INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

U·M·I

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600

Order Number 9202230

**Theoretical calculation of magnetic fields generated by neural
currents**

Ferguson, Archibald Stewart, Ph.D.

Case Western Reserve University, 1991

U·M·I

**300 N. Zeeb Rd.
Ann Arbor, MI 48106**

**THEORETICAL CALCULATION OF MAGNETIC FIELDS
GENERATED BY NEURAL CURRENTS**

by

ARCHIBALD STEWART FERGUSON

**Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy**

**Department of Biomedical Engineering
CASE WESTERN RESERVE UNIVERSITY**

May, 1991

CASE WESTERN RESERVE UNIVERSITY
GRADUATE STUDIES

We hereby approve the thesis of

ARCHIBALD STEWART FERGUSON
candidate for the Ph.D.
degree.*

Signed: Dominique DURAND
(Chairman)
David Kirsch
[Signature]
[Signature]

Date APRIL 24, 1991

*We also certify that written approval has
been obtained for any proprietary material
contained therein.

THEORETICAL CALCULATION OF MAGNETIC FIELDS GENERATED BY NEURAL CURRENTS

Abstract

by

ARCHIBALD STEWART FERGUSON

A new approach to calculating the magnetic field generated by neural currents is developed and applied to specific neural models. The basis for the new model is the calculation of the magnetic field of point sources of current (monopoles) which represents transmembrane currents injected into a volume conductor from patches of membrane. This allows the magnetic field of a neural source to be calculated with much greater accuracy than previous current dipole models.

The magnetic field for a current monopole is initially developed from the known solution for the field of dipole volume currents in both spherical and semi-infinite volume conductors. The same result is then generated from an analysis of Maxwell's equations once a two-part model is adopted for the monopole. By placing two point sources of opposite signs close together, it is possible to simulate a current dipole. It is shown how the monopole solution can be used to readily explain specific characteristics of the dipole field that are otherwise difficult to explain.

The monopole formulation is then applied to calculating an exact solution for the magnetic field of a current filament, a single neuron, and a neural bundle. The conditions under which a current dipole model is an adequate representation are derived. It is shown that the field of a single neuron drops as r^{-2} , compared to r^{-3} for an axon.

An approximate solution for the magnetic field of a monopole in prolate and oblate spheroids is also developed. Although the solution is not exact, it is demonstrated that the mean relative error for the monopole approximation compared to the exact dipole solution is typically less than 5% over an observation surface. The calculation time for the monopole approximation is 9-50 times less than that required for the dipole solution.

*There's a race of men that don't fit in,
A race that can't stay still;
So they break the hearts of kith and kin,
And they roam the world at will.
They range the field and they rove the flood,
And they climb the mountain's crest.
Theirs is the curse of the gypsy blood,
And they don't know how to rest.*

*If they just went straight, they might go far;
They are strong and brave and true;
But they're always tired of the things that are,
And they want the strange and new.
They say: "Could I find my proper groove,
What a deep mark I would make!"
So they chop and change, and each fresh move
Is only a fresh mistake.*

— from "The Men That Don't Fit In" by Robert W. Service

To my parents, Andrew and Ileana Ferguson,
and my advisor, Dr. Dominique Durand.

ACKNOWLEDGEMENTS

After six years spent working on this research, it is impossible to thank all the people who have been a part of the work and made significant contributions towards its final success. However, I wish to express my sincere appreciation and heartfelt gratitude to the following individuals who have been a major part of my life during these years:

Dr. Dominique Durand, my advisor and friend, who believes that life is one endless maze of learning and fun, of wonder and beauty, and of adventure and sport. I am constantly amazed as his desire to learn, his energy to question, his ability to understand, and his compassion for others. I am honored to make claim to being Dr. Durand's first Ph.D. student, and know that I am only the first of many.

My committee members, **Dr. David Ksienski**, **Dr. Robert Collin**, and **Dr. Pedro Diaz** who provided many hours of useful ideas and concepts. I am particularly indebted to Dr. Ksienski for the many late coffees and rice dinners, during which my latest ideas were methodically discussed then destroyed.

My many friends and colleagues at the Applied Neural Control

Laboratory. I learned to appreciate science and was allowed to follow my curiosity by my Master's advisor and laboratory director, **Dr. J.T. Mortimer**. I am forever indebted to my friend and business partner, **Don Atzberger**, for keeping the Macs running, answering questions, restoring crashed disks, and being a fountain of knowledge. I hope my graduation is an incentive for Dr. Durand's next Ph.D. student, **Eddy Warman**, who has worked side-by-side with me over the years. I received an incredible amount of technical support from **Tim Crish** and **Hani Kayyali**, although none of that work ever made it to the dissertation. Maybe in the next one ... Finally, I never received technical help or discussed academic problems with **Nancy Caris**, but I just want her to know that she is my favorite part of ANCL. I will miss her very much when I finally leave.

My very dear friend **Kathy Charlton** who always tried to make me laugh even when she herself was hurting. I tried to use "frostbite" in the dissertation, but it didn't work.

Dr. William P. "Bart" Huebner, the original Green-Bay Packer, who spent a year one semester with me in a class that started my career as a graduate student. I should have known, at that point, that the five year plan was not for us.

My parents, **Andrew and Ileana Ferguson**, who will just have to get used to the fact that I am no longer in school. Thank you for all your love, understanding, and support over the years. Now I will try to settle down.

If any friends or colleagues feel slighted by not being included here, please understand that the dissertation is already past the deadline, the hour is late, and even country-western music is becoming hard to follow.

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	ii
DEDICATION	iv
ACKNOWLEDGEMENTS	v
TABLE OF CONTENTS	viii
LIST OF FIGURES	xii
LIST OF TABLES	xiv
CHAPTER I INTRODUCTION AND HYPOTHESES	1
1.1 Introduction	1
1.2 Statement of Research Hypotheses	4
1.3 Organization of the Dissertation	6
1.4 References	6
CHAPTER II MAGNETIC FIELDS OF CURRENT MONOPOLES IN SPECIAL VOLUME CONDUCTORS	10
2.1 Abstract	11
2.2 Introduction	11
2.3 Theoretical Approach	15
2.4 Sphere	16
2.4.1 Relationship between monopole and dipole solutions	16
2.4.2 Monopole solution and interpretation of results	22
2.5 Semi-Infinite Media	24
2.5.1 Relationship between monopole and dipole solutions	24
2.5.2 Final solution for $\mathbf{B}^m$	28
2.5.3 Interpretation of the results	32
2.6 Discussion	34
2.7 References	40

	<u>Page</u>
CHAPTER III A THEORY OF THE MAGNETIC FIELD FROM CURRENT MONOPOLES	43
3.1 Abstract	44
3.2 Introduction	44
3.3 A Model for a Point Current Source	47
3.4 The Magnetic Field of Point Current Sources	55
3.4.1 The null space of $\mathbf{B}$	55
3.4.2 The spherical conductor	58
3.4.3 Semi-infinite conductor	62
3.5 Discussion	63
3.6 References	65
 CHAPTER IV AN EXACT ANALYSIS OF THE MAGNETIC FIELD OF NEURAL SOURCES IN A SEMI-INFINITE VOLUME CONDUCTOR	 67
4.1 Abstract	68
4.2 Introduction	68
4.3 Elementary Current Filament	73
4.3.1 Intracellular current	75
4.3.2 Extracellular current	81
4.3.3 Error in depth and strength calculations	86
4.3.4 Fields of intracellular versus extracellular dipole currents	90
4.4 Neuronal Models	93
4.4.1 Intracellular current - theoretical approach	96
4.4.2 Extracellular current - theoretical approach	104
4.4.3 Numerical simulations of a single neuron	110
4.4.4 Numerical simulations of neuronal ensemble	116
4.5 Discussion	125
4.6 References	131

	<u>Page</u>
CHAPTER V AN APPROXIMATE MODEL FOR CALCULATING THE MAGNETIC FIELD OF CURRENT SOURCES IN A SPHEROIDAL CONDUCTOR	134
5.1 Abstract	135
5.2 Introduction	136
5.3 Hypothesis	137
5.4 Methods	141
5.5 Results	143
5.5.1 Effect of dipole location	149
5.5.2 Effects of spheroid size	154
5.5.3 Calculation times	157
5.6 Discussion	158
5.7 References	163
CHAPTER VI CONCLUSIONS	164
APPENDIX A DERIVATION OF EQUATIONS (6a) AND (6b) IN CHAPTER II	166
APPENDIX B MAGNETIC FIELD OF HIGHER ORDER POLES	171
APPENDIX C A PROOF THAT PROLATE (OBLATE) SPHEROIDS ARE EQUIPOTENTIAL SURFACES FOR LINE (DISK) CURRENT SOURCES	177
C.1 Prolate Spheroid	177
C.2 Oblate Spheroid	179
APPENDIX D A COMPARISON OF THE MONOPOLE APPROXIMATION AND THE SPHERICAL MODEL FOR CALCULATING THE MAGNETIC FIELD OUTSIDE SPHEROIDS	184

	<u>Page</u>
APPENDIX E FORTRAN SOURCE CODE TO CALCULATE THE DIPOLE SOLUTION IN SPHEROIDAL VOLUMES	198
E.1 Comments on Coding	198
E.2 References	201
APPENDIX F FORTRAN SOURCE CODE TO APPROXIMATE THE DIPOLE SOLUTION IN SPHEROIDAL VOLUMES USING CURRENT MONOPOLES	221
APPENDIX G FORTRAN SOURCE CODE TO CALCULATE THE MAGNETIC FIELD OF A CURRENT MONOPOLE IN SEMI-INFINITE AND SPHERICAL VOLUME CONDUCTORS	228
APPENDIX H FORTRAN SOURCE CODE TO CALCULATE LEGENDRE POLYNOMIALS	248
H.1 Theoretical Approach	249
H.2 References	252
REFERENCES	290

LIST OF FIGURES

<u>Figure</u>	<u>Page</u>
II-1. Schematic of the position of a current monopole in a spherical volume conductor	18
II-2. Equivalent magnetic sources in a sphere	25
II-3. Schematic of the position of a current monopole in a semi-infinite volume conductor	27
II-4. Equivalent magnetic sources in a semi-infinite volume	33
II-5. Schematic diagrams illustrating the use of monopole current sources to explain the observed magnetic fields of certain dipole sources in a spherical volume conductor	37
III-1. Two different models of a point current source	50
III-2. Graphical solution of the general problem in a sphere	60
IV-1. Three different models of dendritic branches	71
IV-2. Schematic diagram of the current and magnetic field orientations of the current filament	74
IV-3. Plots of the relative difference in each component of the magnetic field generated by a current filament and a current dipole positioned in a semi-infinite volume conductor	80
IV-4. Plot of the relative error in calculating source depth (Δz_r) and source strength (ΔP_r) as a function of observation height normalized to source length	89
IV-5. Basis for magnetic source model of neuron	97
IV-6. Plot of voltages and currents of a dendrite in the dentate granule cell layer of a rat hippocampal slice	112
IV-7. Plot of B_{yi} generated by a single dendrite	115

<u>Figure</u>	<u>Page</u>
IV-8. Plot of the relative error in approximating the magnetic field of a dendrite as a function of distance above the dendrite	117
IV-9. Diagram of a dendrite bundle from the granule cell layer of a rat hippocampal brain slice	118
IV-10. Plot of peak values for each component of the magnetic field along different lines	119
IV-11. Plot of peak magnetic field strength generated by the dendrite slice at different field points as a function of distance above the slice	121
V-1. Theoretical basis for computing the magnetic field of a monopole in a spheroidal conductor	139
V-2. How the magnetic field is calculated for the "fan" of line segments in the prolate spheroid	140
V-3. Plots of the magnetic field calculated from the dipole equations	144
V-4. Plot of the relative error in the monopole approximation over the surface of field points	147
V-5. Dipole locations used in the spheroidal conductors	151
V-6. Plot of relative error for the monopole and spherical approximations for different source positions within a prolate spheroid	152
V-7. Relative difference histogram for dipole solution with different volume sizes	155
V-8. Plot of mean and maximum relative difference for different volume sizes	156
V-9. Three orthogonal dipoles can be constructed from four monopoles	161
B-1. Schematic diagram of higher order current sources	172
D-1. A comparison of the monopole approximation and the spherical model for calculating the magnetic field outside spheroids	186

LIST OF TABLES

<u>Table</u>	<u>Page</u>
IV-1. Ratios of operators of the magnetic field generated by extracellular versus intracellular currents	94
V-1. A comparison of the time required to calculate the magnetic field for the exact dipole solution and for the monopole approximation	159

CHAPTER I

INTRODUCTION AND HYPOTHESES

1.1 Introduction

Magnetic field measurements of electrical currents in active tissue are used as a clinical tool for both the diagnosis of neurologic disorders and for improving our understanding of neurological functions [1] - [8]. The magnetic field is a result of current flowing within a conducting medium such as intracellular neuronal current or extracellular volume currents. The goal of magnetic field mapping is to obtain information about the strength, orientation, position and configuration of the current sources. This requires solving an "inverse" problem in which a best-fit analysis may be applied to simple models of current sources and conducting regions in an attempt to duplicate the observed magnetic field pattern. The source which best fits the data should also correlate with neuroanatomical structures to be considered the "real" source.

Solving the inverse problem requires a solution to the "forward" problem (calculating the magnetic field generated by the current source). For example, the "moving dipole solution" (in which a dipole is moved within a volume until the variance in the data is most completely explained) can only be used in volumes for which the magnetic field from a dipole can be predicted. An empirical solution to the forward problem only exists for a current dipole source in semi-

infinite, spherical, and spheroidal conductors [9], [10]. The use of more complex sources or volume conductors requires a numerical approach [11], [12].

The accuracy of the source position determined by the inverse problem depends on the accuracy of the models used for the current source and the presumed shape of the conducting medium [13] - [15]. The particular choice of a source model is dictated by its a) computational simplicity, b) applicability to the underlying sources, and c) ability to explain most of the observed magnetic field. Typically, the neuronal current sources are modelled as current dipoles (regions in which charge moves over a distance short enough to be neglected relative to the observation distance) - or combinations of dipoles - in geometrical volume conductors such as half-spaces, spheres or cylinders [9], [16] - [18]. The magnetic field produced by both a current dipole and its volume currents (i.e. return currents "outside" the dipole) in these volume conductors is well understood.

Although the current dipole is not an anatomically correct model, it is widely used because it often satisfies all three criterion. Dipoles appear to be an adequate model to describe neural activity, particularly at far distances [7], [19], [20], and often account for the variation in the observed data even when a large amount of noise is present [15]. In some instances, a single dipole can explain up to 80% of the measured data [21] while in other cases it is difficult to accurately explain the data without resorting to higher order terms (current

quadrupoles or magnetic dipoles) [12], [22], [23] or using more complex conducting regions [11], [24]. The use of more complex sources is questionable because such configurations tend to have less physical connection to physiological sources [21].

Since the field of a dipole falls off as d^{-2} , whereas the field of quadrupoles, magnetic dipoles, and higher order source terms fall at least as fast as d^{-3} , it is assumed that these sources do not contribute to the magnetic field [20]. In addition, the field generated by source configurations employing disks or annuli composed of dipoles [25], and sheets or lines of dipoles [16], [17] appear to be indistinguishable from the field of a single dipole.

Despite the widespread use of dipole source models, neural current sources are clearly not true dipoles. Their use is justified on the basis that (a) the behavior of the current-generating neuron may approach that of a current dipole at observation distances exceeding a few neuron lengths [20] and (b) magnetic detectors spatially low-pass filter the magnetic field (due to the necessarily large sensing coil area) and thus remove the high-frequency component of the neuron's characteristics [21]. Previous SQUID sensors with bulky dewars could be placed no closer than 10-20 mm from a neural source with a resolution of similar magnitude [26], thereby satisfying both conditions.

Newer sensors have been developed, however, which can be placed 1.5

mm above a current source and are theoretically capable of resolving distinct sources to within $2.5\ \mu\text{m}$ [27]. The source-to-detector distance with these sensors is expected to approach 1 mm [28] which is roughly equivalent to the inter-nodal spacing of $10\ \mu\text{m}$ axons. The increased noise in the new sensors is more than compensated by their proximity to the neural sources. In fact, the image quality can be improved using a noisier sensor when the observation distance is reduced [26]. A suggested rule of thumb [26] is that observation distance should be no greater than the characteristic length of the source. Unfortunately, existing models of sources, such as the current or magnetic dipole, are only valid approximations for neural sources whenever the source-to-observation distance is large or, equivalently, the spatial extent of the source is small [22]. Improving the image quality with smaller distances may, therefore, invalidate the use of the dipole model being used in the inverse problem.

1.2 Statement of Research Hypotheses

The goal of this research is to develop a new model for calculating the magnetic field of currents generated by neural sources in a volume conductor. A patch of membrane, acting as a sink or source for the volume current, can be modelled as a point source of current (or "current monopole") in a conducting volume. This concept has never been used to calculate the magnetic field of a neural source, but has been limited to calculating the potential generated within a conducting volume. This led to the following *working hypotheses*: (1) the

magnetic field of a current monopole in a volume conductor can be determined; (2) the magnetic field of a neural source can be modelled to any degree of accuracy using monopoles; (3) previous models for neural sources can be re-examined to determine the conditions for which they may be inadequate; and (4) an exact solution to the magnetic field of current monopoles can be found for any surface which can be described as an equipotential surface for some current source distribution.

The initial strategy for the first hypothesis was to develop a solution for the magnetic field of a current monopole based on the previously published dipole solution. Due to the complexity of the solution for a spheroidal conductor, this approach was limited to spherical and semi-infinite conductors. This later led to the development of a model for the monopole that allowed the magnetic field to be derived from Maxwell's equations.

The current monopole can be considered the fundamental source from which more complex sources can be derived, including dipoles, quadrupoles and distributed current waveforms. The solution for the field of a monopole was then applied to specific neural models to test the second and third hypotheses. The neural models consisted of a current filament, a core conductor model for an equivalent cable, and a brain slice. This analysis was applied only in the semi-infinite medium to minimize the complexity of the mathematics.

The fourth hypothesis was investigated by extending the monopole formulation from a sphere to prolate and oblate spheroids. These volumes were chosen since the magnetic field generated by dipole volume currents in these volumes is known. The volume current generated by a current dipole is the same as that generated by two closely-spaced monopoles of opposite signs (sink/source combination). The hypothesis was tested by numerically comparing the field of the two monopoles to the known dipole solution.

1.3 Organization of the Dissertation

The following four chapters were written as individual manuscripts to be submitted to peer review journals. Chapter II has already been published as "Magnetic fields of current monopoles in special volume conductors" in IEEE Transactions on Magnetics, vol. MAG-27, pp. 758-767, 1991 with authors A.S. Ferguson and D. Durand. Chapters III-V will be submitted to journals as noted at the beginning of each chapter. Appendix A is the appendix for Chapter II and was also published in IEEE Trans. Magn.. Appendix B will be submitted with Chapter IV, and Appendices C and D will be submitted with Chapter V for publication.

1.4 References

- [1] D.F. Rose, P.D. Smith, and S. Sato, "Magnetoencephalography and epilepsy research," Science, vol. 238, pp. 329-335, 1987.

- [2] C.C. Wood, D. Cohen, B.N. Cuffin, M. Yarita, and Truett Allison, "Electrical sources in human somatosensory cortex: identification by combined magnetic and potential recordings," *Science*, vol. 227, pp. 1051-1053, 1985.
- [3] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.
- [4] Y.C. Okada, L. Kaufman, D. Brenner, and S.J. Williamson, "Modulation transfer functions of the human visual system revealed by magnetic field measurements," *Vision Research*, vol. 22, pp. 319-333, 1982.
- [5] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," *Ann. N.Y. Acad. Sci.*, vol. 425, pp. 722-742, 1984.
- [6] S.J. Williamson and L. Kaufman, "Biomagnetism," *J. Magn. Mag. Mat.*, vol. 22, pp. 129-201, 1981.
- [7] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [8] J. Sarvas, "Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 11-22, 1987.
- [9] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [10] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.
- [11] B.N. Cuffin, "Effects of head shape on EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-37, pp. 44-52, 1990.
- [12] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncitia," *Biophys. J.*, vol. 51, pp. 557-568, 1987.
- [13] D.S. Barth, W. Sutherling, J. Broffman, and J. Beatty, "Magnetic localization of a dipolar current source implanted in a sphere and a human cranium," *EEG. Cl. Neur.*, vol. 63, pp. 260-273, 1986.

- [14] C.J. Stok, J.W.H. Meijs, and M.J. Peters, "Inverse solutions based on MEG and EEG applied to volume conductor analysis," *Phys. Med. Biol.*, vol. 32, pp. 99-104, 1987.
- [15] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [16] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.
- [17] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," *J. Appl. Phys.*, vol. 48, pp. 3971-3980, 1977.
- [18] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 266-272.
- [19] J.P. Wikswo, "Biomagnetic sources and their models." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 1-18.
- [20] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," *Phys. Med. Biol.*, vol. 32, pp.43-51, 1987.
- [21] B.N. Cuffin, T.E. Katila, M. Pelizzzone, and J.P. Wikswo, Jr., "Theoretical models for source localization." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 9-18.
- [22] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," *Phys. Med. Biol.*, vol. 32, pp. 125-131, 1987.
- [23] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," *Phys. Med. Biol.*, vol. 32, pp. 133-142, 1987.
- [24] D.F. Rose, S. Sato, P.D. Smith, and J. White, "Modelling the temporal region in patients with temporal lobe epilepsy," *Phys. Med. Biol.*, vol 32, pp. 59-63, 1987.

- [25] J.C. De Munck, B.W. Van Duk, and H. Spekreijse, "Mathematical dipoles are adequate to describe realistic generators of human brain activity," *IEEE Trans. Biomed. Eng.*, vol. BME-35, pp. 960-966, 1988.
- [26] B.J. Roth, N.G. Sepulveda, and J.P. Wikswo, Jr., "Using a magnetometer to image a two-dimensional current distribution," *J. Appl. Phys.*, vol. 65, pp. 361-372, 1989.
- [27] D.S. Buchanan, D.B. Crum, D. Cox, and J.P. Wikswo, Jr., "Microsquad: a close-spaced four channel magnetometer." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 677-679.
- [28] J.P. Wikswo Jr. and B.J. Roth, "Magnetic determination of the spatial extent of a single cortical current source: a theoretical analysis," *EEG. Cl. Neur.*, vol. 69, 266-276, 1988.

CHAPTER II

MAGNETIC FIELDS OF CURRENT MONOPOLES IN SPECIAL VOLUME CONDUCTOR

In this chapter, the magnetic field of point current sources (monopoles) located in spherical and semi-infinite volume conductors is obtained from previously published solutions for the fields of current dipoles. This chapter has been published in IEEE Trans. Mag, vol. 27, no. 2, pp. 758-767, 1991 with authors A.S. Ferguson and D. Durand. This work was supported in part by the National Science Foundation Presidential Young Investigator Award to D. Durand. The authors are with the Applied Neural Control Laboratory, Department of Biomedical Engineering, Case Western Reserve University, Cleveland, OH 44106.

2.1 Abstract

A theory is presented for deriving the magnetic fields generated by monopoles of current (i.e. point sources of current) located in two volume conductor shapes. The solution is derived for the magnetic field exterior to a semi-infinite conducting volume, and exterior to a conducting spherical volume. In both volumes, the magnetic field of the dipole volume current is shown to be a spatial derivative of the magnetic field due to a current monopole. The magnetic field exterior to the semi-infinite volume conductor is equivalent to that of a semi-infinite line current, starting at the position of the monopole, oriented perpendicular to the surface of the conducting volume, and flowing towards infinity. The field exterior to a conducting sphere is equivalent to that generated by a finite-length line current flowing from the position of the monopole to the center of the sphere. Current monopoles can be used in lieu of dipole current sources to reduce source modelling errors and to better understand the effects of volume currents.

2.2 Introduction

Magnetic field measurements of electrical currents in active tissue are used as a clinical tool for both the diagnosis of neurologic disorders and for improving our understanding of neurological functions [1] - [5]. The magnetic field is a result of current flowing within a conducting medium such as intracellular

neuronal current or extracellular volume currents. The goal of magnetic field mapping is to obtain information about the strength, orientation, position and configuration of the current sources. This requires solving an inverse problem in which a best-fit analysis may be applied to simple models of current sources and conducting regions in an attempt to duplicate the observed magnetic field pattern. The source which best fits the data should also correlate with neuroanatomical structures to be considered the "real" source.

The accuracy of the source position determined by the inverse problem depends on the accuracy of the models used for the current source and the presumed shape of the conducting medium [6] - [8]. Typically, the neuronal current sources are modelled as current dipoles (regions in which charge moves over a distance short enough to be neglected relative to the observation distance) - or combinations of dipoles - in geometrical volume conductors such as half-spaces, spheres or cylinders [9] - [13]. The magnetic field produced by both a current dipole and its volume currents (i.e. return currents "outside" the dipole) in these volume conductors is well understood [11], [14]. Simplifying the current source in this manner, however, may limit the extent to which these sources can replicate the measured magnetic field. In some instances, a single dipole can explain up to 80% of the measured data [15] while in other cases it is difficult to accurately explain the data without resorting to higher order terms (current quadrupoles or magnetic dipoles) [16], [17] or using more complex conducting regions [18]. The use of more complex sources is questionable because such

configurations tend to have less physical connection to physiological sources [15]. Altering the volume conductor and using a more complex shape may be more anatomically correct especially when regions of different conductivity are included and can better account for the observed magnetic fields, although the basic current source is still modelled as a dipole.

Despite the widespread use of dipole source models, neural current sources are clearly not true dipoles. Their use is justified on the basis that (a) the behavior of the current-generating neuron may approach that of a current dipole at observation distances exceeding a few neuron lengths [19] and (b) present magnetic detectors spatially low-pass filter the magnetic field (due to the necessarily large sensing coil area) and thus remove the high-frequency component of the neuron's characteristics [15]. As smaller detectors become available with increased sensitivity, the source-to-detector distance may approach 1 mm [20] which is roughly equivalent to the inter-nodal spacing of 10 μm axons. In such cases, a dipole model would be inaccurate. Moreover, most magnetic field studies only measure the field perpendicular to the surface of the volume conductor since classical theory shows that volume ("return") currents created by a dipole produce only a tangential magnetic field at the surface. Magnetic measurements of cardiac activity indicate that volume currents may account for a large part of the distal magnetic field [21] and recently it has been suggested that measurements of the tangential component may be equally accurate in localizing current sources [26]. The dipole model may not be an accurate

model for larger groups of neurons which resemble a distributed conducting region near the observation point.

This paper provides a new method for calculating the magnetic field generated by extracellular or "volume" currents resulting from transmembrane current of neurons. A patch of membrane, acting as a sink or source for the volume current, can be modelled as a point source of current (or "current monopole") in a conducting volume [22]. The term "monopole" will be used throughout this paper when referring to a single point source, although it is understood that in a real system with balanced currents this source cannot exist alone. Such a source generates a volume current density that radiates equally in all directions in an unbounded medium. The total extracellular current can be obtained from a superposition of monopoles at each membrane patch whose strength is proportional to the transmembrane current. Theoretically, a current monopole source cannot exist by itself without violating continuity of current and therefore a complete model of the neuron must include the intracellular current (which in a sense supplies the current to the monopoles) and the magnetic field generated by this current.

The current monopole can be considered the fundamental source from which more complex sources can be derived, including dipoles, quadrupoles and distributed current waveforms. The equations are derived for the magnetic field produced by the volume currents of a monopole of current situated in two special

volume conductor shapes - the semi-infinite volume and the spherical conductor. The magnetic fields of dipole volume currents in similar volume conductors are shown to be the derivative of the solution for a monopole, causing the dipole solution to be significantly more complex. The magnetic field of the current monopole in either conductor is shown to be equivalent to that produced by a line current. This latter point will be used to explain why some dipole sources are difficult to localize.

2.3 Theoretical Approach

The following two sections provide details on the derivation of the magnetic field $\mathbf{B}$ of current monopoles located in the two special volume conductors. For both conductors, the monopole solution is obtained from the previously published solution for the magnetic field of dipole volume currents in the same conductors [11]. The procedure is similar to that used in electrostatics to calculate the electric field $\mathbf{E}$ of a charge dipole by constructing the charge dipole from two oppositely charged sources for which $\mathbf{E}$ is known. In this case, however, the dipole solution is known and it is possible to work "backwards" to obtain the monopole solution. A current dipole situated in a volume conductor generates a volume "return" current which originates at the head of the dipole and terminates at the dipole's tail; this volume current can be thought to begin and end at current monopoles, the dipole head being considered a point source and the dipole tail being considered a point sink. The dipole volume current is therefore

identical to that produced by two oppositely signed monopoles spaced infinitely close together, and the resulting magnetic field is also the same.

Throughout these sections, cartesian and spherical coordinates will be interchanged - the corresponding coordinate axis being defined by the variables (x,y,z) and (r,θ,ϕ) respectively and the choice of coordinates implied by the use of either set of variables. The use of the subscript "0" denotes source position, thus the monopole and dipole source are located at (x_0,y_0,z_0) or (r_0,ϕ_0,θ_0) . The superscripts "d" and "m" are used to refer to the magnetic field ($\mathbf{B}^d$, $\mathbf{B}^m$) of the volume current of the dipole and monopole respectively. The " Δ " symbol is used to denote small but finite lengths, while the differential symbol "d" is used to denote lengths as the limit approaches zero, i.e.

$$dl = \lim_{\Delta L \rightarrow 0} \Delta L$$

2.4 Sphere

2.4.1 Relationship between monopole and dipole solutions

The magnetic field outside a sphere produced by a dipole inside a conducting sphere has been calculated by Grynszpan [14]. Cuffin and Cohen [11] extended these results and calculated the part of the field due to the dipole alone (the dipole field) and that due to the currents generated in the volume

conductor (volume current field). They obtained the solution for a dipole pointing in the $+x$ direction and located at $r = a$, $\theta = 0^\circ$, $\phi = 0^\circ$ in a spherical conductor centered about the origin. The size of the spherical conductor is arbitrary since it does not enter into the solution other than to limit the position of observation points considered outside the sphere. Although their solution only applies to the tangential component of a dipole, it is a complete solution for the field of a dipole because the radial component produces no external field and because it can be rotated and/or translated to allow for a dipole of any tangential orientation and position within a sphere of any size and location.

A similar approach is used here to obtain the solution for the magnetic field of a monopole. Without loss of generality, a positive current monopole is assumed to be located at $x_0 = 0$, $y_0 = 0$, $z = z_0$ (or $r = r_0$, $\theta = 0^\circ$, $\phi = 0^\circ$) in a conducting sphere centered at the origin. The magnetic field $\mathbf{B}^m$ must, by symmetry, be oriented in the $\mathbf{a}_\phi$ direction (see Fig. II-1(a)) with some magnitude B^m_ϕ . When the monopole is rotated out of this position, the magnetic field is no longer oriented only in the $\mathbf{a}_\phi$ direction. The volume current of a dipole oriented in the $+x$ direction (see Fig. II-1(b)) can, however, be approximated using monopoles rotated out of position by summing the volume current of a positive current monopole whose reference frame is rotated $+\Delta\theta_0$ and a negative current monopole rotated in the opposite direction $(-\Delta\theta_0)$ around the y axis.

Fig. II-1. Schematic of the position of a current monopole in a spherical volume conductor (a). Two monopoles can be used to mimic the volume currents of a dipole in a spherical volume conductor (b).

Superposition implies that a) the net **B** field is the sum of the fields produced by each monopole and b) the **B** field of a negative current monopole must be equal and opposite to that of a positive current monopole. Therefore, at some arbitrary observation point (r, ϕ, θ) outside the volume conductor for sources located at (r_0, ϕ_0, θ_0) , the magnetic fields of the dipole and monopole volume currents are related by

$$\begin{aligned}
 \mathbf{B}^d(r, \theta, \phi, r_0, \theta_0, \phi_0) &= \lim_{2\Delta\theta_0 \rightarrow 0} [\mathbf{B}^m(r, \theta, \phi, r_0, \theta_0 + \Delta\theta_0, \phi_0) - \mathbf{B}^m(r, \theta, \phi, r_0, \theta_0 - \Delta\theta_0, \phi_0)] \\
 &= \lim_{2\Delta\theta_0 \rightarrow 0} [2\Delta\theta_0] \\
 &\quad * \lim_{2\Delta\theta_0 \rightarrow 0} \left[\frac{\mathbf{B}^m(r, \theta, \phi, r_0, \theta_0 + \Delta\theta_0, \phi_0) - \mathbf{B}^m(r, \theta, \phi, r_0, \theta_0 - \Delta\theta_0, \phi_0)}{2\Delta\theta_0} \right] \\
 &= 2d\theta_0 \left(\frac{\partial \mathbf{B}^m(r, \theta, \phi, r_0, \theta_0, \phi_0)}{\partial \theta_0} \right)
 \end{aligned} \tag{1}$$

The length of the dipole depends on the angular separation of the monopoles and, for the orientation being considered here, can be written

$$\Delta L = 2r_0 \sin(\Delta\theta_0)$$

As the angular separation approaches zero, the differential length of the dipole becomes

$$dL = 2r_0 d\theta_0 \tag{2}$$

Equations (1) and (2) yield the final relationship between the **B** field of a dipole

of differential length dL positioned at $r = r_0$, $\theta = 0^\circ$, $\phi = 0^\circ$ and oriented in the $+x$ direction and the $\mathbf{B}$ field of a positive current monopole rotated around the y axis towards the same position as the dipole

$$(\mathbf{B}^d)_{\theta_0=0} = \frac{1}{r_0} dL \left(\frac{\partial \mathbf{B}^m}{\partial \theta_0} \right)_{\theta_0=0} \quad (3)$$

It is important to realize that (3) is specific to this source configuration. If, instead of the system shown in Fig. II-1(b), the dipole position was shifted to lie on the x axis ($r = r_0$, $\theta_0 = 90^\circ$, $\phi = 0^\circ$) but the orientation remained in the $+x$ direction, an analysis similar to that above would yield a very different relationship from (3)

$$(\mathbf{B}^d)_{\theta_0=90} = dL \left(\frac{\partial \mathbf{B}^m}{\partial r_0} \right)_{\theta_0=90} \quad (4)$$

The differential in this case is with respect to r_0 which differs from θ_0 in (3), although it should be clear that in both cases differentiation is taken with respect to the coordinate along which two monopoles must approach each other to approximate a dipole. Likewise, a change in the orientation of the dipole would also change the form of the identity in (3). Consequently, the position and orientation of the dipole and monopole sources are implicit in all subsequent equations and assumed to be as shown in Fig. II-1(b).

Using a primed notation to indicate the magnetic field of a source rotated

some differential amount $d\theta_0$ relative to the fixed coordinates and an unprimed notation for the field of a fixed source, the identity in (3) can be written in spherical coordinates for both the ϕ and θ components of $\mathbf{B}^d$

$$(B_\theta^d)_{\theta_0=0} = \frac{1}{r_0} dL \left(\frac{\partial B_\theta^m}{\partial \theta_0} \right)_{\theta_0=0} \quad (5a)$$

$$(B_\phi^d)_{\theta_0=0} = \frac{1}{r_0} dL \left(\frac{\partial B_\phi^m}{\partial \theta_0} \right)_{\theta_0=0} \quad (5b)$$

It only remains to solve either (5a) or (5b) to obtain a solution for $\mathbf{B}^m$. Although $\mathbf{B}^m$ only contains a ϕ component when the monopole is positioned at $\theta = \phi = 0^\circ$, the rotation of the monopole position about the y axis by the differential amount $d\theta_0$ causes the magnetic field to increasingly assume a θ component. The following identities relate the fields of rotated and fixed sources as observed at a fixed x,y,z point (see Appendix A)

$$\begin{aligned} \left(\frac{\partial B_\theta^m}{\partial \theta_0} \right)_{\theta_0=0} &= \frac{-y \sqrt{x^2 + y^2 + z^2}}{x^2 + y^2} B_\phi^m \\ &= \frac{-\sin\phi}{\sin\theta} B_\phi^m \end{aligned} \quad (6a)$$

$$\begin{aligned}
\left(\frac{\partial B_{\phi}^m}{\partial \theta_0}\right)_{\theta_0=0} &= \left(\frac{\partial B_{\phi}^m}{\partial \theta_0}\right)_{\theta_0=0} \\
&= \left(-z \frac{\partial B_{\phi}^m}{\partial x} + x \frac{\partial B_{\phi}^m}{\partial z}\right)_{\theta_0=0}
\end{aligned} \tag{6b}$$

Eqn (6a) indicates that the change in the θ component of the magnetic field as the source initially rotates around $\theta = 0^\circ$ can be expressed as a function of the ϕ component of the unrotated source. The identity in (6b) verifies that the initial variation in the ϕ component is the same for either a rotating source or a rotating observation point. Substituting (6a) and (6b) into (5a) and (5b) results in the final relationship between the $\mathbf{B}$ field of the dipole and the monopole

$$(B_{\theta}^d)_{\theta_0=0} = \frac{1}{r_0} dL \frac{-\sin\phi}{\sin\theta} (B_{\phi}^m)_{\theta_0=0} \tag{7a}$$

$$(B_{\phi}^d)_{\theta_0=0} = \frac{1}{r_0} dL \left[-z \left(\frac{\partial B_{\phi}^m}{\partial x}\right)_{\theta_0=0} + x \left(\frac{\partial B_{\phi}^m}{\partial z}\right)_{\theta_0=0} \right] \tag{7b}$$

2.4.2 Monopole solution and interpretation of results

Equation (7a) allows us to directly calculate the magnetic field of a current monopole from the solution for a dipole volume field and vice-versa. The dipole solution published by Cuffin and Cohen [11, equation 21] modified to our terminology is

$$B_{\theta}^d = \mu \frac{\sin\phi P_x}{4\pi r^2 \gamma^{1/2}} \left[\frac{\cos\theta}{\sin^2\theta} \left(\cos\theta - \frac{r}{r_0} + \frac{r \gamma^{1/2}}{r_0} \right) + 1 \right] \quad (8a)$$

$$B_{\phi}^d = \mu \frac{\cos\phi P_x}{4\pi r^2 \gamma^{3/2}} \left[\cos\theta - \frac{r_0}{r} - \frac{\gamma}{\sin^2\theta} \left(\cos\theta - \frac{r}{r_0} + \frac{r \gamma^{1/2}}{r_0} \right) \right] \quad (8b)$$

where

$$\gamma = \left[1 - \frac{2r_0 \cos\theta}{r} + \left(\frac{r_0}{r} \right)^2 \right]$$

$P_x = I dL = \text{dipole current moment}$

Substituting the dipole solution (8a) into (7a) yields the monopole solution

$$B_{\phi}^m = \mu \frac{-r_0 I \sin\theta}{4\pi r^2 \gamma^{1/2}} \left[\frac{\cos\theta}{\sin^2\theta} \left(\cos\theta - \frac{r}{r_0} + \frac{r \gamma^{1/2}}{r_0} \right) + 1 \right] \quad (9)$$

This solution for B_{ϕ}^m should also satisfy the differential equation shown in (7b).

After some manipulation, it can be shown that this is true and the B_{ϕ}^d obtained from (7b) is the same as that shown in (8b). Thus, the solution obtained for the monopole satisfies all the conditions established in deriving the equations from the dipole solution.

The magnetic field of a positive current monopole, shown in (9), can be rewritten in cartesian coordinate (setting $z_0 = r_0$ since the source is located on the z axis) as

$$B_{\phi}^m = \mu \frac{I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z - z_0}{\sqrt{x^2 + y^2 + (z - z_0)^2}} - \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \quad (10)$$

The B_ϕ field generated by a line current alone (no volume currents) oriented along the z-axis, in the +z direction, and extending from $z = z_1$ to $z = z_2$ is given by

$$B_\phi = \mu \frac{I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z - z_1}{\sqrt{x^2 + y^2 + (z - z_1)^2}} - \frac{z - z_2}{\sqrt{x^2 + y^2 + (z - z_2)^2}} \right] \quad (1)$$

A comparison of these two equations shows that the $\mathbf{B}$ field of the positive current monopole field is identical to the $\mathbf{B}$ field of a line current extending from the position of the monopole source ($z_1 = z_0$) to the center of the sphere ($z_2 = 0$) (see Fig. II-2(a) and(b)). Similarly, the $\mathbf{B}$ field of a negative current monopole field would be identical to the $\mathbf{B}$ field of a line current extending from the center of the sphere out to the position of the monopole.

2.5 Semi-Infinite Media

2.5.1 Relationship between monopole and dipole solutions

Cuffin and Cohen [11] also derived a solution for the magnetic field of a dipole volume current in a semi-infinite conducting volume for a dipole pointing in the +y direction and located at $x = 0$, $y = 0$, $z = z_0$ ($z_0 < 0$). The volume conductor extends for $z < 0$. Their solution only applies to dipoles oriented tangential to the surface of the semi-infinite space, but is otherwise without loss of generality.

Fig. II-2. Equivalent magnetic sources in a sphere. The magnetic field of a monopole in a spherical volume conductor (a) is equivalent to the field of a line of current extending from the monopole to the center of the sphere (b).

A similar approach to that used in previous section is employed here to obtain the solution for the magnetic field of a monopole. Without loss of generality, a positive current monopole is assumed to be located at $x_0 = 0, y_0 = 0, z = z_0$ ($z_0 < 0$) in a semi-infinite conducting volume $z < 0$. The magnetic field must, by symmetry, be oriented in the $\mathbf{a}_\phi$ direction (see Fig. II-3(a)). The magnitude of the $\mathbf{B}$ field, denoted B_ϕ , clearly depends on x, y, z and z_0 .

The volume current of a dipole oriented in the $+y$ direction (see Fig. II-3(b)) can be approximated by summing the volume current of a positive current monopole whose reference frame is shifted $+\Delta y_0$ and a negative current monopole shifted in the opposite direction $(-\Delta y_0)$. Using the same analysis applied to the spherical conductor, the magnetic fields of the dipole and monopole volume currents are related by

$$\begin{aligned}
 \mathbf{B}^d(x, y, z, x_0, y_0, z_0) &= \lim_{2\Delta y_0 \rightarrow 0} [\mathbf{B}^m(x, y, z, x_0, y_0 + \Delta y_0, z_0) - \mathbf{B}^m(x, y, z, x_0, y_0 - \Delta y_0, z_0)] \\
 &= \lim_{2\Delta y_0 \rightarrow 0} [2\Delta y_0] \\
 &\quad * \lim_{2\Delta y_0 \rightarrow 0} \left[\frac{\mathbf{B}^m(x, y, z, x_0, y_0 + \Delta y_0, z_0) - \mathbf{B}^m(x, y, z, x_0, y_0 - \Delta y_0, z_0)}{2\Delta y_0} \right] \\
 &= 2dy_0 \left(\frac{\partial \mathbf{B}^m(x, y, z, x_0, y_0, z_0)}{\partial y_0} \right) \quad .(12)
 \end{aligned}$$

This identity was derived in vector form, although a separate equation could be obtained for each component of the $\mathbf{B}$ fields. For example, the y component of

Fig. II-3. Schematic of the position of a current monopole in a semi-infinite volume conductor (a). Two monopoles can be used to mimic the volume currents of a dipole (b).

the **B** fields are related by

$$B_y^d(x, y, z, x_0, y_0, z_0) = 2dL \left(\frac{\partial B_y^m(x, y, z, x_0, y_0, z_0)}{\partial y_0} \right) \quad (13)$$

The magnetic field of a dipole volume current is therefore the derivative of the magnetic field of a monopole current. More importantly, the magnetic field of one can be obtained directly from the magnetic field of the other.

2.5.2 Final solution for **B^m**

Cuffin and Cohen [11] obtained the solution for **B^d** in equation (12) for a +y oriented dipole located at (0,0,z₀). More generally, a simple translation of the x,y axis allows their results to be written for a +y oriented dipole located at any point (x₀,y₀,z₀) in the conducting half-space

$$\begin{aligned} B_x^d &= \frac{\mu P_y}{4\pi} \left\{ \frac{(y-y_0)^2 - (x-x_0)^2}{[(x-x_0)^2 + (y-y_0)^2]^2} \right. \\ &\quad \left. - (z-z_0) \frac{[(y-y_0)^2 - (x-x_0)^2] (z-z_0)^2 + 2(y-y_0)^4 + (x-x_0)^2 (y-y_0)^2 - (x-x_0)^4}{[(x-x_0)^2 + (y-y_0)^2]^2 [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \right\} \\ B_y^d &= \frac{\mu (x-x_0) P_y}{4\pi} \left\{ \frac{-2(y-y_0)}{[(x-x_0)^2 + (y-y_0)^2]^2} \right. \\ &\quad \left. + (z-z_0) \frac{(y-y_0) [3(x-x_0)^2 + 3(y-y_0)^2 + 2(z-z_0)^2]}{[(x-x_0)^2 + (y-y_0)^2]^2 [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \right\} \\ B_z^d &= 0 \end{aligned} \quad (14)$$

where $P_y = I dy$ is the current dipole moment.

Using the identity presented in (12), it is possible to obtain the $\mathbf{B}^m$ directly from (14) by integrating with respect to the position variable y_0 . This process is simplified by letting

$$u = \frac{1}{(x-x_0)^2 + (y-y_0)^2}$$

$$v = \frac{(y-y_0)}{(x-x_0)^2 + (y-y_0)^2}$$

$$w = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

and using the differentials of these functions

$$\frac{\partial u}{\partial y_0} = \frac{2(y-y_0)}{[(x-x_0)^2 + (y-y_0)^2]^2}$$

$$\frac{\partial v}{\partial y_0} = \frac{(y-y_0)^2 - (x-x_0)^2}{[(x-x_0)^2 + (y-y_0)^2]^2}$$

$$\frac{\partial w}{\partial y_0} = \frac{(y-y_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}$$

and the differentials of products of these functions

$$\frac{\partial(uw)}{\partial y_0} = w \frac{\partial u}{\partial y_0} + u \frac{\partial w}{\partial y_0}$$

$$= \frac{(y-y_0) [3(x-x_0)^2 + 3(y-y_0)^2 + 2(z-z_0)^2]}{[(x-x_0)^2 + (y-y_0)^2]^2 [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}$$

$$\frac{\partial(vw)}{\partial y_0} = w \frac{\partial v}{\partial y_0} + v \frac{\partial w}{\partial y_0}$$

$$= \frac{[(y-y_0)^2 - (x-x_0)^2] (z-z_0)^2 + 2(y-y_0)^4 + (x-x_0)^2 (y-y_0)^2 - (x-x_0)^4}{[(x-x_0)^2 + (y-y_0)^2]^2 [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}$$

Now the dipole equations in (14) can be written in terms of these functions

$$\begin{aligned} B_x^d &= \frac{\mu P_y}{4\pi} \left\{ \frac{\partial v}{\partial y_0} - (z-z_0) \frac{\partial(vw)}{\partial y_0} \right\} \\ B_y^d &= \frac{\mu (x-x_0) P_y}{4\pi} \left\{ -\frac{\partial u}{\partial y_0} + (z-z_0) \frac{\partial(uw)}{\partial y_0} \right\} \end{aligned} \quad (15)$$

Applying the identity in (13) to this equation is straightforward. Keeping in mind that $P_y = 2IdL$, we can now obtain B_m^y by integrating (15) with respect to y_0

$$\begin{aligned} B_x^m &= \frac{\mu I}{4\pi} \{ v - (z-z_0)vw \} + F(x,y,z,x_0,z_0) \\ B_y^m &= \frac{\mu (x-x_0) I}{4\pi} \{ -u + (z-z_0)uw \} + G(x,y,z,x_0,z_0) \end{aligned} \quad (16)$$

The functions $F(x,y,z,x_0,z_0)$ and $G(x,y,z,x_0,z_0)$ account for the fact that integration with respect to y_0 allows for the possibility of an additional function after integration that does not depend on y_0 . The choice of functions depends on an additional constraint that the magnetic fields generated by the monopole in a

semi-infinite and a spherical volume conductor should be equal as the radius of the sphere approaches infinity.

$$[\mathbf{B}^m(x, y, z, x_0, y_0, z_0)]_{\text{semi-infinite}} = \lim_{\text{curvature} \rightarrow \infty} [\mathbf{B}^m(x, y, z, x_0, y_0, z_0)]_{\text{sphere}} \quad (17)$$

As will be shown later, both $F(x, y, z, x_0, z_0)$ and $G(x, y, z, x_0, z_0)$ must be identically zero in order to preserve continuity of the solution between the two conductors.

The final solution for $\mathbf{B}^m$ is given by

$$\begin{aligned} B_x^m &= \mu \frac{-(y-y_0) I}{4\pi [(x-x_0)^2 + (y-y_0)^2]} \left\{ \frac{(z-z_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - 1 \right\} \\ B_y^m &= \mu \frac{(x-x_0) I}{4\pi [(x-x_0)^2 + (y-y_0)^2]} \left\{ \frac{(z-z_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - 1 \right\} \\ B_z^m &= 0 \end{aligned} \quad (18)$$

If the source is positioned on the z axis such that $x_0=y_0=0$, then the above equation can be written in a more simple form using cylindrical coordinates

$$B_\phi = \mu \frac{I}{4\pi r} \left\{ \frac{(z-z_0)}{\sqrt{r^2 + (z-z_0)^2}} - 1 \right\} \quad r = \sqrt{x^2 + y^2} \quad (19)$$

Note that the monopole solution only has a B_ϕ component, whereas the solution for the dipole located on the z -axis contained both B_ϕ and B_θ components. This is because a single monopole can be situated on the z axis and, by symmetry, generate a simple B_ϕ field whereas the dipole must be constructed from two

monopoles, only one of which may be coincident with the z axis.

2.5.3 Interpretation of the results

The magnetic field generated by a monopole in a semi-infinite volume conductor is equivalent to the field generated by a semi-infinite line of current. The B_ϕ field generated by a line current alone (no volume currents) oriented in the $+z$ direction and extending from $z = z_1$ to $z = -\infty$ is given by

$$B_\phi = \frac{\mu I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z - z_1}{\sqrt{x^2 + y^2 + (z - z_1)^2}} - 1 \right] \quad (20)$$

This can easily be derived from the Biot-Savart law. A comparison of (19) and (20) shows that the $\mathbf{B}$ field of the volume currents of the positive current monopole is identical to the $\mathbf{B}$ field of a line current. In the non-conducting media ($z > 0$), the $\mathbf{B}$ field of the monopole is equivalent to that of a line current extending along the z -axis from $z = z_0$ to $z = -\infty$ (see Fig. II-4(a) and (b)).

For the spherical conductor, the magnetic field was identical to that of a line current extending from the position of the monopole to the center of the sphere. As the radius of curvature of the sphere approaches infinity, the spherical volume approximates a semi-infinite volume. The equivalent line current extends away from monopole source towards infinity at an angle perpendicular to the volume surface. This is identical to the solution for the semi-infinite conductor

Fig. II-4. Equivalent magnetic sources in a semi-infinite volume. The magnetic field in the non-conducting half-space due to a monopole (a) is equivalent to that of a semi-infinite line current (b).

shown in (19), and provides the continuity of solution described in the identity (17). This further justifies setting $F(x,y,z,x_0,z_0)$ and $G(x,y,z,x_0,z_0)$ in (16) at zero.

2.6 Discussion

These results provide an alternate method for calculating the magnetic field produced by current sources in bounded media. In particular, the transmembrane currents of any neural source can be modelled using monopole current sources. The resultant magnetic field from the volume current in bounded media is easily calculated from the equations presented above without having to solve a volume integral. It is perhaps a surprising conclusion that the magnetic field of the monopole is independent of the position of the volume conductor boundary. For both volumes studied, the magnetic field depends only on the monopole position and the center of curvature for the volume conductor.

The magnetic field produced by a monopole in a conducting volume shares many of the characteristics associated with the magnetic field resulting from dipole volume currents in the same conductor. For both the semi-infinite and spherical volume conductors, we can summarize these characteristics as:

- a) the magnetic field is tangential to the conductor surface with no perpendicular component,
- b) in a homogeneous conductor, conductivity does not affect the magnetic

field,

c) inhomogeneities, where the boundary of the inhomogeneity lies parallel to the conductor surface, do not affect the magnetic field (thus planar and concentric spherical inhomogeneities do not effect the semi-infinite and spherical models respectively),

However, several distinct differences exist between the equations for the magnetic field of a monopole and that of a dipole:

- a) The magnetic field of a dipole (resulting from both dipole and dipole volume currents) located in a symmetric volume conductor approaches zero as the dipole is moved to and aligned with an axis of symmetry [14], [24]. The field of a monopole approaches zero only for certain volume conductors and a unique monopole location within the conductor.
- b) The equations for the magnetic field of a monopole current in both the spherical and semi-infinite volume conductors (eqns (9) and (18) respectively) are computationally simpler than the equations for the field of dipole volume currents (eqns (8) and (14) respectively).
- c) The magnetic field of a monopole is equivalent to the field generated by a line of current; no simple equivalent source exists for the field of a dipole volume current.

Recognizing that the magnetic field of a monopole is equivalent to that of a line of current, as shown above, allows us to draw further conclusions about the nature of the magnetic field of particular current sources. For instance, it has

been observed that a current dipole located deep within a spherical conductor generates a magnetic field similar to that of a current quadrupole (or magnetic dipole) because the field drops off with distance quicker than would be explained by an inverse square law [25]. Assuming initially that the current dipole has some "finite length", then the complete magnetic field generated by this source would consist of the field generated by the dipole current and that generated by the returning volume currents (see Fig. II-5(a)). But the volume currents are generated by the "monopolar-like" ends of the dipole, having a positive monopole at the head of the dipole and a negative monopole at the tail. From our earlier analysis, the field of the monopoles is the same as that of line currents extending from the head of the dipole (positive monopole) to the center of the sphere, and another from the center to the tail of the dipole (negative monopole). Coupled with the dipole current, we have now created a triangular-shaped loop of current. As the dipole length shortens and the dipole position approaches the center of the sphere, the current loop more closely approximates a magnetic dipole. This analysis explains why Ilmoniemi [26] suggested using a magnetic dipole to model the source of the magnetic field of a current dipole positioned deep within a spherical conductor. Furthermore, as the dipole is situated deeper within the sphere, the magnetic dipole (or current quadrupole) model is more accurate and, since the field drops off faster with distance compared to a current dipole, we can explain why deeper sources have field strengths that drop with distances quicker than would be explained by a current dipole source [26].

Fig. II-5. Schematic diagrams illustrating the use of monopole current sources to explain the observed magnetic fields of certain current dipole sources in a spherical volume conductor. (a) Diagram illustrating the similarity between the magnetic field of a dipole and that of a loop of current. The dipole's magnetic field is the sum of the field generated by the dipole itself and by the volume currents. The volume currents can be considered to arise from two monopoles positioned at either end of the dipole. The magnetic field of the monopoles is equivalent to the magnetic field of lines of current as shown. The total magnetic field is therefore the magnetic field of a loop of current, approximating a magnetic dipole. (b) Diagram illustrating why the magnetic field of a radially oriented dipole is zero. The dipole volume current results from two monopoles whose magnetic fields are equivalent to those of two lines of current. Because the dipole is radially oriented, these two lines of currents coincide (shown separated in the figure for clarity) and cancel for part of their length. The net equivalent line of current is equal and opposite to the "finite length" dipole current.

Similar geometrical reasoning based on the equivalence between lines of currents and monopoles can be used to further demonstrate specific dipole configurations cannot generate a magnetic field. For instance, it is known that a radially oriented dipole in a sphere produces no magnetic field - this just being a special case of a dipole aligned with an axis of symmetry in a volume conductor [14], [22]. Modelling the dipole volume current as originating from two current monopoles, we can inquire what the summed magnetic field of the monopoles must be (see Fig. II-5(b)). The head of the dipole is modelled as a positive current monopole whose magnetic field was shown earlier to be equivalent to the field of a line of current extending and oriented from the monopole position to the center of the sphere. The tail of the monopole is modelled as a negative current monopole whose magnetic field was shown earlier to be equivalent the field of a line of current extending and oriented from the center of the sphere to the monopole position. Combining these two equivalent current lines results in a net line of current extending from the positive current monopole to the negative current monopole. This generates the same magnetic field as the volume currents of the original dipole. However, note that this "equivalent line of current" is equal and opposite to the original finite length dipole; that is, the magnetic field produced by the volume currents is equal and opposite to the dipole field and results in zero net magnetic field. An equivalent analysis on the semi-infinite media would show that any dipole oriented perpendicular to the boundary surface would produce zero net field.

The magnetic field of the monopole can be used to calculate the magnetic field of more complex current sources located in either of these two conductors and, as such, can be considered to be a fundamental source of magnetic fields from volume currents. Admittedly, previously published solutions for the magnetic field of dipole volume currents were used to obtain the field of a monopole, but the monopole may lend itself more easily to the construction of complex current distributions (see for example the distributions presented in [10]). The magnetic field for any arbitrary distribution of current sources and sinks can also be found without calculating the equivalent dipoles (oriented tangent to the surface of the conductor) that would create similar current flow. This approach also allows neural current generators to be modelled as dipoles of finite length ("bipoles") if the separation of the monopoles does not approach zero but instead remains measurable.

The use of current monopoles allows us to model a greater variety of current sources without requiring the sources to exhibit dipolar behavior. The dipole approach is limited to cases in which the source can honestly be considered a dipole. Generally this is only true if the observation point is sufficiently far from the current source. As smaller magnetic sensors are developed [26], this will cease to be the case. Biological preparations such as myelinated nerve trunks and active brain tissue slices, in which the sensing device can be positioned in close proximity, may be better modelled using monopoles. Monopole theory may then have some implications for the inverse problem. Considering the

greater simplicity of monopole solutions compared to dipole solutions, such a process may not involve great difficulty.

2.7 References

- [1] D.F. Rose, P.D. Smith, and S. Sato, "Magnetoencephalography and epilepsy research," *Science*, vol. 238, pp. 329-335, 1987.
- [2] C.C. Wood, D. Cohen, B.N. Cuffin, M. Yarita, and Truett Allison, "Electrical sources in human somatosensory cortex: identification by combined magnetic and potential recordings," *Science*, vol. 227, pp. 1051-1053, 1985.
- [3] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.
- [4] Y.C. Okada, L. Kaufman, D. Brenner, and S.J. Williamson, "Modulation transfer functions of the human visual system revealed by magnetic field measurements," *Vision Research*, vol. 22, pp. 319-333, 1982.
- [5] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," *Ann. N.Y. Acad. Sci.*, vol. 425, pp. 722-742, 1984.
- [6] D.S. Barth, W. Sutherling, J. Broffman, and J. Beatty, "Magnetic localization of a dipolar current source implanted in a sphere and a human cranium," *EEG. Cl. Neur.*, vol. 63, pp. 260-273, 1986.
- [7] C.J. Stok, J.W.H. Meijs, and M.J. Peters, "Inverse solutions based on MEG and EEG applied to volume conductor analysis," *Phys. Med. Biol.*, vol. 32, pp. 99-104, 1987.
- [8] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [9] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.

- [10] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," J. Appl. Phys., vol. 48, pp. 3971-3980, 1977.
- [11] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.
- [12] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 266-272.
- [13] C. Pantev, M. Hoke, K. Lehnertz, B. Lutkenhoner, G. Anogianakis, and W. Wittkowski, "Tonotopic organization of the human auditory cortex revealed by transient auditory evoked magnetic fields," EEG. Cl. Neur., vol. 69, pp. 160-170, 1988.
- [14] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," Biophys. J., vol. 13, pp. 911-925, 1973.
- [15] B.N. Cuffin, T.E. Katila, M. Pelizzzone, and J.P. Wikswo, Jr., "Theoretical models for source localization." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 9-18.
- [16] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," Phys. Med. Biol., vol. 32, pp. 125-131, 1987.
- [17] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," Phys. Med. Biol., vol. 32, pp. 133-142, 1987.
- [18] D.F. Rose, S. Sato, P.D. Smith, and J. White, "Modelling the temporal region in patients with temporal lobe epilepsy," Phys. Med. Biol., vol. 32, pp. 59-63, 1987.
- [19] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," Phys. Med. Biol., vol. 32, pp. 43-51, 1987.
- [20] J.P. Wikswo Jr. and B.J. Roth, "Magnetic determination of the spatial extent of a single cortical current source: a theoretical analysis," EEG. Cl. Neur., vol. 69, pp. 266-276, 1988.

- [21] M.C. Leifer, J.C. Griffin, D.C. Harrison, and J.P. Wikswo, Jr., "Effect of the heart-lung boundary on the magnetocardiogram," *J. Electr.*, vol. 19, pp. 23-32, 1986.
- [22] P.L. Nunez, Electric Fields of the Brain, New York, Oxford University Press, 1981, Ch. 4, pp. 111-139.
- [23] T.A. Weber and D.J. Macomb, "On the equivalence of the laws of Biot-Savart and Ampere," *Am. J. Phys.*, vol. 57, pp. 57-59, 1989.
- [24] G. Baule and R. McFee, "Theory of magnetic detection of the heart's electrical activity," *J. Appl. Phys.*, vol. 36, pp. 2066-2073, 1965.
- [25] L. Kaufman, "Biomagnetism in the study of brain functions." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 221-234.
- [26] R.J. Ilmoniemi, M.S. Hamalainen, and J. Knuutila, "The forward and inverse problems in the spherical model." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 278-282.

CHAPTER III

A THEORY OF THE MAGNETIC FIELD

FROM CURRENT MONOPOLES

In this chapter, the magnetic field of point current sources (monopoles) located in spherical and semi-infinite volume conductors is derived from first principles. This chapter will be submitted to the Journal of Applied Physics with authors A.S. Ferguson and D. Durand. The authors would like to extend their sincere appreciation to Dr. David Ksienski and Dr. Robert Collin for many stimulating conversations and helpful suggestions during the course of this research. This work was supported in part by the National Science Foundation Presidential Young Investigator Award to D. Durand. The authors are with the Applied Neural Control Laboratory, Department of Biomedical Engineering, Case Western Reserve University, Cleveland, OH 44106.

3.1 Abstract

A method for calculating the magnetic fields of volume currents by using point current sources (monopoles) is derived. A model is developed for a point source that is consistent with Maxwell's equations, in which conduction and displacement currents are separated in each part of the model. It is shown that a model in which the currents are not considered separately violates Maxwell's equations. An arbitrary arrangement of point sources in a conducting volume is shown to generate no magnetic field outside the conductor if the volume surface coincides with an equipotential surface of the sources. This allows such sources to be added to any problem without affecting the magnetic field, and simplifies the calculation of the magnetic field generated by an arbitrary point source in both spherical and semi-infinite volume conductors. The magnetic field generated by current monopoles located within these volume conductors is then derived.

3.2 Introduction

Current sources enclosed in volume conductors generate magnetic fields outside the conductor that can be measured to provide information about the source. This non-invasive technique is capable of providing information about the strength, orientation, and location of the current source. Commonly referred to as the "inverse problem", this procedure is now used to image current flow in living tissue and has important clinical and diagnostic implications [1] - [3].

However, solving the inverse problem requires a solution to the "forward" problem (calculating the magnetic field generated by the current source). An empirical solution to the forward problem only exists for a current dipole source in semi-infinite, spherical, and spheroidal conductors [4], [5]. The use of more complex sources or volume conductors is restricted to numerical approximations [6], [7].

Solving for the magnetic field generated by source currents within a restricted volume is a non-trivial task and has been limited to dipole sources. If the current flow $\mathbf{J}(\mathbf{r}')$ can be fully described within the conducting region Σ , then the law of Biot-Savart can be applied to calculate the field $\mathbf{B}$ at any position $\mathbf{r}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{\Sigma} \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dv' \quad (1)$$

Obtaining an expression for $\mathbf{J}$ requires solving for the potential V everywhere in the volume [3]. Since V is known, rather than solving the volume integral in (1) it may be more efficient to calculate the vector magnetic potential $\mathbf{A}(\mathbf{r})$ using the surface integral formulation published by Geselowitz [8]

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \left[\int_{\Sigma} \frac{\mathbf{J}^i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv' - \sigma \int_S \frac{V(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS \right] \quad (2)$$

from which the magnetic field is readily obtained as

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \quad (3)$$

In eqn. (2) the current source is separated into an impressed current $\mathbf{J}^i$ which is non-ohmic, and an ohmic component whose contribution is reflected in the

surface integral of the potential $V(\mathbf{r}')$ times the outward surface vector $d\mathbf{S}'$. The closed-form solution for the surface potential V of a current dipole exists for only a few conductor shapes [9], [10] and the task of solving (2) and (3) is still non-trivial [4].

A different approach to solving the same problem was recently suggested by Ferguson and Durand [11] whereby any current source can be modelled as a sequence of point current sources or monopoles. For example, the volume currents generated by a current dipole are equivalent to the currents generated by a constant current point source located at the head of the dipole, and a constant current point sink located at the tail of the dipole. Superposition implies the magnetic field of the dipole volume current is simply the sum of the magnetic fields generated by the point source and sink.

This unique approach requires solving for the magnetic field of an unbalanced current source in a restricted volume conductor. We recently demonstrated that the solution can be obtained from the known solution to the dipole problem in a spherical and semi-infinite volume conductor [see Chapter II and ref. 11]. In this paper, we demonstrate that the problem can be solved directly from Maxwell's equations. While developing this approach, it became clear that the choice of a source model was critical. If the point source is merely assumed to represent a point from which current flows at a steady rate, then a paradox occurs in which the assumption of steady-state field conditions is

violated. This model, referred to as the "simple" model, is described in the following section. A satisfactory model is then presented which will be referred to as a "two-part" model. The two-part model is then used to prove that, for specific source configurations and volume conductors, point current sources generate no magnetic field outside the conductor. These elements of the null space of $\mathbf{B}$ can therefore be added to a problem without altering the magnetic field. This approach, which was first suggested for solving electrostatic problems [12], is shown to be useful for solving for the field of point sources which are not elements of the null space. The magnetic field from the point source within both the semi-infinite and spherical volume conductor is derived.

3.3 A Model for a Point Current Source

A point current source can be defined as a point from which conduction current flows at a steady rate in all directions. Initially, the concept appears to be very simple to implement. The analogy of a point charge and a point current source implies the existence of a correspondence between an electrostatic and a steady current problem: i.e. solving for the electric flux density of a charge distribution leads to the solution of currents (and thereby magnetic fields) from a point source distribution. However, this approach leads to a paradox in the formulation of the problem and a violation of Maxwell's equations. This will be demonstrated by considering the currents generated by a point current source in a volume conductor of infinite extent.

The volume conductor is assumed to have conductivity σ , permittivity ϵ and permeability μ which are isotropic, homogenous and time-invariant properties. The following relationships can be defined, regardless of the source, for any conduction current density $\mathbf{J}$, electric field intensity $\mathbf{E}$, electric flux density (or displacement field) $\mathbf{D}$, magnetic field intensity $\mathbf{H}$ and magnetic flux density $\mathbf{B}$ within the conductor

$$\mathbf{J} = \sigma \mathbf{E} \quad (4a)$$

$$\mathbf{D} = \epsilon \mathbf{E} \quad (4b)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (4c)$$

Maxwell's equations which are to be satisfied within the conductor can be expressed in the time-varying differential (or point) form as

$$\nabla \cdot \mathbf{D} = \rho \quad (5a)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (5b)$$

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \quad (5c)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (5d)$$

where ρ is the charge density at any point in the volume conductor. The continuity equation

$$\nabla \cdot \mathbf{J} = \frac{-\partial \rho}{\partial t} \quad (6)$$

can be combined with Maxwell's equation (5d) to produce a more general statement about the conservation of charge

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad (7)$$

After substituting the common notation for displacement current density $\mathbf{J}_d = \partial \mathbf{D} / \partial t$, it becomes clear that conservation of charge is satisfied whenever sources (sinks) of conduction current are superimposed on sinks (sources) of displacement current.

Initially, we assume the problem is steady state, for which a constant current source can be modelled as a point from which conduction current flows equally in all directions (see Fig. III-1(a)). The strength I of the current source is defined as the total current crossing any closed surface that encloses the source. The conduction current density at any point in space is therefore

$$\mathbf{J} = \frac{I}{4\pi r^2} \mathbf{a}_r \quad (8)$$

where the unit vector $\mathbf{a}_r$ is directed from the source to the observation point. The current flows as a result of the electric field $\mathbf{E} = \mathbf{J} / \sigma$ and the electric flux can then be found using eqn. (4b) as

$$\mathbf{D} = \frac{\epsilon}{\sigma} \frac{I}{4\pi r^2} \mathbf{a}_r \quad (9)$$

which is consistent with the initial assumption of steady state fields. However, the surface integral of $\mathbf{D}$ over any closed surface must equal the charge enclosed by eqn. (5a) and, since the source represents a point from which charge is constantly being lost (as conduction current), $\mathbf{D}$ should not be static. Conservation of charge (7) requires that any point with a non-zero divergence of

Fig. III-1. Two different models of a point current source. (a) A simple source which cannot maintain a steady conduction current without violating Maxwell's equation. (b) A two-part model in which (i) the current I is supplied through a thin insulated wire to an uninsulated tip and (ii) the current I is sunk from the insulated end of the wire. The first part of the model contains only conduction current; the second part contains only displacement current. (c) The cross products of dJ and a_R from the two elements are equal and opposite, proving that J or J_d do not generate a magnetic field in an infinite volume conductor.

J must have a corresponding non-zero divergence of $\mathbf{J}_d$ ($\partial \mathbf{D} / \partial t$). Since **J** has a non-zero divergence at the source point, then (7) requires

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{-I}{4\pi r^2} \mathbf{a}_r \quad (10)$$

If **D** varies with time, then **E** and **J** are also time-varying which contradicts the original assumption of steady state fields.

The solution to this paradox is that the conduction current is not completely described by eqn. (8). A complete description of the conduction current can be obtained in an iterative process, beginning with the solution to eqn. (10). Integrating (10) with respect to time yields a solution $\mathbf{D}_1$, and letting $\mathbf{J}_1 = \sigma/\epsilon \mathbf{D}_1$

$$\mathbf{D}_1 = (t + k_1) \frac{-I}{4\pi r^2} \mathbf{a}_r \quad (11)$$

$$\mathbf{J}_1 = \frac{\sigma}{\epsilon} (t + k_1) \frac{-I}{4\pi r^2} \mathbf{a}_r \quad (12)$$

The meaning of the constant of integration k_1 will be examined later. Similar to the development of eqn. (10) earlier, continuity of charge requires that the non-zero divergence of $\mathbf{J}_1$ at the source position requires another displacement current term by (7)

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\sigma}{\epsilon} (t + k_1) \frac{I}{4\pi r^2} \mathbf{a}_r \quad (13)$$

Integrating this equation to get $\mathbf{D}_2$ (and the resultant $\mathbf{J}_2$) produces the next parts in

the iterative process

$$\mathbf{D}_2 = \frac{\sigma}{\varepsilon} \left(\frac{t^2}{2} + k_1 t + k_2 \right) \frac{I}{4\pi r^2} \mathbf{a}_r \quad (14)$$

$$\mathbf{J}_2 = \frac{\sigma^2}{\varepsilon^2} \left(\frac{t^2}{2} + k_1 t + k_2 \right) \frac{I}{4\pi r^2} \mathbf{a}_r \quad (15)$$

The above procedure is repeated indefinitely, until an infinite series is generated describing the conduction current and electric flux density maintained in the conductor. The series for $\mathbf{J}$, whose first three terms are supplied by eqns. (8), (12) and (15) is

$$\begin{aligned} \mathbf{J}_{\text{series}} &= \frac{I}{4\pi r^2} \left[1 - k_1 \left(\frac{\sigma}{\varepsilon} \right) + k_2 \left(\frac{\sigma}{\varepsilon} \right)^2 - k_3 \left(\frac{\sigma}{\varepsilon} \right)^3 + \dots \right] \\ &\quad \cdot \left[1 - \left(\frac{\sigma}{\varepsilon} t \right) + \frac{1}{2} \left(\frac{\sigma}{\varepsilon} t \right)^2 - \frac{1}{3!} \left(\frac{\sigma}{\varepsilon} t \right)^3 + \dots \right] \mathbf{a}_r \\ &= K \frac{I}{4\pi r^2} e^{-(\sigma/\varepsilon)t} \mathbf{a}_r \end{aligned} \quad (16)$$

The series for $\mathbf{D}$ whose first three terms are supplied in (9), (11) and (14) is

$$\mathbf{D}_{\text{series}} = K \frac{\varepsilon}{\sigma} \frac{I}{4\pi r^2} e^{-(\sigma/\varepsilon)t} \mathbf{a}_r \quad (17)$$

The effect of the constants of integration is lumped into a single time-invariant constant K which only acts as a scale factor for both series. It is easily verified that $\mathbf{J}_{\text{series}}$ and $\mathbf{D}_{\text{series}}$ satisfy charge continuity in eqn. (7) and the fundamental definitions in (4a) and (4b).

The conclusion to be drawn from (16) and (17) is that a constant conduction current cannot be maintained in the volume conductor using this simple model. The earlier paradox is resolved once the initial equations are understood to be only the first iterations in an infinite series. The model requires an infinite charge density to be maintained at the point source with a constant drop in the charge over time. However, any charge placed in a volume conductor must necessarily decay exponentially (not constantly) with time. This is easily shown by combining (6) and (5a) into a single equation and noting that $\mathbf{J} = (\sigma/\epsilon)\mathbf{D}$

$$\rho = \frac{-\epsilon}{\sigma} \frac{\partial \rho}{\partial t} \quad (18)$$

The solution describes the charge density at any point in the volume conductor

$$\rho = \rho_0 e^{-(\sigma/\epsilon)t} \quad (19)$$

where ρ_0 is the charge density at time $t=0$. The rate of charge decay is governed by this exponential function, with a decay constant (or relaxation time constant) $\tau = \epsilon/\sigma$ for the volume conductor. Any charge placed in a volume conductor cannot be made to decay at a constant rate and still satisfy Maxwell's equations.

The meaning of the constant K in (16) and (17) can now be explained in terms of the initial charge ρ_0 at the point source. Substituting the equation derived for the series expansion of the current $\mathbf{J}_{\text{series}}$ (16) into the continuity equation (6) and noting the relationship described in (18) for the charge density yields

$$K \cdot I = \frac{\sigma}{\epsilon} \rho_0 \quad (20)$$

This equation describes the initial charge density ρ_0 required at the source based on the current density and the scaling factor K. Clearly, K acts only to scale the charge required at the source to provide the conduction current in (16).

The original assumption that a point current source generates a steady conduction current cannot be satisfied simultaneous with a displacement current. Since neither type of current can just be ignored, the solution is to design a model which permits the effect of each current to be examined separately. To isolate the effect of the conduction current, the displacement current can be removed once a wire is connected to the point source. This effectively supplies the charge to the source and eliminates $\partial \rho / \partial t$ at the source position. The wire can be modelled as an infinitely thin wire of semi-infinite length. The wire current I is oriented towards the source and the wire is "open" ended at the source (supplying current to the volume). The resultant conduction current is given by eqn. (8). The wire, however, introduces its own magnetic field. The added effect of the wire to the magnetic field can then be removed by considering the second part of the model in which the effect of displacement current is examined. The current in the wire is reversed and the end of the wire is insulated at the source point. The insulated end creates a displacement current in the conductor given by (10). This two part model, see Fig. III-1(b), now allows the conduction and displacement currents to be analyzed independently. The two parts of the model will be referred to as the "conduction" and "displacement" parts.

3.4 The Magnetic Field of Point Sources

3.4.1 The null space of $\mathbf{B}$

The null space of the magnetic field $\mathbf{B}$ refers to the family of volume conductors for which one or more point current sources generate no exterior magnetic field. In this section, it is shown that for any arbitrary distribution of point sources, an infinite number of volumes belong to the null space.

A first approach is to consider an infinite volume conductor. The two-part model of the point current source does not generate a magnetic field $\mathbf{B}$ in an infinite volume conductor. This is because a) the currents flowing in the wires are equal and opposite and therefore do not contribute to the field and b) the volume conduction current $\mathbf{J}$ generated by the conduction part of the model is equal and opposite to the volume displacement current $\mathbf{J}_d$ in the other part. It can also be proven that neither $\mathbf{J}$ nor $\mathbf{J}_d$ individually contribute to $\mathbf{B}$. The symmetry of the problem implies this conclusion, although it can be proven more rigorously using the law of Biot-Savart. The contribution $d\mathbf{B}$ of any differential tube of current $d\mathbf{J} = \mathbf{J} \cdot d\mathbf{l} \cdot d\mathbf{s}$ (or $\mathbf{J}_d \cdot d\mathbf{l} \cdot d\mathbf{s}$) to the magnetic field at a point a distance R away from the element is given by

$$d\mathbf{B} = \frac{\mu}{4\pi R^2} \mathbf{J} \times \mathbf{a}_R dl ds \quad (21)$$

where the unit vector $\mathbf{a}_R$ is oriented as shown in Fig. III-1(c). For every element of current, there exists another element with the same current whose cross-product term in (21) will have the opposite sign. The net $\mathbf{B}$ field is zero for any current or superposition of currents whose behavior is described by eqn. (8). If one point source does not generate a magnetic field, then superposition implies this is also true for any number of point sources.

The following theorem can now be proven for a point source, or an arbitrary number of sources, located within a closed volume conductor: *no magnetic field is generated outside a volume conductor by enclosed point current sources if the conductor boundary is chosen to coincide with an equipotential surface of the sources.* An equipotential surface is defined as the loci of all points whose potential at any given time is the same and which encloses all the sources. The important property of the equipotential surface is the lack of tangential currents on the surface. This property, along with the essential boundary conditions at a dielectric, will be used to prove that $\mathbf{B}$ is zero.

Throughout this proof, the volume contained within the conductor will be referred to as region 1, and the region outside as region 2. It is assumed that region 2 has zero conductivity but shares the same permeability and permittivity as region 1. It is also assumed that "current" refers to the summed current of all

sources. At the boundary, Gauss's Law implies that any change in the electric flux crossing the boundary must be compensated by a surface charge distribution

$$\mathbf{D}_{n2} - \mathbf{D}_{n1} = \rho_s \quad (22)$$

where the normal components, denoted with the subscript 'n', are oriented outward from region 1 to region 2. This equation must also hold for time-varying fields

$$\frac{\partial \mathbf{D}_{n2}}{\partial t} - \frac{\partial \mathbf{D}_{n1}}{\partial t} = \frac{\partial \rho_s}{\partial t} \quad (23)$$

The boundary condition on the tangential field is determined by the conservative nature of $\mathbf{E}$ and requires

$$\mathbf{E}_{t1} = \mathbf{E}_{t2} \quad (24)$$

The boundary surface is defined in the problem such that tangential fields do not exist, explicitly satisfying eqn. (24). Since no current flows tangential to the surface, the source conduction current (from the conduction part of the model) is everywhere normal to the surface. This normal (or total) component of $\mathbf{J}$ will cause a surface charge accumulation given by (6). In order to satisfy the boundary condition in eqn. (23), the charge accumulation must cause a displacement current in region 2 given by

$$\frac{\partial \mathbf{D}_{n2}}{\partial t} = \mathbf{J} \quad (25)$$

This now defines the boundary condition on $\mathbf{J}_d$ outside the volume conductor. Simply stated, $\mathbf{J}$ in region 1 becomes $\mathbf{J}_d$ in region 2 with the same normal angle to the surface and the same strength. Since the flow of $\mathbf{J}_d$ and $\mathbf{J}$ are governed by the

same equations, then $\mathbf{J}_d$ in region 2 is exactly the same flow that $\mathbf{J}$ would have assumed if the surface boundary did not exist. The analysis applied to the displacement part of the model yields the conclusion that $\mathbf{J}_d$ is unchanged in region 2 by the volume surface.

The effect of the equipotential surface is to transform conduction current that would normally flow in the infinite conductor to a displacement current in the non-conducting region. The conductor has not changed the overall vector field for the summed current ($\mathbf{J} + \mathbf{J}_d$). Since Maxwell's equation (5d) implies the magnetic field is sensitive only to the sum of $\mathbf{J}$ and $\mathbf{J}_d$, then the magnetic field is not altered from the infinite conductor and is therefore zero for an equipotential surface. It should be noted that the sources need not have uniform strength, and the source strengths can each be scaled by the same scalar multiple without changing the null magnetic field.

3.4.2 The spherical conductor

The theorem developed in the preceding section can be applied to an analysis of the magnetic field generated by a point current source located in a spherical volume conductor. The position of the source is arbitrary but is not located in the center since the surface would then be equipotential and the solution is trivial. The problem is unusual because the point source is not balanced by a corresponding sink, resulting in a time-varying surface charge distribution which

generates a displacement current outside the volume conductor.

This problem can be greatly simplified by balancing the current sources and sinks within the volume. Any number of point sources or sinks can be added to the volume conductor in order to help obtain the solution as long as they do not alter the magnetic field. If the surface is equipotential for some source distribution, those sources can be added to the original problem since they contribute no magnetic field. The total current strength of the added sources can be scaled such that their net current strength is equal and opposite to the original source. The problem is now a balanced current problem and much easier to solve.

For the problem of a non-centered point source of strength I in a spherical conductor, Fig. III-2(a), a corresponding sink of strength $-I$ can be added at the center of the sphere without altering the solution, Fig. III-2(b). The wires connected to the source and sink are arbitrary and are chosen to simplify the problem. Allowing the wires from the added sink to run coincident with the wires of the original source creates an effective cancellation of the magnetic fields generated by these wires since the currents are equal but opposite. The part of the sink's wire that runs beyond the original source wire is chosen to run straight from the source to the sink. The problem now reduces to finding the $\mathbf{B}$ field of the volume currents and two straight wire segments as shown in Fig. III-2(c).

Fig. III-2. Graphical solution of the general problem in a sphere. Details regarding the wire insulation are removed for clarity, but the uninsulated tip is shown with a closed circle and the insulated tip with an open circle. Each wire carries a current I in the direction shown. (a) The original two-part source positioned in the sphere. (b) The addition of a sink at the center does not alter B . (c) After simplification, the problem is reduced to solving the magnetic field of short wire segments.

From symmetry, $\mathbf{B}$ must be oriented around the axis defined by the wire segments and Ampere's law can be used to calculate $\mathbf{B}$

$$\int_L \mathbf{B} \cdot d\mathbf{l} = \int_S (\mathbf{J} + \mathbf{J}_d) \cdot d\mathbf{s} \quad (26)$$

The total current passing through the loop shown in the conduction current part of Fig. III-2(c) is identically zero since the wire current is balanced by the returning conduction current within the volume. The magnetic field generated by the displacement currents and wire segment in Fig. III-2(c) can be calculated using either the law of Biot-Savart (eqn. (1)) or Ampere's law [13]. The general solution, derived from either approach, for the magnetic field of a wire segment running along the z -axis from $z=z_1$ to $z=z_2$ with current running from z_1 to z_2 , will precess around the z -axis and can be described by

$$\mathbf{B} = \mu \frac{I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z - z_1}{\sqrt{x^2 + y^2 + (z - z_1)^2}} - \frac{z - z_2}{\sqrt{x^2 + y^2 + (z - z_2)^2}} \right] \mathbf{a}_\phi \quad (27)$$

where $\mathbf{a}_\phi$ is the field orientation around the z axis and the field point is (x, y, z) .

This result can be written for a point source positioned at a point $z_1=z_0$ on the z axis, located in a spherical volume conductor centered at the origin ($z_2=0$) as

$$\mathbf{B} = \mu \frac{I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z - z_0}{\sqrt{x^2 + y^2 + (z - z_0)^2}} - \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \mathbf{a}_\phi \quad (28)$$

3.4.3 Semi-infinite conductor

The solution of the magnetic field of a point source in the spherical volume can very easily be adapted to find the magnetic field for a point source in a semi-infinite volume conductor. The latter volume is equivalent to a spherical volume as the radius of curvature approaches infinity and the center of the sphere becomes a point at infinity. The magnetic field of the source in the sphere was shown to be equivalent to that of a wire running from the source to the center of the sphere (oriented normal to the surface). The wire which generates the equivalent magnetic field in the semi-infinite volume still originates at the point source and is still oriented orthogonal to the surface, but is now a semi-infinite wire extending into the volume conductor.

Using the same notation as the previous section, we can consider the source to be located at $z=z_0$ on the z -axis in a semi-infinite volume whose surface is defined by a plane of constant z . If the conducting volume extends to $z = -\infty$, then the semi-infinite line that generates the equivalent field extends from $z=z_0$ to $z=-\infty$. The magnetic field is oriented around the z axis and, from eqn. (27) is given by

$$\mathbf{B} = \mu \frac{I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{(z-z_0)}{\sqrt{x^2 + y^2 + (z-z_0)^2}} - 1 \right] \mathbf{a}_\phi \quad (29)$$

The same result for this problem was obtained by Sezginer and Habashy using a method of images [14].

3.5 Discussion

This analysis defines a new fundamental current source and a new family of surfaces for which configurations of this source generate no magnetic field. The current monopole is the fundamental current source because any source which generates volume currents can be modelled using a series of monopoles. The "magnetically silent" nature of monopole configurations in specific volumes results in a new approach for calculating the magnetic field of higher order terms (eg. dipoles) in these volumes. The spherical volume conductor demonstrates the basic concepts resulting from this method. Whenever the volume conductor can be modelled as an equipotential surface for a point source or sink distribution, then the magnetic field of an arbitrary source can be modelled as a) the field generated by open-ended wire segments (and the return currents) running from the point source to the sinks defining the surface, plus b) the field generated by similarly oriented close-ended wire segments. The source/sink distribution which generates the equipotential surface may consist of both sources and sinks whose strengths may all be different. It is only necessary to scale the strengths so the summed strength is equal and opposite to the original point source in order to balance the currents in the volume.

These results can also be applied to obtaining the magnetic fields of the volume currents of dipoles or other higher order current terms. The approach is identical to that used in obtaining the potentials generated by higher order terms in

electrostatic problems [12]. Modelling the volume current of a dipole, for example, is equivalent to placing a point current source at the dipole head and a point current sink at the dipole tail. If $\mathbf{B}(\mathbf{r}, \mathbf{r}')$ is the magnetic field at $\mathbf{r}$ due to the point source at $\mathbf{r}'$, then the magnetic field of the point sink at $\mathbf{r}'+\mathbf{a}$ is $-\mathbf{B}(\mathbf{r}, \mathbf{r}'+\mathbf{a})$. The field of the sink and source together is given as $\mathbf{B}(\mathbf{r}, \mathbf{r}') - \mathbf{B}(\mathbf{r}, \mathbf{r}'+\mathbf{a})$. The field of the dipole results when the length of $\mathbf{a}$ approaches zero and the source strength I approach infinity. Each component of the magnetic field of the dipole (eg. B_x) can therefore be derived from the gradient of each component of monopole magnetic field, the gradient being taken with respect to the source coordinates.

This relationship between the dipole and monopole solutions can also be used to derive the monopole solution from the known dipole solution. Since the dipole solution is a gradient of the monopole solution, it can be integrated with respect to the source coordinates to generate the monopole solution. This approach, fully described in [11] for the spherical and semi-infinite volume conductors, can only provide a monopole solution that is accurate to within a constant of the integration. However, the monopole solutions presented here agree with those derived from the dipole, verifying that the constant of integration was properly chosen in [11].

Although the analysis so far has relied on the addition of sinks and sources for which the surface is equipotential, this requirement can be ignored when solving for the magnetic field of higher order current terms. This is

because higher order current terms are balanced current sources (eg. dipole), which can be broken down into separate unbalanced parts (eg. dipole head and tail). Any arbitrary sources can be added to the volume to help solve one of these parts if an equivalent but opposite source is added when dealing with the other part. In this manner, the effect of the added sources cancel out. This was not possible in the earlier analysis because there was only one part to the problem (a point source) and the added source had to belong to the null space of $\mathbf{B}$.

We are currently working on applications of this theory in spheroidal conductors. The published solutions for the magnetic field of a current dipole in prolate and oblate spheroids consist of double-infinite summations of up to 13 different Legendre polynomial terms [4] and require lengthy computational times. Construction of a dipole solution from the monopole solution seems to offer significant time savings with little loss in accuracy. The extend to which this theory can be applied to other volumes is also being investigated.

3.6 References

- [1] S.J. Williamson and L. Kaufman, "Biomagnetism," J. Magn. Mag. Mat., vol. 22, pp. 129-201, 1981.
- [2] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [3] J. Sarvas, "Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem," Phys. Med. Biol., vol. 32, pp. 11-22, 1987.

- [4] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [5] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.
- [6] B.N. Cuffin, "Effects of head shape on EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-37, pp. 44-52, 1990.
- [7] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncitia," *Biophys. J.*, vol. 51, pp. 557-568, 1987.
- [8] D.B. Geselowitz, "On the magnetic field generated outside an inhomogenous volume conductor by internal current sources," *IEEE Trans. Mag.*, vol. 6, pp. 346-347, 1970.
- [9] E. Frank, "Electric potential produced by two point current sources in a homogeneous conducting sphere," *J. Appl. Phys.*, vol. 23, pp. 1225-1228, 1952.
- [10] P.M. Berry, "Appendix: N,M space harmonics of the oblate spheroid," *Ann. N.Y. Acad. Sci.*, vol. 65, pp. 1126-1134, 1956.
- [11] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," *IEEE Trans. Magn.*, vol. MAG-27, pp. 758-767, 1991.
- [12] R.E. Collin and R. Plonsey, "A note on dipole sources in conducting biological tissues," *Bull. Math. Biol.*, vol. 40, pp. 201-209, 1977.
- [13] T.A. Weber and D.J. Macomb, "On the equivalence of the laws of Biot-Savart and Ampere," *Am. J. Phys.*, vol. 57, pp. 57-59, 1989.
- [14] A. Sezginer and T.M. Habashy, "An image method to compute the static magnetic field due to currents injected into a homogenous, conducting, and magnetically polarizable half-space," *Radio Science*, vol. 23, pp. 41-45, 1988.

CHAPTER IV
AN EXACT ANALYSIS OF THE MAGNETIC FIELD
OF NEURAL SOURCES IN A SEMI-INFINITE
VOLUME CONDUCTOR

In this chapter, exact solutions are derived for the magnetic field generated by current filaments (tubes of current), single neurons, and neuronal ensembles. This chapter will be submitted to IEEE Transactions on Biomedical Engineering with authors A.S. Ferguson and D. Durand. This work was supported in part by the National Science Foundation Presidential Young Investigator Award to D. Durand. The authors are with the Applied Neural Control Laboratory, Department of Biomedical Engineering, Case Western Reserve University, Cleveland, OH 44106.

4.1 Abstract

A theoretical analysis of the magnetic fields generated by current filaments (or finite length tube), passive dendrites, and brain slices located in a semi-infinite conducting medium is presented. Exact solutions of the magnetic field are obtained using current monopoles as models for extracellular current sources. This unique approach leads to the derivation of the conditions whereby the magnetic field of the extracellular and intracellular currents of each source are indistinguishable from those of a current dipole. The relative errors associated with predicting the strength and location of a current filament, using the dipole approximation, are shown to be less than 1% once the observation distance exceeds twice the filament length. The field generated by single dendrites and multiple dendrites in a slice model is shown to be dipolar with an equivalent dipole moment of 0.14 pA-m per dendrite.

4.2 Introduction

Recording magnetic fields of neural currents sources is now an accepted tool in such areas as clinical diagnosis of disease and the study of higher order brain functions [1], [2]. The inverse problem is a fundamental part of this procedure, whereby the underlying neural currents are generated by an assumed source in a specific volume conductor; the problem is to relate the source strength, position and orientation to the observed magnetic field data.

Researchers face an unfortunate paradox in applying the inverse problem. The magnetic field being measured is often extremely weak and falls off with distance as d^{-2} or d^{-3} . It is desirable to position the sensor as close as possible to the neural source to improve the signal to noise ratio. In fact, the image quality can be improved using a noisier sensor when the observation distance is reduced [3]. A suggested rule of thumb [3] is that observation distance should be no greater than the characteristic length of the source. However, existing models of sources, such as the current or magnetic dipole, are only valid approximations for neural sources whenever the source-to-observation distance is large or, equivalently, the spatial extent of the source is small [4]. Improving the image quality with smaller distances may, therefore, invalidate the use of the dipole model being used in the inverse problem.

This problem will become particularly apparent with the advent of newer, smaller sensors. Previous SQUID sensors with bulky dewars could be placed no closer than 10-20 mm from a neural source with a resolution of similar magnitude [3]. Newer sensors can be placed less than 1.5 mm above a current source, and are theoretically capable of resolving distinct sources to within $2.5\ \mu\text{m}$ [5]. The increased noise in these sensors is more than compensated by their proximity to the neural sources. The data provided by these sensors indicates a wealth of information is available about the spatial variation of the source [6]. Previous studies in which a SQUID was placed 17 mm above a cerebellar brain slice [7],

for example, could only estimate the overall dipole strength of the 10 mm² active area; it should now be possible to develop a much better understanding of the brain slice activity because the extent of the active area is well within the resolving power of the new sensors.

The particular choice of a source model is dictated by its a) computational simplicity, b) applicability to the underlying sources, and c) ability to explain most of the observed magnetic field. Although the current dipole (Fig. IV-1) is not an anatomically correct model, it is widely used because it often satisfies all three criteria. The solution for the magnetic field of a current dipole in a volume conductor is known explicitly for semi-infinite, spherical and spheroidal volume conductors [8], [9]. Dipoles appear to be an adequate model to describe neural activity, particularly at far distances [1], [10], [11], and often account for the variation in the observed data even when a large amount of noise is present [12]. When a single dipole cannot properly match the observed data, it is common to use multiple dipole sources, current multipole expansions [13], or more rarely, a numerical approximation of the current waveform [14]. Since the field of a dipole falls off as d^{-2} , whereas the field of quadrupoles, magnetic dipoles, and higher order source terms fall at least as fast as d^{-3} , it is assumed that these sources do not contribute to the magnetic field [11]. Moreover, the field generated by source configurations employing disks or annuli composed of dipoles [15], and sheets or lines of dipoles [16], [17] appear to be indistinguishable from the field of a single dipole.

ANATOMICAL MODEL **ELECTROPHYSIOLOGICAL
MODEL** **BIOMAGNETIC MODEL**

Fig. IV-1. Three different models of dendritic branches.

The conditions which must be satisfied in order to approximate other sources as current dipoles have not been explicitly determined. The new and smaller sensors require a better understanding of the conditions implicit in the use of this source model. The purpose of this paper is to develop exact equations for the magnetic field of neural sources and to derive the conditions for which a simpler model is a valid approximation.

The following sections analyze the magnetic field generated by a variety of current sources, using a novel approach to accurately model the fields of extracellular current. By modelling each patch of membrane as a point current source in a volume conductor, it is possible apply the recently published solution for the magnetic field of point current sources (Chapter II and [18]) to calculate the magnetic field generated by an entire neural membrane with any desired accuracy. The following section presents a detailed analysis of this approach as it applies to a finite-length tube of current located in a semi-infinite conducting volume. The results are used to determine the various conditions necessary for the current tube to be accurately modelled as a dipole. The same methodology and analysis is applied in subsequent sections to passive neural membrane models such as dendrites and whole brain slices.

4.3 Elementary Current Filament

A first step towards developing a more accurate model of a neuronal current source is to permit the source model to have a finite length. This can be approximated by a string of dipoles lying "head-to-tail" in a straight line. The magnetic field generated by this model in a spherical conductor was studied by Cuffin and Cohen [17] and Okada [22]. Their analysis required the use of the complicated dipole solution to model a finite-length tube of current, based on a series of dipoles which inherently are infinitely short sources. The following approach achieves the same objectives, although in a semi-infinite volume conductor, with a drastic reduction in complexity by using the monopole formulation for a true finite-length tube of current.

A finite-length current filament is shown in Fig. IV-2(a). The filament with an overall length of $2\Delta x$ is presumed to be located at some depth in a semi-infinite conducting volume, oriented in the $+x$ direction and extend from $x = -\Delta x$ to $x = +\Delta x$. The filament is centered at the origin and the volume conductor extends to some plane $z > 0$. The position of the surface is arbitrary since its location does not affect the computation of the magnetic field. A current I flows within the filament and is constrained to return within the volume conductor. Because the intent is to use this filament as a model for the current flow due to a portion or an entire axon, dendrite, nerve bundle, or cortical layer, it is convenient for the remainder of this section to use the terms "intracellular" and

Fig. IV-2. Schematic diagram of the current and magnetic field orientations of the current filament. (a) The current filament of length $2\Delta x$ is located at the origin and oriented in the +x direction. The current consists of a volume ("extracellular") and internal ("intracellular") current. (b) The magnetic field B_i generated by the intracellular current lies parallel to the y-z plane. (c) The magnetic field of the extracellular current is equivalent to the summed magnetic fields B_{e-} and B_{e+} generated by two current monopoles embedded in the semi-infinite volume.

"extracellular" current flow to refer to the current flow inside and outside the filament respectively.

4.3.1 Intracellular current

Symmetry suggests that the magnetic field B_i generated by intracellular current must be oriented around the filament as shown in Fig. IV-2(b) so the magnetic flux lines must lie parallel to the y-z plane. The magnitude of B_i , denoted as B_i , is easily obtained by applying the law of Biot-Savart

$$B_i = \frac{\mu I}{4\pi \sqrt{y^2 + z^2}} \left(\frac{x + \Delta x}{\sqrt{(x + \Delta x)^2 + y^2 + z^2}} - \frac{x - \Delta x}{\sqrt{(x - \Delta x)^2 + y^2 + z^2}} \right) \quad (1)$$

By defining the auxiliary function $f_i(x, y, z)$ as

$$f_i(x, y, z) = \frac{\mu I}{4\pi \sqrt{y^2 + z^2}} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \quad (2)$$

then B_i can be written in a more compact form as

$$B_i = f_i(x + \Delta x, y, z) - f_i(x - \Delta x, y, z) \quad (3)$$

A Taylor's series expansion allows any function to be evaluated at some point $x + \Delta x$ based on the knowledge of the function and its derivatives at the point x

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2!} + f^{[3]}(x)\frac{\Delta x^3}{3!} + f^{[4]}(x)\frac{\Delta x^4}{4!} + \dots \quad (4)$$

Applying this expansion to (3) results in

$$B_i = 2 \left\{ \Delta x \frac{\partial f_i(x,y,z)}{\partial x} + \frac{\Delta x^3}{3!} \frac{\partial^3 f_i(x,y,z)}{\partial x^3} + \frac{\Delta x^5}{5!} \frac{\partial^5 f_i(x,y,z)}{\partial x^5} + \mathcal{O}(\Delta x^7) \right\} \quad (5)$$

This is still an exact equation for the magnetic field generated by the intracellular filamentary current of length $2\Delta x$. However, as the length of the filament approaches zero or the observation distance increases, the higher order terms on the right hand side of (5) will decrease in significance. It is therefore instructive to examine each of these terms to understand the conditions under which they may be ignored.

The first term on the right hand side of (5) represents the magnetic field of a dipole located at the origin with dipole strength $P=2I\Delta x$. This occurs as $2\Delta x \rightarrow 0$ and $I \rightarrow \infty$ as long as the product $2I\Delta x$ remains finite. More generally, the magnetic field of a current dipole $\mathbf{P}$ is given by

$$\mathbf{B} = \mu \frac{\mathbf{P} \times \mathbf{a}_r}{4\pi r^2} \quad (6)$$

where $\mathbf{a}_r$ is a unit vector pointing from the dipole to the observation point located a distance r from the dipole. Substituting (2) in to the first term in (5) results in

$$\frac{\mu P}{4\pi d^2} \sin\phi \quad (7)$$

where

$$P = 2\Delta x I$$

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$\sin\phi = \frac{\sqrt{y^2 + z^2}}{d}$$

This clearly satisfies (6) as the magnetic field of a dipole.

The second term on the right hand side of (5) represents an octupole term or, equivalently, the magnetic field of three specially oriented current dipoles. Consider the effect of breaking up the original current filament into three separate filaments, each with the original current I (see Appendix B and Fig. A-1). The first and third filaments extend from $-\Delta x$ to $-\Delta x/2$ and from $+\Delta x/2$ to $+\Delta x$ and have current flowing in the $+x$ direction, similar to the orientation shown in Fig. IV-2(a). The second filament bridges the gap between these two filaments, extending from $-\Delta x/2$ to $+\Delta x/2$, but has current flow directed in the $-x$ direction. The magnetic field of the intracellular currents must still flow, by symmetry, in the pattern shown in Fig. IV-2(b) and has a magnitude B_3 given by

$$\begin{aligned} B_3 &= [f_i(x-\Delta x/2, y, z) - f_i(x-\Delta x, y, z)] + [f_i(x+\Delta x, y, z) - f_i(x+\Delta x/2, y, z)] - \\ &\quad [f_i(x+\Delta x/2, y, z) - f_i(x-\Delta x/2, y, z)] \\ &= f_i(x+\Delta x, y, z) - 2f_i(x+\Delta x/2, y, z) + 2f_i(x-\Delta x/2, y, z) - f_i(x-\Delta x, y, z) \quad (8) \end{aligned}$$

The Taylor's series expansion of each of the terms in (8) yields

$$B_3 = \frac{3}{2} \frac{\Delta x^3}{3!} \frac{\partial^3 f_i(x,y,z)}{\partial x^3} + \frac{15}{8} \frac{\Delta x^5}{5!} \frac{\partial^5 f_i(x,y,z)}{\partial x^5} + \mathcal{O}(\Delta x^7) \quad (9)$$

This series collapses to the octupole solution as $2\Delta x \rightarrow 0$ and $I \rightarrow \infty$ as long as the product $I\Delta x^3/4$ remains finite and equal to the strength of the octupole. It is clear from a comparison of (9) and (5) that the second term in (5) represents the contribution of the octupole whose relative strength (determined by a comparison of the coefficients in (5) and (9)) is 4/3 times the strength of the octupole presented in (8) and (9).

It is now possible to determine the conditions whereby the magnetic field of the current filament is not significantly different from that of a current dipole. In order to show that the higher order terms in (5) may be neglected, it is sufficient to guarantee that for $\Delta x < 1$

$$\Delta x \frac{\partial f_i(x,y,z)}{\partial x} \gg \frac{\Delta x^3}{3!} \frac{\partial^3 f_i(x,y,z)}{\partial x^3} \quad (10)$$

This is similar to finding the minimum distance at which an octupole field may be neglected relative to a dipole field. The octupole field is derived in Appendix B and can be adapted to this configuration by setting $\Delta k = \Delta x/2$

$$B = \frac{3 \mu P \Delta x^2}{8\pi d^4} \sin\phi (5 \cos^2\phi - 1) \quad (11)$$

Weighting the octupole field by 4/3, as determined earlier, and comparing to the dipole field in (7) requires that

$$d^2 \gg 2 \Delta x^2 (5 \cos^2 \phi - 1) \quad (12)$$

which is always satisfied if

$$d \gg \sqrt{8} \Delta x \quad (13)$$

An alternative approach to determine the minimum distance at which the filament is magnetically indifferent from a dipole is to compare the difference between the magnetic fields as a function of distance. Since both fields decrease monotonically with distance from the source, an absolute difference would provide little information. Instead, the relative difference between the magnetic field of the dipole and filament can be computed as

$$\Delta B(z_0) = \text{abs} \left\{ \frac{\max_{\text{all } x,y} [B^d(x,y,z_0) - B^f(x,y,z_0)]}{\max_{\text{all } x,y} [B^f(x,y,z_0)]} \right\} \quad (14)$$

This ratio is the largest difference between the dipole and filament magnetic fields on some observation plane z_0 divided by the largest filament magnetic field on that plane. This ratio can be computed for both B_{yi} and B_{zi} and is shown in Fig. IV-3 as a function of β , the height above the source normalized to source length

$$\beta = \frac{z}{2 \Delta x} \quad (15)$$

Fig. IV-3 also shows the results for this ratio as it applies to the fields generated by extracellular current, which is discussed in the next section. It is evident from Fig. IV-3 that the relative difference drops monotonically as normalized height increase, with the relative difference being less than 1% once the observation

Fig. IV-3. Plots of the relative difference in each component of the magnetic field generated by a current filament and a current dipole positioned in a semi-infinite volume conductor. The dipole and filament are both positioned at the origin and oriented in the +x direction. The relative difference ΔB given by eqn. (14) is defined as the maximum ratio observed on a plane of constant z_0 where the superscript "d" and "f" in eqn. (14) refer to dipole and filament sources respectively. The function $B(x,y,z)$ refers to one of the following functions identified in the plot by different line type:

B_{ye}	B_y due to extracellular current
B_{yi}	B_y due to intracellular current
B_y	B_y due to both intracellular and extracellular current
B_{xe}	B_x due to extracellular current
B_{zi}	B_z due to intracellular current

The ratio ΔB was calculated over an x-y grid of 101 by 101 points. The grid was adjusted so that the point where ΔB was maximized was not at the edge of the grid. The spacing of the grid points was also reduced until the second largest value of ΔB was within 1% of the maximum value so ensure the grid was sufficiently fine to capture the "true" value of the maximum. The abscissa represents the height of the observation plane z_0 relative to the length of the current filament $2\Delta x$.

height is at least 5 times the filament length. The condition established in (13) requires $\beta \gg 1.4$, which guarantees that $\Delta B_i(z_0) \ll 6\%$.

4.3.2 Extracellular current

The extracellular current flow generated by the current filament is equivalent to the current flowing from a point current source positioned at the head ($x = +\Delta x$) of the filament and a point current sink positioned at the tail ($x = -\Delta x$) of the filament (see Fig. IV-2). The magnetic field above a semi-infinite volume conductor generated by a point current source located in the volume conductor has been shown (Chapter II and [18]) to be equivalent to that of a semi-infinite line of current extending from the point current source and oriented away from the conducting surface. The magnetic field at an observation point (x, y, z) generated by a single point current source (or current monopole) located at the origin in a semi-infinite volume conductor is given by

$$B_{\phi}^m(x, y, z) = \frac{\mu I}{4\pi \sqrt{x^2 + y^2}} \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \right] \quad (16)$$

and must, by symmetry, precess around the z axis. The superscript "m" is used to denote the field of a "monopole" source. Eqn. (16) can be expressed for each component as

$$B_{xe}^m(x,y,z) = \frac{-y}{\sqrt{x^2 + y^2}} B_{\phi}^m(x,y,z)$$

$$B_{ye}^m(x,y,z) = \frac{x}{\sqrt{x^2 + y^2}} B_{\phi}^m(x,y,z) \quad (17)$$

where the subscripts "xe" and "ye" indicate the direction of the field and extracellular nature of the generating currents. Unlike the intracellular case, the magnetic field of the extracellular current does not maintain a simple flow pattern for more than one point source. Consequently, the remaining analysis is divided into separate x and y components.

The magnetic field of the filament is the superposition of fields generated by the point source and sink (Fig. IV-1(c)) and is given by

$$B_{xe}(x,y,z) = B_{xe}^m(x-\Delta x,y,z) - B_{xe}^m(x+\Delta x,y,z)$$

$$B_{ye}(x,y,z) = B_{ye}^m(x-\Delta x,y,z) - B_{ye}^m(x+\Delta x,y,z) \quad (18)$$

Using the Taylor's series expansion (4), these equations can be re-written as

$$B_{xe} = -2 \left\{ \Delta x \frac{\partial B_{xe}^m(x,y,z)}{\partial x} + \frac{\Delta x^3}{3!} \frac{\partial^3 B_{xe}^m(x,y,z)}{\partial x^3} + \frac{\Delta x^5}{5!} \frac{\partial^5 B_{xe}^m(x,y,z)}{\partial x^5} + \vartheta(\Delta x^7) \right\} \quad (19a)$$

$$B_{ye} = -2 \left\{ \Delta x \frac{\partial B_{ye}^m(x,y,z)}{\partial x} + \frac{\Delta x^3}{3!} \frac{\partial^3 B_{ye}^m(x,y,z)}{\partial x^3} + \frac{\Delta x^5}{5!} \frac{\partial^5 B_{ye}^m(x,y,z)}{\partial x^5} + \vartheta(\Delta x^7) \right\} \quad (19b)$$

As the length $2\Delta x$ of the filament approaches zero, the higher order terms in the series become negligible. The first term in the series must therefore

represent the magnetic field of the extracellular current of a current dipole located in a semi-infinite volume conductor. Substituting (16) and (17) into (19a) and (19b) and ignoring the higher order terms yields

$$B_{xe}^d = \frac{-\mu (2 I \Delta x) xy}{4\pi [x^2 + y^2]} \left\{ \frac{2}{x^2 + y^2} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \right) + \left(\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \right\} \quad (20a)$$

$$B_{ye}^d = \frac{\mu (2 I \Delta x)}{4\pi [x^2 + y^2]} \left\{ \frac{x^2 - y^2}{x^2 + y^2} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \right) + x^2 \left(\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \right\} \quad (20b)$$

where the current-dipole length product can be replaced by the dipole moment $P_x = 2I\Delta x$. It is easily shown that (20a) and (20b) agree with the published equation for the magnetic field generated by the extracellular current from a current dipole source [8], [18].

Similar to the intracellular analysis, it is interesting to speculate on the conditions whereby the higher order terms in (19a) and (19b) can be neglected. The first terms on the right hand side have been shown to be equivalent to the field generated by a current dipole located at the origin. The analysis presented earlier in developing eqns. (8) and (9) can readily be applied to prove that the second term in (19a) and (19b) is due to a current octupole with the same configuration described by (8). The octupole can be constructed from dipoles each with strength P_x and the overall octupole field would have to be weighted by $4/3$ to account for the coefficients in (20a) and (20b).

However, unlike the intracellular analysis, an exact condition whereby the

octupole field is always less than the dipole field cannot be established due to null zones in the magnetic field of both the current dipole and current filament. For example, both B_{ye} and B_{ye}^d have zero values near the line $y=\pm x$, but slight differences can cause the ratio of the two values to approach zero or ∞ . An alternative approach is to compare the maximum values of B_{xe} and B_{ye} generated by the first and second terms in (19a) and (19b) on a given z plane. The goal is to find a constraint on the distance for which the maximum value attained by the higher order terms in (19a) and (19b) is negligible in comparison to the maximum value of the dipole term.

The magnitude of $\partial B_{ye}^m / \partial x$ is maximized on the z axis ($x=y=0$) and can be obtained by substituting $x=y$ into (20b) and taking the limit as $x \rightarrow 0$

$$\left[\frac{\partial B_{ye}^m(x,y,z)}{\partial x} \right]_{x=y=0} = \frac{-\mu I}{8\pi z^2} \quad (21)$$

The magnitude of $\partial^3 B_{ye}^m / \partial x^3$ can be shown to also achieve a maximum on the z axis which is given by

$$\left[\frac{\partial^3 B_{ye}^m(x,y,z)}{\partial x^3} \right]_{x=y=0} = \frac{9\mu I}{16\pi z^4} \quad (22)$$

Combining these results with (19b) determines the conditions whereby the high order terms in the expansion of B_{ye} can be neglected and the filament can be modelled as a dipole, namely

$$d \gg \frac{\sqrt{3}}{2} \Delta x \quad (23)$$

A similar analysis can be applied to B_{xe} in (19b). The magnitude of $\partial B_{xe}^m / \partial x$ can be shown numerically to reach a maximum at $x=y=0.8283z$ and is given by

$$\left[\frac{\partial B_{xe}^m(x,y,z)}{\partial x} \right]_{x=y=0} = \frac{0.1187 \mu I}{4\pi z^2} \quad (24)$$

The magnitude of $\partial^3 B_{xe}^m / \partial x^3$ can be shown numerically to reach a maximum at $x=0.3981z$, $y=0.5493z$ and is given by

$$\left[\frac{\partial^3 B_{xe}^m(x,y,z)}{\partial x^3} \right]_{x=y=0} = \frac{0.4883 \mu I}{4\pi z^2} \quad (25)$$

The higher order terms in (19a) can therefore be neglected when

$$d \gg 0.828 \Delta x \quad (26)$$

Under this condition, the filament can be modelled as a current dipole for the calculation of B_{xe} .

Similar to the intracellular analysis, eqn. (14) can be applied to each field component to find the minimum distance, normalized to filament length, at which the relative difference between the dipole and filament fields is negligible. Plots of the relative error for B_{xe} and B_{ye} computed with (14) are shown in Fig. IV-3. The conditions established in (23) and (26) requires $\beta \gg 0.41$, which only

guarantees that $\Delta B_i(z_0) \ll 100\%$. However, these conditions were established based on a ratio which set the dipole and higher order terms equal (by taking the ratio of (24) and (25), for example). Using a factor of 10 such that $\beta \gg 4.1$ guarantees that $\Delta B_i(z_0) \ll 1.5\%$. Note the relative error associated with the extracellular current components B_{xe} and B_{ye} is always larger than the intracellular components B_{yi} and B_{zi} .

4.3.3 Error in depth and strength calculations

The depth and strength of a current source can be predicted from a simple analysis of the position and intensity of the maxima of the normal magnetic field (B_z) on a plane above the source [2]. Since only the intracellular current generates a normal field component, this analysis is fairly simple. For both the filament and the dipole current source (with normal field components B_z^f and B_z^d respectively) positioned at the origin and oriented in the +x direction, the field maxima occur on the $x=0$ plane. The location of the maxima is obtained by solving $\partial B / \partial y = 0$ to obtain the relationship between $y_{\max}$ and $z_{\max}$. The maxima for a dipole source whose B field is given by (6) or (7) occurs at the points

$$x_{\max}^d = 0 \qquad y_{\max}^d = \pm \frac{z}{\sqrt{2}} \quad (27)$$

with a peak field strength

$$B_{z\max}^d = \pm \frac{\mu P}{4\pi} \left(\frac{2}{3\sqrt{3} z^2} \right) \quad (28)$$

The maxima of the normal magnetic field generated by the filament can be obtained from (1) and occurs at

$$x_{\max}^f = 0 \quad y_{\max}^f = \pm \frac{z}{2} \frac{1}{A} \left[\sqrt{1 + 8A^2} - 1 \right]^{1/2} \quad (29)$$

where β represents the depth z normalized to the filament length $2\Delta x$ from eqn. (15) and

$$A = \sqrt{\frac{4\beta^2}{4\beta^2 + 1}}$$

At the point $(0, y_{\max}^f, z)$, the peak normal field strength $B_{z\max}^f$ is

$$B_{z\max}^f = \left(\frac{\mu (2 I \Delta x)}{4\pi} \right) \left(\frac{y_{\max}^f}{(y_{\max}^f)^2 + z^2} \right) \frac{1}{\sqrt{(y_{\max}^f)^2 + z^2 + \Delta x^2}} \quad (30)$$

It is easily seen that the filament equations (29) and (30) collapse to the dipole equations (27) and (28) as the length of the filament approaches zero and $P = 2I\Delta x$.

It is now possible to estimate the error that occurs while predicting the depth and strength of a current source that is assumed to be a dipole but is, in fact, a finite length filament. Since the location of the maxima is set by (29), then the assumption that the source is a dipole generates an estimate of the dipole depth, by (27), of

$$z_{\text{est}} = \sqrt{2} |y_{\max}^f| \quad (31)$$

This estimated source depth will always be greater than the actual source depth.

The error in this estimate, $\Delta z = z_{\text{est}} - z$, can be expressed as a relative error (with respect to the actual source depth)

$$\Delta z_r = \frac{z_{\text{est}} - z}{z} \quad (32)$$

Substituting (29) and (31) into this equation, the relative error can be written as

$$\Delta z_r = -1 + \frac{1}{\sqrt{2} A} \left[\sqrt{1 + 8A^2} - 1 \right]^{1/2} \quad (33)$$

A similar analysis can be applied to the prediction of the source strength. Substituting $2y^2 = z^2$ into (28) but noting that y is actually given by (29), then the estimated dipole strength is

$$P_{\text{est}} = \frac{4\pi}{\mu} [3\sqrt{3} (y_{\text{max}}^f)^2] B_{z\text{max}}^f \quad (34)$$

The estimated strength of the dipole is always less than the real strength ($2I\Delta x$) of the current filament. The relative error introduced by this equation is therefore

$$\begin{aligned} \Delta P_r &= \frac{P_{\text{est}} - 2I\Delta x}{2I\Delta x} \\ &= -1 + \frac{3\sqrt{3} \left[\sqrt{1 + 8A^2} - 1 \right]^{3/2}}{\left[\sqrt{1 + 8A^2} + 4A^2 - 1 \right] \left[\sqrt{1 + 8A^2} + 3 \right]^{1/2}} \end{aligned} \quad (35)$$

A plot of the relative error functions (33) and (35) is shown in Fig. IV-4. The relative error in estimating the source strength decreases faster with observation height than the relative error in estimating source depth. Once the

Fig. IV-4. Plot of the relative error in calculating source depth (Δz_r) and source strength (ΔP_r) as a function of observation height normalized to source length.

observation point is positioned at twice the height compared to the source length, then the source position and strength can be estimated with less than 1% error.

4.3.4 Fields of intracellular versus extracellular dipole current

The magnetic flux lines generated by the intracellular dipole current flow parallel to the y-z plane, while those generated by the extracellular current are parallel to the x-y plane. A y-oriented component of the magnetic field is generated by both currents and it is instructive to determine the relative contribution of each current to this component of the overall magnetic field. The relative contributions of the intracellular and extracellular currents to B_y can be compared by setting

$$B_{ye} = k \cdot B_{yi} \quad (36)$$

The proportionality constant k can be obtained from (7) and (20b)

$$k = \frac{-1}{(x^2 + y^2)^2} \left[2x^4 + (x^2 - y^2) \left(y^2 + z^2 - \frac{(x^2 + y^2 + z^2)^{3/2}}{z} \right) \right] \quad (37)$$

Along the z-axis ($x = y = 0$), k reduces to $-1/2$, indicating that the magnetic field due to extracellular currents will be directed opposite to that of the intracellular currents at half the magnitude. This agrees with the dipole results obtained previously [8]. More generally, k also reduces to $-1/2$ at any point on the planes defined by $x = \pm y$, the z axis being a special case where two such planes intersect.

A family of curves can be determined along which the value of k is constant. For example, on the plane defined by $y = 0$, the equation for k reduces to

$$k = \frac{(x^2 + z^2)^{3/2} - z^3}{x^2 z} - 2 \quad y = 0$$

and all lines located on this plane which pass through the origin represent points of constant k . Setting $z=mx$ to define such a line results in

$$k = \frac{(m^2 + 1)^{3/2}}{m} - (m^2 + 2) \quad y = 0 \text{ and } z = mx$$

When $m=0$ (x axis), $k = +\infty$ which is not surprising since the magnetic field due to the intracellular current along the x axis is zero. As m increase, the value of k decreases and reaches zero when

$$z = x \sqrt{\frac{\sqrt{5} - 1}{2}}$$

This line represents part of the null space of the y component of the magnetic field of extracellular currents; i.e. only intracellular current contributes to B_y along this line. Further increase in m produce negative value for k which reaches a limit of $k = -0.5$ as m approaches $+\infty$ (z axis).

The $x = 0$ plane also allows (37) to be simplified as

$$k = 1 + \frac{z^3 - (y^2 + z^2)^{3/2}}{y^2 z} \quad x = 0$$

Lines restricted to this plane which pass through the origin also represents points

of constant k given by

$$k = (m^2 + 1) - \frac{(m^2 + 1)^{3/2}}{m} \quad x = 0 \text{ and } z = my$$

When $m = 0$ (y axis), $k = -\infty$ similar to the effect along the x axis. As m increase, the value of k becomes less negative and reaches $-1/2$ in the limit as m approaches $+\infty$ (z axis). Since k is consistently negative, the magnetic field of the extracellular current does not have a null field in this plane.

Because the values of the ratio k may range continuously from $-\infty$ to $+\infty$, it would appear that the magnetic field generated by the two components of the current are indistinguishable. However, we can take advantage of the fact that magnetic sensors typically employ multiple coils arranged in a gradiometer configuration. Such sensors inherently measure the spatial derivative (or gradient) of the field, the direction of the derivative depending on the orientation of the coils relative to the source. Consequently, if k is defined as the ratio of some differential operators

$$k = \frac{f(B_{ye})}{f(B_{yi})} \quad (38)$$

then it is possible to find an operator that maximizes or minimizes the ratio over a surface domain of interest. If the operator generates a value of k such that $k \ll 1$ over the observation area, then the effect of the extracellular current is significantly reduced. Similarly, if $k \gg 1$ over the same area, then the effect of the intracellular current has been reduced.

A summary of different operators and their effect on the ratio k in (38) is shown in Table IV-1. The magnetic field was calculated at a distance z_0 above the dipole source over an area of interest defined by $-z_0 \leq x \leq z_0$, $-z_0 \leq y \leq z_0$. The ratio k was calculated at 1600 points over this area, from which a cumulative histogram could be constructed defining the probability that a fraction of the area was at or below a particular value of k . The minimum values of k for which 25%, 50%, 75% and 100% of the observation area were below is depicted in Table IV-1. No functions were extremely good at either reducing or increasing k over the entire field of interest. The best function appears to be $(\partial B_{ye}/\partial y) / (\partial B_{yi}/\partial y)$, for which the effect of the extracellular current was reduced by 50% compared to B_{ye}/B_{yi} over the same observation surface.

4.4 Neuronal Models

The core conductor model that is commonly used to model the electrophysiological properties of neurons will be used in this section to calculate the magnetic fields generated by neurons. An excellent review of the development of this model is provided by Rall [26]. Essentially, the core conductor model assumes that neurons can be modelled as a system of long cylinders filled with a conducting material. The cylinder wall represents the neural membrane through which the transmembrane current passes. The "leaky" membrane can be modelled with a parallel combination of membrane resistance and membrane capacitance. Since the transmembrane resistance is usually much

Table IV-1. Ratios of operators of the magnetic field generated by extracellular versus intracellular currents. The values shown are the ratios of the operators $k1/k2$, where $k1$ and $k2$ are listed in the first two columns. Values for the ratio were tabulated at 1600 evenly spaced points on the surface of interest defined by $x \in [1,1]$, $y \in [1,1]$, $z=1$ for a dipole located at the origin and oriented in the +x direction. The dipole is assumed to be located in a semi-infinite conducting medium. B_{ye} and B_{yi} are the y components of the magnetic field generated by the extracellular and intracellular currents respectively. The columns labelled 0% through 100% indicate the values of the ratio $k1/k2$ for that percentile of area as obtained from a cumulative histogram plot of percent area versus $k1/k2$.

$k1^\dagger$	$k2^\dagger$	0%	25%	50%	75%	100%
B_{ye}	B_{yi}	.172	.398	.500	.599	.828
$\partial B_{ye}/\partial x$	$\partial B_{yi}/\partial x$	.562	.739	.841	1.02	1.31
$\partial B_{ye}/\partial y$	$\partial B_{yi}/\partial y$	.000	.104	.198	.269	.438
$\partial B_{ye}/\partial z$	$\partial B_{yi}/\partial z$	.004	.359	.562	.982	2.00
$\partial B_{ye}/\partial t$	$\partial B_{yi}/\partial t$	.000	.443	.529	.965	22.6
$\partial B_{ye}/\partial u$	$\partial B_{yi}/\partial u$	.016	.533	.858	1.33	46.0
$\partial B_{ye}/\partial v$	$\partial B_{yi}/\partial v$	.000	.244	.441	.647	47.0
$\partial B_{ye}/\partial w$	$\partial B_{yi}/\partial w$	.002	.425	.583	1.19	***.†
$\partial^2 B_{ye}/\partial x^2$	$\partial^2 B_{yi}/\partial x^2$	.007	.790	.968	1.23	31.0
$\partial^2 B_{ye}/\partial y^2$	$\partial^2 B_{yi}/\partial y^2$	.003	.110	.219	.434	30.0
$\partial^2 B_{ye}/\partial z^2$	$\partial^2 B_{yi}/\partial z^2$	.008	.486	.930	2.36	191.
$\partial^2 B_{ye}/\partial t^2$	$\partial^2 B_{yi}/\partial t^2$	.001	.412	.543	.985	212.
$\partial^2 B_{ye}/\partial u^2$	$\partial^2 B_{yi}/\partial u^2$	.010	.657	.903	1.08	500.
$\partial^2 B_{ye}/\partial v^2$	$\partial^2 B_{yi}/\partial v^2$	.001	.093	.284	.618	501.
$\partial^2 B_{ye}/\partial w^2$	$\partial^2 B_{yi}/\partial w^2$	.000	.408	.560	.940	786.
$\partial^3 B_{ye}/\partial x^3$	$\partial^3 B_{yi}/\partial x^3$	.037	.808	.948	1.20	22.5
$\partial^3 B_{ye}/\partial y^3$	$\partial^3 B_{yi}/\partial y^3$	.000	.113	.199	.371	23.5
$\partial^3 B_{ye}/\partial z^3$	$\partial^3 B_{yi}/\partial z^3$	.003	.393	.682	1.34	***.†
$\partial^3 B_{ye}/\partial t^3$	$\partial^3 B_{yi}/\partial t^3$	.000	.387	.568	1.06	***.†
$\partial^3 B_{ye}/\partial u^3$	$\partial^3 B_{yi}/\partial u^3$	.005	.655	.918	1.38	180.
$\partial^3 B_{ye}/\partial v^3$	$\partial^3 B_{yi}/\partial v^3$	.001	.146	.374	.865	179.
$\partial^3 B_{ye}/\partial w^3$	$\partial^3 B_{yi}/\partial w^3$	.002	.399	.572	1.05	237.

† The variables r , u , v and w are given by

$$r = \sqrt{x^2 + y^2} \quad u = \sqrt{x^2 + z^2} \quad v = \sqrt{y^2 + z^2} \quad w = \sqrt{x^2 + y^2 + z^2}$$

‡ Values which exceed 999.9 are shown as ***.*

greater than the intracellular resistance, a one-dimensional cable model can be used to predict the flow of current within the cylinder.

Although neurons may contain extensive arborization (branching), it is possible to accurately model the electrical properties of the neuron using just a single cable, often referred to as the "equivalent" cable model. This is based on the assumption that the branching obeys the $3/2$ power rule [26], whereby a parent branch of some diameter d that splits into two daughter branches with diameters d_1 and d_2 maintains the relationship

$$d^{3/2} = d_1^{3/2} + d_2^{3/2}$$

The effect of the two daughter branches on current flow can be lumped into a single cylinder.

In the following sections, the "equivalent" cable model and the fundamental cable equations are used to predict the magnetic field generated by a neuron. It is assumed that the membrane is passive (i.e. membrane properties remain constant and do not change with transmembrane potential). The cable model is used to predict the currents flowing within the neuron, from which the magnetic field can be obtained. The validity of employing the cable model for this purpose is considered in the Discussion.

4.4.1 Intracellular current - theoretical approach

The magnetic field generated by a neuron depends on both the intracellular current and the extracellular current. The basis for the modelling the effects of these currents is depicted in Fig. IV-5. The neuronal cable can be modelled as a series of compartments whose properties are described by an electric analog. From this circuit, both the intracellular and transmembrane current can be computed as a function of position along the neuron. These results can be used in a manner analogous to modelling the electric field generated by active membranes [19], whereby an infinitesimally small length of membrane is modelled as either a ring or disk charge source. The transmembrane current for each segment of the neuron can be modelled as a point source of current, for which the magnetic field is known in special volume conductors (Chapter II and [18]).

The volume conductor in this section is restricted to a semi-infinite space. The passive neuron may be modelled, using linear cable theory, as a one dimensional cable whose electrical properties are determined by intracellular and extracellular resistivity and membrane resistivity and capacitance [20]. If the neuron is oriented along some arbitrary axis ζ , the intracellular current i depends on both the intracellular potential V_i and the intracellular axial resistance per unit length r_a ($\text{k}\Omega/\text{cm}$)

Fig. IV-5. Basis for magnetic source model of neuron. A length of neuron can be divided into segments (a) which can be modelled as individual compartments in an electric cable model (b). The current crossing the membrane for each compartment can be represented as a point current source (c) whose strength is equal to the total transmembrane current for that compartment.

$$-r_a i(\zeta) = \frac{dV_i(\zeta)}{d\zeta} \quad (39)$$

Conservation of charge requires that any change in the intracellular current must be compensated by a change in the transmembrane current such that

$$-i_m(\zeta) = \frac{di(\zeta)}{d\zeta} \quad (40)$$

where i_m is the transmembrane current per unit length.

The neuron is modelled as consisting of a soma (cell body) positioned at the origin and a cylinder of physical length L oriented in the $+x$ direction. The boundary of the semi-infinite volume conductor is assumed to lie parallel to the x - y plane and lie above the neuron ($z>0$). The exact position of the surface is not important since it does not enter into the equations.

The magnetic field of the neuron may be represented as the sum of magnetic fields due to each compartment of the neuron because the system is linear and superposition may be applied. A segment of the neuron, shown in Fig. IV-5(c), consists of a finite length current filament of length $2\Delta x$ centered at x_k ($0 < x_k < L$). The intracellular current in this segment, denoted i_k , generates a magnetic field which is readily obtained by applying the law of Biot-Savart. The total magnetic field of the intracellular dendritic current, evaluated at an arbitrary observation point (x,y,z) involves a summation over all n segments and contains components in both the y and z directions

$$B_{yi} = \frac{-\mu z}{4\pi R^2} \gamma \quad (41a)$$

$$B_{zi} = \frac{\mu y}{4\pi R^2} \gamma \quad (41b)$$

where

$$\gamma = \sum_{k=1}^n i_k \left\{ \frac{(x-x_k) + \Delta x}{\sqrt{[(x-x_k) + \Delta x]^2 + R^2}} - \frac{(x-x_k) - \Delta x}{\sqrt{[(x-x_k) - \Delta x]^2 + R^2}} \right\} \quad (42)$$

$$R^2 = y^2 + z^2$$

Noting that

$$(x - x_{k+1}) + \Delta x = (x - x_k) - \Delta x$$

allows (42) to be written as

$$\begin{aligned} \gamma = & i_0 \frac{(x - x_1) + \Delta x}{\sqrt{[(x-x_1) + \Delta x]^2 + R^2}} - i_L \frac{(x - x_n) - \Delta x}{\sqrt{[(x-x_n) - \Delta x]^2 + R^2}} \\ & + \sum_{k=2}^n (i_k - i_{k-1}) \frac{(x - x_k) + \Delta x}{\sqrt{[(x-x_k) + \Delta x]^2 + R^2}} \end{aligned} \quad (43)$$

The terms i_0 and i_L refer to the intracellular current at the beginning ($x=0$) and end ($x=L$) of the neuron respectively.

As the length $2\Delta x$ of each segment approaches zero, the magnetic field equations (41a) and (41b) become exact representations for the field of the dendritic cable. In this case, the summation term in (43) would become an integral over the length of the neuron and be integrated with respect to x_k . To

avoid confusing the observation variable x with the integrable variable x_k , the dummy variable ζ can be introduced as the variable of integration along the length of the neuron in place of x_k . Using the basic formulation of the cable equation in (40) allows (43) to be written as an integral of the transmembrane current over the length of the neuron such that

$$\lim_{\Delta x \rightarrow 0} \gamma = i_0 \frac{x}{\sqrt{x^2 + R^2}} - i_L \frac{x - L}{\sqrt{(x - L)^2 + R^2}} - \int_0^L i_m \frac{x - \zeta}{\sqrt{(x - \zeta)^2 + R^2}} d\zeta \quad (44)$$

The integral can be broken down using integration by parts

$$\int u dv = uv - \int v du \quad (45)$$

where u and v are defined as

$$u = \frac{x - \zeta}{\sqrt{(x - \zeta)^2 + R^2}} \quad dv = -i_m(\zeta) d\zeta$$

$$du = \frac{-R^2}{[(x - \zeta)^2 + R^2]^{3/2}} d\zeta \quad v = i(\zeta)$$

yielding

$$-\int_0^L i_m \frac{x - \zeta}{\sqrt{(x - \zeta)^2 + R^2}} d\zeta = \left[\frac{(x - \zeta) \cdot i(\zeta)}{\sqrt{(x - \zeta)^2 + R^2}} \right]_{\zeta=0}^{\zeta=L} + R^2 \int_0^L \frac{i(\zeta)}{[(x - \zeta)^2 + R^2]^{3/2}} d\zeta \quad (46)$$

Note that the first term on the right hand side will cancel the first two terms on the right hand side of (44). Integration by parts can be applied once more to the last term in (46) using

$$u = \frac{1}{[(x - \zeta)^2 + R^2]^{3/2}} \quad dv = i(\zeta) d\zeta$$

$$du = \frac{3 (x-\zeta) d\zeta}{[(x-\zeta)^2 + R^2]^{5/2}} \quad v = -\frac{V_i(\zeta)}{r_a}$$

The magnetic field equations can now be written in their final form as

$$B_{yi} = \frac{-\mu z}{4\pi r_a} F(V_i) + \frac{-\mu z}{4\pi r_a} \mathfrak{R} \quad (47a)$$

$$B_{zi} = \frac{-\mu y}{4\pi r_a} F(V_i) + \frac{-\mu y}{4\pi r_a} \mathfrak{R} \quad (47b)$$

where

$$F(V_i) = \left[\frac{V_i(0)}{[x^2 + R^2]^{3/2}} - \frac{V_i(L)}{[(x-L)^2 + R^2]^{3/2}} \right] \quad (47c)$$

$$\mathfrak{R} = \int_0^L \frac{3 (x-\zeta) V_i(\zeta)}{[(x-\zeta)^2 + R^2]^{5/2}} d\zeta \quad (47d)$$

The magnetic field, then, due to intracellular currents in the neuron depends strictly on the intracellular potential at either end of the neuron and a residual function (47d). The residual can be neglected if its magnitude can be shown to be much less than either of the terms in (47c). We can estimate an upper bound on the residual by applying the mean value theorem to (47d)

$$|\mathfrak{R}| = \left| \frac{3 (x-x_m) V_i(x_m) L}{[(x-x_m)^2 + R^2]^{5/2}} \right|$$

where x_m is some point along the neuron such that

$$0 \leq x_m \leq L$$

Since

$$\left| \frac{(x-x_m)}{[(x-x_m)^2 + R^2]^{1/2}} \right| \leq 1$$

then

$$|\mathfrak{R}| \leq \left| \frac{3 V_i(x_m) L}{[(x-x_m)^2 + R^2]^2} \right|$$

The triangle quality suggests that

$$\sqrt{(x-x_m)^2 + R^2} \geq \sqrt{x^2 + R^2} - L$$

and if we further impose the necessary condition that

$$\sqrt{x^2 + R^2} \gg L \quad (48)$$

then the residual term can be reduced to

$$|\mathfrak{R}| \leq \left| \frac{3 V_i(x_m) L}{[x^2 + R^2]^2} \right|$$

Setting $x_{\max}$ to be the point at which V_i reaches its maximum magnitude, then the residual is bounded by

$$|\mathfrak{R}| \leq \left| \frac{3 V_i(x_{\max}) L}{[x^2 + R^2]^2} \right| \quad (49)$$

During the initial phase of excitation, $x_{\max} = 0$ and the upper bound on the residual term given in (49) can be compared to the first term in (47c) to find the conditions under which the residual term can be neglected. This occurs whenever

$$\left| \frac{3 V_i(0) L}{[x^2 + R^2]^2} \right| \ll \left| \frac{V_i(0)}{[x^2 + R^2]^{3/2}} \right|$$

which reduces to the final condition for neglecting the residual term

$$\sqrt{x^2 + R^2} \gg 3L \quad (50)$$

The first term is an expression of the distance from the soma to the observation point. Note that the earlier conditions required in (48) are satisfied whenever the condition in (50) is true, allowing (50) to serve as the necessary and sufficient condition for ignoring the residual term. When (50) is not satisfied, as may be the case with newer sensors, then the complete magnetic field can be computed using (47a) and (47b).

Whenever the observation distance is much greater than 3 neuron lengths away from the neuron, the magnetic field due to the intracellular current depends only on the intracellular potential at the ends of the neuron

$$B_{yi} = \frac{-\mu z}{4\pi r_a} \left[\frac{V_i(0) - V_i(L)}{[x^2 + R^2]^{3/2}} \right] \quad (51a)$$

$$B_{zi} = \frac{\mu y}{4\pi r_a} \left[\frac{V_i(0) - V_i(L)}{[x^2 + R^2]^{3/2}} \right] \quad (51b)$$

The term $[V_i(0) - V_i(L)]/r_a$ has the units of Amps-meters and represents the dipole strength P_x of the neuron. Substituting P_x into the above equations yields the dipole equations exactly, verifying that the condition in (50) is sufficient to allow the neuron to be modelled as a dipole.

4.4.2 Extracellular current - theoretical approach

The current passing out through the dendritic membrane i_m is constrained to flow within the volume conductor and thereby is the source of the extracellular current flow. The current passing out through the somatic membrane i_{soma} also contributes to extracellular current flow and, by conservation of charge, must be equal and opposite to the total current passing out through the dendritic membrane. Equivalently, the somatic current must be equal and opposite to the intracellular current in the neuron at the soma-neuron junction $i(0)$. The two sources of the extracellular current are therefore the transmembrane current i_m and the somatic current i_{soma} .

A ring of membrane with length dx positioned at $(x_k, 0, 0)$ represents a point current source or monopole (see Fig. IV-5(c)) with a current strength of $i_m(x_k) \cdot dx$. The solution for the magnetic field of such a monopole (Chapter II and [18]) consists of only x and y components

$$B_x = \frac{-\mu y}{4\pi [(x-x_k)^2 + y^2]} \left[\frac{z}{\sqrt{(x-x_k)^2 + y^2 + z^2}} - 1 \right] i_m(x_k) dx$$

$$B_y = \frac{\mu (x-x_k)}{4\pi [(x-x_k)^2 + y^2]} \left[\frac{z}{\sqrt{(x-x_k)^2 + y^2 + z^2}} - 1 \right] i_m(x_k) dx$$

The magnetic field due to all dendritic transmembrane currents is the integral of these equations for all x_k where $0 \leq x_k \leq L$. Substituting the dummy variable ζ

for the integration yields

$$B_{xd} = \int_0^L \frac{-\mu y}{4\pi [(x-\zeta)^2 + y^2]} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] i_m(\zeta) d\zeta \quad (52a)$$

$$B_{yd} = \int_0^L \frac{\mu (x-\zeta)}{4\pi [(x-\zeta)^2 + y^2]} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] i_m(\zeta) d\zeta \quad (52b)$$

The magnetic field due to the somatic current is simply the field of a point current source with magnitude $-i(0)$

$$B_{xs} = \frac{-\mu y}{4\pi [x^2 + y^2]} \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \right] i(0) \quad (53a)$$

$$B_{ys} = \frac{\mu x}{4\pi [x^2 + y^2]} \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 1 \right] i(0) \quad (53b)$$

The total magnetic field of the extracellular current is the sum of eqns (52) and (53)

$$B_{xe} = B_{xs} + B_{xd} \quad (54a)$$

$$B_{ye} = B_{ys} + B_{yd} \quad (54b)$$

These equations yield little practical information about the nature of the magnetic field of the extracellular dendritic currents. However, after some manipulation, the equations can be shown to yield similar information to that obtained for the magnetic field of the intracellular current. Integration by parts (45) can be applied to (54)

$$\begin{aligned}
B_{xe} &= B_{xs} + \int_0^L u_{1x}(\zeta) dv_{1x}(\zeta) \\
&= B_{xs} + \left[u_{1x}(L)v_{1x}(L) - u_{1x}(0)v_{1x}(0) - \int_0^L v_{1x}(\zeta) du_{1x}(\zeta) \right] \quad (55a)
\end{aligned}$$

$$\begin{aligned}
B_{ye} &= B_{ys} + \int_0^L u_{1y}(\zeta) dv_{1y}(\zeta) \\
&= B_{ys} + \left[u_{1y}(L)v_{1y}(L) - u_{1y}(0)v_{1y}(0) - \int_0^L v_{1y}(\zeta) du_{1y}(\zeta) \right] \quad (55b)
\end{aligned}$$

where the functions u_{1x} , v_{1x} , u_{1y} and v_{1y} are defined as

$$\begin{aligned}
u_{1x}(\zeta) &= \frac{y}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] \\
dv_{1x}(\zeta) &= \frac{-\mu}{4\pi} i_m(\zeta) d\zeta
\end{aligned}$$

$$\begin{aligned}
u_{1y}(\zeta) &= \frac{(x-\zeta)}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] \\
dv_{1y}(\zeta) &= \frac{\mu}{4\pi} i_m(\zeta) d\zeta
\end{aligned}$$

The corresponding integrals and derivatives to these ancillary functions are easily obtained and, using the fundamental cable equation (40), are given by

$$du_{1x}(\zeta) = \left\{ \frac{2(x-\zeta)y}{[(x-\zeta)^2 + y^2]^2} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] + \frac{(x-\zeta)y}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{3/2}} \right] \right\} d\zeta$$

$$v_{1x}(\zeta) = \frac{\mu}{4\pi} i(\zeta)$$

$$du_{1y}(\zeta) = \left\{ \frac{(x-\zeta)^2 - y^2}{[(x-\zeta)^2 + y^2]^2} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] + \frac{(x-\zeta)^2}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{3/2}} \right] \right\} d\zeta$$

$$v_{1y}(\zeta) = \frac{-\mu}{4\pi} i(\zeta)$$

Note that the product $u_{1x}(0) \cdot v_{1x}(0)$ is exactly equal and opposite to B_{xs} and therefore these terms can be eliminated from (55a). Similarly, the product $u_{1y}(0) \cdot v_{1y}(0)$ is exactly equal and opposite to B_{ys} and therefore these terms can be eliminated from (55b). If the neuron is sealed at $\zeta=L$ or is sufficiently long enough that minimal current flows out that end, then $i(L)=0$ and the product terms $u_{1x}(L) \cdot v_{1x}(L)$ and $u_{1y}(L) \cdot v_{1y}(L)$ can also be eliminated from eqns (55a) and (55b) respectively.

The magnetic field of the extracellular current can now be written as

$$B_{xe} = \frac{-\mu}{4\pi} \int_0^L i(\zeta) du_{1x}(\zeta) \quad (56a)$$

$$B_{ye} = \frac{\mu}{4\pi} \int_0^L i(\zeta) du_{1y}(\zeta) \quad (56b)$$

Integrating by parts yet again where u_2 and dv_2 are defined by

$$\begin{aligned} u_{2x} &= \frac{du_{1x}(\zeta)}{d\zeta} & dv_{2x} &= \frac{-\mu}{4\pi} i(\zeta) d\zeta \\ u_{2y} &= \frac{du_{1y}(\zeta)}{d\zeta} & dv_{2y} &= \frac{\mu}{4\pi} i(\zeta) d\zeta \end{aligned}$$

and the corresponding du_2 and v_2 are

$$\begin{aligned} du_{2x}(\zeta) &= \left\{ \frac{2[3(x-\zeta)^2 - y^2]}{[(x-\zeta)^2 + y^2]^3} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] \right. \\ &\quad + \frac{3(x-\zeta)^2 - y^2}{[(x-\zeta)^2 + y^2]^2} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{3/2}} \right] \\ &\quad \left. + \frac{3(x-\zeta)^2}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{5/2}} \right] \right\} y d\zeta \\ v_{2x}(\zeta) &= \frac{\mu}{4\pi r_a} V_i(\zeta) \end{aligned}$$

$$\begin{aligned}
du_{2y}(\zeta) = & \left\{ \frac{2[(x-\zeta)^2 - 3y^2]}{[(x-\zeta)^2 + y^2]^3} \left[\frac{z}{\sqrt{(x-\zeta)^2 + y^2 + z^2}} - 1 \right] \right. \\
& + \frac{(x-\zeta)^2 - 3y^2}{[(x-\zeta)^2 + y^2]^2} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{3/2}} \right] \\
& \left. + \frac{3(x-\zeta)^2}{[(x-\zeta)^2 + y^2]} \left[\frac{z}{[(x-\zeta)^2 + y^2 + z^2]^{5/2}} \right] \right\} (x-\zeta) d\zeta \\
v_{2y}(\zeta) = & \frac{-\mu}{4\pi r_a} V_i(\zeta)
\end{aligned}$$

yields

$$B_{xe} = u_{2x}(L) \cdot v_{2x}(L) - u_{2x}(0) \cdot v_{2x}(0) - \int_0^L v_{2x}(\zeta) du_{2x}(\zeta) \quad (57a)$$

$$B_{ye} = u_{2y}(L) \cdot v_{2y}(L) - u_{2y}(0) \cdot v_{2y}(0) - \int_0^L v_{2y}(\zeta) du_{2y}(\zeta) \quad (57b)$$

In this form, the magnetic field of the extracellular current depends strictly on the intracellular potential, where the first two terms on the right side are evaluated at the endpoints of the neuron and the last term is a complicated integral along the length of the neuron.

Unlike the residual integral associated with calculating the magnetic field of the intracellular current, it is not possible to show that the residual integral in the above equations can be neglected. The magnetic field of the intracellular current monotonically decreased with increasing distance from the source, and it was possible to determine a minimum distance at which the residual integral term was significantly less than the primary terms. However, B_{xe} and B_{ye} are not

monotonic functions of distance from the source. Field points exist at which either B_{xe} or B_{ye} has a zero value although the residual integral in (57a) or (57b) is non-zero.

The mean value theorem allows the integrals in (57a) and (57b) to be written respectively as $L \cdot v_{2x}(x_m) \cdot du_{2x}(x_m)/d\zeta$ and $L \cdot v_{2y}(x_m) \cdot du_{2y}(x_m)/d\zeta$ where x_m is some point along the neuron ($0 \leq x_m \leq L$). The derivatives $du_{2x}(x_m)/d\zeta$ and $du_{2y}(x_m)/d\zeta$ fall off with a higher order of distance than $u_{2x}(x)$ and $u_{2y}(x)$. From inspection of eqns. (57a) and (57b), the residuals must also fall off with a higher order of distance from the source than do the primary terms. For this reason, the integral will become negligible at some distance above the neuron. The effect of neglecting the residual terms is examined in the next section.

4.4.3 Numerical simulations of a single neuron

The above equations were applied to a neuron modelled with an equivalent dendrite cylinder to calculate the magnetic fields generated by neurons. The following analysis applies to the neuron situated in a semi-infinite volume conductor. This is appropriate for physiological preparations such as brain slices in which a slice of active tissue is immersed in a large conducting bath. A dendrite model from the granule cell layer of a rat hippocampal brain slice was chosen because the properties of this dendrite are well understood and there exists

good agreement between model parameters and observed behavior [21]. The somatic potential obtained experimentally and from a best-fit model by Yuen and Durand [21] for a 400 μm dendrite is shown in Fig. IV-6(a). The morphological and electrical parameters describing the dendrite used for the remainder of this analysis are

Neuronal input resistance	78.6 $\text{M}\Omega$
Dendritic input resistance	118.0 $\text{M}\Omega$
Somatic input resistance	228.0 $\text{M}\Omega$
Dendritic membrane resistivity	6000 $\Omega\text{-cm}^2$
Somatic membrane resistivity	1085 $\Omega\text{-cm}^2$
Cytoplasmic resistivity	170 $\Omega\text{-cm}$
Membrane capacitance	3.4 $\mu\text{F}/\text{cm}^2$
Membrane time constant	20.4 msec
Somatic area	475.0 μm^2
Dendritic radius	1.27 μm
Dendritic length	400 μm
Dendritic electrotonic length	0.88

The somatic potential was applied to a numerical model for which the transmembrane potential, intracellular current, and transmembrane current were obtained by numerical solution of the cable equations. The resultant currents and voltages are shown in Fig. IV-6(b)-(d) for three positions along the dendrite.

The magnetic field of the dendrite was calculated using both the approximate equations (51a,b) and the exact equations (47a,b) for intracellular and extracellular current. An example comparing the approximate and exact solution for calculating B_{yi} using eqn. (47a) is shown in Fig. IV-7. The approximation is not as good an estimate of the magnetic field at 100 μm above the dendrite compared to 1000 μm above the dendrite. The relative error associated

Fig. IV-6. Plot of voltages and currents of a dendrite in the dentate granule cell layer of a rat hippocampal slice. The dendrite model parameters were adjusted to obtain a good fit between the simulated and experimental data for somatic potential (a) in response to 1nA current injection for 0.5 msec. The dendrite is modelled as a single cylinder stretching from $x=0$ (soma position) to $x=400\text{ }\mu\text{m}$. The model can be used to calculate transmembrane potential (b), intracellular current (c), and transmembrane current (d) at different positions within the dendrite.

Fig. IV-6(a).

Fig. IV-6(b).

Fig. IV-6(c).

Fig. IV-6(d).

Fig. IV-7. Plot of B_{yi} generated by a single dendrite. The exact solution was obtained by dividing the dendrite into 80 segments. The approximate solution required only the end potentials. Results are shown for a field point located (a) $100\mu\text{m}$ and (b) $1000\mu\text{m}$ above the soma.

with using the approximate model for the dendrite compared to the exact model can be calculated from the ratio

$$\Delta B(z_0) = \text{abs} \left\{ \frac{\max_{\text{all } x,y} [B_{\text{approx}}(x,y,z_0) - B_{\text{exact}}(x,y,z_0)]}{\max_{\text{all } x,y} [B_{\text{exact}}(x,y,z_0)]} \right\} \quad (58)$$

This ratio compares the maximum difference between the two equations over an observation plane, divided by the maximum value of the exact solution on that plane. This ratio was calculated for all of the individual components of the magnetic field over a range of distances above the dendrite and is shown in Fig. IV-8.

4.4.4 Numerical simulations of a population of neurons

The results for a neuron with an equivalent cylinder are easily adapted to a group of dendrites in a brain slice. The model used in this section, shown in Fig. IV-9, consists of a layer of 700 dendrites each with the properties described in the preceding section. The magnetic field generated by the currents within the slice were computed using the exact equation. The peak values attained during the course of the time-varying waveform are plotted in Fig. IV-10(a),(b) and (c) for lines of field points stretched out in the x,y and z directions respectively. These plots show the drop in the peak magnetic field magnitude as a function of distance in each of three directions. The peak values are shown for each field component in Fig. IV-11(a)-(f) along five different lines of field points oriented in the z direction but starting at different x-y distances from slice.

Fig. IV-8. Plot of the relative error in approximating the magnetic field of a dendrite as a function of distance above the dendrite. Similar to Fig. IV-3, "B" refers to one of the 5 different magnetic field components shown in the figure. B_{exact} is calculated from an 80 segment model of the dendrite. At each distance above the dendrite, the x-y observation plane is divided into a 51 by 51 grid for which the grid spacing is adjusted as described in Fig. IV-3. The slight "bump" in the plot for ΔB_{ye} did not vary with a finer grid and appears to be intrinsic to the equations being used.

Fig. IV-9. Diagram of a dendrite bundle from the granule cell layer of a rat hippocampal brain slice. The origin of the coordinate system is centered on top of the slice, directly above the somatic layer. The somatic layer extends $175\ \mu\text{m}$ on either side of the origin and $200\ \mu\text{m}$ below the origin. The dendrites are oriented in the $+x$ direction and have a length of $400\ \mu\text{m}$. The dendrites are modelled as a regularly spaced matrix of cable models, with a lateral spacing of $10\ \mu\text{m}$ between cylinders or a packing density of 10^4 dendrites/ mm^2 .

Fig. IV-10(a).

Fig. IV-10. Plot of peak values for each component of the magnetic field along different lines. Only components whose values are greater than 10^{-4} pT are shown. The component values are shown along lines defined by (a) $y=0$, $z=1\text{mm}$, (b) $x=0$, $z=1\text{mm}$, and (c) $x=0$, $y=0$.

Fig. IV-10(b).

Fig. IV-10(c).

Fig. IV-11. Plot of peak magnetic field strength generated by the dendrite slice at different field points as a function of distance above the slice. The peak field strength is defined as the peak value of the magnetic field occurring during the excitation waveform. The peaks field is shown for the overall B field magnitude (a), and the individual components B_x (b), B_y (c) and B_z (d). The B_y field can be further broken into the components generated by intracellular current B_{yi} (e) and extracellular current B_{ye} (f). The field is shown for five different x-y points, except when an x-y point generates a negligible field at any height. The plots of B_x and B_z are only shown for the two points at which the field was greater than 10^{-5} pT (note the different vertical scale for B_x). The plots of B_{yi} (e) for the x-y field point (0,10) and (10,0) are coincident and appear as a single line.

Fig. IV-11(a).

Fig. IV-11(b).

Fig. IV-11(c).

Fig. IV-11(d).

Fig. IV-11(e).

Fig. IV-11(f).

The magnitude of the net magnetic field B_{mag} eventually drops as d^{-2} , where d represents the distance from the dendrites. As shown in Fig. IV-10, this is true regardless of the direction along which d is measured. Initially, this could be expected to be a natural result of the nature of the currents in the dendrite, which seem very similar to a pure current dipole. However, the field of a dipole will decrease as d^{-3} when d is oriented along a line parallel to the dipole orientation (see eqn. (7) and vary x while keeping y and z fixed). Along a line of increasing x shown in Fig. IV-10(a) beginning at $y=0$ $z=1\text{mm}$, only B_{yi} and B_{ye} contribute significantly to the overall magnetic field B_{mag} . Despite the drop in the intracellular component B_{yi} as x^{-3} which is predicted by eqn. (51), the extracellular component B_{ye} drops as x^{-2} and is the dominant field component for $x > 3\text{mm}$. Note that B_{ye} initially drops as x^{-3} when approaching a local minima. Along the line of increasing z shown in Fig. IV-10(c) with $x=y=0$, the same field components contribute to the overall field and are related as $B_{yi} \approx -2B_{ye}$. Both components and the resultant overall magnitude B_{mag} decrease as z^{-2} . Along the line of increasing y shown in Fig. IV-10(b), the dominant components B_{ye} and B_{zi} similarly drop with y^{-2} .

4.5 Discussion

A unique method for calculating the magnetic field of neural sources has

been presented, whereby the exact magnetic field of extracellular current was predicted using the concept of current monopoles. Although the technique was only applied to an equivalent cable model, there is no limit to the accuracy that can be obtained in calculating the magnetic field of a neural source. Neurons with extensive branching can be modelled using current monopoles to model the transmembrane current along each branch. The only limit to this approach is the accuracy with which the currents can be computed.

The results obtained for the different sources indicate minimum distances at which the sources can be modelled as current dipoles. Alternatively, these restrictions can be interpreted as setting a maximum limit on the length of a cable which may be considered as one segment. For example, the restriction that $z \gg 1.4 (2\Delta x)$ developed for the intracellular current of the filament can be interpreted as stating a) the observation distance should be much greater than 1.4 times the filament length, or b) the segment length must be much less than $1/1.4$ times the observation distance for the segment to be modelled as a dipole. Similar statements can be made regarding the restriction based on the field of the extracellular current that $z \gg 0.41 (2\Delta x)$.

When the more restrictive of the intracellular and extracellular conditions is satisfied, the filament can be modelled as a dipole. The relative error introduced on an observation plane, shown in Fig. IV-3, is less than 10% when $z = 1.4 (2\Delta x)$ and diminishes as z^{-2} . Since the data in Fig. IV-3 represents a ratio

(eqn. (14)) for which the denominator is decreasing as z^{-2} , then the maximum difference between the fields of the two sources decreases as z^{-4} . The similarity of these two sources is more apparent when the strength and position of the filament is estimated assuming the source to be a dipole. The relative error, Fig. IV-4, was less than 2% for both estimates when $z \gg 1.4 (2\Delta x)$. These findings are similar to those obtained in spherical volume conductor. Using a least squares fit, Okada found little difference between an arc-shaped filament and a dipole for arc lengths up to 20° [22]. The same results were achieved by Cuffin [17] using straight in-line dipoles in a sphere to approximate a filament. Consistent with our findings in eqn. (33), both studies in the sphere show that the dipole model results in an estimated depth that is greater than the actual source depth.

It is unclear how the information contained in the magnetic field can be used to resolve the difference between a dipole and filament source. The comparison of the relative errors in Fig. IV-3 suggests that the extracellular current generates the more distinguishable field. It does not seem possible to separate the B_{yi} and B_{ye} components contained in B_y based on the functions described in Table IV-1. It may be possible to distinguish between the sources based on some other function of the field, such as $\partial^2 B_y / \partial z^2$ which provides richer information for distinguishing between multiple dipole sources [23]. Until the difference can be resolved, the conclusion reached by De Munck [15] appears to be valid: the dipole is an adequate model for predicting source strength and

location, although it does not provide information about the spatial extent of the sources.

If sufficient information is known about an active neuronal population, the magnetic field data can be used to determine specific properties of the neuron besides location, orientation and depth. The dipole approximation in eqns. (51a) and (51b) for the magnetic field due to intracellular current in a single neuron was shown to contain a dipole strength term P_x given by $[V_i(0) - V_i(L)]/r_a$. An intracellular electrode could be used to determine $V_i(0)$, and $V_i(L)$ could be assumed to be zero during the peak B field. If the number of active neurons can be estimated, then the value of r_a can be derived from the estimated dipole strength of the source.

The magnetic field calculated for the dendrite slice drops as d^{-2} compared to the higher order drop (d^{-3}) calculated for axons in the same volume conductor [24]. This is a result of the monophasic nature of the dendritic currents compared to the biphasic nature of the axonal currents. The peak magnetic field for the dendrite, shown in Fig. IV-7, occurs at a time when the intracellular current, shown in Fig. IV-6(c), is oriented in the same direction at all points along the length of the dendrite. Within an axon, the intracellular current exhibits a biphasic waveform along the length of the active fiber [19], [25]. At far distances, the intracellular current of the dendrite will appear as a current dipole while the axon will appear as a current quadrupole. A model of a cell with

bipolar arborization (two dendritic trees spreading out in opposite directions) could be expected to produce results similar to the axonal results, with the field decreasing as d^{-3} .

The cable model is often used to predict the electrical properties of neuronal branches, but is limited in its applications to magnetic modelling. The "equivalent cable" model allows the behavior of a neuron with many branches oriented at any angle to be determined by a single cable. This is based on the assumptions that the branching obeys the $3/2$ power rule discussed earlier, and that the electrotonic length of each branch is the same. If the two daughter branches are oriented in different directions, the magnetic field would decrease. In the extreme case, the branches would be oriented 180° apart and the magnetic field would decrease with an extra order of distance. *The equivalent cylinder model thus represents a prediction of the maximum field that could be expected from the neuron.* The extent to which it can be used clearly depends on the arborization of the neuron.

The net dipole moment for the slice, calculated from Fig. IV-10(c), is 0.1 nA-m. This corresponds to an average dipole moment of .14 pA-m per dendrite, equivalent to a mean current of 0.36 nA along the $400\mu\text{m}$ dendrite length. This can be compared to an equivalent moment of 10 nA-m calculated by Okada [7] for a turtle cerebellum slice. The active tissue in the cerebellum was estimated to be 10 mm^3 , whereas the hippocampal slice simulations assume an active tissue

volume of .028 mm³. Extrapolating from the active volumes would suggest the dendrite model should have a dipole moment of 0.028 nA-m. However, other factors such as packing density, orientation, and arborization of the active tissue should be considered. As indicated above, the slice model represents a maximum bound on the expected field and the actual dipole moment will be somewhat less than predicted and may be closer to 0.028 nA-m.

The analysis derived earlier for the cable model can be easily modified to apply to a single branch of the neuron rather than the entire equivalent cable. The length L used in the equations would refer to the length of the branch, and the observation distance d necessary for assuming dipolar behavior still remains as $d \gg 3L$. The analysis can also be applied to any neural membrane models, such as axons, which obey the basic cable equations. However, the assumption that allowed the residual integral term to be eliminated from eqn. (47a) and (47b) must be examined for the specific transmembrane potential waveform.

The examples presented in this paper represent just a few applications for calculating the magnetic fields of neural sources using current monopoles. The approach seems particularly well suited to calculating the field generated by myelinated axons since the nodes of Ranvier are very nearly point current sources. If the current distribution within a complex dendritic structure is known, then the magnetic field can be accurately predicted using current monopoles. The equations used to calculate the magnetic field of point current

sources (in both semi-infinite and spherical volume conductors) are extremely simple and allow rapid computation of the field generated by complex source distributions.

4.6 References

- [1] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [2] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," *Ann. N.Y. Acad. Sci.*, vol. 425, pp. 722-742, 1984.
- [3] B.J. Roth, N.G. Sepulveda, and J.P. Wikswo, Jr., "Using a magnetometer to image a two-dimensional current distribution," *J. Appl. Phys.*, vol. 65, pp. 361-372, 1989.
- [4] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," *Phys. Med. Biol.*, vol. 32, pp. 125-131, 1987.
- [5] D.S. Buchanan, D.B. Crum, D. Cox, and J.P. Wikswo, Jr., "Microsquid: a close-spaced four channel magnetometer." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 677-679.
- [6] J.P. Wikswo, R.N. Friedman, A.W. Kilroy, J.M. van Egeraat, and D.S. Buchanan, "Preliminary measurements with microsquid." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 681-684.
- [7] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.

- [8] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [9] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.
- [10] J.P. Wikswo, "Biomagnetic sources and their models." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 1-18.
- [11] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," *Phys. Med. Biol.*, vol. 32, pp.43-51, 1987.
- [12] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [13] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," *Phys. Med. Biol.*, vol. 32, pp. 133-142, 1987.
- [14] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncitia," *Biophys. J.*, vol. 51, pp. 557-568, 1987.
- [15] J.C. De Munck, B.W. Van Duk, and H. Spekreijse, "Mathematical dipoles are adequate to describe realistic generators of human brain activity," *IEEE Trans. Biomed. Eng.*, vol. BME-35, pp. 960-966, 1988.
- [16] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.
- [17] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," *J. Appl. Phys.*, vol. 48, pp. 3971-3980, 1977.
- [18] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," *IEEE Trans. Magn.*, vol. MAG-27, pp. 758-767, 1991.
- [19] R. Plonsey, "The active fiber in a volume conductor," *IEEE Trans. Biomed. Eng.*, vol. BME-21, pp. 371-381, 1974.

- [20] J.J.B. Jack, D. Noble, and R.W. Tsien, *Electric Current Flow in Excitable Cells*, Clarendon Press, Oxford, 1983, pp.25-66.
- [21] G.L.F. Yuen and D. Durand, "Reconstruction of hippocampal granule cell electrophysiology by computer simulation," accepted for publication in *Neuroscience*, vol. 41, pp. 411-424, 1991.
- [22] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), *Biomagnetism: Applications and Theory*, Pergamon Press, New York, 1985, pp. 266-272.
- [23] D. Cohen and H. Hosaka, "Part II. Magnetic field produced by a current dipole," *J. Electrocardiology*, vol. 9, pp. 409-417, 1976.
- [24] K.R. Swinney and J.P. Wikswo, Jr., "A calculation of the magnetic field of a nerve action potential," *Biophys. J.*, vol. 32, pp. 719-731, 1980.
- [25] J.P. Barach, B.J. Roth and J.P. Wikswo, Jr., "Magnetic measurements of action currents in a single nerve axon: a core-conductor model," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 136-140, 1985.
- [26] W. Rall, "Core conductor theory and cable properties of neurons." in *Handbook of Physiology - The Nervous System*, American Physiological Society, Bethesda, MD., 1977, sect. 1, vol. 1, part 1, pp. 39-97.

CHAPTER V
AN APPROXIMATE MODEL FOR CALCULATING
THE MAGNETIC FIELD OF CURRENT SOURCES
IN A SPHEROIDAL VOLUME CONDUCTOR

In this chapter, the magnetic field of a current source located in both prolate and oblate spheroidal volume conductors is approximated by a model based on the current monopole theory developed in the preceding chapters. This chapter will be submitted to IEEE Transactions on Biomedical Engineering with authors A.S. Ferguson and D. Durand. This work was supported in part by the National Science Foundation Presidential Young Investigator Award to D. Durand. The authors are with the Applied Neural Control Laboratory, Department of Biomedical Engineering, Case Western Reserve University, Cleveland, OH 44106.

5.1 Abstract

A model is developed to approximate the magnetic field of current dipole sources in spheroidal volume conductors. The model is derived from concepts developed for calculating the magnetic fields of point current sources in spherical and semi-infinite conductors. The relative error between the magnetic field predicted by the model and the field generated by the true dipole solution is less than 5% for a range of source locations and orientations, and volume conductor sizes and shapes. The required calculation time using this model is at least 9 times less than the dipole equations, and may be up to 50 times less depending on the nature of the problem.

5.2 Introduction

The magnetic field of dipole volume currents can be calculated using a combination of known solutions for the magnetic field of current monopoles. The volume currents generated by a dipole are identical to those generated by a current monopole source located at the head of the dipole and a current monopole sink located at the dipole tail. The solution for the magnetic field generated by current monopoles located in semi-infinite and spherical volume conductors was derived from Maxwell's equations [Chapter III]. It was shown that, using the monopole formulation, the equations describing the magnetic field are simpler and easier to calculate [1].

The use of the current monopole formulation is limited to volumes for which the magnetic field is known - specifically semi-infinite and spherical volumes. The dipole solution has also been calculated for prolate and oblate spheroids [2], but it is unreasonable to attempt to derive a monopole solution from the dipole solution due to the complexity of the dipole solution. For example, the calculation of B_x due to a P_ξ oriented dipole in a prolate spheroid involves a double-nested infinite summation of nine different Legendre polynomials and derivatives (see Appendix E). While a monopole solution could be extracted from this equation, it may be too complex to be of any practical value.

A significant drawback to implementing the dipole solution is the programming overhead required to code and test the solution. The calculation times required by the final code may also be exorbitant. An alternative to deriving an exact solution is to obtain an approximate solution using the monopole source. If such a solution could be found, a slight loss in accuracy may be tolerable if the magnetic field could be easily and rapidly calculated.

The development of such an approximation is described in the following section. Based on the model for a monopole in a spherical volume, a hypothesis is presented which forms the basis for the approximate solution in a spheroidal volume conductor. The hypothesis is tested in subsequent sections by comparing

the known dipole solution to an approximation using a monopole sink and source.

5.3 Hypothesis

Two theoretical developments are crucial to the basis of the hypothesis presented in this section. The first is the model for a current monopole. This was developed in Chapter III where it was shown that Maxwell's equations are satisfied by using a two-part model for the monopole. The monopole model consists of (1) an arbitrary wire that is open-ended at the monopole, thereby creating a conduction current source at the monopole position, and (2) another wire identical to the first wire but with current flowing in the opposite direction which is closed-ended at the monopole position, thereby creating a sink for displacement current. The second important development is the proof in Chapter III that a null space exists for which monopole sources generate no magnetic field. It was shown that the null space corresponds to volumes whose surfaces are equipotential surfaces for the monopole sources.

Prolate and oblate spheroids are equipotentials surfaces for a line and disk of uniform current sources respectively (see Appendix C). Consequently, a line or disk of uniform strength current monopoles can be added to these volumes without distorting the magnetic field generated by an arbitrary monopole source. The added monopoles and the arbitrary monopole can all be modelled using the

two-part source model. As was shown in Chapter III for a sphere (see Fig. III-2), the wires associated with the added monopoles can be oriented to cancel out the field of the wire associated with the arbitrary monopole *except for a finite-length wire segment*. In the prolate spheroid, these wire segments run from the monopole source to the line of sources (for which the surface is equipotential), generating a "fan" of line segments; in the oblate spheroid, the segments run from the monopole source to the disk, resulting in a "cone" of segments.

These arrangements of the wire segments are shown in Fig. V-1 and represent an exact model for the magnetic field of a current monopole in a spheroid. The goal is to simplify the analysis of the field generated by these line segments. Returning to the spherical conductor, it was shown that the magnetic field of the open-ended wire segment (conduction current) generated no magnetic field because the field of the segment was equal and opposite to the field generated by the returning volume currents. The remaining closed-ended segment (displacement current) was responsible for the magnetic field of the monopole. A similar approach in the spheroid would assume that the open-ended wire segments can be neglected since they do not contribute significantly to the field. In other words, the field generated by the segments is effectively cancelled by the field of the returning conduction current. The field of the monopole is therefore defined by the field of the remaining closed-ended (displacement) segments.

Fig. V-1. Theoretical basis for computing the magnetic field of a monopole in a spheroidal conductor.

Fig. V-2. How the magnetic field is calculated for the "fan" of line segments in the prolate spheroid. The vector field B generated by each line segment is summed at the observation point P . Only a few representative segments are shown.

This leads to the following hypothesis: *The magnetic field of a current monopole in a prolate or oblate spheroid can be approximated by the magnetic field of a "fan" or "cone" of line segments respectively.* This is equivalent to the statement: the magnetic field of a current monopole in a spheroidal conductor can be approximated by the magnetic field of line segments extending from the monopole to the sources for which the volume surface is equipotential. This process is illustrated in Fig. V-2. The hypothesis cannot be tested directly because there is no known solution for the field generated by point sources in spheroidal conductors. The hypothesis can, however, be tested indirectly by applying it to two monopoles of opposite sign which are spaced slightly apart. This simulates the volume current of a current dipole in a spheroid. Superposition implies that the fields generated by the individual monopoles can be summed, and the resultant field can then be compared to the field of a dipole for which the solution for the magnetic field is known.

5.4 Methods

Simulations were run using compiled Fortran code running on T800 Transputer processor boards installed in an IBM-AT computer. The source code and algorithms for specific routines are included in Appendices E-H. Only the routines which were time-critical (eg. dipole solution - Appendix E), improvements over existing algorithms (eg. Legendre polynomials - Appendix H) or specific to these simulations (eg. monopole calculations - Appendices F,G) are

listed. All calculations employed double precision floating point variables. Special attention was given to coded loops which introduced truncation errors in an attempt to minimize their effects. Transputers were used due to their fast computational speed and the ability to run parallel code on multiple transputers when using the 3L Parallel Fortran compiler. Five transputers were used to run all simulations, producing an actual reduction in processing time of 80% compared to a single transputer. A single transputer is capable of doing floating point calculations approximately 20 times faster than a 8Mhz 80286/80287 combination.

All dipole calculations were carried out using the first 20 n-terms in the equations of Cuffin and Cohen [2], unless otherwise stated. The Legendre polynomials which are an intrinsic part of these equations were similarly calculated using the first 20 terms in a hypergeometric series. The accuracy of these calculations were checked by (a) verifying that the calculated magnetic field exhibited zero divergence at each field point, (b) comparing the solution to the spherical solution as the focal length of the prolate and oblate spheroids approaches zero, and (c) verifying the solutions for dipoles oriented along an axis of revolution with an easily obtained analytic expression. The accuracy of these calculations depended on both the number of n-terms included from the dipole equations and the number of terms included in the hypergeometric series, with no improvement whenever the number of terms in one exceeded the other. Generally, there was little if any change in the magnetic field when using only 10

terms compared to using 20 terms. Cuffin and Cohen similarly found little change between 9 and 10 terms and chose to use 10. However, under asymptotic conditions (approaching singularities in the Legendre polynomials) such as a prolate spheroid that is nearly cylindrical or spherical, the results changed between 9 and 10 terms, but not between 19 and 20. The limit on the number of terms was set by the stability of the numerical solutions, which were stable for up to 30 terms.

The "fan" or "cone" of line segments was split into 36 total segments for both prolate and oblate simulations. No additional accuracy was obtained with a larger number of segments. The separation of the monopoles for approximating a dipole was fixed at .001 (arbitrary units). The results did not vary in most simulations when this number was varied from .05 to .00001.

5.5 Results

Plots of the magnetic field calculated from the dipole equations are shown in Fig. V-3. The magnitude of the field is shown over a surface of observation points for one dipole position in a prolate volume conductor. The three different plots in Fig. V-3 show the field for three different dipole orientations. A plot of the field generated by the monopole approximation is not included because the field is very similar to that shown for the dipole and the differences are not discernable on this type of a plot.

Fig. V-3(a).

Fig. V-3. Plots of the magnetic field calculated from the dipole equations. The magnetic field was calculated on a grid of 80 points on a prolate shell ($C=1.0$, $\eta=1.75$) for a current source located in a prolate spheroidal volume conductor ($C=1.0$, $\eta_a=1.5$). The dipole strength is $(4\pi \cdot 10^{-7})^{-1}$ A-m. The inset plots show a cross-section of the volume conductor (solid line), the shell of observation points (dashed line), and the dipole location ($C=1.0$, $\xi=-.6$, $\eta=1.25$, $\phi=0^\circ$).

Plots are shown for the magnitude of the magnetic field generated by dipoles oriented in the (a) ξ direction, (b) η direction, and (c) ϕ direction as a function of the observation angle ϕ ($0 \leq \phi \leq 360^\circ$) and ξ ($-1 \leq \xi \leq 1$).

Fig. V-3(b).

Fig. V-3(c).

The difference between the monopole approximation and the dipole formulation can be discerned when the relative error is plotted for the surface of field points. The relative error is defined as

$$\text{Relative error} = \frac{|B^d(x,y,z) - B^m(x,y,z)|}{\max_{x,y,z} |B^d(x,y,z)|} \quad (1)$$

where B^d and B^m refer to the magnetic field predicted by the dipole equation and monopole approximation respectively. The denominator of eqn. (1) is the maximum value of B^d over the surface of observation points. It will be clear from the usage whether B^d and B^m refer to a specific component or the magnitude of the magnetic field B .

A plot of the relative error in the magnitude of B is shown in Fig. V-4(a) for a ξ oriented dipole. This dipole orientation is tangential to the surface and parallel to the major axis of the spheroid. The relative error was also calculated at each field point for η and ϕ oriented dipoles (not shown). Since the field points can be considered as defining a grid over an observation surface, then each field point represents some part of an area on this surface. After re-arranging the relative error values in increasing order and considering the grid area at each field point, a cumulative histogram can be determined as shown in Fig. V-4(b). The term "cumulative" refers to the surface area which is integrated and plotted against the relative error. The error that occurs at 50% of the area in Fig. V-4(b), for example, refers to the level of error for which half the observation surface was

Fig. V-4(a).

Fig. V-4. Plot of the relative error in the monopole approximation over the surface of field points. See Fig. V-2 for details. The relative error is shown as a function of surface position for a ξ oriented dipole in (a). This was obtained by taking the absolute difference between the dipole and monopole results at each observation point, divided by the largest dipole value. The data is shown as a histogram for all three dipole orientations in (b). The effect of averaging the field over a disk of points is shown in (c).

Fig. V-4(b).

Fig. V-4(c).

below and half was above. The maximum value of the relative error for the ξ oriented dipole (at 100% of the field area) corresponds to the peak of the surface plot in Fig. V-4(a).

The effect of averaging the magnetic field over the observation surface on the relative error is shown in Fig. V-4(c). This simulates the effect of using a sensor with a large sensing coil. The magnetic field was calculated at each field point and averaged with the field at three neighboring points, the latter chosen to define a plane tangent to the observation surface. The separation of the points was chosen so that the effect of the averaging was similar to using a ring with a radius of 0.2 (relative to the prolate spheroids major axis of 3.0). The averaging had little effect on the relative error since the histogram shown in Fig. V-4(c) is very similar to Fig. V-4(b). This plot suggests that, although a small percentage of the observation area has a relative error exceeding 5%, the area is localized. The error cannot be removed unless the averaging ring is at least as big as this area. Referring to Fig. V-4(a), it can be seen that the greatest errors occur in one specific area, rather than in several distinct points.

5.5.1 Effect of dipole location

The location of the dipole (and monopole approximation) was varied within a prolate and oblate spheroid to determine what, if any, limitations must be placed on using the approximation. In these simulations, the prolate and oblate

volume conductor shapes were fixed and a grid of dipole locations was designed within the conductor, see Fig. V-5. The eccentricity of the spheroids was fixed at $e=0.9$, with a major axis length $a=2$ and a minor axis length $b=0.876$. The observation surface, over which a 24 by 20 grid of observation points was placed, was defined over a sphere with diameter 2.2 (10% larger than major axis of spheroids).

The magnetic field was calculated over the observation surface for three different dipole orientations at each dipole location. The dipole orientations were chosen to be x,y,z rather than ξ,η,ϕ for the prolate or ξ,ζ,ϕ for the oblate spheroid. This prevents the dipole orientation from changing when the volume parameters change, thereby allowing a comparison for different shaped spheroids while maintaining a fixed dipole orientation. For each combination of dipole orientation and position, a relative error histogram was tabulated. The mean, maximum, 75% and 90% values were extracted from the histogram. An example of the effect of dipole position on these values is shown in Fig. V-6; a complete set of data is provided in Appendix D.

The field points were chosen to lie on a spherical surface to make it possible to compare the dipole equations with a similar spherical model. The spherical approximation assumes the volume to be a sphere instead of a spheroid, for which the published solution for the magnetic field of a dipole in a sphere can be used to predict the magnetic field. The relative error between these values and

Fig. V-5. Dipole locations used in the spheroidal conductors. (a) The magnetic field in the prolate spheroid ($C=0.9$, $\eta_a=1.111111111$, $e=0.9$) was calculated for a dipole positioned at each grid point. The grid consisted of constant $x=0$ and varying y as (.01, .05, .1, .15, .2, .25, .3, .35, .4) and z as (0, .1, .2, .3, .4, .5, .6, .7, .8). (b) The magnetic field in the prolate spheroid ($C=0.9$, $\zeta_a=0.484322105$, $e=0.9$) was calculated for a dipole positioned at each grid point. The grid consisted of constant $z=0$ and varying x as (.01, .05, .1, .15, .2, .25, .3, .35, .4) and y as (.01, .1, .2, .3, .4, .5, .6, .7, .8).

Fig. V-6. Plot of relative error for the monopole (solid line) and spherical (dashed line) approximations for different source positions within a prolate spheroid. The dipole is oriented in the $+x$ direction. Shown are the mean (circles) and maximum (no symbols) percent relative error over the observation grid (see text). The abscissa indicates the y position of the dipole. Plots are shown for different dipole z positions: 0.0 (upper-left); 0.1 (upper-center); 0.2 (upper-right); 0.3 (center-left); 0.4 (center); 0.5 (center-right); 0.6 (lower-left); 0.7 (lower-center); and 0.8 (lower-right).

the dipole solution was calculated in a similar fashion by creating a cumulative histogram and extracting the mean, maximum, 75% and 90% values. This is also shown in Fig. V-6 and Appendix D.

In almost all cases, the monopole approximation is better than a simple spherical model for calculating the magnetic field from the dipole in a spheroidal conductor. The only situation in which the spherical approximation may be better than the monopole approximation is for the +y oriented dipole located near the equator in a prolate spheroid (Fig. H-1(b) upper-left). In this case, the dipole is oriented nearly perpendicular to the surface and will generate little or no magnetic field using the spherical approximation. The field predicted by the monopole approximation is evidently larger than the true dipole field and creates a maximum error that exceeds the maximum spherical error.

The mean error generated by the monopole approximation over the observation surface is less than 7% for all dipole locations, orientations, and volumes described in Appendix D. The maximum error at any point on the surface may be quite large (Fig. H-1(a)-(f)), but this is evidently isolated to a small region on the observation surface because the 90th percentile errors (Fig. H-1(g)-(l)) are much smaller.

5.5.2 Effect of spheroid size

The magnetic field predicted by the monopole approximation, unlike the dipole equations, does not depend on the size of the spheroidal conductor. It only depends on the "shape" or focal length of the volume. This is because the "fan" or "cone" of line segments shown in Fig. V-1, which run from the monopole source to a line or disk, are only affected by the focal point of the spheroid. However, the surface terms η_a and ζ_a for the prolate and oblate spheroids respectively cannot be factored out of the published dipole solution. Consequently, the magnetic field determined by the dipole equations for a fixed observation point and current source will vary as the size of the volume conductor changes, while the monopole approximation will remain unchanged.

It is unclear how much the surface term affects the dipole solution, but the effect should be minimal if the monopole approximation is to be valid. An example of the relative difference generated over an observation surface is shown in Fig. V-7 when the major axis of the prolate spheroid varies over the range 3 to 10. The relative difference in this case is defined as the maximum of the relative differences between the field generated by each surface compared to the field generated by the outermost surface. In this example, the relative error never exceeds 3% at any point.

Similar to the previous section, the mean and maximum differences can be

Fig. V-7. Relative difference histogram for dipole solution with different volume sizes. The relative difference is defined as the maximum difference between the fields of the inner volumes and the outermost volume, divided by the maximum field generated by the outermost volume. The inset plot shows a cross-section of the prolate volumes (solid lines) and observation surface (dashed line). The size of the prolate spheroid ($C=1$) varies from $\eta_a=1.5$ to $\eta_a=5.0$. The prolate observation surface is fixed at $\eta = 5.25$. The dipole source is located at $\xi=.9$, $\eta=1.25$, $\phi=0^\circ$, $C=1$.

Fig. V-8. Plot of mean (circles) and maximum (no symbols) relative difference for different volume sizes. Data are shown for prolate and oblate spheroids in the first and second column respectively. The (x,y,z) current source location is fixed at $(0,.25,.4)$ for the prolate and at $(.25,.4,0)$ for the oblate spheroid. The focal point is fixed at $C=0.9$, but the volume surface term is varied so as to increase the semi-major axis of the volume from 1 to 2 (abscissa). The observation surface coincides with the largest volume surface and remains fixed. The difference in the exact dipole equation (solid line) is determined for each volume surface compared to the largest surface. The error in the monopole approximation (dashed line) is determined by comparing the monopole approximation to the exact dipole solution at each volume.

tabulated from the histogram and plotted as a function of volume size. The relative difference in this case refers to the difference between the magnetic field generated by a dipole in an inner volume and that of a dipole in the outermost volume, divided by the largest field value of the outermost volume. This data is shown in Fig. V-8 for both prolate and oblate spheroids as the major axis of the volumes increase by a factor of 2. Despite the large change in the volume size, the maximum relative difference in magnetic field never exceeds 8% and the mean error never exceeds 4%.

The relative error between the monopole approximation and the dipole solution for each volume size is also shown in Fig. V-8. The monopole approximation is invariant with volume size, so the relative error is calculated as the difference between the monopole approximation and the exact dipole solution for each volume conductor size relative to the largest dipole field value for that conductor. The mean error associated with the monopole approximation is nearly the same as the relative difference in the exact dipole equations as the volume size changes. This error is greatest when the volume is much smaller than the observation surface, and drops below 2% when the observation plane lies on the volume surface.

5.5.3 Calculation times

The monopole approximation is a significantly faster method for

calculating the magnetic field of current dipoles in spheroids. Typical computation times are shown in Table V-1. The infinite summation term in the exact dipole solution must be truncated to a finite number of terms, which affects the computational time. In addition, the hypergeometric series used to compute the Legendre polynomials must also be truncated. We found that both series should be truncated at the same limit (i.e. using a 10'th order hypergeometric series when calculating up to 10 terms in the dipole solution) to achieve the greatest accuracy with the least computational time. Using 10 terms, the vector magnetic field calculated from the exact dipole formulation required an average of 0.43 seconds per field point per dipole orientation. The monopole approximation required 0.01 second per field point per dipole orientation using 16 line segments. The approximations for the prolate surface were accurate to 5 digits when only 16 segments were used, and showed less than 0.05% variation for the oblate surface when varying the number of segments from 16 to 100.

5.6 Discussion

The magnetic field calculated with the monopole approximation differed only slightly from the exact solution calculated from the dipole formulation. The "fan" or "cone" of line segments is therefore a sufficient model for the magnetic field of a single monopole in a spheroidal conductor, validating the original hypothesis. The relative difference between the two methods was generally less than 5% for a variety of dipole locations and orientations in both prolate and

Table V-1. A comparison of the time required to calculate the magnetic field for the exact dipole solution and for the monopole approximation. The times shown are the seconds required to generate 100 iterations, each iteration consisting of a calculation of the x,y, and z magnetic field components and the magnitude of the magnetic field. For all simulations, the volume surface is defined by $C=1.0$, $\eta_a=1.5$ (prolate) or $\zeta_a=1.5$ (oblate), with the dipole positioned at (.1,.2,.3) and the observation point fixed at (.3,.3,.3). All position are cartesian (x,y,z).

Dipole calculations are shown for $N=10,20,30$ where N represents the highest degree Legendre polynomial involved in the calculation and also the number of terms included in the hypergeometric calculations of the Legendre polynomial. The calculation times depend on the dipole orientation P . Monopole calculations are shown for different numbers of line segments.

CALCULATION TIMES WITH EXACT DIPOLE EQUATION

<u>Volume conductor</u>	<u>P</u>	<u>N=10</u>	<u>N=20</u>	<u>N=30</u>
<i>Prolate</i>	P_ξ	40.67	195.6	525.1
	P_η	40.58	195.4	524.4
	P_ϕ	33.22	165.3	449.3
<i>Oblate</i>	P_ξ	50.24	232.5	620.1
	P_ζ	52.53	241.4	643.1
	P_ϕ	41.85	199.8	540.2

CALCULATION TIMES WITH MONOPOLE APPROXIMATION

<u>Volume conductor</u>	<u>16 segs</u>	<u>36 segs</u>	<u>64 segs</u>	<u>100 segs</u>
<i>Prolate</i>	0.767	1.718	3.047	4.754
<i>Oblate</i>	1.121	2.554	4.545	7.117

oblate spheroids. The approximation predicted the magnetic field more accurately than the spherical model for almost all dipole positions and orientations studied.

The error that results from using the monopole approximation may be tolerable for solving the inverse problem because the magnetic field can be calculated significantly faster compared to the dipole formulation. The magnetic field generated by the monopole approximation can be calculated at least 9 times faster than the dipole solution. We found that sufficient accuracy could generally be obtained when using only 10 terms in the dipole solution and 16 segments in the monopole approximation. In this case, the latter was approximately 40 times faster.

An additional savings in computational time is realized when the magnetic field must be calculated for three orthogonal dipole orientations. A moving dipole solution, in which the observed data is fit to a dipole whose position, orientation, and strength are constantly updated, may require the calculation of the magnetic fields of dipoles oriented along each axis with each iteration. This requires three complete solutions to the dipole formulation or monopole approximation. However, as shown in Fig. V-9, each additional dipole only requires one additional monopole as long as the dipoles share a common point. The time required to compute the field of three dipoles, using the monopole approximation, is therefore doubled whereas the time to calculate the dipole formulation is tripled. In this case, the monopole approximation reduces the computational time by

Fig. V-9. Three orthogonal dipoles can be constructed from four monopoles.

another factor of $2/3$.

The monopole approximation is independent of volume conductor size, depending only on the focal point of the spheroid. The dipole formulation, by comparison, contains a Legendre polynomial which is a function of the surface term. Surprisingly, varying the surface term does not significantly change the magnetic field calculated from the dipole equations. This could be expected for spheroids whose eccentricity approaches zero, since the size of a sphere does not affect the magnetic field of a dipole. This was also found to be true for prolate and oblate spheroids over a wide range of eccentricities.

Although the monopole solution has been referred to as an approximation throughout this paper, it could be considered an exact solution - but for a different volume. The approximation was based on the premise that the open-ended line segments and corresponding conduction current in Fig. V-1 did not generate a magnetic field, so the magnetic field was solely generated by the closed-ended line segments with displacement current. In other words, the magnetic field of the volume currents were presumed to cancel the magnetic field of the open line segments. This is not exactly correct, but the volume shape could be adjusted until this premise was identically satisfied. The volume would not remain as a spheroid, and the surface adjustments may be specific to the location of the current monopole. Assuming the surface adjustment does not vary when the monopole is shifted by some differential distance, then a dipole could be

constructed from two monopoles in this volume. The magnetic field of the dipole currents could then be calculated exactly from the monopole fields.

Using this idea, the monopole approximation presented in this paper would be considered exact for some unknown volume shape. The dipole solution would be considered as only an approximation because the volume shape has been approximated by a spheroid. This suggestion is not unreasonable, since no *in-vivo* volumes are perfectly spheroidal (eg. head or torso). It still remains to determine what volume shapes can be modelled in this manner and how the shape can be determined.

5.7 References

- [1] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," IEEE Trans. Magn., vol. MAG-27, pp. 758-767, 1991.
- [2] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.

CHAPTER VI

CONCLUSIONS

This research was predicated on the four hypotheses presented in Chapter I. The first hypothesis states that a solution can be found for the magnetic field of a current monopole in a volume conductor. A solution was obtained for the field of a monopole in semi-infinite and spherical volume conductors (Chapters II and III) and shown to be equivalent to that of a line segment of current. The monopole formulation for the semi-infinite volume was applied to three different neuronal models. An exact solution was obtained for the magnetic field of each model, and the conditions under which the model could be simplified were examined. This satisfies the second and third hypotheses stated in Chapter I.

The fourth hypothesis, stating that the monopole formulation can be adapted to special families of surfaces, could not be satisfied exactly. The field solution in both prolate and oblate spheroids was shown to be an approximation of the published dipole field solution. The difference between the two field solutions generally differed by less than 5%. However, the monopole approximation in these volumes is up to 40 times faster computationally and may represent a significant improvement for applying the inverse problem to these volumes.

The methods and techniques presented in the preceding chapters provide

the basis for a new method of calculating the magnetic field generated by neural currents. A monopole represent a patch of membrane from which transmembrane current is sourced or sinked into the surrounding volume conductor. Since the size of the patch is not limited by this model, there is no limit on the accuracy to which the monopole formulation can predict the field generated by neurons. The equations describing the magnetic field of monopoles are also more simple and computationally faster than the corresponding dipole solutions.

APPENDIX A
DERIVATION OF EQUATIONS
(6a) AND (6b) IN CHAPTER II

In this section, it will be necessary to rotate the coordinate system for the monopole to properly compare the **B** field of the monopole with that of the dipole. Primed and unprimed notations are used to indicate the rotated and fixed coordinate systems respectively. For example, if the coordinate system is rotated around the y axis by θ_0 and the new axis are denoted (x',y',z') , then the coordinate transforms are

$$\begin{aligned} x' &= x \cos\theta_0 + z \sin\theta_0 \Leftrightarrow x = x' \cos\theta_0 - z' \sin\theta_0 \Leftrightarrow \frac{\partial x}{\partial \theta_0} = -z \\ y' &= y \Leftrightarrow y = y' \Leftrightarrow \frac{\partial y}{\partial \theta_0} = 0 \\ z' &= -x \sin\theta_0 + z \cos\theta_0 \Leftrightarrow z = x' \sin\theta_0 + z' \cos\theta_0 \Leftrightarrow \frac{\partial z}{\partial \theta_0} = x \end{aligned} \quad (1)$$

If **B** only has a ϕ component in the fixed system, the **B** field can be described in cartesian coordinates in the fixed system as

$$\begin{aligned} B_x &= \frac{-y}{\sqrt{x^2 + y^2}} B_\phi \\ B_y &= \frac{x}{\sqrt{x^2 + y^2}} B_\phi \\ B_z &= 0 \end{aligned} \quad (2)$$

If the current source generating the **B** field is rotated as described above, the **B**

field observed in the fixed system resulting from the source in the rotated system is

$$\begin{aligned} B_{x'} &= B_x \cos\theta_0 + B_z \sin\theta_0 = B_x \cos\theta_0 \\ B_{y'} &= B_y \\ B_{z'} &= -B_x \sin\theta_0 + B_z \cos\theta_0 = -B_x \sin\theta_0 \end{aligned} \quad (3)$$

where the primed notation on the **B** field component indicates the source is rotated with respect to the observation system. Effectively, the rotated source now generates both a ϕ and θ component. This becomes obvious when (3) is written in spherical coordinates

$$\begin{aligned} B_{\theta'} &= (B_{x'} \cos\phi' + B_{y'} \sin\phi') \cos\theta' - B_{z'} \sin\theta' \\ B_{\phi'} &= -B_{x'} \sin\phi' + B_{y'} \cos\phi' \end{aligned} \quad (4)$$

where

$$\begin{aligned} \phi' &= \tan^{-1}\left(\frac{y'}{x'}\right) \\ \theta' &= \cos^{-1}\left(\frac{z'}{\sqrt{(x')^2 + (y')^2 + (z')^2}}\right) \end{aligned}$$

The goal is to obtain (4) as a function of θ_0 so the differential with respect to θ_0 can be obtained. Alternatively, a function of (x,y,z) may also be differentiated with respect to θ_0 using the identities shown in (1) and noting that (x',y',z') are constants since these are the basis vectors of the rotated system. The equations in (4) can be written

$$\begin{aligned}
B_{\theta'} &= \frac{x'z'B_{x'} + y'z'B_{y'} - [(x')^2 + (y')^2]B_{z'}}{\sqrt{(x')^2 + (y')^2} \sqrt{(x')^2 + (y')^2 + (z')^2}} \\
B_{\phi'} &= \frac{-y'B_{x'} + x'B_{y'}}{\sqrt{(x')^2 + (y')^2}}
\end{aligned} \tag{5}$$

Using the identities shown in (1) - (3), we can write (5) entirely in terms of the **B** field due to an unrotated source

$$\begin{aligned}
B_{\theta'} &= B_{\phi} \frac{-x'z'y \cos\theta_0 + y'z'x - [(x')^2 + (y')^2] y \sin\theta_0}{\sqrt{(x')^2 + (y')^2} \sqrt{(x')^2 + (y')^2 + (z')^2} \sqrt{x^2 + y^2}} \\
B_{\phi'} &= B_{\phi} \frac{(y')^2 \cos\theta_0 + (x')^2 \cos\theta_0 - x'z' \sin\theta_0}{\sqrt{(x')^2 + (y')^2} \sqrt{x^2 + y^2}}
\end{aligned} \tag{6}$$

The derivative with respect to the angle of rotation, θ_0 , can now be obtained for each component

$$\begin{aligned}
\frac{\partial B_{\theta'}}{\partial \theta_0} &= \frac{1}{\sqrt{(x')^2 + (y')^2} \sqrt{(x')^2 + (y')^2 + (z')^2}} * \\
&\left\{ \left(\frac{B_{\phi}}{\sqrt{x^2 + y^2}} \right) * [x'z'y \sin\theta_0 + y'z'(-z) - [(x')^2 + (y')^2] y \cos\theta_0] \right. \\
&\quad \left. + [-x'z'y \cos\theta_0 + y'z'x - [(x')^2 + (y')^2] y \sin\theta_0] * \right. \\
&\quad \left. \left[\frac{1}{\sqrt{x^2 + y^2}} \frac{\partial B_{\phi}}{\partial \theta_0} + B_{\phi} \frac{\partial}{\partial \theta_0} \left(\frac{1}{\sqrt{x^2 + y^2}} \right) \right] \right\}
\end{aligned} \tag{7a}$$

$$\begin{aligned}
\frac{\partial B_\phi}{\partial \theta_0} = & \left[\frac{B_\phi}{\sqrt{(x')^2 + (y')^2} \sqrt{x^2 + y^2}} \right] \{ -[(x')^2 + (y')^2] \sin \theta_0 - x'z' \cos \theta_0 \} \\
& + \left[\frac{1}{\sqrt{(x')^2 + (y')^2}} \right] \{ [(x')^2 + (y')^2] \cos \theta_0 - x'z' \sin \theta_0 \} * \\
& \left[\frac{1}{\sqrt{x^2 + y^2}} \frac{\partial B_\phi}{\partial \theta_0} + B_\phi \frac{\partial}{\partial \theta_0} \left(\frac{1}{\sqrt{x^2 + y^2}} \right) \right]
\end{aligned} \tag{7b}$$

The last term in these equations can be evaluated as

$$\begin{aligned}
\frac{\partial}{\partial \theta_0} \left(\frac{1}{\sqrt{x^2 + y^2}} \right) &= \frac{-x \frac{\partial x}{\partial \theta_0} - y \frac{\partial y}{\partial \theta_0}}{(x^2 + y^2)^{3/2}} \\
&= \frac{xz}{(x^2 + y^2)^{3/2}}
\end{aligned}$$

Substituting this back into (7a) and (7b) and noting that the derivative is being evaluated at $\theta_0=0$ results in

$$\begin{aligned}
\left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} &= \frac{1}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} * \\
& \left\{ \left(\frac{B_\phi}{\sqrt{x^2 + y^2}} \right) * [-yz^2 - (x^2 + y^2)y] \right. \\
& \quad \left. + [-xyz + xyz] * \left[\frac{1}{\sqrt{x^2 + y^2}} \left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} + B_\phi \frac{xz}{(x^2 + y^2)^{3/2}} \right] \right\} \\
&= \frac{-y \sqrt{x^2 + y^2 + z^2}}{x^2 + y^2} B_\phi \\
&= \frac{-\sin \phi}{\sin \theta} B_\phi
\end{aligned} \tag{8a}$$

$$\begin{aligned}
\left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} &= \left[\frac{B_\phi}{\sqrt{x^2 + y^2}} \right] [-x'z'] \\
&+ \left[\frac{1}{\sqrt{x^2 + y^2}} \right] [x^2 + y^2] \left[\frac{1}{\sqrt{x^2 + y^2}} \left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} + B_\phi \frac{xz}{(x^2 + y^2)^{3/2}} \right] \\
&= B_\phi \left[\frac{-xz}{x^2 + y^2} + \frac{xz}{x^2 + y^2} \right] + \left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} \\
&= \left(\frac{\partial B_\phi}{\partial \theta_0} \right)_{\theta_0=0} \\
&= \left[-z \left(\frac{\partial B_\phi^m}{\partial x} \right)_{\theta_0=0} + x \left(\frac{\partial B_\phi^m}{\partial z} \right)_{\theta_0=0} \right]
\end{aligned} \tag{8b}$$

APPENDIX B

MAGNETIC FIELD OF HIGHER ORDER POLES

In this section, the equations describing the magnetic field of higher order current terms are derived. The quadrupole field is developed from the dipole field, and the octupole field is then derived from the quadrupole field. The octupole field is also derived direct from the dipole field using a series of Legendre polynomials.

The magnetic field of the current dipole shown in Fig. B-1(a) is described by eqn. (7) in Chapter IV. The values of R, x and d shown in Fig. B-1(a) indicate the distance from the origin to the circle of observation points (which all have the same magnetic field strength due to the symmetry of the problem). Clearly $\sin \phi = R/d$ and $\cos \phi = x/d$. Successively higher order current terms can be fashioned from the basic dipole, with an infinite variety of relative positions and angles of the dipoles which serve as the building blocks of these terms.

One example of a current quadrupole, shown in Fig. B-1(b), is constructed from two oppositely oriented current dipoles each with strength P . The centers of the dipoles are separated by a distance Δk and the resultant magnetic field is given by

$$B = \frac{\mu P}{4\pi} \left(\frac{\sin \phi_2}{d_2^2} - \frac{\sin \phi_1}{d_1^2} \right)$$

Fig. B-1. Schematic diagram of higher order current sources. (a) The magnetic field of a dipole of strength P located at the origin and oriented in the $+x$ direction generates a magnetic field B which precesses around the x axis. (b) Two dipoles separated by a distance Δk , each with strength P , can be configured to generate a quadrupole field. An octupole can be generated from two quadrupoles separated by a distance $2\Delta k$ (c) or from four dipoles positioned as shown in (d).

where ϕ_1 and ϕ_2 are the angles subtended by d_1 and d_2 with the x axis respectively. Noting that $\sin\phi_2 = R/d_2$ and $\sin\phi_1 = R/d_1$ results in

$$B = \frac{\mu P}{4\pi} R \left(\frac{(d_1 - d_2)(d_1^2 + d_1 d_2 + d_2^2)}{d_1^3 d_2^3} \right)$$

Both d_1 and d_2 can be approximated by the distance to the origin d as long as Δk is sufficiently small, and the difference $(d_1 - d_2)$ can be approximated as $-\Delta k \cos\phi$, resulting in the equation for the magnetic field of a current quadrupole

$$B = \frac{-3 \mu P \Delta k}{4\pi d^3} \sin\phi \cos\phi \quad (1)$$

The analysis for the current octupole shown in Fig. B-1(c) proceeds similarly, where the magnetic field can be written, using the quadrupole result, as

$$B = \frac{-3 \mu P \Delta k}{4\pi} \left(\frac{\sin\phi_2 \cos\phi_2}{d_2^3} - \frac{\sin\phi_1 \cos\phi_1}{d_1^3} \right)$$

This can be written, using the trigonometric identities described earlier, as

$$B = \frac{-3 \mu P \Delta k}{4\pi} R \left(\frac{\Delta k (d_1^5 + d_2^5) + x (d_1 - d_2)(d_1^4 + d_1^3 d_2 + d_1^2 d_2^2 + d_1 d_2^3 + d_2^4)}{d_1^5 d_2^5} \right)$$

Since $\cos\phi = x/d = (d_2 - d_1)/(2\Delta k)$, the final equation for the magnetic field of the quadrupole can be approximated (eg. $d_1 d_2^3 = d^4$) as

$$B = \frac{3 \mu P \Delta k^2}{2\pi d^4} \sin\phi (5 \cos^2\phi - 1) \quad (2)$$

The same conclusion could be obtained by expanding the inverse distance relation as an infinite series of Legendre polynomials. This method does not approximate $d_1 d_2$ by d^2 , for example, and can be used to generate an exact equation for the octupole field. The final analysis will be shown to include higher order terms which can be neglected to generate a result identical to that obtained above. In general, the law of cosines can be applied to any triangle with sides of length a, b, c and an angle ϕ subtended by sides a and b such that

$$c^2 = a^2 + b^2 - 2(ab)\cos\phi$$

and, making use of the well-known identity for n 'th order Legendre polynomials $P_n(u)$

$$(1 - 2uz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(u) \quad (3)$$

the inverse distance law can be determined to be

$$\frac{1}{c} = \frac{1}{b} \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n P_n(\cos\phi)$$

A lesser-known identity is the inverse-cubed identity which can be obtained from (3) by differentiating with respect to u , yielding

$$(1 - 2uz + z^2)^{-3/2} = \frac{1}{z} \sum_{n=0}^{\infty} z^n P'_n(u) \quad (4)$$

where $P'_n(u)$ denotes the partial derivative of $P_n(u)$ with respect to u . The first few terms in this series are

$$\begin{array}{ll}
P_0(u) = 1 & P_0'(u) = 0 \\
P_1(u) = u & P_1'(u) = 1 \\
P_2(u) = (3u^2 - 1) / 2 & P_2'(u) = 3u \\
P_3(u) = (5u^3 - 3u) / 2 & P_3'(u) = (3/2)(5u^2 - 1)
\end{array}$$

Substituting the law of cosines into (4) and noting that $P_0'(u) = 0$ yields

$$\frac{1}{c^3} = \frac{1}{a b^2} \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n P_n'(\cos\phi) \quad (5)$$

The equation for the field of a single dipole (eqn. (7) in Chapter IV) can now be applied directly to the configuration of dipoles in Fig. B-1(d)

$$B = \frac{\mu P}{4\pi} R \left(\frac{1}{d_1^3} - \frac{1}{d_2^3} - \frac{1}{d_3^3} + \frac{1}{d_4^3} \right) \quad (6)$$

The expansion for each d_i^{-3} term is based on a triangle whose vertices are the i 'th dipole location, the observation point, and the origin. The angle ϕ in (5) is the angle subtended at the origin vertex. The expansions for d_1^{-3} and d_4^{-3} are similar except for the angles subtended at the origin vertex, ϕ_1 and ϕ_4 respectively, which are related since $\phi_1 + \phi_4 = 180^\circ$. Consequently, $\cos\phi_1 = -\cos\phi_4$ and the terms for even n drop out of the expansion because $P_n'(\cos\phi) = (-1)^{n+1} P_n'(\cos\phi)$ while the terms for odd n are doubled. Therefore,

$$\begin{aligned}
\frac{1}{d_1^3} + \frac{1}{d_4^3} &= \frac{2}{(3\Delta k/2) d^2} \left[\left(\frac{3\Delta k/2}{d} \right) + \left(\frac{3\Delta k/2}{d} \right)^3 \left(\frac{3}{2} \right) (5 \cos^2\phi - 1) + \mathcal{O}(\Delta k^5) \right] \\
\frac{1}{d_2^3} + \frac{1}{d_3^3} &= \frac{2}{(\Delta k/2) d^2} \left[\left(\frac{\Delta k/2}{d} \right) + \left(\frac{\Delta k/2}{d} \right)^3 \left(\frac{3}{2} \right) (5 \cos^2\phi - 1) + \mathcal{O}(\Delta k^5) \right]
\end{aligned}$$

and (6) can easily be shown to reduce to (2) when the higher order terms of order Δk^5 are neglected.

APPENDIX C
A PROOF THAT PROLATE (OBLATE)
SPHEROIDS ARE EQUIPOTENTIAL SURFACES
FOR LINE (DISK) CURRENT SOURCES

C.1 Prolate Spheroid

In this section, it is shown that a line of point current sources with uniform strength will have a family of confocal prolate spheroids as equipotential surfaces. The focal point of the spheroids will be shown to be the ends of the line of current sources.

Two different approaches can be taken to solving this problem. The more difficult approach is to begin with an unknown source distribution, but assume that the equipotential surfaces are prolate spheroids. The current $\mathbf{J}$ must be normal to the surface, but not necessarily uniform over the surface. It could then be shown that, for incremental distances inside this surface, another spheroidal surface exists over which $\mathbf{J}$ is also normal. The two surfaces could then be shown to be confocal and, upon collapsing the surface coordinate to the origin, would degenerate into a line stretching between the focal points.

A more simple approach is to start with the source description and show that the equipotential surfaces must be prolate spheroids. The potential of a point

current source of strength I at a distance R from the source is given by

$$V = \frac{I}{4\pi\sigma R} \quad (1)$$

The potential of a line of currents sources with a total source strength I can be found by integrating the effect of each differential source. The sources are assumed to be placed along the z -axis, stretching from $z = -C$ to $z = +C$, and have differential strength $I dz' / 2C$. The potential is therefore

$$V = \int_{-C}^{+C} \frac{I dz'}{8\pi C\sigma R} \quad (2)$$

where $R = [x^2 + y^2 + (z-z')^2]^{1/2}$. The solution to eqn. (2) is easily shown to be

$$V = \frac{-I}{8\pi C\sigma} \ln \left(\frac{(z-C) + \sqrt{x^2 + y^2 + (z-C)^2}}{(z+C) + \sqrt{x^2 + y^2 + (z+C)^2}} \right) \quad (3)$$

The prolate spheroidal coordinates (ξ, η, ϕ) can be defined in terms of the rectangular coordinates (x, y, z) as

$$\begin{aligned} x &= C\sqrt{(1-\xi^2)(\eta^2-1)} \sin\phi \\ y &= C\sqrt{(1-\xi^2)(\eta^2-1)} \cos\phi \\ z &= C\xi\eta \end{aligned} \quad (4)$$

where $-1 \leq \xi \leq 1$, $1 \leq \eta$, $0 \leq \phi \leq 2\pi$, and C is the length of the foci. The potential at any point (ξ, η, ϕ) given by eqn. (3) can now be written as

$$V = \frac{-I}{8\pi C\sigma} \ln \left(\frac{\eta - 1}{\eta + 1} \right) \quad (5)$$

The potential does not depend on ξ or ϕ , but only depends on η . The significance

of this becomes apparent once the coordinate transformations in eqn. (4) are written as

$$\frac{x^2}{\eta^2 - 1} + \frac{y^2}{\eta^2 - 1} + \frac{z^2}{\eta^2} = C^2$$

from which it can be seen that surfaces of constant η represent prolate spheroids. Therefore, the equipotential surfaces for the line of current sources are prolate spheroids. As the value of η increases towards infinity, the logarithmic term in eqn. (5) can be approximated by $-2/\eta$, and R (in rectangular coordinates) can be approximated as $c\eta$ (in spheroidal coordinates). In this case, eqn. (5) reduces to eqn. (1). This is consistent with the concept that the source should look like a point source at large distances.

C.2 Oblate Spheroid

In this section, it is shown that a circular disk of point current sources with uniform current strength will have, as equipotential surfaces, a family of confocal oblate spheroids. The radius of the source disk will be shown to be equal to the focal distance of the spheroid. The approach that will be taken is to assume that the equipotential surfaces are oblate spheroids and then prove that the current source must be a disk.

The oblate spheroidal system defined by the coordinates (ξ, ζ, ϕ) can be written in terms of the cartesian coordinates (x, y, z) as

$$\begin{aligned}
x &= C\xi\zeta \\
y &= C\sqrt{(1-\xi^2)(1+\zeta^2)} \cos\phi \\
z &= C\sqrt{(1-\xi^2)(1+\zeta^2)} \sin\phi
\end{aligned} \tag{7}$$

where $-1 \leq \xi \leq 1$, $0 \leq \zeta$, $0 \leq \phi \leq 2\pi$, and C is the length of the foci. The metric coefficients are therefore

$$\begin{aligned}
h_\xi &= C\sqrt{\frac{\xi^2 + \zeta^2}{1 - \xi^2}} \\
h_\zeta &= C\sqrt{\frac{\xi^2 + \zeta^2}{1 + \zeta^2}} \\
h_\phi &= C\sqrt{(1-\xi^2)(1+\zeta^2)}
\end{aligned}$$

Surfaces of constant ζ represent oblate spheroids since the transformations in eqn. (7) can be written as

$$\frac{x^2}{\zeta^2} + \frac{y^2}{1 + \zeta^2} + \frac{z^2}{1 + \zeta^2} = C^2 \tag{8}$$

which indicates the major axis occur along the y and z directions, with the minor axis along the x direction.

Laplace's equation can be written for the oblate spheroidal coordinate system as

$$\nabla^2 V = 0 = \frac{1}{h_\xi h_\zeta h_\phi} \left[\frac{\partial}{\partial \xi} \left(\frac{h_\zeta h_\phi}{h_\xi} \frac{\partial V}{\partial \xi} \right) + \frac{\partial}{\partial \zeta} \left(\frac{h_\xi h_\phi}{h_\zeta} \frac{\partial V}{\partial \zeta} \right) + \frac{\partial}{\partial \phi} \left(\frac{h_\xi h_\zeta}{h_\phi} \frac{\partial V}{\partial \phi} \right) \right]$$

Assuming the equipotential surfaces are oblate spheroids, then V can be written

as $V(\zeta)$. After substituting the metric coefficients and integrating with respect to the surface term ζ , the potential is fully described by

$$V = k_1 C \tan^{-1} |\zeta| + k_2 \quad (9)$$

with k_1 and k_2 as constants of integration. The potential must drop as $1/r$ at large distances from the source. If the total current source strength is I , then the potential at infinity is given by eqn. (1). In this case, the potential becomes

$$\lim_{\zeta \rightarrow \infty} V = k_1 C \left(\frac{\pi}{2} - \frac{1}{\zeta} \right) + k_2 = \frac{I}{4\pi\sigma R} \quad (10)$$

In the limit, eqn. (8) indicates that $R = (x^2 + y^2 + z^2)^{1/2}$ can be approximated by $R = C\zeta$. Using this information, the constants k_1 and k_2 can be deduced from eqn (10) to yield the final solution for the potential

$$V = \frac{I}{4\pi\sigma C} \left[\frac{\pi}{2} - \tan^{-1} |\zeta| \right] \quad (11)$$

The Laplacian of the potential in eqn. (11) is everywhere zero except on the surface defined by $\zeta = 0$, so current sources can only exist on this surface. Substituting $\zeta = 0$ into the coordinate transforms in eqn. (7) generates the cartesian coordinates over which the source is defined

$$\begin{aligned} x &= 0 \\ y &= C\sqrt{(1-\xi^2)} \cos\phi \\ z &= C\sqrt{(1-\xi^2)} \sin\phi \end{aligned} \quad (12)$$

Since $1 \geq \xi \geq -1$, the current sources exist on a disk centered at the origin lying on the $x=0$ plane with a radius C .

It only remains to show that the strength of the current sources is uniform over the surface of the disk. The normal component of the current at any oblate spheroidal surface is given by

$$\begin{aligned} \mathbf{J}_\zeta(\zeta) &= -\sigma \frac{1}{h_\zeta} \frac{dV}{d\zeta} \mathbf{a}_\zeta \\ &= \frac{I}{4\pi C^2} \left(\sqrt{\frac{1}{(\xi^2 + \zeta^2)(1 + \zeta^2)}} \right) \mathbf{a}_\zeta \end{aligned}$$

At the disk surface defined by $\zeta = 0$, this reduces to

$$\mathbf{J}_\zeta(0) = \frac{I}{4\pi C^2 \xi} \mathbf{a}_\zeta$$

On the disk, the normal current $d\mathbf{J}_\zeta = \mathbf{J}_\zeta dS$ for a differential surface element $dS = h_\xi h_\phi d\xi d\phi$ is given by

$$d\mathbf{J}_\zeta = \frac{I}{4\pi} \sqrt{1 - \xi^2} d\xi d\phi \mathbf{a}_\zeta$$

Using the following substitutions which are valid on the surface of the disk

$$r = \sqrt{y^2 + z^2} = C \sqrt{1 - \xi^2}$$

$$\mathbf{a}_x = \pm \mathbf{a}_\zeta$$

$$dr = \pm C d\xi$$

allows the differential current density to be written as

$$d\mathbf{J}_x = \pm \frac{I}{4\pi C^2} r dr d\phi \mathbf{a}_x \quad (13)$$

The sign depends on whether the term is evaluated on the upper or lower surface

of the disk. The current density described by eqn. (13) on the disk surface represents a uniform surface current density. The total current I which leaves the surface is split into two parts, one flowing in the $+x$ direction and one flowing in the $-x$ direction. Since the disk surface area is πC^2 , then the current density on either surface must be $I/2\pi C^2$.

APPENDIX D
A COMPARISON OF THE MONOPOLE APPROXIMATION
AND THE SPHERICAL MODEL FOR CALCULATING
THE MAGNETIC FIELD OUTSIDE SPHEROIDS

The following plots show the relative error that occurs when the monopole and spherical approximation are used to estimate the magnetic field generated by a current dipole in a spheroidal conductor. In all plots, the relative error is expressed as a percentage on the ordinate axis. The parameters for these simulations are provided in Chapter V.

The dipole positions are shown in Fig. V-4 for all these plots. The abscissa on each small plot indicates the y position (for prolate volumes) or x position (for oblate volumes) of the current source. The remaining dipole coordinate is fixed for each small plot, but varies over a range for all the plots on one figure. This value represents the z position (for prolate volumes) or y position (for oblate volumes) of the dipole as follows: 0.0 for prolate or 0.01 for oblate (upper-left); 0.1 (upper-center); 0.2 (upper-right); 0.3 (center-left); 0.4 (center); 0.5 (center-right); 0.6 (lower-left); 0.7 (lower-center); and 0.8 (lower-right).

The error associated with the monopole approximation is shown in the plots as a solid line; the error in the spherical approximation is shown as a dashed

line. In Figs. D-1(a)-(f), lines with circles represent the mean error over the observation surface and lines with no symbols represent the maximum value of the error. In Figs. D-1(g)-(l), lines with circles represent the 75th percentile for the relative error over the observation surface and lines with no symbols represent the 90th percentile. These percentiles refer to the values of the relative error for which 75% and 90% of the observation area have lower relative errors.

The relative errors are shown for prolate (Figs. D-1(a)-(c) and Figs. D-1(g)-(i)) and for oblate spheroids (Figs. D-1(d)-(f) and Figs. D-1(j)-(l)). The dipole is oriented in the +x direction (Figs. D-1(a),(d),(g),(j)), +y direction (Figs. D-1(b),(e),(h),(k)), or +z direction (Figs. D-1(c),(f),(i),(l)).

The plots are summarized as follows:

<u>Fig.</u>	<u>Volume</u>	<u>Dipole</u>	<u>Relative errors plotted</u>
(a)	Prolate	+x	mean, maximum
(b)	Prolate	+y	mean, maximum
(c)	Prolate	+z	mean, maximum
(d)	Oblate	+x	mean, maximum
(e)	Oblate	+y	mean, maximum
(f)	Oblate	+z	mean, maximum
(g)	Prolate	+x	75 th and 90 th percentiles
(h)	Prolate	+y	75 th and 90 th percentiles
(i)	Prolate	+z	75 th and 90 th percentiles
(j)	Oblate	+x	75 th and 90 th percentiles
(k)	Oblate	+y	75 th and 90 th percentiles
(l)	Oblate	+z	75 th and 90 th percentiles

Fig. D-1(a).

Fig. D-1(b).

Fig. D-1(c).

Fig. D-1(d).

Fig. D-1(e).

Fig. D-1(f).

Fig. D-1(g).

Fig. D-1(h).

Fig. D-1(i).

Fig. D-1(j).

Fig. D-1(k).

Fig. D-1(l).

APPENDIX E

FORTRAN SOURCE CODE TO CALCULATE THE DIPOLE SOLUTION IN SPHEROIDAL VOLUMES

The following code was used calculate the magnetic field of a current dipole located in both prolate and oblate volume conductors. The theoretical basis and the equations are described in [1]. These routines call subroutines listed in Appendix H to calculate the Legendre polynomials.

The Fortran code was designed according to comply with the ANSI Fortran 77 full language standard. The code was compiled with the Microsoft Fortran Optimizing Compiler v 4.1 to run on IBM-AT computers, and with the 3L Parallel Fortran compiler v2.0 to run in parallel on a network of transputers.

E.1 Comments on Coding

The equations derived in [1] can be modified to minimize computational time and repetitive computation of like-terms. The approach adopted in the following code was to minimize calls to Legendre polynomial routines since these demand almost all the computational time. The programming algorithm will be described here for only one dipole orientation in the prolate spheroid; the extension to other dipole orientations and the oblate spheroid should be obvious and is described in the Fortran source code.

The equation for the $\mathbf{H}$ field generated in the +x direction by the volume currents of a P_ξ oriented dipole, denoted H_{xv} , is given in eqn. (33) of ref. [1] as

$$\begin{aligned}
 H_{xv} = & \frac{P_{\xi_0} \sqrt{1 - \xi_0^2}}{4\pi C^2 \sqrt{\eta_a^2 - 1} \sqrt{\eta_0^2 - \xi_0^2}} \\
 & \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^n (2n+1) (-1)^m \left[\frac{(n-m)!}{(n+m)!} \right]^2 \right. \\
 & \times \cos(m\varphi) P_n^m(\xi) Q_n^m(\eta) P_n^m(\eta_a) \\
 & \times \left[\frac{(n+m)(n-m+1) P_n^{m-1}(\eta_0) P_n^{m-1'}(\xi_0)}{P_n^{m-1'}(\eta_a)} \right. \\
 & \left. \left. - \frac{P_n^{m+1}(\eta_0) P_n^{m+1'}(\xi_0)}{P_n^{m+1'}(\eta_a)} \right] \right. \\
 & \left. - \sum_{n=1}^{\infty} \frac{(2n+1) P_n^1(\eta_0) P_n^{1'}(\xi_0) P_n(\xi) Q_n(\eta) P_n(\eta_a)}{P_n^{1'}(\eta_a)} \right\}
 \end{aligned} \tag{1}$$

The constants K_1 , K_2 and K_3 can be defined as

$$\begin{aligned}
 K_1 &= \frac{P_{\xi_0}}{\pi C^2 \sqrt{\eta_0^2 - \xi_0^2}} \\
 K_2 &= \frac{K_1}{4 \sqrt{\eta_a^2 - 1}} \\
 K_3 &= \frac{K_1 \eta_a}{2 (\eta_a^2 - 1)}
 \end{aligned} \tag{2}$$

and the iteration-dependent variables $A_1(n,m)$ to $A_5(n,m)$ are defined as

$$\begin{aligned}
A_1(n, m) &= (2n + 1) (-1)^m \left[\frac{(n - m)!}{(n + m)!} \right]^2 \\
A_2(n, m) &= P_n^m(\xi) Q_n^m(\eta) P_n^m(\eta_a) \\
A_3(n, m) &= (n + m) (n - m + 1) \times A_4(n, m-1) \\
A_4(n, m) &= \frac{A_5(n, m) \times P_n^m(\eta_0)}{\frac{\partial P_n^m(\eta_a)}{\partial \eta_a}} && \text{if } m \neq n \\
&= 0 && \text{if } m = n \\
A_5(n, m) &= \sqrt{1 - \xi_0^2} \times \frac{\partial P_n^m(\xi_0)}{\partial \xi_0} \text{ if } \xi_0 \neq 1 \\
&= n(n + 1) / 2 && \text{if } \xi_0 = 1 \text{ and } m = 1 \\
&= 0 && \text{if } \xi_0 = 1 \text{ and } m \neq 1
\end{aligned}$$

The equation for H_{xv} and the other components can now be written as

$$\begin{aligned}
H_{xv} &= -K_2 \sum_{n=1}^N \sum_{m=1}^n \cos(m\varphi) \times A_1(n, m) \times A_2(n, m) \times [A_3(n, m) - A_4(n, m+1)] \\
&\quad + K_2 \sum_{n=1}^N (2n + 1) \times A_2(n, 0) \times A_4(n, 1) \\
H_{yv} &= -K_2 \sum_{n=1}^N \sum_{m=1}^n \sin(m\varphi) \times A_1(n, m) \times A_2(n, m) \times [-A_3(n, m) - A_4(n, m+1)] \\
H_{zv} &= K_3 \sum_{n=1}^N \sum_{m=1}^n m \times \sin(m\varphi) \times A_1(n, m) \times A_2(n, m) \times A_4(n, m)
\end{aligned}$$

The $A_3(n, m)$ term relies on previously calculated values of $A_4(n, m-1)$, but does not require more Legendre terms to be calculated. This algorithm therefore only

requires four Legendre polynomials and 2 derivatives of Legendre polynomials to be calculated for each n,m combination in the summation term shown in (1). In the following code, the Fortran variables A1 to A5 are consistent with the variables defined above.

E.2 References

- [1] B.N. Cuffin and D.C. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.

```

c      CONDITIONAL COMPILATION.
c      This module is used for both non-parallel and parallel-worker
c      task. So all write statements are commented out with 'w'
c      which can be compiled only for non-parallel operation.
c      Likewise, the STOP command is also commented out with 'w'.
c      There are no other conditional compilations available for
c      this module.
c
c      UPDATE HISTORY
c
c      23-Mar-91      Check for C=0 (sphere), in which case the
c                    spherical routines are called. A test routine
c                    called DIPOLTST.FOR is available for testing
c                    these things.
c
c      24-Mar-91      Changed qdPnm() routine after testing the
c                    Cuffin and Cohen results shown in eqn (36).
c
c      31-Mar-91      Added BdPoll() which is the same as Bdpole()
c                    except it returns the magnitude as well.
c                    Kept BDpole to maintain backward compatibility.
c                    Return Bx ... Bz = 0 in PrPphi whenever the source
c                    point is located at a pole of spheroid.
c

```

```

c-----
c      Compute the general B field for a dipole current (intracellular) in
c      cartesian coordinates.
c
c      Bdpoll() is the same as Bdpole() except it also returns the
c      magnitude of the B field. Both routines are available to maintain
c      compatibility with routines that call the older Bdpole().
c
c      Let xf,yf,zf = field point
c           xs,ys,zs = source point
c           Px,Py,Pz = dipole moment
c           Bx,By,Bz = B field components
c
c      SUBROUTINE Bdpoll(xf,yf,zf,xs,ys,zs,Px,Py,Pz,Bx,By,Bz,Bm)
c      double precision  xf,yf,zf,xs,ys,zs,Px,Py,Pz,Bx,By,Bz,Bm
c      call Bdpole(xf,yf,zf,xs,ys,zs,Px,Py,Pz,Bx,By,Bz)
c      Bm = dsqrt(Bx*Bx + By*By + Bz*Bz)
c      return
c      end
c
c      SUBROUTINE Bdpole(xf,yf,zf,xs,ys,zs,Px,Py,Pz,Bx,By,Bz)
c
c      double precision  xf,yf,zf,xs,ys,zs,Px,Py,Pz,Bx,By,Bz

```

```

double precision  Rx,Ry,Rz,R,denom

double precision  pi
parameter          (pi = 3.141592653589793d0)

Rx = xf-xs
Ry = yf-ys
Rz = zf-zs
R = dsqrt(Rx*Rx + Ry*Ry + Rz*Rz)

denom = 4.d0 * pi * R * R * R
Bx = ( Py*Rz - Pz*Ry ) / denom
By = ( Pz*Rx - Px*Rz ) / denom
Bz = ( Px*Ry - Py*Rx ) / denom

return
end

```

```

c*****

```

```

c      Computes the B field for a dipole in a prolate spheroid
c      using the equations (33)-(35) of Cuffin and Cohen,
c      IEEE Trans. Biomed. Eng., 24(4):372-381, 1977.

c      The following variables are input to the functions:
c
c      C = constant of transformation (focal point)
c      xi,eta,phi = observation point
c      xi0,eta0,phi0 = source point
c      etaa = eta value corresponding to the surface of the spehroid
c      Pxi,Peta,Pphi = dipole moment in each direction
c      nmax = maximum number of n iterations for summations.

c      Cuffin and Cohen assume that phi0 = 0, so we must first
c      rotate the system to make phi0 = 0. To do this, we work
c      with a phi for the field coordinate that is given by
c      phi (supplied by user) - phi0.
c      At the end, we must rotate the Bx,By,Bz back by the phi0
c      ammount. To do this, we use the routine PrRot().

c      In general, it appears that Cuffin and Cohen are off by
c      a sign in the following routines:
c          PrPxi   eqn (33) for Bx,By
c          PrPeta  eqn (34) for Bx,By
c          PrPxphi eqn (35) for Bx,By,Bz
c      These routine returns the negative of what Cuffin and Cohen
c      show in their paper since I think they are missing a
c      negative sign (except for Bz which seems correct).

```

c*****

c These results are meaningless if the source point is located
c at either pole.

1 SUBROUTINE PrPxi(c,xi0,eta0,phi0,etaa,
 xi,eta,tphi,Pxi,nmax,Bx,By,Bz)

double precision c,xi0,eta0,phi0,etaa

double precision xi,eta,tphi,Pxi

integer nmax

double precision Bx,By,Bz

double precision T1,T2,T3,T4,T5

integer n,m

double precision phi,mphi

double precision pi

parameter (pi = 3.141592653589793d0)

double precision xi02,eta02,etaa2,temp

double precision dneg1,dsq,dfact

external dneg1,dsq,dfact

double precision Pn0,Pnm,dPnm,Qn0,Qnm,dQnm

external Pn0,Pnm,dPnm,Qn0,Qnm,dQnm

double precision qdPnm

external qdPnm

double precision Px,Py,Pz

double precision x,y,z,Bm

c IF PERFECT SPHERE (c=0), call the sphere routine and return

if (c .eq. 0.d0) then

call e2c(Px,Py,Pz,c,xi0,eta0,phi0,1)

call e2c(x,y,z,c,xi,eta,tphi,1)

1 call Bdsphr(Px,Py,Pz,x,y,z,0.d0,Pxi,-pi,0.d0,
 Bx,By,Bz,Bm)

return

endif

c ROTATE SYSTEM BY phi0 AMOUNT

phi = tphi - phi0

c GENERATE SQUARES OF KEY VALUES

xi02 = xi0*xi0

eta02 = eta0*eta0

etaa2 = etaa*etaa

temp = dsqrt(1.d0-xi02)

```

Bx = 0.d0
By = 0.d0
Bz = 0.d0

do 20 n=1,nmax
  T3 = Pnm(n,0,eta0) * qdPnm(-1.d0,temp,n,0,xi0) /
1    dPnm(n,0,etaa)
  T5 = Pnm(n,1,eta0) * qdPnm(-1.d0,temp,n,1,xi0) /
1    dPnm(n,1,etaa)

  do 10 m=1,n
    T1 = (2.d0*n+1.d0) * dnegl(m) *
1      dsq(dfact(n-m)/dfact(n+m))
    T2 = Pnm(n,m,xi) *
1      Qnm(n,m,eta) * Pnm(n,m,etaa)
    T3 = (n+m) * (n-m+1) * T3
    if (m.eq.n) then
      T4 = 0.d0
    else
      T4 = Pnm(n,m+1,eta0) *
1        qdPnm(-1.d0,temp,n,m+1,xi0) /
1        dPnm(n,m+1,etaa)
    endif
    mphi = float(m)*phi
c    write(*,'(2i5,4g15.5)') n,m,dsin(mphi),T1,T2,T5
c    write(*,'(10x,4g15.5)') Bz,dsin(mphi)*m*T1*T2*T5
    Bx = Bx + dcos(mphi)*T1*T2*(T3-T4)
    By = By + dsin(mphi)*T1*T2*(-T3-T4)
    Bz = Bz + dsin(mphi)*m*T1*T2*T5
    T3 = T5
    T5 = T4
c    write(*,'(10x,4g15.5)') Bz

10    continue

    T1 = (2.d0*n+1.d0) * Pnm(n,1,eta0) *
1      qdPnm(-1.d0,temp,n,1,xi0) * Pn0(n,xi) *
2      Qn0(n,eta) * Pn0(n,etaa) / dPnm(n,1,etaa)
    Bx = Bx - T1

20    continue

    T1 = Pxi / ( pi * c*c * dsqrt(eta02-xi02) )
    T2 = T1 / ( 4.d0 * dsqrt(etaa2-1.d0) )
    Bx = -Bx * T2
    By = -By * T2
    Bz = Bz * T1 * etaa / ( 2.d0 * (etaa2-1.d0) )

c    ROTATE Bx,By back by phi0 amount

```



```

call PrRot(Bx,By,phi0)

end

c*****
c      Same as above, but change the following:
c          sqrt(1-xi02) ---> sqrt(eta02-1)
c          dPnm(xi0) ---> Pnm(xi0)
c          Pnm(eta0) ---> dPnm(eta0)
c      and the sign in any calls to qdPnm changes from -1 to +1.
c
1      SUBROUTINE PrPeta(c,xi0,eta0,phi0,etaa,
           xi,eta,tphi,Peta,nmax,Bx,By,Bz)

           double precision c,xi0,eta0,phi0,etaa
           double precision xi,eta,tphi,Peta
           integer nmax
           double precision Bx,By,Bz

           double precision T1,T2,T3,T4,T5
           integer n,m
           double precision phi,mphi

           double precision pi
           parameter (pi = 3.141592653589793d0)

           double precision xi02,eta02,etaa2,temp

           double precision dneg1,dsq,dfact
           external dneg1,dsq,dfact
           double precision Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
           external Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
           double precision qdPnm
           external qdPnm

           double precision Px,Py,Pz
           double precision x,y,z,Bm

c      IF PERFECT SPHERE (c=0), call the sphere routine and return

           if (c .eq. 0.d0) then
               call e2c(Px,Py,Pz,c,xi0,eta0,phi0,1)
               call e2c(x,y,z,c,xi,eta,tphi,1)
               call Bdsphr(Px,Py,Pz,x,y,z,0.d0,Peta,0.d0,pi/2.d0,
1                  Bx,By,Bz,Bm)
               return
           endif

c      ROTATE SYSTEM BY phi0 AMOUNT

```

```

phi = tphi - phi0

c  GENERATE SQUARES OF KEY VALUES
   xi02 = xi0*xi0
   eta02 = eta0*eta0
   etaa2 = etaa*etaa
   temp = dsqrt( eta02-1.d0 )

   Bx = 0.d0
   By = 0.d0
   Bz = 0.d0

   do 20 n=1,nmax
      T3 = Pnm(n,0,xi0) * qdPnm(1.d0,temp,n,0,eta0) /
1         dPnm(n,0,etaa)
      T5 = Pnm(n,1,xi0) * qdPnm(1.d0,temp,n,1,eta0) /
1         dPnm(n,1,etaa)

      do 10 m=1,n
         T1 = (2.d0*n+1.d0) * dneg1(m) *
1            dsq(dfact(n-m)/dfact(n+m))
         T2 = Pnm(n,m,xi) *
1            Qnm(n,m,eta) * Pnm(n,m,etaa)
         T3 = (n+m) * (n-m+1) * T3
         if (m.eq.n) then
            T4 = 0.d0
         else
            T4 = Pnm(n,m+1,xi0) * qdPnm(1.d0,temp,n,m+1,eta0) /
1            dPnm(n,m+1,etaa)
         endif
         mphi = float(m)*phi
         Bx = Bx + dcos(mphi)*T1*T2*(T3-T4)
         By = By + dsin(mphi)*T1*T2*(-T3-T4)
         Bz = Bz + dsin(mphi)*m*T1*T2*T5
         T3 = T5
         T5 = T4

10      continue

      T1 = (2.d0*n+1.d0) * Pnm(n,1,xi0) *
1         qdPnm(1.d0,temp,n,1,eta0) * Pn0(n,xi) *
2         Qn0(n,eta) * Pn0(n,etaa) / dPnm(n,1,etaa)
      Bx = Bx - T1

20      continue

      T1 = Peta / ( pi * c*c * dsqrt(eta02-xi02) )
      T2 = T1 / ( 4.d0 * dsqrt(etaa2-1.d0) )
      Bx = -Bx * T2
      By = -By * T2

```

```

Bz = Bz * T1 * etaa / ( 2.d0 * (etaa2-1.d0) )

c    ROTATE Bx,By back by phi0 amount
c    call PrRot(Bx,By,phi0)

end

c*****
c    This returns 0.d0 for the B field if the source point is located
c    at either pole.

1    SUBROUTINE PrPphi(c,xi0,eta0,phi0,etaa,
        xi,eta,tphi,Pphi,nmax,Bx,By,Bz)

        double precision c,xi0,eta0,phi0,etaa
        double precision xi,eta,tphi,Pphi
        integer          nmax
        double precision Bx,By,Bz

        double precision T1,T2,T3,T4,T5
        integer          n,m
        double precision phi,mphi

        double precision pi
        parameter        (pi = 3.141592653589793d0)

        double precision xi02,eta02,etaa2

        double precision dneg1,dsq,dfact
        external          dneg1,dsq,dfact
        double precision Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
        external          Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
        double precision qdPnm
        external          qdPnm

        double precision Px,Py,Pz
        double precision x,y,z,Bm

c    IF PERFECT SPHERE (c=0), call the sphere routine and return

        if (c .eq. 0.d0) then
            call e2c(Px,Py,Pz,c,xi0,eta0,phi0,1)
            call e2c(x,y,z,c,xi,eta,tphi,1)
            call Bdsphr(Px,Py,Pz,x,y,z,0.d0,Pphi,-pi/2.d0,0.d0,
1            Bx,By,Bz,Bm)
            return
        endif

c    ROTATE SYSTEM BY phi0 AMOUNT
c    phi = tphi - phi0

```

```

c      GENERATE SQUARES OF KEY VALUES
      xi02 = xi0*xi0
      eta02 = eta0*eta0
      etaa2 = etaa*etaa

c      IF XI0 = +/- 1, THEN WE ARE AT THE POLES OF THE SPHEROID
c      FOR WHICH THE SOLUTION COLLAPSES ... SO JUST RETURN ZERO

      if (xi02 .eq. 1.d0) then
        Bx = 0.d0
        By = 0.d0
        Bz = 0.d0
        return
      endif

      Bx = 0.d0
      By = 0.d0
      Bz = 0.d0

      do 20 n=1,nmax
        T3 = 0.d0
        T5 = Pnm(n,1,eta0) * Pnm(n,1,xi0) / dPnm(n,1,etaa)

        do 10 m=1,n
          T1 = (2.d0*n+1.d0) * dneg1(m) *
1             dsq(dfact(n-m)/dfact(n+m))
          T2 = Pnm(n,m,xi) *
1             Qnm(n,m,eta) * Pnm(n,m,etaa)
          T3 = (n+m) * (n-m+1) * T3
          if (m.eq.n) then
            T4 = 0.d0
          else
            T4 = (m+1) * Pnm(n,m+1,eta0) * Pnm(n,m+1,xi0) /
1             dPnm(n,m+1,etaa)
          endif
          mphi = float(m)*phi
          Bx = Bx + dsin(mphi)*T1*T2*(T3-T4)
          By = By + dcos(mphi)*T1*T2*(T3+T4)
          Bz = Bz + dcos(mphi)*m*T1*T2*T5
          T3 = T5
          T5 = T4
10       continue

          T1 = (2.d0*n+1.d0) * Pnm(n,1,eta0) *
1             Pnm(n,1,xi0) * Pn0(n,xi) *
2             Qn0(n,eta) * Pn0(n,etaa) / dPnm(n,1,etaa)
          By = By + T1

```

```

20      continue

      T1 = Pphi / ( pi * c*c * dsqrt( (eta02-1.d0)*(1.d0-xi02) ) )
      T2 = T1 / ( 4.d0 * dsqrt(etaa2-1.d0) )
      Bx = -Bx * T2
      By = -By * T2
      Bz = -Bz * T1 * etaa / ( 2.d0 * (etaa2-1.d0) )

c      ROTATE Bx,By back by phi0 amount
      call PrRot(Bx,By,phi0)

      end

c*****
c      Function to take care of the condition specified in
c      equation (36) in Cuffin and Cohen's paper.
c      If x = 1.d0, then return the value
c          signx*n*(n+1)/2 if m=1
c          0 otherwise
c      else return
c          sqrt(1-x**2) * dPnm(n,m,x)
c      where sqrt(1-x**2) is supplied by the calling routine in xmag.
c
c 24-Mar-91      Modified this to return a positive results everytime
c 24-Mar-91      except when n is odd and x is -1.d0

      DOUBLE PRECISION FUNCTION qdPnm(signx,xmag,n,m,x)

      double precision  signx,xmag,x
      integer           n,m

      double precision  dPnm
      external          dPnm

      if ( x .eq. 1.d0 ) then
        if (m.eq.1) then
c 24-Mar-91      qdPnm = signx * n * (n+1.d0) / 2.d0
                  qdPnm = n * (n+1.d0) / 2.d0
        else
                  qdPnm = 0.d0
        endif
      elseif ( x .eq. -1.d0 ) then
        if (m.eq.1) then
                  qdPnm = n * (n+1.d0) / 2.d0
                  if (mod(n,2).eq.1) qdPnm = -qdPnm
        else
                  qdPnm = 0.d0
        endif
      else
        qdPnm = xmag * dPnm(n,m,x)
      
```

```
endif
return
end
```

```
c*****
c      Rotates Bx, By back by phi0 amount.
c      The equations are:
c          x = K sin(phi0)
c          y = L cos(phi0)
c      where K,L are irrelevant constants w.r.t phi0.
```

```
SUBROUTINE PrRot(Bx,By,phi0)
```

```
double precision Bx,By,phi0
double precision Bt,sinphi,cosphi
```

```
sinphi = dsin(phi0)
cosphi = dcos(phi0)
Bt = Bx
```

```
Bx = By*sinphi + Bt*cosphi
By = Bt*cosphi - Bx*sinphi
```

```
return
end
```

```
c*****
c      Computes the B field for a dipole in a oblate spheroid
c      using the equations (39)-(41) of Cuffin and Cohen. The "zeta"
c      term is replaced by "eta" in this discussion.

c      The following variables are input to the functions:
c
c      C = constant of transformation (focal point)
c      xi,eta,phi = observation point
c      xi0,eta0,phi0 = source point
c      etaa = eta value corresponding to the surface of the spehroid
c      Pxi,Peta,Pphi = dipole moment in each direction
c      nmax = maximum number of n iterations for summations.

c      Cuffin and Cohen assume that phi0 = 0, so we must first
c      rotate the system to make phi0 = 0. To do this, we work
c      with a phi for the field coordinate that is given by
c      phi (supplied by user) minus phi0.
c      At the end, we must rotate the Bx,By,Bz back by the phi0
c      ammount. To do this, we use the routine ObRot().

c      We can ignore the use of complex variables because, for complex x,
```

```

c          P(n,m,x) / dP(n,m,x) = real
c          j * P(n,m,x) * Q(n,m,x) = real
c      and all the equations can be broken into real numbers times
c      combinations of the above.
c      This assumes that, for complex x (x=ju) we can write
c          dP(n,m,x) = d/du(P(n,m,x))
c      rather than
c          dP(n,m,x) = d/dx(P(n,m,x))
c      The Legendre functions were modified on 10-11-90 to
c      account for this.
c      What it means for this code is that we always have j*j in
c      the result which means we just put a -1 sign on the final
c      output.
c
c      CORRECTIONS TO CUFFIN AND COHENS PAPER:
c
c          ObPxi   Changed divisor on Bx from 8 to 2.
c                  Changed sign on Bx.
c
c          ObPeta  Changed divisor on Bx from 8 to 2.
c                  Changed sign on Bx.
c
c          ObPphi  Changed sign on Bx.
c
c*****
c      These results are meaningless if the source point is located
c      at either pole.
c
c      SUBROUTINE ObPxi(c,xi0,eta0,phi0,etaa,
1          xi,eta,tphi,Pxi,nmax,Bx,By,Bz)
c
c      double precision  c,xi0,eta0,phi0,etaa
c      double precision  xi,eta,tphi,Pxi
c      integer           nmax
c      double precision  Bx,By,Bz
c
c      double precision  T1,T2,T3,T4,T5
c      integer           n,m
c      double precision  phi,mphi
c
c      double precision  pi
c      parameter         (pi = 3.141592653589793d0)
c
c      double precision  xi02,eta02,etaa2,temp
c
c      double precision  dnegl,dsq,dfact
c      external          dnegl,dsq,dfact
c
c      double precision  Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
c      external          Pn0,Pnm,dPnm,Qn0,Qnm,dQnm

```

```

double precision cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
external         cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
integer          jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
external         jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
double precision jpower
external         jpower

double precision qdPnm
external         qdPnm

double precision Px,Py,Pz
double precision x,y,z,Bm

c      IF PERFECT SPHERE (c=0), call the sphere routine and return

      if (c .eq. 0.d0) then
            call e2c(Px,Py,Pz,c,xi0,eta0,phi0,2)
            call e2c(x,y,z,c,xi,eta,tphi,2)
            call Bdsphr(-Pz,Py,Px,-z,y,x,0.d0,Pxi,-pi,0.d0,
1              Bz,By,Bx,Bm)
            Bz = -Bz
            return
      endif

c      ROTATE SYSTEM BY phi0 AMOUNT
      phi = tphi - phi0

c      GENERATE SQUARES OF KEY VALUES
      xi02 = xi0*xi0
      eta02 = eta0*eta0
      etaa2 = etaa*etaa
      temp = dsqrt( 1.d0-xi02 )

      Bx = 0.d0
      By = 0.d0
      Bz = 0.d0

      do 20 n=1,nmax
            T3 = cPnm(n,0,eta0) * qdPnm(-1.d0,temp,n,0,xi0) /
1              cdPnm(n,0,etaa)
            T5 = cPnm(n,1,eta0) * qdPnm(-1.d0,temp,n,1,xi0) /
1              cdPnm(n,1,etaa)

            do 10 m=1,n
                  T1 = (2.d0*n+1.d0) * dneg1(m) *
1                  dsq(dfact(n-m)/dfact(n+m))
                  T2 = Pnm(n,m,xi) *
1                  cQnm(n,m,eta) * cPnm(n,m,etaa)
                  T3 = (n+m) * (n-m+1) * T3
            enddo 10
      enddo 20

```



```

        if (m.eq.n) then
            T4 = 0.d0
        else
            T4 = cPnm(n,m+1,eta0) *
1             qdPnm(-1.d0,temp,n,m+1,xi0) /
1             cdPnm(n,m+1,etaa)
        endif
        mphi = float(m)*phi
        Bz = Bz + dcos(mphi)*T1*T2*(T3-T4)
        By = By + dsin(mphi)*T1*T2*(-T3-T4)
        Bx = Bx + dsin(mphi)*m*T1*T2*T5
        T3 = T5
        T5 = T4
10      continue

        T1 = (2.d0*n+1.d0) * cPnm(n,1,eta0) *
1         qdPnm(-1.d0,temp,n,1,xi0) * Pn0(n,xi) *
2         cQn0(n,eta) * cPn0(n,etaa) / cdPnm(n,1,etaa)
        Bz = Bz - T1

20      continue

        T1 = Pxi / ( pi * c*c * dsqrt(xi02+eta02) )
        T2 = T1 / (4.d0 * dsqrt(1.d0+etaa2))
        Bz = -Bz * T2
        By = -By * T2
        Bx = Bx * T1 * etaa / ( 2.d0 * (1.d0+etaa2) )

c      ROTATE Bx,By back by phi0 amount
c      call ObRot(By,Bz,phi0)

        end

c*****
c      Same as above, but change the following:
c          sqrt(1-xi02) ---> sqrt(1+eta02)
c          dPnm(xi0) ---> Pnm(xi0)
c          cPnm(eta0) ---> cdPnm(eta0)
c      and thus calls to qdPnm change to qcdPnm since the argument
c      in the dPnm() is complex.

1      SUBROUTINE ObPeta(c,xi0,eta0,phi0,etaa,
           xi,eta,tphi,Peta,nmax,Bx,By,Bz)

        double precision  c,xi0,eta0,phi0,etaa
        double precision  xi,eta,tphi,Peta
        integer           nmax
        double precision  Bx,By,Bz

```

```

double precision  T1,T2,T3,T4,T5
integer           n,m
double precision  phi,mphi

double precision  pi
parameter         (pi = 3.141592653589793d0)

double precision  xi02,eta02,etaa2,temp

double precision  dneg1,dsq,dfact
external          dneg1,dsq,dfact
double precision  Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
external          Pn0,Pnm,dPnm,Qn0,Qnm,dQnm

double precision  cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
external          cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
integer           jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
external          jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
double precision  jpower
external          jpower

double precision  qcdPnm
external          qcdPnm

double precision  Px,Py,Pz
double precision  x,y,z,Bm

c      IF PERFECT SPHERE (c=0), call the sphere routine and return

      if (c .eq. 0.d0) then
            call e2c(Px,Py,Pz,c,xi0,eta0,phi0,2)
            call e2c(x,y,z,c,xi,eta,tphi,2)
            call Bdsphr(-Pz,Py,Px,-z,y,x,0.d0,Peta,0.d0,pi/2.d0,
1          Bz,By,Bx,Bm)
            Bz = -Bz
            return
      endif

c      ROTATE SYSTEM BY phi0 AMOUNT
      phi = tphi - phi0

c      GENERATE SQUARES OF KEY VALUES
      xi02 = xi0*xi0
      eta02 = eta0*eta0
      etaa2 = etaa*etaa
      temp = dsqrt( 1.d0 + eta02 )

      Bx = 0.d0
      By = 0.d0

```

```

Bz = 0.d0

do 20 n=1,nmax
  T3 = Pnm(n,0,xi0) * qcdPnm(-1.d0,temp,n,0,eta0) /
1    cdPnm(n,0,etaa)
  T5 = Pnm(n,1,xi0) * qcdPnm(-1.d0,temp,n,1,eta0) /
1    cdPnm(n,1,etaa)

  do 10 m=1,n
    T1 = (2.d0*n+1.d0) * dneg1(m) *
1      dsq(dfact(n-m)/dfact(n+m))
    T2 = Pnm(n,m,xi) *
1      cQnm(n,m,eta) * cPnm(n,m,etaa)
    T3 = (n+m) * (n-m+1) * T3
    if (m.eq.n) then
      T4 = 0.d0
    else
1      T4 = Pnm(n,m+1,xi0) * qcdPnm(-1.d0,temp,n,m+1,eta0) /
        cdPnm(n,m+1,etaa)
    endif
    mphi = float(m)*phi
    Bz = Bz + dcos(mphi)*T1*T2*(T3-T4)
    By = By + dsin(mphi)*T1*T2*(-T3-T4)
    Bx = Bx + dsin(mphi)*m*T1*T2*T5
    T3 = T5
    T5 = T4

10    continue

    T1 = (2.d0*n+1.d0) * Pnm(n,1,xi0) *
1      qcdPnm(-1.d0,temp,n,1,eta0) * Pn0(n,xi) *
2      cQn0(n,eta) * cPn0(n,etaa) / cdPnm(n,1,etaa)
    Bz = Bz - T1

20    continue

    T1 = Peta / ( pi * c*c * dsqrt(xi02+eta02) )
    T2 = T1 / (4.d0 * dsqrt(1.d0+etaa2))
    Bz = -Bz * T2
    By = -By * T2
    Bx = Bx * T1 * etaa / ( 2.d0 * (1.d0+etaa2) )

c    ROTATE Bx,By back by phi0 amount
c    call ObRot(By,Bz,phi0)

end

c*****
c    These results are meaningless if the source point is located
c    at either pole, so return 0.d0 values.

```

```

SUBROUTINE ObPphi(c,xi0,eta0,phi0,etaa,
1      xi,eta,tphi,Pphi,nmax,Bx,By,Bz)

double precision c,xi0,eta0,phi0,etaa
double precision xi,eta,tphi,Pphi
integer          nmax
double precision Bx,By,Bz

double precision T1,T2,T3,T4,T5
integer          n,m
double precision phi,mphi

double precision pi
parameter        (pi = 3.141592653589793d0)

double precision xi02,eta02,etaa2

double precision dneg1,dsq,dfact
external         dneg1,dsq,dfact
double precision Pn0,Pnm,dPnm,Qn0,Qnm,dQnm
external         Pn0,Pnm,dPnm,Qn0,Qnm,dQnm

double precision cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
external         cPn0,cPnm,cdPnm,cQn0,cQnm,cdQnm
integer          jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
external         jPn0,jPnm,jdPnm,jQn0,jQnm,jdQnm
double precision jpower
external         jpower

double precision qdPnm
external         qdPnm

double precision Px,Py,Pz
double precision x,y,z,Bm

c      IF PERFECT SPHERE (c=0), call the sphere routine and return

      if (c .eq. 0.d0) then
            call e2c(Px,Py,Pz,c,xi0,eta0,phi0,2)
            call e2c(x,y,z,c,xi,eta,tphi,2)
            call Bdsphr(-Pz,Py,Px,-z,y,x,0.d0,Pphi,pi/2.d0,0.d0,
1              Bz,By,Bx,Bm)
            Bz = -Bz
            return
      endif

c      ROTATE SYSTEM BY phi0 AMOUNT
      phi = tphi - phi0

```

```

c      GENERATE SQUARES OF KEY VALUES
      xi02 = xi0*xi0
      eta02 = eta0*eta0
      etaa2 = etaa*etaa

c      IF XI0 = +/- 1, THEN WE ARE AT THE POLES OF THE SPHEROID
c      FOR WHICH THE SOLUTION COLLAPSES ... SO JUST RETURN ZERO

      if (xi02 .eq. 1.d0) then
          Bx = 0.d0
          By = 0.d0
          Bz = 0.d0
          return
      endif

      Bx = 0.d0
      By = 0.d0
      Bz = 0.d0

      do 20 n=1,nmax
          T3 = 0.d0
          T5 = cPnm(n,1,eta0) * Pnm(n,1,xi0) / cdPnm(n,1,etaa)

          do 10 m=1,n
              T1 = (2.d0*n+1.d0) * dneg1(m) *
1              dsq(dfact(n-m)/dfact(n+m))
              T2 = Pnm(n,m,xi) *
1              cQnm(n,m,eta) * cPnm(n,m,etaa)
              T3 = (n+m) * (n-m+1) * T3
              if (m.eq.n) then
                  T4 = 0.d0
              else
1              T4 = (m+1) * cPnm(n,m+1,eta0) * Pnm(n,m+1,xi0) /
                  cdPnm(n,m+1,etaa)
              endif
              mphi = float(m)*phi
              Bz = Bz + dsin(mphi)*T1*T2*(T3-T4)
              By = By + dcos(mphi)*T1*T2*(T3+T4)
              Bx = Bx + dcos(mphi)*m*T1*T2*T5
              T3 = T5
              T5 = T4

10          continue

              T1 = (2.d0*n+1.d0) * cPnm(n,1,eta0) *
1              Pnm(n,1,xi0) * Pn0(n,xi) *
2              cQn0(n,eta) * cPn0(n,etaa) / cdPnm(n,1,etaa)
              By = By + T1

20          continue

```

```

T1 = Pphi / ( pi * c*c * dsqrt( (1.d0+eta02)*(1.d0-xi02) ) )
T2 = T1 / ( 4.d0 * dsqrt(1.d0+etaa2) )
Bz = -Bz * T2
By = -By * T2
Bx = -Bx * T1 * etaa / ( 2.d0 * (1.d0+etaa2) )

c      ROTATE Bx,By back by phi0 amount
c      call ObRot(By,Bz,phi0)

c      end

c*****
c      Function to take care of the condition specified in
c      equation (36) in Cuffin and Cohen's paper when x is imaginary.
c
c      Calculates xmag * cdPnm(n,m,x) but it does not seem to matter
c      if x is zero, since the value of xmag is 1 and cdPnm(n,m,0)
c      is easily calculated.

c      DOUBLE PRECISION FUNCTION qcdPnm(signx,xmag,n,m,x)

c      double precision  signx,xmag,x
c      integer           n,m

c      double precision  cdPnm
c      external          cdPnm

c      if ( x .eq. 0.d0 ) then
c      if (m.eq.1) then
c          qcdPnm = signx * n * (n+1.d0) / 2.d0
c      else
c          qcdPnm = 0.d0
c      endif
c      else
c          qcdPnm = xmag * cdPnm(n,m,x)
c      endif
c      return
c      end

c*****
c      Rotates By, Bz back by phi0 amount.
c      The equations are:
c          y = L cos(phi0)
c          z = K sin(phi0)
c      where K,L are irrelevant constants w.r.t phi0.

c      SUBROUTINE ObRot(By,Bz,phi0)

```

```
double precision By,Bz,phi0
double precision Bt,sinphi,cosphi
```

```
sinphi = dsin(phi0)
cosphi = dcos(phi0)
Bt = Bz
```

```
Bz = By*sinphi + Bt*cosphi
By = By*cosphi - Bt*sinphi
```

```
return
end
```

```
c*****
```

```
c      Returns the value of +/- 1.d0 depending on the number
c      of j terms supplied. Essentially, it returns a multiplier
c      based on sqrt(-1) to the j'th power.
```

```
c
c      The following is returned:
c          mod(j,4) = 0 returns +1  (= 0*j)
c          mod(j,4) = 1 returns +1  (= j)
c          mod(j,4) = 2 returns -1  (= j*j = -1)
c          mod(j,4) = 3 returns -1  (= j*j*j = -1*j)
```

```
DOUBLE PRECISION FUNCTION jpower(j)
```

```
integer          j,jmod
```

```
jmod = mod(j,4)
if (jmod.le.1) then
    jpower = 1.d0
else
    jpower = -1.d0
endif
end
```

APPENDIX F
FORTRAN SOURCE CODE TO APPROXIMATE
THE DIPOLE SOLUTION IN SPHEROIDAL VOLUMES
USING CURRENT MONOPOLES

The following code was designed to calculate the magnetic field of a current dipole located in both prolate and oblate volume conductors by using the monopole approximation. The theoretical basis for these routines is described in Chapter V, in which the magnetic field is described as a summation of monopole fields. These routines call subroutines described in Appendix G to evaluate the magnetic field of single monopoles.

The Fortran code was designed according to comply with the ANSI Fortran 77 full language standard. The code was compiled with the Microsoft Fortran Optimizing Compiler v 4.1 to run on IBM-AT computers, and with the 3L Parallel Fortran compiler v2.0 to run in parallel on a network of transputers.


```

c      Routines to do calculate magnetic field using
c      monopole approximation for prolate and oblate ellipsoid.
c      TEST routine at end of this file.

c      UPDATE HISTORY
c      16-Mar-91      CONDITIONAL COMPILATION.
c      This module is used for both non-parallel and parallel-worker
c      task. So all write statements are commented out with 'w'
c      which can be compiled only for non-parallel operation.
c      Likewise, the STOP command is also commented out with 'w'.
c      There are no other conditional compilations available for
c      this module.

c      23-Mar-91      Double-checked to make sure there is no problem
c                      when a sphere is used. Looks fine.

c      14-Apr-91      Changed it so that the routines call Bdsphr()
c                      if the number of segments is zero or less
c                      than one.

c-----
c      The B field is computed based on a line of current sources
c      running from xs0,ys0,zs0 to xs1,ys1,zs1, with ns number
c      of sources. The monopole sits at xm,ym,zm and has
c      strength S, so each of the line of sources has strength -S/ns.
c      The field point at which the B field is being calculated
c      is given by xf,yf,zf.

c      Given the above info, this routine returns the complete
c      B field from the monopole and the associated line of
c      current sources.
c      This calls Bmsphr() which can be found in the Bfield1 module.

1      SUBROUTINE PrMono(xs0,ys0,zs0,xs1,ys1,zs1,xm,ym,zm,S,ns,
           xf,yf,zf,Bx,By,Bz)

           double precision  xs0,ys0,zs0,xs1,ys1,zs1
           double precision  xm,ym,zm,S,xf,yf,zf,Bx,By,Bz
           integer           ns

           double precision  xs,ys,zs,cs
           double precision  xscale,yscale,zscale

           double precision  xfs,yfs,zfs,xms,yms,zms
           double precision  Bxt,Byt,Bzt,Bmag

           integer           i

c      IF ONLY USING ONE OR LESS SEGMENTS ... SIMPLIFY

```

```

if (ns .le. 1) then

    cs = S

    xs = (xs0 + xs1) / 2.d0
    ys = (ys0 + ys1) / 2.d0
    zs = (zs0 + zs1) / 2.d0

    xms = xm - xs
    yms = ym - ys
    zms = zm - zs

    xfs = xf - xs
    yfs = yf - ys
    zfs = zf - zs

    call Bmsphr(xms,yms,zms,xfs,yfs,zfs,cs,
1      Bx,By,Bz,Bmag)

    return
endif

xscale = (xs1-xs0) / ns
yscale = (ys1-ys0) / ns
zscale = (zs1-zs0) / ns

cs = S / ns

Bx = 0.d0
By = 0.d0
Bz = 0.d0

do 100 i=1,ns

    xs = xs0 + xscale*(i-0.5d0)
    ys = ys0 + yscale*(i-0.5d0)
    zs = zs0 + zscale*(i-0.5d0)

c      If the monopole and this part of the line coincide,
c      do not do this one.

    if (xs.ne.xm .or. ys.ne.ym .or. zs.ne.zm) then

c          Using the source as the reference, do a simple
c          translation to make this the origin. The B field
c          is unchanged by such a process.

        xms = xm - xs
        yms = ym - ys
        zms = zm - zs

```

```

        xfs = xf - xs
        yfs = yf - ys
        zfs = zf - zs

        call Bmsphr(xms,yms,zms,xfs,yfs,zfs,cs,
1          Bxt,Byt,Bzt,Bmag)

        Bx = Bx + Bxt
        By = By + Byt
        Bz = Bz + Bzt

        endif

100    continue

        return
        end

```

```

c-----
c      Routine to do monopole stuff for oblate ellipsoid.
c
c      The B field is computed based on a disk of current sources
c      sitting in the y-z plane, centered at the origin with a
c      radius equal to C. The disk is split into ns number
c      of sources. The monopole sits at xm,ym,zm and has
c      strength S.
c
c      The field point at which the B field is being calculated
c      is given by xf,yf,zf.
c
c      Basically, we want to split the disk into equal areas with
c      equal current strengths. To do this, we assume that we
c      will split the disk into N concentric rings. We split
c      the i'th ring into 4*(2*i-1) sectors, giving a total number
c      of 4*(N**2) sectors on the disk. So the value of N is chosen
c      to be the smallest number such that 4*(N**2) is at least
c      as big as ns. In other words, ns specifies the smallest
c      number of sources to be used and defines N
c      N = sqrt(ns) / 2
c      If the sectors in the i'th ring are spaced theta(i) degrees
c      apart, then they are started theta(i-1)/2 degrees away from
c      the first sector in the previous ring, so that sectors do
c      not become aligned too obviously.
c
SUBROUTINE ObMono(C,xm,ym,zm,S,ns,xf,yf,zf,Bx,By,Bz)

```

```
double precision C,xm,ym,zm,S,xf,yf,zf,Bx,By,Bz
integer ns
```

```
double precision xs,ys,zs,cs
```

```
double precision xfs,yfs,zfs,xms,yms,zms
double precision Bxt,Byt,Bzt,Bmag
```

```
integer i
```

```
integer nring,iring
integer itheta,ntheta
```

```
double precision offset,dtheta,theta,radius
```

```
double precision pi
```

```
parameter (pi = 3.141592653589793d0)
```

c IF ONLY USING ONE OR LESS SEGMENTS ... SIMPLIFY

```
if (ns .le. 1) then
```

```
cs = S
```

```
xs = 0.d0
```

```
ys = 0.d0
```

```
zs = 0.d0
```

```
xms = xm - xs
```

```
yms = ym - ys
```

```
zms = zm - zs
```

```
xfs = xf - xs
```

```
yfs = yf - ys
```

```
zfs = zf - zs
```

```
1 call Bmsphr(xms,yms,zms,xfs,yfs,zfs,cs,
            Bx,By,Bz,Bmag)
```

```
return
```

```
endif
```

c FIND SMALLEST NUMBER OF RINGS THAT WE CAN USE

```
nring = int( sqrt(float(ns)) ) / 2
```

```
if ( (nring*nring*4) .lt. ns ) nring = nring+1
```

c CALCULATE THE STRENGTH OF EACH SECTOR

```
cs = S / (nring*nring*4)
```

```

c      write(*,*) 'Number of rings: ',nring
c      write(*,*) 'Each source strength is ',cs

      Bx = 0.d0
      By = 0.d0
      Bz = 0.d0

      offset = 0.d0
      dtheta = 0.d0

c      area = 0
c      sumc = 0

      do 100 iring = 1,nring
         offset = offset + dtheta/2.d0

         ntheta = 4*(2*iring-1)
         dtheta = 2.d0*PI/float(ntheta)

         radius = C * (float(iring)-0.5d0)/float(nring)

         do 100 itheta=0,ntheta-1

            theta = offset + float(itheta)*dtheta

            xs = 0.d0
            ys = radius * dcos(theta)
            zs = radius * dsin(theta)

c            If the monopole and this part of the disk coincide,
c            do not do this one.

            if (xs.ne.xm .or. ys.ne.ym .or. zs.ne.zm) then

c                Using the source as the reference, do a
c                simple translation to make this the origin.
c                The B field is unchanged by such a process.

                xms = xm - xs
                yms = ym - ys
                zms = zm - zs

                xfs = xf - xs
                yfs = yf - ys
                zfs = zf - zs

                call Bmsphr(xms,yms,zms,xfs,yfs,zfs,cs,
                           Bxt,Byt,Bzt,Bmag)

1          Bx = Bx + Bxt

```

By = By + Byt
Bz = Bz + Bzt

endif

100 continue

return
end

```
c      TEST ROUTINE
c      double precision  C,xm,ym,zm,S,xf,yf,zf,Bx,By,Bz
c      integer          ns
c
c      xm = 0.
c      ym = 0.
c      zm = 0.
c      xf = 1.
c      yf = 1.
c      zf = 1.
c
c10    write(*,*) 'Enter radius,current,number of sources'
c      read(*,*) C,S,ns
c
c      call ObMono(C,xm,ym,zm,S,ns,xf,yf,zf,Bx,By,Bz)
c      goto 10
c      end
c
```

APPENDIX G
FORTRAN SOURCE CODE TO CALCULATE
THE MAGNETIC FIELD OF A CURRENT
MONOPOLE IN SEMI-INFINITE AND
SPHERICAL VOLUME CONDUCTORS

The following code was designed to calculate the magnetic field of a current monopole located in either a semi-infinite or spherical volume conductor. The theoretical basis for these routines is described in Chapters II and III.

The Fortran code was designed according to comply with the ANSI Fortran 77 full language standard. The code was compiled with the Microsoft Fortran Optimizing Compiler v 4.1 to run on IBM-AT computers, and with the 3L Parallel Fortran compiler v2.0 to run in parallel on a network of transputers.

```

c      CONDITIONAL COMPILATION.
c      This module is used for both non-parallel and parallel-worker
c      task. So all write statements are commented out with 'w'
c      which can be compiled only for non-parallel operation.
c      Likewise, the STOP command is also commented out with 'w'.
c      There are no other conditional compilations available for
c      this module.

c*****
c      FUNCTIONS TO RETURN B FIELD AT A GIVEN POINT IN SPACE

c      In all functions, we have
c          xs,ys,zs Source location
c          xf,yf,zf Field location
c          c          Source current strength
c          a          Angle of dipole (where appropriate)
c          Bx,By,Bz   Returned B field in x,y,z direction
c          Bmag       Magnitude of B field

c      Notation for added variables:
c          xd,yd,zd Difference in source, field points
c          rxy       Radius on x-y plane
c          rxyz      Radius to point
c          ?n        New point, after or before rotation

c      Always assume that, for a slice, the conducting region is the
c      region with  $z < 0$ . For a sphere, it can be anything but the radius
c      of the observation point must be greater than that of the source.
c      The center of the sphere is always at 0,0,0.
c

c      NOTE ... ALL VALUES ARE RETURNED WITHOUT THE SCALING BY THE
c      PERMEABILITY CONSTANT  $\mu_0$ .

c*****
c***** MONOPOLE VOLUME CURRENTS *****
c*****

c-----
c      Returns B field of monopole in slice conductor

c      SUBROUTINE Bmslce(xs,ys,zs,xf,yf,zf,c,Bx,By,Bz,Bmag)

c      double precision xs,ys,zs,xf,yf,zf,c,Bx,By,Bz,Bmag

c      double precision pi
c      parameter (pi = 3.141592653589793d0)
c      double precision xd,yd,zd
c      double precision r,r2

```



```

xd = xf - xs
yd = yf - ys
zd = abs(zf) - zs
r2 = xd*xd + yd*yd
r = sqrt(r2)

if (zs.gt.0.) then
w      write(*,*) char(7), 'ERROR in Bmslce ... zs > 0'
w      write(*,'(1x,a,3g15.5)') 'xf,yf,zf = ',xf,yf,zf
w      write(*,'(1x,a,3g15.5)') 'xs,ys,zs = ',xs,ys,zs
w      write(*,'(1x,a,1g15.5)') 'c = ',c
w      write(*,*) 'Setting Bx,By,Bz,Bmag = 0'
      Bx = 0.
      By = 0.
      Bz = 0.
      Bmag = 0.
      return
endif

if (r.eq.0.d0) then
      Bx = 0.
      By = 0.
      Bz = 0.
      Bmag = 0.
      return
endif

Bmag = (c / (4.d0*pi*r)) * ( zd/sqrt(r2+zd*zd) - 1.d0 )
Bx = -(yd/r) * Bmag
By = (xd/r) * Bmag
Bz = 0.d0
Bmag = abs(Bmag)
return
end

c-----
c      Returns B field of monopole in a spherical conductor

SUBROUTINE Bmsphr(xs,ys,zs,xf,yf,zf,c,Bx,By,Bz,Bmag)

double precision  xs,ys,zs,xf,yf,zf,c,Bx,By,Bz,Bmag

double precision  pi
parameter          (pi = 3.141592653589793d0)
double precision  zdn
double precision  r,r2

double precision  rsxyz,rsxy,costh,sinth,cosph,sinph

```

```
double precision  xfn,yfn,zfn,rfxyn2,rfxyn
double precision  Bxn,Byn,Bzn
double precision  Bxnn,Bynn,Bznn
double precision  Bp
```

```
c      Get angle of monopole source ... don't need
c      exact theta, phi but need to get cos(), sin() functions.
```

```
rsxyz = sqrt(xs*xs + ys*ys + zs*zs)
rsxy  = sqrt(xs*xs + ys*ys)
costh = zs / rsxyz
sinth = rsxy / rsxyz
if (rsxy.eq.0.d0) then
    cosph = 1.d0
    sinph = 0.d0
else
    cosph = xs / rsxy
    sinph = ys / rsxy
endif
```

```
c      Rotate field point back through theta,phi
c      to get new field points
```

```
xfn = (xf*cosph + yf*sinph)*costh - zf*sinth
yfn = -xf*sinph + yf*cosph
zfn = (xf*cosph + yf*sinph)*sinth + zf*costh
```

```
rfxyn2 = (xfn*xfn + yfn*yfn)
rfxyn = sqrt(rfxyn2)
```

```
if (rfxyn.eq.0.d0) then
    Bx = 0.
    By = 0.
    Bz = 0.
    Bmag = 0.
    return
endif
```

```
c      Calculate what B-phi (Bp) is for back-rotated points,
c      with the source sitting at x=0,y=0,z=rsxyz
```

```
zdn = zfn - rsxyz
```

```
1      Bp = (c / (4.d0*pi*rfxyn)) * ( zdn/sqrt(rfxyn2+zdn*zdn) -
      zfn/sqrt(rfxyn2+zfn*zfn) )
```

```
Bxn = -(yfn/rfxyn) * Bp
Byn = (xfn/rfxyn) * Bp
Bzn = 0.d0
```

```

c      Now rotate around by theta, phi degrees
c      Do theta rotation first, then phi rotation

      Bxnn = Bxn*costh + Bzn*sinth
      Bynn = Byn
      Bznn = -Bxn*sinth + Bzn*costh

      Bx = Bxnn*cosph - Bynn*sinph
      By = Bxnn*sinph + Bynn*cosph
      Bz = Bznn

      Bmag = abs(Bp)

900    format(1x,a,3f10.4)
      return
      end

c*****
c*****  DIPOLE VOLUME CURRENTS  *****
c*****

c-----
c      Returns B field of dipole in slice conductor.
c      If length=l is non-zero, then we use two monopoles and sum
c      the results.

      SUBROUTINE Bdslice(xs,ys,zs,xf,yf,zf,l,c,a,Bx,By,Bz,Bmag)

      double precision  xs,ys,zs,xf,yf,zf,l,c,a,Bx,By,Bz,Bmag

      double precision  pi
      parameter          (pi = 3.141592653589793d0)
      double precision  xd,yd,zd
      double precision  xdn,ydn,zdn
      double precision  xdn,ydn,zdn
      double precision  r,r2
      double precision  t1,t2,t3,t4
      double precision  xdn2,ydn2,zdn2
      double precision  Bxn,Byn
      double precision  Bxnn,Bynn

c      Data variables added for non-zero length dipole

      double precision  Bx1,Bx2,By1,By2,Bz1,Bz2,junk
      double precision  x1,x2,y1,y2,z1,z2,c1,c2

      if (zs.gt.0.d0 .or. zf.lt.0.d0) then
          Bx = 0.
          By = 0.
          Bz = 0.

```

```

        Bmag = 0.
        return
    endif

c    If non-zero length, then use monopoles to simulate the dipole

    if (l.ne.0.d0) then

        xd = (l/2.d0) * cos(a)
        yd = (l/2.d0) * sin(a)

        x1 = xs + xd
        y1 = ys + yd
        z1 = zs
        c1 = c

        x2 = xs - xd
        y2 = ys - yd
        z2 = zs
        c2 = -c

        call Bmslce(x1,y1,z1,xf,yf,zf,c1,Bx1,By1,Bz1,junk)
        call Bmslce(x2,y2,z2,xf,yf,zf,c2,Bx2,By2,Bz2,junk)

        Bx = Bx1 + Bx2
        By = By1 + By2
        Bz = Bz1 + Bz2
        Bmag = sqrt(Bx*Bx + By*By + Bz*Bz)
        return
    endif

c    Get offset (lateral shift in x-y plane) for shift in source.
c    Also get the value (zf-zs) corresponding to the (d+z) in
c    Cuffin and Cohen's paper.

    xd = xf - xs
    yd = yf - ys
    zd = zf - zs

    if (xd.eq.0.d0 .and. yd.eq.0.d0) then

c        Cuffin and Cohen show equation for limit
c        as r approaches zero. They calculate Bx for
c        a +y dipole, so we use
c        By = -Bx

        Bxn = 0.
        Byn = c / (8.d0*pi*zd*zd)

c        Now finish the rotation by the angle = a.

```

```

                                goto 100
endif

c      Further rotate the field points around the angle = -a.
c      Why? The dipole is oriented in the phi=+a direction.
c      By rotating everything -a, we can align the dipole
c      back with the x-axis.

      xdn = xd*cos(a) + yd*sin(a)
      ydn = -xd*sin(a) + yd*cos(a)
      zdn = zd

      xdn2 = xdn*xdn
      ydn2 = ydn*ydn
      zdn2 = zdn*zdn

      r2 = xdn2 + ydn2
      r = sqrt(r2)

c      Now we have a dipole positioned at 0,0,zs and we want to
c      know the magnetic field at the point zxn,zyn,zf.
c
c      Calculate B field for dipole oriented in +x direction
c      i.e. unrotated dipole. Note that Cuffin's results are
c      for a dipole pointed in +y direction. To make point in
c      +x direction, basically rotate by phi = -90 degrees which
c      is the same as setting:
c
c
c          x = y
c          y = -x
c
c          OURS          Cuffin/Cohen
c          Bx(x,y,z) =   By(y,-x,z)
c          By(x,y,z) =   -Bx(y,-x,z)
c
c

      t1 = xdn2 + ydn2 + zdn2
      t2 = sqrt(t1)
      t3 = t1*t2
      t4 = xdn2 - ydn2

      Byn = -(c / (4.d0*pi*r2*r2*t3)) *
1         ( t4*t3 +
2         zdn*(-t4*zdn2 - 2.d0*xdn2*xdn2 - xdn2*ydn2 + ydn2*ydn2) )

      Bxn = -(c*xdn*ydn / (4.d0*pi*r2*r2*t3)) *
1         ( -2.d0*t3 + zdn*(2.d0*zdn2 + 3.d0*ydn2 + 3.d0*xdn2) )

```

```

c      Come here once you have calculated Bxn, Byn
100    continue

      Bmag = sqrt( Bxn*Bxn + Byn*Byn )

c      Rotate around z-axis by the angle phi=a

      Bx = Bxn*cos(a) - Byn*sin(a)
      By = Bxn*sin(a) + Byn*cos(a)
      Bz = 0.d0

      return
      end

c-----
c      Returns B field of dipole in a spherical conductor.
c      If non-zero length, then we rotate the dipole origin then
c      use monopole routines.
c      The angle a is the angle of the dipole with respect to the
c      x axis after the source point is rotated onto the z-axis.
c      The angle aa is the angle the dipole makes with respect
c      to the x-y plane (0 = parallel to x-y plane and therefor
c      parallel to the surface, pi/2 = perpendicular to the
c      surface and pointed outwards).
c
c      AS IT WORKS NOW, THE 'aa' ANGLE IS ONLY USED BY THE PURE
c      DIPOLE (l=0) CASE. In this case, the dipole is considered
c      to have a strength given by c*cos(aa) which lies parallel
c      to the surface, and a strength c*sin(aa) which lies perpendicular
c      to the surface.

      SUBROUTINE Bdsphr(xs,ys,zs,xf,yf,zf,l,c,a,aa,Bx,By,Bz,Bmag)

      double precision  xs,ys,zs,xf,yf,zf,l,c,a,aa
      double precision  Bx,By,Bz,Bmag

      double precision  pi
      parameter          (pi = 3.141592653589793d0)
      double precision  zdn

      double precision  rsxyz,rsxy,costh,sinth,cosph,sinph
      double precision  xfn,yfn,zfn
      double precision  xfnn,yfnn,zfnn
      double precision  rfxyzn,rfxyn,costhn,sinthn,cosphn,sinphn
      double precision  cosaa,sinaa,cperp,cpar
      double precision  Rx,Ry,Rz,R,denom
      double precision  Btn,Bpn
      double precision  Bxn,Byn,Bzn

```

```

double precision  Bxnn,Bynn,Bznn
double precision  Bxnnn,Bynnn,Bznnn
double precision  gm,gmsqrt,gm32,t1,t2

c      Data variables added for non-zero length dipole

double precision  Bx1,Bx2,By1,By2,Bz1,Bz2,junk
double precision  x1,x2,y1,y2,z1,z2,c1,c2
double precision  xd,yd

c      Data variables added for monopole approximation when on z-axis.

double precision  ltemp,ctemp

c      Get angle of dipole source ... don't need
c      exact theta, phi but need to get cos(), sin() functions.

cosaa = cos(aa)
sinaa = sin(aa)
cperp = c * sinaa
cpar = c * cosaa

rsxyz = sqrt(xs*xs + ys*ys + zs*zs)
rsxy = sqrt(xs*xs + ys*ys)
costh = zs / rsxyz
sinth = rsxy / rsxyz
if (rsxy.eq.0.d0) then
    cosph = 1.d0
    sinph = 0.d0
else
    cosph = xs / rsxy
    sinph = ys / rsxy
endif

c      Rotate field point back through theta,phi
c      to get new field points

xfn = (xf*cosph + yf*sinph)*costh - zf*sinth
yfn = -xf*sinph + yf*cosph
zfn = (xf*cosph + yf*sinph)*sinth + zf*costh

c      The source can now be considered to be sitting at
c      x=0,y=0,z=rsxyz, and the field point at xfn,yfn,zfn.

c      If non-zero length, then use monopoles to simulate the dipole

if (l.ne.0.d0) then

    xd = (l/2.d0) * cos(a)
    yd = (l/2.d0) * sin(a)

```

```

x1 = xd
y1 = yd
z1 = rsxyz
c1 = c

x2 = -xd
y2 = -yd
z2 = rsxyz
c2 = -c

call Bmsphr(x1,y1,z1,xfn,yfn,zfn,c1,Bx1,By1,Bz1,junk)
call Bmsphr(x2,y2,z2,xfn,yfn,zfn,c2,Bx2,By2,Bz2,junk)

Bxnn = Bx1 + Bx2
Bynn = By1 + By2
Bznn = Bz1 + Bz2

else

c      Further rotate the field points around the angle = -a.
c      Why? The dipole is oriented in the phi=+a direction.
c      By rotating everything -a, we can align the dipole
c      back with the x-axis.

      xfn = xfn*cos(a) + yfn*sin(a)
      yfn = -xfn*sin(a) + yfn*cos(a)
      zfn = zfn

c      For the new field points, convert them into spherical
c      coordinates. Again, only need the cos() sin() functions.

      rfxyn = sqrt( xfn*xfn + yfn*yfn + zfn*zfn )
      rfxyn = sqrt( xfn*xfn + yfn*yfn )

      costhn = zfn / rfxyn
      sinthn = rfxyn / rfxyn
      if (rfxyn.eq.0.d0) then
         cosphn = 1.d0
         sinphn = 0.d0
      else
         cosphn = xfn / rfxyn
         sinphn = yfn / rfxyn
      endif

      if (abs(sinthn) .lt. 1.7e-3) then

c      If within 0.1 degree of z axis, then say we are on
c      the axis. Use this method because round-off errors
c      allow us to be very close to the axis without it being

```



```

c      very obvious.
c      If on z axis, can only have a Bx,By component. Just
c      use a monopole approximation where we fake a finite-
c      length dipole. Set the length at .001 times the
c      distance out to the dipole and change the current-
c      moment into current.

      ltemp = .001 * rsxyz
      ctemp = cpar / ltemp

w      write(*,*) char(7), 'Bdsphr ... r = 0'
w      write(*,'(1x,a,3g15.5)') 'xf,yf,zf = ',xf,yf,zf
w      write(*,'(1x,a,3g15.5)') 'xs,ys,zs = ',xs,ys,zs
w      write(*,'(1x,a,2g15.5)') 'c,a = ',c,a
w      write(*,*) 'Using monopole approximation with'
w      write(*,'(1x,a,2g15.5)') 'l,c = ',ltemp,ctemp

      xd = (ltemp/2.d0)
      yd = 0.

      x1 = xd
      y1 = yd
      z1 = rsxyz
      c1 = ctemp

      x2 = -xd
      y2 = -yd
      z2 = rsxyz
      c2 = -ctemp

      call Bmsphr(x1,y1,z1,xfnn,yfnn,zfnn,c1,
1          Bx1,By1,Bz1,junk)
      call Bmsphr(x2,y2,z2,xfnn,yfnn,zfnn,c2,
1          Bx2,By2,Bz2,junk)

      Bxn = Bx1 + Bx2
      Byn = By1 + By2
      Bzn = Bz1 + Bz2

else

c      Do not need to do any monopole approximation.
c      The source can now be considered to be sitting at
c      x=0,y=0,z=rsxyz, and the field point at xfnn,yfnn,zfnn.
c      Now, the B field from Cuffin and Cohen for a dipole
c      oriented in the +x direction can be found.
c      Calculate what Bmag is for back-rotated points, with
c      the source sitting at x=0,y=0,z=rsxyz.

```

```

t1 = rsxyz / rfxyzn
t2 = rfxyzn / rsxyz
gm = 1.d0 - (2.d0*rsxyz*costhn/rfxyzn) + t1*t1
gmsqrt = sqrt(gm)
gm32 = gm * gmsqrt

1 Btn = (sinphn*cpar / (4.d0*pi*rfxyzn*rfxyzn*gmsqrt)) *
2   ( ( costhn/(sinthn*sinthn))*(costhn - t2 + t2*gmsqrt) )
   + 1.d0 )

1 Bpn = (cosphn*cpar / (4.d0*pi*rfxyzn*rfxyzn*gm32)) *
2   ( ( -gm/(sinthn*sinthn))*(costhn - t2 + t2*gmsqrt) )
   + costhn - t1 )

Bxn = -sinphn*Bpn + costhn*cosphn*Btn
Byn = cosphn*Bpn + costhn*sinphn*Btn
Bzn = -sinthn*Btn

endif

c Add in the component of the magnetic field
c generated by the perpendicular dipole component.
c This is just the opposite of the field generated by
c the dipole itself.

Rx = xfnn
Ry = yfnn
Rz = zfnn - rsxyz
R = dsqrt(Rx*Rx + Ry*Ry + Rz*Rz)

denom = 4.d0 * pi * R * R * R
Bxn = Bxn + ( cperp*Ry ) / denom
Byn = Byn - ( cperp*Rx ) / denom

c Now rotate around by theta, phi degrees, and dipole angle=a
c Do phi=angle for the dipole itself first.

Bxnn = Bxn*cos(a) - Byn*sin(a)
Bynn = Bxn*sin(a) + Byn*cos(a)
Bznn = Bzn

endif

c Continue with theta rotation for dipole position,
c then phi rotation.

Bxnnn = Bxnn*costh + Bznn*sinth
Bynnn = Bynn
Bznnn = -Bxnn*sinth + Bznn*costh

```

```

Bx = Bxnnn*cosph - Bynnn*sinph
By = Bxnnn*sinph + Bynnn*cosph
Bz = Bznnn

Bmag = sqrt(Bx*Bx + By*By + Bz*Bz)

w900  format(1x,a,3f10.4,2g15.5)

      return
      end

c-----
c      Converts B field in x,y,z coordinates to Btheta,Bp,Br
c      based on field point xf,yf,zf. Steps:
c          1. Project the vector Bx,By,Bz onto the axis
c             vectors xaxs(),yaxs(),zaxs().
c          2. Based on the field point xy,yf,zf and the
c             origin xorg,yorg,zorg calculate Btheta,Bp,Br.
c      This now returns a modified Bx,By,Bz based on the projection
c      of the original vector on the xaxs,yaxs,zaxs vectors.
c
c      Assume that xaxs,yaxs,zaxs are all normalized unit vectors
c      in the x,y,z direction desired.
c
1      SUBROUTINE Bconv(xf,yf,zf,Bx,By,Bz,Br,Bt,Bp,
           xorg,yorg,zorg,xaxs,yaxs,zaxs)

      double precision  xf,yf,zf,Bx,By,Bz,Br,Bp,Bt
      double precision  xorg,yorg,zorg,xaxs(3),yaxs(3),zaxs(3)

      double precision  rfxy,rfxyz
      double precision  costh,sinth,cosph,sinph
      double precision  t1,Bxtemp,Bytemp,Bztemp
      double precision  xfnew,yfnew,zfnew

c      Project onto the xaxs,yaxs,zaxs coordinate system

      Bxtemp = Bx*xaxs(1) + By*yaxs(1) + Bz*zaxs(1)
      Bytemp = Bx*yaxs(2) + By*yaxs(2) + Bz*zaxs(2)
      Bztemp = Bx*yaxs(3) + By*yaxs(3) + Bz*zaxs(3)

      Bx = Bxtemp
      By = Bytemp
      Bz = Bztemp

c      The new field point is just shifted from the origin:

      xfnew = xf - xorg
      yfnew = yf - yorg

```

```

zfnew = zf - zorg

c      Get angle of field point ... don't need
c      exact theta, phi but need to get cos(), sin() functions.

rxyz = sqrt(xfnew*xfnew + yfnew*yfnew + zfnew*zfnew)
rxy = sqrt(xfnew*xfnew + yfnew*yfnew)

if (rxyz.eq.0.d0) then
    Br = sqrt(Bx*Bx + By*By + Bz*Bz)
    Bt = 0.d0
    Bp = 0.d0
    return
endif

costh = zfnew / rxyz
sinth = rxy / rxyz
if (rxy.eq.0.d0) then
    cosph = 1.d0
    sinph = 0.d0
else
    cosph = xfnew / rxy
    sinph = yfnew / rxy
endif

c      Put Bx,By,Bz into Br,Bt,Bp

t1 = Bx*cosph + By*sinph
Br = t1*sinth + Bz*costh
Bt = t1*costh - Bz*sinth
Bp = -Bx*sinph + By*cosph

c      write(*,'(1x,a,3g15.5)') 'xf,yf,zf = ',xf,yf,zf
c      write(*,'(1x,a,3g15.5)') 'Bx,By,Bz = ',Bx,By,Bz
c      write(*,'(1x,a,3g15.5)') 'Br,Bt,Bp = ',Br,Bt,Bp

return
end

c*****
c***** INTRACELLULAR CURRENTS *****
c*****

c-----
c      Returns B field of monopole's intracellular currents by assuming
c      the current "c" flows from xs0,ys0,zs0 to xs1,ys1,zs1. Note the
c      "c" may be negative to indicate the current flows in the other
c      direction.

SUBROUTINE Bmintr(xs0,ys0,zs0,xs1,ys1,zs1,

```

```

1      xf0,yf0,zf0,c,Bx,By,Bz,Bmag)

double precision  xs0,ys0,zs0,xs1,ys1,zs1,xf0,yf0,zf0
double precision  c,Bx,By,Bz,Bmag

double precision  pi
parameter          (pi = 3.141592653589793d0)
double precision  zdn
double precision  r,r2

double precision  xs,ys,zs
double precision  xf,yf,zf
double precision  rsxyz,rsxy,costh,sinth,cosph,sinph
double precision  xfnn,yfnn,zfnn
double precision  xfn,yfn,zfn,rfxyn2,rfxyn
double precision  Bxn,Byn,Bzn
double precision  Bxnn,Bynn,Bznn
double precision  Bp

c      Convert two monopoles into a tube of current from 0,0,0 to
c      xs,ys,zs by laterally shifting xs0,ys0,zs0 to origin and
c      adjusting current.

      xs = xs1 - xs0
      ys = ys1 - ys0
      zs = zs1 - zs0

c      Shift field point similarly to shifting sources laterally

      xf = xf0 - xs0
      yf = yf0 - ys0
      zf = zf0 - zs0

c      Get angle of monopole source ... don't need
c      exact theta, phi but need to get cos(), sin() functions.

      rsxyz = sqrt(xs*xs + ys*ys + zs*zs)
      rsxy = sqrt(xs*xs + ys*ys)
      costh = zs / rsxyz
      sinth = rsxy / rsxyz
      if (rsxy.eq.0.d0) then
         cosph = 1.d0
         sinph = 0.d0
      else
         cosph = xs / rsxy
         sinph = ys / rsxy
      endif

c      Rotate field point back through theta,phi
c      to get new field points

```

```

xfn = (xf*cosph + yf*sinph)*costh - zf*sinth
yfn = -xf*sinph + yf*cosph
zfn = (xf*cosph + yf*sinph)*sinth + zf*costh

rfxyn2 = (xfn*xfn + yfn*yfn)
rfxyn = sqrt(rfxyn2)

if (rfxyn.eq.0.d0) then
c      write(*,*) char(7), 'Bmintr ... r = 0'
      Bx = 0.
      By = 0.
      Bz = 0.
      Bmag = 0.
      return
endif

c      Calculate what B-phi is for back-rotated points, with the source
c      sitting at x=0,y=0,z=rsxyz

      zdn = zfn - rsxyz

1      Bp = (c / (4.d0*pi*rfxyn)) * ( zfn/sqrt(rfxyn2+zfn*zfn) -
      zdn/sqrt(rfxyn2+zdn*zdn) )

      Bxn = -(yfn/rfxyn) * Bp
      Byn = (xfn/rfxyn) * Bp
      Bzn = 0.d0

c      Now rotate around by theta, phi degrees
c      Do theta rotation first, then phi rotation

      Bxnn = Bxn*costh + Bzn*sinth
      Bynn = Byn
      Bznn = -Bxn*sinth + Bzn*costh

      Bx = Bxnn*cosph - Bynn*sinph
      By = Bxnn*sinph + Bynn*cosph
      Bz = Bznn

      Bmag = abs(Bp)

      return
end

c-----
c      Returns B field of the intracellular currents of a dipole
c      Set ROTATE = TRUE if we rotate the field point back to the
c      origin x,y=0,0 ; otherwise we do a shift back to this point.

```

- c (Normally set ROTATE true only if in a sphere).
 c If the dipole length is non-zero, we are doing the same
 c as for two monopoles, hence we call Bmintr.

SUBROUTINE Bdintr(xs,ys,zs,xf,yf,zf,l,c,a,ROTATE,Bx,By,Bz,Bmag)

double precision xs,ys,zs,xf,yf,zf,l,c,a,Bx,By,Bz,Bmag
 logical ROTATE

double precision pi
 parameter (pi = 3.141592653589793d0)
 double precision zdn

double precision rsxyz,rsxy,costh,sinth,cosph,sinh
 double precision xfn,yfn,zfn
 double precision xfnn,yfnn,zfnn
 double precision Bxn,Byn,Bzn
 double precision Bxnn,Bynn,Bznn
 double precision Bxnnn,Bynnn,Bznnn
 double precision t1,t2,t3,t4

- c Data variables added for non-zero length dipole

double precision junk
 double precision x1,x2,y1,y2,z1,z2
 double precision xd,yd

if (ROTATE) then

- c Get angle of dipole source ... don't need
 c exact theta, phi but need to get cos(), sin() functions.

```
rsxyz = sqrt(xs*xs + ys*ys + zs*zs)
rsxy = sqrt(xs*xs + ys*ys)
costh = zs / rsxyz
sinth = rsxy / rsxyz
if (rsxy.eq.0.d0) then
    cosph = 1.d0
    sinph = 0.d0
else
    cosph = xs / rsxy
    sinph = ys / rsxy
endif
```

- c Rotate field point back through theta,phi
 c to get new field points

```
xfn = (xf*cosph + yf*sinph)*costh - zf*sinth
yfn = -xf*sinph + yf*cosph
```

```

zfn = (xf*cosph + yf*sinph)*sinth + zf*costh

else

c      If not rotating, then do a lateral shift in the field point
c      and place the source z value (rsxyz) at the real zs.

      rsxyz = zs
      xfn = xf - xs
      yfn = yf - ys
      zfn = zf

endif

c      The source can now be considered to be sitting at
c      x=0,y=0,z=rsxyz, and the field point at xfn,yfn,zfn.

c      If non-zero length, then use monopoles to simulate the dipole
c      with current "c" flowing from x2,y2,z2 to x1,y1,z1

      if (l.ne.0.d0) then

          xd = (l/2.d0) * cos(a)
          yd = (l/2.d0) * sin(a)

          x1 = xd
          y1 = yd
          z1 = rsxyz

          x2 = -xd
          y2 = -yd
          z2 = rsxyz

          call Bmintr(x2,y2,z2,x1,y1,z1,xfn,yfn,zfn,c,
1              Bxnn,Bynn,Bznn,junk)

      else

c      Further rotate the field points around the angle = -a.
c      Why? The dipole is oriented in the phi=+a direction.
c      By rotating everything -a, we can align the dipole
c      back with the x-axis.

          xfnn = xfn*cos(a) + yfn*sin(a)
          yfnn = -xfn*sin(a) + yfn*cos(a)
          zfnn = zfn

c      The source can now be considered to be sitting at
c      x=0,y=0,z=rsxyz, and the field point at xfnn,yfnn,zfnn.
c      Now, the B field from Cuffin and Cohen for a dipole
c      oriented in the +x direction can be found.

```



```

t1 = zfnr - rsxyz
t2 = (xfnr*xfnr + yfnr*yfnr + t1*t1)
t3 = sqrt(t2)
t4 = (c / (4.d0*pi*t2*t3))

w      if (t3.eq.0.d0) then
        write(*,*) char(7), 'Bdintr ... r = 0'
        Bx = 0.
        By = 0.
        Bz = 0.
        Bmag = 0.
        return
      endif

      Bxn = 0.d0
      Byn = -t1 * t4
      Bzn = yfnr * t4

c      Now rotate around by theta, phi degrees and angle=a.
c      Do phi=angle for the dipole itself first.

      Bxnn = Bxn*cos(a) - Byn*sin(a)
      Bynn = Bxn*sin(a) + Byn*cos(a)
      Bznn = Bzn

endif

c      Now rotate around by theta, phi degrees and angle=a.
c      Continue with theta rotation for dipole position,
c      then phi rotation.

      if (ROTATE) then

        Bxnnn = Bxnn*costh + Bznn*sinth
        Bynnn = Bynn
        Bznnn = -Bxnn*sinth + Bznn*costh

        Bx = Bxnnn*cosph - Bynnn*sinph
        By = Bxnnn*sinph + Bynnn*cosph
        Bz = Bznnn

      else

        Bx = Bxnn
        By = Bynn
        Bz = Bznn

      endif

```

```
Bmag = sqrt(Bx*Bx + By*By + Bz*Bz)
w900  format(1x,a,3f10.4,2g15.5)
      return
      end
```

APPENDIX H

FORTRAN SOURCE CODE TO CALCULATE LEGENDRE POLYNOMIALS

The following code was designed to calculate Legendre and associated Legendre polynomials of the first and second kind, and derivatives of each of these functions. The Fortran code was designed according to comply with the ANSI Fortran 77 full language standard. The code was compiled with the Microsoft Fortran Optimizing Compiler v 4.1 to run on IBM-AT computers, and with the 3L Parallel Fortran compiler v2.0 to run in parallel on a network of transputers.

Although commercial packages exists to calculate Legendre polynomials, we found the accuracy of such packages to be suspect. Mathematica, for example, uses an algorithm that is unstable for polynomials for degree $n > 10$ and unstable whenever x approaches a singularity. In addition, the processing time is too slow for our needs. The following routines were developed to achieve sufficient accuracy for polynomials up to degree $n=20$, and to remain stable and accurate at points very close to the singularities. Processing time was only considered in the final writing of the do-loops and other code. All computations and result use double precision floating point variables.

H.1 Theoretical Basis

The standard iterative approach to calculating Legendre polynomials was inadequate for our need because such an approach becomes unstable at higher order terms. The standard iterative schemes [1] apply equally to polynomials of the first ($P_n^m(x)$) or second ($Q_n^m(x)$) kind, so we will use $U_n^m(x)$ to refer to either polynomial in the following discussion. An iterative scheme in only n can be developed

$$(n-m)U_n^m(x) = (2n-1) \cdot x \cdot U_{n-1}^m(x) - (n+m-1) \cdot U_{n-2}^m(x) \quad (1)$$

as well as an iterative scheme in m

$$U_n^m(x) = \frac{-2(m-1) \cdot x}{\sqrt{x^2-1}} U_n^{m-1}(x) - (n-m-2)(n+m-1) \cdot U_n^{m-2}(x) \quad (2)$$

Using this approach, it is only necessary to have four "seed" functions to begin the iterative scheme to find the polynomial result for any n, m . Since the values of $P_n^m(x)$ increase with increasing n whenever x is imaginary or greater than 1, this approach is usually stable. However, values for $Q_n^m(x)$ decrease with increasing n and the iterative method generates a loss of significant digits even with $n < 10$. The iterative scheme for $P_n^m(x)$ can also become unstable with higher m , so an alternate method had to be developed.

One solution is to reverse the iterative scheme such that higher order "seed" terms (say for $n=20$) are calculated and then iterate towards lower values

of n . However, the time required to compute the higher order "seed" terms is the same time that it would take to compute the desired term directly. Consequently, for any terms where $n > 2$, the polynomial was computed directly without using an iterative scheme.

Higher order terms were calculated directly by expressing the Legendre polynomial as a function of a hypergeometric series [2]. The hypergeometric series is defined as

$$\begin{aligned} F(\alpha, \beta, \gamma, x) &= 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k) \Gamma(\beta+k) \Gamma(\gamma)}{k! \Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma+k)} x^k \end{aligned} \quad (3)$$

where the Gamma function is defined as

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1) \cdots (x+n)}$$

The hypergeometric series is absolutely convergent if $\text{abs}(x) < 1$. One example of using this to calculate a Legendre polynomial was derived by MacRobert [2, pp. 288] for $P_n^m(z)$ where z may be imaginary or real

$$P_n^m(z) = \frac{1}{\Gamma(1-m)} \left(\frac{z+1}{z-1} \right)^{m/2} F\left(-n, n+1; 1-m; \frac{1-z}{2}\right) \quad (4)$$

If n is a positive integer, which is generally the case in our simulations, the Gamma function reduces to

$$\Gamma(n+1) = n!$$

The hypergeometric series in eqn. (4) will not converge for all z , so (4) is only valid for $\text{abs}(z-1) < 2$. When $\text{abs}(z) > 1$, an alternate expression derived by Hobson [3, pp. 114] can be used

$$P_n^m(z) = \frac{2^{-m-2n} (2n)!}{n! (n-m)!} \xi^{-(m+n)/2} (z^2-1)^{-m/2} F\left(-n-m, \frac{1}{2}-m; \frac{1}{2}-n; \xi\right) \quad (5)$$

where the intermediate variable ξ is defined as

$$\xi = \left(z - \sqrt{z^2-1}\right)^2$$

We found this equation did not converge as fast as (4) in the region for which both equations are valid, so eqn. (4) was used whenever possible.

Similar functions of the hypergeometric series can be found for polynomials of the second kind, $Q_n^m(z)$, for complex z . Depending on the value of z , one series may converge faster than another. When $\text{abs}(z) \leq 3$, the following equation was found to converge fastest [1], [3, pp.286]

$$Q_n^m(z) = \frac{(-1)^m \Gamma\left(\frac{1}{2}\right) \Gamma(n+m+1)}{\Gamma\left(n+\frac{3}{2}\right) \sqrt{2} (z^2-1)^{1/4} \left(z+\sqrt{z^2-1}\right)^{n+1/2}} \times F\left(\frac{1}{2}+m, \frac{1}{2}-m; n+\frac{3}{2}; -\frac{z-\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right) \quad (6)$$

For all other z , the following equation was used [1], [2, pp. 116], [3, pp. 195]

$$Q_n^m(z) = \frac{(-1)^m \Gamma\left(\frac{1}{2}\right) \Gamma(n+m+1)}{\Gamma\left(n+\frac{3}{2}\right) 2^{n+1} (z^2-1)^{m/2} z^{n-m+1}} \\ \times F\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}; n+\frac{3}{2}; z^{-2}\right) \quad (7)$$

The final difficulties in dealing with complex versus real values of x , negative values of n or m , calculating derivatives of polynomials, and singularities around $x=0$ and $x=\pm 1$ are fully explained in the following documentation. In some cases, there is no "correct" solution and the decision on how to deal with a specific case was based on generating results that were consistent with commercial software packages such as Mathematica.

H.2 References

- [1] L.J. Briggs and A.N. Lowan, Tables of Associated Legendre Functions, Columbia University Press, New York, 1945.
- [2] T.M. MacRobert, Spherical Harmonics, Pergamon Press, Oxford, 1967.
- [3] E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Chelsea Publishing Co., New York, 1955.

```

c      CONDITIONAL COMPILATION.
c      This module is used for both non-parallel and parallel-worker
c      task. So all write statements are commented out with 'w'
c      which can be compiled only for non-parallel operation.
c      Likewise, the STOP command is also commented out with 'w'.
c      There are no other conditional compilations available for
c      this module.

c      TEST ROUTINE IS COMMENTED OUT AT THE END OF THIS FILE

c      Some routines supply warning statements, but the default
c      is NOT to supply any warnings, and just return a zero
c      value when something is undefined. To turn warnings
c      on, call the routine PQwarn(.TRUE.), and to turn them
c      off call PQwarn(.FALSE.).

c      Subroutines to return the value of the Legendre
c      and the associated Legendre Polynomial.

c      REFERENCES #####
c
c      An excellent reference for this stuff is W.R. Smythe,
c      Static and Dynamic Electricity, McGraw-Hill Book Company, Inc.,
c      New York, 1939 QC518 .S55 1939.
c
c      Also see these books:
c
c      E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics,
c      University Press, Cambridge, 1955.
c
c      T.M. MacRobert, Spherical Harmonics, Pergamon Press,
c      Oxford, 1967.
c
c      L.J. Briggs and A.N. Lowan, Tables of Associated Legendre
c      Functions, Columbia Univ. Press., National Bureau of Standards,
c      1945.
c
c      M. Abramowitz and I.A. Stegun (eds), Handbook of Mathematical
c      Functions, Dover Publications, New York, 1972.
c
c      #####
c
c      The values of m,n are limited to positive integers.

c      The user should only call the routines Pnm,Qnm,Pn0,Qn0,dPnm,dQnm
c      or a routine with these names prefixed by the letter "c" or "j".
c      In general, these routines expect a value of X which is either
c      going to be less than unity, or greater than unity (thus
c      setting the variables SMALL or BIG) but is nonetheless

```


c a real number. Routines prefixed by the letter "c" interpret
c X as an entirely imaginary number (not sent as an imaginary number
c but rather a double precision number). These routines then
c return a double precision number, but again it may be either
c completely real or completely imaginary. This can be determined
c by calling the routines with a "j" prefix which return a
c single precision number either (0,1) or (1,0) which tells
c if the "c" result should be interpreted as imaginary or
c real respectively.
c
c Routines beginning with the prefix "i" are internal to this
c module and should not be called by the user.
c

c Update History:
c 29-Oct-90 ASF Added the routines described in THE TABLES
c OF ASSOCIATED LEGENDRE FUNCTIONS to calculate
c Qnm because the simple iterative scheme
c used originally would break down at higher
c values of n. Rather than use a backwards
c iterative scheme, we just use the equations
c directly. Also set Qnm() to flip values
c only when $x < 0$, not when $x = 0$. This was
c originally included to satisfy Mathematica's
c results, but the TABLES book has a positive
c value for Qnm at $x = 0$.
c

c 10-29-90 ASF Changed derivative formula to use
c $P(n,m,x)$ and $P(n,m-1,x)$ instead of
c $P(n,m+1,x)$. Also did it for $Q(n,m+1,x)$.
c Allow negative m in Pnm, Qnm.
c

c 11-04-90 ASF Modified Fnm() to use the variable "x2k"
c instead of calling x^{**k} everytime.
c Changed all double**integer to
c double**double since the old version
c produces an error.
c In a routine where the first double may
c be less than zero, the exponent must
c be an integer or else the result
c from the transputer is *****.
c

c 11-08-90 ASF Changed all $1.d0-x*x$ or $x*x-1.d0$ to
c the form $(1.d0-x)*(1.d0+x)$ etc. I think
c this will create less error.
c

c 11-09-90 ASF Added dfact2() so that we don't waste
c time or accuracy in factorial comps.
c Reorganized all computations to reduce
c chance for overflow error.
c

```

c
c      11-13-90      ASF      Speeded up Fnm() routine.
c
c      16-Mar-91      ASF      Changed implicit integer abs() to iabs()
c                                for linking with stand alone library.
c                                OOPS ... this did not work since it cannot
c                                find iabs().  Changed to if() statement.
c
c      26-Mar-91      Fixed up dPnm, dQnm to return the correct value
c                                if m=0. Fixed bug in dPnm, dQnm when x=-1 (used
c                                to bomb).

```

```

c#####
c      USER CALLABLE ROUTINES ... REAL X
c#####

```

```

c*****
c      Turn warnings ON (TRUE) or OFF (FALSE)

      SUBROUTINE PQwarn(seting)

      logical          seting

      logical          XREAL,XCMPLX,BIG,SMALL,WARN
      integer          ITMAX
      common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      WARN = seting
      return
      end

```

```

c*****
c      Allows the user to change the value of ITMAX

      SUBROUTINE PQmax(seting)

      integer          seting

      logical          XREAL,XCMPLX,BIG,SMALL,WARN
      integer          ITMAX
      common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      ITMAX = seting
      return
      end

```

```

c*****

```

```
c      Returns the value of the Legendre polynomial of the first
c      kind for real X
```

```
DOUBLE PRECISION FUNCTION Pn0(n,x)
```

```
integer          n
double precision x
```

```
external          iPnm
double precision iPnm
```

```
call rsetX(x)
Pn0 = iPnm(n,0,x)
return
end
```

```
c*****
```

```
c      Returns the value of the Legendre polynomial of the second
c      kind for real X
```

```
DOUBLE PRECISION FUNCTION Qn0(n,x)
```

```
integer          n
double precision x
```

```
external          iQnm
double precision iQnm
```

```
c      write(*,*) 'In Qn0 n,x = ',n,x
c      call rsetX(x)
c      Qn0 = iQnm(n,0,x)
c      return
c      end
```

```
c*****
```

```
c      Returns the value of the associated Legendre polynomial of the first
c      kind for real X
```

```
DOUBLE PRECISION FUNCTION Pnm(n,m,x)
```

```
integer          n,m
double precision x
```

```
external          iPnm
double precision iPnm
```

```
call rsetX(x)
Pnm = iPnm(n,m,x)
return
end
```

```

c*****
c      Returns the value of the associated Legendre polynomial of the second
c      kind for real X

      DOUBLE PRECISION FUNCTION Qnm(n,m,x)

      integer          n,m
      double precision x

      external          iQnm
      double precision iQnm

c      write(*,*) 'In Qnm n,m,x = ',n,m,x
      call rsetX(x)
      Qnm = iQnm(n,m,x)
      return
      end

c*****
c      Returns d/dx of the associated Legendre polynomial of the first
c      kind for real X

      DOUBLE PRECISION FUNCTION dPnm(n,m,x)

      integer          n,m
      double precision x

      double precision idPnm
      external          idPnm

      call rsetX(x)
      dPnm = idPnm(n,m,x)
      return
      end

c*****
c      Returns d/dx of the associated Legendre polynomial of the second
c      kind for real X

      DOUBLE PRECISION FUNCTION dQnm(n,m,x)

      integer          n,m
      double precision x

      double precision idQnm
      external          idQnm

      call rsetX(x)

```

```

dQnm = idQnm(n,m,x)
return
end

```

```

c#####
c      USER CALLABLE ROUTINES ... IMAGINARY X
c#####

```

```

c*****

```

```

c      Returns the value of the Legendre polynomial of the first
c      kind for real X

```

```

      DOUBLE PRECISION FUNCTION cPn0(n,x)

```

```

      integer                n
      double precision  x

```

```

      external      iPnm
      double precision  iPnm

```

```

      call csetX()
      cPn0 = iPnm(n,0,x)
      return
      end

```

```

c*****

```

```

c      Returns the value of the Legendre polynomial of the second
c      kind for real X

```

```

      DOUBLE PRECISION FUNCTION cQn0(n,x)

```

```

      integer                n
      double precision  x

```

```

      external      iQnm
      double precision  iQnm

```

```

      call csetX()
      cQn0 = iQnm(n,0,x)
      return
      end

```

```

c*****

```

```

c      Returns the value of the associated Legendre polynomial of the first
c      kind for real X

```

```

      DOUBLE PRECISION FUNCTION cPnm(n,m,x)

```

```

integer                n,m
double precision  x

external            iPnm
double precision  iPnm

call csetX()
cPnm = iPnm(n,m,x)
return
end

c*****
c      Returns the value of the associated Legendre polynomial of the second
c      kind for real X

      DOUBLE PRECISION FUNCTION cQnm(n,m,x)

integer                n,m
double precision  x

external            iQnm
double precision  iQnm

call csetX()
cQnm = iQnm(n,m,x)
return
end

c*****
c      Returns d/dx of the associated Legendre polynomial of the first
c      kind for real X

      DOUBLE PRECISION FUNCTION cdPnm(n,m,x)

integer                n,m
double precision  x

double precision  idPnm
external            idPnm

call csetX()
cdPnm = idPnm(n,m,x)
return
end

c*****
c      Returns d/dx of the associated Legendre polynomial of the second
c      kind for real X

```

DOUBLE PRECISION FUNCTION cdQnm(n,m,x)

```

integer          n,m
double precision x

double precision idQnm
external         idQnm

call csetX()
cdQnm = idQnm(n,m,x)
return
end

```

```

c#####
c  USER CALLABLE ROUTINES ... REAL X
c#####

```

```

c*****
c  Return the value 0 or 1 depending on whether the associated
c  routine returns a real or imaginary value respectively.

```

```

INTEGER FUNCTION jPn0(n)
integer          n
jPn0 = mod(n,2)
end

```

```

INTEGER FUNCTION jQn0(n)
integer          n
jQn0 = 1-mod(n,2)
end

```

```

INTEGER FUNCTION jPnm(n)
integer          n
jPnm = mod(n,2)
end

```

```

INTEGER FUNCTION jQnm(n)
integer          n
jQnm = 1-mod(n,2)
end

```

```

INTEGER FUNCTION jdPnm(n)
integer          n
c 11-10-90      jdPnm = 1-mod(n,2)
jdPnm = mod(n,2)
end

```

```

INTEGER FUNCTION jdQnm(n)

```

```

integer          n
c 11-10-90      jdQnm = mod(n,2)
               jdQnm = 1-mod(n,2)
               end

c#####
c      INTERNAL (NON-CALLABLE) ROUTINES
c#####

c*****
c      Returns the value of the associated Legendre polynomial of
c      the first kind for real X.
c      Compute Pnm() for abs(x), then if x is negative
c      set Pnm() = (-1)**(n+m) * Pnm()
c
c      The routine rsetX or csetX must already have been called.
c
c      Normally, the result is continuous across x=0 so it doesn't
c      matter how we handle x=0. But the result for Qnm() is
c      discontinuous across jx=0 (i.e. imaginary x). To agree
c      with Mathematica, we treat x=0 as if it were negative.
c      Thus, we do something at the end if x.le.0.d0 (but we
c      cannot write it as xin.le.0.d0 because it doesn't
c      act on the case when x=0.d0 ... it must be broken
c      into two case .lt. and .eq.).
c
c      Special care must be taken with x=0. If it is complex, for Pnm
c      no problem. Otherwise, it only defined for specific combinations
c      of n and m. For Qnm, it basically produces infinity for any n,m
c      so we just warn the user, then return zero.
c
c      See Hobson 1955 page 99 for relation on P(n,m,x) when m < 0.

DOUBLE PRECISION FUNCTION iPnm(n,minp,xinp)

integer          n,m,minp
double precision  xinp,x

double precision  P00,P10,P11,P21,dfact2
double precision  PnmHy1,PnmHy2
external          P00,P10,P11,P21,dfact2
external          PnmHy1,PnmHy2

c      double precision  err,rerr,rval

logical          XREAL,XCMLPX,BIG,SMALL,WARN
integer          ITMAX

```



```

common /LEGNDR/      XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      x = dabs(xinp)
c      m = iabs(minp)
      m = minp
      if (minp.lt.0) m=-minp

      if (XREAL .and. x .eq. 1.d0) then
        if (m.gt.n) then
          iPnm = 0.d0
        elseif (m.eq.0) then
          iPnm = 1.d0
        elseif (m.eq.1 .and. (n.eq.1 .or. n.eq.2)) then
          iPnm = 0.d0
        else
c          UNDEFINED ... SET TO ZERO WITH A WARNING
          iPnm = 0.d0
cw      if (WARN) write(*,*)
cw      1      'Pnm() undef for n,m,x = ',n,m,xinp
          endif
        elseif (m.gt.n) then
          iPnm = 0.d0
        elseif (m.eq.0) then
          if (n.eq.0) then
            iPnm = P00()
          elseif (n.eq.1) then
            iPnm = P10(x)
          else
            if (XREAL) then
              iPnm = PnmHy1(n,m,x)
            else
              iPnm = PnmHy2(n,m,x)
            endif
          endif
        elseif (m.eq.1) then
          if (n.eq.1) then
            iPnm = P11(x)
          elseif (n.eq.2) then
            iPnm = P21(x)
          else
            if (XREAL) then
              iPnm = PnmHy1(n,m,x)
            else
              iPnm = PnmHy2(n,m,x)
            endif
          endif
        else
c      M is bigger than 1 but less than or equal to N.
c      Thus we guarantee that n is at least 2.

```

```

        if (XREAL) then
            iPnm = PnmHy1(n,m,x)
        else
            iPnm = PnmHy2(n,m,x)
        endif

    endif

    if (xinp.lt.0.d0 .or. xinp.eq.0.d0) then
        if (mod(n+m,2).eq.1) iPnm = -iPnm
    endif

    if (minp.lt.0) then
        iPrm = iPnm * dfact2((n-m),(n+m))
        if (SMALL .and. mod(m,2).eq.1) iPnm = -iPnm
    endif

c    if (XCMLPX) then
c        rval = PnmItr(n,minp,xinp)
c        err = rval - iPnm
c        if (rval .ne. 0.d0) then
c            rerr = dabs(err / rval)
c        else
c            rerr = dabs(err)
c        endif
c        if (rerr .gt. 1.d-8) then
c            write(*,'(2i4,5g14.6)') n,minp,xinp,
c 1          rval,iPnm,err,rerr
c            write(*,*) PnmHy2(n,m,x),
c 1          PnmHy2(n,minp,xinp)
c        endif
c        return
c    endif

    return
end

c*****
c    Returns the value of the associated Legendre polynomial of
c    the first kind using the iterative method.
c    Compute Pnm() for abs(x), then if x is negative
c    set Pnm() = (-1)**(n+m) * Pnm()
c
c    The routine rsetX or csetX must already have been called.
c
c    Normally, the result is continuous across x=0 so it doesn't

```

```

c      matter how we handle  $x=0$ . But the result for  $Q_{nm}()$  is
c      discontinuous across  $jx=0$  (i.e. imaginary  $x$ ). To agree
c      with Mathematica, we treat  $x=0$  as if it were negative.
c      Thus, we do something at the end if  $x.le.0.d0$  (but we
c      cannot write it as  $xin.le.0.d0$  because it doesn't
c      act on the case when  $x=0.d0$  ... it must be broken
c      into two case  $.lt.$  and  $.eq.$ ).
c
c      Special care must be taken with  $x=0$ . If it is complex, for  $P_{nm}$ 
c      no problem. Otherwise, it only defined for specific combinations
c      of  $n$  and  $m$ . For  $Q_{nm}$ , it basically produces infinity for any  $n,m$ 
c      so we just warn the user, then return zero.
c
c      See Hobson 1955 page 99 for relation on  $P(n,m,x)$  when  $m < 0$ .

DOUBLE PRECISION FUNCTION PnmItr(n,minp,xinp)

integer                n,minp,m
double precision      xinp,x

double precision      R(0:2),S(0:2)

double precision      P00,P10,P11,P21,dfact2
external              P00,P10,P11,P21,dfact2

logical                XREAL,XCMPLX,BIG,SMALL,WARN
integer                ITMAX
common /LEGNDR/        XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

x = dabs(xinp)
c      m = iabs(minp)
m = minp
if (minp.lt.0) m=-minp
c      write(*,*) 'PnmItr m,minp = ',m,minp
if (XREAL .and. x .eq. 1.d0) then
    if (m.gt.n) then
        PnmItr = 0.d0
    elseif (m.eq.0) then
        PnmItr = 1.d0
    elseif (m.eq.1 .and. (n.eq.1 .or. n.eq.2)) then
        PnmItr = 0.d0
    else
c      UNDEFINED ... SET TO ZERO WITH A WARNING
        PnmItr = 0.d0
cw      if (WARN) write(*,*)
cw 1      'Pnm() undef for n,m,x = ',n,m,xinp
    endif
elseif (m.gt.n) then
    PnmItr = 0.d0
elseif (m.eq.0) then

```

```

    if (n.eq.0) then
      PnmItr = P00()
    elseif (n.eq.1) then
      PnmItr = P10(x)
    else
      R(0) = P00()
      R(1) = P10(x)
      call Item(2,n,m,x,R,'P')
      PnmItr = R(2)
    endif
  elseif (m.eq.1) then
    if (n.eq.1) then
      PnmItr = P11(x)
    elseif (n.eq.2) then
      PnmItr = P21(x)
    else
      R(0) = P11(x)
      R(1) = P21(x)
      call Item(3,n,m,x,R,'P')
      PnmItr = R(2)
    endif
  else
c      M is bigger than 1 but less than or equal to N.
c      Thus we guarantee that n is at least 2.
c      Get starting term for m = 0

      R(0) = P00()
      R(1) = P10(x)
      call Item(2,n,0,x,R,'P')
      S(0) = R(2)

c      Get starting term for m = 1

      if (n.eq.2) then
        S(1) = P21(x)
      else
        R(0) = P11(x)
        R(1) = P21(x)
        call Item(3,n,1,x,R,'P')
        S(1) = R(2)
      endif

      call Item(n,m,x,S)
      PnmItr = S(2)

    endif

    if (xinp.lt.0.d0 .or. xinp.eq.0.d0) then
      if (mod(n+m,2).eq.1) PnmItr = -PnmItr
    endif

```

```

if (minp.lt.0) then
    PnmItr = PnmItr * dfact2( (n-m),(n+m) )
    if (SMALL .and. mod(m,2).eq.1) PnmItr = -PnmItr
endif

return
end

```

c*****

```

c      Returns the value of the associated Legendre polynomial of
c      the second kind for real X.
c      We calculate the value of Qnm using either an exact formula
c      (for small n,m), a truncated form involving the hypergeometric
c      series, or iterate using one of these earlier methods for
c      seed values. Whenever the hypergeometric series is used, it
c      is only valid when:
c          X is real and has magnitude greater than 1
c          X is complex
c      I may eventually find the hypergeometric series for real |x|<1,
c      but I don't need it just now.
c
c      We do not worry about m>n since it doesn't get used in my
c      current application. This produces a warning to the user.
c
c      Compute Qnm() for abs(x), then if x is negative
c      set Qnm() = (-1)**(n+m+1) * Qnm()
c
c      We use the following algorithms:
c
c          x = 1          Undefined
c
c          m=0      n=0,1  Use exact formula
c                  n > 1   Use truncated hypergeometric series
c
c          m=1      n=0,1  Use exact formula
c                  n > 1   Use truncated hypergeometric series
c
c          m>1      all n   Use truncated hypergeometric series
c
c      See Hobson 1955 page 99,196 for relation on Q(n,,m,x) when m < 0.

```

DOUBLE PRECISION FUNCTION iQnm(n,minp,xinp)

integer n,m,minp

double precision xinp,x

double precision R(0:2),S(0:2)

```

double precision  Q00,Q10,Q01,Q11,Q1415,dfact2
external          Q00,Q10,Q01,Q11,Q1415,dfact2

logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      write(*,*) 'In iQnm n,minp,xinp = ',n,minp,xinp
      x = dabs(xinp)
c      m = iabs(minp)
      m = minp
      if (minp.lt.0) m=-minp
c      write(*,*) 'In iQnm n,m,x = ',n,m,x
c      write(*,*) 'iQnm m,minp = ',m,minp

      if ( (XREAL .and. x .eq. 1.d0) .or. m .gt. n) then
        iQnm = 0.d0
        if (WARN) write(*,*)
cw      1      'Qnm() undef for n,m,x = ',n,m,xinp
        return

      elseif (m.eq.0) then
        if (n.eq.0) then
          iQnm = Q00(x)
        elseif (n.eq.1) then
          iQnm = Q10(x)
        else
          iQnm = Q1415(n,m,x)
        endif

      elseif (m.eq.1) then
        if (n.eq.0) then
          iQnm = Q01(x)
        elseif (n.eq.1) then
          iQnm = Q11(x)
        else
          iQnm = Q1415(n,m,x)
        endif

      else
        iQnm = Q1415(n,m,x)

      endif

      if (xinp.lt.0.d0) then
        if (mod(n+m+1,2).eq.1) iQnm = -iQnm
      endif

      if (minp.lt.0) then

```

```

        iQnm = iQnm * dfact2( (n-m),(n+m) )
        if (SMALL .and. mod(m,2).eq.1) iQnm = -iQnm
    endif

    return
end

c*****
c      Returns d/dx of the associated Legendre polynomial of the first
c      kind.
c      If it has a problem with any value (x=1 for example), then
c      we just do a numerical derivative based on evaluating
c      Pnm() at x+dx,x-dx.
c
c
c      DOUBLE PRECISION FUNCTION idPnm(n,m,x)
c
c      integer          n,m
c      double precision x
c
c      double precision dx
c      parameter        (dx = 1.d-7)
c
c      double precision Pnm1,Pnm2,temp
c
c      double precision iPnm
c      external         iPnm
c
c      integer          jPnm
c      external         jPnm
c
c      logical          XREAL,XCMLPX,BIG,SMALL,WARN
c      integer          ITMAX
c      common /LEGNDR/  XREAL,XCMLPX,BIG,SMALL,WARN,ITMAX
c
c 10-29-90      Pnm1 = iPnm(n,m,x)
c 10-29-90      Pnm2 = iPnm(n,m+1,x)
c
c      Pnm1 = m * iPnm(n,m,x)
c      Pnm2 = (n-m+1) * (n+m) * iPnm(n,m-1,x)
c
c      if (XREAL) then
c        if (x .eq. 1.d0) then
c          goto 900
c        elseif (SMALL) then
c          temp = (1.d0-x)*(1.d0+x)
c          idPnm = Pnm1*(-x*m/temp) - Pnm2/dsqrt(temp)
c 10-29-90
c          if (temp.eq.0.d0) goto 900
c          idPnm = Pnm1*(x/temp) + Pnm2/dsqrt(temp)

```

```

        else
            temp = (x-1.d0)*(x+1.d0)
c 10-29-90      idPnm = Pnm1*(x*m/temp) + Pnm2/dsqrt(temp)
            if (temp.eq.0.d0) goto 900
            idPnm = Pnm1*(-x/temp) + Pnm2/dsqrt(temp)
        endif
    else
        temp = x*x+1.d0
c 10-29-90      idPnm = Pnm1*(x*m/temp) + Pnm2/dsqrt(temp)
c 11-10-90      if (jPnm(n).eq.0) idPnm = -idPnm
            idPnm = Pnm1*(-x/temp) + Pnm2/dsqrt(temp)
    endif
    return

c      Come here if you must do a numerical derivative

900      idPnm = (iPnm(n,m,x+dx) - iPnm(n,m,x-dx)) / (2.d0 * dx)
    return

end

c*****
c      Returns d/dx of the associated Legendre polynomial of the second
c      kind.
c      If X is real but equal to 1, returns 0.

      DOUBLE PRECISION FUNCTION idQnm(n,m,x)

      integer                n,m
      double precision      x

      double precision      Qnm1,Qnm2,temp

      double precision      iQnm
      external              iQnm

      integer                jQnm
      external              jQnm

      logical                XREAL,XCMPLX,BIG,SMALL,WARN
      integer                ITMAX
      common /LEGNDR/       XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      if (XREAL .and. x.eq.1.d0) goto 900

      temp = (1.d0-x)*(1.d0+x)
c 10-29-90      Qnm1 = iQnm(n,m,x)
c 10-29-90      Qnm2 = iQnm(n,m+1,x)
      Qnm1 = m * iQnm(n,m,x)

```



```

Qnm2 = (n-m+1) * (n+m) * iQnm(n,m-1,x)

if (XREAL) then
  if (SMALL) then
    temp = (1.d0-x)*(1.d0+x)
c 10-29-90    idQnm = Qnm1*(-x*m/temp) - Qnm2/dsqrt(temp)
    if (temp.eq.0.d0) goto 900
    idQnm = Qnm1*(x/temp) + Qnm2/dsqrt(temp)
  else
    temp = (x-1.d0)*(x+1.d0)
c 10-29-90    idQnm = Qnm1*(x*m/temp) + Qnm2/dsqrt(temp)
    if (temp.eq.0.d0) goto 900
    idQnm = Qnm1*(-x/temp) + Qnm2/dsqrt(temp)
  endif
else
  temp = x*x+1.d0
c 10-29-90    idQnm = Qnm1*(x*m/temp) + Qnm2/dsqrt(temp)
c 11-10-90    if (iQnm(n).eq.0) idQnm = -idQnm
    if (temp.eq.0.d0) goto 900
    idQnm = Qnm1*(-x/temp) + Qnm2/dsqrt(temp)
endif
return

c      Come here if there is a problem with the routine.

900    idQnm = 0.d0
      return

end

c#####
c      ITERATIVE ROUTINES
c#####

c*****
c      Iterate along the values of "n".
c
c      For the P series, iterate towards increasing n from
c      n0 to n1, assuming that
c          R(0) = P(n0-2)
c          R(1) = P(n0-1)
c      The calling routine must guarantee that n0 is not greater
c      than n1.
c
c      For the Q series, iterate towards decreasing n from
c      n0 to n1, assuming that
c          R(0) = P(n0+2)
c          R(1) = P(n0+1)

```

```

c      The calling routine must guarantee that n1 is not greater
c      than n0.
c
c      These routines varies if X is complex, in which case
c      the scheme is
c           $u(n,m) = A*j*u(n-1,m) - B*u(n-2,m)$ 
c      for the Q series, u(n,m) is imaginary for n even.
c      Thus, for the P series, we subtract A*u(n-1,m) when we are
c      calculating u(n,m) where n is even.
c
c * * * * *
c
      SUBROUTINE itern(n0,n1,m,x,R,ittype)

      integer                n0,n1,m
      double precision      x,R(0:2)
      character              itype

      integer                n

      logical                XREAL,XCMPLX,BIG,SMALL,WARN
      integer                ITMAX
      common /LEGNDR/        XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      if (itype.eq.'P') then

        if (XREAL) then
          do 10 n=n0,n1
            R(2) = ((2*n-1)*x*R(1) - (n+m-1)*R(0)) / (n-m)
            R(0) = R(1)
            R(1) = R(2)
10          continue

        else
          do 20 n=n0,n1
            if (mod(n,2).eq. 0) then
              R(2) = ((1-2*n)*x*R(1) - (n+m-1)*R(0)) / (n-m)
            else
              R(2) = ((2*n-1)*x*R(1) - (n+m-1)*R(0)) / (n-m)
            endif
            R(0) = R(1)
            R(1) = R(2)
20          continue
        endif

      elseif (itype.eq.'Q') then

        if (XREAL) then

```

```

do 110 n=n0,n1,-1
    R(2) = ((2*n+3)*x*R(1) - (n+2-m)*R(0)) / (n+m+1)
    R(0) = R(1)
    R(1) = R(2)
110    continue

    else
do 120 n=n0,n1,-1
    if (mod(n,2) .eq. 0) then
        R(2) = ((2*n+3)*x*R(1) - (n+2-m)*R(0)) / (n+m+1)
    else
        R(2) = (-2*n+3)*x*R(1) - (n+2-m)*R(0) / (n+m+1)
    endif
    R(0) = R(1)
    R(1) = R(2)
120    continue
endif

endif
return
end

c*****
c    Iterate along the values of "m".
c    Assumes that R(0), R(1) have been pre-loaded.
c    No checking is done on n or m.
c    The routine varies only slightly for complex values or
c    real values greater than 1 in magnitude.

SUBROUTINE item(n,mmax,x,R)

integer                n,mmax
double precision      x,R(0:2)

integer                m
double precision      temp

logical                XREAL,XCMLPX,BIG,SMALL,WARN
integer                ITMAX
common /LEGNDR/        XREAL,XCMLPX,BIG,SMALL,WARN,ITMAX

if (XREAL) then

    if (SMALL) then
        temp = x / dsqrt((1.d0-x)*(1.d0+x))
        do 10 m=2,mmax
            R(2) = -2.d0*(m-1)*temp*R(1) -
1              (n-m+2)*(n+m-1)*R(0)
            R(0) = R(1)
            R(1) = R(2)

```

```

10          continue

          elseif (BIG) then
            temp = x / dsqrt((x-1.d0)*(x+1.d0))
            do 20 m=2,mmax
              R(2) = -2.d0*(m-1)*temp*R(1) +
1              (n-m+2)*(n+m-1)*R(0)
              R(0) = R(1)
              R(1) = R(2)
20          continue
          endif

          else
            temp = x / dsqrt(x*x+1.d0)
            do 30 m=2,mmax
              R(2) = -2.d0*(m-1)*temp*R(1) +
1              (n-m+2)*(n+m-1)*R(0)
              R(0) = R(1)
              R(1) = R(2)
30          continue
          endif
          return
          end

c#####
c      SEED FUNCTIONS TO START THE ITERATIVE SCHEMES
c#####

c*****
c      Seed function values which form the basis of
c      the iterative routines. The general form is
c      Pnm(x) or Qnm(x)

      DOUBLE PRECISION FUNCTION P00()
      P00 = 1.d0
      end

      DOUBLE PRECISION FUNCTION P10(x)
      double precision  x
      P10 = x
      end

      DOUBLE PRECISION FUNCTION P11(x)
      double precision  x
      logical           XREAL,XCMPLX,BIG,SMALL,WARN
      integer           ITMAX
      common /LEGNDR/   XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
      if (XREAL) then
        if (SMALL) then
          P11 = -dsqrt((1.d0-x)*(1.d0+x))

```

```

        else
            P11 = dsqrt((x-1.d0)*(x+1.d0))
        endif
    else
        P11 = dsqrt(x*x+1.d0)
    endif
end

```

DOUBLE PRECISION FUNCTION P21(x)

```

double precision x
logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
if (XREAL) then
    if (SMALL) then
        P21 = -3.d0 * x * dsqrt((1.d0-x)*(1.d0+x))
    else
        P21 = 3.d0 * x * dsqrt((x-1.d0)*(x+1.d0))
    endif
else
        P21 = -3.d0 * x * dsqrt(x*x+1.d0)
endif
end

```

DOUBLE PRECISION FUNCTION Q00(x)

```

double precision x
logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
double precision dacot
external         dacot
if (XREAL) then
    if (SMALL) then
        Q00 = dlog( (1.d0+x)/(1.d0-x) ) / 2.d0
    else
        Q00 = dlog( (x+1.d0)/(x-1.d0) ) / 2.d0
    endif
else
        Q00 = -dacot(x)
endif
end

```

DOUBLE PRECISION FUNCTION Q10(x)

```

double precision x
logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
double precision dacot
external         dacot
if (XREAL) then

```

```

        if (SMALL) then
            Q10 = x * dlog((1.d0+x)/(1.d0-x)) / 2.d0 - 1.d0
        else
            Q10 = x * dlog((x+1.d0)/(x-1.d0)) / 2.d0 - 1.d0
        endif
    else
        Q10 = x * dacot(x) - 1.d0
    endif
end

DOUBLE PRECISION FUNCTION Q01(x)
double precision x
logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
if (XREAL) then
    if (SMALL) then
        Q01 = -1.d0 / dsqrt((1.d0-x)*(1.d0+x))
    else
        Q01 = -1.d0 / dsqrt((x-1.d0)*(x+1.d0))
    endif
else
        Q01 = 1.d0 / dsqrt(x*x+1.d0)
endif
end

DOUBLE PRECISION FUNCTION Q11(x)
double precision x
double precision temp
logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
double precision dacot
external         dacot
if (XREAL) then
    if (SMALL) then
        temp = 1.d0 - x*x
        Q11 = -dsqrt(temp) *
1          (dlog((1.d0+x)/(1.d0-x))/2.d0 + x/temp)
    else
        temp = x*x - 1.d0
        Q11 = dsqrt(temp) *
1          (dlog((x+1.d0)/(x-1.d0))/2.d0 - x/temp)
    endif
else
        temp = x*x + 1.d0
        Q11 = dsqrt(temp) *
1          (dacot(x) - x/temp)
    endif
end

```

```

c#####
c  ROUTINES FOR TRUNCATED HYPERGEOMETRIC SERIES
c#####

c-----
c  MacRobert's equation on page 288 for  $|x-1| < 2$ 
c  and another for higher x.
c  This routine converges pretty fast, and faster than the
c  next routine.

      DOUBLE PRECISION FUNCTION PnmHy1(n,minp,xinp)

      integer          n,minp
      double precision  xinp

      integer          m
      double precision  x

      double precision  dfact,dfact2,Fnm
      external          dfact,dfact2,Fnm

      double precision  a,b,c,d,temp

      logical           XREAL,XCMLPX,BIG,SMALL,WARN
      integer           ITMAX
      common /LEGNDR/   XREAL,XCMLPX,BIG,SMALL,WARN,ITMAX

      x = dabs(xinp)
      m = iabs(minp)
      m = minp
      if (minp.lt.0) m=-minp

      a = m-n
      b = m+n+1.d0
      c = m+1.d0
      d = (1.d0-x) / 2.d0

      if (XREAL) then

         if (SMALL) then
            temp = 1.d0-(x*x)
         else
            temp = (x*x)-1.d0
         endif
         PnmHy1 = ((dfact2(n+m,n-m)/dfact(m)) / (2.d0**m)) *
1          Fnm(a,b,c,d,ITMAX) * (temp**dble(m/2.d0))

      else

```

```

cw          write(*,*) 'PNM undef for complex x'
          return

endif

if (SMALL .and. mod(m,2).eq.1) PnmHy1 = -PnmHy1

if (xinp.lt.0.d0 .or. xinp.eq.0.d0) then
    if (mod(n+m,2).eq.1) PnmHy1 = -PnmHy1
endif

if (minp.lt.0) then
    PnmHy1 = PnmHy1 * dfact2(n-m,n+m)
    if (SMALL .and. mod(m,2).eq.1) PnmHy1 = -PnmHy1
endif

return
end

c-----
c      Hobson's equation 52 on page 114. This is valid for
c      any x that is not real and inbetween -1,1.
c      After playing with this, it does not converge as fast
c      for real x as the other formula. So use this only
c      for complex x.

DOUBLE PRECISION FUNCTION PnmHy2(n,minp,xinp)

integer          n,minp
double precision xinp

integer          m
double precision x

double precision dfact,dfact2,Fnm
external         dfact,dfact2,Fnm

double precision a,b,c,d,temp,z

logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      x = dabs(xinp)
c      m = iabs(minp)
c      m = minp
c      if (minp.lt.0) m=-minp

a = -n-m
b = 0.5d0 - m

```



```

c = 0.5d0 - n

if (XREAL .and. BIG) then
    temp = dsqrt( (x-1.d0)*(x+1.d0) )
    z = x - temp
    d = z*z
elseif (XCMLPX) then
    temp = dsqrt( x*x + i.d0 )
    z = x-temp
    d = -z*z
else
    write(*,*) 'INVALID x in PnmHy2 ',n,m,x
    continue
endif

PnmHy2 = ((dfact2(2*n,n)/dfact(n-m))/(2.d0**dble(m+2.d0*n))) *
1      Fnm(a,b,c,d,itmax) /
1      ( (z**(m+n)) * (temp**dble(m)) )

c      PnmHy2 = dfact(2*n) * Fnm(a,b,c,d,itmax) /
c 1      ( dfact(n) * dfact(n-m) * (2.d0**dble(m+2.d0*n)) *
c 2      (z**(m+n)) * (temp**dble(m)) )

if (XCMLPX) then
    if (mod(m,2).eq.1) PnmHy2 = -PnmHy2
    if (mod(n,4).eq.1 .or. mod(n,4).eq.2) PnmHy2 = -PnmHy2
endif

if (xinp.lt.0.d0 .or. xinp.eq.0.d0) then
    if (mod(n+m,2).eq.1) PnmHy2 = -PnmHy2
endif

if (minp.lt.0) PnmHy2 = PnmHy2 * dfact2(n-m,n+m)

return
end

```

```

c
c      Computes Qnm() using the equation 14,15 given in THE TABLES
c      OF ASSOCIATED LEGENDRE POLYNOMIALS on page xix.
c      If x > 3, use equation 14. For x <=3, use equation 15.
c      The number of iterations, set by ITMAX, can be set at
c      the value of 15 to produce 10 accurate digits.
c
c      X MUST BE STRICTLY POSITIVE AND GREATER THAN 1
c

```

DOUBLE PRECISION FUNCTION Q1415(n,m,x)

```

integer                n,m
double precision      x

double precision      Qnm14,Qnm15
external              Qnm14,Qnm15

logical               XREAL,XCMPLX,BIG,SMALL,WARN
integer              ITMAX
common /LEGNDR/      XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      write(*,*) 'In Q1415 n,m,x = ',n,m,x
      if (SMALL) then
        Q1415 = 0.d0
      cw      if (WARN)
      cw 1      write(*,*) 'ERROR IN Q1415, undef for x = ',x

      elseif (x .le. 3.d0) then
        Q1415 = Qnm15(n,m,x)

      else
        Q1415 = Qnm14(n,m,x)

      endif

      if (XCMPLX .and. mod(n,4).le.1) Q1415 = -Q1415

      end

c-----
c      Returns the value obtained from Equation (14)

DOUBLE PRECISION FUNCTION Qnm14(n,m,x)
integer                n,m
double precision      x

double precision      a,b,c,d

double precision      Anm,Bnm,Fnm
external              Anm,Bnm,Fnm

logical               XREAL,XCMPLX,BIG,SMALL,WARN
integer              ITMAX
common /LEGNDR/      XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      write(*,*) 'In Q14 n,m,x = ',n,m,x
      a = (n-m+2) / 2.d0
      b = (n-m+1) / 2.d0
      c = n + 1.5d0
      d = 1.d0 / (x*x)
      if (XCMPLX) d = -d

```

```
Qnm14 = Anm(n,m,x) * Fnm(a,b,c,d,itmax)
end
```

c-----
c Returns the value obtained from Equation (15)

```
DOUBLE PRECISION FUNCTION Qnm15(n,m,x)
```

```
integer n,m
```

```
double precision x
```

```
double precision a,b,c,t,temp
```

```
double precision Anm,Bnm,Fnm
```

```
external Anm,Bnm,Fnm
```

```
logical XREAL,XCMPLX,BIG,SMALL,WARN
```

```
integer ITMAX
```

```
common /LEGNDR/ XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
```

c write(*,*) 'In Q15 n,m,x = ',n,m,x

```
a = 0.5d0 + m
```

```
b = 0.5d0 - m
```

```
c = n + 1.5d0
```

```
if (XREAL) then
```

```
temp = dsqrt(x*x - 1.d0)
```

```
else
```

```
temp = dsqrt(x*x + 1.d0)
```

```
endif
```

```
t = (x - temp) / (2.d0 * temp)
```

```
Qnm15 = Bnm(n,m,x) * Fnm(a,b,c,-t,itmax)
```

```
end
```

c-----
c Returns the value for A() in equation (14)

```
DOUBLE PRECISION FUNCTION Anm(n,m,x)
```

```
integer n,m
```

```
double precision x
```

```
double precision dfact,dneg1,doddf
```

```
external dfact,dneg1,doddf
```

```
double precision temp
```

```
logical XREAL,XCMPLX,BIG,SMALL,WARN
```

```

integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      write(*,*) 'In Anm n,m,x = ',n,m,x
      if (XREAL) then
            temp = dsqrt(x*x - 1.d0)
      else
            temp = dsqrt(x*x + 1.d0)
      endif

      Anm = dneg1(m) * (dfact(n+m)/doddf(2*n+1)) /
1      ( (temp**dble(m)) * (x**dble(n-m+1)) )
      end

```

c-----
c Returns the value for B() in equation (15)

```

DOUBLE PRECISION FUNCTION Bnm(n,m,x)
integer          n,m
double precision x

double precision dfact,dneg1,doddf
external         dfact,dneg1,doddf

double precision temp

logical          XREAL,XCMPLX,BIG,SMALL,WARN
integer          ITMAX
common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

c      write(*,*) 'In Bnm n,m,x = ',n,m,x
      if (XREAL) then
            temp = dsqrt(x*x - 1.d0)
      else
            temp = dsqrt(x*x + 1.d0)
      endif

      Bnm = dneg1(m) * (dfact(n+m)/doddf(2*n+1)) * (2.d0**dble(n+1)) /
1      ( dsqrt(2.d0) * dsqrt(temp) * ((x+temp)**dble(n+0.5)) )
      end

```

c-----
c Returns the value of the hypergeometric function.

```

DOUBLE PRECISION FUNCTION Fnm(a,b,c,x,itmax)

double precision a,b,c,x
integer          itmax

```

```

integer          k
double precision temp,dblk,x2k

c      write(*,*) 'In Fnm a,b,c,x,itmax = ',a,b,c,x,itmax
      Fnm = 1.d0
      x2k = 1.d0
      temp = 1.d0

      do 10 k=1,itmax
          dblk = k-1
          temp = temp * ((a+dblk) * (b+dblk) / (c+dblk))
          x2k = x2k * x / float(k)
          Fnm = Fnm + temp * x2k
10     continue

      end

c#####
c      ASSORTED MISCELLANEOUS FUNCTIONS AND SUBROUTINES
c#####

c*****
c      Sets common variables corresponding to the value of x.
c      These common variables help the above routines decide
c      how to handle real or complex values.

c * * * * *

      SUBROUTINE rsetx(x)

      double precision  x

      logical          XREAL,XCMPLX,BIG,SMALL,WARN
      integer          ITMAX
      common /LEGNDR/  XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

      XREAL = .TRUE.
      XCMPLX = .FALSE.
      if (x .lt. 1.d0 .and. x .gt. -1.d0) then
          SMALL = .TRUE.
          BIG = .FALSE.
      else
          SMALL = .FALSE.
          BIG = .TRUE.
      endif
      return
      end

```

c * * * * *

SUBROUTINE csetx()

logical XREAL,XCMPLX,BIG,SMALL,WARN
integer ITMAX
common /LEGNDR/ XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX

XREAL = .FALSE.
XCMPLX = .TRUE.
SMALL = .FALSE.
BIG = .TRUE.
return
end

c*****

c Calculate the arccot of abs(x). This is not normally supplied
c by intrinsic routines, so it must be synthesized

DOUBLE PRECISION FUNCTION dacot(x)

double precision x

double precision pi
parameter (pi = 3.141592653589793d0)
double precision pid2
parameter (pid2 = pi/2.d0)

dacot = pid2 - datan(x)
end

c*****

BLOCK DATA PQinit

logical XREAL,XCMPLX,BIG,SMALL,WARN
integer ITMAX
common /LEGNDR/ XREAL,XCMPLX,BIG,SMALL,WARN,ITMAX
data WARN /.TRUE./
data ITMAX /15/
end

c*****

c BASIC ROUTINES TO DO SIMPLE FUNCTIONS

c*****

```

c      Assorted functions
c      prefix "i" = integer
c      prefix "r" = single precision real
c      prefix "s" = double precision real

c*****
c      Returns the factorial of n

      INTEGER FUNCTION ifact(n)

      integer      n
      integer      i

      ifact = 1
      if (n.ge.13) then
cw        write(*,*) 'ERROR in ifact ... n too big: ',n
      elseif (n.gt.1) then
        do 10 i=2,n
          ifact = ifact*i
10        continue
      endif
      end

c*****

      REAL FUNCTION rfact(n)

      integer      n
      integer      i

      rfact = 1.
      if (n.gt.1) then
        do 10 i=2,n
          rfact = rfact*i
10        continue
      endif
      end

c*****

      DOUBLE PRECISION FUNCTION dfact(n)

      integer      n
      integer      i

      dfact = 1.d0
      if (n.gt.1) then
        do 10 i=2,n
          dfact = dfact*i
10        continue

```

```

endif
end

c*****
c    Computes dfact(n1)/dfact(n2)

DOUBLE PRECISION FUNCTION dfact2(n1in,n2in)

integer      n1in,n2in
integer      n1,n2
integer      i

n1 = max(n1in,1)
n2 = max(n2in,1)

dfact2 = 1.d0
if (n1.gt.n2) then
    do 10 i=n2+1,n1
        dfact2 = dfact2 * i
10    continue
elseif (n1.lt.n2) then
    do 20 i=n1+1,n2
        dfact2 = dfact2 * i
20    continue
dfact2 = 1.d0 / dfact2
endif
end

c*****
c    Returns 1 * 3 * 5 * ... * n
c    where n must be odd.
c    Essentially, an odd factorial.

DOUBLE PRECISION FUNCTION doddf(n)

integer      n
integer      i

if ( mod(n,2) .ne. 1) then
cw    write(*,*) 'ERROR in doddf ... passed even n = ',n
    stop
endif

doddf = 1.d0
if (n.gt.1) then
    do 10 i=3,n,2
        doddf = doddf*i
10    continue

```



```
endif
end
```

```
c*****
```

```
c      Returns the square of x
```

```
      REAL FUNCTION rsq(x)
```

```
      real      x
```

```
      rsq = x*x
```

```
      end
```

```
      DOUBLE PRECISION FUNCTION dsq(x)
```

```
      double precision x
```

```
      dsq = x*x
```

```
      end
```

```
c*****
```

```
c      Computes (-1)**m
```

```
      INTEGER FUNCTION ineg1(m)
```

```
      integer      m
```

```
      if ( mod(m,2).eq.0 ) then
```

```
          ineg1 = 1.0
```

```
      else
```

```
          ineg1 = -1.0
```

```
      endif
```

```
      end
```

```
c*****
```

```
      REAL FUNCTION megl(m)
```

```
      integer      m
```

```
      integer      ineg1
```

```
      external ineg1
```

```
      megl = ineg1(m)
```

```
      end
```

```
      DOUBLE PRECISION FUNCTION dneq1(m)
```

```
      integer      m
```

```
      integer      ineg1
```

```
      external ineg1
```

```
      dneq1 = ineg1(m)
```

```
      end
```

```
c*****
```

```
c      Computes the value of an indexed point
```

```
      DOUBLE PRECISION FUNCTION interp(lo,hi,n,i)
```

```
      double precision lo,hi
```

```

integer          n,i

if (n.le.1) then
    interp = lo
else
    interp = lo + (hi-lo)*float(i-1)/float(n-1)
endif
end

c#####
cc      TEST ROUTINE
c#####
c
c      integer          n,m,lo,hi
c      double precision x,dx,tem,err1,err2
c
c      double precision Pnm,dPnm,cPnm,cdPnm
c      external        Pnm,dPnm,cPnm,cdPnm
c      double precision Qnm,dQnm,cQnm,cdQnm
c      external        Qnm,dQnm,cQnm,cdQnm
c
c      integer          jPnm,jdPnm
c      external        jPnm,jdPnm
c      integer          jQnm,jdQnm
c      external        jQnm,jdQnm
c
c1      write(*,*) 'Enter 1 - Give out Pnm,Qnm for various n,m,x'
c      write(*,*) ' 2 - Give out cPnm,cQnm for various n,m,x'
c      write(*,*) ' 3 - Test the d/dx formulae for real x'
c      write(*,*) ' 4 - Test the d/dx formulae for imag x'
c      read(*,*) n
c      goto (100,200,300,400) n
c
c100    write(*,*) 'Enter nmlo,nmhi,x'
c      read(*,*) lo,hi,x
c      write(*,*) 'Table of Pnm'
c      do 110 n=lo,hi
c          write(*, '(5g15.5)') (Pnm(n,m,x),m=lo,hi)
c110    continue
c      write(*,*) 'Table of Qnm'
c      do 120 n=lo,hi
c          write(*, '(5g15.5)') (Qnm(n,m,x),m=lo,hi)
c120    continue
c      goto 100
c
c
c200    write(*,*) 'Enter nmlo,nmhi,x'
c      read(*,*) lo,hi,x
c      write(*,*) 'Table of Pnm'
c      do 210 n=lo,hi

```

```

c          write(*,'(i2,1x,12g15.5)') jPnm(n),(cPnm(n,m,x),m=lo,hi)
c210 continue
c          write(*,*) 'Table of Qnm'
c          do 220 n=lo,hi
c              write(*,'(i2,1x,12g15.5)') jQnm(n),(cQnm(n,m,x),m=lo,hi)
c220 continue
c          goto 200
c
c
c300 write(*,*) 'Enter nmlo,nmhi,x,dx'
c          read(*,*) lo,hi,x,dx
c          do 310 n=lo,hi
c              do 310 m=lo,hi
c
c                  tem = (Pnm(n,m,x+dx) - Pnm(n,m,x-dx)) / (2.*dx)
c                  write(*,'(2i3,2g15.5)') n,m,dPnm(n,m,x),tem
c
c                  err1 = 0
c                  err2 = 0
c                  if (dPnm(n,m,x).ne.0) then
c                      err1 = 100. * abs(dPnm(n,m,x)-tem) / dPnm(n,m,x)
c                  endif
c
c                  if (tem.ne.0) then
c                      err2 = 100. * abs(dPnm(n,m,x)-tem) / tem
c                  endif
c                  write(*,'(36x,2g15.5)') err1,err2
c
c310 continue
c          pause
c          do 320 n=lo,hi
c              do 320 m=lo,hi
c
c                  tem = (Qnm(n,m,x+dx) - Qnm(n,m,x-dx)) / (2.*dx)
c                  write(*,'(2i3,2g15.5)') n,m,dQnm(n,m,x),tem
c
c                  err1 = 0
c                  err2 = 0
c                  if (dQnm(n,m,x).ne.0) then
c                      err1 = 100. * abs(dQnm(n,m,x)-tem) / dQnm(n,m,x)
c                  endif
c
c                  if (tem.ne.0) then
c                      err2 = 100. * abs(dQnm(n,m,x)-tem) / tem
c                  endif
c                  write(*,'(36x,2g15.5)') err1,err2
c
c320 continue
c          goto 300
c

```

```

c
c400 write(*,*) 'Enter nmlo,nmhi,x,dx'
c      read(*,*) lo,hi,x,dx
c      do 410 n=lo,hi
c        do 410 m=lo,hi
c
c          tem = (cPnm(n,m,x+dx) - cPnm(n,m,x-dx)) / (2.*dx)
c          if (jPnm(n).eq.0) tem = -tem
c          write(*,'(3i3,2g15.5)') n,m,jdPnm(n),cdPnm(n,m,x),tem
c
c          err1 = 0
c          err2 = 0
c          if (cdPnm(n,m,x).ne.0) then
c            err1 = 100. * abs(cdPnm(n,m,x)-tem) / cdPnm(n,m,x)
c          endif
c
c          if (tem.ne.0) then
c            err2 = 100. * abs(cdPnm(n,m,x)-tem) / tem
c          endif
c          write(*,'(39x,2g15.5)') err1,err2
c
c410 continue
c      pause
c      do 420 n=lo,hi
c        do 420 m=lo,hi
c
c          tem = (cQnm(n,m,x+dx) - cQnm(n,m,x-dx)) / (2.*dx)
c          if (jQnm(n).eq.0) tem = -tem
c          write(*,'(3i3,2g15.5)') n,m,jdQnm(n),cdQnm(n,m,x),tem
c
c          err1 = 0
c          err2 = 0
c          if (cdQnm(n,m,x).ne.0) then
c            err1 = 100. * abs(cdQnm(n,m,x)-tem) / cdQnm(n,m,x)
c          endif
c
c          if (tem.ne.0) then
c            err2 = 100. * abs(cdQnm(n,m,x)-tem) / tem
c          endif
c          write(*,'(39x,2g15.5)') err1,err2
c
c420 continue
c      goto 400
c      end
c

```

REFERENCES

Chapter I

- [1] D.F. Rose, P.D. Smith, and S. Sato, "Magnetoencephalography and epilepsy research," *Science*, vol. 238, pp. 329-335, 1987.
- [2] C.C. Wood, D. Cohen, B.N. Cuffin, M. Yarita, and Truett Allison, "Electrical sources in human somatosensory cortex: identification by combined magnetic and potential recordings," *Science*, vol. 227, pp. 1051-1053, 1985.
- [3] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.
- [4] Y.C. Okada, L. Kaufman, D. Brenner, and S.J. Williamson, "Modulation transfer functions of the human visual system revealed by magnetic field measurements," *Vision Research*, vol. 22, pp. 319-333, 1982.
- [5] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," *Ann. N.Y. Acad. Sci.*, vol. 425, pp. 722-742, 1984.
- [6] S.J. Williamson and L. Kaufman, "Biomagnetism," *J. Magn. Mag. Mat.*, vol. 22, pp. 129-201, 1981.
- [7] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [8] J. Sarvas, "Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 11-22, 1987.
- [9] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [10] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.

- [11] B.N. Cuffin, "Effects of head shape on EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-37, pp. 44-52, 1990.
- [12] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncitia," *Biophys. J.*, vol. 51, pp. 557-568, 1987.
- [13] D.S. Barth, W. Sutherling, J. Broffman, and J. Beatty, "Magnetic localization of a dipolar current source implanted in a sphere and a human cranium," *EEG. Cl. Neur.*, vol. 63, pp. 260-273, 1986.
- [14] C.J. Stok, J.W.H. Meijs, and M.J. Peters, "Inverse solutions based on MEG and EEG applied to volume conductor analysis," *Phys. Med. Biol.*, vol. 32, pp. 99-104, 1987.
- [15] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [16] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.
- [17] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," *J. Appl. Phys.*, vol. 48, pp. 3971-3980, 1977.
- [18] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 266-272.
- [19] J.P. Wikswo, "Biomagnetic sources and their models." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 1-18.
- [20] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," *Phys. Med. Biol.*, vol. 32, pp. 43-51, 1987.
- [21] B.N. Cuffin, T.E. Katila, M. Pelizzzone, and J.P. Wikswo, Jr., "Theoretical models for source localization." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 9-18.

- [22] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," *Phys. Med. Biol.*, vol. 32, pp. 125-131, 1987.
- [23] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," *Phys. Med. Biol.*, vol. 32, pp. 133-142, 1987.
- [24] D.F. Rose, S. Sato, P.D. Smith, and J. White, "Modelling the temporal region in patients with temporal lobe epilepsy," *Phys. Med. Biol.*, vol. 32, pp. 59-63, 1987.
- [25] J.C. De Munck, B.W. Van Duk, and H. Spekreijse, "Mathematical dipoles are adequate to describe realistic generators of human brain activity," *IEEE Trans. Biomed. Eng.*, vol. BME-35, pp. 960-966, 1988.
- [26] B.J. Roth, N.G. Sepulveda, and J.P. Wikswo, Jr., "Using a magnetometer to image a two-dimensional current distribution," *J. Appl. Phys.*, vol. 65, pp. 361-372, 1989.
- [27] D.S. Buchanan, D.B. Crum, D. Cox, and J.P. Wikswo, Jr., "Microsquid: a close-spaced four channel magnetometer." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 677-679.
- [28] J.P. Wikswo Jr. and B.J. Roth, "Magnetic determination of the spatial extent of a single cortical current source: a theoretical analysis," *EEG. Cl. Neur.*, vol. 69, 266-276, 1988.

Chapter II

- [1] D.F. Rose, P.D. Smith, and S. Sato, "Magnetoencephalography and epilepsy research," *Science*, vol. 238, pp. 329-335, 1987.
- [2] C.C. Wood, D. Cohen, B.N. Cuffin, M. Yarita, and Truett Allison, "Electrical sources in human somatosensory cortex: identification by combined magnetic and potential recordings," *Science*, vol. 227, pp. 1051-1053, 1985.

- [3] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.
- [4] Y.C. Okada, L. Kaufman, D. Brenner, and S.J. Williamson, "Modulation transfer functions of the human visual system revealed by magnetic field measurements," *Vision Research*, vol. 22, pp. 319-333, 1982.
- [5] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," *Ann. N.Y. Acad. Sci.*, vol. 425, pp. 722-742, 1984.
- [6] D.S. Barth, W. Sutherling, J. Broffman, and J. Beatty, "Magnetic localization of a dipolar current source implanted in a sphere and a human cranium," *EEG. Cl. Neur.*, vol. 63, pp. 260-273, 1986.
- [7] C.J. Stok, J.W.H. Meijs, and M.J. Peters, "Inverse solutions based on MEG and EEG applied to volume conductor analysis," *Phys. Med. Biol.*, vol. 32, pp. 99-104, 1987.
- [8] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [9] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.
- [10] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," *J. Appl. Phys.*, vol. 48, pp. 3971-3980, 1977.
- [11] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [12] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 266-272.

- [13] C. Pantev, M. Hoke, K. Lehnertz, B. Lutkenhoner, G. Anogianakis, and W. Wittkowski, "Tonotopic organization of the human auditory cortex revealed by transient auditory evoked magnetic fields," *EEG. Cl. Neur.*, vol. 69, pp. 160-170, 1988.
- [14] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.
- [15] B.N. Cuffin, T.E. Katila, M. Pelizzzone, and J.P. Wikswo, Jr., "Theoretical models for source localization." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 9-18.
- [16] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," *Phys. Med. Biol.*, vol. 32, pp. 125-131, 1987.
- [17] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," *Phys. Med. Biol.*, vol. 32, pp. 133-142, 1987.
- [18] D.F. Rose, S. Sato, P.D. Smith, and J. White, "Modelling the temporal region in patients with temporal lobe epilepsy," *Phys. Med. Biol.*, vol. 32, pp. 59-63, 1987.
- [19] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," *Phys. Med. Biol.*, vol. 32, pp. 43-51, 1987.
- [20] J.P. Wikswo Jr. and B.J. Roth, "Magnetic determination of the spatial extent of a single cortical current source: a theoretical analysis," *EEG. Cl. Neur.*, vol. 69, pp. 266-276, 1988.
- [21] M.C. Leifer, J.C. Griffin, D.C. Harrison, and J.P. Wikswo, Jr., "Effect of the heart-lung boundary on the magnetocardiogram," *J. Electr.*, vol. 19, pp. 23-32, 1986.
- [22] P.L. Nunez, Electric Fields of the Brain, New York, Oxford University Press, 1981, Ch. 4, pp. 111-139.
- [23] T.A. Weber and D.J. Macomb, "On the equivalence of the laws of Biot-Savart and Ampere," *Am. J. Phys.*, vol. 57, pp. 57-59, 1989.
- [24] G. Baule and R. McFee, "Theory of magnetic detection of the heart's electrical activity," *J. Appl. Phys.*, vol. 36, pp. 2066-2073, 1965.

- [25] L. Kaufman, "Biomagnetism in the study of brain functions." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 221-234.
- [26] R.J. Ilmoniemi, M.S. Hamalainen, and J. Knuutila, "The forward and inverse problems in the spherical model." In H. Weinberg, G. Stroink, and T. Katila (Eds.), Biomagnetism: Applications and Theory, Pergamon Press, New York, 1985, pp. 278-282.

Chapter III

- [1] S.J. Williamson and L. Kaufman, "Biomagnetism," J. Magn. Mag. Mat., vol. 22, pp. 129-201, 1981.
- [2] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [3] J. Sarvas, "Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem," Phys. Med. Biol., vol. 32, pp. 11-22, 1987.
- [4] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.
- [5] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," Biophys. J., vol. 13, pp. 911-925, 1973.
- [6] B.N. Cuffin, "Effects of head shape on EEG's and MEG's," IEEE Trans. Biomed. Eng., vol. BME-37, pp. 44-52, 1990.
- [7] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncytia," Biophys. J., vol. 51, pp. 557-568, 1987.
- [8] D.B. Geselowitz, "On the magnetic field generated outside an inhomogenous volume conductor by internal current sources," IEEE Trans. Mag., vol. 6, pp. 346-347, 1970.

- [9] E. Frank, "Electric potential produced by two point current sources in a homogeneous conducting sphere," J. Appl. Phys., vol. 23, pp. 1225-1228, 1952.
- [10] P.M. Berry, "Appendix: N,M space harmonics of the oblate spheroid," Ann. N.Y. Acad. Sci., vol. 65, pp. 1126-1134, 1956.
- [11] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," IEEE Trans. Magn., vol. MAG-27, pp. 758-767, 1991.
- [12] R.E. Collin and R. Plonsey, "A note on dipole sources in conducting biological tissues," Bull. Math. Biol., vol. 40, pp. 201-209, 1977.
- [13] T.A. Weber and D.J. Macomb, "On the equivalence of the laws of Biot-Savart and Ampere," Am. J. Phys., vol. 57, pp. 57-59, 1989.
- [14] A. Sezginer and T.M. Habashy, "An image method to compute the static magnetic field due to currents injected into a homogenous, conducting, and magnetically polarizable half-space," Radio Science, vol. 23, pp. 41-45, 1988.

Chapter IV

- [1] G.L. Romani, "Fundamentals on Neuromagnetism." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 33-46.
- [2] L. Kaufman, Y. Okada, J. Tripp, and H. Weinberg, "Evoked neuromagnetic fields," Ann. N.Y. Acad. Sci., vol. 425, pp. 722-742, 1984.
- [3] B.J. Roth, N.G. Sepulveda, and J.P. Wikswo, Jr., "Using a magnetometer to image a two-dimensional current distribution," J. Appl. Phys., vol. 65, pp. 361-372, 1989.
- [4] T. Katila, R. Maniewski, M. Makijarvi, J. Nenonen, and P. Siltanen, "On the accuracy of source localisation in cardiac measurements," Phys. Med. Biol., vol. 32, pp. 125-131, 1987.

- [5] D.S. Buchanan, D.B. Crum, D. Cox, and J.P. Wikswo, Jr., "Microsquid: a close-spaced four channel magnetometer." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 677-679.
- [6] J.P. Wikswo, R.N. Friedman, A.W. Kilroy, J.M. van Egeraat, and D.S. Buchanan, "Preliminary measurements with microsquid." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 681-684.
- [7] Y.C. Okada, M. Lauritzen, and C. Nicholson, "Magnetic field associated with neural activities in an isolated cerebellum," *Brain Res.*, vol. 412, pp. 151-155, 1987.
- [8] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," *IEEE Trans. Biomed. Eng.*, vol. BME-24, pp. 372-381, 1977.
- [9] F. Grynszpan and D.B. Geselowitz, "Model studies of the magnetocardiogram," *Biophys. J.*, vol. 13, pp. 911-925, 1973.
- [10] J.P. Wikswo, "Biomagnetic sources and their models." In S.J. Williamson, M. Hoke, G. Stroink and M. Kotani (Eds.), Advances in Biomagnetism, Plenum Press, New York, 1989, pp. 1-18.
- [11] Y. Okada, M. Lauritzen, and C. Nicholson, "MEG source models and physiology," *Phys. Med. Biol.*, vol. 32, pp.43-51, 1987.
- [12] B.N. Cuffin, "The role of model and computational experiments in the biomagnetic inverse problem," *Phys. Med. Biol.*, vol. 32, pp. 33-42, 1987.
- [13] R.S. Gonnelli and M. Agnello, "Inverse problem solution in cardiomagnetism using a current multipole expansion of the primary sources," *Phys. Med. Biol.*, vol. 32, pp. 133-142, 1987.
- [14] N.G. Sepulveda and J.P. Wikswo, Jr., "Electric and magnetic fields from two-dimensional anisotropic bisyncytia," *Biophys. J.*, vol. 51, pp. 557-568, 1987.

- [15] J.C. De Munck, B.W. Van Duk, and H. Spekreijse, "Mathematical dipoles are adequate to describe realistic generators of human brain activity," *IEEE Trans. Biomed. Eng.*, vol. BME-35, pp. 960-966, 1988.
- [16] B.N. Cuffin, "A comparison of moving dipole inverse solutions using EEG's and MEG's," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 905-910, 1985.
- [17] B.N. Cuffin and D. Cohen, "Magnetic fields produced by models of biological current sources," *J. Appl. Phys.*, vol. 48, pp. 3971-3980, 1977.
- [18] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," *IEEE Trans. Magn.*, vol. MAG-27, pp. 758-767, 1991.
- [19] R. Plonsey, "The active fiber in a volume conductor," *IEEE Trans. Biomed. Eng.*, vol. BME-21, pp. 371-381, 1974.
- [20] J.J.B. Jack, D. Noble, and R.W. Tsien, *Electric Current Flow in Excitable Cells*, Clarendon Press, Oxford, 1983, pp. 25-66.
- [21] G.L.F. Yuen and D. Durand, "Reconstruction of hippocampal granule cell electrophysiology by computer simulation," accepted for publication in *Neuroscience*, vol. 41, pp. 411-424, 1991.
- [22] Y. Okada, "Discrimination of localized and distributed current dipole sources and localized single and multiple sources." In H. Weinberg, G. Stroink, and T. Katila (Eds.), *Biomagnetism: Applications and Theory*, Pergamon Press, New York, 1985, pp. 266-272.
- [23] D. Cohen and H. Hosaka, "Part II. Magnetic field produced by a current dipole," *J. Electrocardiology*, vol. 9, pp. 409-417, 1976.
- [24] K.R. Swinney and J.P. Wikswo, Jr., "A calculation of the magnetic field of a nerve action potential," *Biophys. J.*, vol. 32, pp. 719-731, 1980.
- [25] J.P. Barach, B.J. Roth and J.P. Wikswo, Jr., "Magnetic measurements of action currents in a single nerve axon: a core-conductor model," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 136-140, 1985.

- [26] W. Rall, "Core conductor theory and cable properties of neurons." in Handbook of Physiology - The Nervous System, American Physiological Society, Bethesda, MD., 1977, sect. 1, vol. 1, part 1, pp. 39-97.

Chapter V

- [1] A.S. Ferguson and D. Durand, "Magnetic fields of current monopoles in special volume conductors," IEEE Trans. Magn., vol. MAG-27, pp. 758-767, 1991.
- [2] B.N. Cuffin and D. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.

Appendix E

- [1] B.N. Cuffin and D.C. Cohen, "Magnetic fields of a dipole in special volume conductor shapes," IEEE Trans. Biomed. Eng., vol. BME-24, pp. 372-381, 1977.

Appendix H

- [1] L.J. Briggs and A.N. Lowan, Tables of Associated Legendre Functions, Columbia University Press, New York, 1945.
- [2] T.M. MacRobert, Spherical Harmonics, Pergamon Press, Oxford, 1967.
- [3] E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Chelsea Publishing Co., New York, 1955.