1}{2}(p_1+p_2) = \frac{1}{2}(a_{11}+a_{22})$. If $a_{22} \geq a_{11}$ then the zero ($=a_{22}$) will be center or right of center. Conversely, if $a_{11} \geq a_{22}$ then the zero will be center or left of center. If $a_{11}=a_{22}$ then it will be center.

Theorem 5.4: For Type II systems ($c_2=0$), there is no zero ($s_0=-\infty$). (Proof omitted).

Theorem 5.5: For Type III systems, the zero is bounded above by the upper bound for Type I systems ($c_2 \rightarrow 0$), and below by the lower bound for Type II systems ($c_1 \rightarrow 0$) (i.e., there is no lower bound). (Proof omitted.)

Thus, for Type III systems, if it is known that $a_{11} \geq a_{22}$ then the upper bound is $\frac{1}{2}(p_1 + p_2)$; otherwise it is p_1 .

5.3 Discrete Time ARMAX Conversion

The continuous time (c.t.) conditions from the previous section can be converted to discrete time (d.t.) under an implementation mapping. We assume the outputs will be sampled at evenly spaced times ($0, T, 2T, \dots$) with a negligible delay between the sampling and the availability of the measurement. We also assume the inputs will be administered using a zero order hold (ZOH) at these same times. These assumptions lead to mapping by step invariance (i.e., ZOH):

$$\frac{Y(z)}{U(z)} = \left[\frac{z-1}{z} \right] \mathcal{Z} \left\{ \frac{Y(s)}{U(s)} \frac{1}{s} \right\} \quad (5.5)$$

where z is the Z-transform operator, and $Z\{.\}$ is the Z-transform of $\{.\}$. Writing (5.3) in time constant form,

$$\frac{Y(s)}{U(s)} = \frac{K(\tau s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (5.6)$$

where, for all types, $\tau_1 = -1/p_1$, and $\tau_2 = -1/p_2$; for Types I and III, $\tau = -1/s_0$, $K = b_1 c_1 \tau_1 \tau_2 / \tau$; for Type II, $\tau = 0$ and $K = b_1 c_2 \tau_1 \tau_2 a_{11}$. For Type I, $\tau_1 > \tau > \tau_2 > 0$; for Type II, $\tau_1 > \tau_2 > 0$; and for Type III, $\tau_1 > \tau_2 > 0$ and $\tau_1 > \tau \geq 0$. Converting to d.t.,

$$\frac{Y(z)}{U(z)} = K \frac{[\beta(r_1 - r_2) + (1 - r_2)]z + \beta(r_2 - r_1) + (r_2 - 1)r_1}{(z - r_1)(z - r_2)} \quad (5.7)$$

where $\beta = -\frac{\tau_1 - \tau}{\tau_1 - \tau_2}$, $r_1 = e^{-T/\tau_1}$, $r_2 = e^{-T/\tau_2}$, and K is as

defined above. Clearly, the d.t. poles are at r_1 and r_2 , while the zero is at

$$z_0 = \frac{\beta(r_1 - r_2) + (1 - r_2)r_1}{\beta(r_1 - r_2) + (1 - r_2)}. \quad (5.8)$$

We now map the earlier c.t. theorems to d.t. Results are summarized in Table 5.II.

Theorem 5.6: For Types I, II, and III, the resulting poles are real, and $0 < r_2 < r_1 < 1$.

Proof: The proof follows directly from the definitions of r_1 , r_2 , $\tau_1(p_1)$, $\tau_2(p_2)$, and Theorem 5.1.

This theorem means the d.t. system is BIBO stable, as would be expected.

For the remaining theorems, we need the following lemmas.

Lemma 5.1: Define $f_1(x) = \frac{1-e^{-x}}{x}$, then $f_1(x_1) > f_1(x_2)$

if $x_2 > x_1 > 0$.

Proof: The proof is in the Appendix.

Lemma 5.2: For Type I, II, and III systems, the denominator of z_0 is greater than zero (i.e., $B(r_1-r_2)+(1-r_2) > 0$).

Proof: The proof is in the Appendix.

Theorem 5.7: For Type I systems, $r_2 < z_0 < r_1$.

Proof: There are two parts. By Lemma 5.2:

$$\begin{aligned} \text{(i)} \quad & [\beta(r_1-r_2)+(1-r_2)]r_2 < \beta(r_1-r_2) + (1-r_2)r_1 \\ \text{(ii)} \quad & \beta(r_1-r_2) + (1-r_2)r_1 < [\beta(r_1-r_2)+(1-r_2)]r_1 \end{aligned}$$

Proceeding with part (i), group terms with β on the left, all else on the right:

$$\beta(r_1-r_2)(1-r_2) > -(r_1-r_2)(1-r_2)$$

or, $\beta > -1$, which is true by definition of β and Theorem 5.2.

For part (ii), again group terms with β on the left, all else on the right:

$$\beta(r_1-r_2)(1-r_1) < 0$$

or, $\beta < 0$, which is true by definition of β and Theorem 5.2.

As for the continuous time case, further restrictions can be placed on the zero if the ordering of a_{11} and a_{22} is known.

Theorem 5.8: The d.t. zero's supremum or infimum for Type I systems for arbitrary T may be further restricted:

- 5.8.1: If $a_{22} \geq a_{11}$ then $z_0 \geq (r_1 r_2)^{\frac{1}{2}}$.
 5.8.2: If $a_{11} \geq a_{22}$ then $z_0 \leq \frac{1}{2}(r_1 + r_2)$.
 5.8.3: If $a_{11} = a_{22}$ then $(r_1 r_2)^{\frac{1}{2}} \leq z_0 \leq \frac{1}{2}(p_1 + p_2)$.

Proof: 5.8.1 and 5.8.2 are in the Appendix. 5.8.3 follows directly from 5.8.1 and 5.8.2.

Remark: Given T , the explicit upper or lower bound can be calculated by inverse mapping r_1 to τ_1 , r_2 to τ_2 , then solving for z_0 (eq. (5.8)) at the boundary condition, $\tau = 2\tau_1\tau_2/(\tau_1+\tau_2)$ (i.e, $\beta = -\tau_1/(\tau_1+\tau_2)$). Theorem 5.8.3 in this case collapses to an equality constraint.

Theorem 5.9: For Type II systems, the zero is bounded by a supremum and infimum for arbitrary T :

$$-1 \leq z_0 \leq 0.$$

Proof: The proof is in the Appendix.

Remark: Given T , an explicit value for z_0 may be calculated by inverse mapping the d.t. poles, then forward mapping to z_0 with the boundary condition $\tau = 0$ (i.e, $\beta = -\tau_1/(\tau_1-\tau_2)$). Thus, 5.9 reduces to an equality constraint.

Theorem 5.10: For Type III systems, the d.t. zero is bounded above by the upper bound for Type I systems ($c_2 \rightarrow 0$), and below by the lower bound for Type II systems ($c_1 \rightarrow 0$). (Proof omitted).

Remark: For Type III systems, if it is known that $a_{11} \geq a_{22}$, then the supremum is $\frac{1}{2}(r_1 + r_2)$; otherwise it is r_1 . For all cases, the infimum is -1. As before, given T, the explicit upper bound for the case $a_{11} \geq a_{22}$ may be calculated, as can the explicit lower bound for all cases.

5.4 Constrained Identification

Adaptive control based on a time series ARMAX model (autoregressive moving average model with exogenous inputs) consists of two parts, an identifier and a controller. Consider the ARMAX model

$$\begin{aligned} y(t) &= a_1 y(t-1) + a_2 y(t-1) + \dots + a_{n_1} y(t-n_1) \\ &\quad + b_1 u(t-1) + b_2 u(t-2) + \dots + b_{n_2} u(t-n_2) \quad (5.9) \\ &\quad + e(t) \\ &= \phi^T(t-1)\theta + e(t), \end{aligned}$$

where $n_1 \geq n_2$, and where the vectors θ and $\phi(t-1)$ are defined by

$$\begin{aligned} \theta &\triangleq [a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}]^T \quad (5.10) \\ \phi(t-1) &\triangleq [y(t-1), \dots, y(t-n_1), u(t-1), \dots, u(t-n_2)]^T. \end{aligned}$$

Generally, the parameter vector θ is unknown, and is to be estimated.

Assume the following linear constraints apply to θ :

$$\begin{aligned} m_i^T \theta &= k_i, & i &= 1, \dots, n_4 \\ l_i^T \theta &\leq c_i, & i &= 1, \dots, n_5 \end{aligned} \quad (5.11)$$

where m_i and l_i are p length vectors and k_i and c_i are scalars. For ease of notation, we may rewrite (5.11) as

$$\begin{aligned} M \theta &= K \\ L \theta &\leq C \end{aligned} \quad (5.12)$$

where K and C are n_4 and n_5 length vectors made up of the k_i and c_i respectively, and M and L are $n_4 \times p$ and $n_5 \times p$ matrices defined as

$$\begin{aligned} M &= [m_1 | m_2 | \dots | m_{n_4}]^T \\ L &= [l_1 | l_2 | \dots | l_{n_5}]^T \end{aligned} \quad (5.13)$$

At time t , we may solve for the unconstrained least squares estimate, θ_f (f for "free"), by recursive least squares (e.g., see [6]):

$$\begin{aligned} \theta_f(t) &= \theta_f(t-1) \\ &+ \frac{P(t-2)\phi(t-1) \cdot [y(t) - \phi(t-1)^T \theta_f(t-1)]}{\alpha(t-1) + \phi(t-1)^T P(t-2)\phi(t-1)} \end{aligned} \quad (5.14)$$

$$P(t-1) = \frac{1}{\alpha(t-1)} \cdot \left[P(t-2) - \frac{P(t-2)\phi(t-1)\phi(t-1)^T P(t-2)}{\alpha(t-1) + \phi(t-1)^T P(t-2)\phi(t-1)} \right] \quad (5.15)$$

where $P(t)$ is proportional to the parameter error covariance matrix (it is the inverse of the data-covariance matrix), and $\alpha(t)$ is a scalar forgetting factor. The recursive least squares estimate minimizes the sum of the squared prediction errors:

$$\min_{\hat{\theta}(t)} J = \frac{1}{2} \sum_{k=1}^t w_k (y(k) - \phi^T(k-1)\hat{\theta}(t))^2, \quad (5.16)$$

where $\hat{\theta}$ is the vector of parameters to be estimated, and the w_k are defined by the $\alpha(t)$.

In real-time, we can solve for $\hat{\theta}$ subject to the constraints in (5.12) by formulating the problem as the unconstrained estimate plus correction terms:

$$\theta_c(t) = \theta_f(t) - P(t-1)M^T \mu(t) - P(t-1)L^T \lambda(t) \quad (5.17)$$

where θ_c is the constrained solution, and $\mu(t)$ and $\lambda(t)$ are vectors of Lagrange multipliers associated with, respectively, the equality and inequality constraints in (5.12). The multipliers are readily determined using quadratic programming techniques. We use an enhanced

version of positive semi-definite complementary linear programming. Details of the algorithm are in Chapter 3.

5.5 Models of Methotrexate Distribution

Methotrexate (amethopterin) is a folate antagonist that shows marked consistent antitumor effects in animals and humans [14]. Since it is toxic to normal cells, it is important to maintain as little drug concentration as possible in sensitive healthy tissues while maintaining effective levels in tumors [22].

Bischoff, Dedrick, and Zaharko (1970) have proposed a simplified second order compartmental model to predict the plasma and gut lumen concentrations of MTX (methotrexate) in rodents and, with appropriately scaled compartments, in man [1] [22]. Their model includes a 5 minute delay in excretion from the plasma to the gut lumen, and a constant resorption term (irrespective of concentration) in the gut lumen to reflect saturation of an active transport mechanism. The resorption term has a delay of between 2 and 3 hours [1].

Northrop and Woodruff (1986), however, report that the MTX plasma response can be modeled as a Type I

compartmental system with negligible transport delays (compared to the sampling times) [13]. Therefore, we fit a Type I model to plasma measurements of two human subjects after an intravenous (iv) MTX injection of 1 mg per kg body weight (data from [7], Chart 1). A Nelder-Mead Simplex algorithm (from the PC-MATLAB nonlinear toolbox [15]) was used to minimize the sum of the squared prediction errors. The resulting transfer function for the plasma concentration is:

$$P1(s) = \frac{223.78(14.683s + 1)}{(5.2105s + 1)(148.41s + 1)}, \quad (5.18)$$

where the MTX plasma concentration is in $\mu\text{g/ml}$ and the iv infusion rate is in mg/kg/min . If we interpret the plasma concentration response as the first compartment in a two compartment model, the second compartment would consist of the rest of the body. Bischoff, Dedrick, and Zaharko (1970) have shown that the liver and the gut lumen are the major components affecting the second compartment. Zaharko et al. (1971) subsequently published predictions of the gut lumen concentration in man following a 1 mg/kg bolus injection of MTX [22]. Though not specifically our "second" compartment, we used this data to determine the final parameter of our two-compartment model. The resulting steady state ratio of plasma concentration to concentration in the second

compartment, liver and gut lumen (LGL), is 0.108. This value is approximately the steady state plasma to liver ratio (0.1) reported in [1]. The resulting transfer function for this compartment is

$$LGL1(s) = \frac{2075.6}{(5.2105s + 1)(148.41s + 1)}, \quad (5.19)$$

where the MTX LGL concentration is in $\mu\text{g}/\text{mg}$ and the iv infusion rate is in $\text{mg}/\text{kg}/\text{min}$.

Northrop and Woodruff also estimated parameters for the plasma concentration of a Type I model (which they claim represents the "typical" patient response) [13]. Assuming their original transfer function is for a 70 kg patient, we can normalize it to:

$$P2(s) = \frac{295.39(104.41s + 1)}{(56.880s + 1)(314.40s + 1)}, \quad (5.20)$$

where the MTX plasma concentration is in $\mu\text{g}/\text{ml}$ and the iv infusion rate is in $\text{mg}/\text{kg}/\text{min}$. Then, assuming the same steady state ratio exists between the plasma and Compartment II, the LGL transfer function would be

$$LGL2(s) = \frac{2739.8}{(56.880s + 1)(314.40s + 1)}, \quad (5.21)$$

where the MTX LGL concentration is in $\mu\text{g}/\text{mg}$ and the iv infusion rate is in $\text{mg}/\text{kg}/\text{min}$.

For illustration purposes, we shall assume that intra-patient variability can account for the differences between (5.18) and (5.20), and (5.19) and (5.21). This variability can be attributed to poor renal circulation or compromised liver function [5], [16]. The impulse responses for each compartment are plotted in Figures 5.2a (plasma) and 5.2b (LGL). The observed plasma concentrations of the two human subjects used to fit (5.18) and (5.19) are also included in Figure 5.2a.

5.6 Controller Setup

We shall now use these models to test adaptive control in which we restrict the identified ARMAX model to the compartmental subspace. As a challenge to the controller, we will mimic the potential time-varying behavior of our assumed MTX system by ramping from one model to the other.

Following the example in [13], we shall sample and control every 5 minutes. This time period is within the recommended range of $1/15$ to $1/4$ of the 90% settling time of the fastest transient of interest. However, since the typical assay method for determining plasma concentrations of methotrexate depends on the transit

time of the blood sample through a high pressure liquid chromatography column, the output measurements are delayed by three minutes [13]. We shall therefore assume that the output measurements are delayed by one sampling period. Since this assay method is precise, we shall assume the measurements are noise free [13].

Given a desired output trajectory, $y^*(t)$, the predictive controller CAMAC may be used to calculate the control input [19]. At each sampling time, CAMAC attempts to specify the step input that would drive the output to its desired value in K sampling intervals. K must be greater than or equal to the input/output delay d . For $K = d$, the control specifies the input for minimum variance control, while for $K \geq \tau_{\text{set}}$ (the settling time), the control specifies the steady state input. Thus, CAMAC determines the next input by minimizing the expected K -step ahead control error:

$$\min_u \mathcal{E} \left\{ \left[y^*(t+K) - y(t+K) \right]^2 \right\} \quad (5.22)$$

As we have shown, second order linear compartmental models have their zeros strictly inside the unit circle. For such systems, minimum variance control can typically be used. However, we have the physical constraint that

drug cannot be removed once it is injected. Thus, we must adjust K until the control does not depend on negative inputs. In preliminary simulations, we have found that, for Compartment I (plasma), K can equal d (5 minutes), while for Compartment II (LGL), K should be greater than 50 minutes.

Parameter constraints were determined as follows. Since Northrop and Woodruff (1986) claim their model (5.20) is "typical", we shall assume it is, and that (5.18) is due to random drift over time. We can also assume that an equal but opposite drift may occur. We may do the same with the LGL compartment. Tables 5.III and 5.IV list the resulting models for each compartment. If we allow for additional drifts and modeling error, these plants can be used to set bounds for the constraints. The constraint bounds that we used for each compartment are listed in Tables 5.V and 5.VI. Table 5.VII lists the constraint equations used to implement the bounds.

Since we know that $a_{22} > a_{11}$, we can use the conditions from Theorem 5.8.1 to constrain the zero for Type I control (plasma concentration), and Theorem 5.9 to constrain the zero for Type II control (LGL concentration). However, these constraints require

knowledge of the poles, which is not available until after estimation. Therefore, at each time step we use an iterative approach to polish the zero constraint:

Step 1. Set large bounds for the zero. For Type I, set $z_{0 \min} = r_{2 \min}$ and $z_{0 \max} = r_{1 \max}$. For Type II, set $z_{0 \min} = -1$ and $z_{0 \max} = 0$.

Step 2. Constrain the free parameters.

Step 3. Establish new bounds based on the constrained parameters. For Type I, calculate $z_{0 \min}$ from (5.8) with $\beta = -\tau_1/(\tau_1 + \tau_2)$, and set $z_{0 \max} = r_{1 \max}$. For Type II, calculate $z_{0 \max}$ from (5.8) with $\beta = -\tau_1/(\tau_1 - \tau_2)$.

Step 4. Test for convergence. The convergence test for Type I is

$$z_{0 \min}/(1+\epsilon) \leq -b_2/b_1 \leq z_{0 \max}/(1+\epsilon).$$

(where ϵ is a small positive value to relax the convergence criteria). The convergence test for Type II is,

$$z_0/(1+\epsilon) \leq -b_2/b_1 \leq z_0/(1+\epsilon).$$

If the criteria are met, then return with the current constrained estimates. Otherwise, go to Step 2.

If, when polishing, the zero constraint converges to an empty space, or if the projection becomes numerically unstable, or if some maximum number of iterations have been performed, then the last feasible solution is returned. The last feasible solution may be the solution from the previous sampling interval. In our simulations,

convergence was rapid (never more than 4 iterations with $\epsilon = 0.02$).

5.7 Control Examples

We have previously demonstrated that start-up can be improved by using constraints (see [18]). Other well known techniques exist that can also improve start-up (such as open loop probing, or starting with a conservative fixed controller and switching to the adaptive controller when parameters converge). We shall therefore conclude that start-up is not a problem, and instead concentrate on other facets of control.

Example 1: Control of a Time-Varying Type I System. In our first control example, we compare the unconstrained identifier with the constrained identifier for the adaptive control of plasma MTX concentration when the patient response changes over time from eq. (5.20) to (5.18). To avoid start-up differences between the constrained and unconstrained identifiers, the parameter estimates are set to correct values for the initial plant (5.20). Control is then started with a target concentration of 1 $\mu\text{g/ml}$ MTX, and with K set for minimum variance control (5 minutes). After 25 minutes of

control, the initial plant's time constants and gain linearly ramp to the final plant over a 60 minute interval. The transition is complete at 85 minutes into the simulation. After the transition begins, the desired plasma concentration is changed to 5 $\mu\text{g/ml}$ at time $t_{5 \mu\text{g/ml}}$ minutes. To avoid random differences between the controllers, we performed a series of simulations in which the setpoint was changed at progressively later times ($t_{5 \mu\text{g/ml}} = \{60, 65, 70, 75, 80, 85, \text{ and } 90 \text{ minutes}\}$). The resulting MTX plasma concentrations, MTX infusion rates, parameter estimates, and z-domain estimates for the unconstrained and constrained controllers are plotted side by side in Figures 5.3 and 5.4. Descriptive statistics (ensemble averages) are listed in Table 5.VIII.

Typically, the initial response following the change in setpoint is an overshoot of about 2.8 $\mu\text{g/ml}$ followed by several decaying oscillations. There is little difference in the controllers' initial input and output measures whether or not the parameter estimates are constrained (compare Figure 5.3 a and b with Figure 5.4 a and b). However, after the initial oscillations die down, residual oscillations can remain if parameter constraints are not used. In fact, the plasma concentration for one of the unconstrained simulations

never settled to within 3% of the desired level, while another took 185 minutes (Figure 5.3a). When constraints were imposed, the longest settling time was only 60 minutes (Figure 5.4a). There was always at least a 20 minute improvement when constraints were imposed. Hence, the mean settling time with constraints was almost three times faster than it was without constraints (see Table 5.VIII). Furthermore, when constraints were used, the mean squared control error for the last 150 minutes was nearly 80 times better than when constraints were not used (see Table 5.VIII).

The identification is also significantly better when constraints are imposed. The mean squared parameter bias with constraints is less than half the value without constraints (Table 5.VIII). Both consistency and accuracy of the parameter estimates are improved. The consistency and accuracy of the (Z-domain) poles, zeros and steady state gain are also improved when constraints are used. Indeed, the first pole and the steady state gain are more than an order of magnitude better when constraints are used (see Table 5.VIII).

Example 2: Control of a Time-Invariant Type I System. In this example, we study the long-term behavior of adaptive

identification and control with and without constraints on the parameters. To avoid numerical instability in the estimator, the identification is usually turned off when the information content of the data is low [6]. However, to study the long-term low-excitation effects, we shall keep the identification turned on. To further exacerbate the situation, we shall set the exponential forgetting factor to a low value ($\alpha=0.7$). While this example may be unrealistic, it illustrates potential problems that can occur when the level of excitation is low. It also demonstrates the extra margin of safety provided by constraints.

We used equation (5.18) for the plant. Parameter constraints are the same as for Example 1 (Table 5.V). Again, to avoid start-up issues, the initial parameter estimates are set to their correct values. Control is started with a target plasma concentration of 1 $\mu\text{g/ml}$ MTX with K set for minimum variance control (5 minutes). After 800 minutes of control, the target concentration is changed to 5 $\mu\text{g/ml}$ MTX. The resulting MTX plasma concentrations, MTX infusion rates, parameter estimates, and z -domain estimates for the unconstrained and constrained controllers are plotted side by side in Figures 5.5 and 5.6. Descriptive statistics are listed in Table 5.IX.

The gain of the unconstrained controller became near singular at the change in setpoint. A large input was specified, producing an overshoot of 42.23 $\mu\text{g/ml}$. When constraints are used, the gain cannot become singular. Hence, for the constrained controller, the overshoot following the setpoint change was significantly less (5.18 $\mu\text{g/ml}$) (compare Figures 5.5a and 5.6a).

As was found for Example 1, the identification is significantly better when constraints are imposed. Both consistency and accuracy of the parameter estimates are improved (compare Figures 5.5 c-f with Figures 5.6 c-f). Consistency and accuracy of the (Z-domain) poles, zeros and steady state gain of the constrained model are also improved (compare Figures 5.5 g-j with Figures 5.6 g-j). Overall, the mean squared parameter bias was more than an order of magnitude better when constraints were used (see Table 5.IX). Due to the large excitation from the overshoot, it would be expected that the unconstrained estimator would produce better estimates after the setpoint change than the constrained estimator would. Nevertheless, the parameter estimates are remarkably similar.

Example 3: Control of a Time-Varying Type II System. In this control example, we compare the unconstrained identifier with the constrained identifier for the adaptive control of the liver and gut lumen (LGL) compartment when, as in Example 1, the patient response changes over time.

To avoid start-up differences between the constrained and unconstrained identifiers, the parameter estimates are set to the correct values for the initial plant (5.21). Control starts with a target LGL concentration of 10 $\mu\text{g}/\text{mg}$ MTX, and with K set to 60 minutes (12 steps). After 125 minutes of control, the initial plant transfer function coefficients linearly ramp to the final plant (5.19) over a 60 minute interval. The transition is complete at 185 minutes into the simulation. After adequate time for recovery, the desired LGL concentration is changed to 50 $\mu\text{g}/\text{mg}$ ($t = 625$ minutes). The resulting MTX LGL concentrations, MTX infusion rates, parameter estimates, and z-domain estimates for the unconstrained and constrained controllers are plotted side by side in Figures 5.7 and 5.8. Descriptive statistics are listed in Table 5.X.

During the transition, the gain for the unconstrained controller became singular twice, so that

an infinite input was called for (Figure 5.7a, $t = 145$ and 155). However, in Examples 1 and 2, an input greater than 1 mg/kg/min was never truly needed. Therefore, we limited the input infusion rates to 1 mg/kg/min (thus avoiding a catastrophically large dosage). Even so, 1 mg/kg/min was too large in this instance, and resulted in a large control error (see Peak Error, Table 5.X).

The controller with constrained parameters, on the other hand, showed only minor control errors during and after the transition period (see Figure 5.7a). Furthermore, recovery of the LGL concentration to within 3% of the desired level was 115 minutes sooner with constraints than without (Table 5.X).

Without constraints, all the parameter estimates moved in the wrong direction early in the plant transition period (Figures 5.7 c-f). The estimated steady state gain reflects this problem clearly (Figure 5.7g). First, it became negative, then large positive. Also, the poles moved outside the unit circle, then back in (Figures 5.7 h and i), while the zero moved from less than -1 to greater than 2.5 and back to less than -1 (Figure 5.7j). With constraints, the estimated steady state gain (Figure 5.8g) remained close to its true value and the parameters moved in the correct direction

(Figures 5.8 c-f). After the large excitation, one would expect the unconstrained parameter estimates to be better than the constrained estimates. However, the improvements were only minor, as both controllers performed nearly identically for the final setpoint change (compare Figures 5.7a with 5.8a).

5.8 Discussion

There is much information that can be extracted from the second order compartmental structure, as summarized in Tables 5.I and 5.II. The information, though theoretical, has important practical use. For example, Theorems 5.8 through 5.10 specify the maximum possible upper bound and the minimum possible lower bound for the zero of these systems. The actual upper and lower bounds are within these outer bounds, and can be determined for any sampling interval T . Depending on the certainty of the compartmental assumptions, the resulting ARMAX constraints can be tightened or relaxed. If we are only weakly confident in the compartmental model, then the outer bounds in Theorems 5.8 through 5.10 should be used. If, as in our Examples, we are strongly confident in our assumptions, then the actual bounds based on T should be

used. The compartmental assumptions thus provide a basis upon which to build a system of constraints.

Given a set of constraints, probably the most important benefit for ARMAX-based adaptive controllers is the increased margin of safety. In Example 1, the constrained identifier produced much better parameter estimates than the unconstrained identifier, although this improvement did not produce a correspondingly large improvement in control (see Table 5.VIII). The improvement did show up, however, in the elimination of oscillations after recovery (compare Figures 5.4 a and b with Figures 5.5 a and b). The steady state oscillations produced by the unconstrained controller indicate potential problems. It may be that later disturbances or setpoint changes could cause overshoot, undershoot, or additional oscillations.

Indeed, in Example 2 using the same system, we clearly demonstrated that a poorly identified system can result in the administration of an inordinately large, possibly toxic dosage (Figure 5.5b). Poor identification thus opens the possibility for a dangerous situation to occur. While nothing can completely eliminate this possibility, the use of constraints can significantly reduce the likelihood (see, for example, Figure 5.6b).

It may be argued that various ad hoc strategies already exist to eliminate each of the problems induced in Examples 1 and 2. While this may be true, ad hoc strategies only treat symptoms of a problem, whereas constrained identification treats the problem causing the symptoms. There will almost certainly be other unanticipated problems resulting from poor identification. Unless each problem is specifically addressed by an ad hoc strategy, the control performance will suffer. Since constrained identification improves the identification, it is likely that any such problem (not just the ones we anticipated) will be reduced or eliminated. Thus, constraints on the estimated parameters provide an important safety net for the controller.

Another advantage of constrained identification is that it does not interfere with the use of other performance enhancers and safety features. Indeed, in Chapter 4 (Paper 1), we showed that the combination of constrained identification with other identification techniques can improve the control more than either alone. Hence, excitation monitors can and should be used alongside the constrained identifier to eliminate long-term low-excitation effects. Constraints on the inputs or weights on the change in control should be used to

eliminate overshoot and oscillations. And, as shown in Chapter 4, re-identification should be used when system changes are detected (for example, we used the variable forgetting factor routine in [6, pg. 227], due to Fortuescue). In general, we have found that constrained identification enhances the usefulness of these techniques and others.

While Examples 1 and 2 demonstrate the improved safety resulting from the use of constraints, Example 3 demonstrates the potential for greatly improved control. In this instance, the system information allowed us to reduce the degrees of freedom of the parameter estimates by one. This reduction is equivalent to reducing the number of estimated parameters by one. It is well known that least squares identifiers converge more quickly and more accurately when the number of parameters is reduced. Thus it is not surprising that the control is significantly better when constraints are imposed. Indeed, even when the unconstrained controller produced large excitation (ideal conditions for identification), the constrained identifier still produced parameters that were nearly as good.

5.9 Conclusions

We have developed linear constraints that delineate the linear time-series model subspace corresponding to linear second order compartmental models. For the discrete-time adaptive control of second order compartmental-like systems (specifically a human model of the pharmacokinetics of methotrexate distribution), we have shown that restriction of the ARMAX model parameters to the compartmental subspace improves both the consistency and accuracy of the estimated model parameters. This improvement in turn can lead to significantly improved control. Most importantly for biomedical applications, the use of constraints, if they are true and valid, increases the safety of these controllers. We therefore conclude that constrained identification is a useful addition for the safe and effective adaptive control of biomedical systems.

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Appendix: Miscellaneous Proofs

Lemma 5.1: Define $f_1(x) = \frac{1-e^{-x}}{x}$, then $f_1(x_1) > f_1(x_2)$

if $x_2 > x_1 > 0$.

Proof: $f_1(x)$ should be strictly monotonic decreasing:

$$\frac{d}{dx}[f_1] = \frac{(1+x)e^{-x} - 1}{x^2} < 0 \quad (x > 0).$$

Multiplying both sides by $-x^2 e^x$, this becomes

$e^x - (1+x) > 0$. Expanding e^x and canceling terms, we get

$$\frac{x^2}{2!} + \frac{x^3}{3!} + \dots > 0,$$

which is obviously true for positive x , Q.E.D.

Lemma 5.2: For Type I, II, and III systems, the denominator of $z_0 = B(r_1 - r_2) + (1 - r_2) > 0$.

Proof: Substitute in the definition of $B (= -\frac{\tau_1 - \tau}{\tau_1 - \tau_2})$

and multiply through by $-(\tau_1 - \tau_2)$:

$$(\tau_1 - \tau)(r_1 - r_2) - (1 - r_2)(\tau_1 - \tau_2) < 0.$$

Expand, cancel like terms, and re-group:

$$\tau_1(1 - r_1) > \tau_2(1 - r_2).$$

Dividing both sides by T , we obtain

$$\frac{1 - r_1}{T/\tau_1} > \frac{1 - r_2}{T/\tau_2},$$

which is in the form of Lemma 5.1 for $x_1 = T/\tau_1$ and $x_2 = T/\tau_2$ ($x_2 > x_1 > 0$), Q.E.D.

Theorem 5.8.1: For Type I systems, if $a_{22} \geq a_{11}$ then $z_0 \geq (r_1 r_2)^{\frac{1}{2}}$.

Proof: Substitute in the definition of z_0 and multiply by its denominator (which is positive by Lemma 5.2):

$$[\beta(r_1 - r_2) + (1 - r_2)](r_1 r_2)^{\frac{1}{2}} \leq \beta(r_1 - r_2) + (1 - r_2)r_1.$$

Isolate $-\beta$ on the left and substitute in its definition:

$$\frac{\tau_1 - \tau}{\tau_1 - \tau_2} \leq \frac{[r_1 - (r_1 r_2)^{\frac{1}{2}}](1 - r_2)}{(r_1 - r_2)[1 - (r_1 r_2)^{\frac{1}{2}}]}.$$

Clearly, the worst case is for $\tau = \frac{2\tau_1 \tau_2}{\tau_1 + \tau_2}$. Substituting

it in,

$$\frac{\tau_1}{\tau_1 + \tau_2} \leq \frac{[r_1 - (r_1 r_2)^{\frac{1}{2}}](1 - r_2)}{(r_1 - r_2)[1 - (r_1 r_2)^{\frac{1}{2}}]}.$$

Multiplying by each denominator (both positive), we obtain

$$\tau_1 (r_1 - r_2)[1 - (r_1 r_2)^{\frac{1}{2}}] \leq (\tau_1 + \tau_2)[r_1 - (r_1 r_2)^{\frac{1}{2}}](1 - r_2).$$

Expand, cancel like terms, and re-group:

$$\tau_1 [(r_1 r_2)^{\frac{1}{2}} - r_2](1 - r_1) \leq \tau_2 [r_1 - (r_1 r_2)^{\frac{1}{2}}](1 - r_2).$$

Multiply both sides by $T[\tau_1 \tau_2 (1 - r_1)(1 - r_2)]^{-1}$:

$$\left(\frac{T}{\tau_2}\right) \frac{[(r_1 r_2)^{\frac{1}{2}} - r_2]}{(1 - r_2)} \leq \left(\frac{T}{\tau_1}\right) \frac{[r_1 - (r_1 r_2)^{\frac{1}{2}}]}{(1 - r_1)}.$$

Let $x_1 = T/\tau_1$ and $x_2 = T/\tau_2$ ($x_2 > x_1 > 0$), and define

$$a = \frac{x_1}{1 - e^{-x_1}} \text{ and } b = \frac{x_2}{1 - e^{-x_2}}. \text{ Now,}$$

$$b[(r_1 r_2)^{\frac{1}{2}} - r_2] \leq a[r_1 - (r_1 r_2)^{\frac{1}{2}}].$$

Collect $(r_1 r_2)^{\frac{1}{2}}$ terms on the left, and all else on the right:

$$(a + b)(r_1 r_2)^{\frac{1}{2}} \leq ar_1 + br_2.$$

Since both sides are positive, we can square both sides and retain the inequality. Thus we obtain

$$(a^2 + b^2)r_1 r_2 \leq a^2 r_1^2 + b^2 r_2^2.$$

By rearranging as follows:

$$a^2 r_1 (r_1 - r_2) \geq b^2 r_2 (r_1 - r_2),$$

we can eliminate the common divisor $(r_1 - r_2) (>0)$, so that

$a^2 r_1 \geq b^2 r_2$. Substituting in the definition of a , b , r_1

and r_2 , we arrive at the relation

$$\frac{x_1^2}{(1 - e^{-x_1})} e^{-x_1} \geq \frac{x_2^2}{(1 - e^{-x_2})} e^{-x_2}.$$

Define $f_2(x) = \frac{x^2}{1 - e^{-x}} e^{-x}$, so that the relation

becomes $f_2(x_1) \geq f_2(x_2)$ for $x_1 > x_2 > 0$. For it to be true,

$f_2(x)$ must be monotonic decreasing for $x > 0$. Therefore,

we must show that

$$\frac{d}{dx}[f_2(x)] = \frac{x e^{-x}}{(1 - e^{-x})^3} [(2 - x) - (2 + x)e^{-x}] \leq 0 \quad (x > 0).$$

All the terms left of the brackets are positive, so they can be eliminated. Rearranging, we obtain

$$\frac{2 - x}{2 + x} \leq e^{-x}.$$

This is obviously true for $x \geq 2$, so we only need to prove the case for $0 < x < 2$. By re-writing in reciprocal form and taking the natural logarithm of both sides, we get

$$\ln \left[\frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x} \right] \geq x.$$

Now we can substitute in the identity

$$\ln \left[\frac{1+y}{1-y} \right] = 2 \left[y + \frac{y^3}{3} + \frac{y^5}{5} + \dots \right] \quad (|y| < 1)$$

($|\frac{1}{2}x| < 1$), and rearrange:

$$\frac{(\frac{1}{2}x)^3}{3} + \frac{(\frac{1}{2}x)^5}{5} + \dots \geq 0.$$

This is obviously true for positive x , Q.E.D.

Theorem 5.8.2: For Type I systems, if $a_{11} \geq a_{22}$ then $z_0 \leq \frac{1}{2}(r_1 + r_2)$.

Proof: Substitute in the definition of z_0 and multiply by its denominator (which is positive by Lemma 5.2):

$$\beta(r_1 - r_2) + (1 - r_2)r_1 \leq [\beta(r_1 - r_2) + (1 - r_2)](r_1 + r_2)/2.$$

Isolate $-\beta$ on the left and substitute in its definition:

$$\frac{\tau_1 - \tau}{\tau_1 - \tau_2} \geq \frac{1 - r_2}{2 - (r_1 + r_2)}.$$

Clearly, the worst case is for $\tau = \frac{2\tau_1\tau_2}{\tau_1+\tau_2}$. Substituting it in,

$$\frac{\tau_1}{\tau_1+\tau_2} \geq \frac{1-r_2}{2-(r_1+r_2)}.$$

Multiplying by the common denominator (>0), eliminating like terms and rearranging,

$$\tau_1(1-r_1) \geq \tau_2(1-r_2).$$

Multiplying by T^{-1} , we obtain

$$\frac{1-r_1}{T/\tau_1} \geq \frac{1-r_2}{T/\tau_2},$$

which is in the form of Lemma 5.1 for $x_1 = T/\tau_1$ and $x_2 = T/\tau_2$ ($x_2 > x_1 > 0$), Q.E.D.

Theorem 5.8.9: For Type II systems, the zero is bounded by a supremum and infimum for arbitrary T :

$$-1 \leq z_0 \leq 0.$$

Proof: There are two parts. By Lemma 5.2:

- (i) $-\beta(r_1-r_2) + (1-r_2) \leq \beta(r_1-r_2) + (1-r_2)r_1$
- (ii) $\beta(r_1-r_2) + (1-r_2)r_1 \leq 0.$

Proceeding with part (i), first collect terms with β on the left and all else on the right:

$$-2\beta(r_1-r_2) \leq (1-r_2)(1+r_1).$$

Substituting in the definition of β with $\tau = 0$, and multiplying by $(\tau_1 - \tau_2)$, we get

$$2\tau_1(r_1 - r_2) \leq (1 - r_2)(1 + r_1)(\tau_1 - \tau_2).$$

Separate τ_1 terms from τ_2 terms:

$$\tau_2(1 + r_1)(1 - r_2) \leq \tau_1(1 + r_2)(1 - r_1),$$

and multiply by $T[\tau_1 \tau_2(1 - r_2)(1 - r_1)]^{-1}$:

$$\left(\frac{T}{\tau_1}\right)\left(\frac{1 + r_1}{1 - r_1}\right) \leq \left(\frac{T}{\tau_2}\right)\left(\frac{1 + r_2}{1 - r_2}\right).$$

Letting $x_1 = T/\tau_1$ and $x_2 = T/\tau_2$ ($x_2 > x_1 > 0$), define

$$f_3(x) = x \left(\frac{1 + e^{-x}}{1 - e^{-x}} \right) \text{ so that the relation becomes}$$

$f_3(x_1) \leq f_3(x_2)$. We need to show that $f_3(x)$ is monotonic increasing for $x > 0$, thus

$$\frac{d}{dx}[f_3(x)] = \frac{1 - e^{-2x} - 2xe^{-x}}{(1 - e^{-x})^2} \geq 0 \quad (x > 0).$$

Multiply through by $\frac{1}{2}e^x(1 - e^{-x})^2$ so that

$$\frac{1}{2}(e^x - e^{-x}) - x \geq 0.$$

Expanding e^x and e^{-x} and summing terms, we obtain

$$\frac{x^3}{3!} + \frac{x^5}{5!} + \dots \geq 0,$$

which is obviously true for positive x . This completes part (i).

Proceeding with part (ii), by substituting in the definition of β with $\tau = 0$, and multiplying by $(\tau_1 - \tau_2)$, we get

$$\tau_1(r_1 - r_2) + (\tau_1 - \tau_2)(1 - r_2)r_1 \leq 0.$$

Expanding, canceling like terms, and rearranging,

$$\tau_1 r_2 (1 - r_1) \leq \tau_2 r_1 (1 - r_2),$$

which, by multiplying by $[T r_1 r_2]^{-1}$, becomes

$$\frac{r_1^{-1} - 1}{T/\tau_1} \leq \frac{r_2^{-1} - 1}{T/\tau_2}.$$

Letting $x_1 = T/\tau_1$ and $x_2 = T/\tau_2$ ($x_2 > x_1 > 0$), define

$f_4(x) = \frac{e^x - 1}{x}$, so that the relation becomes

$f_4(x_1) \leq f_4(x_2)$. We need to show that $f_4(x)$ is monotonic increasing for $x > 0$:

$$\frac{d}{dx}[f_4(x)] = \frac{x e^x - e^x + 1}{x^2} \geq 0 \quad (x > 0).$$

Multiply by x^2 and rearrange terms:

$$xe^x \geq e^x - 1.$$

Expanding e^x and subtracting x from both sides, we arrive at

$$x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots \geq \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

which is true for positive x , since each term on the left is greater than the corresponding term on the right.

This completes part (ii) and finishes the proof.

Figure 5.1: Linear second order compartmental model.

Figure 5.2: Response to a bolus injection of 1 mg/kg MTX into the plasma compartment. (a) Concentrations in the plasma compartment. P1 is the response of equation (5.18) (which was fit to the observed data "x"), and P2 is the response of equation (5.20) (which was reported in [13]). (b) Concentrations in the combined liver and gut lumen compartment. LGL1 is the response of equation (5.19) (from the same model as P1), and LGL2 is the response of equation (5.21) (from the same model as P2).

Figure 5.3: Example 1. Unconstrained identification and control of a time-varying Type I system. (a) MTX concentration in the plasma (dotted line is the target concentration); (b) MTX infusion rate into the plasma compartment; (c)-(f) estimated ARMAX parameters, a_1 , a_2 , b_1 , and b_2 (the dotted line in each case is the actual value); (g)-(j) estimated steady state gain, $\text{Re}\{r_1\}$, $\text{Re}\{r_2\}$, and z_0 (the dotted line in each case is the actual value). This figure is continued on subsequent pages.

Figure 5.4: Example 1. Constrained identification and control of a time-varying Type I system. Organized as in Figure 5.3 over subsequent pages.

Figure 5.3 (continued)

Figure 5.4 (continued)

Figure 5.3 (continued)

Figure 5.4 (continued)

Figure 5.5: Example 2. Long-term unconstrained identification and control of a Type I system. (a) MTX concentration in the plasma (dotted line is the target concentration); (b) MTX infusion rate into the plasma compartment; (c)-(f) estimated ARMAX parameters, a_1 , a_2 , b_1 , and b_2 (the dotted line in each case is the actual value); (g)-(j) estimated steady state gain, $\text{Re}\{r_1\}$, $\text{Re}\{r_2\}$, and z_0 (the dotted line in each case is the actual value). This figure is continued on subsequent pages.

Figure 5.6: Example 2. Long-term constrained identification and control of a Type I system. Organized as in Figure 5.5 over subsequent pages.

Figure 5.5 (continued)

Figure 5.6 (continued)

Figure 5.5 (continued)

Figure 5.6 (continued)

Figure 5.7: Example 3. Unconstrained identification and control of a time-varying Type II system. (a) MTX concentration in the liver/gut lumen (dotted line is the target concentration); (b) MTX infusion rate into the plasma compartment; (c)-(f) estimated ARMAX parameters, a_1 , a_2 , b_1 , and b_2 (the dotted line in each case is the actual value); (g)-(j) estimated steady state gain, $\text{Re}\{r_1\}$, $\text{Re}\{r_2\}$, and z_0 (the dotted line in each case is the actual value). This figure is continued on subsequent pages.

Figure 5.8: Example 3. Constrained identification and control of a time-varying Type II system. Organized as in Figure 5.7 over subsequent pages.

Figure 5.7 (continued)

Figure 5.8 (continued)

Figure 5.7 (continued)

Figure 5.8 (continued)

Table 5.I: Summary of continuous time theorems.

THEOREM	MODEL TYPE	ADDITIONAL ASSUMPTIONS [*]	RESULTS
II.1	I,II,III		Poles are real, distinct, and negative. System is BIBO stable.
II.2	I		$p_1 < s_0 < p_1 < 0$
II.3	I	$a_{22} \geq a_{11}$	$\frac{1}{2}(p_1+p_2) \leq s_0 \leq p_1$
		$a_{11} \geq a_{22}$	$p_1 \leq s_0 \leq \frac{1}{2}(p_1+p_2)$
		$a_{22} = a_{11}$	$s_0 = \frac{1}{2}(p_1+p_2)$
II.4	II		There is no zero ($s_0 \rightarrow -\infty$)
II.5	III	$a_{22} \geq a_{11}$	$-\infty \leq s_0 \leq p_1$
		$a_{11} \geq a_{22}$	$-\infty \leq s_0 \leq \frac{1}{2}(p_1+p_2)$
		$a_{22} = a_{11}$	$-\infty \leq s_0 \leq \frac{1}{2}(p_1+p_2)$

^{*} For all Types (I, II, and III), $a_{ii} < 0$, $a_{ij} > 0$ ($i \neq j$) by definition.

Table 5.II: Summary of discrete time theorems.

THEOREM	MODEL TYPE	ADDITIONAL ASSUMPTIONS [†]	RESULTS
III.1	I,II,III		Poles are real, and $0 < r_2 < r_1 < 1$
III.2	I		$r_2 < z_0 < r_1$
III.3 [†]	I	$a_{22} \geq a_{11}$	$(r_1 r_2)^{\frac{1}{2}} \leq z_0 < r_1$
		$a_{11} \geq a_{22}$	$r_2 < z_0 \leq \frac{1}{2}(r_1 + r_2)$
		$a_{22} = a_{11}$	$(r_1 r_2)^{\frac{1}{2}} \leq z_0 \leq \frac{1}{2}(r_1 + r_2)$
III.4 [†]	II		$-1 \leq z_0 \leq 0$
III.5 [†]	III	$a_{22} \geq a_{11}$	$-1 \leq z_0 \leq r_1$
		$a_{11} \geq a_{22}$	$-1 \leq z_0 \leq \frac{1}{2}(r_1 + r_2)$
		$a_{22} = a_{11}$	$-1 \leq z_0 \leq \frac{1}{2}(r_1 + r_2)$

[†] For all Types (I, II, and III), $a_{ii} < 0$, $a_{ij} > 0$ ($i \neq j$) by definition.

[†] Given T, a more restrictive upper or lower bound may be calculated. See text.

Table 5.III: Type I models of the plasma compartment.

		P1 [†]	P2 [†]	P3 [‡]
	G_{ss}	233.78	295.39	366.99
Continuous Time Parameters	τ	14.683	104.41	194.13
	τ_1	148.41	314.40	480.39
	τ_2	5.2105	56.880	108.55
Discrete Time ARMAX Parameters	a_1	1.3499	1.9001	1.9446
	a_2	-0.37036	-0.90140	-0.94509
	b_1	16.056	8.3879	6.7278
	b_2	-11.482	-7.9958	-6.5568
Z-domain Parameters	r_1	0.96687	0.98422	0.98965
	r_2	0.38305	0.91585	0.95498
	z_0	0.71513	0.95324	0.97457

[†] P1 is our fit to two human subjects (equation (5.18)).

[†] P2 is the model from [13] (equation (5.20)).

[‡] We assumed P1 is due to a random drift in the time constants and steady state gain from P2 (the so called "typical" human response). We constructed P3 using an equal but opposite random drift for the purpose of setting constraint bounds.

Table 5.IV: Type II models of the combined liver and gut lumen compartment.

		LGL1 [*]	LGL2 [†]	LGL3 [‡]
	G_{ss}	2075.6	2739.8	3404.0
Continuous Time Parameters	τ	0	0	0
	τ_1	148.41	314.40	480.39
	τ_2	5.2105	56.880	108.55
Discrete Time ARMAX Parameters	a_1	1.3499	1.9001	1.9446
	a_2	-0.37036	-0.90140	-0.94509
	b_1	24.670	1.8503	0.80080
	b_2	17.753	1.7874	0.78587
Z-domain Parameters	r_1	0.96687	0.98422	0.98965
	r_2	0.38305	0.91585	0.95498
	z_0	-0.71960	-0.96599	-0.98135

^{*} LGL1 is our estimated second compartment (equation (5.19)).

[†] LGL2 is the second compartment from [13] (equation (5.21)).

[‡] As we did for P1, P2, and P3, we assumed LGL1 is due to a random drift in the time constants and steady state gain from LGL2 (the so called "typical" human response). We constructed LGL3 using an equal but opposite random drift for the purpose of setting constraint bounds.

Table 5.V: Constraint bounds (based on models P1 and P3 in Table 5.III) for control of [MTX] in the plasma compartment.

	MINIMUM	MAXIMUM	INCREMENTAL [†]
$G_{ss} (*)$	200	400	5%
a_1	0.9	2.0	FR
a_2	-1.0	-0.1	FR
b_1	6.0	18.0	FR
b_2	-13.5	-6.0	FR
$r_1, r_2 (*)$	0.3	0.992	FR
$z_0 (*)^{\ddagger}$	0.3	0.992	FR

FR = Full Range.

[†] Constraints for the starred quantities were developed in Chapter 4 (Paper 1), and are listed in Table 5.VII for convenience.

[‡] Incremental bounds are based on the constrained estimates from the previous sampling time.

[§] The specified bounds are the initial limits. The actual zero constraints (inequalities) are iteratively refined, since they depend on the identified poles (see text).

Table 5.VI: Constraint bounds (based on models LGL1 and LGL3 in Table 5.IV) for control of [MTX] in the combined liver and gut lumen compartment.

	MINIMUM	MAXIMUM	INCREMENTAL [†]
$G_{ss} (*)$	1825	3650	5%
a_1	0.9	2.0	FR
a_2	-1.0	-0.1	FR
b_1	0.1	30.0	FR
b_2	0.1	25.0	FR
$r_1, r_2 (*)$	0.3	0.992	FR
$z_0 (*)^{\ddagger}$	-0.992	-0.001	FR

FR = Full Range.

* Constraints for the starred quantities were developed in Chapter 4 (Paper 1), and are listed in Table VII for convenience.

† Incremental bounds are based on the constrained estimates from the previous sampling time.

‡ The specified bounds are the absolute limits for the zero location. These bounds are initially imposed as inequality constraints. The actual zero constraint (equality) is iteratively refined, since it depends on the identified poles (see text).

Table 5.VII: Constraint equations.

Individual Parameters	$\theta_i \geq \theta_{i_{\min}}$ $\theta_i \leq \theta_{i_{\max}}$
Steady State Gain [†]	$\sum b_i + G_{s_{\min}} \sum a_i \geq G_{s_{\min}}$ $\sum b_i + G_{s_{\max}} \sum a_i \leq G_{s_{\max}}$
Minimum and Maximum Positive Real Poles (Approximate)	$-r_{\max}^2 a_1 + (r_{\min} - 2r_{\max}) a_2 \leq -r_{\max}^2 r_{\min}$ $r_{\max} a_1 + a_2 \leq r_{\max}^2$ $a_2 \leq 0$
Minimum and Maximum Zero [†]	$\text{sign}(G_s) * [b_1 z_{0_{\min}} + b_2 \leq 0]$ $\text{sign}(G_s) * [b_1 z_{0_{\max}} + b_2 \geq 0]$
Zero Location	$b_1 z_0 + b_2 = 0$

[†] This constraint is valid only when all poles lie within the unit circle (e.g., $r_{\max} < 1$).

[†] This constraint is valid only when all poles and zeros lie within the unit circle (e.g., $r_{\max} < 1$, $z_{0_{\min}} > -1$, and $z_{0_{\max}} < +1$).

Table 5.VIII: Ensemble averages for Example 1, control of a time-varying Type I system (plasma [MTX]).

		Constrained	Unconstrained
MSE ($\mu\text{g/ml}$) ²		0.4094 (0.0001) [†]	0.4114 (0.0079) [†]
Mean Infusion Rate $\pm$ Std (mg/kg/min)		0.0312 $\pm$ 0.0571 (0.0223 $\pm$ 0.0004) [†]	0.0316 $\pm$ 0.0576 (0.0224 $\pm$ 0.0041) [†]
Over-shoot ($\mu\text{g/ml}$)	Max	3.6572	3.5180
	Mean $\pm$ Std	2.8370 $\pm$ 0.5642	2.6973 $\pm$ 0.6229
3% Set-ling Time (min)	Max	60	∞
	Mean $\pm$ Std	40.0 $\pm$ 11.18	113.6 $\pm$ 91.3
MSE (θ)		6.3611	14.4619
MSE (a_1)		0.0515	0.1342
MSE (a_2)		0.0453	0.1095
MSE (b_1)		5.4321	5.1151
MSE (b_2)		19.916	52.489
MSE (G_{ss})		252.5	12000.
MSE (r_1)		0.0002	0.0038
MSE (r_2)		0.0466	0.1161
MSE (z_0)		0.0589	0.1827

MSE = Mean Squared Error.

[†] Last 150 minutes.

Table 5.IX: Measurements of interest for Example 2, control of a time-invariant Type I system (plasma [MTX]).

	Constrained	Unconstrained
Peak Overshoot (%)	103.68	844.56
MSE (Plasma [MTX]) ($\mu\text{g/ml}$) ²	0.4061	22.7851
Total Input (mg/kg/min)	12.366	19.007
MSE (θ)	3.9678	54.0997
MSE (a_1)	0.0236	2.1866
MSE (a_2)	0.0224	2.0826
MSE (b_1)	6.9370	45.483
MSE (b_2)	8.8882	166.65
MSE (G_{ss})	0.0001	0.0000
MSE (r_1)	0.0001	1.9619
MSE (r_2)	0.0240	0.3953
MSE (z_0)	0.0186	1.0435

MSE = Mean Squared Error.

Table 5.X: Measurements of interest for Example 3, control of a time-varying Type II system (liver and gut lumen [MTX]).

		Constrained	Unconstrained
Peak Error For First 100 Minutes of Disturbance ($\mu\text{g}/\text{mg}$)		-7.308	55.981
Total Input For First 100 Minutes of Disturbance (mg/kg)		0.65	10.54
MSE (LGL [MTX]) ($\mu\text{g}/\text{mg}$) ²	Total Time	66.429	176.858
	Before Setpoint Change	13.052	229.727
MSE (θ)		55.413	46.114
MSE (a_1)		0.0595	0.0314
MSE (a_2)		0.0599	0.0325
MSE (b_1)		31.207	22.794
MSE (b_2)		24.087	23.256
MSE (G_{ss})		19,044.	582,180.
MSE (r_1)		0.0017	0.0041
MSE (r_2)		0.0775	0.0229
MSE (z_0)		0.0166	0.0863

MSE = Mean Squared Error.

CHAPTER 6:

(PAPER 3)

PRELIMINARY STUDIES IN THE ADAPTIVE CONTROL OF MEAN ARTERIAL BLOOD PRESSURE USING CONSTRAINED IDENTIFICATION

6.1 Introduction

Our early work in adaptive control of cardiovascular variables (see, for example, [11], [14], [15], and [13]) led us to design and evaluate a real-time constrained identifier in Chapters 3 and 4. In Chapter 5, we proved its utility for the control of a linear second order compartmental system. In this paper, we begin to explore a more complex, non-linear and time-varying system: the automated control of mean arterial blood pressure using sodium nitroprusside, a vasodilator. We will test our controller on two simplified models of this system, and then prepare to control a more complex model.

The first model is a time-varying ARMAX model that mimics a disturbance seen in prior animal data [12]. The second model, a first order linear continuous-time model developed in [10], has been used previously to test other controllers [9], [7]. By controlling these two models we will have demonstrated the feasibility of our algorithm for controlling mean arterial pressure. The third, a more complex physiologically based model, gives us the ability to determine practical constraints over a variety of realistic conditions, such as during concomitant drug therapy [17]. By linearizing the model at various operating points, we will be able to develop novel constraints for use with our controller.

Sodium nitroprusside (SNP) is a potent vasodilator that is used clinically during hypertensive emergencies to lower blood pressure, during surgery to decrease bleeding, and during recovery to stabilize blood pressure and to decrease blood seepage from graft sites [2]. Since reliable blood pressure sensors are available, the control of MAP (mean arterial blood pressure) has been popular for testing a variety of controllers. From a clinical point of view, it is desirable to automate the control of MAP so that the anesthesiologist or attending nurse can be free to concentrate on more demanding tasks. From a theoretical point of view, the MAP system exhibits

time-varying behavior and non-linearities. Hence it is an excellent system to test new designs.

Koivo (1981) was one of the first to adaptively control MAP in the anesthetized dog [6]. Slate and Sheppard (1982) were fast on his heels, using an adaptive PID with gain scheduling [9]. Arnsparger (1983) and Stern (1985) further explored adaptive control of MAP [1], [11]. Stern compared performance of the self-tuning regulator with that of a highly trained anesthesiologist, and found that they were quite similar.

Kaufman, Roy, and Xu (1984) used a model reference adaptive controller to control MAP (MRAC) [5]. MRACs are similar to the controllers used here, except that the objective of the control is to get the output of the system to follow the trajectory of a secondary model's output. Hence, they are not optimal in the sense of trying to minimize a cost function. Furthermore, the advent of receding horizon controllers similar to the one used in this dissertation obviate the need for MRACs.

An interesting outgrowth of MRACs is the Multiple Model Adaptive Controller (MMAC). MMACs pre-specify a bank of system models. The object during control is to choose the model or combination of models that best fit

the data. The advantages and disadvantages of this type of controller have been discussed earlier. He, et al. (1985) applied an MMAC to the regulation of MAP [4], while Yu, et al. (1987) applied it to the regulation of arterial oxygen concentration [16].

Neat, Kaufman, and Roy (1989) combined MRAC, MMAC, fuzzy control and expert systems to control MAP in a time-varying model. This approach was designed to handle uncertainties in the patient responses by using an expert system to match the dynamic structure of the plant with the best control scheme [7].

Voss, et al. (1987) used the MAP system as a test bed for a new type of predictive controller, CAMAC [14]. The same year, using this algorithm, they were the first to demonstrate feasibility of automated concomitant drug therapy by simultaneously adaptively controlling MAP and cardiac output in anesthetized dogs [15].

Timmons, et al. (1991) developed an approach to identify a drifting baseline during adaptive control. They demonstrated improvements over other identification techniques during the adaptive control of MAP in anesthetized dogs [13].

Adaptive control offers the potential for nearly optimal control. These controllers identify an input/output model of the controlled process, then use it to modify the controller parameters either directly or indirectly. However, input/output models are not physiologically based, hence noise and non-linearities may corrupt the parameter estimates.

By using a priori knowledge about the system, linear constraints can be developed specifying allowable ranges for the model parameters. In Chapter 4, constraints were developed to impose bounds on each of the parameters, on the minimum and maximum settling time, on certain pole and zero locations, on the steady state gain (patient sensitivity), and on the baseline level (the unforced MAP). The constraint equations are summarized in Table 6.I. We shall develop additional constraints in this chapter.

Using the algorithm previously developed in Chapters 3 and 4, these constraints can be imposed in real-time to improve the identified model, thus decreasing the controller's susceptibility to errors due to noise and non-linearities.

Given the constraints and algorithm, we will demonstrate improved control in simulation. For illustration purposes, a time-varying ARMAX model ([12]) will be used to simulate our first "patient". Another model, developed in [10], will be used to simulate our second "patient". Finally, a recently developed non-linear model ([17]) shall be used to develop additional constraints.

6.2 Constrained Identification

A time series ARMAX model (autoregressive moving average model with exogenous inputs) can be used to model the MAP/SNP dose response characteristics. Consider the ARMAX model

$$\begin{aligned}
 y(t) &= a_1 y(t-1) + a_2 y(t-1) + \dots + a_{n_1} y(t-n_1) \\
 &\quad + b_1 u(t-d) + b_2 u(t-d-1) + \dots + b_{n_2} u(t-d+1-n_2) \quad (6.1) \\
 &\quad + e(t) \\
 &= \phi^T(t-1)\theta + e(t),
 \end{aligned}$$

where $n_1 \geq n_2$, d is the input/output delay, and where the vectors θ and $\phi(t-1)$ are defined by

$$\begin{aligned} \theta \triangleq [a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}]^T \\ \phi(t-1) \triangleq [y(t-1), \dots, y(t-n_1), \\ u(t-d), \dots, u(t-d+1-n_2)]^T. \end{aligned} \quad (6.2)$$

Generally, the parameter vector θ is unknown, and is to be estimated.

Assume the following linear constraints apply to θ :

$$\begin{aligned} m_i^T \theta &= k_i, & i &= 1, \dots, n_4 \\ l_i^T \theta &\leq c_i, & i &= 1, \dots, n_5 \end{aligned} \quad (6.3)$$

where m_i and l_i are p length vectors and k_i and c_i are scalars. For ease of notation, we may rewrite (6.3) as

$$\begin{aligned} M \theta &= K \\ L \theta &\leq C \end{aligned} \quad (6.4)$$

where K and C are n_4 and n_5 length vectors made up of the k_i and c_i respectively, and M and L are $n_4 \times p$ and $n_5 \times p$ matrices defined as

$$\begin{aligned} M &= [m_1 | m_2 | \dots | m_{n_4}]^T \\ L &= [l_1 | l_2 | \dots | l_{n_5}]^T \end{aligned} \quad (6.5)$$

At time t , we may solve for the unconstrained least squares estimate, θ_f (f for "free"), by recursive least squares (e.g., see [3]):

$$\begin{aligned} \theta_f(t) = & \theta_f(t-1) \\ & + \frac{P(t-2)\phi(t-1) \cdot [y(t) - \phi(t-1)^T \theta_f(t-1)]}{\alpha(t-1) + \phi(t-1)^T P(t-2)\phi(t-1)} \end{aligned} \quad (6.6)$$

$$\begin{aligned} P(t-1) = & \frac{1}{\alpha(t-1)} \cdot \left[P(t-2) - \right. \\ & \left. \frac{P(t-2)\phi(t-1)\phi(t-1)^T P(t-2)}{\alpha(t-1) + \phi(t-1)^T P(t-2)\phi(t-1)} \right] \end{aligned} \quad (6.7)$$

where $P(t)$ is proportional to the parameter error covariance matrix (it is the inverse of the data-covariance matrix), and $\alpha(t)$ is a scalar forgetting factor. The recursive least squares estimate minimizes the sum of the squared prediction errors:

$$\min_{\hat{\theta}(t)} J = \frac{1}{2} \sum_{k=1}^t w_k (y(k) - \phi^T(k-1) \hat{\theta}(t))^2, \quad (6.8)$$

where $\hat{\theta}$ is the vector of parameters to be estimated, and the w_k are defined by the $\alpha(t)$.

In real-time, we can solve for $\hat{\theta}$ subject to the constraints in (6.4) by formulating the problem as the unconstrained estimate plus correction terms:

$$\theta_c(t) = \theta_f(t) - P(t-1)M^T \mu(t) - P(t-1)L^T \lambda(t) \quad (6.9)$$

where θ_c is the constrained solution, and $\mu(t)$ and $\lambda(t)$ are vectors of Lagrange multipliers associated with,

respectively, the equality and inequality constraints in (6.4). The multipliers are readily determined using quadratic programming techniques. We use an enhanced version of positive semi-definite complementary linear programming. Details of the algorithm are in Chapter 3.

6.3 Control of a Linear Time-Varying ARMAX Model

6.3.1 The ARMAX Model

In modeling the MAP/SNP system, the $y(t)$ in eq. (6.1) would correspond to measurements of the MAP at time t , while the $u(t)$ would correspond to the SNP infusion rate at time t . To form a time-varying ARMAX model of the MAP/SNP system, we sampled the MAP and infused a random SNP rate into an anesthetized dog every 30 seconds. Using recursive least squares (eqs. (6.6) and (6.7)), we then tracked the parameter changes over time. In one instance, an unknown disturbance occurred to the system. Before to the disturbance, the parameters converged to one set of values. After the disturbance, the parameters re-converged to a different set of values. The primary difference between these two models was a three-fold change in the steady state gain [12].

To approximate the disturbance in simulation, we therefore defined a *before* model as the model before the disturbance, and an *after* model as the same model with the steady state gain increased by a factor of 3. These models are listed in Table 6.II. Their step responses are plotted in Figure 6.1. For the following control example, we reproduce the unknown disturbance in simulation by ramping the parameters between the *before* and *after* model. A zero mean gaussian white noise ($\sigma^2=4$ mmHg²) was added to the output. To obtain a resting MAP, we added a background level of 130 mmHg.

6.3.2 Controller Setup

For system identification, we followed the previous example in [13]. Assuming a 30 second sampling interval, we identified 2 autoregressive terms (*a*'s) and 3 exogenous input terms (*b*'s) with an input/output delay of 1 sample. We used the FI approach from [13] to estimate the background level. The FI approach separates out low frequency components from the input and output data. Once removed, the dynamic ARMAX parameters are estimated. Then, the FI approach uses the low frequency components of the data to estimate the background. This approach

requires that the data be low pass filtered. We used a five point averager to do the filtering.

For the constrained identification, we imposed constraints on the minimum and maximum individual parameters, the minimum and maximum settling time (hence stability was constrained), the maximum radius of the zero's, the minimum and maximum steady state gain, and the minimum and maximum background level (the unforced MAP). Incremental bounds were placed on the steady state gain and the background, so that the allowable change at any one sampling time was restricted. The constraint bounds and their incremental limits are listed in Table 6.III.¹

The predictive controller CAMAC was used to generate the SNP infusion rates [14]. At each sampling time, CAMAC attempts to specify the step input that would drive the output to its desired value in K sampling intervals. K must be greater than or equal to the input/output delay d . For $K = d$, the control specifies the input for minimum variance control, while for $K \geq \tau_{\text{set}}$ (the settling time), the control specifies the steady state input.

¹ Since this work pre-dated that of Chapter 4, re-identification upon system changes, as suggested in that Chapter, was not implemented.

Thus, CAMAC determines the next input by minimizing the expected K-step ahead control error:

$$\min_u \mathcal{E} \left\{ \left[y^*(t+K) - y(t+K) \right]^2 \right\} \quad (6.10)$$

where $y^*(t)$ is the desired output at time t .

Since both MAP models have their zeros strictly inside the unit circle, minimum variance control can be used. However, we have the physical constraint that drug cannot be removed once it is injected. Thus, we must adjust K until the control does not depend on negative inputs. In preliminary simulations, we have found that K equal to $d+2$ ($1\frac{1}{2}$ minutes) is a good choice.

6.3.3 Simulation

Two simulations were run to compare adaptive control with and without the constrained identifier. These are plotted in Figure 6.2. To avoid initial open loop probing, we started control with preset parameter estimates. These initial parameters were selected to over-estimate the reactivity of the system, and are listed in Table 6.II. The initial target MAP was specified as 100 mmHg, a 30 mmHg drop from the background

level. To allow fast identification during start-up, the incremental bounds on steady state gain were relaxed to allow larger changes than normal (see Table 6.III). After $2\frac{1}{2}$ minutes of control, the incremental bounds are reset to normal. After 15 minutes of additional control, the transition from the before model to the after model began unknown to the controller ($t=17\frac{1}{2}$ minutes, 6.2, first bar). The transition was complete in $2\frac{1}{2}$ minutes ($t=20$ minutes). The transition time is indicated in Figure 6.2 by vertical bars. Control was then allowed to continue for $17\frac{1}{2}$ minutes, after which the setpoint was changed to 80 mmHg ($t=37\frac{1}{2}$ minutes). As for start-up, the incremental constraints on the steady state gain were relaxed to allow larger changes than normal. After $2\frac{1}{2}$ minutes, they were returned to normal ($t=40$ minutes), and control was continued for 15 minutes more. Summary statistics are listed in Table 6.IV.

Three important points are demonstrated by these simulations. First, start-up with the constrained identifier produces smoother and better control than without (note that the unconstrained controller (Figure 6.2 left) momentarily turns off the infusion of SNP at $t=4$ minutes). Second, large oscillations occur during and following the model transition period for the unconstrained controller, while only small oscillations

occur for the constrained controller. Furthermore, residual slow oscillations remain for the unconstrained controller until the setpoint change. These later oscillations are completely removed when the constraints are used. Finally, without constraints, the controller produces a large overshoot at the final setpoint change ($t=37\frac{1}{2}$), and the MAP momentarily becomes very low (62.4 mmHg). When constraints are used, however, the MAP is driven to the new setpoint with only minor overshoot. Overall, for nearly the same amount of drug infused, when constraints are used the mean squared control error is reduced by 30% (see Table 6.IV).

6.4 Control of a Linear Continuous Time Model

Slate and Sheppard (1979) developed a linear model of the human MAP/SNP dose response characteristics [10]. It has one pole and no zero, with two delays in the inputs (the first to reflect the input/output delay, the second to reflect a supposed re-circulation effect):

$$\frac{\Delta MAP(s)}{\Delta SNP(s)} = K_s e^{-T_i s} \cdot \frac{(1 + c e^{-T_c s})}{(1 + \tau s)} \quad (6.11)$$

where s is the Laplace operator, $K_s = -0.72 \text{ mmHg} \cdot \text{ml}^{-1} \cdot \text{hr}$, $T_i = 30 \text{ seconds}$, $T_c = 45 \text{ seconds}$, $c = 0.4$, and $\tau = 50$

seconds. We defined this continuous time model as SL1. For illustration purposes, define SL2 as the same model, but with K_s doubled (this value is well within the natural variabilities of the human MAP/SNP system). The sampled impulse response for SL1 is shown in Figure 6.3 for a 20 second sampling interval. Note that, at this sampling rate, there is a delay of two samples before an effect is seen. If the continuous time system is mapped to a discrete time ARMAX model, 1 autoregressive term (a_1) and 4 exogenous input terms (b 's) with a delay (d) of 2 samples (40 seconds) will be needed to capture the behavior of this model (see Table 6.V, models SL1 and SL2). Therefore, in the following control example, 1 a and 4 b 's are estimated. The controller's prediction horizon was set to 8 steps (2 2/3 minutes). This horizon predicts beyond the transient effects caused by the zeros, hence zero constraints were not necessary. The constraint bounds used for control are listed in Table 6.V. To avoid startup differences between the constrained and unconstrained controller, the correct initial parameters were specified (Table 6.V, SL1).

A fourth order Runge-Kutta algorithm was used to integrate the continuous time plant. The plant background level was set at 140 mmHg. The control objective was to lower and maintain the MAP at 100 mmHg.

For the first half of the simulation, SL1 was used as the "patient". As a challenge to the controller, after $33 \frac{2}{3}$ minutes the steady state gain was doubled (unknown to the controller) (i.e., the "patient" was switched to model SL2 in Table 6.V). The resulting control is plotted in Figure 6.4.

Shortly after the change in the plant's steady state gain, the gain of the unconstrained controller became large (the controller had identified the system as unstable) so that a large SNP infusion rate was administered. This action caused the MAP to fall dangerously low (~ 190 mmHg) (Figure 6.4, top plot, dashed line, $t=37$ minutes).² The constrained controller, on the other hand, smoothly reduced the input to bring the MAP back to the target level. Despite an almost immediate reduction in the infusion rate, the MAP still dropped to 73 mmHg (Figure 6.4, top plot, dotted line, $t=37$ minutes). The controller performance could probably have been improved further if the least squares forgetting factor had been adjusted (as suggested in Chapter 4). Nevertheless, this example demonstrates the improved safety possible with constraints. The variance

² This model is a linear approximation to the human response. This unrealistic MAP indicates one of the limitations of the model. Nevertheless, this model is adequate for illustration purposes.

of the control error was 1236 mmHg² without constraints, and 83.4 mmHg² with constraints. The constraints thus reduced the variance 15 fold.

6.5 Constraints for a Physiologically Based Model

Recently, Xue, Timmons, and Katona (1991) developed a physiologically based model of the cardiovascular system [17]. It incorporates a modified Windkessel model of the circulation and a general pharmacokinetic/pharmacodynamic (PK/PD) model of drug uptake and distribution. When the PK/PD parameters are adjusted to reflect the known effect of SNP on the peripheral resistance and vascular compliance (see Tables 6.VI and 6.VII), the resulting MAP behavior is strikingly realistic. The steady state effect of SNP on the PK/PD CV (pharmacokinetic/pharmacodynamic cardiovascular) model is illustrated in Figure 6.5. We have also plotted the bolus response in Figure 6.6. Since dobutamine (an inotrope) is often used concomitantly with SNP, PK and PD parameters for the cardiovascular model were determined for this drug also [17]. Finally, the model is capable of simulating chronic and acute heart failure, hypertension, and a healthy heart. In this study, we consider the healthy heart only.

To determine constraints for the healthy model, we linearized the model about several operating points. We probed the system with a pseudorandom binary input sequence (SNP infusion rate) with and without a concomitant infusion of dobutamine (see Table 6.VIII). We used a sampling period of 30 seconds. Once input and output data was obtained, we used the Matlab Identification Toolbox to determine the best model structure [8]. The model selection was based on a log loss function, the autocorrelation of the residuals, the crosscorrelation of the residuals with the input, and the potential for pole/zero cancellation given a range of ± 3 standard deviations on the poles and zeros. After consideration of these criteria, we determined that the best overall ARMAX model structure consisted of 2 autoregressive terms (a's) and 3 exogenous input terms (b's) with a delay of 2 samples (1 minute). Table 6.VIII lists the estimated model parameters given this structure.

Given these ARMAX parameter estimates, we could then construct ranges for each parameter of the model. These ranges are listed in Table 6.IX (excluding the a's). By forming a scatter plot of the estimated autoregressive terms (a's) surrounded by ± 3 standard deviations, we were able to construct a novel constraint region for the a's (see Figure 6.7, top plot). The corner points for the

constraint region are listed in Table 6.X. In the bottom right plot in Figure 6.7, we have translated these constraints to the Z-plane.

6.6 Discussion

At several operating points of the PK/PD CV model, the best model structure had a delay of only 1 sampling period (as opposed to the overall best model structure which had a delay of 2 sampling periods). As the operating point or the patient condition changes, it would thus be useful to be able to adjust the number of delays. This adjustment could be done if 4 b -parameters were estimated. Then, given the parameter error covariance matrix from (6.7), the first and last b -parameter could be tested to see if it were statistically different from zero. Depending on the outcome of this test, the first or last b -parameter could be set to zero using an equality constraint.

In summary, we have tested our constrained identifier in two simplified models of the MAP/SNP system. For each model tested, we showed significant improvements in the control when constraints were imposed on the identification. For the final, physiologically

based system, we did not have the opportunity to test our controller; nevertheless, we were able to construct novel constraints on the a -parameters.

Additional work will be required to test these constraints during control of the PK/PD CV model. Once this goal is reached, the next step would be to analyze input/output data from animals with the intention of refining the constraints and validating the PK/PD CV model. Once accomplished, constrained identification should be tested during the control of an anesthetized animal. Each iterative refinement in the constraints and in our techniques will hopefully bring us closer to our ultimate goal of producing a clinically useful controller.

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STEP RESPONSE

Figure 6.1: Step responses of ARMAX MAP/SNP dog models (deviation from background level).

SIMULATION

Unconstrained Constrained

Figure 6.2: Unconstrained (left) vs. constrained (right) identification during control of the ARMAX MAP/SNP dog models. The top trace in each is the simulated MAP (jagged line) with the setpoint superimposed (straight line). The bottom trace is the SNP infusion rate. The vertical bars indicate when the model parameters were ramped from the before model to the after model.

Figure 6.3: Discretized impulse response of SL1.

Figure 6.4: Unconstrained (dashed) vs. constrained (dotted) identification during control of SL1/SL2. Top trace is the simulated MAP, bottom trace is the SNP infusion rate. Initially, SL1 is simulated. At $t = 33 \frac{2}{3}$ minutes, the model parameters for SL2 are switched in. The MAP for the unconstrained controller falls to -190 mmHg shortly after the model switch.

Figure 6.5: The steady state SNP effect on the PK/PD CV model (reproduced from [17]).

Figure 6.6: Response of the PK/PD CV model with a healthy heart to a bolus injection of 250 µg SNP. The top left plot is MAP in mmHg, the top right plot is cardiac output (co), and the bottom left plot is heart rate in beats per minute. Notice that the double dip in the cardiac output resembles SL1's MAP waveform. This waveform may be due to the renin-angiotensin reflex in the model.

A constraints to Z-plane for [232] ARX structure

Figure 6.7: Constraint region for the a -parameters. Top: Scatter plot of the estimated a 's ("x") at different operating points of the PK/PD CV model (from Table 6.VIII). The ellipses (solid lines) show ± 3 standard deviations. The dashed line shows the constraint region (see Table 6.X). Parameters lying above the dotted parabola correspond to positive real poles while those lying below it correspond to complex poles. Bottom Left: The constraint region and the positive real parabola. Bottom Right: The translation of the constraint region to the z -plane.

Table 6.I: Constraint equations.

Individual Parameters	$\theta_i \geq \theta_{i_{\min}}$ $\theta_i \leq \theta_{i_{\max}}$
Minimum and Maximum Positive Real Poles (Approximate)	$-r_{\max}^2 a_1 + (r_{\min} - 2 r_{\max}) a_2 \leq -r_{\max}^2 r_{\min}$ $r_{\max} a_1 + a_2 \leq r_{\max}^2$ $a_2 \leq 0$
Zero Magnitude ^{†,‡}	$\text{sign}(G_{ss}) \cdot [-b_1 z_{\theta_{\max}}^2 - b_2 z_{\theta_{\max}} - b_3] \leq 0$ $\text{sign}(G_{ss}) \cdot [-b_1 z_{\theta_{\max}}^2 + b_2 z_{\theta_{\max}} - b_3] \leq 0$ $\text{sign}(G_{ss}) \cdot [-b_1 z_{\theta_{\max}}^2 + b_3] \leq 0$
Steady State Gain [†]	$\sum b_i + G_{ss_{\min}} \sum a_i \geq G_{ss_{\min}}$ $\sum b_i + G_{ss_{\max}} \sum a_i \leq G_{ss_{\max}}$
Background ^{†,‡}	$\hat{A}(q^{-1})(y_p(t) - Y_{\theta_{\min}}) \leq \hat{B}(q^{-1})(u_p(t) - U_1)$ $\hat{A}(q^{-1})(y_p(t) - Y_{\theta_{\max}}) \geq \hat{B}(q^{-1})(u_p(t) - U_1)$

[†] This constraint is valid only when all poles lie within the unit circle (e.g., $r_{\max} < 1$).

[†] This constraint additionally requires knowledge of the sign of G_{ss} .

[‡] This constraint additionally requires use of the FI (see text).

Table 6.II: ARMAX models of the MAP/SNP dose response based on data from an anesthetized dog. The initial parameters used for control startup are also listed.

		Before [†]	After [†]	Initial [‡]
ARMAX Parameters	a_1	0.7	0.7	0.3
	a_2	-0.06	-0.06	-0.0225
	b_1	-0.072	-0.216	-0.2
	b_2	0.036	0.108	-0.1
	b_3	N/A	N/A	0.0
	d	1	1	1
Steady State Characteristics	G_{ss}	-0.1	-0.3	-0.415
	Y_0	130	130	130
Z-domain Parameters	r_1	0.1	0.1	0.15
	r_2	0.6	0.6	0.15
	z_1	0.5	0.5	0.0
	z_2	N/A	N/A	-0.5

[†] Model parameters identified before an unknown disturbance in an anesthetized dog.

[†] Model parameters identified after the unknown disturbance in an anesthetized dog.

[‡] Initial parameters used for control startup.

Table 6.III: Constraint bounds for identification during control of the ARMAX dog MAP/SNP model in Table 6.II.

	MIN	MAX	INC Normal [†]	INC Startup [†]
a_1	-2.0	2.0	FR	FR
a_2	-2.0	2.0	FR	FR
b_1	-0.4	0.2	FR	FR
b_2	-0.4	0.2	FR	FR
b_3	-0.4	0.2	FR	FR
$G_{ss} (*)$	-1.0	-0.04	20%	50%
$Y_0 (*)$	120	140	2.5	25
$r_1, r_2 (*)$	0.4	0.7	FR	FR
$z_1, z_2 (*)$ ‡	0	0.7	FR	FR

FR = Full Range.

* Constraints for the starred quantities were developed in Chapter 4 (Paper 1), and are listed in Table 6.I for convenience.

† Incremental bounds are based on the constrained estimates from the previous sampling time.

‡ The zero constraints limit the magnitude of the zeros.

Table 6.IV: Descriptive statistics: Control of the ARMAX dog MAP/SNP model.

		Unconstrained	Constrained
Mean Squared Control Error (mmHg ²)	First 10 minutes	133.37	119.13
	From model change to second setpoint [*]	31.29	8.80
	From second setpoint to end [†]	25.89	16.78
	Overall	44.21	30.63
Total infused SNP (mg)		10.00	10.12
Peak overshoot (second setpoint) (mmHg)		17.6	4.2

MSE = Mean Squared Error.

^{*} t = 18 to 37½ minutes.

[†] t = 38 to 55 minutes.

Table 6.V: Discrete time equivalent models of SL1 and SL2, and constraint bounds used for identification during control.

	ARMAX Models		Constraint Bounds [‡]	
	SL1	SL2 [†]	MIN	MAX
a_1	0.672	0.672	0.6	0.8
b_1	-0.104	-0.208	-3.0	0.0
b_2	-0.133	-0.266	-3.0	0.0
b_3	-0.0163	-0.0326	-3.0	0.0
b_4	-0.0774	-0.1548	-3.0	0.0
G_{ss}	-1.0	-2.0	-8.0	-0.5
Y_0	140.	140.	NC	NC

NC = Not constrained.

[‡] The constraints imposing steady state gain was developed in Chapter 4 (Paper 1), and is listed in Table 6.I for convenience. It requires that the model poles be constrained to lie within the unit circle. The bounds on a_1 correspond to settling time bounds of 2 2/3 minutes (min) and 6 minutes (max).

[†] SL2 poles and zeros are identical to SL1, although the steady state gain has been doubled.

Table 6.VI: PK/PD parameters for the CV model.[†]

	SNP	DOB
T_d (sec)	30.0	30.0
τ	5.0	15.0
INP_{in} ($\mu g/sec$)	0.5	1.0
INP_{out} ($\mu g/sec$)	5.0	60.0
EFF_{Ra}	-0.95	-0.2
EFF_{Rv}	-0.2	0.0
EFF_{V0a}	0.2	0.0
EFF_{V0v}	0.22	0.0
EFF	0.0	2.5
EFF_{RR}	0.0	0.3

[†] Parameters as defined in [17].

Table 6.VII: CV model parameters for the healthy heart.[†]

$K = 15$
$daft_1 = 250$
$bvhr = 1.0$
$vol = 3300$

[†] Parameters as defined in [17].

Table 6.VIII: Linearized ARMAX models of the PK/PD CV model.

	CV1	CV2	CV3	CV4	CV5	CV6
DOB ($\mu\text{g/s}$)	10	0	10	0	0	0
SNP (PRBS) ($\mu\text{g/s}$)	2 ± 2	2 ± 2	$2 \pm \frac{1}{2}$	$2 \pm \frac{1}{2}$	$\frac{1}{2} \pm \frac{1}{2}$	$5 \pm \frac{1}{2}$
MAP (mean) (mmHg)	82.2	78.7	81.1	77.4	99.9	63.5
a_1	1.036	1.168	1.324	1.308	0.872	1.330
a_2	-0.258	-0.362	-0.481	-0.478	-0.119	-0.542
b_1	-2.385	-2.019	-2.537	-2.182	-1.027	-0.741
b_2	-0.938	-1.013	-0.315	-0.850	-0.629	-0.314
b_3	0.275	0.510	0.745	0.793	-0.081	0.375
r_1	0.617	$0.584 + 0.146i$	$0.662 + 0.206i$	$0.654 + 0.225i$	0.703	$0.665 + 0.316i$
r_2	0.419	$0.584 - 0.146i$	$0.662 - 0.206i$	$0.654 - 0.225i$	0.169	$0.665 - 0.316i$
z_1	-0.589	-0.813	-0.608	-0.828	-0.428	-0.954
z_2	0.196	0.311	0.484	0.439	-0.184	0.530
G_{ss}	-13.69	-12.97	-13.43	-13.15	-7.03	-3.21

DOB = dobutamine, s = seconds.

Table 6.IX: Suggested individual parameter bounds for use with the PK/PD CV model (Tables 6.VI and 6.VII).

	MIN	MAX
b_1	-3.0	-0.2
b_2	-1.2	0.1
b_3	-0.25	0.1
zeros [†]	0	0.975
G_{ss} [‡]	-16.5	-2.65

[†] The a-parameters are bounded by the constraint region specified in the following table (Table 6.X).

[†] The zero constraints can only impose a maximum magnitude since the zeros span zero. The identified system must be constrained to be stable, and the sign of G_{ss} must be known (and constrained) for these bounds to be imposed.

[‡] The constraints that impose steady state gain (see Table 6.I) require that the poles be constrained to lie within the unit circle.

Table 6.X: Suggested corner points for the constraint region (polygon) surrounding the a-parameters.

$C1 = (0.695, 0)$ $C2 = (0.772, 0)$ $C3 = (1.383, -0.506)$ $C4 = (1.383, -0.622)$
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[†] Points are in the form (a_1, a_2) .

CHAPTER 7 :

CONCLUDING REMARKS

While much has been accomplished, much still remains to be done. At this point, we are poised and ready to tackle a wide range of other "biomedical" systems. In this chapter, we close with a section on future directions. However, before we delve into the future, we will first summarize the past and discuss a few additional points here in the present.

7.1 Summary

We originally hypothesized in Chapter 1 that we could improve adaptive control by improving system identification. To test this hypothesis, we first had to develop the means for imposing known information on the identification. During the search for a suitable real-time algorithm, we chanced upon a new quadratic programming technique. We described it in detail in Chapter 3, where we also proved its validity (hence our rather lengthy treatment).

Having succeeded in finding a suitable real-time quadratic program solver, our next step was to develop an identifier that could impose arbitrary linear constraints. Thus in Chapter 4 (Paper 1) we formulated a quadratic program to do this, solved it, and attached it to a controller. But before we could test our identifier, we needed to put *a priori* knowledge into a usable form. Thus, we developed linear constraints to impose such commonly available information as stability, steady state gain, settling time, baseline, and certain conditions on the poles and zeros. Given these constraints, we then tested our enhanced identifier in simulation by controlling an established benchmark model.

The resulting control performance was impressive. The additional knowledge imposed by the constraints eliminated several problems seen in the unconstrained controller. For example, momentary instabilities were eliminated and disturbance recovery was faster. Furthermore, we found that, upon changes in the plant, the adaptation gain of the identifier could be temporarily increased without adversely affecting the control performance. This gain adjustment allowed us to re-identify the system faster and more accurately than we could without constraints. Using this technique, we were

able to improve the control performance by more than a factor of 100 in some instances.

In Chapter 4, we also demonstrated that the optimally constrained predictor generally leads to better control than the (suboptimal) orthogonally constrained predictor. This result was not expected. A theorem in [1] states that the orthogonal projector will always improve the parameter estimates, whereas the optimal projector will not. However, the proof of the theorem assumes the inputs are uncorrelated with the system noise. These conditions certainly do not arise in closed loop. It is also possible that the noise conditions necessary for the orthogonal projector to outperform the optimal projector never arise in practice.

Also from Chapter 4, we discovered a phenomenon we called the "balloon affect". If one end of a balloon is squeezed, the other end will expand. Similarly, if only one part of a model is constrained, modeling error may be exaggerated in another. If control happens to be sensitive to errors in the unconstrained part, then control may be worse than if the constraints had not been used. This problem may be solved by using a different controller or by adding constraints reflecting the stability conditions. In general, it has been our

experience that if constraints are developed for one part of a model, then similar constraints should be developed for the rest.

Finally, in Chapter 4, we found that untrue constraints can lead to disaster. We discussed several instances in which a seemingly innocuous assumption can lead to incorrect constraints. Thus, it is important to carefully scrutinize *a priori* assumptions about a plant, especially if the plant is non-linear or time-varying.

From our theoretical treatment of constrained identification and control in Chapter 4, we moved to a more practical application in Chapter 5. Many drug delivery systems are based on low order pharmacokinetic/pharmacodynamic models, thus we thought that these systems would provide a practical test-bed for our controller, as well as illustrate to the biomedical community the potential benefits of parameter-constrained adaptive control. To convert our *a priori* knowledge to a usable form, we converted a general second order compartmental model to ARMAX form. We were then able to identify certain conditions on the poles and zero, which we translated into constraints on the ARMAX parameters. To our surprise, when we finished this exercise we found that our constraints applied to a much wider class of

systems. This class can be put into compartmental form by a similarity transform, hence we defined the class as "compartmental-like" systems.

One problem we encountered was that the constraints imposing the zero conditions depended on the location of the constrained poles. Since we could not know the location of the poles before we constrained them, we resorted to a polishing routine to determine the zero constraints. In this routine, we iteratively refined the zero constraints until they converged. Convergence was fast, as it took less than four iterations to meet our 2% error criterion.

These simulations confirmed what we had found in Chapter 4, namely that constrained identification can dramatically improve the control in some instances, and that it improves the identification in almost all instances. Most importantly, for applications to compartmental-like systems, we found that constraints reduce the likelihood of anomalies, and thus improve safety.

In Chapter 6, we initiated a study into another practical application: the control of mean arterial pressure using sodium nitroprusside, a vasodilator. We

successfully controlled the mean arterial pressure (MAP) of two simplified models of this system. The improvements were again dramatic in one example, and significant in the other. In this same chapter, we analyzed a complex physiologically based model of the cardiovascular system. We were able to identify a set of equivalent ARMAX models at various operating points. These equivalent models showed specific ranges for the some of the parameters. We were thus able to develop novel constraints for these parameters. However, this study was incomplete, as we did not have the opportunity to test the effect of the new constraints on the control. In the section on future directions, we discuss the steps necessary to complete this study.

7.2 Discussion

There are several items we did not discuss in the previous chapters. First and foremost, we must discuss the conditions under which constrained identification can hurt control. Also we need to discuss a numerical problem that we encountered in our early work. We will also re-visit our earlier discussion on equality constraints in Chapter 2, then close with a few remarks

on our enhanced complementary linear programming algorithm.

There are several conditions for which constraints do not help the control. We already mentioned that untrue constraints usually degrade controller performance. We have also encountered a few instances in which true constraints degraded the controller performance. Although none of the infractions were great, these instances were crucial for the development of our thinking. They led us to discover several properties of constrained identification for adaptive control, and thus merit discussion here. Each instance occurred after a sudden change in the plant.

In one case, before the plant changed, the constrained parameters were closer to the true plant values than the unconstrained parameters were. It just so happened that the unconstrained parameters were closer to what the true values would be after the change. Hence, after the plant change, the unconstrained parameters were better than the constrained parameters. As would be expected in this instance, control was better with the unconstrained parameters than it was with the constrained parameters during the transient period immediately following the plant change.

In another example, incongruous input and output data from the transition period violated the compartmental modeling conditions. The constrained model predicted that the drug concentration was decaying so rapidly that it would become negative (clearly impossible). Even though the unconstrained parameters were much worse, the unconstrained model happened to make a better prediction. While the constrained parameters were always closer to the actual plant than the unconstrained parameters, they clearly did not capture the time-varying behavior of the system. Thus the constraints, which were developed for time-invariant systems, temporarily became invalid.

A third instance occurred during our early investigative work, and taught us the importance of re-identification. In this instance, the unconstrained controller became momentarily unstable early after the plant change. The control error did not become excessively large, yet it was large enough to re-identify the system. The constrained controller, on the other hand, smoothly increased the input until the control error went to zero (hence the constrained identifier did not get a chance to re-identify the system). At a subsequent change in the setpoint, the controller with the unconstrained identifier smoothly and quickly drove

the system to the new operating point. The constrained identifier, on the other hand, had to re-learn the system while its controller attempted to drive the output to the new setpoint. In the end, the mean squared control error was larger for the constrained system than it was for the unconstrained system, even though the constrained controller never became unstable.

These examples illustrate that constraints are not a panacea for control. There will always be instances when either the constraints become invalid, or by chance, conditions happen to work against the constrained system. Since constraints limit the worst case model, at least catastrophic errors will be avoided. In each of the above instances, it should be noted that the unconstrained parameters could have been much worse, but that by luck they just happened to be better.

Nevertheless, these instances clearly demonstrate the need for a high level supervisory algorithm that monitors the patient and controller and makes adjustments as needed. Fortunately, the constraints themselves (or related conditions) often indicate impending problems. For example, we know that drug concentration cannot be negative. When the predictor violates this condition, we know something is wrong. Several strategies could be

used to remedy this problem. The simplest solution would be to limit the change in the drug infusion rate at any one step. Alternately (or simultaneously), the negative output prediction could be used to trigger a safety mechanism that would decrease the drug infusion rate and begin re-identification. On the other hand, if it is known that the system is jump linear, then the control task and modeling should be re-formulated to use modern jump linear systems theory. Indeed, another application of constrained identification might be for the control of jump linear systems, since, in the above examples at least, it clearly detected system jumps.

Another issue that we must discuss comes from a costly mistake we made early in our investigations. With some embarrassment, we re-count it here. We made the assumption that small constraint errors were not important. We therefore ignored constraint errors (in the form of Euclidean distances) that were smaller than 0.01. However, we did not take into account the effects of data normalizations or the conditioning of the simplex tableau. It is straightforward to show that, depending on the eigenvalues of the complementary simplex tableau constructed from the set of active constraints, a small constraint error could result in a large change in the parameter estimates, or conversely, a large constraint

error could result in a small change in the parameter estimates. Thus, full numerical precision is necessary. In all subsequent studies, we used double precision arithmetic to its fullest precision.

Moving to another topic, we wish to point out once more the advantage of using mixed equality and inequality constraints over using equality constraints alone. We originally conjectured that a mixed equality and inequality constrained estimator would be far more flexible and useful than a simple equality constrained estimator. In retrospect, the closest we came to exact knowledge of a pole or zero location (and hence an equality constraint) was the condition on the zero of a Type II compartmental model in Chapter 5 (Paper 2). The resulting equality constraint represented very useful knowledge, but it depended on the simultaneous imposition of several other inequality constraints (to force the poles to be positive and real). Otherwise, the equality constraint would have been unenforceable. Hence, not once have we encountered a stand-alone equality constraint. Thus our supposition has been confirmed by experience.

Finally, we return to our complementary linear programming algorithm. Our algorithm is ideally suited

for time-limited applications, such as real-time identification and control. For example, in all our simulations (executed on a 16 MHz IBM AT compatible computer), the constraining algorithm (including the constraint polishing time) never took more than 2 seconds. Typically, less than $\frac{1}{2}$ second was required.

For non-time critical applications, our complementary pivoting algorithm is probably as good as any other, if not better. For example, with a simple modification, our algorithm can be used for high accuracy solutions: once the active set of inequality constraints is known, the problem can be re-formulated as an equivalent equality constrained-problem, then re-solved using an accurate matrix inversion algorithm. Very high accuracy, on the other hand, requires specialized algorithms. Nevertheless, these solvers typically need a good "first estimate", hence our algorithm is still useful [2].

7.3 Future Directions

There are several directions where one may proceed from here. The study of the adaptive control of MAP in Chapter 6 should be extended. The constraints developed

for the pharmacokinetic/pharmacodynamic cardiovascular model need to be tested in the control of this model. If their usefulness is established for this model, the next step would be to determine if, with appropriate scaling, they also apply to a live animal model, such as an anesthetized dog. Eventually, we would want to test whether the constraints apply to humans also. If so, then the next step would be to develop the controller so that it could be used clinically to improve patient care.

However, before we could test our controller in a clinical setting, safety features would have to be built into it. These safety features would include the development of a supervisory algorithm to monitor the performance of the identifier and controller. It would enforce prescribed limits on the maximum and minimum infusion rate and the total drug delivered. Additionally, it would sound an alarm if the MAP wanders out of preset bounds.

Another safety feature that should be included concerns the numerical stability of the recursive least squares P -matrix. If this matrix should become ill-conditioned, instability may occur in the identifier. Therefore the P -matrix should be kept well conditioned. Most conditioning algorithms operate by replacing the P -

matrix with a linear combination of itself and the identity matrix. Interestingly, from the point of view of constraining the parameters, this modification can be thought of as combining the optimal projector with the orthogonal projector. Thus, while we have shown that the orthogonal projector by itself can significantly degrade the controller performance, it may be that when combined with the optimal projector it is not so bad. That is, enough should be added to the optimal projector to keep it from becoming ill-conditioned, but not so much should be added that it affects the estimated parameters and the control.

While striving to make the controller safe for the control of MAP, we could also explore other applications. For example, with minor modifications, the constrained identifier could be used for the adaptive control of anesthesia level or blood sugar level. Many other possibilities exist besides these biomedical ones.

Another area that would probably benefit from constrained identification is concomitant drug therapy. For example, many patients receiving sodium nitroprusside also receive other drugs such as dobutamine (an inotrope). Thus, a reasonable goal would be to automate the infusion of both (or more) drugs. Voss, Katona, and

Chizeck (1987) successfully automated the simultaneous control of MAP and cardiac output in anesthetized dogs using sodium nitroprusside and dobutamine [3]. Since this system could be decoupled into two triple-input single output systems for identification purposes [3], our constrained identifier could be used without modification. Many of the constraints developed in Chapter 4 could also be used without modification. The stability constraints and other related constraints for second order systems (such as positive real poles) could be used without change. Steady state gain (for each of the decoupled inputs) could also be imposed. Other constraints, such as background level, might be more difficult to impose, and would require further work.

In addition to using constraints to improve the identification and control, they could also be used to detect system anomalies, as mentioned in Chapter 4. When the system is well behaved, the constraints (if they are true) should not be invoked. If they are invoked, then something may be happening to the system. A measure of the seriousness of the situation would be the projection costs. We have found that noise on the system, as well as minor non-linearities, may cause minor constraint violations, and hence, minor projection costs. When a significant system change occurs, however, the projection

costs can rise dramatically. The constraints could thus be a useful system monitor, and should therefore be considered for further study in this regard.

As mentioned above, the control of jump linear systems might also benefit from the use of constraints, since one of the important tasks when controlling such a system is to determine if and when the system changes. The projection costs might be a useful indicator here too.

Finally, our enhanced positive semi-definite complementary linear programming (CLP) technique has many potential uses. For example, limits are often placed on the maximum drug infusion rates and the total amount of drug infused over a period of time. These types of conditions can be written as constraints on the inputs, as opposed to constraints on the model parameters. For single-input single-output systems, these conditions simply require the infusion rates to be clipped when they exceed the given bounds. For multiple-input multiple-output systems, however, simple saturation can no longer be used. Instead, the control problem must be formulated as a quadratic program subject to the input constraints. While we have only used CLP to solve for the parameter

estimates, it could be used to solve for this problem also.

7.4 Conclusions

Optimally constrained identification for adaptive control can dramatically improve controller performance, especially when combined with re-identification. Common information can thus be used advantageously to improve the control. Other specific information is certainly helpful, as the more knowledge that can be imposed, the better the control. However, constraint assumptions need careful scrutiny, especially if they involve approximations or affect other constraints.

Additionally, we have presented a novel enhancement to positive semi-definite complementary linear programming that can significantly reduce the number of iterations require for a solution. The enhancement requires only a minor modification to existing code.

Most importantly for biomedical applications, the appropriate use of constraints increases the safety of adaptive control. Constrained identification is thus a

useful addition for the safe and effective adaptive control of biomedical systems.

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