CYCLIC BEHAVIOR OF HOLOMORPHIC FUNCTIONS ON A RUNGE REGION

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ABSTRACT

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Let $\Omega \subset \mathbb{C}^N$ be a Runge region and let $H(\Omega)$ denote the Fréchet space of holomorphic functions on Ω . In this dissertation, we explore the cyclic behavior of various operators defined on $H(\Omega)$. First, we provide extensions of some earlier results regarding nonscalar continuous linear operators on $H(\Omega)$ commuting with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Specifically, we demonstrate that all such operators are hypercyclic and share a dense set of common cyclic vectors. Motivated by our results, we introduce a class of finite sets of Fréchet space operators patterned after the partial differentiation operators, called backward multi-shifts, and show that any nonscalar operator in the commutant of such a finite set is supercyclic. Lastly, we demonstrate the existence of a dense set of common cyclic vectors for all nonscalar operators in the commutant of a *weighted differentiation operator* $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ defined by $B_\lambda f(z) = \frac{d}{dz}[f(\lambda z)]$, where $\lambda \neq 0$. To do so, we need to make use of the method of spectral synthesis, as well as more classical techniques for different values of λ .

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TABLE OF CONTENTS

	Page
CHAPTER 1 INTRODUCTION	1
CHAPTER 2 OPERATORS COMMUTING WITH DIFFERENTIATION	4
2.1 Hypercyclicity	4
2.2 Common Cyclicity	7
CHAPTER 3 INTRODUCING BACKWARD MULTI-SHIFTS	16
3.1 Preliminary Results	16
3.2 Supercyclicity	25
CHAPTER 4 EXAMPLES OF BACKWARD MULTI-SHIFTS	31
4.1 Partial Differentiation	31
4.2 Weighted Partial Differentiation	32
4.3 Double Sequence Spaces	36
CHAPTER 5 COMMON CYCLICITY AND WEIGHTED DIFFERENTIATION	39
5.1 Introduction	39
5.2 Weighted Differentiation and Spectral Synthesis	39
5.3 Contracting Differentiation	49
5.4 Conclusion	62
BIBLIOGRAPHY	65

CHAPTER 1 INTRODUCTION

If X is a metrizable topological vector space and $T : X \rightarrow X$ is a continuous linear operator, then T is said to be *hypercyclic* if there exists a vector $x \in X$ such that its T -orbit $\text{Orb}(T, x) := \{x, Tx, T^2x, \dots\}$ is dense in X . Any such vector x is said to be a *hypercyclic vector* of T . A much weaker form of hypercyclicity is that of cyclicity: T is said to be *cyclic* if there exists a vector $x \in X$ such that the closed linear span of its T -orbit is the whole space X ; that is, $\overline{\text{span}} \text{Orb}(T, x) = X$, or equivalently, $\{p(T)x : p \text{ is a polynomial}\}$ is dense in X . Any such vector x is said to be a *cyclic vector* of T . In between hypercyclicity and cyclicity lies the concept of supercyclicity: T is said to be *supercyclic* if there exists a vector $x \in X$ such that the set of all scalar multiples of vectors in its T -orbit is dense in X ; that is, $\overline{\mathbb{C} \cdot \text{Orb}(T, x)} = X$.

Clearly, every hypercyclic operator is supercyclic and every supercyclic operator is cyclic. These operators have been extensively studied and there are many classical results. For example, every unilateral weighted backward shift on l^p with $1 \leq p < \infty$ is supercyclic; see Hilden and Wallen [11]. In the special case that the weights of such a shift are a constant $\lambda > 1$, the shift is hypercyclic; see Rolewicz [15].

There are classical examples that take place in the Fréchet spaces $H(\Omega)$ of holomorphic functions on a region Ω in $\mathbb{C}$ with the compact-open topology; that is, the topology of uniform convergence on compact subsets of Ω . For instance, Birkoff [4] and MacLane [12] showed that the operators of translation and differentiation, respectively, are hypercyclic on the Fréchet space $H(\mathbb{C})$ of entire functions. These results were strengthened by Godefroy and Shapiro [10], who further showed that nonscalar operators on $H(\mathbb{C}^N)$ that commute with translation, or equivalently with differentiation, are hypercyclic.

Note that translation operators, unlike differentiation, need not be well defined on $H(\Omega)$ for a general region $\Omega \subset \mathbb{C}^N$. Shapiro [20] later showed that if D is the differentiation operator and P is a nonconstant polynomial then $P(D) : H(\Omega) \rightarrow H(\Omega)$ is hypercyclic if and only if $\Omega \subset \mathbb{C}$ is simply connected. A harmonic version of Shapiro's result was obtained by Chan, Kadel, and

Pinheiro [8].

Of course, if Ω is not simply connected then we can take a point ζ in a bounded component of $\mathbb{C} \setminus \Omega$ and observe that $\frac{1}{z - \zeta}$ is not in the closure of the range of the differentiation operator $D : H(\Omega) \rightarrow H(\Omega)$. This follows from the uniqueness of the Laurent series representation of a holomorphic function on an annulus. Thus, when Ω is simply connected, the denseness of the polynomials in $H(\Omega)$ plays an important role in the aforementioned work of Shapiro [20]. In the present paper, we study regions Ω in $\mathbb{C}^N$ for which the denseness of polynomials in $H(\Omega)$ is translated into the following terminology.

Definition 1.1. A region Ω in $\mathbb{C}^N$ is called a *Runge region* if the algebra of polynomials $\mathcal{P}$ is dense in the Fréchet space $H(\Omega)$ of holomorphic functions on Ω .

Most of this dissertation is dedicated to the study of operators on the Fréchet space $H(\Omega)$ of holomorphic functions on a Runge region $\Omega \subset \mathbb{C}^N$. In Chapter 2 we focus on nonscalar operators on $H(\Omega)$ that commute with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. In Section 2.1, we show that every such operator is hypercyclic; see Theorem 2.3. Our result extends not only the aforementioned work of Godefroy and Shapiro [10] to a several-variable setting, but also the work of L. Bernal-González [3], who proved the result for a subclass of such operators on $H(\Omega)$. In Section 2.2, we show that the collection of all such operators has a dense set of common cyclic vectors; see Corollary 2.9. If $\mathcal{T}$ denotes a set of continuous linear operators on a metrizable topological vector space X , and x is a cyclic vector for each operator $T \in \mathcal{T}$, then we say that x is a *common cyclic vector* for $\mathcal{T}$. Specifically, in Theorem 2.8 we provide a new proof for a result of Bourdon and Shapiro [5], who considered the case $\Omega = \mathbb{C}^N$, and we also extend their result to the space $H(\Omega)$, where $\Omega \subset \mathbb{C}^N$ is any Runge region.

In Chapter 3, we develop the new machinery of a *backward multi-shift* as given in Definition 3.2. A backward multi-shift can be thought of as a multi-operator or multi-dimensional analogue of a generalized backward shift, studied by Godefroy and Shapiro [10]. We first obtain several preliminary results regarding backward multi-shifts in Section 3.1 that are analogous to those of generalized backward shifts in [10]; see Lemma 3.3, Corollary 3.4, and Corollary 3.5. Then, in

Section 3.2 we provide a backward multi-shift analogue of a result of Müller [14, Theorem 9], who answered in the positive a question in [10] that asked whether every nonscalar operator commuting with a generalized backward shift is supercyclic; see Theorem 3.9.

Chapter 4 continues our work on backward multi-shifts, as we provide several examples of backward multi-shifts and apply Theorem 3.9 to obtain multiple corollaries. In Section 4.1, we obtain that if Ω is a Runge region in $\mathbb{C}^N$ and L is a nonscalar continuous linear operator commuting with each partial differentiation operator $\frac{\partial}{\partial z_k}$, then L is supercyclic; see Corollary 4.1. Furthermore, in Section 4.2 we prove a $\mathbb{C}^N$ analogue of a result of Aron and Markose [1] and Fernández and Hallack [9], who proved that for given scalars $\lambda \neq 0$ and b the weighted differentiation $B : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ defined by

$$Bf(z) = \frac{d}{dz}[f(\lambda z + b)] = \lambda f'(\lambda z + b)$$

is hypercyclic if $|\lambda| \geq 1$; see Corollary 4.7. For our last example, in Section 4.3 we apply Theorem 3.9 on double sequence spaces $\mathcal{L}_q$ to show that any nonscalar operator commuting with the two backward shifts on $\mathcal{L}_q$ is supercyclic; see Corollary 4.11. Our result expands the study of Shu, Wang, and Zhao [21] from the two backward shifts to nonscalar operators commuting with them.

Finally, in Chapter 5 we return to the study of common cyclicity, this time considering nonscalar linear operators on $H(\mathbb{C})$ that commute with weighted differentiation operators of the form

$$B_\lambda f(z) = \frac{d}{dz}[f(\lambda z)] = \lambda f'(\lambda z).$$

We show that, given any such $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, there exists a dense set of common cyclic vectors for all nonscalar operators in the commutant of B_λ ; see Theorem 3.2. To do so, we proceed with two different techniques. In Section 5.2, we borrow the spectral synthesis techniques of Bourdon and Shapiro [5] to tackle the case when $|\lambda| \geq 1$. Then, in Section 5.3 we borrow the techniques of Wogen [22] in the case when $0 < |\lambda| < 1$.

CHAPTER 2 OPERATORS COMMUTING WITH DIFFERENTIATION

2.1 Hypercyclicity

In 1952, MacLane [12] showed that the differentiation operator D given by $Df(z) = f'(z)$ is hypercyclic on the Fréchet space $H(\mathbb{C})$ of entire functions. Godefroy and Shapiro [10, Theorem 5.1, Proposition 5.2] in 1991 greatly generalized MacLane's result to hold for any nonscalar operator $L : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$ that commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Shapiro [20, Theorem 3.2] showed in 1998 that a nonscalar operator $L : H(\Omega) \rightarrow H(\Omega)$ commuting with the differentiation D is hypercyclic if $\Omega \subset \mathbb{C}$ is simply connected. A year later, Bernal-González [3, Theorem 8c] extended the result of Godefroy and Shapiro to nonscalar operators $\phi(D) : H(\Omega) \rightarrow H(\Omega)$ where $\phi : \mathbb{C}^N \rightarrow \mathbb{C}$ is an entire function of subexponential type, D is given by $D = \frac{\partial}{\partial z_1} \dots \frac{\partial}{\partial z_N}$, and Ω is a Runge region in $\mathbb{C}^N$. The operator $\phi(D)$ is defined using the power series representation of ϕ and notation $D^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial z_N}\right)^{\alpha_N}$ for any N -tuple $\alpha = (\alpha_1, \dots, \alpha_N)$ of nonnegative integers. To continue this line of research, we show in Theorem 2.3 below that Bernal-González's result indeed holds for all nonscalar operators $L : H(\Omega) \rightarrow H(\Omega)$ that commute with each partial differentiation operator.

To begin, we need the following definition, which is the multi-dimensional version of [20, Definition 4.2].

Definition 2.1. Let $\Omega \subset \mathbb{C}^N$ be a region and $L : H(\Omega) \rightarrow H(\Omega)$ be a continuous linear operator that commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Given $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$, denote $e_\alpha(z) = \exp(z_1\alpha_1 + \dots + z_N\alpha_N)$ for all $z = (z_1, \dots, z_N) \in \mathbb{C}^N$. Note that e_α is an entire function and $e_\alpha \in H(\Omega)$. In addition,

$$\frac{\partial}{\partial z_k}(Le_\alpha) = \left(L \frac{\partial}{\partial z_k}\right) e_\alpha = \alpha_k Le_\alpha, \text{ whenever } 1 \leq k \leq N.$$

and thus e_α is an eigenvector of L . Therefore, we can define

$$Le_\alpha = F_L(\alpha)e_\alpha.$$

The function $F_L : \mathbb{C}^N \rightarrow \mathbb{C}$ is called the *characteristic function* of L .

As it turns out, the characteristic function F_L is a holomorphic function that provides crucial properties of L that we need in our discussion. To explain that, we first need the following generalization of [20, Lemma 4.5], which was presented for the one-variable case. We provide a proof for the several-variable case for the sake of completeness of our exposition. In the proof, we use the symbol $\|f\|_K = \sup\{|f(z)| : z \in K\}$ for any compact subset K of Ω .

Lemma 2.2. *Let Ω be a region in $\mathbb{C}^N$ and $L : H(\Omega) \rightarrow H(\Omega)$ be a nonscalar continuous linear operator that commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Then its characteristic function F_L is an entire function of exponential type.*

Proof. Given any $\epsilon > 0$ and compact set $K \subset \Omega$, denote a neighborhood $N(K, \epsilon)$ of zero in $H(\Omega)$ as

$$N(K, \epsilon) := \{f \in H(\Omega) : \|f\|_K < \epsilon\}.$$

Since L is continuous, the set $L^{-1}(N(K, \epsilon))$ is also a neighborhood of zero. Thus there exists $\delta > 0$ and a compact subset $J \subset \Omega$ such that $L(N(J, \delta)) \subset N(K, \epsilon)$. Thus if $f \in H(\Omega)$, then

$$\frac{\delta f}{\|f\|_J} \in N(J, \delta), \text{ and so } \left\| L \left(\frac{\delta f}{\|f\|_J} \right) \right\|_K < \epsilon.$$

Taking $\epsilon = 1$ and setting $A = 1/\delta$, we obtain

$$\|Lf\|_K \leq A\|f\|_J \text{ for all } f \in H(\Omega). \quad (2.1.1)$$

By taking $f = e_\alpha$ and $K = \{\xi\} \subset \Omega$ we can restate 2.1.1 as.

$$|F_L(\alpha)e_\alpha(\xi)| = |(Le_\alpha)(\xi)| = \|Le_\alpha\|_K \leq A\|e_\alpha\|_J \leq Ae^{C|\alpha|},$$

where $C = \max\{|z| : z \in J\}$. It follows that $|F_L(\alpha)| \leq Ae^{C|\alpha|}/|e_\alpha(\xi)|$, and so if we let $B = C + |\xi|$, then $|F_L(\alpha)| \leq Ae^{B|\alpha|}$. Hence, F_L is an entire function of exponential type. $\square$

Using Lemma 2.2 and borrowing arguments from Godefroy and Shapiro [10] and Shapiro [20, Corollary 4.7], we now prove the main result of this section.

Theorem 2.3. *Let Ω be a Runge region in $\mathbb{C}^N$. If $L : H(\Omega) \rightarrow H(\Omega)$ is a nonscalar continuous linear operator that commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$, then L is hypercyclic.*

Proof. By Lemma 2.2, the characteristic function F_L of L is an entire function of exponential type, so by [10, Remark 5.3] the power series representation of $F_L(D)$ defines a continuous linear operator on $H(\mathbb{C}^N)$ and hence on $E = \{f|_\Omega : f \in H(\mathbb{C}^N)\}$. Since $F_L(D)$ and $L|_E$ are continuous operators on E that agree on the dense set $\{e_\alpha|_\Omega : \alpha \in \mathbb{C}^N\}$, we have

$$L = F_L(D) \text{ on } E. \tag{2.1.2}$$

Note that by [10, Theorem 5.1, Proposition 5.2], $F_L(D)$ defines a hypercyclic operator on $H(\mathbb{C}^N)$. Thus there exists an entire function $f \in H(\mathbb{C}^N)$ such that $\text{Orb}(F_L(D), f)$ is dense in $H(\mathbb{C}^N)$. Hence for any polynomial p , subset K of Ω , and $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|F_L(D)^n f - p\|_K < \epsilon$. Since Ω is a Runge region, this implies that $\text{Orb}(F_L(D), f|_\Omega)$ is dense in $H(\Omega)$. Since $L = F_L(D)$ on E , this implies that $\text{Orb}(L, f|_\Omega)$ is dense in $H(\Omega)$ and thus L is hypercyclic. $\square$

Observe that while Lemma 2.2 holds for any region Ω , the argument for Theorem 2.3 fails if Ω is not a Runge region. This leads to the question of whether Theorem 2.3 holds for any domain of holomorphy Ω in $\mathbb{C}^N$.

2.2 Common Cyclicity

Common cyclic vectors were first studied by Wogen [22], who proved that there is a dense set of vectors in the Hardy space H^2 each of which is a cyclic vector for the adjoint of any nonscalar multiplication operator $M_\phi : H^2 \rightarrow H^2$ defined by $M_\phi f = \phi f$ where $\phi \in H^\infty$. This result was generalized by Chan [6] to a wide class of Banach spaces of analytic functions, including the Bergman space. Chan [7] also demonstrated the existence of a dense set of common cyclic vectors for the family of nonscalar operators that are polynomials in differentiation on the Fréchet space $H(\mathbb{C}^N)$; we present Chan's later result below, using $\mathbb{Z}^+$ to denote the set of all nonnegative integers.

Theorem 2.4. *Let $N \geq 1$. If $\alpha = (a_1, \dots, a_N) \in \mathbb{Z}^+ \times \dots \times \mathbb{Z}^+$ is a multi-index, $D^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{a_1} \dots \left(\frac{\partial}{\partial z_N}\right)^{a_N}$ denotes a partial differentiation operator on $H(\mathbb{C}^N)$. Denote the family of nonscalar operators that are polynomials in D as*

$$\mathcal{P}(D) := \{p(D) : p \text{ is a nonconstant polynomial}\}.$$

Then the set of all common cyclic vectors for $\mathcal{P}(D)$ is nonempty and in fact dense in $H(\mathbb{C}^N)$.

Later, the technique of spectral synthesis was used by Bourdon and Shapiro [5] to extend Chan's results to a larger class of analytic function spaces. In particular, their technique applies to nonscalar operators commuting with each partial differentiation operator $\frac{\partial}{\partial z_k}$ on the Fréchet space $H(\mathbb{C}^N)$ of entire functions, where $N \in \mathbb{N}$ and $1 \leq k \leq N$. Marín Jr. and Seubert [13] proved that various collections of cyclic diagonal operators on $H(\mathbb{C})$ have dense sets of common cyclic vectors, where a diagonal operator is a continuous linear operators having the monomials z^n as eigenvectors for all $n \in \mathbb{Z}^+$. Their result was later extended to more general collections of cyclic diagonal operators by Seubert [19]. Meanwhile, Ross and Wogen [16, 17] investigated the common cyclicity of various collections of multiplication operators on the Hilbert Space $L^2(\mu)$ for various measures μ on $\mathbb{C}$ or the unit circle $\mathbb{T}$.

In this section, we provide an alternative proof of Bourdon and Shapiro's [5] result regarding

nonscalar operators commuting with each partial differentiation operator $\frac{\partial}{\partial z_k}$; furthermore, we extend their result from $H(\mathbb{C}^N)$ to $H(\Omega)$, where $\Omega \subset \mathbb{C}^N$ is a Runge region. That is, we show that the conclusion of Theorem 2.4 still holds if we replace $\mathbb{C}^N$ by a Runge region Ω in $\mathbb{C}^N$ and the family $\mathcal{P}(D)$ with the family $\text{Com}_D(\Omega)$ of all nonscalar continuous linear operators on $H(\Omega)$ that commute with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$.

First, in the following lemma we paraphrase the classification of the operators in $\text{Com}_D(\mathbb{C}^N)$ given in [10, Proposition 5.2, Remark 5.3], which shows that an operator commuting with each partial differentiation operator on $H(\mathbb{C}^N)$ can be represented as a power series of D .

Lemma 2.5. *A continuous linear operator $L : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$ commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$, if and only if*

$$L = \sum_{i_N=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \alpha_{i_1, i_2, \dots, i_N} D_1^{i_1} D_2^{i_2} \cdots D_N^{i_N},$$

where $\alpha_{i_1, i_2, \dots, i_N} \in \mathbb{C}$, for all of which there exists some $C > 0$ depending only on L with

$$|\alpha_{i_1, \dots, i_N}| \leq \frac{C^{i_1 + \dots + i_N}}{i_1! \cdots i_N!}.$$

For notational simplicity, we focus on the case where $N = 2$. In this case, Lemma 2.5 states that L has the form $L = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_{i,j} D_1^i D_2^j$, where there exists $C > 0$ such that $|\alpha_{i,j}| \leq \frac{C^{i+j}}{i!j!}$. In order to proceed, we need a variant of [7, Lemma], which was used by Chan to prove [7, Theorem]. The lemma makes use of a projection map, which we define for the $N = 2$ case as part of the following remark, which also contains conventions regarding multi-indices. These conventions can be found in [7]. Though these conventions are specific to the case where $N = 2$, similar conventions can be made for other integers $N \geq 3$. These conventions are also used in Section 3.2.

Remark 2.6. We define a partial ordering on $\mathbb{Z}^+ \times \mathbb{Z}^+$ as follows: for $\alpha = (m, n)$ and $\beta = (p, q)$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, write $\alpha < \beta$ or $\beta > \alpha$ when either (i) $m + n < p + q$ or (ii) $m + n = p + q$ and $m < p$. We also write $\alpha \leq \beta$ (or $\beta \geq \alpha$) to mean $\alpha < \beta$ or $\alpha = \beta$. The relation $\leq$ is a linear ordering on

$\mathbb{Z}^+ \times \mathbb{Z}^+$. It is easy to check that if $\alpha < \gamma$ and $\alpha + \beta - \gamma \in \mathbb{Z}^+ \times \mathbb{Z}^+$, then $\alpha + \beta - \gamma < \beta$. Also, we use the notation $\alpha! = m!n!$.

Associated with this ordering, there is a natural mapping $J : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ so that $J(0, 0) = 0$ and $J(\alpha) < J(\beta)$ whenever $\alpha < \beta$. More precisely, given $\alpha = (m, n)$ one can easily check that one example of J is given by

$$J(\alpha) = \frac{|\alpha|}{2}(|\alpha| + 1) + m,$$

where $|\alpha| = m + n$.

For each $\delta \in \mathbb{Z}^+ \times \mathbb{Z}^+$, define a linear projection map $Q_\delta : H(\mathbb{C}^2) \rightarrow H(\mathbb{C}^2)$ onto $\overline{\text{span}}\{z^\alpha : \alpha \geq \delta\}$ by $Q_\delta(f) = \sum_{\alpha \geq \delta} a_\alpha z^\alpha$ if $f(z) = \sum_\alpha a_\alpha z^\alpha$.

Similar maps and linear orderings can also be given for multi-indices of higher orders. To state the lemma, we use the notation $\|f\|_R = \sup\{|f(z)| : |z| \leq R\}$, where $f \in H(\mathbb{C}^N)$, as we have done in Section 2.1.

Lemma 2.7. *Suppose $S : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$ is a nonscalar continuous linear operator that commutes with each partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$, such that $S = D^\rho + \sum_{\sigma > \rho} u_\sigma D^\sigma$ for some $u_\sigma \in \mathbb{C}$ and a nonzero multi-index ρ . Let $f(z) = \sum_\alpha a_\alpha z^\alpha \in H(\mathbb{C}^N)$. Let $R > 1$ and let δ any multi-index such that $a_{\rho+\delta} \neq 0$. Then*

$$\left\| \frac{\delta!}{(\rho + \delta)!a_{\rho+\delta}} Q_\delta(Sf) - z^\delta \right\|_R \leq \frac{\delta!}{(\rho + \delta)!|a_{\rho+\delta}|} \left(1 + \sum_{\sigma > \rho} |u_\sigma| \right) \sum_{\alpha > \rho+\delta} |a_\alpha| \alpha! R^{|\alpha|}.$$

Proof. We provide a proof for the case where $N = 2$ for notational simplicity. Other cases can be proved with the same argument.

The proof of [7, Lemma] demonstrates that if ρ is nonzero then

$$\left\| Q_\delta(D^\rho f) - \frac{(\rho + \delta)!a_{\rho+\delta}}{\delta!} z^\delta \right\|_R \leq \sum_{\alpha > \rho+\delta} |a_\alpha| \alpha! R^{|\alpha|}, \quad (2.2.1)$$

and also that, for all $\beta > \rho$,

$$\|Q_\delta(D^\beta f)\|_R \leq \sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|}. \quad (2.2.2)$$

Enumerate all multi-indices $\sigma > \rho$ as $\{\sigma_k\}_{k=1}^\infty$. For notational simplicity, denote the coefficient u_{σ_k} of the operator S by $u_k = u_{\sigma_k}$, and so the partial sums S_k of the operator S become

$$S_n = D^\rho + \sum_{k=1}^n u_k D^{\sigma_k}. \quad (2.2.3)$$

Then

$$\begin{aligned} & \left\| \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} Q_\delta(S_n f) - z^\delta \right\|_R \\ &= \left\| \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} Q_\delta(D^\rho f) - z^\delta + \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} Q_\delta \left(\sum_{k=1}^n u_k D^{\sigma_k} f \right) \right\|_R, \quad \text{by (2.2.3)} \\ &= \left| \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} \right| \cdot \left\| Q_\delta(D^\rho f) - \frac{(\rho + \delta)! a_{\rho + \delta}}{\delta!} z^\delta + Q_\delta \left(\sum_{k=1}^n u_k D^{\sigma_k} f \right) \right\|_R \\ &\leq \left| \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} \right| \left(\left\| Q_\delta(D^\rho f) - \frac{(\rho + \delta)! a_{\rho + \delta}}{\delta!} z^\delta \right\|_R + \left\| Q_\delta \left(\sum_{k=1}^n u_k D^{\sigma_k} f \right) \right\|_R \right) \\ &\leq \left| \frac{\delta!}{(\rho + \delta)! a_{\rho + \delta}} \right| \left(\sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|} + \left\| Q_\delta \left(\sum_{k=1}^n u_k D^{\sigma_k} f \right) \right\|_R \right), \quad \text{by (2.2.1)} \\ &\leq \frac{\delta!}{(\rho + \delta)! |a_{\rho + \delta}|} \left(\sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|} + \sum_{k=1}^n |u_k| \cdot \|Q_\delta(D^{\sigma_k} f)\|_R \right) \\ &\leq \frac{\delta!}{(\rho + \delta)! |a_{\rho + \delta}|} \left(\sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|} + \sum_{k=1}^n |u_k| \sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|} \right), \quad \text{by (2.2.2)} \\ &= \frac{\delta!}{(\rho + \delta)! |a_{\rho + \delta}|} \left(1 + \sum_{k=1}^n |u_k| \right) \sum_{\alpha > \rho + \delta} |a_\alpha| \alpha! R^{|\alpha|}. \end{aligned}$$

Since the sum $\sum_{\sigma > \rho} |u_\sigma|$ is convergent by Lemma 2.5, we have $\sum_{k=1}^n |u_k| \leq \sum_{\sigma > \rho} |u_\sigma|$, so we have

$$\left\| \frac{\delta!}{(\rho + \delta)!a_{\rho+\delta}} Q_\delta(S_n f) - z^\delta \right\|_R \leq \frac{\delta!}{(\rho + \delta)!|a_{\rho+\delta}|} \left(1 + \sum_{\sigma > \rho} |u_\sigma| \right) \sum_{\alpha > \rho+\delta} |a_\alpha| \alpha! R^{|\alpha|}. \quad (2.2.4)$$

Taking the limit of both sides of (2.2.4) as $n \rightarrow \infty$ and using continuity of both Q_δ and $\|\cdot\|_R$, we obtain the desired inequality:

$$\left\| \frac{\delta!}{(\rho + \delta)!a_{\rho+\delta}} Q_\delta(Sf) - z^\delta \right\|_R \leq \frac{\delta!}{(\rho + \delta)!|a_{\rho+\delta}|} \left(1 + \sum_{\sigma > \rho} |u_\sigma| \right) \sum_{\alpha > \rho+\delta} |a_\alpha| \alpha! R^{|\alpha|}.$$

□

We are now ready to prove the main theorem of this section. Again, our theorem is an extension of a result of Chan [7, Theorem], presented as Theorem 2.4 above, and was originally demonstrated by Bourdon and Shapiro [5] with a very different proof.

Theorem 2.8. *Let $\text{Com}_D(\mathbb{C}^N)$ be the family of all nonscalar continuous linear operators L on $H(\mathbb{C}^N)$ that commute with each of the partial differentiation operators $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Then the set of all common cyclic vectors for $\text{Com}_D(\mathbb{C}^N)$ is nonempty and in fact dense in $H(\mathbb{C}^N)$.*

Proof. We present the proof for the case $N = 2$ for notational simplicity. The proof for larger values of N follows from the same argument.

Define $\mathcal{F}$ to be the set of all entire functions $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \in H(\mathbb{C}^2)$ such that $a_{\alpha} \neq 0$ for all multi-indices α , and furthermore for all $C, R > 1$,

$$\lim_{|\delta| \rightarrow \infty} \frac{C^{|\delta|}}{|a_{\delta}|} \sum_{\alpha > \delta} |a_{\alpha}| \alpha! R^{|\alpha|} = 0. \quad (2.2.5)$$

Observe that $\mathcal{F}$ is nonempty. One example can be constructed by inductively choosing

$$0 < a_{\alpha} < \frac{1}{\alpha! 2^{J(\alpha)} |\alpha|^{2|\alpha|}} \min_{\delta < \alpha} |a_{\delta}| \quad (2.2.6)$$

where $J(\alpha)$ is as defined in Remark 2.6. Then (2.2.5) is satisfied because for all $C, R > 1$, if $|\delta| > \max\{C, R\}$, we get

$$\frac{C^{|\delta|}}{|a_\delta|} \sum_{\alpha > \delta} |a_\alpha| \alpha! R^{|\alpha|} \leq \left(\frac{C}{|\delta|}\right)^{|\delta|} \sum_{\alpha > \delta} \frac{1}{2^{J(\alpha)}} \left(\frac{R}{|\alpha|}\right)^{|\alpha|} \leq \sum_{\alpha > \delta} \frac{1}{2^{J(\alpha)}} \rightarrow 0 \text{ as } |\delta| \rightarrow \infty.$$

Note further that it is clear that $\mathcal{F}$ is dense in the algebra $\mathcal{P}$ of polynomials. To see why, let $p(z) = \sum_{\alpha} p_{\alpha} z^{\alpha} \in \mathcal{P}$ with finitely many nonzero p_{α} , and consider $g_k(z) = \sum_{\alpha} a_{\alpha}^{(k)} z^{\alpha}$ where $a_{\alpha}^{(k)} = p_{\alpha}$ if $p_{\alpha} \neq 0$, and $a_{\alpha}^{(k)} = a_{\alpha}$ if $p_{\alpha} = 0$, where a_{α} is defined by (2.2.6). Then clearly each g_k satisfies (2.2.5) and hence $g_k \in \mathcal{F}$, and also $g_k \rightarrow p$ as $k \rightarrow \infty$. Since $\mathcal{P}$ is dense in $H(\mathbb{C}^2)$, this implies that $\mathcal{F}$ is dense in $H(\mathbb{C}^2)$.

To prove our theorem, we now show that every vector $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ in $\mathcal{F}$ is a cyclic vector for any operator $L \in \text{Com}_D(\mathbb{C}^N)$. To begin, we first observe that a nonzero scalar multiple of a cyclic operator is cyclic, with identical sets of cyclic vectors. Secondly, we observe that for any vector g and any constant c , we have

$$\{p(L)g : p \text{ is a polynomial}\} = \{p(L+c)g : p \text{ is a polynomial}\}.$$

Hence, g is a cyclic vector for L if and only if g is a cyclic vector for $L+c$. Therefore, we can assume without loss of generality that L is given by $L = \sum_{\sigma \geq \rho} u_{\sigma} D^{\sigma}$ for some nonzero multi-index ρ and $u_{\sigma} \in \mathbb{C}$ with $u_{\rho} = 1$. Let $Y = \{p(L)f : p \text{ is a polynomial}\}$. To show that f is a cyclic vector for L , it suffices to show that $z^{\delta} \in \overline{Y}$ for every multi-index δ . We do this by induction on $J(\delta)$ as defined in Remark 2.6.

Base case: $J(\delta) = 0$; that is, $\delta = (0, 0)$. So we want to show that $1 \in \overline{Y}$. Note that $u_{\rho} = 1$ and

$$L^n = \left(D^{\rho} + \sum_{\sigma > \rho} u_{\sigma} D^{\sigma} \right)^n = D^{n\rho} + \sum_{\beta > n\rho} v_{\beta} D^{\beta},$$

where

$$v_\beta = \sum_{\substack{\sigma_1 + \sigma_2 + \dots + \sigma_n = \beta \\ \sigma_i \geq \rho}} u_{\sigma_1} u_{\sigma_2} \dots u_{\sigma_n}.$$

Therefore, we have

$$1 + \sum_{\beta > n\rho} |v_\beta| \leq 1 + \sum_{\beta > n\rho} \sum_{\substack{\sigma_1 + \dots + \sigma_n = \beta \\ \sigma_i \geq \rho}} |u_{\sigma_1}| \dots |u_{\sigma_n}| = \left(1 + \sum_{\sigma > \rho} |u_\sigma|\right)^n. \quad (2.2.7)$$

Thus for any $R > 1$,

$$\begin{aligned} & \left\| \frac{1}{(n\rho)!a_{n\rho}} L^n f - 1 \right\|_R \\ & \leq \frac{1 + \sum_{\beta > n\rho} |v_\beta|}{(n\rho)!|a_{n\rho}|} \sum_{\alpha > n\rho} |a_\alpha| \alpha! R^{|\alpha|}, \quad \text{by Lemma 2.7 with } S = L^n \text{ and } \delta = (0, 0) \\ & \leq \frac{(1 + \sum_{\sigma > \rho} |u_\sigma|)^n}{(n\rho)!|a_{n\rho}|} \sum_{\alpha > n\rho} |a_\alpha| \alpha! R^{|\alpha|}, \quad \text{by (2.2.7)} \\ & = \frac{C^{|\alpha|}}{(n\rho)!|a_{n\rho}|} \sum_{\alpha > n\rho} |a_\alpha| \alpha! R^{|\alpha|}, \quad \text{where } C = \left(1 + \sum_{\sigma > \rho} |u_\sigma|\right)^{1/|\rho|} < \infty \text{ by Lemma 2.5} \\ & \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \text{by (2.2.5).} \end{aligned}$$

Thus $1 \in \overline{Y}$, completing the base case.

Induction step: suppose that $z^\theta \in \overline{Y}$ for all θ with $J(\theta) \leq k$; that is, $z^\theta \in \overline{Y}$ for all $\theta < \zeta$ for some multi-index ζ . We want to show that $z^\zeta \in \overline{Y}$.

Since $L^n f \in \overline{Y}$ and $z^\theta \in \overline{Y}$ for all $\theta < \zeta$, we have, for some $w_\gamma \in \mathbb{C}$,

$$Q_\zeta(L^n f) = \sum_{\gamma \geq n\rho + \zeta} w_\gamma z^{\gamma - n\rho} = \left(L^n f - \sum_{\gamma < n\rho + \zeta} w_\gamma z^{\gamma - n\rho} \right) \in \overline{Y}. \quad (2.2.8)$$

Applying Lemma 2.7 with $\delta = \zeta$ and $S = L^n$, we obtain

$$\begin{aligned}
& \left\| \frac{\zeta!}{(n\rho + \zeta)!a_{n\rho+\zeta}} Q_\zeta(L^n f) - z^\zeta \right\|_R \\
& \leq \frac{\zeta!(1 + \sum_{\beta > n\rho} |v_\beta|)}{(n\rho + \zeta)!|a_{n\rho+\zeta}|} \sum_{\alpha > n\rho+\zeta} |a_\alpha| \alpha! R^{|\alpha|} \\
& \leq \frac{\zeta!(1 + \sum_{\sigma > \rho} |u_\sigma|)^n}{(n\rho + \zeta)!|a_{n\rho+\zeta}|} \sum_{\alpha > n\rho+\zeta} |a_\alpha| \alpha! R^{|\alpha|}, \quad \text{by (2.2.7)} \\
& \leq \frac{\zeta! C^{|n\rho+\zeta|}}{(n\rho + \zeta)!|a_{n\rho+\zeta}|} \sum_{\alpha > n\rho+\zeta} |a_\alpha| \alpha! R^{|\alpha|}, \quad \text{where } C = \left(1 + \sum_{\sigma > \rho} |u_\sigma|\right)^{1/|\rho|} < \infty \text{ by Lemma 2.5} \\
& \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \text{by (2.2.5).}
\end{aligned}$$

By (2.2.8), this implies that $z^\zeta \in \overline{Y}$, completing the induction step and hence the proof. $\square$

Using some of the preliminary results of Section 2.1, we can extend Theorem 2.8 to operators commuting with partial differentiation on the space $H(\Omega)$, where $\Omega \subset \mathbb{C}^N$ is a Runge region. The key observation to make is that any such operator $L : H(\Omega) \rightarrow H(\Omega)$ can be identified with its characteristic function (see Definition 2.1), which by Lemma 2.2 is an entire function of exponential type. Thus, L also defines an operator on $H(\mathbb{C}^N)$.

Corollary 2.9. *Let $N \in \mathbb{N}$ and let $\Omega \subset \mathbb{C}^N$ be a Runge region. Let $\text{Com}_D(\Omega)$ be the family of all nonscalar continuous linear operators L on $H(\Omega)$ that commute with each of the partial differentiation operators $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Then the set of all common cyclic vectors for $\text{Com}_D(\Omega)$ is nonempty and in fact dense in $H(\Omega)$.*

Proof. Let $L \in \text{Com}_D(\Omega)$. By the proof of Theorem 2.3, L can also be considered as an operator on $H(\mathbb{C}^N)$. By Theorem 2.8, there exists a set $X \subset H(\mathbb{C}^N)$ of common cyclic vectors for $\text{Com}_D(\mathbb{C}^N)$. Thus any vector $f_0 \in X$ is a cyclic vector for $L : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$; that is,

$$\{p(L)f_0 : p \text{ is a polynomial}\} \quad (2.2.9)$$

is dense in $H(\mathbb{C}^N)$. Since Ω is a Runge region, $H(\mathbb{C}^N)$ is dense in $H(\Omega)$. It follows that

$$\{p(L)f_0|_{\Omega} : p \text{ is a polynomial}\}$$

is dense in $H(\Omega)$. Since $L \in \text{Com}_D(\Omega)$ is arbitrary, this shows that $f_0|_{\Omega}$ is a common cyclic vector for $\text{Com}_D(\Omega)$, hence

$$X_{\Omega} := \{f_0|_{\Omega} : f_0 \in X\}$$

is a set of common cyclic vectors for $\text{Com}_D(\Omega)$.

To see that X_{Ω} is dense in $H(\Omega)$, recall that X is dense in $H(\mathbb{C}^N)$ by Theorem 2.8. Since Ω is a Runge region, this implies that X_{Ω} is dense in $H(\Omega)$, finishing the proof. $\square$

We remark again that, while Theorem 2.8 is not a new result, having been originally proven by Bourdon and Shaprio [5], the proof we present above, which borrows techniques from Chan [7], is very different from their method of spectral synthesis. The method of spectral synthesis is used to tackle the common cyclicity of different operators in Section 5.2.

CHAPTER 3 INTRODUCING BACKWARD MULTI-SHIFTS

3.1 Preliminary Results

Godefroy and Shapiro [10] provided the following Definition 3.1 of a generalized backward shift as a generalization of unilateral weighted backward shifts on a Banach sequence space. One example of a generalized backward shift is the adjoint $M_{z-\alpha}^*$ of the multiplication operator $M_{z-\alpha} : A^2(\Omega) \rightarrow A^2(\Omega)$ on the Bergman space $A^2(\Omega)$, where $\alpha \in \mathbb{C}$; see [10, page 239]. Another example is the weighted differentiation operator $B : f \mapsto \frac{d}{dz}[f(\lambda z + b)]$ on the Fréchet space $H(\mathbb{C})$, where $\lambda, b \in \mathbb{C}$ with $\lambda \neq 0$; see Proposition 4.3 below.

Definition 3.1. A continuous linear operator B on a Fréchet space X is called a *generalized backward shift* if it satisfies the following conditions:

(GBS 1) $\dim \ker B = 1$.

(GBS 2) $\bigcup_{n=1}^{\infty} \ker B^n$ is dense in X .

Note that differentiation on $H(\mathbb{C})$ is another example of a generalized backward shift. However, the situation for N variables is a lot more complicated when $N \geq 2$. On $H(\mathbb{C}^N)$, no partial differentiation operator $\frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$, is a generalized backward shift because it has too large of a kernel. Inspired by the shifting structure of partial differentiation on $H(\mathbb{C}^N)$ and seeking to generalize that structure, we present the following Definition 3.2.

Due to the complexity of N variables, a direct generalization of (GBS 1) and (GBS 2) as (BMS 1), (BMS 2), and (BMS 3) below is not sufficient for our purpose. We need to introduce a new condition (BMS 4), in which the operator T_k may be interpreted as an integral operator in the case that the operators $B_1, \dots, B_N$ are taken to be partial differentiation operators on the Fréchet space $X = H(\mathbb{C}^N)$.

Definition 3.2. We call a finite set of bounded linear operators $\{B_1, B_2, \dots, B_N\}$ on a Fréchet space X a *backward multi-shift* if it satisfies the following conditions:

$$\text{(BMS 1)} \quad \dim \bigcap_{k=1}^N \ker B_k = 1.$$

$$\text{(BMS 2)} \quad M := \bigcup_{n=1}^{\infty} \bigcap_{k=1}^N \ker B_k^n \text{ is dense in } X.$$

$$\text{(BMS 3)} \quad B_i B_j = B_j B_i \text{ for all } i, j \in \{1, 2, \dots, N\}.$$

(BMS 4) For each k with $1 \leq k \leq N$, there exists a (not necessarily continuous) linear operator

$T_k : M \rightarrow M$ such that

$$B_k T_k = I|_M \text{ where } I : X \rightarrow X \text{ denotes the identity operator,} \quad (3.1.1)$$

$$T_i T_j = T_j T_i \text{ for all } i, j \in \{1, 2, \dots, N\}, \text{ and} \quad (3.1.2)$$

$$T_i B_j|_M = B_j T_i \text{ whenever } i \neq j. \quad (3.1.3)$$

We remark that the operator $T_i B_j|_M$ in (3.1.3) makes sense because B_j takes M to M by (BMS 2).

In 2010, Müller [14, Theorem 5] proved that any nonscalar operator that commutes with a generalized backward shift is supercyclic. Our goal for this chapter is to prove an analogue of this result for backward multi-shifts. To do that, we first require several lemmas analogous to [10, Proposition 3.3]. The first lemma we now obtain gives a precise description of $\bigcap_{k=1}^N \ker B_k^n$ that appears in (BMS 2). This lemma is generalized in Corollary 3.5.

Lemma 3.3. *Suppose that $\{B_1, \dots, B_N\}$ is a backward multi-shift on a Fréchet space X . Let $x_0 \in \bigcap_{k=1}^N \ker B_k$ be a nonzero vector. Then for all $n \geq 2$,*

$$\bigcap_{k=1}^N \ker B_k^n = \text{span}\{T_1^{j_1} T_2^{j_2} \dots T_N^{j_N} x_0 : 0 \leq j_1, \dots, j_N < n\}. \quad (3.1.4)$$

Furthermore, the set $\{T_1^{j_1}T_2^{j_2}\dots T_N^{j_N}x_0 : 0 \leq j_1, \dots, j_N < n\}$ is linearly independent and so

$$\dim \bigcap_{k=1}^N \ker B_k^n = n^N.$$

Proof. For notational simplicity, we only provide a proof for the case where $N = 2$. Other cases can be proved with the same argument.

We first inductively show the linear independence of the vectors $T_1^i T_2^j x_0$, where $0 \leq i < n$ and $0 \leq j < n$.

Base case: $n = 2$. We want to show that the vectors x_0 , $T_1 x_0$, $T_2 x_0$, and $T_1 T_2 x_0$ are linearly independent. To that end, let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\alpha x_0 + \beta T_1 x_0 + \gamma T_2 x_0 + \delta T_1 T_2 x_0 = 0.$$

If we apply $B_1 B_2$ to this equality, we obtain $\delta x_0 = 0$ so that $\delta = 0$. Therefore, our equality becomes

$$\alpha x_0 + \beta T_1 x_0 + \gamma T_2 x_0 = 0.$$

Applying B_2 to this, we obtain $\gamma x_0 = 0$ so that $\gamma = 0$. Therefore, our equality becomes

$$\alpha x_0 + \beta T_1 x_0 = 0.$$

Applying B_1 to this, we obtain $\beta x_0 = 0$, so that $\beta = 0$. Therefore, our equality becomes $\alpha x_0 = 0$ so that $\alpha = 0$ as well, and hence $\{x_0, T_1 x_0, T_2 x_0, T_1 T_2 x_0\}$ is a linearly independent set.

Induction step: suppose that $\{T_1^i T_2^j x_0 : 0 \leq i < n, 0 \leq j < n\}$ is a linearly independent set, where $n \geq 2$. Let $\lambda_{i,j} \in \mathbb{C}$ such that

$$\sum_{i=0}^n \sum_{j=0}^n \lambda_{i,j} T_1^i T_2^j x_0 = 0. \quad (3.1.5)$$

Applying $B_1 B_2$ to (3.1.5), we obtain

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} T_1^{i-1} T_2^{j-1} x_0 = 0.$$

Therefore, the induction hypothesis implies that $\lambda_{i,j} = 0$ for $i \neq 0 \neq j$. It follows that (3.1.5) becomes

$$\lambda_{0,0} x_0 + \sum_{i=1}^n \lambda_{i,0} T_1^i x_0 + \sum_{j=1}^n \lambda_{0,j} T_2^j x_0 = 0. \quad (3.1.6)$$

Applying B_1 to (3.1.6), we obtain

$$\sum_{i=1}^n \lambda_{i,0} T_1^{i-1} x_0 = 0.$$

Therefore, the induction hypothesis implies that $\lambda_{i,0} = 0$ for $1 \leq i \leq n$. Similarly, we apply B_2 to (3.1.6) to obtain

$$\sum_{j=1}^n \lambda_{0,j} T_2^{j-1} x_0 = 0,$$

and the induction hypothesis implies that $\lambda_{0,j} = 0$ for $1 \leq j \leq n$. Therefore, (3.1.5) becomes

$$\lambda_{0,0} x_0 = 0,$$

so $\lambda_{0,0} = 0$ also. Therefore, the set $\{T_1^i T_2^j x_0 : 0 \leq i < n+1, 0 \leq j < n+1\}$ is linearly independent, completing our induction.

To finish the whole proof of our lemma, it remains to show $\ker B_1^n \cap \ker B_2^n = \text{span}\{T_1^i T_2^j x_0 : 0 \leq i, j < n\}$. From (BMS 4) and the fact that $x_0 \in \ker B_1 \cap \ker B_2$, it is obvious that $\text{span}\{T_1^i T_2^j x_0 : 0 \leq i, j < n\} \subset \ker B_1^n \cap \ker B_2^n$. We now show the reverse containment by induction on n .

Base case: $n = 2$. We are to show that

$$\ker B_1^2 \cap \ker B_2^2 \subset \text{span}\{x_0, T_1 x_0, T_2 x_0, T_1 T_2 x_0\}.$$

To that end, let $x \in \ker B_1^2 \cap \ker B_2^2$. Then $B_1^2 x = 0 = B_2^2 x$. Therefore by (BMS 3), $B_1 B_2 x \in \ker B_1 \cap \ker B_2$, so by (BMS 1) there exists $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} B_1 B_2 x &= \lambda x_0, & \text{or equivalently,} \\ B_1 B_2 (x - \lambda T_1 T_2 x_0) &= 0; & \text{that is,} \\ x - \lambda T_1 T_2 x_0 &\in \ker B_1 B_2. \end{aligned}$$

By (BMS 3), this implies that

$$\begin{aligned} B_2 x - \lambda T_1 x_0 &\in \ker B_1, \text{ and similarly} \\ B_1 x - \lambda T_2 x_0 &\in \ker B_2. \end{aligned} \tag{3.1.7}$$

Observe that by (3.1.3) $T_1 x_0 \in \ker B_2$ and $T_2 x_0 \in \ker B_1$. Furthermore, the vector x is chosen so that $x \in \ker B_1^2 \cap \ker B_2^2$, and hence

$$\begin{aligned} B_2 x - \lambda T_1 x_0 &\in \ker B_2, \text{ and} \\ B_1 x - \lambda T_2 x_0 &\in \ker B_1. \end{aligned} \tag{3.1.8}$$

Putting (3.1.7) and (3.1.8) together, we have

$$\begin{aligned} B_2 x - \lambda T_1 x_0 &\in \ker B_1 \cap \ker B_2, \text{ and} \\ B_1 x - \lambda T_2 x_0 &\in \ker B_1 \cap \ker B_2. \end{aligned}$$

Therefore, there exist $\alpha, \beta \in \mathbb{C}$ with

$$\begin{aligned} B_2 x - \lambda T_1 x_0 &= \alpha x_0, \text{ and} \\ B_1 x - \lambda T_2 x_0 &= \beta x_0. \end{aligned}$$

By (3.1.1) and (3.1.3), we can rewrite the above two equations as

$$\begin{aligned} B_2(x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0) &= 0, \text{ and} \\ B_1(x - \lambda T_1 T_2 x_0 - \beta T_1 x_0) &= 0. \end{aligned}$$

In other words,

$$\begin{aligned} x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0 &\in \ker B_2, \text{ and} \\ x - \lambda T_1 T_2 x_0 - \beta T_1 x_0 &\in \ker B_1. \end{aligned} \tag{3.1.9}$$

Moreover, $\beta T_1 x_0 \in \ker B_2$ and $\alpha T_2 x_0 \in \ker B_1$ by (3.1.3), so it follows from (3.1.9) that

$$x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0 - \beta T_1 x_0 \in \ker B_2, \text{ and} \tag{3.1.10}$$

$$x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0 - \beta T_1 x_0 \in \ker B_1. \tag{3.1.11}$$

Combining (3.1.10) and (3.1.11), we have

$$x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0 - \beta T_1 x_0 \in \ker B_1 \cap \ker B_2.$$

Therefore, there exists $\gamma \in \mathbb{C}$ such that

$$x - \lambda T_1 T_2 x_0 - \alpha T_2 x_0 - \beta T_1 x_0 = \gamma x_0.$$

That is,

$$x = \gamma x_0 + \beta T_1 x_0 + \alpha T_2 x_0 + \lambda T_1 T_2 x_0,$$

thus proving the desired containment and finishing the base case of our induction.

Induction step: suppose inductively that for some $n \geq 2$,

$$\ker B_1^n \cap \ker B_2^n \subset \text{span}\{T_1^i T_2^j x_0 : 0 \leq i, j < n\}.$$

Let $x \in \ker B_1^{n+1} \cap \ker B_2^{n+1}$. Then by (BMS 3), $B_1^n B_2^n x \in \ker B_1 \cap \ker B_2$, so there exists $\lambda \in \mathbb{C}$ such that $B_1^n B_2^n x = \lambda x_0$. By (3.1.1) and (3.1.3), this means that

$$x - \lambda T_1^n T_2^n x_0 \in \ker B_1^n B_2^n.$$

Furthermore, by operating B_1^n and B_2^n respectively to the above vector and using (3.1.1) and (3.1.3), we obtain

$$\begin{aligned} B_1^n x - \lambda T_2^n x_0 &\in \ker B_2^n, \quad \text{and} \\ B_2^n x - \lambda T_1^n x_0 &\in \ker B_1^n. \end{aligned} \tag{3.1.12}$$

In fact, since $B_1 B_1^n x = B_1^{n+1} x = 0$ by our assumption on x and $B_1(\lambda T_2^n x_0) = \lambda T_2^n B_1 x_0 = 0$ by (3.1.3), we have

$$B_1(B_1^n x - \lambda T_2^n x_0) = 0, \tag{3.1.13}$$

and so by (3.1.12)

$$B_1^n x - \lambda T_2^n x_0 \in \ker B_1^n \cap \ker B_2^n.$$

Similarly we obtain

$$B_2(B_2^n x - \lambda T_1^n x_0) = 0, \tag{3.1.14}$$

and so

$$B_2^n x - \lambda T_1^n x_0 \in \ker B_1^n \cap \ker B_2^n.$$

Therefore, using the induction hypothesis we have that there exist $y_1, y_2 \in \text{span}\{T_1^i T_2^j x_0 : 0 \leq$

$i, j < n\}$ such that

$$B_1^n x - \lambda T_2^n x_0 = y_1, \text{ and} \quad (3.1.15)$$

$$B_2^n x - \lambda T_1^n x_0 = y_2. \quad (3.1.16)$$

By (3.1.13) and (3.1.14), we have $B_1 y_1 = 0$ and $B_2 y_2 = 0$. Since we have proved that the set $\{T_1^i T_2^j x_0 : 0 \leq i, j < n\}$ is linearly independent, by (3.1.1) and (3.1.3) this implies that

$$y_1 \in \text{span}\{T_2^j x_0 : 0 \leq j < n\}, \text{ and} \quad (3.1.17)$$

$$y_2 \in \text{span}\{T_1^i x_0 : 0 \leq i < n\}. \quad (3.1.18)$$

Hence by (3.1.3),

$$T_1^n y_1 \in \ker B_2^n, \text{ and} \quad (3.1.19)$$

$$T_2^n y_2 \in \ker B_1^n. \quad (3.1.20)$$

Observe that by (3.1.15) and (3.1.16), we have

$$x - \lambda T_1^n T_2^n x_0 - T_1^n y_1 \in \ker B_1^n, \text{ and} \quad (3.1.21)$$

$$x - \lambda T_1^n T_2^n x_0 - T_2^n y_2 \in \ker B_2^n. \quad (3.1.22)$$

It follows from (3.1.19), (3.1.20), (3.1.21), and (3.1.22) that

$$x - \lambda T_1^n T_2^n x_0 - T_1^n y_1 - T_2^n y_2 \in \ker B_1^n \cap \ker B_2^n.$$

Thus by our induction hypothesis, there exists $y_3 \in \text{span}\{T_1^i T_2^j x_0 : 0 \leq i, j < n\}$ so that

$$x - \lambda T_1^n T_2^n x_0 - T_1^n y_1 - T_2^n y_2 = y_3.$$

Upon rearranging terms, we have

$$x = \lambda T_1^n T_2^n x_0 + T_1^n y_1 + T_2^n y_2 + y_3,$$

which is in $\text{span}\{T_1^i T_2^j x_0 : 0 \leq i, j < n + 1\}$ by (3.1.17) and (3.1.18). This completes the induction process and thus the whole proof. $\square$

Equation (3.1.4) in the conclusion of Lemma 3.3 gives a precise description of the dense set M in (BMS 2). Hence, we can restate (BMS 2) as the following corollary.

Corollary 3.4. *Let $\{B_1, \dots, B_N\}$ be a backward multi-shift on an infinite dimensional, separable Fréchet space X , and let $x_0 \in \bigcap_{k=1}^N \ker B_k$ be a nonzero vector. Then*

$$\text{span}\{T_1^{j_1} T_2^{j_2} \dots T_N^{j_N} x_0 : j_1, \dots, j_N \geq 0\}$$

is dense in X .

As another corollary of Lemma 3.3, we now show that the conclusion (3.1.4) of the lemma can be improved.

Corollary 3.5. *Suppose that $\{B_1, \dots, B_N\}$ is a backward multi-shift on an infinite dimensional, separable Fréchet space X . Let $x_0 \in \bigcap_{k=1}^N \ker B_k$ be a nonzero vector. Then for all $n_1, n_2, \dots, n_N \in \mathbb{N}$,*

$$\bigcap_{k=1}^N \ker B_k^{n_k} = \text{span}\{T_1^{j_1} T_2^{j_2} \dots T_N^{j_N} x_0 : 0 \leq j_k < n_k \text{ for each } k\}.$$

Proof. We only present a proof for the case where $N = 2$ for notational simplicity. The same argument works for the case where $N \geq 3$. Let $m, n \in \mathbb{N}$. It is obvious by (3.1.1) and (3.1.3) that $\text{span}\{T_1^i T_2^j x_0 : 0 \leq i < m, 0 \leq j < n\} \subset \ker B_1^m \cap \ker B_2^n$.

To show the opposite inclusion, suppose without loss of generality that $m \leq n$ and let $x \in \ker B_1^m \cap \ker B_2^n \subset \ker B_1^n \cap \ker B_2^n$. Then by Lemma 3.3, the vector x is in the linear span of

$\{T_1^i T_2^j x_0 : 0 \leq i, j < n\}$, which is a linearly independent set. Since $x \in \ker B_1^m$, it must be that $x \in \text{span}\{T_1^i T_2^j x_0 : 0 \leq i < m, 0 \leq j < n\}$. $\square$

This concludes our preliminary discussion on backward multi-shifts. In the next section, we apply the above results to obtain supercyclicity results analogous to [14, Theorem 5].

3.2 Supercyclicity

With the results of Section 3.1, we are now ready to discuss supercyclicity results in the setting of backward multi-shifts. For that we require the Supercyclicity Criterion, defined by H. N. Salas in [18].

Lemma 3.6. *A continuous linear operator $T : X \rightarrow X$ on an infinite dimensional, separable Fréchet space X is supercyclic if it satisfies the following Supercyclicity Criterion: there exist an increasing sequence of integers (n_k) , two dense sets $D_1, D_2 \subset X$, and a map $S : D_2 \rightarrow D_2$ such that*

(SC 1) *TS is the identity map on D_2 .*

(SC 2) *$\lim_{k \rightarrow \infty} p_V(T^{n_k}x)p_V(S^{n_k}y) = 0$ for all $x \in D_1$ and $y \in D_2$ and all convex, balanced neighborhoods V of zero, where $p_V(z) = \inf\{t : t \geq 0 \text{ and } z \in tV\}$ is the Minkowski functional associated with V .*

The next lemma provides a sufficient condition for an operator to be mixing. We therefore also present the definition of mixing. It is well-known that every mixing operator is hypercyclic.

Definition 3.7. Let X be a topological vector space. A continuous linear operator $T : X \rightarrow X$ is said to be topologically *mixing* if for any pair U, V of non-empty open subsets of X one can find an integer $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

For an operator to be mixing, we provide a sufficient condition in the following lemma, which is a variant of a result of Müller [14, Theorem 1], who used it to prove much of the results of [14], including his aforementioned result [14, Theorem 5] regarding the commutant of a generalized backward shift. The lemma is taken from [2, Theorem 2.2].

Lemma 3.8. *Let T be a continuous linear operator on a topological vector space X . Let $I : X \rightarrow X$ denote the identity operator. Suppose that*

$$\text{span} \left\{ \bigcup_{n=1}^{\infty} \ker(T - \lambda I)^n \cap \text{Im}(T - \lambda I)^n : |\lambda| = 1 \right\}$$

is dense in X . Then T is mixing.

Making use of the conventions in Remark 2.6, we are now ready to state and prove the main result of this chapter. Our result is a multiple operator analogue of Müller [14, Theorem 5], who proved that every nonscalar operator commuting with a generalized backward shift is supercyclic, answering a question raised by Godefroy and Shapiro [10].

Theorem 3.9. *Let $\{B_1, \dots, B_N\}$ be a backward multi-shift on an infinite dimensional, separable Fréchet space X . If $A : X \rightarrow X$ is a nonscalar continuous linear operator that commutes with each B_k , then A is supercyclic.*

Furthermore, if $\bigcap_{k=1}^N \ker B_k \not\subset \ker A$, then there exists $\lambda \in \mathbb{C}$ such that λA is hypercyclic. On the contrary, if $\bigcap_{k=1}^N \ker B_k \subset \ker A$, then $I + A$ is mixing.

Proof. We need only to present the case where $N = 2$ for notational simplicity, as the exact same argument works for the case where $N \geq 3$.

Let $\{B_1, B_2\}$ be a backward multi-shift with respective densely defined right inverses T_1, T_2 on M as given in (BMS 4) of Definition 3.2, and let x_0 denote a nonzero vector in X such that $\ker B_1 \cap \ker B_2 = \text{span}\{x_0\}$ per (BMS 1). Let $A : X \rightarrow X$ be a continuous linear operator such that $AB_1 = B_1A$ and $AB_2 = B_2A$. Then $A(\ker B_1^m \cap \ker B_2^n) \subset \ker B_1^m \cap \ker B_2^n$ for all $m, n \in \mathbb{N}$. Thus by Corollary 3.5, given any $m, n \geq 0$ we can express

$$A(T_1^m T_2^n x_0) = \sum_{i=0}^m \sum_{j=0}^n \beta_{m,n,i,j} T_1^i T_2^j x_0, \quad (3.2.1)$$

for some $\beta_{m,n,i,j} \in \mathbb{C}$. Furthermore, we have for all $l, k \geq 0$,

$$\begin{aligned}
\sum_{i=0}^m \sum_{j=0}^n \beta_{m,n,i,j} T_1^i T_2^j x_0 &= A(T_1^m T_2^n x_0) \\
&= AB_1^l B_2^k (T_1^{m+l} T_2^{n+k} x_0) \\
&= B_1^l B_2^k A(T_1^{m+l} T_2^{n+k} x_0) \\
&= B_1^l B_2^k \sum_{i=0}^{m+l} \sum_{j=0}^{n+k} \beta_{m+l,n+k,i,j} T_1^i T_2^j x_0 \\
&= \sum_{i=l}^{m+l} \sum_{j=k}^{n+k} \beta_{m+l,n+k,i,j} T_1^{i-l} T_2^{j-k} x_0.
\end{aligned}$$

Since by Lemma 3.3 the vectors $T_1^i T_2^j x_0$ are linearly independent, we must have $\beta_{m,n,i,j} = \beta_{m+l,n+k,i+l,j+k}$ for all $m, n, i, j, l, k \in \mathbb{Z}^+$. Hence, the coefficients $\beta_{m,n,i,j}$ in (3.2.1) depend only on $m - i$ and $n - j$. Therefore, there exists $\{\alpha_{i,j}\} \subset \mathbb{C}$ such that the formula (3.2.1) for A becomes

$$A(T_1^m T_2^n x_0) = \sum_{i=0}^m \sum_{j=0}^n \alpha_{i,j} T_1^{m-i} T_2^{n-j} x_0 \quad (m, n \in \mathbb{Z}^+). \quad (3.2.2)$$

Since A is not a scalar multiple of the identity, there exists a multi-index $(a, b) \neq (0, 0)$ such that

$$\begin{aligned}
\alpha_{a,b} &\neq 0, \quad \text{and} \\
\alpha_{i,j} &= 0 \quad \text{for all multi-indices } (i, j) \text{ with } (0, 0) \neq (i, j) < (a, b).
\end{aligned} \quad (3.2.3)$$

To continue the proof of our theorem, we proceed in two different cases. We now consider the first case.

Case A: $\alpha_{0,0} \neq 0$. First consider the case where

$$\alpha_{0,0} = 1. \quad (3.2.4)$$

We show that A is in fact mixing. Set $S := A - \alpha_{0,0}I$. Then $S = A - I$ by (3.2.4), so by (3.2.2) $Sx_0 = 0$. Therefore, if we let $c = n + m + 1$, then $S^c(T_1^m T_2^n x_0) = 0$. It follows that if we let

$M = \text{span}\{T_1^i T_2^j x_0 : i, j \geq 0\}$, then

$$M \subset \bigcup_{k=1}^{\infty} \ker S^k. \quad (3.2.5)$$

Now we proceed to show that $SM = M$. Clearly $SM \subset M$ by (3.2.2). We now show the reverse inclusion by using induction on $J(i, j)$ to show that $T_1^i T_2^j x_0 \in SM$ for all $i, j \geq 0$, where $J(i, j)$ is as defined in Remark 2.6.

Base case: for a, b satisfying (3.2.3), we have

$$\begin{aligned} S(\alpha_{a,b}^{-1} T_1^a T_2^b x_0) &= A(\alpha_{a,b}^{-1} T_1^a T_2^b x_0) - I(\alpha_{a,b}^{-1} T_1^a T_2^b x_0) \\ &= \left(\sum_{i=0}^a \sum_{j=0}^b \alpha_{a,b}^{-1} \alpha_{i,j} T_1^{a-i} T_2^{b-j} x_0 \right) - \alpha_{a,b}^{-1} T_1^a T_2^b x_0, \quad \text{by (3.2.2)} \\ &= \alpha_{a,b}^{-1} T_1^a T_2^b x_0 + \alpha_{a,b}^{-1} \alpha_{a,b} T_1^{a-a} T_2^{b-b} x_0 - \alpha_{a,b}^{-1} T_1^a T_2^b x_0, \quad \text{by (3.2.3) and (3.2.4)} \\ &= x_0. \end{aligned}$$

Induction step: we inductively assume that $T_1^i T_2^j x_0 \in SM$ for all multi-indices $(i, j) < (m, n)$ for some multi-index $\sigma = (m, n)$. We need to show that $T_1^m T_2^n x_0 \in SM$. Using the new notation $\sigma = (m, n)$, we denote $T^\sigma = T_1^m T_2^n$, and so we are to show $T^\sigma x_0 \in SM$. Letting $\delta = (a, b)$, by (3.2.2) we have

$$\begin{aligned} S(\alpha_{a,b}^{-1} T_1^{a+m} T_2^{b+n} x_0) &= (A - I)(\alpha_\delta^{-1} T^{\delta+\sigma} x_0) \\ &= \sum_{\substack{\gamma \neq (0,0) \\ \delta+\sigma-\gamma \in \mathbb{Z}^+ \times \mathbb{Z}^+}} \alpha_\delta^{-1} \alpha_\gamma T^{\delta+\sigma-\gamma} x_0. \end{aligned} \quad (3.2.6)$$

Note that the summation is over all nonzero multi-indices γ with $\delta + \sigma - \gamma \in \mathbb{Z}^+ \times \mathbb{Z}^+$. By (3.2.3), if $(0, 0) \neq \gamma < \delta$ then $\alpha_\gamma = 0$. Therefore, (3.2.6) becomes

$$S(\alpha_{a,b}^{-1} T_1^{a+m} T_2^{b+n} x_0) = T^\sigma x_0 + \sum_{\substack{\gamma > \delta \\ \delta+\sigma-\gamma \in \mathbb{Z}^+ \times \mathbb{Z}^+}} \alpha_\delta^{-1} \alpha_\gamma T^{\delta+\sigma-\gamma} x_0. \quad (3.2.7)$$

By Remark 2.6, $\gamma > \delta$ implies $\delta + \sigma - \gamma < \sigma = (m, n)$. Therefore, the summation in (3.2.7) is in $\text{span}\{T_1^i T_2^j x_0 : (i, j) < (m, n)\}$. The induction hypothesis then implies that there exists $z \in M$ such that

$$Sz = \sum_{\substack{\gamma > \delta \\ \delta + \sigma - \gamma \in \mathbb{Z}^+ \times \mathbb{Z}^+}} \alpha_\delta^{-1} \alpha_\gamma T^{\delta + \sigma - \gamma} x_0.$$

Therefore $S(\alpha_{a,b}^{-1} T_1^{a+m} T_2^{b+n} x_0 - z) = T_1^m T_2^n x_0$ and hence $T_1^m T_2^n x_0 \in SM$. By induction this statement is true for all multi-indices (m, n) , thus $M \subset SM$ and hence $SM = M$.

It follows that $S^k M = M$ for all $k \in \mathbb{N}$. This, along with (3.2.5), implies that

$$M \subset \bigcup_{k=1}^{\infty} \ker S^k \cap M = \bigcup_{k=1}^{\infty} \ker S^k \cap S^k M \subset \bigcup_{k=1}^{\infty} \ker S^k \cap \text{Im } S^k.$$

By Corollary 3.4, the set M is dense in X . Therefore, $A = S + I$ is mixing by Lemma 3.8.

If $\alpha_{0,0} \neq 0$, then the previous argument shows that the operator $\alpha_{0,0}^{-1} A$ is mixing. Since any nonzero scalar multiple of a mixing operator is supercyclic, A is supercyclic.

Case B: $\alpha_{0,0} = 0$. We first observe that, by (3.2.2),

$$\text{span}\{x_0\} = \ker B_1 \cap \ker B_2 \subset \ker A \text{ if and only if } \alpha_{0,0} = 0.$$

We now use the Supercyclicity Criterion to show that A is supercyclic. By repeating the argument for establishing (3.2.6) and (3.2.7) with the operator S in the argument replaced by our operator A , we observe that $AM = M$. Hence the restriction operator $A|_M : M \rightarrow M$ is surjective. Therefore, there exists a (possibly discontinuous) linear operator $F : M \rightarrow M$ such that $AF = I_M$. The above argument for establishing (3.2.5) also shows that the dense set M is contained in $\bigcup_{n=1}^{\infty} \ker A^n$; indeed, it shows that for all $x \in M$, $A^n x = 0$ for large enough n . Therefore, for all convex, balanced neighborhoods V of zero, the Minkowski functional $p_V(A^n x)$ is zero for large enough n . We therefore have

$$\lim_{n \rightarrow \infty} p_V(A^n x) p_V(F^n y) = 0,$$

for all $x, y \in M$ and for all convex, balanced neighborhoods V of zero. Thus A satisfies the Supercyclicity Criterion, so A is supercyclic by Lemma 3.6.

Finally, note that the same argument presented in Case A demonstrates that the operator $A + I$ is mixing by Lemma 3.8. □

We now move to Chapter 4, where we present various examples of backward multi-shifts, showing that they satisfy the criterion of Definition 3.2. We are then able to apply Theorem 3.9 and obtain supercyclicity results in each example.

CHAPTER 4 EXAMPLES OF BACKWARD MULTI-SHIFTS

4.1 Partial Differentiation

We now present examples of backward multi-shifts. Our first corollary and example is a direct application of Theorem 3.9 to the setting of Chapter 2. It is, however, a weaker result than Theorem 2.3.

Corollary 4.1. *Let $\Omega \subset \mathbb{C}^N$ be a Runge region. Suppose that $L : H(\Omega) \rightarrow H(\Omega)$ is a nonscalar continuous linear operator that commutes with each partial differentiation operator $D_k = \frac{\partial}{\partial z_k}$, where $1 \leq k \leq N$. Then L is supercyclic.*

Proof. In light of Theorem 3.9, it suffices for us to show that $\{D_1, \dots, D_N\}$ is a backward multi-shift. It is a basic result of several complex variables that each D_k is a continuous linear operator on $H(\Omega)$, thus establishing (BMS 3). It is obvious that $\bigcap_{k=1}^N \ker D_k = \text{span}\{1\} = \mathbb{C}$, which has dimension one. Thus (BMS 1) is satisfied.

Note also that

$$\text{span}\{z_1^{j_1} \dots z_N^{j_N} : 0 \leq j_1, \dots, j_N < n\} = \bigcap_{k=1}^N \ker D_k^n.$$

Thus $\bigcup_{n=1}^{\infty} \bigcap_{k=1}^N \ker D_k^n$ is the set $\mathcal{P}$ of all N -variable polynomials. Since Ω is a Runge region, this implies that $\{D_1, \dots, D_N\}$ satisfies (BMS 2).

For each $1 \leq k \leq N$, define $T_k : \mathcal{P} \rightarrow \mathcal{P}$ to be the antidifferentiation with respect to z_k , with the constant of integration being zero. Thus $D_k T_k$ is the identity on $\mathcal{P}$. It is straightforward to verify that $D_i T_j = T_j D_i$ for all $i \neq j$, so (BMS 4) is satisfied. $\square$

We conclude this section with an open question. Again, the supercyclicity result of Corollary 4.1 is weaker than the hypercyclicity result of Theorem 2.3. Can the techniques of Chapter 3, [7], and Section 2.2 be used to prove supercyclicity analogues of Theorem 2.4, Theorem 2.8, or Corollary 2.9? In particular, does there exist a holomorphic function f that is a supercyclic vector for all nonscalar operators commuting with each partial differentiation operator on $H(\mathbb{C}^N)$?

4.2 Weighted Partial Differentiation

For this section, we consider some operators on $H(\mathbb{C}^N)$ that do not commute with differentiation and so do not fall under Theorem 2.3; rather, we show that Theorem 3.9 can be applied instead.

Before we continue our discussion on N variables, we make a few remarks on the one-variable case. The following definition was provided by Aron and Markose in [1].

Definition 4.2. Let $\lambda, b \in \mathbb{C}$ with $\lambda \neq 0$. A continuous linear operator $B : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is said to be a *weighted differentiation operator* if

$$Bf(z) = \frac{d}{dz}[f(\lambda z + b)] = \lambda f'(\lambda z + b). \quad (4.2.1)$$

Note that the functions $\lambda z + b$ are the conformal maps of $\mathbb{C}$ and so a weighted differentiation operator is a composition of two natural maps: a composition with a conformal map followed by differentiation. It turns out that weighted differentiation, like differentiation itself, is a generalized backward shift.

Proposition 4.3. *If $B : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is a weighted differentiation operator, then B is a generalized backward shift as in Definition 3.1.*

Proof. It is easy to see that $\dim \ker B = 1$ because $Bf = 0$ if and only if f is constant. Next we show $M = \bigcup_{n=1}^{\infty} \ker B^n$ is dense in $H(\mathbb{C})$. To do this, we show that the dense set $\mathcal{P}$ of polynomials is contained in M . It suffices to show that $z^n \in M$ for all $n \in \mathbb{Z}^+$, which we do by induction. The case when $n = 0$ follows from our argument above.

Inductively suppose that $z^k \in M$ for all $0 \leq k \leq n$, where $n \geq 0$. Then $Bz^{n+1} = (n+1)\lambda(\lambda z + b)^n \in \text{span}\{z^k : 0 \leq k \leq n\}$, so we have $Bz^{n+1} \in M$, and hence $z^{n+1} \in M$ by the definition of M . Thus, by induction $\bigcup_{n=1}^{\infty} \ker B^n$ is dense in $H(\mathbb{C})$ and hence B is a generalized backward shift. □

Aron and Markose [1] and Fernández and Hallack [9] proved that a weighted differentiation operator B is hypercyclic if $|\lambda| \geq 1$. Since weighted differentiation operators are generalized backward shifts, the following corollary immediately follows from the results of Godefroy and Shapiro [10, Theorem 3.6(b)] and Müller [14, Theorem 5].

Corollary 4.4. *Let B be a weighted differentiation operator on $H(\mathbb{C})$. If $A : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is a nonscalar continuous linear operator commuting with B , then A is supercyclic.*

Furthermore, if $\ker B \subset \ker A$, then $I + A$ is mixing.

Note that the operators A in Corollary 4.4 are non-convolution operators on $H(\mathbb{C})$. Thus Corollary 4.4 continues the study of hypercyclicity and supercyclicity results on the convolution operators studied by Godefroy and Shapiro [10]. We now turn our attention to the case of N variables. Inspired by Aron and Markose [1], we define an N -variable analogue of Definition 4.2.

Definition 4.5. Let $N \geq 2$ and $1 \leq k \leq N$. Let $\lambda, b \in \mathbb{C}$ such that $\lambda \neq 0$. Given $z = (z_1, \dots, z_k, \dots, z_N) \in \mathbb{C}^N$ denote $w = (z_1, \dots, \lambda z_k + b, \dots, z_N)$, which is obtained by replacing one variable z_k in $(z_1, \dots, z_N)$ by $\lambda z_k + b$ and keeping the rest of them unchanged. An operator $B : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$ defined by

$$Bf(z) = \frac{\partial}{\partial z_k} f(w) = \frac{\partial}{\partial z_k} f(z_1, \dots, \lambda z_k + b, \dots, z_N)$$

is called a *weighted partial differentiation operator*.

As applications of Definition 3.2 and Theorem 3.9, we provide N -variable analogues of Proposition 4.3 and Corollary 4.4. These results also parallel Corollary 4.1.

Proposition 4.6. *If for each $1 \leq k \leq N$, B_k is a weighted partial differentiation operator on $H(\mathbb{C}^N)$ with respect to z_k , then $\{B_1, \dots, B_N\}$ is a backward multi-shift on $H(\mathbb{C}^N)$.*

Proof. First we show $\dim \bigcap_{k=1}^N \ker B_k = 1$. Let $f \in \bigcap_{k=1}^N \ker B_k$. Then

$$0 = B_k f(z) = \frac{\partial}{\partial z_k} f(z_1, \dots, \lambda_k z_k + b_k, \dots, z_N), \text{ whenever } 1 \leq k \leq N,$$

or equivalently $\frac{\partial f}{\partial z_k} = 0$ for all $1 \leq k \leq N$. This follows easily from the chain rule:

$$B_k f(z) = \lambda_k \frac{\partial f}{\partial z_k}(z_1, \dots, \lambda z_k + b, \dots, z_N). \quad (4.2.2)$$

Thus f is a constant function. Conversely, it is easy to see that if f is constant then $f \in \bigcap_{k=1}^N \ker B_k$.

Thus $\bigcap_{k=1}^N \ker B_k$ is one-dimensional, establishing (BMS 1).

Note that $B_i B_j = B_j B_i$ for each $i \neq j$, establishing (BMS 3). To establish (BMS 4) we need to define the right inverse T_k of B_k on the set $\mathcal{P}$ of all N -variable polynomials by the operator linearly extended from the formula

$$T_k(z_1^{j_1} \dots z_k^{j_k} \dots z_N^{j_N}) = \frac{z_1^{j_1} \dots z_{k-1}^{j_{k-1}} \left(\frac{z_k - b_k}{\lambda_k} \right)^{1+j_k} z_{k+1}^{j_{k+1}} \dots z_N^{j_N}}{1 + j_k} \quad \text{for all } j_1, \dots, j_N \in \mathbb{Z}^+. \quad (4.2.3)$$

It is obvious that $T_i T_j = T_j T_i$ and $B_i T_j = T_j B_i$ whenever $i \neq j$. Moreover, it is a simple calculation to show that $B_k T_k(z_1^{j_1} \dots z_N^{j_N}) = z_1^{j_1} \dots z_N^{j_N}$, thus establishing (BMS 4).

To finish the proof, it remains to establish (BMS 2) by demonstrating that $M := \bigcup_{n=1}^{\infty} \bigcap_{k=1}^N \ker B_k^n = \mathcal{P}$. To show $\mathcal{P} \subset M$, it suffices to note that, for any given $f(z) = z_1^{j_1} \dots z_N^{j_N}$, we have $B_k^n f(z) = 0$ for all $1 \leq k \leq N$ if $n = \max\{j_1, \dots, j_N\} + 1$.

Conversely, let $f \in M$. Then there exists some $n \in \mathbb{N}$ such that $f \in \bigcap_{k=1}^N \ker B_k^n$. It is a straightforward calculation to iterate (4.2.2) and obtain that if $\lambda_k \neq 1$ then

$$0 = B_k^n f(z) = \lambda_k^{n(n+1)/2} \frac{\partial^n f}{\partial z_k^n} \left(z_1, \dots, \lambda_k^n z_k + \frac{1 - \lambda_k^n}{1 - \lambda_k} b_k, \dots, z_N \right), \quad \text{whenever } 1 \leq k \leq N. \quad (4.2.4)$$

Furthermore, if $\lambda_k = 1$, then

$$0 = B_k^n f(z) = \frac{\partial^n f}{\partial z_k^n}(z_1, \dots, z_k + n b_k, \dots, z_N), \quad \text{whenever } 1 \leq k \leq N. \quad (4.2.5)$$

This implies that $\frac{\partial^n f}{\partial z_k^n}(z) = 0$ for all $z \in \mathbb{C}^N$, and hence that f is a polynomial. Thus $M \subset \mathcal{P}$, so $M = \mathcal{P}$. Since $\mathcal{P}$ is dense, this establish (BMS 2) and thus $\{B_1, \dots, B_N\}$ is a backward multi-shift on $H(\mathbb{C}^N)$. $\square$

Corollary 4.7. *Let $\{B_1, \dots, B_N\}$ be a set of weighted partial differentiation operators on $H(\mathbb{C}^N)$. If $A : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$ is a nonscalar continuous linear operator commuting with each B_k for $1 \leq k \leq N$, then A is supercyclic.*

Furthermore, if $\bigcap_{k=1}^N \ker B_k \subset \ker A$, then $I + A$ is mixing.

Proof. This follows immediately from Theorem 3.9 and Proposition 4.6. $\square$

As with the one-variable case that we have previously commented on, Corollary 4.7 provides us with hypercyclicity and supercyclicity results for non-convolution operators for the $\mathbb{C}^N$ case.

The weighted differentiation and weighted partial differentiation operators discussed above are defined on $H(\mathbb{C})$ or $H(\mathbb{C}^N)$ but not on $H(\Omega)$ for any region Ω that is not closed under the transformation taking z_k to $\lambda_k z_k + b_k$. As an example of a Runge region where weighted differentiation is defined, we conclude this section by considering “rotated differentiation” operators on polydiscs $\mathbb{D}^N$.

Let $\theta \in [0, 2\pi)$. Then the operator $R : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ defined by $Rf(z) = \frac{d}{dz}f(e^{i\theta}z)$ is a generalized backward shift as given in Definition 3.1. More generally, let $\theta_1, \dots, \theta_N \in [0, 2\pi)$. For each $1 \leq k \leq N$, define $R_k : H(\mathbb{D}^N) \rightarrow H(\mathbb{D}^N)$ by

$$R_k f(z_1, \dots, z_N) = \frac{\partial}{\partial z_k} f(z_1, \dots, e^{i\theta_k} z_k, \dots, z_N).$$

Then $\{R_1, \dots, R_N\}$ is a backward multi-shift.

We leave it to the reader to verify that the proofs of Propositions 4.3 and 4.6 apply essentially unaltered to this new setting. The only major observation to make is that, for all $z_k \in \mathbb{D}$, $e^{i\theta_k} z_k \in \mathbb{D}$ and $\frac{z_k}{e^{i\theta_k}} = e^{-i\theta_k} z_k \in \mathbb{D}$.

4.3 Double Sequence Spaces

For the final section of this chapter, we turn our attention away from Fréchet spaces of holomorphic functions and towards spaces of double sequences, as given in Definition 4.8 below. It should come to no surprise that double sequence spaces provide a natural setting for backward multi-shifts (see Definition 3.2), given that separable Banach sequence spaces l^p admit unilateral weighted backward shifts. On these double sequence spaces, Shu, Wang, and Zhao [21] studied the universality results of the leftward and backward shift operators given by Definition 4.9. We now expand their work using Theorem 3.9.

Definition 4.8. Let $1 \leq q < \infty$. Define the Banach Space

$$\mathcal{L}_q := \left\{ x = \{x_{m,n}\}_{m,n \in \mathbb{Z}^+} = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \dots \\ x_{1,0} & x_{1,1} & x_{1,2} & \dots \\ x_{2,0} & x_{2,1} & x_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : x_{m,n} \in \mathbb{C}, \sum_{m,n \in \mathbb{Z}^+} |x_{m,n}|^q < \infty \right\}$$

with the norm

$$\|x\|_q = \left(\sum_{m,n \in \mathbb{Z}^+} |x_{m,n}|^q \right)^{1/q}. \quad (4.3.1)$$

These spaces can be thought of as two-dimensional analogues of sequence spaces l_q .

Definition 4.9. Let $x \in \mathcal{L}_q$ as in Definition 4.8. Define operators $B_1, B_2 : \mathcal{L}_q \rightarrow \mathcal{L}_q$ by

$$B_1 x := \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & \dots \\ x_{2,0} & x_{2,1} & x_{2,2} & \dots \\ x_{3,0} & x_{3,1} & x_{3,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad B_2 x := \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & \dots \\ x_{1,1} & x_{1,2} & x_{1,3} & \dots \\ x_{2,1} & x_{2,2} & x_{2,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then B_1 is called the *upward shift* and B_2 is called the *leftward shift*.

Note it is obvious that B_1 and B_2 commute. Furthermore, it is easy to check that B_1 and B_2 are continuous linear operators on $\mathcal{L}_q$ with $\|B_1\| = \|B_2\| = 1$. The pair of operators $\{B_1, B_2\}$ provides another example of a backward multi-shift.

Proposition 4.10. *The pair of upward and leftward shift operators $\{B_1, B_2\}$ is a backward multi-shift on $\mathcal{L}_q$, for $1 \leq q < \infty$.*

Proof. We are to verify that (BMS 1), (BMS 2), (BMS 3), and (BMS 4) are satisfied. Let $x \in \ker B_1 \cap \ker B_2$. Since $B_1 x = 0$, it follows from Definition 4.9 that $x_{m,n} = 0$ when $m \geq 1$ and $n \geq 0$. Similarly, since $B_2 x = 0$ we have $x_{m,n} = 0$ when $m \geq 0$ and $n \geq 1$. Thus $x_{m,n} = 0$ except possibly for $x_{0,0}$. Thus

$$\ker B_1 \cap \ker B_2 = \left\{ x = \begin{pmatrix} x_{0,0} & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : x_{0,0} \in \mathbb{C} \right\},$$

so $\dim(\ker B_1 \cap \ker B_2) = 1$, establishing (BMS 1).

It is clear from the definition of $\|\cdot\|_q$ in (4.3.1) that the set X_0 defined by

$$X_0 = \{x \in \mathcal{L}_q : x_{i,j} \neq 0 \text{ for at mostly finitely many } (i,j)\}$$

is dense in $\mathcal{L}_q$. Clearly, if $x \in X_0$, then there exists some $N \in \mathbb{N}$ with $x \in \ker B_1^N \cap \ker B_2^N$. Thus $\bigcup_{n=1}^{\infty} (\ker B_1^n \cap \ker B_2^n)$ contains the dense set X_0 . Conversely, it is clear that $\bigcup_{n=1}^{\infty} (\ker B_1^n \cap \ker B_2^n)$ is contained in X_0 so that the two sets are indeed the same, establishing (BMS 2). It is clear that B_1 and B_2 commute, so (BMS 3) is also satisfied.

Define operators $T_1, T_2 : X_0 \rightarrow X_0$ by

$$T_1 x = \begin{pmatrix} 0 & 0 & 0 & \dots \\ x_{0,0} & x_{0,1} & x_{0,2} & \dots \\ x_{1,0} & x_{1,1} & x_{1,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad T_2 x = \begin{pmatrix} 0 & x_{0,0} & x_{0,1} & \dots \\ 0 & x_{1,0} & x_{1,1} & \dots \\ 0 & x_{2,0} & x_{2,1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So T_1 is the downward shift and T_2 is the rightward shift. Clearly, $B_1 T_1 x = x$ and $B_2 T_2 x = x$ for all vectors $x \in X_0$. Furthermore, it is obvious that T_1 and T_2 commute, that T_1 and B_2 commute, and that T_2 and B_1 commute, proving (BMS 4). Thus $\{B_1, B_2\}$ is a backward multi-shift. $\square$

Shu, Wang, and Zhao proved various universality results of the pair $\{\lambda_1 B_1, \lambda_2 B_2\}$ for certain complex constants $\lambda_1, \lambda_2 \in \mathbb{C}$; see [21, Theorem 3.3, Theorem 3.4, Corollary 3.5, Theorem 3.6]. For example, [21, Theorem 3.3] states that $\{Sx : S \in \mathcal{F}\}$ is dense in $\mathcal{L}_q$ for some $x \in \mathcal{L}_q$ if and only if $\max\{|\lambda_1|, |\lambda_2|\} > 1$, where $\mathcal{F} = \{B_1^i B_2^j : i, j \in \mathbb{N}\}$; this condition is defined in [21] as hypercyclicity of the pair of operators $\{\lambda_1 B_1, \lambda_2 B_2\}$. After showing $\{B_1, B_2\}$ is a backward multi-shift, we can readily apply Theorem 3.9 to obtain the following result.

Corollary 4.11. *Let $\{B_1, B_2\}$ denote the upward and leftward shifts on $\mathcal{L}_q$, where $1 \leq q < \infty$. Let $A : \mathcal{L}_q \rightarrow \mathcal{L}_q$ be a nonscalar continuous linear operator commuting with both B_1 and B_2 . Then A is supercyclic.*

Furthermore, if $(\ker B_1 \cap \ker B_2) \subset \ker A$, then $I + A$ is mixing.

We conclude this section with a question. In [21, Section 4], various universality results are proven regarding the pair (W_1, W_2) on $\mathcal{L}_q$, where W_1 is a weighted upward shift and W_2 is a weighted leftward shift such that W_1 and W_2 do not commute. Does the conclusion of Corollary 4.11 hold for such a pair of operators? If the answer is positive, then this would suggest the possibility that Theorem 3.9 might hold even if we drop (BMS 2), (3.1.2), and/or (3.1.3) from Definition 3.2, even though our proof does rely on these conditions.

CHAPTER 5 COMMON CYCLICITY AND WEIGHTED DIFFERENTIATION

5.1 Introduction

In this chapter, we combine the topic of Section 2.2 with the setting of Section 4.2 and consider the common cyclicity of operators on $H(\mathbb{C})$ that commute with weighted differentiation operators; for the purpose of this chapter, a continuous linear operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, where $\lambda \neq 0$, is said to be a *weighted differentiation operator* if, for all $f \in H(\mathbb{C})$, we have

$$B_\lambda f(z) = \frac{d}{dz}[f(\lambda z)] = \lambda f'(\lambda z).$$

These operators were first studied by Aron and Markose [1], who proved that B_λ is hypercyclic if and only if $|\lambda| \geq 1$.

Let $\lambda \neq 0$ and denote the set of all nonscalar continuous linear operators commuting with B_λ as $\text{Com}(B_\lambda)$. We are to show that there exists a dense set of common cyclic vectors for $\text{Com}(B_\lambda)$; see Theorem 3.2. Surprisingly, we are not able to use one technique to prove the result for all values of λ . In Section 5.2 we borrow the techniques of Bourdon and Shapiro [5] in the case when $|\lambda| \geq 1$; see Proposition 5.5. Then, in Section 5.3 we borrow the techniques of Wogen [22] in the case when $0 < |\lambda| < 1$; see Proposition 5.10.

5.2 Weighted Differentiation and Spectral Synthesis

Let $|\lambda| \geq 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. We borrow the techniques of Bourdon and Shapiro [5, Theorem 4.2] and make use of their Eventual Spectral Synthesis (ESS) Theorem that is stated as follows.

Lemma 5.1. *There is a sequence $\{\alpha_j\}_1^\infty \subset \mathbb{C}$ of points convergent to a point in $\mathbb{C}$, such that the image sequence $\{\phi(\alpha_j)\}_1^\infty$ is an **eventual spectral synthesis** sequence for every nonconstant holomorphic function $\phi : \mathbb{C} \rightarrow \mathbb{C}$; that is, whenever $\{\beta_j\}_1^\infty$ is an absolutely summable sequence*

of complex numbers for which

$$\sum_{j=1}^{\infty} \beta_j \phi(\alpha_j)^n = 0 \text{ for all integers } n \geq 0,$$

then there exists $N \in \mathbb{N}$ such that $\beta_j = 0$ for all $j \geq N$.

In order to make use of the ESS Theorem, we need to establish several results about the eigenvectors of weighted differentiation operators. These results are analogues of well-known elementary results regarding the differentiation operator $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ given by $Df = f'$. The lemma below is analogous to the fact that, for each $0 \neq \alpha \in \mathbb{C}$, the function $e^{\alpha z}$ is an eigenvector of D with eigenvalue α .

Lemma 5.2. *Let $|\lambda| \geq 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. Let $\alpha \in \mathbb{C}$ be nonzero. Then α is an eigenvalue of B_λ with eigenvector $f_0 = 1$ if $\alpha = 0$, and eigenvector*

$$f_\alpha = \sum_{n=0}^{\infty} c_{\alpha,n} z^n := \sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{n(n+1)/2}} z^n \quad (5.2.1)$$

if $\alpha \neq 0$.

Proof. It is obvious that $B_\lambda f_0 = 0$. For all nonzero $\alpha \in \mathbb{C}$, note that the function f_α defines an entire function, so $f_\alpha \in H(\mathbb{C})$. This can be seen using the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|c_{\alpha,n+1}|}{|c_{\alpha,n}|} = \lim_{n \rightarrow \infty} \frac{|\alpha|}{|\lambda|^{n+1}(n+1)} \rightarrow 0. \quad (5.2.2)$$

To see that f_α is an eigenvector of B_λ with eigenvalue α , simply observe that, by continuity of B_λ ,

for all $\alpha \neq 0$ we have

$$\begin{aligned}
B_\lambda f_\alpha &= B_\lambda \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{n(n+1)/2}} z^n \right) \\
&= D \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{n(n+1)/2}} (\lambda z)^n \right) \\
&= D \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{-n+n(n+1)/2}} z^n \right) \\
&= \sum_{n=1}^{\infty} \frac{\alpha^n}{(n-1)! \lambda^{n(n-1)/2}} z^{n-1} \\
&= \alpha \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)! \lambda^{n(n-1)/2}} z^{n-1} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{n(n+1)/2}} z^n \\
&= \alpha f_\alpha.
\end{aligned}$$

□

Observe that the proof of Lemma 5.2 fails for the case $0 < |\lambda| < 1$; in that case the limit in (5.2.2) is infinity, and hence f_α no longer defines an entire function. In fact, B_λ has no eigenvectors or eigenvalues when $0 < |\lambda| < 1$. It is crucial to our present techniques that the eigenvectors of B_λ be “abundant” in some way. The exact way (for the case $|\lambda| \geq 1$) is provided by the following lemma, which is analogous to [20, 2.7 Density Theorem].

Lemma 5.3. *Let $|\lambda| \geq 1$ and define f_α as in (5.2.1). If $\Lambda \subset \mathbb{C}$ has an accumulation point in $\mathbb{C}$, then $\text{span}\{f_\alpha : \alpha \in \Lambda\}$ is dense in $H(\mathbb{C})$.*

Proof. Let $\Lambda \subset \mathbb{C}$ have an accumulation point. Suppose that $L : H(\mathbb{C}) \rightarrow \mathbb{C}$ is a linear functional that annihilates $\text{span}\{f_\alpha : \alpha \in \Lambda\}$. Using the Hahn-Banach Theorem and the Riesz Representation theorem, we obtain a complex measure μ with compact support K such that

$$L(f) = \int_K f d\mu, \text{ for all } f \in H(\mathbb{C}). \quad (5.2.3)$$

Define a function $F : \mathbb{C} \rightarrow \mathbb{C}$ by

$$F(\alpha) := L(f_\alpha) = \int_K f_\alpha d\mu. \quad (5.2.4)$$

Observe that $f_\alpha(z) = \hat{f}(\alpha z)$ for all $\alpha \neq 0$, where

$$\hat{f}(z) = \sum_{n=0}^{\infty} \frac{1}{n! \lambda^{n(n+1)/2}} z^n.$$

Thus

$$\hat{f}^{(k)}(z) = \sum_{n=k}^{\infty} \frac{z^{n-k}}{(n-k)! \lambda^{n(n+1)/2}},$$

and

$$\hat{f}^{(k)}(\alpha z) = \sum_{n=k}^{\infty} \frac{\alpha^{n-k} z^{n-k}}{(n-k)! \lambda^{n(n+1)/2}}.$$

Differentiating under the integral sign yields that

$$\frac{d^k}{d\alpha^k} F(\alpha) = \int_K \frac{d^k}{d\alpha^k} \hat{f}(\alpha z) d\mu(z) = \int_K z^k \hat{f}^{(k)}(\alpha z) d\mu(z), \quad (5.2.5)$$

and so F is entire. Moreover, by (5.2.4) we have $F(\alpha) = 0$ for all $\alpha \in \Lambda$. Since Λ has an accumulation point in $\mathbb{C}$, the Uniqueness Principle implies that F is identically zero, and so by (5.2.5) we obtain

$$0 = F^{(k)}(0) = \int_K z^k \hat{f}^{(k)}(0) d\mu(z) = \frac{1}{\lambda^{k(k+1)/2}} \int_K z^k d\mu(z).$$

Therefore, by (5.2.3) we have

$$0 = \int_K z^k d\mu(z) = L(z^k), \text{ for all } k \in \mathbb{N}.$$

Thus L annihilates the algebra $\mathcal{P}$ of all complex polynomials. Since L is continuous, and $\mathcal{P}$ is dense in $H(\mathbb{C})$ by Runge's Theorem, this implies that L is identically zero, which in turn implies

that $\text{span}\{f_\alpha : \alpha \in \Lambda\}$ is dense in $H(\mathbb{C})$. $\square$

For the last of our preliminary lemmas regarding weighted differentiation operators, we show that the eigenvectors of B_λ are also eigenvectors of all operators in $\text{Com}(B)$. The proof of this result borrows arguments from [20, Lemma 4.5]. In particular, we make use of an entire function that is defined in a similar way as the characteristic function defined in [20, Definition 4.2].

Lemma 5.4. *Let $|\lambda| \geq 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. Let $A \in \text{Com}(B_\lambda)$. Then there exists a nonconstant entire function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ of exponential type such that $Af_\alpha = \phi(\alpha)f_\alpha$ for all $\alpha \in \mathbb{C}$.*

Proof. Let $A \in \text{Com}(B_\lambda)$. First we define $\phi : \mathbb{C} \rightarrow \mathbb{C}$. Let $\alpha \in \mathbb{C}$ and define f_α as in (5.2.1). Then $B_\lambda f_\alpha = \alpha f_\alpha$ by Lemma 5.2, so

$$B_\lambda(Af_\alpha) = A(B_\lambda f_\alpha) = \alpha(Af_\alpha).$$

Thus the function Af_α satisfies $B_\lambda y = \alpha y$; that is, it satisfies the differential equation

$$\lambda y'(\lambda z) = \alpha y(z).$$

Thus there exists some scalar $\phi(\alpha) \in \mathbb{C}$ such that

$$Af_\alpha = \phi(\alpha)f_\alpha.$$

Thus we can define a function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ by setting $\phi(\alpha)$ as above for all $\alpha \in \mathbb{C}$. In light of the above equation, it now remains to show that ϕ is a nonconstant entire function of exponential type.

Given any $\epsilon > 0$ and compact set $K \subset \mathbb{C}$, denote a neighborhood $N(K, \epsilon)$ of zero in $H(\mathbb{C})$ as

$$N(K, \epsilon) := \{f \in H(\mathbb{C}) : \|f\|_K < \epsilon\}.$$

Since A is continuous, the set $A^{-1}(N(K, \epsilon))$ is also a neighborhood of zero. Thus there exists

$\delta > 0$ and a compact subset $J \subset \mathbb{C}$ such that $A(N(J, \delta)) \subset N(K, \epsilon)$. Thus if $f \in H(\mathbb{C})$, then

$$\frac{\delta f}{2\|f\|_J} \in N(J, \delta), \text{ and so } \left\| A \left(\frac{\delta f}{2\|f\|_J} \right) \right\|_K < \epsilon.$$

Taking $\epsilon = 1$ and setting $C = 2/\delta$, we obtain

$$\|Af\|_K \leq C\|f\|_J, \text{ for all } f \in H(\mathbb{C}).$$

If we take $f = f_\alpha$ and $K = \{0\}$, and set $E = \max\{|z| : z \in J\}$, then we can restate the above inequality as

$$\begin{aligned} |\phi(\alpha)| &= |\phi(\alpha)f_\alpha(0)| = |(Af_\alpha)(0)| \\ &= \|Af_\alpha\|_K \\ &\leq C\|f_\alpha\|_J \\ &= C \sup \left\{ \left| \sum_{n=0}^{\infty} \frac{\alpha^n}{n! \lambda^{n(n+1)/2}} z^n \right| : z \in J \right\} \\ &\leq C \sup \left\{ \sum_{n=0}^{\infty} \frac{|\alpha|^n |z|^n}{n! |\lambda|^{n(n+1)/2}} : z \in J \right\} \\ &= C \sum_{n=0}^{\infty} \frac{(E|\alpha|)^n}{n! |\lambda|^{n(n+1)/2}} \\ &\leq C \sum_{n=0}^{\infty} \frac{(E|\alpha|)^n}{n!} \\ &= Ce^{E|\alpha|}. \end{aligned}$$

Thus $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function of exponential type.

Moreover, ϕ is nonconstant. To see why, by means of contradiction suppose that there exists $\beta \in \mathbb{C}$ such that $\phi(\alpha) = \beta$ for all $\alpha \in \mathbb{C}$. By the definition of ϕ we therefore have $Af_\alpha = \beta f_\alpha$ for all $\alpha \in \mathbb{C}$. By Lemma 5.3 and the continuity of A , this implies that $Af = \beta f$ for all $f \in H(\mathbb{C})$; that is, A is a scalar multiple of the identity operator. Since $\text{Com}(B_\lambda)$ is defined to be the set of all

nonscalar continuous linear operators commuting with B_λ , this is a contradiction. Thus ϕ is in fact a nonconstant entire function of exponential type. $\square$

Observe that Lemma 5.4 together with Lemma 5.3 show that any operator in $\text{Com}(B_\lambda)$ has dense range when $|\lambda| \geq 1$. We can now present and prove our main result for this section.

Proposition 5.5. *Let $|\lambda| \geq 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. There is a dense set of common cyclic vectors for $\text{Com}(B_\lambda)$.*

Proof. We start by demonstrating the existence of one such common cyclic vector. Let $\{\alpha_j\}_1^\infty$ be the sequence given by Lemma 5.1. Let $\{b_j\}_1^\infty \subset \mathbb{C}$ nonzero with $\sum_{j=1}^\infty |b_j| < \infty$. Consider

$$f := \sum_{j=1}^\infty b_j f_{\alpha_j}.$$

Claim: $f \in H(\mathbb{C})$.

Proof of claim: let $K \subset \mathbb{C}$ be compact, and $\epsilon > 0$. If we let $E = \max\{|z| : z \in K\}$, then the argument given in the proof of Lemma 5.4 yields that

$$\|f_\alpha\|_K \leq e^{E|\alpha|}. \quad (5.2.6)$$

Since $\alpha_j \rightarrow \alpha_0$ for some $\alpha_0 \in \mathbb{C}$ by Lemma 5.1, there exists some $R > 0$ with

$$|\alpha_j| < R \text{ for all } j \in \mathbb{N}. \quad (5.2.7)$$

Since $\{b_j\}_1^\infty$ converges absolutely, we can choose N large enough so that

$$\sum_{i=N+1}^\infty |b_j| e^{ER} < \epsilon. \quad (5.2.8)$$

Therefore,

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} b_j f_{\alpha_j} - \sum_{j=1}^N b_j f_{\alpha_j} \right\|_K &= \left\| \sum_{j=N+1}^{\infty} b_j f_{\alpha_j} \right\|_K \\
&\leq \sum_{j=N+1}^{\infty} |b_j| \cdot \|f_{\alpha_j}\|_K \\
&\leq \sum_{j=N+1}^{\infty} |b_j| e^{E|\alpha_j|}, && \text{by (5.2.6)} \\
&\leq \sum_{j=N+1}^{\infty} |b_j| e^{ER}, && \text{by (5.2.7)} \\
&< \epsilon, && \text{by (5.2.8).}
\end{aligned}$$

Thus $\sum_{j=1}^n b_j f_{\alpha_j} \rightarrow f$ in $H(\mathbb{C})$, and so $f \in H(\mathbb{C})$, finishing the proof of our claim.

Let $A \in \text{Com}(B_\lambda)$, and set $Y := \text{span}\{p(A)f : p \text{ is a polynomial}\}$. To show that f is a cyclic vector for A , we must show that Y is dense in $H(\mathbb{C})$. Since $H(\mathbb{C})$ is a locally convex topological vector space and Y is a linear manifold in $H(\mathbb{C})$, to show that Y is dense it suffices to show that the only continuous linear functional on $H(\mathbb{C})$ that vanishes on Y is identically zero. To that end, let $L : H(\mathbb{C}) \rightarrow \mathbb{C}$ be a continuous linear functional that annihilates Y . By the Riesz Representation Theorem, there exists a Borel measure μ with compact support K such that

$$Lg = \int_K g \, d\mu, \text{ for all } g \in H(\mathbb{C}).$$

So, in particular, since L annihilates Y , for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
0 &= L(A^n f) = \int_K A^n f \, d\mu \\
&= \int_K A^n \left(\sum_{j=1}^{\infty} b_j f_{\alpha_j} \right) d\mu \\
&= \int_K \left(\sum_{j=1}^{\infty} A^n(b_j f_{\alpha_j}) \right) d\mu, && \text{by the continuity of } A \\
&= \int_K \left(\sum_{j=1}^{\infty} b_j \phi(\alpha_j)^n f_{\alpha_j} \right) d\mu, && \text{by Lemma 5.4} \\
&= \sum_{j=1}^{\infty} \left(\int_K b_j \phi(\alpha_j)^n f_{\alpha_j} \, d\mu \right), && \text{by the Fubini-Tonelli Theorem} \\
&= \sum_{j=1}^{\infty} \left(b_j \int_K f_{\alpha_j} \, d\mu \right) \phi(\alpha_j)^n.
\end{aligned}$$

Moreover, since $\alpha_j \rightarrow \alpha_0$, we have $f_{\alpha_j} \rightarrow f_0$ in $H(\mathbb{C})$, and so there exists $M > 0$ such that $|\int_K f_{\alpha_j} \, d\mu| < M$ for all $j \in \mathbb{N}$. Thus, since $\{b_j\}_1^\infty$ is an absolutely summable sequence, so is $\{b_j \int_K f_{\alpha_j} \, d\mu\}_1^\infty$. Therefore, since ϕ is a nonconstant entire function by Lemma 5.4, by Lemma 5.1 we have that there exists $N \in \mathbb{N}$ such that

$$b_j \int_K f_{\alpha_j} \, d\mu = 0 \text{ for all } j \geq N.$$

Since $b_j \neq 0$ for all $j \in \mathbb{N}$, this implies that

$$L f_{\alpha_j} = \int_K f_{\alpha_j} \, d\mu = 0 \text{ for all } j \geq N.$$

Thus L annihilates $\text{span}\{f_{\alpha_j} : j \geq N\}$, which is dense in $H(\mathbb{C})$ by Lemma 5.3, and hence L must then be identically zero. Hence, Y is dense in $H(\mathbb{C})$ and thus f is a cyclic vector for any A in $\text{Com}(B_\lambda)$.

We now finish the proof of this theorem by showing that the set F defined by

$$F := \{Af : A \text{ is a continuous linear operator on } H(\mathbb{C}) \text{ such that } AB_\lambda = B_\lambda A\}$$

is a dense set of common cyclic vectors for $\text{Com}(B_\lambda)$, where $f \in H(\mathbb{C})$ is the common cyclic vector for $\text{Com}(B_\lambda)$ whose existence is shown above. That is, F is the set of all entire functions of the form Af where $A \in \text{Com}(B_\lambda)$ or A is a scalar multiple of the identity operator. Observe that if A commutes with B_λ and p is a polynomial, then $p(A)$ commutes with B_λ also. Therefore, the dense set $\{p(A)f : p \text{ is a polynomial}\}$ is contained in F , and hence F is also dense.

It remains to show that if T and A are continuous linear operators commuting with B_λ and T is nonscalar, then Af is a cyclic vector for T . Observe that A and T commute; this is so because, running the identities $TB_\lambda z^k = B_\lambda T z^k$ and $AB_\lambda z^k = B_\lambda A z^k$ through all $k \in \mathbb{Z}^+$, we obtain sequences of scalars $\{c_k\}_0^\infty$ and $\{d_k\}_0^\infty$ such that, for all polynomials $p \in \mathcal{P}$, we have

$$\begin{aligned} Tp &= \sum_{k=0}^{\infty} c_k B_\lambda^k p, \text{ and} \\ Ap &= \sum_{k=0}^{\infty} d_k B_\lambda^k p. \end{aligned}$$

Therefore, if p is a polynomial of degree n , then

$$\begin{aligned} TAp &= \left(\sum_{k=0}^n c_k B_\lambda^k \right) \left(\sum_{j=0}^n d_j B_\lambda^j \right) p = \sum_{k=0}^n \sum_{j=0}^n c_k d_j B_\lambda^{k+j} p \\ &= \sum_{j=0}^n \sum_{k=0}^n c_k d_j B_\lambda^{k+j} p = \left(\sum_{j=0}^n d_j B_\lambda^j \right) \left(\sum_{k=0}^n c_k B_\lambda^k \right) p = TAp \end{aligned}$$

Thus $TA = AT$ on the algebra $\mathcal{P}$ of polynomials, so by Runge's Theorem A and T commute on all of $H(\mathbb{C})$. Therefore, we have

$$\text{span Orb}(T, Af) = A(\text{span Orb}(T, f)). \quad (5.2.9)$$

Since f is a common cyclic vector for $\text{Com}(B_\lambda)$, we have that the set $\text{span Orb}(T, f)$ is dense. Moreover, if A is a scalar multiple of the identity then A has dense range; otherwise A has dense range by Lemmas 5.4 and 5.3. Either way, $A(\text{span Orb}(T, f))$ is dense in $H(\mathbb{C})$. Therefore, by (5.2.9) we have that $\text{span Orb}(T, Af)$ is dense in $H(\mathbb{C})$. That is, Af is a cyclic vector for T , finishing the proof. $\square$

Observe again that the techniques of Section 5.2 do not hold for weighted differentiation operators B_λ with $0 < |\lambda| < 1$. Therefore, it is necessary to consider alternative methods if we wish to extend Proposition 5.5 to the case when $0 < |\lambda| < 1$.

5.3 Contracting Differentiation

To complement the results in Section 5.2 for B_λ when $|\lambda| \geq 1$, in this section we study B_λ when $0 < |\lambda| < 1$. The methods of this section are entirely unlike those of the previous section; here we follow the techniques of Wogen [22]. In order to proceed, we require Hadamard's Three Circles Theorem.

Lemma 5.6. *Let $f \in H(\mathbb{C})$. Then for all $r_1 \leq r_2 \leq r_3$ with $r_1 \neq r_3$, we have*

$$\ln \|f\|_{r_2} \leq \frac{\ln(r_3/r_2)}{\ln(r_3/r_1)} \ln \|f\|_{r_1} + \frac{\ln(r_2/r_1)}{\ln(r_3/r_1)} \ln \|f\|_{r_3}.$$

We use Hadamard's Three Circles Theorem in order to find a bound on $\|Af\|_R$, presented below, where $A \in \text{Com}(B_\lambda)$.

Lemma 5.7. *Let $0 < |\lambda| < 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. For any function $f \in H(\mathbb{C})$, we define a corresponding set*

$$\mathcal{F} := \{B_\lambda^n(f - p_k) : n, k \in \mathbb{Z}^+\}, \quad (5.3.1)$$

where p_k denotes the sum of the first k terms of the power series of f . Then for all $A \in \text{Com}(B_\lambda)$

and $R \geq 1/(1 - |\lambda|)$, there exists $Q > 0$ and $m \in \mathbb{N}$ such that, for all $g \in \mathcal{F}$, we have

$$\|Ag\|_R < Q \sqrt[m]{\|g\|_R}.$$

Proof. Let $R \geq 1/(1 - |\lambda|)$. We first find bounds on $\|B_\lambda^n h\|_R$ for all $h \in H(\mathbb{C})$. To that end, let $h \in H(\mathbb{C})$ and $|z_0| \leq R$. Then, since $|\lambda| < 1$, we also have $|\lambda^n z_0| < R$ for all $n \in \mathbb{N}$. Thus by the Cauchy Estimates,

$$\begin{aligned} |(B_\lambda^n h)(z_0)| &= |\lambda^{n(n+1)/2} h^{(n)}(\lambda^n z_0)| \\ &= |\lambda|^{n(n+1)/2} |h^{(n)}(\lambda^n z_0)| \\ &\leq \frac{n! |\lambda|^{n(n+1)/2}}{((1 - |\lambda|^n)R)^n} \|h\|_R \\ &\leq n! |\lambda|^{n(n+1)/2} \|h\|_R. \end{aligned}$$

Since $|z_0| \leq R$ is arbitrary, we have shown

$$\|B_\lambda^n h\|_R \leq n! |\lambda|^{n(n+1)/2} \|h\|_R, \text{ for all } h \in H(\mathbb{C}). \quad (5.3.2)$$

When $n = 1$, we have $\|B_\lambda h\|_R \leq \lambda \|h\|_R < \|h\|_R$. Inductively, we have

$$\|B_\lambda^n h\|_R \leq \lambda \|B_\lambda^{n-1} h\|_R \leq \dots \leq |\lambda|^n \|h\|_R < \|h\|_R, \text{ for all } h \in H(\mathbb{C}). \quad (5.3.3)$$

We now let $A \in \text{Com}(B_\lambda)$. Since A is continuous, there exists $C_1, M > 0$ such that, for all $h \in H(\mathbb{C})$, we have

$$\|Ah\|_R \leq C_1 \|h\|_M. \quad (5.3.4)$$

Without loss of generality, we may assume that there exists $m \in \mathbb{N}$ such that $M = e^{m-1}R$; this is

because (5.3.4) remains true no matter how much we increase M .

Fix $f = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$ and the corresponding set $\mathcal{F}$ as in (5.3.1). Then applying Lemma 5.6 with $r_1 = R$, $r_2 = M = e^{m-1}R$, and $R_3 = eM = e^m R$, we obtain, for all $g \in \mathcal{F}$,

$$\ln \|g\|_M \leq \frac{\ln e}{\ln e^m} \ln \|g\|_R + \frac{\ln e^{m-1}}{\ln e^m} \ln \|g\|_{eM}.$$

That is,

$$\ln \|g\|_M \leq \ln \sqrt[m]{\|g\|_R} + \ln \left(\|g\|_{eM}^{\frac{m-1}{m}} \right) = \ln \left(\sqrt[m]{\|g\|_R} \cdot \|g\|_{eM}^{\frac{m-1}{m}} \right).$$

In other words,

$$\|g\|_M \leq \sqrt[m]{\|g\|_R} \cdot \|g\|_{eM}^{\frac{m-1}{m}}. \quad (5.3.5)$$

We now proceed to show that there is an upper bound on $\|g\|_{eM}^{\frac{m-1}{m}}$ in (5.3.5). Since $g \in \mathcal{F}$, there exists $n, k \in \mathbb{Z}^+$ such that $g = B_\lambda^n(f - p_k)$. Therefore, by (5.3.3), we have

$$\begin{aligned} \|g\|_{eM} &= \|B_\lambda^n(f - p_k)\|_{eM} \\ &< \|f - p_k\|_{eM} \\ &\leq \|f\|_{eM} + \|p_k\|_{eM} \\ &\leq \sum_{j=0}^{\infty} |a_j| (eM)^j + \sum_{j=0}^{k-1} |a_j| (eM)^j \\ &\leq 2 \sum_{j=0}^{\infty} |a_j| (eM)^j \\ &< \infty, \end{aligned} \quad \text{since } f \text{ is entire.}$$

Thus, there exists $C_2 > 0$ such that $\|g\|_{eM}^{\frac{m-1}{m}} \leq C_2$ for all $g \in \mathcal{F}$, and so (5.3.5) becomes

$$\|g\|_M \leq C_2 \sqrt[m]{\|g\|_R}.$$

Combining this with (5.3.4) and setting $Q := C_1 C_2$, we obtain the desired inequality for all $g \in \mathcal{F}$:

$$\|Ag\|_R \leq Q \sqrt[n]{\|g\|_R}.$$

□

We remark that the proof of Lemma 5.7 fails for the case $|\lambda| \geq 1$. In that case, it is no longer true that if $|z_0| \leq R$ then $|\lambda^n z_0| < R$, and hence we are unable to use the Cauchy Estimates to obtain (5.3.2) and (5.3.3).

In order to make use of the above bound in Lemma 5.7, we need a nice way to write $A \in \text{Com}(B_\lambda)$. Our way of writing A hinges on Proposition 4.3, which states that B_λ is a generalized backward shift. Therefore, to rewrite A , we can use a result of Godefroy and Shapiro [10, Proposition 3.4]. We present a variant of their result below. Note that, although [10, Proposition 3.4] was originally presented in the Banach Space case, it in fact holds true in Fréchet spaces like $H(\mathbb{C})$.

Lemma 5.8. *Let $B : X \rightarrow X$ be a generalized backward shift on a separable Fréchet space X , and denote $Y := \bigcup_{n=1}^{\infty} \ker B^n$. Suppose further that B has a continuous right inverse T . Let $A \in \text{Com}(B)$. Denote $Y_n := \ker B^n$. Then $AY_n \subset Y_n$ for every $n \in \mathbb{N}$. Moreover, if we let $r = \min\{k - 1 : Ax_k \neq 0\}$, then there exists an operator $A_r \in \text{Com}(B)$ such that $A = B^r A_r$ and A_r takes each Y_n isomorphically onto Y_n ; in fact, A_r is given by AT^r .*

Furthermore, There exists a scalar $a \in \mathbb{C}$, $0 \neq b \in \mathbb{C}$, $r \in \mathbb{N}$ and $S \in \text{Com}(B)$ such that

$$A = aI + bB^r + B^{r+1}S. \tag{5.3.6}$$

Proof. The first part of the statement of Lemma 5.8 is largely just [10, Proposition 3.4], so we present it without proof. The only problem appears to be the formula $A_r = AT^r$, because the existence of a right inverse T of B is not assumed in [10]; nevertheless, that formula is obtained explicitly in the proof of [10, Proposition 3.4]. Note also that in [10] the operator A_r is only defined

on the dense set $Y = \bigcup_{n=1}^{\infty} \ker B^n$, but because of the assumption that T is continuous, AT^r is also continuous and hence the operator A_r is defined on all of X .

To prove (5.3.6), let $A \in \text{Com}(B)$. Since $AY_1 \subset Y_1$, there exists $a \in \mathbb{C}$ such that $Ax_1 = a$. Thus $A - aI$ is a nonzero operator in $\text{Com}(B)$ satisfying $(A - aI)x_1 = 0$, so we can apply the first part of Lemma 5.8 to $A - aI$ to obtain $r \in \mathbb{N}$ and an operator $A_r \in \text{Com}(B)$ such that $A - aI = B^r A_r$ and A_r takes Y_1 isomorphically onto Y_1 . Hence there exists nonzero $b \in \mathbb{C}$ such that $A_r x_1 = b$. If $A_r - bI = 0$, then we can choose $S = 0$ to finish the proof. Otherwise, we can apply the first part of Lemma 5.8 to obtain $k \in \mathbb{N}$ and an operator $V \in \text{Com}(B)$ such that $A_r - bI = B^k V$. hence we conclude that

$$A - aI = B^r A_r = B^r (B^k V + bI) = bB^r + B^{r+k} V = bB^r + B^{r+1} (B^{k-1} V).$$

□

Note that while Lemma 5.8 holds for any generalized backward shift B on a Fréchet space X , we only need it for weighted differentiation operators $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ given by $B_\lambda f(z) = \lambda f'(\lambda z)$. Furthermore, as a consequence of Lemma 5.8 and Lemma 5.7, we can also nicely rewrite A^n for all $n \in \mathbb{N}$ and $A \in$.

Corollary 5.9. *Let $0 < |\lambda| < 1$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. Let $A \in \text{Com}(B_\lambda)$. Suppose that a, b and r are given as in Lemma 5.8 with $a = 0$ and $b = 1$. Then for all $n \in \mathbb{N}$, there exists an operator $S_n \in \text{Com}(B_\lambda)$ such that*

$$A^n - B_\lambda^{nr} = S_n B_\lambda^{nr+1}.$$

Moreover, given $f \in H(\mathbb{C})$, $\mathcal{F}$ defined as in (5.3.1), and $R \geq 1/(1 - |\lambda|)$, there exists $C > 1$ and $m \in \mathbb{N}$ such that for all $g \in \mathcal{F}$ with $\|g\|_R < 1$, we have

$$\|S_n g\|_R \leq C^n \|g\|_R^{1/m^n}.$$

Proof. We first must show that Lemma 5.8 applies to B_λ when $0 < |\lambda| < 1$. By Proposition 4.3, B_λ is a generalized backward shift, so it remains to show that B_λ has a continuous right inverse $T_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. We claim that such a T_λ is given by

$$T_\lambda f(z) := \int_0^{z/\lambda} f(\zeta) d\zeta,$$

where the integral is taken along the straight line from 0 to z/λ . That is, $T_\lambda f(z)$ is a primitive of f evaluated at z/λ . It is obvious that T_λ is linear, and it is straightforward to check that $B_\lambda T_\lambda = I$ on $H(\mathbb{C})$. Furthermore, T_λ is continuous because if $f \in H(\mathbb{C})$, $R > 0$, and $|z_0| \leq R$, then

$$|(T_\lambda f)(z_0)| = \left| \int_0^{z_0/\lambda} f(\zeta) d\zeta \right| \leq \|f\|_{R/|\lambda|} |z_0/\lambda| \leq \frac{R}{|\lambda|} \|f\|_{R/|\lambda|},$$

and hence

$$\|T_\lambda f\|_R \leq \frac{R}{|\lambda|} \|f\|_{R/|\lambda|}.$$

Turning to the proof of the statement of Corollary 5.9, the additional assumptions $a = 0$ and $b = 1$ imply that there exists $r \in N$ and $S \in \text{Com}(B_\lambda)$ such that

$$A = B_\lambda^r + B_\lambda^{r+1}S.$$

Raising both sides to the n -th power and using the Binomial Theorem, we obtain

$$A^n = B_\lambda^{nr} + \sum_{j=0}^{n-1} \binom{n}{j} B_\lambda^{rj} (B_\lambda^{r+1}S)^{n-j},$$

which can be rearranged as

$$A^n - B_\lambda^{nr} = B_\lambda^{nr+1}S_n,$$

where

$$S_n = \sum_{j=0}^{n-1} \binom{n}{j} B_\lambda^{n-j-1} S^{n-j}. \quad (5.3.7)$$

Obviously $S_n \in \text{Com}(B_\lambda)$. Let $f \in H(\mathbb{C})$ and define $\mathcal{F}$ as in (5.3.1). Let $R \geq 1/(1 - |\lambda|)$ and $g \in \mathcal{F}$ with $\|g\|_R < 1$. To finish the proof, we must estimate the seminorm $\|S_n g\|_R$. Since $S \in \text{Com}(B_\lambda)$, by Lemma 5.7 there exists $Q > 0$ such that $\|Sg\|_R \leq Q \sqrt[m]{\|g\|_R}$. Without loss of generality we may assume that $Q \geq 1$, and hence we inductively obtain

$$\|S^k g\|_R \leq Q^k \|g\|_R^{1/m^k} \text{ for all } k \in \mathbb{N}. \quad (5.3.8)$$

Therefore, for all $g \in \mathcal{F}$ with $\|g\|_R < 1$, we have

$$\begin{aligned} \|S_n g\|_R &\leq \sum_{j=0}^{n-1} \binom{n}{j} \|B_\lambda^{n-j-1} S^{n-j} g\|_R, && \text{by (5.3.7)} \\ &\leq \sum_{j=0}^{n-1} \binom{n}{j} |\lambda|^{n-j-1} \|S^{n-j} g\|_R, && \text{by (5.3.3)} \\ &\leq \sum_{j=0}^{n-1} \binom{n}{j} \|S^{n-j} g\|_R, && \text{since } |\lambda| < 1 \\ &\leq \sum_{j=0}^{n-1} \binom{n}{j} Q^{n-j} \|g\|_R^{1/m^{n-j}}, && \text{by (5.3.8)} \\ &\leq \sum_{j=0}^{n-1} \binom{n}{j} Q^{n-j} \|g\|_R^{1/m^n}, && \text{since } \|g\|_R < 1 \\ &= (1 + Q)^n \|g\|_R^{1/m^n} \end{aligned}$$

□

Using Corollary 5.9 and some of the inequalities presented in our lemmas, we can now present and prove our main result for this section.

Proposition 5.10. *Let $0 < |\lambda| < 1$ and consider the weighted differentiation operator B_λ :*

$H(\mathbb{C}) \rightarrow H(\mathbb{C})$. There is a dense set of common cyclic vectors for $\text{Com}(B_\lambda)$.

Proof. Let E denote the set of all entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ of the form $f = \sum_{n=0}^{\infty} a_n z^n$, where $\{a_n\}_0^\infty \subset \mathbb{C} \setminus \{0\}$ such that, for all $C, R > 1$ and $m \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \frac{C^n}{|a_n|} \sum_{k=n+1}^{\infty} |a_k| R^k = 0, \text{ and} \quad (5.3.9)$$

$$\lim_{n \rightarrow \infty} \frac{C^{n^2}}{|a_n|} \left(\sum_{k=n+1}^{\infty} |a_k| R^k \right)^{1/m^n} = 0. \quad (5.3.10)$$

We remark that E is nonempty because one may take $a_0 = a_1 = 1$ and $a_{n+1} = \left(\frac{a_n}{n^{n^2+n}} \right)^{n^n}$, for $n \in \mathbb{N}$. Obviously f defined in this way yields an entire function. To see that (5.3.9) is satisfied, let $C, R > 1$. Then, for all $n \in \mathbb{N}$ with $n > 2R$ and $n^n > C$, we have

$$\begin{aligned} \frac{C^n}{|a_n|} \sum_{k=n+1}^{\infty} |a_k| R^k &= \frac{C^n}{a_n} \sum_{k=n+1}^{\infty} \frac{(a_{k-1})^{k-1^{k-1}} R^k}{(k-1)^{((k-1)^2+k-1)(k-1)^{k-1}}} \\ &\leq \frac{C^n}{a_n} \sum_{k=n+1}^{\infty} \frac{a_{k-1} R^k}{(k-1)^{(k-1)^2+k-1}}, && \text{because } a_{k-1} \leq 1 \\ &\leq \frac{C^n}{a_n} \sum_{k=n+1}^{\infty} \frac{a_n R^k}{n^{n^2+k-1}}, && \text{because } \{a_n\}_0^\infty \text{ is decreasing} \\ &\leq \frac{C^n}{n^{n^2}} \sum_{k=n+1}^{\infty} \frac{R^k}{n^{k-1}} \\ &= \left(\frac{C}{n^n} \right)^n \sum_{k=n+1}^{\infty} \frac{R^k}{n^{k-1}} \\ &< \sum_{k=n+1}^{\infty} \frac{R^k}{n^{k-1}}, && \text{because } n^n > C \\ &< R \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}}, && \text{because } n > 2R \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

To see that (5.3.10) is satisfied, also let $m \in \mathbb{N}$. Suppose that $n > \max\{C, 2R, m\}$. Because

$\sum_{k=n+1}^{\infty} a_k R^k$ converges absolutely and $(a+b)^{1/m^n} \leq a^{1/m^n} + b^{1/m^n}$ for all $a, b > 0$, we have

$$\begin{aligned}
\frac{C^{n^2}}{|a_n|} \left(\sum_{k=n+1}^{\infty} |a_k| R^k \right)^{1/m^n} &\leq \frac{C^{n^2}}{a_n} \sum_{k=n+1}^{\infty} a_k^{1/m^n} R^{k/m^n} \\
&\leq \frac{C^{n^2}}{a_n} \sum_{k=n+1}^{\infty} a_k^{1/m^n} R^k, && \text{because } R > 1 \\
&= \frac{C^{n^2}}{a_n} \sum_{k=n+1}^{\infty} \left(\frac{a_{k-1}}{(k-1)^{(k-1)^2+k-1}} \right)^{\frac{(k-1)^{k-1}}{m^n}} R^k \\
&\leq \frac{C^{n^2}}{a_n} \sum_{k=n+1}^{\infty} \frac{a_{k-1} R^k}{(k-1)^{(k-1)^2+k-1}}, && \text{because } n > m \\
&\leq \frac{C^{n^2}}{a_n} \sum_{k=n+1}^{\infty} \frac{a_n R^k}{n^{n^2+k-1}} \\
&= \left(\frac{C}{n} \right)^{n^2} \sum_{k=n+1}^{\infty} \frac{R^k}{n^{k-1}} \\
&< \sum_{k=n+1}^{\infty} \frac{R^k}{n^{k-1}}, && \text{because } n > C \\
&< R \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}}, && \text{because } n > 2R \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Thus E is indeed nonempty. Moreover, E is in fact dense in $H(\mathbb{C})$ because the algebra $\mathcal{P}$ of polynomials in z is dense in $H(\mathbb{C})$ and if $f \in E$, $p \in \mathcal{P}$, and $n \in \mathbb{N}$, then $p + \frac{f}{n} \in E$.

To proceed with our proof, let $f \in E$. Also let $A \in \text{Com}(B_\lambda)$. We must show that f is a cyclic vector of A ; that is, if $Y := \{p(A)f : p \text{ is a polynomial}\}$, then Y is dense in X . Because $\mathcal{P}$ is dense in $H(\mathbb{C})$, it suffices to show that $z^n \in Y$ for all $n \geq 0$ by mathematical induction.

By Lemma 5.8, we can write $A = aI + bB_\lambda^r + B_\lambda^{r+1}S$, and hence we can rewrite Y as

$$Y = \left\{ p \left(\frac{A - aI}{b} \right) f : p \text{ is a polynomial} \right\} = \left\{ p \left(B_\lambda^r + \frac{1}{b} B_\lambda^{r+1} S \right) f : p \text{ is a polynomial} \right\}.$$

Therefore, without loss of generality we may assume that the operator A takes the form $A = B^r + B^{r+1}S$ for some positive integer $r \in \mathbb{N}$ and some operator $S \in \text{Com}(B)$. Let $R \geq 1/(1-|\lambda|)$. Then, by Corollary 5.9, there exists $m \in \mathbb{N}$, $C > 1$ and a sequence $\{S_n\}_1^\infty \subset \text{Com}(B_\lambda)$ such that for all $g \in \mathcal{F}$ with $\|g\|_R < 1$ where $\mathcal{F}$ is defined as in (5.3.1), we have

$$A^n - B_\lambda^{nr} = S_n B_\lambda^{nr+1}, \text{ and} \quad (5.3.11)$$

$$\|S_n g\|_R \leq C^n \|g\|_R^{1/m^n}. \quad (5.3.12)$$

We now start our mathematical induction.

Base case: we wish to show that the constant function 1 is in Y . To demonstrate this, we show that, for all $R \geq 1/(1-|\lambda|)$, as $n \rightarrow \infty$ we have

$$\left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} A^n f - 1 \right\|_R \rightarrow 0.$$

By the triangle inequality, we can rewrite the above seminorm as

$$\begin{aligned} & \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} A^n f - 1 \right\|_R \\ & \leq \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} (A^n - B_\lambda^{nr}) f \right\|_R + \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} B_\lambda^{nr} f - 1 \right\|_R. \end{aligned} \quad (5.3.13)$$

We show that each of the two summands in the right side of (5.3.13) goes to zero as $n \rightarrow \infty$. Note that by (5.3.3) we have $\|B_\lambda^{nr+1} f\|_R < 1$ for sufficiently large n . Therefore, for the first summand

we have

$$\begin{aligned}
& \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} (A^n - B_\lambda^{nr}) f \right\|_R \\
&= \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} S_n B_\lambda^{nr+1} f \right\|_R, & \text{by (5.3.11)} \\
&\leq \frac{C^n}{|a_{nr}|(nr)! |\lambda|^{nr(nr+1)/2}} \|B_\lambda^{nr+1} f\|_R^{1/m^n}, & \text{by (5.3.12)} \\
&= \frac{C^n}{|a_{nr}|(nr)! |\lambda|^{nr(nr+1)/2}} \left\| B_\lambda^{nr+1} \sum_{k=nr+1}^{\infty} |a_k| z^k \right\|_R^{1/m^n} \\
&< \frac{C^n}{|a_{nr}|(nr)! |\lambda|^{nr(nr+1)/2}} \left\| \sum_{k=nr+1}^{\infty} |a_k| z^k \right\|_R^{1/m^n}, & \text{by (5.3.3)} \\
&\leq \left(\frac{C}{|\lambda|} \right)^{(nr)^2} \cdot \left(\frac{1}{|a_{nr}|} \right) \cdot \left(\sum_{k=nr+1}^{\infty} |a_k| R^k \right)^{1/m^n} \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty, & \text{by (5.3.10).}
\end{aligned}$$

For the second summand of (5.3.13), we have

$$\begin{aligned}
\left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} B_\lambda^{nr} f - 1 \right\|_R &= \left\| \frac{1}{a_{nr}(nr)! \lambda^{nr(nr+1)/2}} B_\lambda^{nr} \sum_{k=nr+1}^{\infty} a_k z^k \right\|_R \\
&\leq \frac{1}{|a_{nr}|} \left\| \sum_{k=nr+1}^{\infty} a_k z^k \right\|_R, & \text{by (5.3.2)} \\
&\leq \frac{1}{|a_{nr}|} \sum_{k=nr+1}^{\infty} |a_k| R^k \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty, & \text{by (5.3.9).}
\end{aligned}$$

This finishes the proof of the base case.

Induction step: suppose that $1, z, z^2, \dots, z^{j-1} \in Y$ for some $j \in \mathbb{N}$. We wish to show that $z^j \in Y$. Since Y is a subspace of $H(\mathbb{C})$, our induction hypothesis implies that Y contains the subspace $Y_{j-1} := \text{span}\{1, z, z^2, \dots, z^{j-1}\}$. Furthermore, Y also contains $A^n f$ by definition, and so

it contains the set $\alpha A^n f + Y_{j-1}$ for every scalar α . Therefore, by (5.3.11),

$$\alpha B_\lambda^{nr} f + \alpha S_n B_\lambda^{nr+1} f + Y_{j-1} \subset Y. \quad (5.3.14)$$

The term $B_\lambda^{nr} f$ in (5.3.14) can be rewritten as

$$B_\lambda^{nr} f = \sum_{k=nr}^{nr+j-1} \frac{k! a_k \lambda^{nr(nr+1)/2}}{(k-nr)!} (\lambda^{nr} z)^{k-nr} + B_\lambda^{nr} \left(\sum_{k=nr+j}^{\infty} a_k z^k \right). \quad (5.3.15)$$

Similarly, the term $S_n B_\lambda^{nr+1} f$ in (5.3.14) can be rewritten as

$$S_n \left(\sum_{k=nr+1}^{nr+j} \frac{k! a_k \lambda^{(nr+1)(nr+2)/2}}{(k-nr-1)!} (\lambda^{nr+1} z)^{k-nr-1} \right) + S_n B_\lambda^{nr+1} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right). \quad (5.3.16)$$

Observe that the first term on the right-hand side of (5.3.15) is in Y_{j-1} . In addition, so is the first term on the right-hand side of (5.3.16), as $S_n Y_{j-1} \subset Y_{j-1}$ by Lemma 5.8. Therefore, from (5.3.14), we deduce that, for all scalars α , we have

$$\alpha B_\lambda^{nr} \left(\sum_{k=nr+j}^{\infty} a_k z^k \right) + \alpha S_n B_\lambda^{nr+1} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right) \in Y. \quad (5.3.17)$$

We now show that $z^j \in Y$ by putting

$$\alpha = \alpha_n = \frac{j!}{a_{nr+j}(nr+j)! \lambda^{jnr+nr(nr+1)/2}} \quad (5.3.18)$$

into the expression in (5.3.17) and showing that the resulting expression goes to z^j as $n \rightarrow \infty$.

Indeed, letting $R \geq 1/(1 - |\lambda|)$, by the triangle inequality we have

$$\begin{aligned}
& \left\| \alpha_n B_\lambda^{nr} \left(\sum_{k=nr+j}^{\infty} a_k z^k \right) + \alpha_n S_n B_\lambda^{nr+1} \left(\sum_{k=rn+j+1}^{\infty} a_k z^k \right) - z^j \right\|_R \\
&= \left\| \alpha_n B_\lambda^{nr} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right) + \alpha_n S_n B_\lambda^{nr+1} \left(\sum_{k=rn+j+1}^{\infty} a_k z^k \right) \right\|_R \\
&\leq \left\| \alpha_n B_\lambda^{nr} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right) \right\|_R + \left\| \alpha_n S_n B_\lambda^{nr+1} \left(\sum_{k=rn+j+1}^{\infty} a_k z^k \right) \right\|_R.
\end{aligned} \tag{5.3.19}$$

As in the base case, we now show that each of the two summands in (5.3.19) goes to zero as $n \rightarrow \infty$. For the first summand, we have

$$\begin{aligned}
\left\| \alpha_n B_\lambda^{nr} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right) \right\|_R &\leq |\alpha_n| (nr)! |\lambda|^{nr(nr+1)/2} \left\| \sum_{k=nr+j+1}^{\infty} a_k z^k \right\|_R, \quad \text{by (5.3.2)} \\
&= \frac{j!(nr)!}{|a_{nr+j}|(nr+j)! |\lambda|^{jnr}} \left\| \sum_{k=nr+j+1}^{\infty} a_k z^k \right\|_R, \quad \text{by (5.3.18)} \\
&< j! \frac{|\lambda^{-j}|^{nr+j}}{|a_{nr+j}|} \sum_{k=nr+j+1}^{\infty} |a_k| R^k \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty, \quad \text{by (5.3.9).}
\end{aligned}$$

For the second summand in (5.3.19), note that $B_\lambda^{nr+1} \sum_{k=rn+j+1}^{\infty} a_k z^k = B_\lambda^{nr+1}(f - p_{rn+j+1})$, where p_k denotes the sum of the first k terms of the power series of f , and hence we have $B_\lambda^{nr+1} \sum_{k=rn+j+1}^{\infty} a_k z^k \in \mathcal{F}$. Observe also that by (5.3.3), we have $\left\| B_\lambda^{nr+1} \sum_{k=rn+j+1}^{\infty} a_k z^k \right\|_R < 1$ for

sufficiently large n . Therefore, for sufficiently large n ,

$$\begin{aligned} & \left\| \alpha_n S_n B_\lambda^{nr+1} \left(\sum_{k=nr+j+1}^{\infty} a_k z^k \right) \right\|_R \\ & \leq |\alpha_n| C^n \left\| B_\lambda^{nr+1} \sum_{k=nr+j+1}^{\infty} a_k z^k \right\|_R^{1/m^n}, \end{aligned} \quad \text{by (5.3.12)}$$

$$< |\alpha_n| C^n \left\| \sum_{k=nr+j+1}^{\infty} a_k z^k \right\|_R^{1/m^n}, \quad \text{by (5.3.3)}$$

$$\begin{aligned} & < \frac{C^n j!}{|a_{nr+j}| \cdot |\lambda|^{jnr+nr(nr+1)/2}} \left(\sum_{k=nr+j+1}^{\infty} |a_k| R^k \right)^{1/m^n}, \quad \text{by (5.3.18)} \\ & \leq \left(\frac{C}{|\lambda|} \right)^{(nr+j)^2} \cdot \left(\frac{j!}{|a_{nr+j}|} \right) \cdot \left(\sum_{k=nr+j+1}^{\infty} |a_k| R^k \right)^{1/m^n} \end{aligned}$$

$$\longrightarrow 0 \text{ as } n \longrightarrow \infty, \quad \text{by (5.3.10).}$$

This finishes our induction, and thus the whole proof. $\square$

Observe again that the techniques of Section 5.3 do not hold for weighted differentiation operators B_λ with $|\lambda| \geq 1$. Hence, we the techniques of Section 5.2 are necessary in order to extend Proposition 5.10 to the case when $|\lambda| \geq 1$.

5.4 Conclusion

We collect here some final comments on the techniques of this chapter. We start by noting that, by combining Proposition 5.5 and Proposition 5.10, we obtain the following theorem.

Theorem 5.11. *Let $\lambda \neq 0$ and consider the weighted differentiation operator $B_\lambda : H(\mathbb{C}) \rightarrow H(\mathbb{C})$. There is a dense set of common cyclic vectors for $\text{Com}(B_\lambda)$.*

We remark once more on the apparent necessity of using two separate techniques in order to derive Theorem 5.11. In the case when $|\lambda| \geq 1$ that we consider in Section 5.2, we have a wealth of eigenvectors which allow us to use the spectral synthesis method of Bourdon and Shapiro [5]. This is no longer true in the case when $0 < |\lambda| < 1$, as we then we have no eigenvectors and so cannot

use this method. Fortunately, in that case Wogen's [22] original techniques for demonstrating common cyclicity work. Here, the apparent difficulty lies in the fact that $H(\mathbb{C})$ is a Fréchet space and not a Hilbert or Banach space, and so we have a collection of seminorms instead of a norm. However, as we have seen, the Cauchy Estimates together with Hadamard's Three Circles Theorem enable us to provide an appropriate bound for a given seminorm $\|\cdot\|_R$, in effect allowing us to treat the seminorm as if it were a norm. Yet, the application of the Cauchy Inequalities relies on the contracting nature of B_λ when $0 < |\lambda| < 1$, and so fails when $|\lambda| \geq 1$.

To conclude this dissertation, we raise a question based on Theorem 5.11. In Section 4.2, we consider operators of the form $Bf(z) = \lambda f'(\lambda z + b)$, where λ and b are nonzero scalars. One might ask if Theorem 5.11 remains true if we permit b to be nonzero. It is clear that this is the case when $0 < |\lambda| < 1$; indeed, the whole machinery of Section 5.3 remains unchanged except that the assumption $R \geq 1/(1 - |\lambda|)$ in Lemma 5.7 and Corollary 5.9 must be replaced by the assumption $R \geq \frac{|b| + 1}{1 - |\lambda|}$; this is because, if $0 < |\lambda| < 1$ and $|z_0| \leq R$, then $R \geq \frac{|b| + 1}{1 - |\lambda|}$ implies that $|\lambda z_0 + b| \leq |\lambda|R + |b| < R$, and by iterating this we have $\left| \lambda^n z_0 + \frac{1 - \lambda^n}{1 - \lambda} b \right| < R$ for all $n \in \mathbb{N}$. Therefore, by the Cauchy Estimates we have

$$\begin{aligned} |(B_\lambda^n f)(z_0)| &= \left| \lambda^{n(n+1)/2} f^{(n)} \left(\lambda^n z_0 + \frac{1 - \lambda^n}{1 - \lambda} b \right) \right| \\ &\leq \frac{n! |\lambda|^{n(n+1)/2}}{(R(1 - |\lambda|) - |b|)^n} \|f\|_R \\ &\leq n! |\lambda|^{n(n+1)/2} \|f\|_R. \end{aligned}$$

We are thus able to recover (5.3.2) and (5.3.3) and proceed with the remaining proofs of Section 5.3 without further changes. However, for the case when $|\lambda| \geq 1$, it is not obvious if Proposition 5.5 can be extended for nonzero b ; the difficulty here lies in determining what the new appropriate eigenvectors are for such an operator, if any.

Finally, in Section 4.2 we extend the definition of a weighted differentiation operator to the N -variable case ($N \geq 2$) and introduced weighted partial differentiation operators. This leads

to the question: can a result analogous to Theorem 5.11 be proven regarding weighted partial differentiation operators?

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