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El-Bialy, Ahmed Mohamed, Ph.D.

Case Western Reserve University, 1990

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CONTROL OF MULTIPLICATIVE DISCRETE-TIME SYSTEMS

by

AHMED M. EL-BIALY

**SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY**

Thesis Advisor: Dr. HOWARD J. CHIZECK

**Department of Systems Engineering
CASE WESTERN RESERVE UNIVERSITY
August, 1990**

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CONTROL OF MULTIPLICATIVE DISCRETE - TIME SYSTEMS

Abstract

by

AHMED M. EL-BIALY

This dissertation is concerned with the formulation and solution of control problems associated with discrete time nonlinear systems. More specifically, we are considering a class of *multiplicative discrete-time nonlinear systems*, whose outputs are expressed in terms of products of linear, piecewise linear, or nonlinear subsystems. This class of systems appears to have potential application in the modeling and control of wide biomedical systems. In particular it is appropriate for electrically stimulated muscles, for use in *neural prostheses* for the rehabilitation of paralyzed individuals.

In this research, new nonlinear controllers are going to be designed, based upon a combination of a recently developed technique called **exact linearization** and both state space optimal as well as classical control methods. The exact linearization technique is used to transform the nonlinear control problem into a linear one, through a nonlinear feedback and a change in the state coordinates. Finding the equivalent linear system is a first step towards the nonlinear controller design. The second step is to design this controller

in the linear space. Once the feedback law is obtained, the third step is to map back this control law to the original nonlinear space.

In this study, we derive the discrete time exact linearization, prove the required necessary and sufficient conditions, for its application, and apply it to the class of multiplicative systems. We then propose a family of objective functions that, under the exact linearization technique, yield easily computed optimal laws. Both finite and infinite time horizon optimal nonlinear compensators result for the class of multiplicative systems under consideration.

We also consider the application of the classical linear control methods for the linearized models. This results in methods for the design of relatively robust nonlinear compensators.

We introduce another two applications for the exact linearization technique. First, we apply the technique to solve the nonlinear first conditions of optimality of the quadratic nonlinear control problem. Second, we investigate problems involving the control of the nonlinear systems subject to equality and inequality constraints.

Finally, these results are applied to the control of electrically stimulated muscles. A discrete time model is used to simulate a double - muscle joint system. For this system all of the controller designs developed here are tested and compared (via computer simulations). The controllers are tested in trials with and without input and output simulated noise, as well as in trials having parameters variations. In these simulations we tested three types of controllers; a PID, a linearized quadratic and an open loop controller. The

results of these simulations indicate that the linearized quadratic controllers are less sensitive to environmental noise than the others. On the other hand, the results of testing the use of incorrect parameters for the controller design show that the open loop controllers are best if parameter variations are small, while for large parameter variations the linearized quadratic controllers are better.

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CHAPTER ONE

INTRODUCTION

1.1 THESIS OVERVIEW

This dissertation is concerned with the formulation and solution of control problems associated with discrete time nonlinear systems. More specifically, we are considering a class of *multiplicative discrete-time nonlinear systems*, whose outputs are expressed in terms of products of linear, piecewise linear, or nonlinear subsystems. This class of systems appears to have potential application in the modeling and control of wide biomedical systems. In particular it is appropriate for electrically stimulated muscles, for use in *neural prostheses* for the rehabilitation of paralyzed individuals.

This chapter, which introduces the nonlinear control problems under study, is organized as follows. Section 1.2 contains the thesis objectives and the main contributions of this research. The multiplicative systems are defined in section 1.3. Section 1.4 includes the major application example for the class of nonlinear systems under study; this section introduces the three factor model for the electrically stimulated muscles. Finally, an outline of the contents of this thesis is given in section 1.5.

1.2 THESIS OBJECTIVES

In this research, we are going to consider discrete time nonlinear systems of the form:

$$\begin{aligned} y(k) &= h(x(k)) \\ x(k+1) &= f(x(k), u(k)) \end{aligned} \tag{1.1}$$

where $x(0) = x_o$ is given, and

$$\begin{aligned} y &\in \mathbb{R}^p \\ x &\in \mathbb{R}^N \\ u &\in \mathbb{R}^m. \end{aligned} \tag{1.2}$$

To control the previous nonlinear system we are going to apply a technique called **exact linearization** [Monaco and Normand-Cyrot, 1987]. This technique accomplishes input - output linearization of the nonlinear system by changes in the state coordinate and through the use of nonlinear feedback. This allows the control of the nonlinear systems using the well developed tools of linear system theories.

When applying this technique to the system of (1.1), we obtain the following linear system:

$$\begin{aligned} y(k) &= c^t z(k) \\ z(k+1) &= Az(k) + Bv(k) \end{aligned} \tag{1.3}$$

This is accomplished* with the state coordinate change; $z(k) = T(x(k))$, and the feedback $v(k) = \gamma(x(k), u(k))$.

* See chapter two for a brief survey and detailed explanation of the exact linearization

In this research, new nonlinear controllers, based upon a combination of exact linearization and both state space optimal as well as classical control methods, are going to be designed. Specifically, the research includes:

- 1- The independent derivation and extension of the discrete-time **exact linearization** methods, reported in [Monaco and Normand-Cyrot 1987] and the application of these methods to the class of multiplicative systems under consideration.
- 2- The design of nonlinear controllers for the linearized systems obtained in (1). This includes:
 - a) Applying linear quadratic control theory methods to the linearized models. This control problem is formulated by solving for the input $v(k)$ that minimizes the following objective function:

$$\begin{aligned}
 J(v(k)) = \min_v & \left[\frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \right. \\
 & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\
 & \left. + v^t(k) R(k) v(k) \right]
 \end{aligned} \tag{1.4}$$

subject to:

$$\begin{aligned}
 z(k+1) &= Az(k) + Bv(k) \\
 y(k) &= c^t z(k)
 \end{aligned} \tag{1.5}$$

$$\begin{aligned}
 y &\in \mathbb{R}^p \\
 z &\in \mathbb{R}^N \\
 u &\in \mathbb{R}^m
 \end{aligned} \tag{1.6}$$

where $z(k) = T(x(k))$, $v(k) = \gamma(x(k), u(k))$ and T, γ are the linearizing state and feedback maps; $z(k)$ is the state of the corresponding exact linearized system. This tracking control problem is formulated in the context of driving the system to a specified output trajectory $Y_{ref}(k)$, where it is assumed that the output $\{y(k)\}$ is measured exactly at each time k . Here Q and R are symmetric time varying matrices; Q is assumed to be positive semidefinite and R is positive definite.

This controller allows us to exploit the richly developed linear quadratic optimal control problem theory, for finite and infinite time horizons. The result is a class of easily computed optimal laws that, when translated back into the nonlinear model representation comprise both finite and infinite time optimal nonlinear compensators. Necessary and sufficient conditions for existence and uniqueness are also obtained. The main advantage of this controller is the asymptotic stability of the resulting closed loop nonlinear system.

- b) The application of classical linear control system methods for controller design (eg., pole placement). These methods are applied to the exactly linearized system of (1). The resulting controller designs, obtained in these two steps, can be implemented by nonlinear components. These designs can thus be endowed with the qualitative features (eg., zero steady state error, good transient

response) of good classical linear system controllers. However, disturbance rejection and other robustness properties of the corresponding linear controllers are not fully inherited.

- 3- Optimal control problems involving quadratic costs (in state and control) for the nonlinear system are formulated and solved. The solution method uses both the Calculus of Variations and the Exact Linearization methods and results in an optimal nonlinear feedback law. This approach has more naturally motivated objective functions than the method of (2a), but involves a more complex feedback solution. Unfortunately a large number of technical conditions must be satisfied for this method to be applied. This controller considers the nonlinear system together with the following quadratic objective function:

$$\begin{aligned}
 J(u) = \min_u [& \frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \\
 & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\
 & + u^t(k) R(k) u(k)] \quad (1.7)
 \end{aligned}$$

subject to the model dynamics described in eq (1.1), where

$$\begin{aligned}
 y & \in \mathbb{R}^p \\
 x & \in \mathbb{R}^N \\
 u & \in \mathbb{R}^m \quad (1.8)
 \end{aligned}$$

It is assumed that the output $\{y(k)\}$ is measured exactly at each time k . One goal of the controller is to make $\{y(k)\}$ follows $\{Y_{ref}(k)\}$. A second goal is to minimize dissipated energy; this leads to the $u^t(k)R(k)u(k)$ term. Quadratic penalties on $u(k)$ and $[y(k) - Y_{ref}(k)]$ are used for

reasons of mathematical simplicity. A penalty is also assessed on the final output error at the end of the movement $k = N$. Here Q and R are symmetric time varying matrices; Q is assumed to be positive semidefinite and R is positive definite. Note that the Q and R of this problem formulation need not to be the same as in (2).

- 4- Initial investigation into the problems involving control of these non-linear systems subject to inequality constraints on states, inputs and outputs are considered. Insight is obtained regarding how the constraints and exact linearization interact. As in the linear system case, numerical methods are required to obtain specific solutions. Unfortunately, exact linearization appears to generate extra constraints, and to make the problem harder when constraints are present.

The technique reported by Jacobson and Lele [1969], for the solution of constraint control problems will be discussed. This technique is based on adding slack variables to the inequality constraints; through increasing the dimension of the system constraints are removed. Thus exact linearization is applied to obtain a problem of controlling a linear system with input constraints.

The remainder of this chapter contains a description of the nonlinear, discrete-time multiplicative systems and application examples of this class of systems, which will be used throughout the thesis.

1.3 MULTIPLICATIVE SYSTEMS

The results in this thesis hold for the class of multiplicative systems described by:

$$\Sigma_m : y(k) = h_1(X_1(k))h_2(X_2(k))\dots h_\beta(X_\beta(k)) \quad (1.9)$$

and

$$\begin{aligned} X_1(k+1) &= f_1(X_1(k), u(k)) \\ X_2(k+1) &= f_2(X_2(k), u(k)) \\ &\vdots \\ X_\beta(k+1) &= f_\beta(X_\beta(k), u(k)) \end{aligned} \quad (1.10)$$

where, at each time k ;

$$\begin{aligned} y &\in \mathbb{R} \\ X_i &\in \mathbb{R}^{n_i} \\ u &\in \mathbb{R}^m \end{aligned} \quad (1.11)$$

and

$$n_1 + n_2 + \dots + n_\beta = N \quad (1.12)$$

Figure 1.1 shows a block diagram for this system. Note that there are parallel, possibly nonlinear discrete-time subsystems, each with output X_i , ($i = 1, \dots, \beta$). Output functions h_i map each X_i into quantities which are then multiplied together, to obtain output $y(k)$.

An example for such system, is the electrically stimulated muscles, which is the main motivation of this research. The muscle model can be represented by a three factor discrete-time model [Geng, 1989], of the class

1.1 The Multiplicative System

described in (1.9)-(1.12). We will use this model as an example, for possibly practical application of control methods derived in this thesis. In the next section, we are going to present the three factor muscle model in detail.

1.4 THE THREE FACTOR MODEL CLASS AND ELECTRICALLY STIMULATED MUSCLES

1.4.1. INTRODUCTION

The class of multiplicative discrete - time nonlinear models under study appears to have relevance in the representation of electrically stimulated muscles. The class of optimal control laws is motivated by the need of controlling muscles, through electrical stimulation (for rehabilitation), in patients with spinal cord injuries. In this section, the model of the muscles

to be considered in this research will be discussed, and a generalized version will be given and used.

1.4.2 MUSCLE MODEL

Over the previous decades, three structurally different basic types of models have been proposed to describe the muscle-joint systems [Winters and Stark, 1987]. The first is based on an input/output analysis of a certain task; this leads to a second order differential equation for each muscle. The second approach is based on Hill's model [1938]; it results in higher order, nonlinear ordinary differential equation for each muscle. The third method is based on the detailed biological and mechanical structure of muscles. It leads to systems of complex ordinary or partial differential equations [Zajac, 1983]. Simple models lack depth in accurately describing this system. The complex models are excessively detailed, and require the identification of large number of unknown parameters (that are difficult, if not impossible, to simultaneously measure experimentally).

The model structure adopted in this work is based on the work of Hatze [1977], which was an extension to the classic structure of Hill. The model used in this work is the middle ground between the two approaches. The model, due to Crago and Chizeck, and is first referred to in [Chizeck, 1989]. Parameter identification of this model has been accomplished using nonlinear least squares methods [Geng, 1989] and [Shue, 1990]. It has been used in control designs (in animal models) by [Veltink et al, 1989a,b and c].

According to Hatze's work, the developed force in the muscle can be expressed as the product of the three factors: **activation**, **length-tension**,

and force-velocity. As shown in Figure 1.2, we can represent this in discrete time by:

$$y(k) = A(k)LT(k)FV(k), \quad (1.13)$$

where,

$y(k)$ = output force at time k ,

$A(k)$ = activation state at time k ,

$LT(k)$ = length-tension factor at time k , and,

$FV(k)$ = force-velocity factor at time k .

Each of these factors is described below.

Fig. 1.2 Muscle Model

a) Activation Dynamics System

Force is generated by muscle fiber activation. This can be represented by a dynamic system which depends on both the muscle length and activation input (stimulation). Bernotas et al. [1986], proposed the following second

order discrete-time model for the force activation dynamics of the muscle, which was shown to be useful in the real-time identification of electrically stimulated muscles under isometric (constant length) conditions:

$$A(k) = a_1 A(k-1) + a_2 A(k-2) + bu(k-1). \quad (1.14)$$

In this model, muscles are electrically stimulated by a train of pulses which have constant frequency and constant amplitude. The input sequence u denotes the pulse width. The **discrete-time** model was proposed because of the all-or-none behavior of the electrically stimulated muscles, that makes them essentially independent of the stimulus pulse shape. The muscle time constant is short, compared to the pulse width and inter-pulse intervals. Discrete-time models have been used for control [e.g., Bernotas et al, 1987; Inbar, 1986] because they facilitate computer controller design and implementation.

In Geng [1989], eq. (1.14) is extended, to make a_1 and a_2 linear functions of the muscle length:

$$A(k) = (a_1 + c_1 \phi(k))A(k-1) + (a_2 + c_2 \phi(k))A(k-2) + bu(k-1), \quad (1.15)$$

where

$\phi(k)$ =muscle length at time k , and,

$u(k-1)$ =the activation input at time $k-1$

(i.e., recruitment level).

b) The Length-Tension Factor

The length-tension factor can be experimentally obtained by measuring the isometric steady state muscle force for different length values. Figure 1.3 shows a typical length-tension curve. This relation has been approximated by Hatze [1977] as:

$$LT(k) = \exp \frac{-(\phi(k) - \underline{a})^2}{\underline{b}^2}, \quad (1.16)$$

where $\underline{b}$ is a constant quantity and $\underline{a}$ is the rest-length (the length of the muscle at rest position). In [Geng, 1989], the length-tension relation is approximated by:

$$LT(k) = e_1 \phi(k) - e_o, \quad (1.17)$$

where e_o is the (*unrealizable*) length at which no force is produced.

c) The Force-Velocity factor

The force-velocity relationship depends upon the rate of change of muscle length (refer to Figure 1.4). Note that this relationship is different for muscle lengthening and shortening. Hill [1938] proposed the following force-velocity relationship:

$$(T + \mathbf{a})(v + \mathbf{b}) = (T_o + \mathbf{a})\mathbf{b}, \quad (1.18)$$

where

T	:	tension,
T_o	:	isometric tension (force),
v	:	velocity,
v_{max}	:	maximum velocity at zero tension,
$\mathbf{a}$ and $\mathbf{b}$	:	constants.

Fig.1.3 Length-Tension Curve [from McMahon 1984]

This relation can be approximated by the function:

$$\frac{T}{T_0} = \begin{cases} 1 + v \frac{s_1}{v_{max}} & \text{in case of lengthening, i.e., } v > 0 \\ 1 + v \frac{s_2}{v_{max}} & \text{in case of shortening, i.e., } v < 0 \end{cases} \quad (1.19)$$

where s_1 and s_2 are the rate of change of the force with respect to the velocity for $v > 0$ and $v < 0$, respectively.

1.4.3 Identification And Use of The Multiplicative Muscle Model

Geng [1989] used a sequential nonlinear least square identification algorithm [Goodwin and Sin, 1984] to identify this eight-parameter model as

Fig. 1.4 Force-Velocity Curve [from McMahon 1984]

well as a simplified six-parameter version (neglecting c_1 and c_2). The eight parameter version did not predict output forces significantly better than the six parameter version, in animal trials. Shue [1990] used the same identification algorithm as [Geng, 1989] in the six parameters version of (1.15), (1.17) and (1.19), setting $c_1 = c_2 = 0$.

Veltink et al [1989a,b and c] used a simplified version of the six parameter model to control a cat soleus muscle. This *ad hoc* method was used to first experimentally obtain parameter estimates. The resulting model was then

inverted and used both with and without a PID control loop, in feedback controllers.

1.4.4. A Generalized Muscle Model

For our purposes, we consider a generalized version of this three factor model. We generalize (1.15), (1.17) and (1.19) by letting $LT(k)$ and $FV(k)$ be continuous, non-zero functions. We no longer restrict the parameters $a_1(\phi)$ and $a_2(\phi)$ of the activation factor to be linear in the length of the muscle. This leads to the following model:

$$\begin{aligned} y(k) &= A(k)LT(k)FV(k), \\ A(k) &= a_1(\phi)A(k-1) + a_2(\phi)A(k-2) + bu(k-1). \end{aligned} \tag{1.20}$$

For control of this muscle model, equation (1.20) can be rewritten using state space notation, with the following correspondences:

$$\begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \triangleq \begin{pmatrix} A(k-1) \\ A(k) \\ \phi(k-1) \end{pmatrix}$$

$$h_1(k) \triangleq c^t x(k)$$

$$h_2(k) \triangleq LT(k)$$

$$h_3(k) \triangleq FV(k)$$

$$\phi(k) \triangleq \text{the muscle length at time } k$$

$$u(k) \triangleq \text{the activation input at time } k$$

where, as discussed in the previous section:

- $h_2(\cdot)$ is an affine function of the input ϕ :

$$h_2(\cdot) = e_1\phi(k) - e_o \tag{1.21}$$

where e_0 and e_1 are constants.

- $h_3(\cdot)$ is piecewise linear in the differences of input ϕ (with offset):

$$\begin{aligned} h_3(\cdot) &= 1 + e_3 [\phi(k) - \phi(k-1)] \\ &= 1 + e_3 [\phi(k) - x_3(k)] \end{aligned} \quad (1.22)$$

where,

$$e_3 = \begin{cases} s_1 & \text{if } \phi(k) > \phi(k-1) \\ s_2 & \text{if } \phi(k) < \phi(k-1) \end{cases} \quad (1.23)$$

which leads to the force-velocity factor in eq. (1.19), where $h_3(k) = \frac{T}{T_0}$ and $\frac{s_1}{v_{max}}$, $\frac{s_2}{v_{max}}$ correspond to s_1 and s_2 respectively.

Therefore, at each discrete time $k = 0, 1, \dots$, the output can be expressed as follows:

$$\begin{aligned} y(k) &= x_2(k) h_2(\phi(k)) h_3(\phi(k), x_3(k)) \\ \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} &= \begin{pmatrix} \bar{A}(\phi(k)) & 0 \\ 0 & 1 \end{pmatrix} x(k-1) + \begin{pmatrix} \bar{b} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(k-1) \\ \phi(k-1) \end{pmatrix} \\ X_3(k) &= \phi(k-1) \end{aligned} \quad (1.24)$$

where,

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ a_2(\phi) & a_1(\phi) \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 0 \\ b \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.25)$$

At each time k ,

$$y \in \mathbb{R}$$

That is, the system is nonlinear in input ϕ and linear in input u .

1.4.5 Dynamic Constraints: (Loads on Muscles)

In building the above muscle model, it was assumed that the muscle length can be changed mechanically, by stretching or shortening. However,

often we are interested in controlling muscles that are acting against some physical load. This load may be dynamic or static; it relates the muscle length (or angle) to the output force (or torque).

In this section we discuss the concatenation of loads to the muscle model. Two types of loads that are considered: internal loads and external loads. Internal loads result from the muscle and the biomechanical system that the muscle is a part of. External loads result from the outside environment. Figure 1.5 shows the muscle model with the two types of loads. The muscle output force in this figure is the summation of both load forces, i.e.

$$F = F_{in} + F_{ex} \quad (1.26)$$

However, it should be noted that only the external force, F_{ex} , is measurable. If we assume that the external load is large enough, we can neglect the internal load, and thus assume that $F = F_{ex}$. If not, then both loads must be considered. The forces generated by the muscles through interaction with the loads result in the position changes $\phi(k)$. Figure 1.6 shows the same system as Figure 1.5, if the internal load is neglected.

In general, the load may be unknown. If it is unknown then position must be measured and fed back to the controller, at each time step (or predicted, based upon a model of the load). The external load may be assumed static, (e.g., a simple spring), or it may be dynamic.

We consider two types of muscle-load attachments; direct load attachments and joint attachment. By direct load attachment we mean that the muscle is connected directly to the load, as shown in Figure 1.7. In this case, the

Fig. 1.5 Muscle + Both Loads

Fig. 1.6 Muscle + Ext. Loads

muscle directly exerts the force acting on the load. In joint attachment, the muscle is connected to the load through a joint. In this case, the muscle produces a force that is transformed to a torque, which then acts on the load. Figure 1.8 shows this case.

Fig. 1.7 Direct Attachment [adapted from McMahon 1984]

Fig. 1.8 Joint Attachment

1.4.6. Example I: Single Muscle and Known Dynamic External Load

In this section, we describe the simplest example systems that will be used in this work to evaluate controller performances. Assume that the load

is external, attached directly to the muscle and is described by a linear, second order system

$$y(t) = M \ddot{\phi}(t) + D \dot{\phi}(t) + K \phi(t) \quad (1.27)$$

That is, we represent an external load of three parallel elements: a linear dashpot D , a linear stiffness K , and a mass term M . When this load is discretized, we obtain the discrete time model:

$$\phi(i+1) = l_1 \phi(i) + l_2 \phi(i-1) + l_3 y(i) \quad (1.28)$$

In order to deal with the control problem that results from muscle action on this external linear, time invariant dynamic load, the load model is combined with the muscle model. If we assume, for the moment, that the sampling interval i of (1.28) is the same as for the model, then we get:

$$\begin{aligned} y(k) &= A(k)LT(k)FV(k), \\ A(k) &= a_1(\phi)A(k-1) + a_2(\phi)A(k-2) + bu(k-1), \\ \phi(k+1) &= l_1 \phi(k) + l_2 \phi(k-1) + l_3 y(k). \end{aligned} \quad (1.29)$$

Of course, we may have to sample the load equation at a higher rate (to avoid aliasing). In that case, we will let the muscle sampling interval k be an integer multiple of load interval i , and we will write (1.29) in an appropriate order difference equation.

1.4.7. Example II: Two-Muscles Model Under Dynamic External Load

A second problem configuration that will be considered as a test case for the theoretical work is given here. It consists of two muscles, acting

against each other and on a common second order linear dynamic load. Figure 1.9 illustrates this situation. A block diagram representation is given in Figure 1.10. From (1.20) the dynamics of this problem setup can be described as follows:

Fig. 1.9 An Elbow Joint (from Zhang et al 1984)

Let

F_f = the force exerted by the flexor muscle,

F_e = the force exerted by the extensor muscle,

$$g_f(k) = LT_f(k)FV_f(k),$$

$$g_e(k) = LT_e(k)FV_e(k),$$

Then the dynamics of the flexor muscle are:

$$F_f(k) = A_f(k)g_f(k), \quad (1.30)$$

$$A_f(k) = a_{1f}(\phi)A_f(k-1) + a_{2f}(\phi)A_f(k-2) + p_f(u(k-1)). \quad (1.31)$$

Fig. 1.10 Block Diagram For Two-Muscles Model

and *dynamics of the extensor muscle* are:

$$F_e(k) = A_e(k)g_e(k), \quad (1.32)$$

$$A_e(k) = a_{1e}(\phi)A_e(k-1) + a_{2e}(\phi)A_e(k-2) + p_e(u(k-1)). \quad (1.33)$$

where ϕ is the angular displacement; ϕ is related geometrically to the muscle length [Veltink et al, 1989a,b and c]. The input maps p_f and p_e are approximate maps [Crago et al, 1986] that decide how much each muscle is stimulated as a function of controller output. These two maps can be approximated by linear, overlapping functions in the control output (u). The overlapping interval is used to increase stiffness of the joint.

As in the earlier example problem setup, a discretized second order linear load is assumed for the load with the same sampling rate as the muscles.

Thus we have the following equation:

$$\phi(k+1) = l_1\phi(k) + l_2\phi(k-1) + l_3(F_f(k) - F_e(k)). \quad (1.34)$$

However, l_1 , l_2 and l_3 are not the same as in (1.28), because of the different type of muscle-load attachments.

1.5. THESIS OUTLINE AND WORK ACCOMPLISHED

The major accomplishments of the work reported here are as follows:

- 1- The application of the exact linearization to the class of nonlinear systems under study, and the muscle model as an example. This is the first step towards the nonlinear controller design. The results of this work are reported in chapter two.
- 2- The design of a controller based on the exactly linearized system which includes:
 - (a) Extending the asymptotic stability solutions of the linear quadratic controllers to the nonlinear systems. This will yield an optimal control law for the nonlinear tracking problem, that has the same asymptotic characteristics of the linear optimal feedback laws.
 - (b) Applying the classical controller design methods to linearized system. Chapter three contains the corresponding results.
- 3- Computational evaluation of the classical vs. optimal controllers after linearization. This is done through computer simulation for the muscle model; results are given in chapter four.

- 4- The design of a feedback law for the nonlinear system with quadratic costs that combines the exact linearization with calculus of variation. This provides closed form a feedback law to the optimal linear controllers. A solution for the constrained control problem is also presented.

CHAPTER TWO

APPLYING “EXACT LINEARIZATION”

2.1 INTRODUCTION

This chapter considers the application of the “*exact linearization*” techniques to the class of multiplicative discrete-time nonlinear systems described in chapter one. By “*exact linearization*”, it is meant that the linearization process is not an approximation, in the sense of a Taylor’s series expansion. This terminology was coined for continuous time systems; it is less precise when applied to discrete time systems that arise from sampling continuous systems, where intersampling intervals are unaccessable. The controller design approaches that will be pursued in the following chapters require this exact linearization. However, it should be noted that not all systems can be linearized. Therefore, necessary and sufficient conditions on the models for the application of exact linearization will be discussed in this chapter. The application of these conditions and techniques to the muscle and load examples of chapter one will used to demonstrate our results.

This chapter is organized as follows; section 2.2 gives a brief survey of the exact linearization technique. The notations and definitions employed, are introduced in section 2.3. Section 2.4 presents the so called “*partial*

exact linearization". Full exact linearization is discussed in section 2.5. In sections 2.6 and 2.7, the exact linearization technique is applied to multiplicative systems and to the muscle model. Finally, section 2.8 contains some concluding remarks.

2.2 A BRIEF SURVEY OF THE EXACT LINEARIZATION TECHNIQUE

The idea of **exact linearization** is to take a nonlinear system and to obtain from it, by use of (nonlinear) state feedback and coordinate changes, a controllable linear system that can achieve the same outputs as the original system. By exact linearization, the nonlinear system can be controlled using linear control methods [Kailath 1980 and Kuo 1987]. To do this, we obtain the controller for the "exact linearized" system, and then apply the corresponding inverted input to the nonlinear system. That is, instead of dealing with the nonlinearity of the problem directly, we can apply the well developed tools of the linear system theory to the linearized system.

Exact linearization problems have attracted a great deal of attention, for fields continuous and discrete-time systems. Krener in [1973] studied the case of linearization by state coordinate change without feedback, for continuous-time systems without outputs. Many authors have studied linearization and approximate linearization of nonlinear continuous-time systems by feedback and state coordinate changes [Brockett 1978, Su 1982, Hunt et al 1983a,b and Krener 1984]. Hunt et al [1986] solved the same problem with outputs.

In discrete-time, the linearization problem has been recently addressed by three different groups; Monaco and Normand-Cyrot [1983-1988], Grizzle [1985-1986] and Lee et al [1986-1988]. The first approach, of Monaco and Normand-Cyrot [1983a,b], studied the immersion of the discrete time nonlinear system into a linear one, such that both systems have the same input-output behavior. They also considered [1984] the invariant distribution and the decoupling problem, the digital control of sampled nonlinear systems and their exact linearization [1985, 1986, 1988a,b]. They used a power series expansion to represent discrete time nonlinear systems, and proved necessary and sufficient conditions for their theories [1983a,b, 1984]. The second group (Grizzle [1985a,b and 1986a]), considered the decoupling problem of nonlinear systems by employing results of differential geometry. He obtained necessary and sufficient conditions for decoupling. Later in [1986b], he derived the necessary and sufficient conditions for the linearization of multi-input nonlinear systems without outputs. Jakubczyk [1987] obtained, in parallel to Grizzle, linearization conditions for the multi-input case. His work is based on the properties of distributions, defined on nonlinear state space functions. Grizzle and Kokotovic [1988] considered the problem of feedback linearization of sampled data. They proposed a multirate sampling scheme to assure the linearizability of the resulting discrete time systems.

The third school, of Lee and Marcus [1986], considered both local and approximate linearization methods for single and multi-input systems without outputs. They derived sufficient conditions for exact linearization and necessary and sufficient conditions for approximate linearization of the

multi-input multi-output case. They used Taylor's series expansion formula to prove their results. In [Lee and Marcus 1987], they also derived the necessary and sufficient conditions for local linearization of input-output systems, using a Volterra series expansion to prove their results. In [Lee et al 1987a,b] the authors solved the following four problems*, for **single-input single-output** discrete-time nonlinear systems:

- (i) linearization of a system without outputs, by state coordinate changes;
- (ii) linearization of a system without outputs, by state coordinate changes and by feedback;
- (iii) linearization of an input-output system by state coordinate changes;
- (iv) linearization of an input-output system by state coordinate changes and by feedback.

Their results were proved **locally** around operating points, using a successive composition of the nonlinear state space function. Finally they considered the linearization of the discretized (sampled) continuous systems [1987c and 1988a,b].

Our interest here is to apply the exact linearization technique to the multiplicative systems of section 1.2. Therefor, in the following sections we establish the input output linearization technique in a manner similar to that found in [Monaco and Normand-Cyrot 1987]. However, unlike earlier work, we derive **global** conditions on the nonlinear system for the application of the exact linearization for **multi-input, multi-output** case, using a somewhat different, simpler approach to obtain the required feedback law for the linearization. We define for discrete time nonlinear systems the

* For a summary of Lee et al [1987a,b] refer to appendix A.

zero dynamics that Byrnes and Isidori [1986] developed for continuous time systems. This result, linearizing only the controlled and observed parts of the nonlinear system, yields the *partial linearization* as defined in [Akhrif, 1989] (as compared to full or state linearization of systems without outputs).

The results obtained are a relatively straightforward generalization, but are needed for the application to the muscle model problems that motivate this research. A benefit of our approach is that the results are derived without the need to use the theoretical complexities of differential geometry and distributions (as in Grizzle [1986b]) or the need to carry out tedious computations (as needed in Lee and Marcus [1986, 1987]).

2.3 DEFINITIONS AND PRELIMINARIES

This section introduces the notation and definitions employed in this study of discrete time nonlinear systems.

Definition 2.1

We will consider a nonlinear discrete time system to be a five-tuple $\Sigma(X, Y, U, f, h)$, where X (the state space), Y (the output space) and U (the input space) are differential manifolds of finite dimensions N , p , and m respectively;

$$f : X \times U \rightarrow X$$

$$h : X \longrightarrow Y$$

are smooth (i.e., C^∞) maps, which describe the system as follows: at each time $k = 0, 1, \dots$; we have:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \tag{2.1}$$

with initial condition $x(0) = x_0$.

Example 2.1 (Discrete Time Linear System)

Consider the discrete time system in the previous example. Taking $f(x(k), u(k)) = Ax(k) + B u(k)$ and $h(x(k)) = c^t x(k)$, for $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $B : \mathbb{R}^m \rightarrow \mathbb{R}^N$ and $c^t : \mathbb{R}^N \rightarrow \mathbb{R}^p$ linear operators, we will indicate the corresponding discrete time linear system by $\Sigma^L(\mathbb{R}^N, \mathbb{R}^p, \mathbb{R}^m, (A, b), c^t)$.

Definition 2.2

Consider the open sets $U_1, U_2 \subseteq U$. A smooth function $\gamma : X \times U_1 \rightarrow U_2$ is said to be **u-invertible** if there exists a map denoted by $\gamma^{-1} : X \times U_2 \rightarrow U_1$ such that $\gamma^{-1}(x, \gamma(x, u)) = u, \forall x \in X$ and $u \in U_1$. It is **locally u-invertible** at (x_o, u_o) if this holds in a neighborhood $N_{x_o} \times U_{u_o}$ around (x_o, u_o) .

We will use v to indicate elements of U_2 in the following discussion, for example $v = \gamma(x, u)$.

Definition 2.3

Consider the two nonlinear systems $\Sigma_i(X, Y, U, f_i, h_i); i = 1, 2$. Σ_1 and Σ_2 are said to be **equivalent** iff $\exists$ the pair (T, γ) ; T (invertible) and γ (u-invertible) that define an **equivalence relation** according to:

$$(f_1, h_1) \rightarrow (T(f_1(T^{-1}, \gamma^{-1}), h_1(T^{-1})) \quad (2.2)$$

where:

$$\begin{aligned} f_2(z(k), v(k)) &= T(f_1(T^{-1}(z(k)), \gamma^{-1}(T^{-1}(z(k), v(k))))) \\ h_2(z(k)) &= h_1(T^{-1}(z(k))) \end{aligned} \quad (2.3)$$

Definition 2.4

Two discrete time systems are said to be **state equivalent** if there exists a smooth state coordinate change around $0 \in \mathbb{R}^N$ which transforms one into the other.

Definition 2.5

A discrete time nonlinear system of the form (2.1) is said to be **linearizable by state coordinate change** if it is state equivalent to a reachable linear system.

Definition 2.6

A discrete time nonlinear system of the form (2.1) is said to be **linearizable by state coordinate change and feedback** if there exists a smooth nonlinear feedback $u = \gamma^{-1}(x, v)$ such that the closed loop system is linearizable by state coordinate change.

To illustrate the exact linearization technique let us consider a discrete time system $\Sigma(X, Y, U, f, h)$ which can be represented as follows:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k)) \\ x(0) &= x_0 \end{aligned} \tag{2.4}$$

Fig. 2.1 The nonlinear system $\Sigma(X, Y, U, f, h)$

Now, if there exists a u-invertible function $\gamma^{-1} : X \times U \rightarrow U$, we can rewrite (2.4) as an input – output equivalent system described by:

$$\begin{aligned} x(k+1) &= f(x(k), \gamma^{-1}(x(k), v(k))) \\ v(k) &= \gamma(x(k), u(k)) \\ y(k) &= h(x(k)) \\ x(0) &= x_0 \end{aligned} \tag{2.5}$$

Fig. 2.2 The nonlinear system $\Sigma(X, Y, U, f^\gamma, h)$

We will define

$$f^\gamma(x(k), v(k)) \triangleq f(x(k), \gamma^{-1}(x(k), v(k))) : X \times V \rightarrow X$$

where $V \triangleq \gamma(X, U) \subset U$.

Let us denote the subsystem (inside dashed lines) in Figure 2.2 as $\Sigma^s(X, Y, U, f^\gamma, h)$. It is obtained from $\Sigma(X, Y, U, f, h)$ by the choice of the u -invertible $\gamma(x, u)$.

Suppose we change the coordinate of the state space X , by applying invertible map T . Let:

$$z = T(x) \tag{2.6}$$

We could rewrite $\Sigma(X, Y, U, f^\gamma, h)$ as follows:

$$\begin{aligned}
 z(k+1) &= T[f(T^{-1}(z(k)), \gamma^{-1}(T^{-1}(z(k)), v(k)))] \\
 &= T[f^\gamma(T^{-1}(z(k)), v(k))] \\
 v(k) &= \gamma(T^{-1}(z(k)), u(k)) \\
 y(k) &= h(T^{-1}(z(k))) \\
 z(0) &= T(x(0)) = T(x_0)
 \end{aligned} \tag{2.7}$$

Fig. 2.3 The nonlinear system Σ with the state map T

Once again, inside the dashed line in Fig. 2.3 is input-output equivalent system to $\Sigma^s(X, Y, U, f^\gamma, h)$

If we define

$$\begin{aligned}
 f^T(z(k), v(k)) &\triangleq T[f^\gamma(T^{-1}(z(k)), v(k))] \\
 h^T(z(k)) &\triangleq h(T^{-1}(z(k))),
 \end{aligned} \tag{2.8}$$

then we can define the system $\Sigma(X, Y, U, f^T, h^T)$ which is input-output equivalent to $\Sigma^s(X, Y, U, f^\gamma, h)$.

We say that $\Sigma(X, Y, U, f, h)$ is **feedback linearizable** if we can represent f^T, h^T as linear functions, i.e.:

$$\begin{aligned} f^T(z(k), v(k)) &= Az(k) + Bv(k) \\ h^T(z(k)) &= c^t z(k) \end{aligned} \quad (2.9)$$

That is, we have $\Sigma^L(Z, Y, V, (A, B), c^t) \equiv \Sigma(X, Y, U, f^T, h^T)$. It is input-output equivalent to $\Sigma^s(X, Y, U, f^T, h)$.

Fig. 2.4 The linearized system $\Sigma(X, Y, U, f^T, h^T)$

The nonlinear system $\Sigma(X, Y, U, f, h)$ will be **equivalent** to the linear system $\Sigma^L(Z, Y, V, (A, B), c^t)$ if it has the above representation. Note that we need the existence of invertible T and u -invertible γ .

Remark

Σ equivalent to Σ^L means also that the subsystem Σ^s is input-output equivalent to Σ^L .

In the following we will take:

$y \in Y$ is a p -dimensional output vector

$x \in X$ is an N -dimensional state vector

$z \in Z$ is an N -dimensional state vector

$u \in U$ is an m -dimensional input vector

$v \in V$ is an m -dimensional input vector

A is an $N \times N$ linear system matrix

B is an $N \times m$ linear input matrix

c^t is a $p \times N$ linear output matrix

Definition 2.7

Suppose that $\Sigma(X, Y, U, f, h)$ is equivalent to $\Sigma^L(Z, Y, V, (A, B), h^T)$ that is,

$$f^T(z(k), v(k)) = Az(k) + Bv(k) \quad (2.10)$$

but $h^T = h[T^{-1}(z(k))]$ is not necessary linear. Then the linearization is called a **state-exact linearization**. However, if h^T is linear,

$$h^T(z(k)) = c^t z(k) \quad (2.11)$$

then it is called **input-output exact linearization**.

2.4 PARTIAL EXACT LINEARIZATION

In the previous section we introduced definitions of equivalent systems, and according to definition 2.6, if we know the coordinate change T

and the feedback γ , it then possible to find out if the nonlinear system is equivalent to a linear one or not. In another words, we can determine if it is exactly linearizable or not. However, since not all nonlinear systems are exactly linearizable, then the question is: what conditions should the nonlinear system satisfy to be exactly linearizable? In particular, what are the conditions for the existence of T and γ ?

In this section, we will derive these conditions for input-output exact linearization. These results are similar to those of [Monaco and Normand-Cyrot 1987]. This work is analogous to the work of Byrnes and Isidori [1986] and that of Akhrif [1989] for continuous time systems. We will consider, the **partial exact linearization** and then the **full exact linearization** for both single-input single-output (SISO) and multi-input multi-output (MIMO) systems.

Definition 2.8

If the dimension of the linear system is of the same dimension as the original one, then the linearization is called **full exact linearization**. If the dimension of the equivalent linear system is less or equal, it is called **partial exact linearization**. For example, if only the observed and controlled part of the state space of the nonlinear system is linearized, then it is a **partial exact linearization**.

Before proceeding to the next section, we will define the transition functions of the discrete time nonlinear systems as follows:

Definition 2.9

Consider the discrete-time nonlinear systems $\Sigma(X, Y, U, f, h)$. At each time step k , we define the r -step transition functions $\phi^r : X \times U^r \rightarrow X$, (for $r = 0, 1, \dots$) from state $x(k) \in X$ to state $x(k + r)$, given the input sequence $u(k), \dots, u(k + r - 1)$ by the following recursive relation:

$$\begin{aligned} \phi^0 &\triangleq x(k) \\ \phi^r(x(k), u(k), \dots, u(k + r - 1)) &\triangleq \overbrace{f(f(\dots f(x(k), u(k)), \dots), u(k + r - 1))}^{r \text{ times}} \end{aligned} \quad (2.12)$$

For convenience we will denote ϕ^r for $\phi^r(x(k), u(k), \dots, u(k + r - 1))$. The output $y(k + r)$ can then be written as follows:

$$y(k + r) = h(\phi^r) \quad (2.13)$$

2.4.1 SISO Partial Exact Linearization

In this section, the conditions on the discrete time nonlinear systems for input output linearization are given. These conditions are proved in terms of the relative degree of the nonlinear system, which is defined as follows:

Definition 2.10 :

The system $\Sigma(\mathbb{R}^N, \mathbb{R}^p, \mathbb{R}^m, f, h)$ has a relative degree $r(k)$ at x_o , ($x_o \in \mathbb{R}^N$ if there exists a neighborhood N_{x_o} of x_o such that for all $x \in N_{x_o}$ ($\forall x_o \in \mathbb{R}^N$):

$$1. \frac{\partial h(x(k+s))}{\partial x(k+s)} \Big|_{\phi^s} \frac{\partial \phi^s}{\partial u(k)} = 0 \quad \forall s < r(k)$$

$$2. \frac{\partial h(x(k+r))}{\partial x(k+r)} \Big|_{\phi^r} \frac{\partial \phi^r}{\partial u(k)} \Big|_{x_0} \neq 0$$

Example 2.2

Consider the linear system $\Sigma^L(\mathbb{R}^N, \mathbb{R}^p, \mathbb{R}^m, (A, b), c^t)$ given in example 2.2:

$$y(k) = h(x(k)) = c^t x(k)$$

$$x(k+1) = f(x(k), u(k)) = Ax(k) + bu(k)$$

and assume that the system has a relative degree $r = 3$;

Then at $s = 1$

$$\phi^1 = Ax(k) + bu(k)$$

$$y(k+1) = h(x(k+1)) = c^t x(k+1) = c^t [Ax(k) + bu(k)]$$

$$\begin{aligned} \frac{\partial y(k+1)}{\partial u(k)} &= 0 = \frac{\partial c^t x(k+1)}{\partial x(k+1)} \Big|_{\phi^1} \frac{\partial c^t [Ax(k) + bu(k)]}{\partial u(k)} \\ &= c^t b \end{aligned}$$

Similarly at $s = 2$

$$\phi^2 = A^2 x(k) + Ab u(k) + bu(k+1)$$

$$\frac{\partial y(k+2)}{\partial u(k)} = 0 = c^t Ab$$

Finally at $r = 3$

$$\frac{\partial y(k+3)}{\partial u(k)} = c^t A^2 b \neq 0$$

The relative degree is the first time that the input $u(k)$ applied at time k appears explicitly in the output. In other words, it is the **time delay** of the system. It should be noted from the definition that:

$$y(k+s) = h(\phi^s(x, 0)) \quad \forall s < r$$

That is, $y(k+s)$ is **not** explicitly a function of any of the inputs

$$\{u(k), \dots, u(k+s-1)\} \forall s < r.$$

It should also be noted that, the relative degree is not constant; it is time varying and varies with each operating point x_o . This time varying characteristic of the relative degree is therefor different than that defined in [Monaco and Norman-Cyrot 1987].

If we test the time delay for inputs applied at $k = 0$ and if we restrict the input to be impulses, then the relative degree will be the **characteristic number** as defined by Lee et al [1987 a,b] (see appendix A), and the **relative index** as in Lee and Marcus [1987].

Theorem 2.1 :

For the single-input single-output system $\Sigma(X, Y, U, f, h)$:

1. The relative degree is the **minimum** time delay required in order to have an u-invertible feedback law. It is also the order of the equivalent linear system.
2. Σ is exactly linearizable and equivalent to Σ^L if there exists a finite relative degree r such that:

$$\text{i. } \frac{\partial h(x(k+s))}{\partial x(k+s)} \Big|_{\phi^s} \frac{\partial \phi^s}{\partial u(k)} = 0 \quad \forall s < r(k)$$

$$\text{ii. } \frac{\partial h}{\partial x} \Big|_{\phi^r} \frac{\partial \phi^r}{\partial u(k)} \Big|_{x_o} \neq 0$$

Furthermore the linearizing equivalent relation is:

$$\begin{aligned} z_1(k) &= T_1(x(k)) = h[x(k)] \\ z_2(k) &= T_2(x(k)) = h[f(x(k), u(k))] = h(\phi^1) \\ &\vdots \\ z_r(k) &= T_r(x(k)) = h(\phi^{r-1}) \end{aligned}$$

while the feedback law is:

$$v(k) = h(\phi^r).$$

Proof :

We prove this theorem by introducing an algorithm to obtain the feedback linearization law. This algorithm depends on getting the relative degree of the nonlinear system.

To illustrate the importance of the relative degree for input-output exact linearization, let us consider the output $y(k)$ at different time steps k :

$$\text{at time } k: \quad y(k) = h(x(k))$$

$$\text{at time } k+1: \quad y(k+1) = h(f(x(k), u(k)))$$

$$\frac{\partial y(k+1)}{\partial u(k)} = \frac{\partial h(x(k+1))}{\partial x(k+1)} \Big|_{f(x(k), u(k))} \frac{\partial f(x(k), u(k))}{\partial u(k)} \quad (2.14)$$

If the last equation is nonzero then $r = 1$ and there is an u-invertible feedback law $v : \mathbf{R}^{N+1} \rightarrow \mathbf{R}$:

$$v(k) = h(f(x(k), u(k)))$$

such that the new equivalent linear system will become:

$$y(k+1) = v(k). \quad (2.15)$$

But if the derivative (2.14) is zero vector:

$$\frac{\partial h(x(k+1))}{\partial x(k+1)} \Big|_{f(x(k), u(k))} \frac{\partial f(x(k), u(k))}{\partial u(k)} = 0$$

then we test for $r = 2$; i.e. for

$$y(k+2) = h(f(f(x(k), u(k)))) = h(\phi^2)$$

and we need:

$$\frac{\partial y(k+2)}{\partial u(k)} = \frac{\partial h(x(k+2))}{\partial x(k+2)} \Big|_{\phi^2} \frac{\partial \phi^2}{\partial u(k)} \neq 0 \quad (2.16)$$

Then the invertible feedback linearization law is:

$$v(k) = h(\phi^2)$$

and the equivalent linear system is:

$$y(k+2) = v(k) \quad (2.17)$$

If (2.16) does not hold, we then consider $r = 3$, and so forth, until the relative degree is obtained.

If the system has a finite relative degree r then

$$\frac{\partial y(k+r)}{\partial u(k)} = \frac{\partial h(x(k+r))}{\partial x(k+r)} \Big|_{\phi^r} \frac{\partial \phi^r}{\partial u(k)} \neq 0 \quad (2.18)$$

and the system Σ in (2.1), is equivalent to the following linear system:

$$y(k+r) = v(k)$$

with the linearizing feedback law:

$$v(k) = h(\phi^r) \quad (2.19)$$

and this proves (1.) of the theorem, while the second condition comes from the definition of the relative degree.

According to the previous proof, we obtain the state coordinate change directly while getting the relative degree, without the need to solve a system of partial differential equations as in the continuous time exact linearization. This is a major advantage for the discrete time over the continuous time exact linearization.

In Akhrif [1989], input-output continuous time exact linearization was referred to as *partial linearization* problem because the r functions

$$\{h, h(\phi^1), \dots, h(\phi^{r-1})\}$$

can be used to define the r -states coordinate transformation

$$T(x) = [z_1, z_2, \dots, z_r]^t.$$

If $r < N$ then the $(N-r)$ transformation functions $T_{r+1}, \dots, T_N$ can be found such that $T = [T_1, \dots, T_N]^t$ is invertible. Moreover it is possible to choose these $N-r$ functions such that the remaining coordinates z_i , $r+1 \leq i \leq N$ are not explicitly functions in the input $u(\cdot)$; i.e.:

$$\frac{\partial z_i(k)}{\partial u(k)} = 0 \quad \forall k \text{ and } r+1 \leq i \leq N$$

Therefore,

$$z_i(k+1) = T_i(x(k+1)) = T_i[f(x(k), u(k))] = z_{i+1}(k) \quad \forall i < r-1$$

$$z_r(k+1) = v(k)$$

(2.20)

and since the remaining coordinates are not explicit functions of u , then we can write:

$$z_i(k+1) = \psi_i(z(k)) \quad r+1 \leq i \leq N \quad (2.21)$$

However, it should be noted that the previous $N - r$ states are those internal states unobserved by the output function $h(x)$. These states are called the **zero dynamics** by Byrnes and Isidori [1986] for continuous time systems. They play a role similar to that of the zeros of the transfer function in the linear systems, corresponding to the internal behavior of the system. We call these states the **r-step zero dynamics** because, when the input is forced to be zero then after $r + 1$ time steps, the output is driven to zero. This implies that the linearized part of the system (the first r -states) is zero. Thus, the internal behavior will be governed by the remaining $N - r$ states that satisfy:

$$z_i(k+1) = \psi_i(0, z_j(k)) \quad r+1 \leq i, j \leq N$$

A more complete study of the zero dynamics of discrete time nonlinear systems is given in Monaco and Normand-Cyrot [1987]. For our use, it is important when choosing the remaining coordinate maps $(T_{r+1}, \dots, T_N)$ to have at least asymptotically stable internal states. It should be noted also that exact linearization is responsible only for getting the equivalent linear system. The zero dynamics and their stability are not explicitly considered. Note that only the controllable - observable states are linearized. In the literature, full exact linearization for discrete-time systems is studied; i.e., the whole state space is assumed to be controllable and observable. The

previous theorem thus involves weaker conditions for linearization, for the input output problem.

Example 2.3 :

Consider the system:

$$\begin{aligned} y(k) &= x_3 \\ x(k+1) &= \begin{pmatrix} x_2^2(k) \\ u(k) \\ (x_1(k) + x_2(k))^3 \\ x_4^3(k) \end{pmatrix} \end{aligned}$$

Testing for the relative degree, we get:

$$y(k) = h(x(k)) = x_3$$

$$y(k+1) = h(\phi^1) = (x_1(k) + x_2(k))^3$$

$$y(k+2) = h(\phi^2) = (x_2^2(k) + u(k))^3$$

It is obvious that $r = 2$. The system is linearizable by:

$$v(k) = (x_2^2(k) + u(k))^3 \longleftrightarrow u(k) = \sqrt[3]{v(k)} - x_2^2(k)$$

and

$$z_1(k) = x_3(k)$$

$$z_2(k) = (x_1(k) + x_2(k))^3$$

where:

$$z_3(k) = x_1(k)$$

$$z_4(k) = x_4(k)$$

This will lead to the input-output linear system:

$$y(k) = z_1(k)$$

$$z_1(k+1) = z_2(k)$$

$$z_2(k+1) = v(k)$$

and

$$z_3(k+1) = \sqrt[3]{z_2(k)} - z_3^2(k)$$

$$z_4(k+1) = z_4^3(k)$$

Finally the inverse of the coordinate mapping is:

$$x_1(k) = z_3(k)$$

$$x_2(k) = z_1(k)$$

$$x_3(k) = \sqrt[3]{z_2(k)} - z_3^2(k)$$

$$x_4(k) = z_4(k).$$

2.4.2 MIMO Partial Exact Linearization

Let us consider the nonlinear system $\Sigma(\mathbb{R}^N, \mathbb{R}^m, \mathbb{R}^m, f, h)$, i.e., $x \in \mathbb{R}^N$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. We will consider the case $p = m$ (number of inputs = number of outputs), and let us define the relative degree vector $\{r_1, \dots, r_m\}$;

Definition 2.11 :

A multivariable nonlinear system $\Sigma(X, Y, U, f, h)$ has a relative degree vector $\{r_1, \dots, r_m\}$ at x_o if $\forall x$ in the neighborhood N_{x_o} :

$$1. \frac{\partial h_i(x(k+s))}{\partial x(k+s)} \Big|_{\phi^s} \frac{\partial \phi^s}{\partial u_j} = 0 \quad \forall s < r_i \text{ and } \forall 1 \leq i, j \leq m$$

2. the linearization matrix:

$$\begin{pmatrix} \frac{\partial h_1(x(k+r_1))}{\partial x(k+r_1)} \Big|_{\phi^{r_1}} \frac{\partial \phi^{r_1}}{\partial u_1(k)} & \dots & \frac{\partial h_1(x(k+r_1))}{\partial x(k+r_1)} \Big|_{\phi^{r_1}} \frac{\partial \phi^{r_1}}{\partial u_m(k)} \\ \vdots & & \vdots \\ \frac{\partial h_m(x(k+r_m))}{\partial x(k+r_m)} \Big|_{\phi^{r_m}} \frac{\partial \phi^{r_m}}{\partial u_1(k)} & \dots & \frac{\partial h_m(x(k+r_m))}{\partial x(k+r_m)} \Big|_{\phi^{r_m}} \frac{\partial \phi^{r_m}}{\partial u_m(k)} \end{pmatrix}$$

is nonsingular.

It is obvious that r_i is the order of the i 's decoupled linear subsystem and the resulting equivalent linear system is:

$$\begin{pmatrix} y_1(k + r_1) \\ \vdots \\ y_m(k + r_m) \end{pmatrix} = \begin{pmatrix} v_1(k) \\ \vdots \\ v_m(k) \end{pmatrix} \quad (2.22)$$

Theorem 2.2:

The multi-input multi-output system $\Sigma(\mathbb{R}^N, \mathbb{R}^m, \mathbb{R}^m, f, h)$ is linearizable if there exist a **relative degree vector** $\{r_1, \dots, r_m\}$, where r_i is the minimum time delay required in order to have an u-invertible feedback law for each output y_i . It is also the order of the corresponding equivalent linear system.

Proof :

The proof for this theorem is similar to the SISO problem. Considering each output y_i individually and by obtain the relative degree for each output with respect to each input u_j , $j = 1 \dots, m$. Nonsingularity of the linearization matrix, ensures that there exists at least one input for each output to be used for the linearizing u-invertible feedback law.

Each subsystem in equation (2.22) can be represented in an equivalent

state space Brunovsky form [1970]:

$$\begin{aligned}\Sigma^L : y(k) &= \begin{pmatrix} c_{r_1} & & 0 \\ & \ddots & \\ 0 & & c_{r_m} \end{pmatrix} z(k) \\ z(k+1) &= \begin{pmatrix} A_{r_1} & & 0 \\ & \ddots & \\ 0 & & A_{r_m} \end{pmatrix} z(k) + \begin{pmatrix} b_{r_1} & & 0 \\ & \ddots & \\ 0 & & b_{r_m} \end{pmatrix} v(k)\end{aligned}\quad (2.23)$$

$$\text{where } A_{r_i} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix} \in \mathbb{R}^{r_i \times r_i} \quad b_{r_i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{r_i}$$

$$\text{and } c_{r_i} = (1 \ 0 \dots 0) \in \mathbb{R}^{1 \times r_i}$$

If $r = r_1 + \dots + r_m < N$ then it is always possible to find $N - r$ functions $T_{r+1}, \dots, T_N$ such that $T = [T_1, \dots, T_N]^t$ is invertible, and to choose these $N - r$ functions in a way that the rest $N - r$ states are not functions in v :

$$\begin{aligned}z(k+1) &= Az(k) + Bv(k) \\ z_i(k+1) &= \psi_i(z(k)) \quad \forall i = r+1, \dots, N.\end{aligned}\quad (2.24)$$

2.5. FULL (STATE) EXACT LINEARIZATION

In this section, we will study input-output exact linearization, by assuming the system is state linearizable [Lee et al 87a,b] and the test for conditions on the output function $h(\cdot)$. This will give input-output (but full) linearization, where the whole state space is linearized. This assures the

stability of the overall system. We will examine some special linearization problems that involve full linearization.

Definition 2.12 (Nonlinear difference equation):

The single-input single-output nonlinear discrete time system may also be represented by the three tuple $\Sigma_d(Y, U, d)$, where Y (the output space) $\in \mathbb{R}$ and U (the input space) $\in \mathbb{R}$ and the **nonlinear difference equation**: $d : Y^{N+1} \times U^{m+1} \rightarrow Y$ is a smooth map which describe the system as follows:

$$y(k+1) = d(\underline{y}(k), \underline{u}(k)) \quad (2.25)$$

where

$$\underline{y}(k) = [y(k), y(k-1), \dots, y(k-N)]$$

$$\underline{u}(k) = [u(k), u(k-1), \dots, u(k-m)]$$

Lemma 2.1 :

If the single input single output nonlinear systems Σ_d is represented by its difference equation then the relative degree is defined to be the smallest integer r such that:

1. $\frac{\partial d(\underline{y}(k), \underline{u}(k))}{\partial \underline{u}(k-s)} = 0 \quad s \leq r$
2. $\frac{\partial d(\underline{y}(k), \underline{u}(k))}{\partial \underline{u}(k-r+1)} \neq 0$

Proof :

The relative degree, as defined in the previous section, is the first instant of time the output is affected by an input applied at time k . In another words, the input applied at time k will affect the output at $k + r$, or input applied at time $k + 1 - r$ will affect the output at time $k + 1$; this proves (2). Furthermore, due to this input output delay, the output at time $k + 1$ will not be affected by any input applied in the previous r -time steps to that moment, which prove (1).

Lemma 2.2 :

If the single input single output nonlinear systems Σ_d is represented by its difference equation

$$y(k + 1) = d(\underline{y}(k), \underline{u}(k))$$

where

$$\underline{y}(k) = [y(k), y(k - 1), \dots, y(k - N)]$$

$$\underline{u}(k) = [u(k), u(k - 1), \dots, u(k - m)]$$

and has a relative degree r , then this system is linearizable with the following feedback:

$$v(k - r + 1) = d(\underline{y}(k), \underline{u}(k)) \quad (2.26)$$

Proof :

- a) Let $r \leq N$ be the relative degree. Then $\frac{\partial d}{\partial u(k-r+1)} \neq 0$ and $\frac{\partial d}{\partial u(k-s)} = 0 \forall s \leq r$ and d is $u(k-r+1)$ -invertible. That is, there exists a function $\tilde{d}: \mathbb{R}^{N+1} \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}$ such that:

$$u(k-r+1) = \tilde{d}(\underline{y}(k), y(k+1), u(k-r), u(k-r-1), \dots, u(k-m)) \quad (2.27)$$

with the feedback law:

$$v(k-r+1) = d(\underline{y}(k), \underline{u}(k)) \quad (2.28)$$

the equivalent system is:

$$y(k) = v(k-r) \quad (2.29)$$

- b) Assume that Σ_d is exactly linearizable with the feedback law:

$$v(k-r+1) = d(\underline{y}(k), \underline{u}(k)) \quad (2.30)$$

Then d is $u(k-r+1)$ -invertible and

$$\frac{\partial d}{\partial u(k-r+1)} \neq 0$$

Since both systems Σ_d and the equivalent linear system have the same input output delay, then

$$\frac{\partial d}{\partial u(k-s)} = 0 \quad \forall s \leq r$$

In the following discussion we consider the following two types of systems:

1. Discrete time nonlinear systems with output $\Sigma^o(\mathbb{R}^N, \mathbb{R}^p, \mathbb{R}^m, f, h)$ as in (2.1), and
2. The same system without output, i.e.,

$$\Sigma^i : x(k+1) = f(x(k), u(k)) \quad (2.31)$$

Theorem 2.3 (parallel to [Lee et al 87a,b] Theorem 8)

If the single input discrete-time system without output Σ^i is linearizable by state coordinate mapping $z = T(x)$ and the feedback $u = \gamma^{-1}(x, v)$, then the single-output system Σ^o is also linearizable iff $\frac{\partial h \circ T^{-1}}{\partial v(k-r)} \neq 0$, where r is the relative degree, $v(k)$ is the input of the state linearizable system.

Proof

Consider the discrete time nonlinear system:

$$\begin{aligned} \Sigma^o : y(k) &= h(x(k)) \\ x(k+1) &= f(x(k), u(k)) \end{aligned} \quad (2.32)$$

and let:

$$\Sigma^i : x(k+1) = f(x(k), u(k)) \quad (2.33)$$

be linearizable by $T(x)$ and $u = \gamma^{-1}(x, v)$. The equivalent linear system is

$$z(k+1) = Az(k) + bv(k) \quad (2.34)$$

where

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

That is

$$\begin{aligned}
 z_1(k) &= v(k - N) \\
 z_2(k) &= v(k - N + 1) \\
 &\vdots \\
 z_n(k) &= v(k - 1)
 \end{aligned} \tag{2.35}$$

or

$$z_i(k) = v(k - n + i - 1)$$

Therefor, substituting for $x(k)$ by $T^{-1}(z(k))$ in $y(k) = h(x(k))$ will give:

$$y(k) = h(T^{-1}(z(k)))$$

then substituting $v(k - n + i - 1)$ for each $z_i(k)$ for all i , we get the difference equation:

$$y(k) = h \circ T^{-1}(v(k - n), v(k - n + 1), \dots, v(k - 1)) \tag{2.36}$$

Now let r be the relative degree, Form Lemma 2.1, if the system $\frac{\partial h \circ T^{-1}}{\partial v(k-r)} \neq 0$ then $h \circ T^{-1}$ is u-invertible and $y(k) = w(k - r)$ is the equivalent linear system.

Theorem 2.4

If the multi-input discrete-time system without output Σ^i is linearizable by the state coordinate mapping $z = T(x)$ and the u-invertible feedback $u = \gamma(x, v)$, then the single output system Σ^o is also linearizable iff $\exists$ at least one input $u_i \in \mathbb{R}^m$ has a corresponding relative degree r_i such that $\frac{\partial h \circ T^{-1}}{\partial v_{r_i}(k-s)} \neq 0$, for some finite integer s .

Proof

Consider the discrete time nonlinear system:

$$\begin{aligned}\Sigma^o : y(k) &= h(x(k)) \\ x(k+1) &= f(x(k), u(k))\end{aligned}\tag{2.37}$$

and let:

$$\Sigma^i : x(k+1) = f(x(k), u(k))\tag{2.38}$$

be linearizable by $T(x)$ and $u = \gamma^{-1}(x, v)$. The equivalent linear system is

$$z(k+1) = Az(k) + bv(k)\tag{2.39}$$

where A and B are in the Brunovsky canonical forms. Each subsystem is single-input linear system of dimension r_i . The output $y(k)$ can be obtained as a function in the new linear inputs as follows:

$$y(k) = h(x(k)) = h \circ T^{-1}(z_1, \dots, z_m)\tag{2.40}$$

and $z_i = \begin{pmatrix} z_{i1} \\ \vdots \\ z_{ir_i} \end{pmatrix} \in \mathbb{R}^{r_i}$. As in the previous theorem, we can substitute for each z_{ij} by $v_{r_i}(k - r_i + j - 1)$ in $y(k)$. Then

$$y(k) = h(x(k)) = h \circ T^{-1}(v_{r_1}(k - r_1), \dots, v_{r_m}(k - 1))\tag{2.41}$$

and the first instant any of these inputs affects the output corresponds to r_i :

$$\frac{\partial y(k)}{\partial v_{r_i}(k - s)} \neq 0.\tag{2.42}$$

Similar results for multi-input, multi-output systems follows from Theorems 2.5 and 2.6.

2.6 APPLICATION OF EXACT LINEARIZATION TO MULTIPLICATIVE SYSTEMS

In this section we apply discrete-time exact linearization to the case of multiplicative systems described in (1.9)-(1.12) and shown in Fig. 2.5.

Fig. 2.5 General Multiplicative Systems

$$\begin{aligned} y(k) &= h(x(k)) \\ &= h_1(X_1(k)) h_2(X_2(k)) \dots h_\beta(X_\beta(k)) \end{aligned} \quad (2.43)$$

$$X_i(k+1) = f_i(X_i(k), u(k)) \quad i = 1, 2, \dots, \beta \quad (2.44)$$

$$y \in \mathbf{R}$$

$$X_i \in \mathbf{R}^{n_i}$$

$$u \in \mathbf{R}^m$$

where $n_1 + n_2 + \dots + n_\beta = N$

Lemma 2.4 :

This family of multiplicative systems is **invariant under subsystems linearization**. That is, each subsystem individually is exactly linearizable then the whole system is still multiplicative.

Proof :

Consider the nonlinear multiplicative system described by (2.43)-(2.44). Assume each subsystem is linearized by:

$$\begin{aligned} T_i : \mathbb{R}^{n_i} &\rightarrow \mathbb{R}^{r_i}; Z_i = T_i(X_i) \\ \gamma : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^m; v = \gamma(X, u) \end{aligned} \quad (2.45)$$

Then

$$\begin{aligned} y(k) &= (h_1 \circ T_1^{-1}(Z_1)) (h_2 \circ T_2^{-1}(Z_2)) \dots (h_\beta \circ T_\beta^{-1}(Z_\beta)) \\ y(k) &= \tilde{h}_1(Z_1) \tilde{h}_2(Z_2) \dots \tilde{h}_\beta(Z_\beta) \end{aligned} \quad (2.46)$$

Lemma 2.5 :

If the following conditions are satisfied, then the multiplicative system described by (2.43) and (2.44) is exactly linearizable:

a) the subsystems are completely decoupled, i.e.,

$$X_i(k+1) = f_i(X_i(k), u_i(k)) \quad \forall i = 1, \dots, \beta$$

where

$$X_i \in \mathbb{R}^{n_i}$$

$$u_i \in \mathbb{R}^{m_i}$$

and $n_1 + \dots + n_\beta = N$ and $m_1 + \dots + m_\beta = m$ (however for simplicity assume $m_1 = m_2 = \dots = m_\beta = 1$).

- b) each subsystem is linearizable; i.e., there exists a state coordinate change and a feedback:

$$T_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{r_i}; Z_i = T_i(X_i)$$

$$\gamma_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i}; v_i = \gamma_i(X_i, u_i)$$

Proof :

Since each subsystem is linearizable and decoupled from the other subsystems, then from theorem 2.5 it is only required to find one inputs v_i and a finite integer s such that

$$\frac{\partial y(k)}{\partial v_i(k-s)} = \frac{\partial h \circ T^{-1}}{\partial v_i(k-s)} \neq 0. \quad (2.47)$$

However,

$$y(k) = (h_1 \circ T_1^{-1}(Z_1)) (h_2 \circ T_2^{-1}(Z_2)) \dots (h_\beta \circ T_\beta^{-1}(Z_\beta))$$

$$y(k) = \tilde{h}_1(Z_1) \tilde{h}_2(Z_2) \dots \tilde{h}_\beta(Z_\beta) \quad (2.48)$$

Since each subsystem is in Brunovsky canonical form, from (2.35) we can express each Z_i in (2.48) in terms of v_i :

$$y(k) = \prod_{j=1}^{\beta} \tilde{h}_j(v_j(k-n_j) \dots v_j(k-1)) \quad (2.49)$$

Let $\underline{v}_j = \{v_j(k-n_j), \dots, v_j(k-1)\}$. Then

$$\frac{\partial y(k)}{\partial v_i(k-s)} = \left[\prod_{j=1}^{i-1} \tilde{h}_j(\underline{v}_j) \right] \left[\prod_{j=i+1}^{\beta} \tilde{h}_j(\underline{v}_j) \right] \left[\frac{\partial h_i(T_i^{-1})}{\partial T_i^{-1}} \right]^t \left[\frac{\partial T_i^{-1}}{\partial Z_{n_i-s+1}(k)} \Big|_{v_i(k-s)} \right]$$

Note that:

- 1- The first and the second terms are non-zero scalars; otherwise $y(k) = 0 \forall k$.

2- The third term is a vector

$$\left[\frac{\partial h_i(T_i^{-1})}{\partial T_i^{-1}}\right]^t = \left[\frac{\partial h_i}{\partial X_{i1}} \cdots \frac{\partial h_i}{\partial X_{in_i}}\right] \Big|_{X_i = T_i^{-1}(Z_i)}$$

and $X_i = \{X_{i1}, \dots, X_{in_i}\}$.

3- The fourth term is a vector:

$$\left[\frac{\partial T_i^{-1}}{\partial Z_{n_i-s+1}(k)}\right] = \left[\frac{\partial T_{i1}}{\partial Z_{n_i-s+1}} \cdots \frac{\partial T_{in_i}}{\partial Z_{n_i-s+1}}\right] \Big|_{z_{n_i-s+1} = v_i(k-s)}$$

where $T_i = \{T_{i1}, \dots, T_{in_i}\}$. However, there exists integer values $i \in \{1, \dots, \beta\}$ such that:

$$\frac{\partial h_i}{\partial X_i} \neq 0$$

Otherwise, no states will be observable. This means that

$$\frac{\partial h_i}{\partial X_{ij}} \neq 0$$

Furthermore, for those X_{ij} states there is also exists an integer s_i such that:

$$\frac{\partial h_i}{\partial X_{ij}} \Big|_{z_{n_i-s+1}} \frac{\partial T_{is_i}^{-1}}{\partial Z_{n_i-s+1}} \neq 0$$

for some finite values s_i . Now let $s = \min\{s_i\}$ for all i described above then

$$\frac{\partial y}{\partial v_i(k-s)} \neq 0 \quad (2.50)$$

2.7 APPLICATION OF EXACT LINEARIZATION TO THE MUSCLE MODELS

In this section the discrete-time exact linearization will be applied to both models described in chapter one: the single-muscle and the two muscles models.

2.7.1 Single-Muscle Model Without Load

The muscle model (1.20) and (1.21) can be represented in the following difference equation:

$$y(k) = g(k) \left[a_1(\phi) \frac{y(k-1)}{g(k-1)} + a_2(\phi) \frac{y(k-2)}{g(k-2)} + bu(k-1) \right] \quad (2.51)$$

where $g(k) = LT(k) FV(k)$

a) According to Lemma 2.2, this system is linearizable by the feedback

$$v(k-1) = g(k) \left[a_1(\phi) \frac{y(k-1)}{g(k-1)} + a_2(\phi) \frac{y(k-2)}{g(k-2)} + bu(k-1) \right] \quad (2.52)$$

if b and $g(k) \neq 0 \quad \forall k$.

b) Another way to linearize this muscle model is to find a feedback law illustrated by Figure 2.6:

$$u(k) = \alpha(x, \phi) + \beta(x, \phi, v(k)) \quad (2.53)$$

and a state map;

$$z(k) = T(x(k), \phi(k)), \quad (2.54)$$

Fig. 2.6 Linearized Muscle Model

such that the muscle model;

$$\begin{aligned}
 \Sigma_1 : y(k) &= c^t x(k) \, h_2(\phi(k)) \, h_3(\phi(k), X_3(k)) \\
 X_1(k) &= \begin{pmatrix} 0 & 1 \\ a_2(\phi) & a_1(\phi) \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u(k-1) \\
 X_3(k) &= \phi(k-1)
 \end{aligned} \tag{2.55}$$

is equivalent to the linear system:

$$\begin{aligned}
 \Sigma_2 : y(k) &= \hat{c}^t z(k) \\
 z(k) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(k-1) \\ z_2(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(k-1).
 \end{aligned} \tag{2.56}$$

Lemma 2.5 :

The muscle model Σ_1 is equivalent to the linear system Σ_2 if and only if:

$$\left. \begin{aligned} h_2(\phi) &\neq 0 \\ h_3(\phi, X_3) &\neq 0 \end{aligned} \right\} \quad \forall k = 0, \dots.$$

Moreover, if we define:

$$g(k) = h_2(\phi(k)) \ h_3(\phi(k), \phi(k-1)) \quad (2.57)$$

and

$$\underline{a}^t = (a_2(\phi) \ a_1(\phi)) \quad (2.58)$$

then the transformation can be made with the state map:

$$z(k) = \begin{pmatrix} g(k-1) & 0 \\ 0 & g(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \quad (2.59)$$

and the feedback law is

$$u(k-1) = -\frac{\langle \underline{a}, x(k-1) \rangle}{b} + \frac{1}{bg(k)}v(k-1), \quad (2.60)$$

That is,

$$\alpha(\cdot) = -\frac{\langle \underline{a}, x \rangle}{b}$$

$$\beta(\cdot) = \frac{1}{bg(k)}v(k-1)$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Proof :

Given in appendix B.

2.7.2 Single-Muscle and known, Dynamic External Load

In this section the model for single muscle with a second order discrete time load (2.29) will be considered. The dynamic of the single-muscle equation is described by:

$$y(k) = g(k)[a_1(\phi)\frac{y(k-1)}{g(k-1)} + a_2(\phi)\frac{y(k-2)}{g(k-2)} + b u(k-1)] \quad (2.61)$$

For application convenience we will change our notation for the output $y(k)$ to $f(k)$ to indicate the output force:

$$f(k) = g(k)[a_1(\phi)\frac{f(k-1)}{g(k-1)} + a_2(\phi)\frac{f(k-2)}{g(k-2)} + bu(k-1)] \quad (2.62)$$

The second order load is:

$$\phi(k+1) = l_1\phi(k) + l_2\phi(k-1) + l_3f(k) \quad (2.63)$$

Solving both equations and then incrementing k one step, we get:

$$\begin{aligned} \phi(k+2) &= l_1\phi(k+1) + l_2\phi(k) \\ &+ l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)] \end{aligned} \quad (2.64)$$

where

$$O(k+1) \triangleq \frac{\phi(k+1) - l_1\phi(k) - l_2\phi(k-1)}{l_3} \quad (2.65)$$

From Lemma 2.2, this system is linearizable by the feedback law:

$$\begin{aligned} v(k) &= l_1\phi(k+1) + l_2\phi(k) \\ &+ l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)] \end{aligned} \quad (2.66)$$

if b , l_3 and g all are non zero, and the equivalent linear system is:

$$\phi(k+2) = v(k) \quad (2.67)$$

Finally the inverse linearization is:

$$u(k) = \frac{v(k) - l_1\phi(k+1) - l_2\phi(k)}{l_3bg(k+1)} - [a_1(\phi)\frac{O(k+1)}{bg(k)} + a_2(\phi)\frac{O(k)}{bg(k-1)}] \quad (2.68)$$

2.7.3 Two-Muscle Model

In this section we will apply the exact linearization technique to the two-muscles model. From the previous chapter the difference equations for the two muscles (1.30)-(1.34) can be modeled as follows:

the flexors difference equation is:

$$F_f(k) = g_f(k)[a_{1f}\frac{F_f(k-1)}{g_f(k-1)} + a_{2f}\frac{F_f(k-2)}{g_f(k-2)} + p_f(u(k-1))] \quad (2.69)$$

and the extensor difference equation is:

$$F_e(k) = g_e(k)[a_{1e}\frac{F_e(k-1)}{g_e(k-1)} + a_{2e}\frac{F_e(k-2)}{g_e(k-2)} + p_e(u(k-1))] \quad (2.70)$$

We are going to assume that a_{if} and a_{ie} are constants; furthermore, let $g(k) = LT(k) FV(k)$. Finally we can assume the following second order discrete-time load equation:

$$\phi(k+1) = l_1\phi(k) + l_2\phi(k-1) + l_3(F_f(k) - F_e(k)) \quad (2.71)$$

Substituting (2.69) and (2.70) in (2.71) will yield:

$$\begin{aligned}\phi(k+1) = & l_1\phi(k) + l_2\phi(k-1) + l_3g_f(k)\left[a_{1f}\frac{F_f(k-1)}{g_f(k-1)} + a_{2f}\frac{F_f(k-2)}{g_f(k-2)}\right] \\ & - l_3g_e(k)\left[a_{1e}\frac{F_e(k-1)}{g_e(k-1)} + a_{2e}\frac{F_e(k-2)}{g_e(k-2)}\right] \\ & + l_3\{g_f(k)p_f(u(k-1)) - g_e(k)p_e(u(k-1))\}\end{aligned}\quad (2.72)$$

Incrementing this equation one time step we get:

$$\begin{aligned}\phi(k+2) = & l_1\phi(k+1) + l_2\phi(k) \\ & + l_3g_f(k+1)\left[a_{1f}\frac{F_f(k)}{g_f(k)} + a_{2f}\frac{F_f(k-1)}{g_f(k-1)}\right] \\ & - l_3g_e(k+1)\left[a_{1e}\frac{F_e(k)}{g_e(k)} + a_{2e}\frac{F_e(k-1)}{g_e(k-1)}\right] \\ & + l_3\{g_f(k+1)p_f(u(k)) - g_e(k+1)p_e(u(k))\}\end{aligned}\quad (2.73)$$

and the feedback will be:

$$\begin{aligned}v(k) = & l_1\phi(k+1) + l_2\phi(k) \\ & + l_3g_f(k+1)\left[a_{1f}\frac{F_f(k)}{g_f(k)} + a_{2f}\frac{F_f(k-1)}{g_f(k-1)}\right] \\ & - l_3g_e(k+1)\left[a_{1e}\frac{F_e(k)}{g_e(k)} + a_{2e}\frac{F_e(k-1)}{g_e(k-1)}\right] \\ & + l_3\{g_f(k+1)p_f(u(k)) - g_e(k+1)p_e(u(k))\}\end{aligned}\quad (2.74)$$

Finally this feedback is u-invertible iff:

$$\frac{\partial v(k)}{\partial u(k)} \neq 0$$

That is,

$$g_f(k+1)\frac{\partial p_f(u(k))}{\partial u(k)} \neq g_e(k+1)\frac{\partial p_e(u(k))}{\partial u(k)}\quad (2.75)$$

and the equivalent linear system is:

$$\phi(k+2) = v(k).\quad (2.76)$$

2.8 CONCLUSIONS

In this chapter the input - output exact linearization technique for discrete - time systems was introduced and discussed. First, partial exact linearization was introduced by Theorems 2.1 and 2.2. Then full state linearization was solved in Theorems 2.3 and 2.4. The technique was then applied to the muscle models described in chapter one. Necessary and sufficient conditions for linearization were given and proved. In the following chapters, the design of controllers based upon exact linearized systems will be discussed and applied to the examples systems described above.

CHAPTER THREE

CONTROL DESIGN FOR THE EXACTLY LINEARIZED SYSTEM

3.1 INTRODUCTION

In chapter two, the exact linearization technique for discrete time nonlinear systems was developed. Finding the equivalent linear system is a first step towards the nonlinear controller design. The second step is to design this controller in the linear space. Once the feedback law is obtained, the third step is to map back this control law to the original nonlinear space (inverse linearization). In this chapter, we consider the design of controllers for the linearized system, using linear quadratic and classical controller design techniques.

This chapter is organized as follow. Section 3.2 contains development of the linearized quadratic controller; in this section we propose an objective function that, when applying the exact linearization, results in a linear quadratic control problem. Finally, in section 3.3, a controller is designed based on the classical control methods.

3.2 Linearized Quadratic Problem

In this section, we design controllers for the discrete time nonlinear systems, by applying the available results of linear quadratic control theory. This will allow us to exploit the advantages of this type of control, such as **asymptotic stability, infinite time horizons solutions, uniqueness and existence properties, etc..** Since such optimal control laws are obtained by minimizing an appropriate quadratic cost objective function, in this chapter we must study the effects of linearization on the original objective function.

In this section we are going to present the problem formulation, and illustrate how the exact linearization technique interacts with the objective function. In section 3.2.2, an alternative objective function is proposed, with its corresponding problem formulation. The nonlinear optimal control law is obtained in section 3.2.3. Section 3.2.4 applies some results from linear quadratic theory to the nonlinear systems. The quadratic solution is applied to class of the multiplicative systems in section 3.2.5. In section 3.2.6, the muscle model is used as example.

3.2.1. Problem Formulation

To study the effects of exact linearization on the objective function, and to see how the nonlinearities of the dynamics are moved to the quadratic

objective function, let us consider the following tracking control problem:

$$\begin{aligned}
 J(u) = \min_u & \left[\frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \right. \\
 & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\
 & \left. + u^t(k) R(k) u(k) \right] \quad (3.1)
 \end{aligned}$$

subject to the dynamic constraints of the nonlinear discrete time system given in (2.1):

$$\begin{aligned}
 \Sigma : \quad y(k) &= h(x(k)) \\
 x(k+1) &= f(x(k), u(k)) \quad (3.2)
 \end{aligned}$$

where $x(0) = x_0$ is given. Let us assume that the nonlinear discrete time system (3.2) is exactly linearizable, with the following mappings:

$$\begin{aligned}
 z(k) &= T(x(k)) \\
 v(k) &= \gamma(x(k), u(k)) \quad (3.3)
 \end{aligned}$$

When substituting for $u(k)$ in (3.1), using (3.3), we obtain the following tracking control problem:

$$\begin{aligned}
 J(u) = \min_u & \left[\frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \right. \\
 & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\
 & \left. + \gamma^{-1}(T^{-1}(z(k)), v(k))^t R(k) \gamma^{-1}(T^{-1}(z(k)), v(k)) \right] \quad (3.4)
 \end{aligned}$$

under the linear dynamic constraints:

$$\begin{aligned}
 \Sigma^L : \quad y(k) &= c^t z(k) \\
 z(k+1) &= Az(k) + Bv(k) \quad (3.5)
 \end{aligned}$$

Thus the new problem becomes quadratic in the error, but it is nonlinear in the input $v(k)$ and the new linear states $z(k)$. Unfortunately, objective

function (3.4) is not well suited for analytic solution. Therefore, we consider another quadratic function that, after linearization, becomes a linear quadratic control problem.

3.2.2. Problem Reformulation

Let us consider the problem of finding $v(k)$ that minimizes the following objective function:

$$\begin{aligned} J(v) = \min_v & \left[\frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \right. \\ & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\ & \left. + v^t(k) R(k) v(k) \right] \end{aligned} \quad (3.6)$$

under the linear dynamic constraints (3.5).

From (3.3), this problem is the same as finding $u(k)$ that minimizes the following objective function:

$$\begin{aligned} J(u) = \min_u & \left[\frac{1}{2} (y(N) - Y_{ref}(N))^t H(N) (y(N) - Y_{ref}(N)) \right. \\ & + \frac{1}{2} \sum_{k=0}^{N-1} (y(k) - Y_{ref}(k))^t Q(k) (y(k) - Y_{ref}(k)) \\ & \left. + \gamma^t(x(k), u(k)) R(k) \gamma(x(k), u(k)) \right] \end{aligned} \quad (3.7)$$

under the nonlinear dynamic constraints (3.2).

Therefore, if Σ is the nonlinear system as described in (3.2) and Σ^L is the equivalent linear system as given in (3.5), and it is required to find u that minimizes (3.7) under the nonlinear dynamic constraints of Σ , then the solution will be equivalent to finding the input v that minimizes (3.6) under the linear dynamic constraints Σ^L .

3.2.3. Solution of Quadratic Control Problem

The linear problem of the previous section, has the well known recursive solution:

$$v(N-1) = -F(N-1)z(N-1) = -F(N-1)T(x(N-1)), \quad (3.8)$$

$$\begin{aligned} v(N-k) &= -F(N-k)z(N-k) - \tilde{F}(N-k) \\ &= -F(N-k)T(x(N-k)) - \tilde{F}(N-k) \end{aligned} \quad (3.9)$$

where

$$F(N) = \tilde{F}(N-1) = 0$$

and

$$F(N-k-1) = [R + B^t P(N-k)B]^{-1} B^t P(N-k)A \quad (3.10)$$

$$\tilde{F}(N-k-1) = -[R + B^t P(N-k)B]^{-1} B^t \bar{Q}(N-k) \quad (3.11)$$

$$\begin{aligned} \bar{Q}(N-k-1) &= cQY_{ref}(N-k-1) - F^t(N-k-1)R\tilde{F}(N-k-1) \\ &\quad + (A - BF(N-k-1))^t \\ &\quad [P(N-k)B\tilde{F}(N-k-1) + \bar{Q}(N-k)] \end{aligned} \quad (3.12)$$

$$\begin{aligned} P(N-k) &= [A - BF(N-k)]^t P(N-k+1) [A - BF(N-k)] \\ &\quad + (F(N-k)^t R F(N-k)) + cQc^t \end{aligned} \quad (3.13)$$

with terminal conditions

$$P(N) = cH(N)c^t$$

$$\bar{Q}(N-1) = cQY_{ref}(N-1).$$

The solution can be mapped back to the nonlinear system by substituting (3.8) and (3.9) in

$$u(k) = \gamma^{-1}(x(k), v(k)) \quad (3.14)$$

This leads to the following optimal control law:

$$u(N-1) = \gamma^{-1}(x(N-1), -F(N-1)T(x(N-1))) \quad (3.15)$$

and, recursively, to

$$u(N-k) = \gamma^{-1}(x(N-k), -F(N-k)T(x(N-k) - \tilde{F}(N-k))) \quad (3.16)$$

3.2.4. Results From The LQ Theory That Can Be Applied

The main advantage of exact linearization is to exploit the richly developed linear quadratic optimal control problem theory, for finite and infinite time horizons. In this section, we recall some results of the linear quadratic control theory that can be applied. A full discussion of discrete-time LQ theory appears in [Mahmoud and Singh, 1984].

a) Infinite time regulator problem

Consider the regulator problem; i.e., let $Y_{ref}(k) = 0 \forall k$. If $H(N)$ is zero in eq. (3.1), then let $\underline{P}$ be the steady state solution for (3.13):

$$\underline{P}(k) = \lim_{N \rightarrow \infty} P(N-k) \quad (3.17)$$

If the system $[A, B, c]$ is stabilizable and detectable, then $\underline{P}$ exists and is the unique, positive - semidefinite matrix solution of the algebraic Riccati equation:

$$\underline{P} = A^t \underline{P} A + Q - A^t \underline{P} B [B^t \underline{P} B + R]^{-1} B^t \underline{P} A \quad (3.18)$$

and;

$$J_{min} = z(0)^t \underline{P}(0) z(0)$$

That is,

$$J_{\min} = T^t(x(0))\underline{P}(0)T(x(0)) \quad (3.19)$$

The steady state input to the linear system is:

$$v^*(k) = -[R + B^t \underline{P} B]^{-1} B^t \underline{P} A z(k) \quad (3.20)$$

That is, the steady state input of the nonlinear system is obtained by:

$$u^*(k) = \gamma^{-1}[x(k), -[R + B^t \underline{P} B]^{-1} B^t \underline{P} A T(x(k))] \quad (3.21)$$

b) Asymptotically Stable, Time Invariant Closed Loop Dynamics

If the pair $[A, D]$ is completely observable, where D is any matrix such that $Q = D^t D$, then $\underline{P}$ is the unique positive definite solution of (3.18).

This solution yield the closed loop dynamics

$$z(k+1) = [A - B[R + B^t \underline{P} B]^{-1} B^t \underline{P} A]z(k) \quad (3.22)$$

This closed loop system is asymptotically stable. This leads to an input-output, asymptotically stable equivalent nonlinear system, since both systems (linear and nonlinear) share the same output.

3.2.5. A General Solution For Multiplicative Systems

Let the following two systems be equivalent:

(1.) the multiplicative system as defined in (1.9)-(1.12); i.e,

$$\Sigma_m : y(k) = h_1(X_1(k))h_2(X_2(k)) \cdots h_\beta(X_\beta(k)) \quad (3.23)$$

$$\begin{aligned}
X_1(k+1) &= A_1(\phi)X_1(k) + b_1u(k) \\
X_2(k+1) &= A_2(\phi)X_2(k) + b_2u(k) \\
&\vdots \\
X_\beta(k+1) &= A_\beta(\phi)X_\beta(k) + b_\beta u(k)
\end{aligned} \tag{3.24}$$

where, at each time k ;

$$y \in \mathbb{R}$$

$$X_i \in \mathbb{R}^{n_i}$$

$$u \in \mathbb{R}^m$$

and

(2.) the linear system:

$$\begin{aligned}
\Sigma^L : \quad y(k) &= c^t z(k) \\
z(k+1) &= Az(k) + Bu(k)
\end{aligned} \tag{3.25}$$

Let the equivalence relations be:

$$\begin{aligned}
x(k) &= T^{-1}(z(k)) \\
u(k) &= \gamma^{-1}(T^{-1}(z(k)), v(k))
\end{aligned} \tag{3.26}$$

Then following the arguments of the previous section, it can be shown that the solutions of P_1 in (3.6) and P_2 in (3.7) are the same. The solution is given by the recursive equations (3.8)-(3.13).

3.2.6. Example: Muscle Model

In this section we apply these results to the tracking control problem for the muscle model. We will consider the single muscle and known dynamic external load case, as given in section 1.4.6.

As in to (2.64), (2.65), the single muscle and with known dynamic external load is described by:

$$\begin{aligned}\phi(k+2) = & l_1\phi(k+1) + l_2\phi(k) \\ & + l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)]\end{aligned}\quad (3.27)$$

where

$$O(k+1) = \frac{\phi(k+1) - l_1\phi(k) - l_2\phi(k-1)}{l_3} \quad (3.28)$$

and

$$g(k) = h_2(\phi(k))h_3(\phi(k), \phi(k-1)) \quad (3.29)$$

This system is linearizable by the feedback law:

$$\begin{aligned}v(k) = & l_1\phi(k+1) + l_2\phi(k) \\ & + l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)]\end{aligned}\quad (3.30)$$

if b , l_3 and g all are non zero. The inverse linearization is:

$$\begin{aligned}u(k) = & \frac{v(k) - l_1\phi(k+1) - l_2\phi(k)}{l_3bg(k+1)} \\ & - [a_1(\phi)\frac{O(k+1)}{bg(k)} + a_2(\phi)\frac{O(k)}{bg(k-1)}]\end{aligned}\quad (3.31)$$

The equivalent linear system is:

$$\phi(k+2) = v(k) \quad (3.32)$$

which can be represented in a state space realization by letting:

$$x(k) \triangleq [\phi(k) \quad \phi(k+1)]^t \quad (3.33)$$

Thus, the linear quadratic tracking problem can be reformulated as the minimization of the following tracking objective function:

$$J(v) = \min_{v(k)} \left[\frac{1}{2} \sum_{k=0}^{N-1} (x(k) - r(k))^t Q (x(k) - r(k)) + v^2(k) R \right] \quad (3.34)$$

subject to:

$$x(k+1) = Ax(k) + bv(k) \quad (3.35)$$

$$x(k) \in \mathbb{R}^2$$

where

$$x(k) = [\phi(k) \quad \phi(k+1)]^t$$

$$r(k) = [\phi_{ref}(k) \quad \phi_{ref}(k+1)]^t$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $\underline{Q} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$.

The optimal control law for this system is:

$$v(k) = \frac{q}{(R+q)} \phi_{ref}(k+1) \quad (3.36)$$

Therefor, from (3.31):

$$\begin{aligned} u(k) = \frac{1}{l_3 bg(k+1)} & \left[\frac{q}{(R+q)} \phi_{ref}(k+1) - l_1 \phi(k+1) - l_2 \phi(k) \right] \\ & - [a_1(\phi) \frac{O(k+1)}{bg(k)} + a_2(\phi) \frac{O(k)}{bg(k-1)}] \end{aligned} \quad (3.37)$$

3.3 LINEARIZED CLASSICAL CONTROLLER

In this section we apply certain classical controller design methods to the linearized system. By classical, we mean non-optimal, (e.g. PID, pole placement, phase-lag, phase-lead or lead-lag compensators). These compensators are designed based on pre-assigned control performance specifications. In this section we formulate the control problem for the muscle model.

3.3.1. Control System Specification

Consider the nonlinear discrete time system of (2.1):

$$\begin{aligned}\Sigma : \quad y(k) &= h(x(k)) \\ x(k+1) &= f(x(k), u(k))\end{aligned}\tag{3.38}$$

where $x(0) = x_0$ is given. Let this nonlinear system be equivalent to the following linear system:

$$\begin{aligned}\Sigma^L : \quad y(k) &= c^t z(k) \\ z(k+1) &= Az(k) + Bv(k)\end{aligned}\tag{3.39}$$

where

$$\begin{aligned}z(k) &= T(x(k)) \\ v(k) &= \gamma(x(k), u(k))\end{aligned}\tag{3.40}$$

Furthermore, for simplicity, let us consider only single input, single output systems.

Classical controller design for the linear system (3.39) is achieved by defining first some desired characteristics, or performance specification to the closed loop system. These characteristics are usually given in terms of steady state accuracy, transient response, relative stability, or sensitivity to

change in system parameters and to disturbance rejection. In [Phillips and Nagle 1990, Ackerman 1985 and Åström and Wittenmark 1984] a complete study for each of these previous performance specifications is given.

3.3.2. Problem Formulation

In this section, the example case of single-muscle with known dynamic external load will be used to illustrate these design procedures. The position tracking problem discussed in chapter one is approached, solved using different types of controllers. First, a PID controller is applied. Then, an open loop controller based on the inverse of the linear system dynamics will be tested. These controllers will be designed in the linear space and then mapped back to the original nonlinear space using the inverse linearization. Simulation results using the control laws obtained here are presented in chapter four.

In the muscle model with load, given in chapter two, by (2.64), (2.65), and (2.68), the inverse linearization is given by:

$$u(k) = \frac{1}{l_3 b g(k+1)} [v(k) - l_1 \phi(k+1) - l_2 \phi(k)] - [a_1(\phi) \frac{O(k+1)}{b g(k)} + a_2(\phi) \frac{O(k)}{b g(k-1)}] \quad (3.41)$$

where

$$\begin{aligned} O(k+1) &= \frac{1}{l_3} [\phi(k+1) - l_1 \phi(k) - l_2 \phi(k-1)] \\ g(k) &= h_2(\phi(k)) h_3(\phi(k), \phi(k-1)) \end{aligned} \quad (3.42)$$

a) PID Controller

Recall that the transfer function for the PID controller for the continuous time systems is:

$$D(s) = k_p + \frac{k_i}{s} + k_d s \quad (3.43)$$

and that a discretized version in the z -domain can be described by:

$$D(z) = k_p + k_i \frac{T}{2} \left[\frac{z+1}{z-1} \right] + k_d \left[\frac{z-1}{Tz} \right]$$

$$D(z) = G_a \frac{(z - \xi_1)(z - \xi_2)}{(z^2 - z)}$$

where G_a , ξ_1 and ξ_2 are the gain and the zeroes of the controller that can be chosen to satisfy various transient and steady state criteria.

The difference equation of this controller is:

$$v(k) = v(k-1) + G_a[e(k) - (\xi_1 + \xi_2)e(k-1) + \xi_1\xi_2e(k-2)] \quad (3.44)$$

where $e(k)$ is the output error at time k :

$$e(k) = \phi_{ref}(k) - \phi(k). \quad (3.45)$$

this controller is to be mapped back to the nonlinear domain using equation (2.68), thus:

$$u(k) = \frac{1}{l_3 bg(k+1)} [v(k-1) + G_a(e(k) - (\xi_1 + \xi_2)e(k-1) + \xi_1\xi_2e(k-2))$$

$$- l_1\phi(k+1) - l_2\phi(k)]$$

$$- [a_1(\phi) \frac{O(k+1)}{bg(k)} + a_2(\phi) \frac{O(k)}{bg(k-1)}] \quad (3.46)$$

b) Open Loop Controller

This controller is based on inverting the linear system dynamics which results in the following controller:

$$v(k) = \phi_{ref}(k+2) \quad (3.47)$$

Thus:

$$u(k) = \frac{1}{l_3 b g(k+1)} [\phi_{ref}(k+2) - l_1 \phi(k+1) - l_2 \phi(k)] \\ - [a_1(\phi) \frac{O(k+1)}{b g(k)} + a_2(\phi) \frac{O(k)}{b g(k-1)}] \quad (3.48)$$

CHAPTER FOUR

SIMULATION of CONTROLLERS USING EXACT LINEARIZATION

4.1 INTRODUCTION

In this chapter we simulate and test the performance of the three controllers designed in chapter three; namely the PID, the linearized quadratic and the open loop controllers.

Three types of simulations are conducted in this chapter. First, we test the three controllers in a noise free environment. Then we add noise to the inputs and outputs of the system. Finally, we test and compare the performance of each controller when the model parameters are not precisely known.

Figure 4.1 shows a block diagram for the muscle control problem under consideration, where $w(k)$ and $e(k)$ are the input and the output noises.

The two muscle model with load case of (2.73) is simulated, using the following parameter values [Veltink et al a,b and c 1990]:

$$\begin{aligned}a_{1f} &= a_{1e} = 0.8 \\ a_{2f} &= a_{2e} = a_{1f}^2 = -0.64\end{aligned}\tag{4.1}$$

Fig. 4.1. Example of Muscle Model Controller

$$g_f(k) = g_e(k) = [e_1 \phi(k) - e_0][1 + e_3(\phi(k) - \phi(k-1))] \quad (4.2)$$

where

$$e_0 = 0.008 \quad (4.3)$$

$$e_1 = 1$$

$$e_3 = \begin{cases} -0.03 & \text{if } \phi(k) > \phi(k-1) \\ -0.008 & \text{if } \phi(k) < \phi(k-1) \end{cases} \quad (4.4)$$

and

$$p_f(u(k)) = bu(k) \quad (4.5)$$

$$p_e(u(k)) = 0.8bu(k)$$

where

$$b = 0.86 \quad (4.6)$$

$$l_1 = 0.73$$

$$l_2 = -0.7 \quad (4.7)$$

$$l_3 = -26.73$$

For the design of the PID controller (3.44), the two zeros ξ_1 and ξ_2 are placed at (0.255,0.244), so that the closed loop system will have a phase margin of 55° at $W_1 = 1.728$. For the linearized quadratic controller (3.36), we use $R = 1$ and $q = 8$. The open loop controller is simulated using the control law of (3.47).

In the simulations, the controller seeks to make the joint follow a certain angle trajectory, $\phi_{ref}(k)$. The simulation is run for 4. sec. and the sampling period is assumed to be 0.04 seconds.

4.2 NOISE FREE ENVIRONMENT SIMULATION

Figures 4.2-4.3 show the results of simulating the three controllers when $w(k)$ and $e(k)$, are assumed to be zero (no noise). For each controller, Figure (a) shows both ϕ_{ref} and ϕ , while Figure (b) shows the output torque exerted on the load. Figures (c) and (d) shows the error ($\phi_{ref}(k) - \phi(k)$) and the corresponding input $u(k)$. Figures (e) and (f) show the output torque and the corresponding input for the flexor muscle, while Figures (g) and (h) give the torque and input of the extensor muscle. Finally an overall measure for the controller performance is computed as the root square error given by:

$$E_{rms} = \sqrt{\sum_{k=0}^{N-1} (\phi_{ref}(k) - \phi(k))^2} \quad (4.8)$$

Figure 4.2 shows the results of simulating the PID controller. This controller shows a ramp error of 30%, as seen in Figure 4.2-a, and has

zero steady state error to a step input. This controller has an error of $Erms = 41.85^\circ$.

Figure 4.3 shows the results of simulating the linearized quadratic controller. In Figure 4.3-a, there is step steady state error of $\frac{q}{q+R}$. This is because it has no integral action and there is a cost on the input. The phase shift results in a one sample delay. This controller show an $Erms$ error of 37.46° .

Figure 4.4 shows the results of simulating the open loop controller. For this no-noise case, this controller shows the best error: $Erms = 0.0^\circ$. The system follows exactly the reference trajectory, as both start from the same value. If the system has an initial value that is different from the reference trajectory, this controller takes two time steps to reach steady state. It has zero steady state error to step and ramp inputs.

4.3 ROBUSTNESS OF EXACT LINEARIZATION

In this section investigate the relative performance of the controllers in the presence of input or output noises. We consider Gaussian white noises with zero means, at different standard deviations (STD).

In Figure 4.5 (input noise environment), the PID controller is the worst controller. Its $Erms$ increases at a higher rate than those of the other controllers (eg., at $STD=1.$, the PID shows an $Erms = 325^\circ$, while the linearized quadratic and the open loop controllers show values around 200° and 220° respectively). For very low noises, the open loop controller is best.

Fig. 4.2 PID Controller

Fig. 4.3 LQ Controller

Fig. 4.4 Open Loop Controller

However as the noise increases, the linearized quadratic becomes better. At $\text{STD}=1.7$, the open loop controller shows a peak of $E_{rms} = 650^\circ$.

Figure 4.6 shows the results of simulating the system with output noise. The system in this case shows less sensitivity to noise than the previous input noise case. For example, at $\text{STD}=3.5$ the system shows an E_{rms} that is less than 500° for all the controllers. For output noise, the three control approaches are roughly equivalent (except at low noise).

Figure 4.7, shows the results of simulating the system with both input and output noises. Once more, the PID controller shows the worst sensitivity to noise.

From these simulations the linearized quadratic controller seems to be best, unless the noises are very small.

4.4 SENSITIVITY OF EXACT LINEARIZATION

In this section we examine what happens if the controllers are designed according to incorrect, fixed model parameters. In particular, we consider the performance of the three controllers when the parameters in equation (2.73) are estimated incorrectly, with a fixed percentage error that takes values between (-30% and 30%). Figures 4.8 - 4.14 show the results of varying the parameters, a_1 , a_2 , b , e_0 , e_1 , s_1 and s_2 .

The open loop controller shows the best results for small single parameter variations. The linearized quadratic controller is best for moderate parameter variations. Note that it has its best E_{rms} at a incorrect value of the parameters; at this value, the controller steady state error is reduced.

Fig. 4.5 The PID, LQR and Open Loop Controllers with Input Noise.

-- b --

-- a --

-- c --

Fig. 4.6 The PID, LQR and Open Loop Controllers with Output Noise.

-- b --

-- a --

-- c --

Fig. 4.7 The PID, LQR and Open Loop Controllers with Input - Output Noise.

-- b --

-- a --

-- c --

Fig. 4.8 The PID, LQR and Open Loop Controllers with Variation in Par. a_1 .

-- b --

-- c --

-- d --

Fig. 4.9 The PID, LQR and Open Loop Controllers with Variation in Par. α_1

Fig. 4.10 The PID, LQR and Open Loop Controllers with Variation in Par. b.

Fig. 4.11 The PID, LQR and Open Loop Controllers with Variation in Par. ϵ_a .

Fig. 4.12 The PID, LQR and Open Loop Controllers with Variation in Par. e_1 .

Fig. 4.13 The PID, LQR and Open Loop Controllers with Variation in Par. θ_1 .

Fig. 4.14 The PID, LQR and Open Loop Controllers with Variation in Par. s_2 .

Except for the Figure 4.9 (parameter a_2), the linearized controllers have a minimum *Erms* at parameter error +10%.

The performance of the PID controller is not sensitive to variation in the model parameters, but it has a minimum *Erms* value that is generally higher than those of the other controllers.

4.5 CONCLUSIONS

In this chapter, the controller designs were tested and simulated, as applied to the two muscle with load system. Three tests were made; a noise free simulation, a input and /or output noise environment and finally an incorrect model parameter was assumed and simulated. The linearized quadratic controller proved to have better performance in noisy environments or when the parameter error exceed 10% than the other two controllers.

It is also recommended to use the open loop controller when we have errors in the model parameters that does not exceed 10%.

CHAPTER FIVE

ADDITIONAL APPLICATIONS OF EXACT LINEARIZATION

5.1. INTRODUCTION

In this chapter we consider two additional applications of the exact linearization technique. First, we apply exact linearization to solve the nonlinear Hamiltonian first order optimality conditions; these conditions arise in the solution of nonlinear quadratic control problems. Second, we briefly investigate problems involving the control of nonlinear systems subject to inequality constraints on states, inputs and outputs.

This chapter is organized as follows. Section 5.2 addresses the nonlinear quadratic controller. In section 5.3, the results of studying the interactions of exact linearization and constrained control problems is given.

5.2 NONLINEAR QUADRATIC CONTROLLER

5.2.1. Introduction

In chapter three, we applied the linear quadratic control methods to exactly linearized models. This resulted in objective functions in the nonlinear domain that are difficult to interpret. In this section, we instead consider problems with quadratic costs in the nonlinear domain. The solution to the quadratic tracking control problem will be obtained using the **Calculus of Variations** to obtain the first order optimality conditions. Then the derived nonlinear system, with dimension $2n$, is linearized using the **exact linearization** technique. An analytical solution for the problem is then obtained.

Although the resulting solution lacks the convenient recursive computational structure of the Riccati equation of LQ control, it gives us the ability to minimize a well motivated, meaningful quadratic objective function. This work is in the spirit of work for analytic nonlinear systems in [Yoshida and Loparo 1989]. Similar results for the continuous-time problem are much simpler than for the discrete time case. These are given in appendix C.

5.2.2. Problem Formulation

Consider a single input, nonlinear discrete time system. That is linear in the input;

$$\Sigma : x(k+1) = f(x(k)) + g(x(k))u(k) \quad (5.1)$$

with given $x(0) = x_0$ and

$$x \in \mathbb{R}^N$$

$$u \in \mathbb{R}$$

Here $f(\cdot)$ and $g(\cdot)$ are smooth functions. We wish to track a function $r(k)$, by imposing the quadratic objective function:

$$\begin{aligned} J = & \frac{1}{2}[x(N)-r(N)]^t H[x(N) - r(N)] \\ & + \frac{1}{2} \left[\sum_{k=0}^{N-1} [x(k) - r(k)]^t Q[x(k) - r(k)] \right. \\ & \left. + u(k)^t R u(k) \right] \end{aligned} \quad (5.2)$$

under the nonlinear dynamic constraints Σ , where $x(N)$, $r(N)$ are the final values of at time N .

5.2.3. Solution I

Proposition 5.1 :

If the following three statements hold:

(a) if we can solve for $\lambda(k+1)$ in the equation:

$$\begin{aligned} 0 = & \lambda(k) + Q(x(k) - r(k)) + \left(\frac{\partial f}{\partial x} \right)^t \lambda(k+1) \\ & - \left[\frac{\partial(g^t(x)\lambda(k+1))}{\partial x} \right]^t R^{-1} g^t(x) \lambda(k+1) \end{aligned}$$

that is, $\lambda(k+1)$ can be expressed as:

$$\lambda(k+1) = \psi^\lambda(x(k), \lambda(k), r(k)); \quad (5.3)$$

- (b) the following nonlinear system is state equivalent to a linear system, (i.e., it can be linearized by only state coordinate change, without feedback, see Lee et al [1987] and the summary in appendix A):

$$\begin{pmatrix} x(k+1) \\ \lambda(k+1) \end{pmatrix} = \begin{pmatrix} f(x) - g(x)R^{-1}g(x)^t\psi^\lambda(x(k), \lambda(k), r(k)) \\ \psi^\lambda(x(k), \lambda(k), r(k)) \end{pmatrix} \quad (5.4)$$

and it is equivalent to the linear system:

$$z(k+1) = Az(k) + br(k) \quad (5.5)$$

where $A \in \mathbb{R}^{2n \times 2n}$ and $b \in \mathbb{R}^{1 \times 2n}$

- (c) if A in (5.5) is nonsingular,

Then:

there exists a map $\tilde{T} : x(k) \rightarrow \lambda(k)$, and an optimal control input u^* that has the form:

$$u^*(k) = -R^{-1}g^t(x(k))\tilde{T}(x(k))$$

where $\lambda(k)$ is the costate.

Proof

The Hamiltonian equation for this problem is given by [Kirk 1970]:

$$H = \frac{1}{2}[x - r]^t Q[x - r] + \frac{1}{2}u^t R u + \lambda^t(k+1)[f + gu] \quad (5.6)$$

The first order optimality conditions are

$$\begin{aligned}
\frac{\partial H}{\partial u(k)} &= Ru(k) + g^t(x(k))\lambda(k+1) \\
\frac{\partial H}{\partial \lambda(k+1)} &= x(k+1) = f(x(k)) + g(x(k))u(k) \\
-\frac{\partial H}{\partial x(k)} &= \lambda(k) = -Q(x(k) - r(k)) - \left(\frac{\partial f}{\partial x}\right)^t \lambda(k+1) \\
&\quad - \left[\frac{\partial(g^t(x)\lambda(k+1))}{\partial x}\right]^t u(k)
\end{aligned} \tag{5.7}$$

which yields:

$$u^*(k) = -R^{-1}g^t(x(k))\lambda(k+1) \tag{5.8}$$

$$x(k+1) = f(x(k)) - g(x(k))R^{-1}g^t(x(k))\lambda(k+1) \tag{5.9}$$

$$\begin{aligned}
\lambda(k) &= -Qx(k) - \left(\frac{\partial f}{\partial x(k)}\right)^t \lambda(k+1) + Qr(k) \\
&\quad + \left[\frac{\partial(g^t(x(k))\lambda(k+1))}{\partial x(k)}\right]^t R^{-1}g^t(x(k))\lambda(k+1)
\end{aligned} \tag{5.10}$$

Suppose that we can solve for $\lambda(k+1)$ in the previous equation, obtaining the solution:

$$\lambda(k+1) = \psi^\lambda(x(k), \lambda(k), r(k))$$

Then the equations (5.8)-(5.10) will yield:

$$u^* = -R^{-1}g^t(x)\psi^\lambda(x(k), \lambda(k), r(k)) \tag{5.11}$$

$$x(k+1) = f(x) - g(x)R^{-1}g^t(x)\psi^\lambda(x(k), \lambda(k), r(k)) \tag{5.12}$$

$$\lambda(k+1) = \psi^\lambda(x(k), \lambda(k), r(k)) \tag{5.13}$$

The next step is to linearize, by state coordinate change, the nonlinear system (5.12 and 5.13) to be equivalent to a linear one, so that both should

have the same input $r(k)$. If this can be done, then there exists a linear system with the states $z(k)$ and a transformation $z(k) = T(x(k), \lambda(k))$ such that:

$$z(k+1) = Az(k) + br(k) \quad (5.14)$$

If A is nonsingular then

$$z(k) = A^{-(N-k)}z(N) - \sum_{i=0}^{N-k-1} A^{-i-1}br(k+i) \quad (5.15)$$

where N is the final time. Then a solution for the feedback law can be derived according to the following steps:

Step 1: Since $r(k)$ is known, (5.15) can be evaluated as a function in $z(N)$ only:

$$z(k) = \psi^z(z(N)). \quad (5.16)$$

Step 2: But $z(N) = T(x(N), \lambda(N))$. Thus

$$z(k) = \psi^z \circ T(x(N), \lambda(N)). \quad (5.17)$$

Step 3: From the boundary condition of the original system

$$\lambda(N) = Hx(N) - Hr(N) \quad (5.18)$$

Step 4: Since $r(N)$ is known, we then have:

$$z(k) = \psi^z \circ T(x(N), Hx(N) - Hr(N)) = \psi^z \circ \bar{T}(x(N)) \quad (5.19)$$

Step 5: Since T is invertible (from the linearization condition), there exist T_1, T_2 such that:

$$x(k) = T_1(z(k)) \quad (5.20)$$

$$\lambda(k) = T_2(z(k)). \quad (5.21)$$

Step 6: Since

$$x(k) = T_1(z(k)) = T_1 \circ \psi^z \circ \bar{T}(x(N)) \quad (5.22)$$

and since the original system is assumed to be controllable then:

$$\begin{aligned} x(N) &= (T_1 \circ \psi^z \circ \bar{T})^{-1}(x(k)) \\ &= (\psi^z)^{-1}(x(k)). \end{aligned} \quad (5.23)$$

Step 7: Finally,

$$\begin{aligned} \lambda(k) &= T_2(z(k)) \\ &= T_2 \circ \psi^z \circ \bar{T}(x(N)) \\ &= T_2 \circ \psi^z \circ \bar{T} \circ (\psi^z)^{-1}(x(k)) \\ &= \bar{T}(x(k)) \end{aligned} \quad (5.24)$$

This can be substituted in the optimal control feedback law:

$$\begin{aligned} u^*(k) &= -R^{-1}g(x(k))^t \bar{T}(x(k+1)) \\ &= -R^{-1}g(x(k))^t \bar{T}(f(x(k)) + g(x(k))u^*(k)) \end{aligned} \quad (5.25)$$

Solving for u^* in the previous equation we get:

$$u^*(k) = -R^{-1}g(x(k))^t \bar{T}(x(k))$$

5.2.4 Solution II

In the previous discussion it is required that the obtained A matrix in Equation (5.14) be nonsingular. Otherwise we can not express $z(k)$ as a function in $z(N)$. However, it is tedious to computationally confirm that A is singular. This motivates an alternate proof of the previous results.

We first, introduce the following *backward exact linearization*.

Definition 5.1

The nonlinear system Σ is said to be **backwardly exact linearizable** by a state coordinate change, into a linear system Σ^L , iff there exists an analytic map

$$T : \mathbb{R}^N \rightarrow \mathbb{R}^{N^L}; z(k) = T(x(k))$$

such that $x \in \mathbb{R}^N$ and $T(x) = x^L \in \mathbb{R}^{N^L}$. With the same initial conditions, systems Σ_x and $\Sigma_{x^L}^L$ have the same input output behavior.

Lemma 5.1

Let us define the backward system as

$$x(k) = f(x(k+1), u(k))$$

This system is backwardly linearized by state map $T : \mathbb{R}^N \rightarrow \mathbb{R}^{N^L}$ iff,

$$\psi_x(u_{N-1}, \dots, u_0) \triangleq f(f(\dots f(x, u_{N-1}), \dots), u_1), u_0)$$

$$\mathcal{F}_x(u_N, \dots, u_0) \triangleq f(f(\dots f(x, u_N), \dots), u_1), u_0)$$

we have:

- (1) $(\psi_0)_*|_{\mathbf{u}=0}$ is isomorphism, and
- (2) $[\frac{\partial}{\partial \mathbf{u}_i}, \ker(\mathcal{F}_0)_*] \subset \ker(\mathcal{F}_0)_*$ for $0 \leq i \leq N$.

Proof

For the proof of this theorem see Lee, et al (Theorem 4 [1987]).

We can use the previous result to prove the following:

Proposition 5.2

If the following two conditions hold:

- (a) we can solve for $x(k)$ in the following equation

$$x(k+1) - f(x(k)) + g(x(k))R^{-1}g^t(x(k))\lambda(k+1) = 0 \quad (5.27)$$

with $x(k) \triangleq \psi^x(x(k+1), \lambda(k+1))$

- (b) the nonlinear system (5.28) is backwardly exact linearizable, (i.e., can be linearized by state coordinate map only, without feedback)

$$\begin{pmatrix} x(k) \\ \lambda(k) \end{pmatrix} = \begin{pmatrix} \psi^x(x(k+1), \lambda(k+1)) \\ Q(r(k) - x(k)) - (\frac{\partial f}{\partial x})^t \lambda(k+1) - [\frac{\partial(g^t(x)\lambda(k+1))}{\partial x}]^t u(k) \end{pmatrix} \quad (5.28)$$

and it is equivalent to the linear system

$$z(k) = Az(k+1) + br(k)$$

Then

- 1- There exists a map $\tilde{T} : x(k) \rightarrow \lambda(k)$.
- 2- An optional control input u^* that has the form

$$u^*(k) = -R^{-1}g^t(x)\tilde{T}(x(k))$$

Proof

The Hamiltonian equation of the system is given by:

$$H = \frac{1}{2}[x - r]^t Q[x - r] + \frac{1}{2}u^t R u + \lambda^t(k+1)[f + gu] \quad (5.29)$$

The first order optimality conditions are:

$$\begin{aligned} \frac{\partial H}{\partial u(k)} &= Ru(k) + g^t(x(k))\lambda(k+1) \\ \frac{\partial H}{\partial \lambda(k+1)} &= x(k+1) = f(x(k)) + g(x(k))u(k) \\ -\frac{\partial H}{\partial x(k)} &= \lambda(k) = -Q(x(k) - r(k)) - \left(\frac{\partial f}{\partial x}\right)^t \lambda(k+1) \\ &\quad - \left[\frac{\partial(g^t(x)\lambda(k+1))}{\partial x}\right]^t u(k) \end{aligned} \quad (5.30)$$

which yield:

$$u^*(k) = -R^{-1}g^t(x(k))\lambda(k+1) \quad (5.31)$$

$$x(k+1) = f(x(k)) - g(x(k))R^{-1}g^t(x(k))\lambda(k+1) \quad (5.32)$$

$$\begin{aligned} \lambda(k) &= +Qr(k) - Qx(k) - \left(\frac{\partial f}{\partial x(k)}\right)^t \lambda(k+1) \\ &\quad + \left[\frac{\partial(g^t(x(k))\lambda(k+1))}{\partial x(k)}\right]^t R^{-1}g^t(x(k))\lambda(k+1) \end{aligned} \quad (5.33)$$

Substituting for u^* in the last equation we get:

$$x(k+1) = f(x(k)) - g(x(k))R^{-1}g^t(x(k))\lambda(k+1)$$

If this equation is $x(k)$ -invertible, we can express $x(k)$ as:

$$x(k) = \psi^x(x(k+1), \lambda(k+1))$$

Then we obtain the nonlinear system (5.28). If this system is linearizable by state coordinate maps only (without feedback) to

$$z(k) = Az(k+1) + br(k)$$

then we do not need to have a nonsingular matrix A , as in the previous proposition.

5.2.5 Conclusion And Discussion

In this section, we obtained a solution for the nonlinear discrete time tracking problem. Necessary and sufficient conditions for this existence of the solution were also established. Although the problem was formulated as a tracking problem, a similar solution for the nonlinear regulator control problem can be obtained. This technique yields gives solutions for the special class of nonlinear systems that satisfies the conditions of Theorem 5.2.

5.3 NONLINEAR CONSTRAINED CONTROL PROBLEM

5.3.1. Introduction

In this section, constrained control problems associated with the nonlinear systems and exact linearization will be considered. How constraints and exact linearization interact will be investigated. We consider, for simplicity, only single-input, single-output (SISO) systems.

5.3.2. Problem Formulation

Consider the **SISO** discrete time nonlinear system as described in (2.10) $\Sigma_1(\mathbb{R}^N, \mathbb{R}, \mathbb{R}, f, h)$, where:

$$\begin{aligned}\Sigma_1 : \quad y(k) &= h(x) \\ x(k+1) &= f(x(k), u(k))\end{aligned}\tag{5.33}$$

and its equivalent (**SISO**) linear system $\Sigma_2(\mathbb{R}^N, \mathbb{R}, \mathbb{R}, (A, b), c^t)$:

$$\begin{aligned}\Sigma_2 : \quad y(k) &= c^t z(k) \\ z(k+1) &= Az(k) + Bu(k)\end{aligned}\tag{5.34}$$

with the equivalence relation

$$\begin{aligned}x(k) &= T^{-1}(z(k)) \\ u(k) &= \gamma^{-1}(T^{-1}(z(k)), v(k))\end{aligned}\tag{5.35}$$

Let us also assume that the nonlinear system is subject to the following equality and inequality constraints:

$$\underline{h}_i(x, y, u) = 0 \quad i = 1, \dots, s\tag{5.36}$$

$$\underline{g}_i(x, y, u) \leq 0 \quad i = 1, \dots, l\tag{5.37}$$

for some integers s and l .

If these constraints are mapped to the linear system under the equivalence relations (5.35), we obtain:

$$\underline{h}_i(T^{-1}(z), y, \gamma^{-1}(T^{-1}(z), v)) = 0 \quad i = 1, \dots, s\tag{5.38}$$

$$\underline{g}_i(T^{-1}(z), y, \gamma^{-1}(T^{-1}(z), v)) \leq 0 \quad i = 1, \dots, l\tag{5.39}$$

5.3.3. Additional Constraints Due to Exact Linearization

In general, two additional types of constraints may be introduced in the linear quadratic problem by:

- a- The state transformation: additional constraints may arise due to the fact that the coordinate transformation must be a diffeomorphism,
- b- The feedback conditions: constraints may be imposed by the fact that the feedback must be u-invertible*.

The following examples will illustrate these two types of constraints:

Example 5.1 :

Consider the nonlinear scalar system ($x \in \mathbb{R}$):

$$x(k+1) = \frac{x(k)}{1 + x(k)u(k)}$$

with the state transformation:

$$z(k) = \frac{1}{x(k)}$$

The equivalent linear system is:

$$z(k+1) = z(k) + u(k)$$

Here $z(k) \neq 0$ is the new constraint that ensures that, this transformation is invertible.

* See definition 2.2

Example 5.2 :

Consider the nonlinear scalar system ($x \in \mathbb{R}$):

$$x(k+1) = \exp(u(k))$$

with the feedback law:

$$v(k) = \exp(u(k))$$

The equivalent linear system is:

$$z(k+1) = v(k)$$

Here $v(k)$ must always be ≥ 0 for the inversion to exist.

In some cases however, the original constraints (5.36)-(5.37) may disappear in the process of linearization, as in the following example:

Example 5.3 :

Consider the nonlinear scalar system ($x \in \mathbb{R}$):

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= \sin^{-1}(u(k)) \end{aligned} \tag{5.40}$$

with the input constraints $|u(k)| \leq 1$. Such system can be linearized by the feedback law:

$$v(k) = \sin^{-1}(u(k)) \tag{5.41}$$

The equivalent linear system is :

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= v(k) \end{aligned}$$

Note that the constraints are now not necessary, because v can take any value without violating the original constraint on u .

5.3.4. Problem Reduction

We next use exact linearization to transform the nonlinear control problem with state inequality constraints to a linear problem with input constraints. We prove that the obtained linear system will have a dimension less than or equal to that of the original one.

The proposed algorithm, in this section is based on the work of Jacobson and Lele [1969], where they considered adding slack variables to the inequality constraints to transform the nonlinear problem to another unconstrained problem with higher dimension. In Jacobson and Lele [1969], the authors solved the continuous time case when the number of inputs equal the number of inequality constraints. In this section, we derive an analogous technique for the discrete time systems and combine it with exact linearization to linearize and reduce the dimension of the resulting system. This will yield a linear system with input constraints.

5.3.4.1 Transformation Algorithm

Let us consider the nonlinear system (2.1)

$$y(k) = h(x(k)) \quad (5.42)$$

$$x(k+1) = f(x(k), u(k))$$

where $x(0) = x_0$ is given and

$$y \in \mathbf{R}$$

$$x \in \mathbf{R}^N$$

$$u \in \mathbf{R}$$

Assume that this system is subject to the following state inequality constraint

$$\underline{g}^0(x) \leq 0 \quad (5.43)$$

As in [Jacobson and Lele 1969], (5.43) can be converted into an equality constraint by introducing the slack variable $\alpha_0(\cdot)$, as follows:

$$\underline{g}^0(x(k)) + \alpha_0^2(k) = 0 \quad (5.44)$$

If (5.44) is enforced at all $k = 0, \dots, N-1$, and since $\alpha_0^2(k)$ is always nonnegative, then (5.43) holds automatically.

By "r" successive compositions, the following set of equations can be obtained

$$\begin{aligned} \underline{g}^1(x(k)) + \alpha_1^2(k) &= \underline{g}^0(f(x, u)) + \alpha_0^2(k+1) = 0 \\ \underline{g}^2(x(k)) + \alpha_2^2(k) &= \underline{g}^1(f(x, u)) + \alpha_1^2(k+1) = 0 \\ &\vdots \\ \underline{g}^r(x(k)) + \alpha_r^2(k) &= \underline{g}^{r-1}(f(x, u)) + \alpha_{r-1}^2(k+1) = 0 \end{aligned} \quad (5.45)$$

where r is the least nonnegative integer such that

$$\frac{\partial \underline{g}^r(x(k))}{\partial u(k)} \neq 0 \quad (5.46)$$

Using the last equation in (5.45), we can solve for the control u to obtain

$$u(k) = G(x(k), \alpha_r(k)) \quad (5.47)$$

Using this feedback, and treating $\{\alpha_0, \alpha_1, \dots, \alpha_{r-1}\}$ as additional state variables, the following unconstrained problem, with $\alpha_r(k)$ as the new control, is obtained:

$$y(k) = h(x(k))$$

$$\begin{aligned}
x(k+1) &= f(x(k), G(x(k), \alpha_r(k))) \\
\alpha_0(k+1) &= \alpha_1(k) \\
&\vdots \\
\alpha_{r-1}(k+1) &= \alpha_r(k)
\end{aligned} \tag{5.48}$$

with known $x(0)$ and unknown $\alpha_i(0)$, $i = 1, \dots, r-1$. However, $\alpha_i(0)$ can be chosen to satisfy (5.44) and the first $r-1$ equations in (5.45). Finally, for any input $\alpha_r(k)$, and $\forall k = 0, \dots, N-1$, (5.44) and (5.45) will be satisfied and produce an admissible trajectory.

Obviously, the required conditions for this algorithm to be applied is that $\bar{g}^0$ and f must be (at least once) differentiable.

5.3.4.2 Applying Exact Linearization

Let us assume that the nonlinear system of (5.42) is linearizable. That is, there exists a finite relative degree r such that*

$$\frac{\partial y(k+r)}{\partial u(k)} \neq 0 \tag{5.49}$$

Proposition 5.3

If the nonlinear system (5.42) is exactly linearizable, then the nonlinear system represented by (5.48) is also linearizable.

* See Section 2.4.1

Proof

According to Theorem 2.1, we can prove the linearization of (5.48) if we find a relative degree r for $y(k)$ such that

$$\begin{aligned} \frac{\partial y(k+r)}{\partial \alpha_r(k)} &\neq 0 \\ \frac{\partial y(k+s)}{\partial \alpha_r(k)} &= 0 \quad \forall s < r \end{aligned} \quad (5.50)$$

Now, according to (5.49) we have $r \geq r$. Therefore, for $r = r$:

$$\frac{\partial y(k+r)}{\partial \alpha_r(k)} = \frac{\partial y(k+r)}{\partial u(k)} \Big|_{u=G} \frac{\partial G}{\partial \alpha_r(k)} \quad (5.51)$$

Since the first term in the RHS is non-zero at $r = r$, we must show that

$$\frac{\partial G(x(k), \alpha_r(k))}{\partial \alpha_r(k)} \neq 0 \quad (5.52)$$

which is satisfied since u in (5.47) is obtained from the last equation in (5.48) (by the inverse function theorem). We represent the resulting equivalent linear system by:

$$\begin{aligned} y(k) &= c^t z(k) \\ z(k+1) &= Az(k) + bv(k) \end{aligned} \quad (5.53)$$

To obtain the α_r -invertible* relation (5.47), there will exist, in general, constraints on the input v in the linear domain.

* See definition 2.2 for α_r -invertible.

5.4 EXAMPLE: MUSCLE MODEL

Let us consider the single muscle and known dynamic external load, as given in section 1.6.6. According to (2.64), (2.65) in chapter two, the single muscle and known dynamic external load are described by:

$$\begin{aligned}\phi(k+2) = & l_1\phi(k+1) + l_2\phi(k) \\ & + l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)]\end{aligned}\quad (5.54)$$

where

$$O(k+1) = \frac{\phi(k+1) - l_1\phi(k) - l_2\phi(k-1)}{l_3} \quad (5.55)$$

and

$$g(k) = h_2(\phi(k))h_3(\phi(k), \phi(k-1)) \quad (5.56)$$

Let us impose the following inequality angle (length) constraint:

$$\phi^2(k) - deg^2 \leq 0 \quad (5.57)$$

where deg is some specified angle value.

Then (5.45) can be obtained as follows:

$$\begin{aligned}\phi^2(k) - deg^2 + \alpha_0^2(k) &= 0 \\ \phi^2(k+1) - deg^2 + \alpha_1^2(k) &= 0 \\ \phi^2(k+2) - deg^2 + \alpha_2^2(k) &= 0\end{aligned}\quad (5.58)$$

Thus

$$\begin{aligned}-deg^2 + \alpha_2^2(k) + \{l_1\phi(k+1) + l_2\phi(k) \\ + l_3g(k+1)[a_1(\phi)\frac{O(k+1)}{g(k)} + a_2(\phi)\frac{O(k)}{g(k-1)} + bu(k)]\} = 0\end{aligned}\quad (5.59)$$

We can obtain the required input of (5.48) to be:

$$u(k) = \frac{\pm\sqrt{deg^2 - \alpha_2^2(k)} - l_1\phi(k+1) - l_2\phi(k)}{l_3bg(k+1)} - [a_1(\phi)\frac{O(k+1)}{bg(k)} + a_2(\phi)\frac{O(k)}{bg(k-1)}] \quad (5.60)$$

iff $\alpha_2^2(k) \leq deg^2$

This will result in:

$$\phi(k+2) = \pm\sqrt{deg^2 - \alpha_2^2(k)} \quad (5.61)$$

which is linearizable with the following feedback law:

$$v(k) = \pm\sqrt{deg^2 - \alpha_2^2(k)} \quad (5.62)$$

and the $\alpha_2(k)$ -invertible relation is:

$$\alpha_2^2(k) = \pm\sqrt{deg^2 - v^2(k)} \quad (5.63)$$

This will hold as long the following condition is satisfied:

$$v^2(k) \leq deg^2 \quad \forall k = 0, \dots, N-1 \quad (5.64)$$

5.5 CONCLUSIONS

In this chapter, the exact linearization technique was applied to solve the nonlinear quadratic tracking control problem. The solution obtained lacks the simplicity and the recursive structure of the Riccati equation and it has the disadvantage of very strong existence conditions. However, its derivation is straight forward and its objective function has a meaningful

physical interpretation. It was shown in this chapter that exact linearization may add extra constraints to a problem, when we transform it into a linear quadratic problem with nonlinear static constraints. To simplify such the problems for numerical solution, we introduced an algorithm to reduce the dimension of the system from N to its relative degree r , and to transform inequality constraints from state type constraints to input constraints. Finally, we applied these results to the muscle example.

CHAPTER SIX

SUGGESTIONS FOR FUTURE WORK

The controller design methods presented in this dissertation achieve successful regulation and tracking the exactly linearized systems. However, the exact linearization technique assumes complete and exact knowledge of the system (model) parameters, which is not true for practical application. In general, nonlinear systems parameters must be estimated. The robustness of exact linearization controllers, and the development of adaptive exact linear controllers are open topics. Preliminary results on adaptive exact linearization can be found in [Monaco and Normand-Cyrot 1988]

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APPENDIX A

EXACT LINEARIZATION METHOD

A.1 EXACT LINEARIZATION

The idea behind Exact linearization is to linearize a nonlinear control system by state feedback and coordinate change (map).

In this appendix the work of Lee and Marcus [86, 87a-b] is summarized. They solved the following four problems for discrete - time systems:

- (i) Linearization of a system without output, by state coordinate change,
- (ii) Linearization of a system without output, by state coordinate change and feedback,
- (iii) Linearization of an input-output system, by state coordinate change, and,
- (iv) Linearization of an input-output system, by state coordinate change and feedback.

Throughout this section Σ_x will denote a single - input nonlinear discrete - time system, where:

$$\Sigma_x : x(k+1) = f(x(k), u(k)) \quad (A.1)$$

and Σ_o will refer to the same dynamics with output:

$$\begin{aligned} \Sigma_o : y(k) &= h(x(k)) \\ x(k+1) &= f(x(k), u(k)) \end{aligned} \quad (A.2)$$

Let

$$\psi_x = f(f(\cdots (f(x, u_N), \cdots), u_2), u_1),$$

and

$$\mathcal{F}_x = f(f(\cdots (f(x, u_{N+1}), \cdots), u_2), u_1)$$

Also let $\hat{f}^k$ denotes the impulse response at time k .

Definition:

The smallest integer ρ such that $\frac{\partial h(\hat{f}^\rho)}{\partial \mathbf{u}} \neq 0$ is defined to be the **characteristic number** of Σ_o . It is the first instant of time at which the output is affected by the input at time $k = 0$.

Lee and Marcus [86 and 87 a,b] linearized the discrete - time system Σ_x in (A.1) locally, and proved that Σ_x is locally linearizable by state coordinate change iff:

- (1) $(\psi_o)_*|_{\mathbf{u}=0}$ is isomorphism, and
- (2) $[\frac{\partial}{\partial \mathbf{u}_i}, \ker(\mathcal{F}_o)_*] \subset \ker(\mathcal{F}_o)_*$ for $1 \leq i \leq N + 1$.

Furthermore, $T = (\psi_0)^{-1}$ is a linearization coordinate change.

They proved that the second condition in the previous theory is satisfied iff:

- (1) $\hat{f}_*^i(\frac{\partial}{\partial \mathbf{u}})$ is well-defined for $i = 1, 2, \cdots, N + 1$, and
- (2) $[\hat{f}_*^i(\frac{\partial}{\partial \mathbf{u}}), \hat{f}_*^j(\frac{\partial}{\partial \mathbf{u}})] = 0$ for $1 \leq i, j \leq N + 1$

For the linearization of the same system by state coordinate change and feedback, they proved that it is possible to linearize the system locally iff:

- (1) $(\psi_o)_*|_{\mathbf{u}=0}$ is isomorphism, and

$$(2) \left[\frac{\partial}{\partial u_i}, \ker(\mathcal{F}_o)_* \right] \subset \ker(\mathcal{F}_o)_* + \text{span}\left\{ \frac{\partial}{\partial u_i} \right\} \quad \text{for } 1 \leq i \leq N-1.$$

Lee and Marcus also solved for the linearization of the discrete - time systems with output via state coordinate change only. They obtained the following linearization conditions:

$$(1) \left\{ \left(\frac{\partial f}{\partial u} \right)_{0,0}, \left(\frac{\partial f}{\partial x} \right)_{0,0} \left(\frac{\partial f}{\partial u} \right)_{0,0}, \dots, \left(\frac{\partial f}{\partial x} \right)_{0,0}^{N-1} \left(\frac{\partial f}{\partial u} \right)_{0,0} \right\} \text{ are linearly independent,}$$

(2)

$$\left\{ \left(\frac{\partial h}{\partial x} \right)_{x=0}, \left(\frac{\partial h}{\partial x} \right)_{x=0} \left(\frac{\partial f}{\partial x} \right)_{0,0}, \dots, \left(\frac{\partial h}{\partial x} \right)_{x=0} \left(\frac{\partial f}{\partial x} \right)_{0,0}^{N-1} \right\}$$

are linearly independent, and

$$(3) \frac{\partial}{\partial u}(h \circ \hat{f}^i) = \text{constant} \quad 1 \leq i \leq N, \text{ and,}$$

$$(4) h \circ \hat{f}^N(x, 0) \in \text{span}\{h(x), h \circ f(x, 0), \dots, h \circ \hat{f}^{N-1}(x, 0)\}.$$

Finally, they proved that system Σ_o can be locally linearized by state coordinate change and feedback if and only if:

$$(1) \left\{ \left(\frac{\partial f}{\partial u} \right)_{0,0}, \left(\frac{\partial f}{\partial x} \right)_{0,0} \left(\frac{\partial f}{\partial u} \right)_{0,0}, \dots, \left(\frac{\partial f}{\partial x} \right)_{0,0}^{N-1} \left(\frac{\partial f}{\partial u} \right)_{0,0} \right\} \text{ are linearly independent,}$$

$$(2) \frac{\partial}{\partial u}(h \circ \hat{f}^0)_{0,0} \neq 0,$$

$$(3) \tilde{f}_*^i \left(\frac{\partial}{\partial v_i} \right) \text{ is a well-defined smooth vector field on an open neighborhood of } 0 \in R^N \text{ for } 1 \leq i \leq N+1 \text{ where } \tilde{f}(x, v) = f(x, g(x, v)) \text{ and } h \circ \hat{f}(x, g(x, v)) = v.$$

APPENDIX B

MUSCLE MODEL LINEARIZATION

The idea behind the exact linearization is to find a feedback law and a state map to transfer the nonlinear system into a linear one. For the muscle model, it is required to find a feedback law;

$$u(k) = \alpha(x) + \beta(x, v(k)) \quad (B.1)$$

and a state map;

$$z(k) = T(x(k)), \quad (B.2)$$

such that the general muscle model;

$$\begin{aligned} \Sigma_1 : y(k) &= c^t x(k) \quad f_2(\phi(k)) \quad f_3(\phi(k), \phi(k-1)) \\ x(k) &= \begin{pmatrix} 0 & 1 \\ a_2(\phi) & a_1(\phi) \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u(k-1) \end{aligned} \quad (B.3)$$

is equivalent to the linear system:

$$\begin{aligned} \Sigma_2 : y(k) &= \hat{c}^t z(k) \\ z(k) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(k-1) \\ z_2(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(k-1). \end{aligned} \quad (B.4)$$

In the following lemma, it will be proved that systems (B.3) and (B.4) are equivalent, and the state transformation and the feedback are given. Linearization conditions are also obtained.

Lemma B.1 :

The muscle model Σ_1 is equivalent to the linear system Σ_2 if and only if:

$$\left. \begin{array}{l} f_2(\phi) \neq 0 \\ f_3(\phi) \neq 0 \end{array} \right\} \quad \forall k = 0, \dots, N-1.$$

Moreover, if we define:

$$g(k) = f_2(\phi(k)) f_3(\phi(k), \phi(k-1))$$

and

$$\underline{a}^t = (a_2(\phi) \quad a_1(\phi)) \quad (B.5)$$

then the transformation can be made with the state map:

$$z(k) = \begin{pmatrix} g(k-1) & 0 \\ 0 & g(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \quad (B.6)$$

and the feedback law:

$$u(k-1) = -\frac{\langle \underline{a}, x(k-1) \rangle}{b} + \frac{1}{bg(k)}v(k-1), \quad (B.7)$$

That is,

$$\begin{aligned} \alpha(\cdot) &= -\frac{\langle \underline{a}, x \rangle}{b} \\ \beta(\cdot) &= \frac{1}{bg(k)}v(k-1) \end{aligned} \quad (B.8)$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Proof :

Necessary condition: Since $y(k)$ is linear function in $x(k)$, where

$$y(k) = x_2(k)g(k),$$

and since $y(k)$ is also linear function in $z(k)$, where

$$y(k) = c^t z(k),$$

$$c^t z(k) = x_2(k)g(k)$$

then the map between $z(k)$ and $x(k)$ (if it exists), is linear:

$$z(k) = T(\phi)x(k)$$

$$z(k) = \begin{pmatrix} T_{11}(k) & T_{12}(k) \\ T_{21}(k) & T_{22}(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \quad (B.9)$$

From Σ_2 and (B.9):

$$\begin{aligned} y(k) &= x_2(k)g(k) = \hat{c}_1 z_1(k) + \hat{c}_2 z_2(k) \\ &= \hat{c}_1 [T_{11}x_1 + T_{12}x_2] + \hat{c}_2 [T_{21}x_1 + T_{22}x_2] \\ &= x_1 [\hat{c}_1 T_{11} + \hat{c}_2 T_{21}] + x_2 [\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \end{aligned} \quad (B.10)$$

Since $y(\cdot)$ is not an explicit function in x_1 , then the coefficient of x_1 equals zero (since x_1 and x_2 are independent variables):

$$[\hat{c}_1 T_{11} + \hat{c}_2 T_{21}] = 0 \quad (B.11)$$

Therefore,

$$y(k) = x_2(k)g(k) = x_2(k)[\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \quad (B.12)$$

and

$$[\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] = g(k) \quad (B.13)$$

The difference equation of system Σ_2 can be computed as follows:

$$\begin{aligned} y(k) &= \hat{c}^t z(k) = \hat{c}_1 z_1(k) + \hat{c}_2 z_2(k) \\ &= \hat{c}_1 z_2(k-1) + \hat{c}_2 z_2(k) \\ &= \hat{c}_1 v(k-2) + \hat{c}_2 v(k-1) \end{aligned} \quad (B.14)$$

From (B.12) and (B.14) we get:

$$\hat{c}_1 v(k-2) + \hat{c}_2 v(k-1) = x_2(k)[\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \quad (B.15)$$

Substituting the nonlinear state equation of x_2 , from Σ_1 , in (B.15) we get:

$$\hat{c}_1 v(k-2) + \hat{c}_2 v(k-1) = [\langle \underline{a}, x(k-1) \rangle + bu(k-1)][\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \quad (B.16)$$

where $\underline{a}^t = (a_2(\phi) \quad a_1(\phi))$

Referring to Figure 9:

$$u(k-1) = \alpha(x(k-1)) + \beta(x(k-1), v(k-1)) \quad (B.17)$$

Substituting (B.17) in (B.16);

$$\begin{aligned} \hat{c}_1 v(k-2) + \hat{c}_2 v(k-1) = [\langle \underline{a}, x(k-1) \rangle + b(\alpha(x(k-1)) \\ + b\beta(x(k-1), v(k-1)))] [\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \end{aligned} \quad (B.18)$$

Equating all terms that are function in $v(\cdot)$ we get:

$$\hat{c}_1 v(k-2) + \hat{c}_2 v(k-1) = [b\beta(x(k-1), v(k-1))][\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \quad (B.19)$$

Equating all terms that are not function in $v(\cdot)$ we get:

$$0 = [\langle \underline{a}, x(k-1) \rangle + b\alpha(x(k-1))][\hat{c}_1 T_{12} + \hat{c}_2 T_{22}] \quad (B.20)$$

Since the RHS of (B.19) is not an explicit function in $v(k-2)$, then

$$\hat{c}_1 = 0. \quad (B.21)$$

From (B.13) and (B.21);

$$T_{22} = \frac{g(k)}{\hat{c}_2} \quad (B.22)$$

assuming $\hat{c}_2 \neq 0$. However, we can let $\hat{c}_2 = 1$.

Since the LHS of (B.19) is not a function in $x(\cdot)$, then β is a function only in $v(k-1)$; i.e, (B.19) will become:

$$\hat{c}_2 v(k-1) = \hat{c}_2 T_{22} b \beta(v(k-1)) \quad (B.23)$$

From (B.22) and (B.23) and if $g \neq 0, \forall k = 0 \dots N-1$, then:

$$\beta(v(k-1)) = \frac{1}{bg(k)} v(k-1) \quad (B.24)$$

From (B.20), (B.21) and also if $g \neq 0, \forall k = 0 \dots N-1$ we obtain

$$\alpha(k) = -\frac{< \underline{a}, x >}{b} \quad (B.25)$$

From (B.11) and (B.21) we get $T_{21} = 0$. Therefore, the state map will be:

$$\begin{aligned} z_1(k) &= T_{11}x_1(k) + T_{12}x_2(k) \\ z_2(k) &= T_{22}x_2(k) \end{aligned} \quad (B.26)$$

Since $T_{22} = g$ and $g \neq 0, \forall k = 0 \dots N-1$, then

$$x_2(k) = \frac{z_2(k)}{T_{22}} = \frac{z_2(k)}{g(k)} \quad (B.27)$$

but

$$x_1(k) = x_2(k-1) = \frac{z_2(k-1)}{g(k-1)} \quad (B.28)$$

and

$$z_1(k) = z_2(k-1)$$

so

$$x_1(k) = \frac{z_1(k)}{g(k-1)} \quad (B.29)$$

or

$$z_1(k) = x_1(k)g(k-1) = T_{11}x_1(k) + T_{12}x_2(k)$$

so $T_{12} = 0$ and

$$T_{11}(k) = g(k-1). \quad (B.30)$$

Sufficient conditions: If $g(k) \neq 0$ then by direct substitution both systems can be proved to be equivalent.

APPENDIX C

NONLINEAR QUADRATIC CONTROLLER

CONTINUOUS - TIME PROBLEM

C.1. PROBLEM FORMULATION

Suppose that it is required to track a function $r(t)$ according to the following quadratic objective function:

$$J = \frac{1}{2}[x(t_f) - r(t_f)]^t H[x(t_f) - r(t_f)] + \frac{1}{2} \left[\int_{t_0}^{t_f} [x(t) - r(t)]^t Q[x(t) - r(t)] + u(t)^t R u(t) \right] dt \quad (C.1)$$

under the nonlinear dynamic constraints:

$$S_1 : \quad \dot{x}(t) = f(x) + g(x)u(t) \quad (C.2)$$

C.2 SOLUTION

Theorem :

- If the nonlinear system S_1 is controllable, and
- if the nonlinear system :

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f(x) - g(x)R^{-1}g(x)^t \lambda(t) \\ -Qx - \left(\frac{\partial f}{\partial x}\right)^t \lambda(t) + \left[\frac{\partial(g^t(x)\lambda(t))}{\partial x}\right]^t R^{-1}g^t(x)\lambda(t) \end{pmatrix} + \begin{pmatrix} 0 \\ Q \end{pmatrix} r(t) \quad (C.3)$$

can be immersed into a linear one then:

- 1- there exists a map $\tilde{T} : x(t) \rightarrow \lambda(t)$, and
- 2- an optimal control input u^* exists and has the form:

$$u^* = -R^{-1}g^t(x)\tilde{T}(x(t))$$

where $\lambda(t)$ is the costate.

Proof

The Hamiltonian equation of the system is given by:

$$H = \frac{1}{2}[x - r]^t Q[x - r] + \frac{1}{2}u^t R u + \lambda^t[f + g u] \quad (C.4)$$

The first order optimality conditions are:

$$\begin{aligned} \frac{\partial H}{\partial u} &= R u + g^t \lambda \\ \frac{\partial H}{\partial \lambda} &= \dot{x} = f(x) + g(x)u(t) \\ -\frac{\partial H}{\partial x} &= \dot{\lambda} = -Qx - \left(\frac{\partial f}{\partial x}\right)^t \lambda(t) - \left[\frac{\partial(g^t(x)\lambda(t))}{\partial x}\right]^t u(t) + Qr \end{aligned} \quad (C.5)$$

which yield:

$$\begin{aligned} u^* &= -R^{-1}g^t(x)\lambda(t) \\ \dot{x} &= f(x) - g(x)R^{-1}g^t(x)\lambda(t) \\ \dot{\lambda} &= -Qx - \left(\frac{\partial f}{\partial x}\right)^t \lambda(t) + \left[\frac{\partial(g^t(x)\lambda(t))}{\partial x}\right]^t R^{-1}g^t(x)\lambda(t) + Qr(t) \end{aligned} \quad (C.6)$$

The next step is to immerse the nonlinear system into a linear one; i.e. to find an equivalent linear system so that both should have the same input $r(t)$. If this condition is satisfied, then there exists a linear system $z(t)$, a transformation $z(t) = T(x, \lambda)$ such that:

$$\dot{z}(t) = A z(t) + b r(t) \quad (C.7)$$

and

$$z(t) = \phi(t, t_f)z(t_f) + \int_t^{t_f} \phi(t, \tau)br(\tau)d\tau \quad (C.8)$$

where $\phi(t, t_f) = e^{A(t-t_f)}$

Then a solution for the feedback law can be derived according to the following steps:

Step 1: Since $r(t)$ is known, the previous formula can be evaluated to be function only in $z(t_f)$; i.e.,

$$z(t) = \psi_z(z(t_f)). \quad (C.9)$$

Step 2: But $z(t_f) = T(x(t_f), \lambda(t_f))$, then

$$z(t) = \psi_z \circ T(x(t_f), \lambda(t_f)). \quad (C.10)$$

Step 3: From the boundary condition of the original system:

$$\lambda(t_f) = Hx(t_f) - Hr(t_f) \quad (C.11)$$

Step 4: Therefor:

$$z(t_f) = T(x(t_f), Hx(t_f) - Hr(t_f)) = T(x(t_f), r(t_f)) \quad (C.12)$$

and $r(t_f)$ is known, so:

$$z(t_f) = T(x(t_f)) \quad (C.13)$$

$$z(t) = \psi_z \circ T(x(t_f)). \quad (C.14)$$

Step 5: Since T is invertible (immersion condition), then there exists T_1, T_2 such that:

$$x(t) = T_1(z(t)) \quad (C.15)$$

$$\lambda(t) = T_2(z(t)). \quad (C.16)$$

Step 6: Since

$$\begin{aligned} x(t) &= T_1(z(t)) = T_1 \circ \psi_z \circ T(x(t_f)) \\ &= \psi_z(x(t_f)), \end{aligned} \quad (C.17)$$

and since the original system is controllable then:

$$\begin{aligned} x(t_f) &= (T_1 \circ \psi_z \circ T)^{-1}(x(t)) \\ &= \psi_z^{-1}(x(t)). \end{aligned} \quad (C.18)$$

Step 7: Finally,

$$\begin{aligned} \lambda(t) &= T_2(z) \\ &= T_2 \circ \psi_z \circ T(x(t_f)) \\ &= T_2 \circ \psi_z \circ T \circ \psi_z^{-1}(x(t)) \\ &= \tilde{T}(x) \end{aligned} \quad (C.19)$$

and can be substituted in the optimal control feedback law:

$$u^* = -R^{-1}g(x)^t \tilde{T}(x). \quad (C.20)$$

- 1- The resulting control law depends on the ability of finding an equivalent linear system (immersion conditions).
- 2- The next step is to include the direct evaluation of the necessary and sufficient conditions for the immersion.