No. 392

THE LATTICE OF L-GROUP VARIETIES

Jo E. Smith

A Dissertation

Submitted to the Graduate College of Bowling Green
State University in partial fulfillment of
the requirements for the degree of

DEGREE OF PHILOSOPHY

August 1976

Approved by Doctoral Committee

W. Charles Hollenhorst Advisor
Department of Mathematics

Conner Duncan
Graduate College Representative

Charles Johnson

David B. Merritt

AMW Giam.

BOWLING GREEN UNIV. LIBRARY

ABSTRACT

For any type of abstract algebra, a variety is an equationally defined class of such algebras. Varieties were first introduced in the 30's and since then have been used successfully as a tool in the study of group theory. More recently, varieties of lattice-ordered groups (ℓ -groups) have been found to be of interest, with various results obtained concerning particular varieties of ℓ -groups. A more comprehensive study of ℓ -group varieties was made by Martinez, in which the set of all such varieties was shown to form a lattice $\mathbf{L}$ under set inclusion with a compatible associative multiplication operation. The purpose of this dissertation is to continue the study of this lattice.

Certain of the Scrimger varieties $\mathcal{S}_n$ ($n \in \mathbb{N}$) had been shown to be minimal non-abelian varieties in $\mathbf{L}$. In this work, it is shown that these varieties can be used to produce varieties minimal with respect to properly containing various other varieties in $\mathbf{L}$. Also discussed are the relations among the $\mathcal{S}_n$'s, and it is established that all infinite collections of $\mathcal{S}_n$'s have the same least upper bound in $\mathbf{L}$.

Martinez has also classified ℓ -groups using torsion classes, a generalization of the idea of varieties. It is proved here that $\mathbf{L}$ is not a sublattice of the lattice of torsion classes, and conditions are given for when joins in the two lattices will differ.

Finally, an inductive construction is given for nested families of ℓ -groups and corresponding varieties, which generalize the Scrimger varieties. These new varieties are examined in some detail and used to prove that in general $\mathcal{S}_n \neq \mathcal{L}_n$ (the variety of ℓ -groups for which n th powers commute).

378.771
B7do
no. 392

TABLE OF CONTENTS

	Page
CHAPTER I	1
Introduction	
CHAPTER II.	4
Properties of Varieties	
CHAPTER III	11
Properties of the Scrimger Varieties	
CHAPTER IV.	27
Varieties and Torsion Classes	
CHAPTER V	32
A New Family of Varieties	
BIBLIOGRAPHY.	44

CHAPTER I

INTRODUCTION

For any type of abstract algebra, a variety is an equationally defined class of such algebras. Varieties were essentially introduced by Garrett Birkhoff in the 1930's and in the 1950's began to reappear extensively in group theory papers. Since that time, a great deal of work on varieties of groups has been done, much of which is collected in Hanna Neumann's book *Varieties of Groups* [12].

A lattice-ordered group (ℓ -group) is a group with a partial order that is compatible with the group operation and which assures the existence of both the least upper bound (join) $g \vee f$ and the greatest lower bound (meet) $g \wedge f$ for any pair g, f of elements. Equivalently, an ℓ -group is a group together with two binary operations $\vee$ and $\wedge$ under which it is a lattice, and such that $x(g \vee f)y = (xgy) \vee (xfy)$ and dually. Interpreted this way, an ℓ -group may satisfy certain equational laws which involve both the group and lattice operations.

As the study of varieties of groups had proved such a useful tool in group theory, it is natural that varieties of ℓ -groups would be of interest to those concerned with ℓ -group theory. The first works to appear in this area mainly dealt with specific ℓ -group varieties. For example, Weinberg's work on abelian ℓ -groups [14] led to the conclusion that the abelian variety $\mathcal{A}$ is the smallest non-trivial ℓ -group variety. Martinez [9] considered a family of varieties $\mathcal{L}_n$ ($n \in \mathbb{N}$)

defined by the equation $x^n y^n = y^n x^n$. Scrimger [13], working within the $\mathcal{L}_n$'s, showed that certain varieties $\mathcal{S}_p \subseteq \mathcal{L}_p$ cover $\mathcal{A}$ for all primes p , i.e. no variety lies between $\mathcal{A}$ and $\mathcal{S}_p$. Holland [6] proved that the variety $\mathcal{N}$ of normal-valued ℓ -groups contains every other ℓ -group variety but the variety of all ℓ -groups. A more comprehensive study of ℓ -group varieties was made by Martinez [10], in which the set $\mathbf{L}$ of all ℓ -group varieties was shown to form a lattice, using set containment as the partial order, with a compatible associative multiplication operation.

This thesis is an attempt to extend the work on the structure of $\mathbf{L}$ and to further investigate the role of the $\mathcal{S}_n$'s within this lattice. Chapter II deals mainly with background material and conditions for the join-irreducibility of a variety within $\mathbf{L}$.

In Chapter III the Scrimger varieties $\mathcal{S}_n$ ($n \in \mathbb{N}$) are discussed, as well as the ℓ -groups G_n which generate them. Relations among the $\mathcal{S}_n$'s are discussed, and it is proved that for any variety $\mathcal{V}$ which does not contain $\mathcal{S}_p$ (p is prime), $\mathcal{V} \vee \mathcal{S}_p$ covers $\mathcal{V}$ within $\mathbf{L}$ (Proposition 3.8). Proposition 3.11 and Theorem 3.13 deal with the fact that the sublattice of $\mathbf{L}$ generated by $\{\mathcal{S}_p \mid p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes, but that there is only one variety in $\mathbf{L}$ which is an infinite join of $\mathcal{S}_n$'s. Also contained in this chapter are necessary and sufficient conditions for an ℓ -group to be ℓ -isomorphic to some G_n .

Chapter IV begins with a discussion of an alternate means of classifying ℓ -groups, called torsion classes. A torsion class, as defined by Martinez [11], is a class of ℓ -groups which is closed with

respect to convex ℓ -subgroups, ℓ -homomorphic images, and joins of convex ℓ -subgroups. Holland [7] has shown that varieties of ℓ -groups are also torsion classes. Hence, Theorem 4.4 deals with conditions under which the smallest variety containing a collection of varieties is distinct from the smallest torsion class containing the same collection of varieties, and Proposition 4.5 demonstrates the prevalence of these conditions in $\mathbf{L}$. Thus, $\mathbf{L}$ is far from being a sublattice of the lattice of all torsion classes.

In Chapter V an inductive construction of a nested family of ℓ -groups $G_{n_1} \subseteq G_{n_1, n_2} \subseteq \dots \subseteq G_{n_1, \dots, n_m} \subseteq \dots$ is given for any sequence of natural numbers $\{n_i\}$, where the initial group G_{n_1} is a Scrimger group. The ideal structure of these ℓ -groups is identified (Lemma 5.1), as well as certain varieties of interest to which these ℓ -groups belong (Theorem 5.2 and Proposition 5.5). Using this information, it is possible to show that in general $\mathcal{S}_n \neq \mathcal{L}_n$. The relations among the varieties $\mathcal{S}_{n_1, \dots, n_m}$ generated by $G_{n_1, \dots, n_m}$ are described in Theorems 5.9 and 5.11. The last section deals with a continuation of the construction noted above to an ℓ -group $G_{\{n_i\}}$, which is the supremum of the nested family

$$G_{n_1} \subseteq G_{n_1, n_2} \subseteq \dots \subseteq G_{n_1, \dots, n_m} \subseteq \dots$$

CHAPTER II

PROPERTIES OF VARIETIES

§1. Preliminaries

$(G, \cdot, \leq)$ is a *partially ordered group* (po-group) if $(G, \cdot)$ is a group, $(G, \leq)$ is a partially ordered set, and for any elements $g, h \in G$ such that $g \leq h$, $f \cdot g \leq f \cdot h$ and $g \cdot f \leq h \cdot f$ for all $f \in G$. A po-group G is a *lattice ordered group* (ℓ -group) if for each pair of elements $g, h \in G$, there exists both $g.\ell.b.\{g, h\}$ and $g.\ell.u.b.\{g, h\}$ in G . If such elements exist, they must be unique; hence we may adopt the notation $g \wedge h = g.\ell.b.\{g, h\}$ and $g \vee h = g.\ell.u.b.\{g, h\}$ without ambiguity. If G is a po-group whose partial order is in fact a total order, then G is called an *o-group*. Hence, all o-groups are also ℓ -groups.

A subgroup H of an ℓ -group G is called an *ℓ -subgroup* if $h \wedge k \in H$ and $h \vee k \in H$ for all elements $h, k \in H$. It is a *convex ℓ -subgroup* if it satisfies the additional property that when $h < g < k$ for $h, k \in H$, then $g \in H$. Normal convex ℓ -subgroups are called *ℓ -ideals* of G . A function $\phi : G \rightarrow H$ where G and H are ℓ -groups is an *ℓ -homomorphism* if it is a group homomorphism such that $(g \wedge h)\phi = g\phi \wedge h\phi$ and $(g \vee h)\phi = g\phi \vee h\phi$ for all $g, h \in G$. It is possible to show that ℓ -ideals are kernels of ℓ -homomorphisms and conversely.

There are two common methods of forming products (sums) of ℓ -groups. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of ℓ -groups, then $\prod G_\lambda$ (ΣG_λ) will

denote the *cardinal product (sum)* of the G_λ 's. It is the direct product (sum) of the G_λ 's as groups with the order defined componentwise, i.e. for $g, h \in \prod G_\lambda$, $g \leq h$ if $g_\lambda \leq h_\lambda$ for all $\lambda \in \Lambda$. If G is an ℓ -group and H is an o-group, then $G \overset{\leftarrow}{\times} H$ will denote the *lex product* of G and H . It is again the direct product of G and H as groups, with the order defined by $(g_1, h_1) \leq (g_2, h_2)$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$ if $h_1 < h_2$ or $h_1 = h_2$ and $g_1 \leq g_2$. An ℓ -group G is a *subdirect product* of ℓ -groups $\{G_\lambda \mid \lambda \in \Lambda\}$ if G is ℓ -embeddable in $\prod G_\lambda$ such that its image in $\prod G_\lambda$ projects onto G_λ for each $\lambda \in \Lambda$. Equivalently, there exist ℓ -ideals N_λ in G for each $\lambda \in \Lambda$ such that $\bigcap N_\lambda = \{e\}$ (where e is the group identity) and $G/N_\lambda \approx G_\lambda$ [4]. An ℓ -group G is *subdirectly irreducible* if whenever G is a subdirect product of ℓ -groups $\{G_\lambda \mid \lambda \in \Lambda\}$, there exists $\lambda_0 \in \Lambda$ such that G and G_{λ_0} are ℓ -isomorphic. This corresponds to G having a smallest non-trivial ℓ -ideal [4].

§2. General Properties of $\mathbf{L}$

For a set X , an ℓ -group $F \supseteq X$ is called a *free ℓ -group generated by X* if for any ℓ -group G and any function $\phi : X \rightarrow G$, there is a unique extension of ϕ to an ℓ -homomorphism $\phi' : F \rightarrow G$. Both the existence and the uniqueness of a free ℓ -group for any set X has been shown [4]. For the algebra of ℓ -groups, a *word* will be considered to be an element w of $F_{\aleph_0}$, the free ℓ -group over a countable alphabet $\{x_1, x_2, x_3, \dots\}$. The word w is a *law* in an

ℓ -group G if $w\alpha = e$ for every ℓ -homomorphism $\alpha : F_{N_0} \rightarrow G$. This is equivalent to producing only e by substituting into w all possible combinations of elements of G . A *variety* of ℓ -groups is then the class of all ℓ -groups for which a given set of words are laws. Equivalently, a variety of ℓ -groups is a class of ℓ -groups which is closed with respect to ℓ -subgroups, ℓ -homomorphic images, and cardinal products by Birkhoff's Theorem [4].

Some examples of well known varieties of ℓ -groups are as follows:

$\{\{e\}\} \equiv$ the *trivial* variety, which is defined by the word $w = x_1$.

$\mathcal{A} \equiv$ the variety of *abelian* ℓ -groups, which is defined by the word $w = x_1 x_2 x_1^{-1} x_2^{-1}$.

$\mathcal{R} \equiv$ the variety of *representable* ℓ -groups, which is defined by the word $w = (x_1 \wedge (x_2^{-1} x_1^{-1} x_2)) \vee e$.

$\mathcal{W} \equiv$ the variety of *weakly abelian* ℓ -groups, which is defined by the word $w = [(x_1 \vee e)^2 x_2^{-1} (x_1 \vee e)^{-1} x_2] \wedge e$.

$\mathcal{N} \equiv$ the variety of *normal-valued* ℓ -groups, which is defined by the word $w = [(x_1 \vee e)^2 (x_2 \vee e)^2 (x_1 \vee e)^{-1} (x_2 \vee e)^{-1}] \wedge e$.

$\mathcal{L} \equiv$ the variety of *all* ℓ -groups, which is defined by the word $w = x_1 x_1^{-1}$.

We shall use $\mathbf{L}$ to denote the set of all ℓ -group varieties.

It can be easily verified that the intersection of any collection $\{\mathcal{V}_\lambda \mid \lambda \in \Lambda\}$ of ℓ -group varieties is again a variety. Therefore, by considering set containment as a partial order on $\mathbf{L}$, it is possible

to define the lattice operation $\wedge$ on $\mathbf{L}$, i.e. $\wedge \mathcal{V}_\lambda = \bigcap \mathcal{V}_\lambda$. Since for such a collection, $\mathcal{V}_\lambda \leq \mathcal{L}$ for each $\lambda \in \Lambda$, we may define $\vee \mathcal{V}_\lambda = \bigcap \{\mathcal{U} \in \mathbf{L} \mid \mathcal{U} \geq \mathcal{V}_\lambda, \text{ for all } \lambda \in \Lambda\}$. It can be shown that $G \in \vee \mathcal{V}_\lambda$ iff G is an ℓ -homomorphic image of a subdirect product of ℓ -groups from the $\mathcal{V}_\lambda$'s [10]. These definitions for $\vee$ and $\wedge$ suffice to make $\mathbf{L}$ a complete lattice.

In a similar manner, it is possible to define what is meant by the variety generated by an ℓ -group G , letting $\mathcal{V}(G) = \bigcap \{\mathcal{U} \in \mathbf{L} \mid G \in \mathcal{U}\}$. We may also consider a type of associative multiplication on $\mathbf{L}$, where for $\mathcal{U}, \mathcal{V} \in \mathbf{L}$, $G \in \mathcal{U} \cdot \mathcal{V}$ iff G has an ℓ -ideal H with $H \in \mathcal{U}$ and $G/H \in \mathcal{V}$ (i.e. G is an extension of an ℓ -group in $\mathcal{U}$ by one in $\mathcal{V}$). It should be noted that this notation is consistent with that of Neumann [12], while Martinez [10] reverses the roles of the factors $\mathcal{U}, \mathcal{V}$ in his notation.

It has been shown by Martinez [10] that with respect to the above operations, $\mathbf{L}$ is a dually-Brouwer lattice, i.e.

$\mathcal{U} \vee (\wedge \mathcal{V}_\lambda) = \wedge (\mathcal{U} \vee \mathcal{V}_\lambda)$, and that the following distributive laws hold:

$$(D1) \quad (\wedge \mathcal{V}_\lambda) \cdot \mathcal{U} = \wedge (\mathcal{V}_\lambda \cdot \mathcal{U}) \text{ for } \mathcal{U}, \mathcal{V}_\lambda \in \mathbf{L}.$$

$$(D2) \quad (\mathcal{V}_1 \vee \mathcal{V}_2) \cdot \mathcal{U} = \mathcal{V}_1 \cdot \mathcal{U} \vee \mathcal{V}_2 \cdot \mathcal{U} \text{ for } \mathcal{U}, \mathcal{V}_1, \mathcal{V}_2 \in \mathbf{L}.$$

It should also be noted that the multiplication on $\mathbf{L}$ is compatible with the partial order, i.e. if $\mathcal{U} \leq \mathcal{V}$, then $\mathcal{W}_1 \cdot \mathcal{U} \cdot \mathcal{W}_2 \leq \mathcal{W}_1 \cdot \mathcal{V} \cdot \mathcal{W}_2$ for all varieties $\mathcal{U}, \mathcal{V}, \mathcal{W}_1, \mathcal{W}_2 \in \mathbf{L}$.

Since $\mathbf{L}$ is a lattice, it is of interest to attempt to characterize join-irreducible elements. An element $\mathcal{V} \in \mathbf{L}$ is *join-irreducible* if whenever $\mathcal{V} = \vee \mathcal{U}_\lambda$ for some collection $\{\mathcal{U}_\lambda \in \mathbf{L} \mid \lambda \in \Lambda\}$, then

$\mathcal{V} = \mathcal{U}_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. While the general case has not been solved, it is possible to give a condition which guarantees *finite* join-irreducibility. In order to discuss this, we must first consider the following proposition due to Martinez [10].

2.1. Proposition: If $\mathcal{V} = \mathcal{V}_1 \vee \mathcal{V}_2$ for $\mathcal{V}_1, \mathcal{V}_2 \in \mathbf{L}$, then $G \in \mathcal{V}$ iff G is a subdirect product of ℓ -groups in $\mathcal{V}_1$ and $\mathcal{V}_2$, or equivalently, iff there exist ℓ -ideals A_1 and A_2 in G such that $A_1 \cap A_2 = \{e\}$ and $G/A_1 \in \mathcal{V}_1$, $G/A_2 \in \mathcal{V}_2$.

Let G and H be ℓ -groups. Due to a theorem of Holland [5], there exist chains Ω_1 and Ω_2 such that G and H can be considered as ℓ -groups of order-preserving permutations (ℓ -permutation groups) on Ω_1 and Ω_2 respectively. It should be noted that an order-preserving permutation g on a chain Γ is considered positive if $\gamma g \geq \gamma$ for all $\gamma \in \Gamma$. Using this representation for G and H , let $(W, \Omega) = (G, \Omega_1) \text{Wr} (H, \Omega_2)$ be the order-preserving permutation group on $\Omega = \Omega_1 \times \Omega_2$, where all elements of W are of the form (θ, h) for $h \in H$ and $\theta : \Omega_2 \rightarrow G$, and the action of W on Ω is defined by $(\alpha_1, \alpha_2)(\theta, h) = (\alpha_1 \theta(\alpha_2), \alpha_2 h)$ for any $(\alpha_1, \alpha_2) \in \Omega$. It can then be verified that the group operation on W is such that $(\theta, h)(\phi, k) = (\psi, hk)$, where for $\alpha \in \Omega_2$, $\psi(\alpha) = \theta(\alpha) \cdot \phi(\alpha h)$. Using this multiplication on W and the usual ordering of order-preserving permutations discussed above, W is an ℓ -group [8], and is called the *Wreath product* of G with H . Within W , $W' = \{(\theta, h) \mid \theta(\alpha) = e_G \text{ for all but a finite subset of } \Omega_2\}$ forms an ℓ -subgroup which also acts as an ℓ -permutation group on $\Omega = \Omega_1 \times \Omega_2$.

This ℓ -subgroup is called the *wreath product* of G with H and denoted by $(W', \Omega) = (G, \Omega_1) \text{wr} (H, \Omega_2)$.

When an ℓ -group G is itself totally ordered, it is possible to consider G as an ℓ -permutation group on itself, often using right translation as the action. In such a case, G is usually preferred to the more cumbersome notation (G, G) ; so the notation $G \text{Wr} (H, \Omega_2)$, $(G, \Omega_1) \text{Wr} H$, $G \text{Wr} H$, etc. make sense in this context and will be used when appropriate.

2.2 Theorem: *A variety $\mathcal{V} \in \mathbf{L}$ is finitely join-irreducible if $\mathcal{V}$ is closed with respect to Wreath or wreath products.*

PROOF: By way of contradiction, suppose that $\mathcal{V}$ is finitely join-reducible. Then $\mathcal{V} = \mathcal{U} \vee \mathcal{W}$ for some $\mathcal{U}, \mathcal{W} \in \mathbf{L}$ such that $\mathcal{U} < \mathcal{V}$ and $\mathcal{W} < \mathcal{V}$. Since every ℓ -group can be expressed as a subdirect product of subdirectly irreducible ℓ -groups [4], the subdirectly irreducible ℓ -groups of a variety generate the variety. Thus, the strict containments above assure us of the existence of subdirectly irreducible ℓ -groups $H \in \mathcal{V} \setminus \mathcal{U}$ and $K \in \mathcal{V} \setminus \mathcal{W}$. Then there exist chains Ω_1 and Ω_2 such that (H, Ω_1) and (K, Ω_2) are transitive ℓ -permutation groups [5], i.e. a group G acts *transitively* on a set Γ if for all $\gamma, \delta \in \Gamma$, there is an element $g \in G$ such that $\gamma g = \delta$. $(W, \Omega) = (H, \Omega_1) \text{Wr} (K, \Omega_2)$, or similarly for wr , is also a transitive ℓ -permutation group. If W contains non-trivial disjoint ℓ -ideals A_1 and A_2 , then there exist $\alpha, \beta \in \Omega$ such that $\alpha h_1 > \alpha$ for some $h_1 \in A_1^+$ and $\beta h_2 > \beta$ for some $h_2 \in A_2^+$. By the transitivity of W , there exists $g \in W$ such that $\alpha g = \beta$. Thus, $g^{-1} h_1 g \in A_1$

and $\beta g^{-1}h_1g = \alpha h_1g > \alpha g = \beta$; but then $g^{-1}h_1g \in A_1$ and $h_2 \in A_2$ both move β , which is impossible for disjoint ℓ -ideals. Hence W contains no pairs of non-trivial disjoint ℓ -ideals. However, since $\mathcal{V}$ is closed under Wreath (wreath) products, $W \in \mathcal{V} = \mathcal{U} \vee \mathcal{W}$. By Proposition 2.1, there must exist ℓ -ideals A_1 and A_2 of W with $A_1 \cap A_2 = \{e\}$ and $W/A_1 \in \mathcal{U}$, $W/A_2 \in \mathcal{W}$. By the argument above, one of these ℓ -ideals must be trivial. If $A_1 = \{e\}$, then $W \approx W/A_1 \in \mathcal{U}$; and since H is ℓ -embeddable in W , $H \in \mathcal{U}$, contradicting the choice of H . If $A_2 = \{e\}$, then $W \approx W/A_2 \in \mathcal{W}$. But since K is an ℓ -homomorphic image of W , then $K \in \mathcal{W}$, which contradicts our choice of K . Hence, $\mathcal{V}$ must be finitely join-irreducible. //

2.3 Corollary: If $\mathcal{V}^2 = \mathcal{V} \cdot \mathcal{V} = \mathcal{V}$ ($\mathcal{V}$ is idempotent in $\mathbf{L}$), then $\mathcal{V}$ is finitely join-irreducible.

PROOF: If $\mathcal{V}^2 = \mathcal{V}$, then $\mathcal{V}$ is closed under finite extensions.

Since a Wreath product of G with H is an extension of $(\prod G)$ by H , $\mathcal{V}$ must be closed with respect to Wreath products. Hence, it is finitely join-irreducible by the preceding theorem. //

2.4 Corollary: $\mathcal{N}$ is finitely join-irreducible in $\mathbf{L}$.

PROOF: Martinez [10] has shown that $\mathcal{N}$ is idempotent. //

CHAPTER III

PROPERTIES OF THE SCRIMGER VARIETIES

It is known that $\mathcal{A}$ is the smallest non-trivial variety in $\mathbf{L}$ [14, Cor. 5]. Its unique position within $\mathbf{L}$ therefore makes it a likely candidate for useful generalizations. One type of generalization of $\mathcal{A}$ is the countable collection of varieties $\mathcal{L}_n$ for $n \in \mathbb{N}$, where $\mathcal{L}_n$ is the variety of ℓ -groups for which $w = x_1^n x_2^n x_1^{-n} x_2^{-n}$ is a law. Obviously, $\mathcal{L}_1 = \mathcal{A}$ and $\mathcal{L}_m \leq \mathcal{L}_n$ if m divides n . In order to further investigate the properties of these varieties, it is useful to consider a specific collection of ℓ -groups, first described by Martinez [9]. They may be defined as follows:

$$G_n = \{(\phi, k) \in \mathbf{ZWrZ} \mid \phi(i) = \phi(j) \text{ if } i \equiv j \pmod{n}\}$$

for any $n \in \mathbb{N}$. It may be easily verified that such a set forms an ℓ -subgroup of $\mathbf{ZWrZ}$.

Another representation of G_n which often proves more informative is

$$G_n = \begin{pmatrix} n-1 \\ \Pi & \mathbf{Z} \\ 0 \end{pmatrix} \overset{\leftarrow}{\times}_{\theta} \mathbf{Z},$$

a lexicographically ordered splitting extension of $\begin{pmatrix} n-1 \\ \Pi & \mathbf{Z} \\ 0 \end{pmatrix}$ by $\mathbf{Z}$

with multiplication defined by:

$$\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \begin{pmatrix} \ell \\ (s_0, \dots, s_{n-1}) \end{pmatrix} = \begin{pmatrix} k + \ell \\ (r_0 + s_{0+k}, \dots, r_{n-1} + s_{n-1+k}) \end{pmatrix}$$

where $r_i, s_j, k, \ell \in \mathbf{Z}$ and all subscripts are computed $(\text{mod } n)$. Using

this representation, we have that for any $r_0, \dots, r_{n-1}, k \in \mathbb{Z}$,

$$\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix}^n = \begin{pmatrix} nk \\ (t_0, \dots, t_{n-1}) \end{pmatrix}$$

where $t_i = r_i + r_{i+k} + r_{i+2k} + \dots + r_{i+(n-1)k}$ for $i = 0, 1, \dots, n-1$ and again all subscripts computed (mod n). Since the noncommutativity of multiplication in G_n depends solely on the fact that multiplication permutes the second n -tuple but not the first, assuring that the second n -tuple is permuted through a multiple of n steps would essentially alleviate the commutativity problem. This is precisely what happens when we restrict ourselves to the n th power elements of G_n ; hence, $G_n \in \mathcal{L}_n$.

Some other facts worth noting about the G_n 's are as follows:

- (1) G_n is subdirectly irreducible for any $n \in \mathbb{N}$.
- (2) $G_n \in \mathcal{A}^2$ for any $n \in \mathbb{N}$.
- (3) For $n, m \in \mathbb{N}$, $G_m \in \mathcal{L}_n$ if and only if $m|n$.

This last fact makes it apparent then that:

- (4) $\mathcal{L}_m \leq \mathcal{L}_n$ if and only if $m|n$, and containment is proper if and only if m properly divides n .

We define the *Scrimger variety* $\mathcal{S}_n = \mathcal{V}(G_n)$, for $n \in \mathbb{N}$.

3.1 Proposition: For any $m, n > 1$

- (a) $\mathcal{S}_m \vee \mathcal{S}_n < \mathcal{S}_{m \cdot n}$
- (b) $\mathcal{L}_m \vee \mathcal{L}_n < \mathcal{L}_{m \cdot n}$.

PROOF: (a) Clearly, G_m and G_n are ℓ -isomorphic to ℓ -subgroups of $G_{m \cdot n}$. Thus, $\mathcal{S}_m, \mathcal{S}_n \leq \mathcal{S}_{m \cdot n}$ and $\mathcal{S}_m \vee \mathcal{S}_n \leq \mathcal{S}_{m \cdot n}$. If

$G_{m \cdot n} \in \mathcal{S}_m \vee \mathcal{S}_n$, then by Proposition 2.1 $G_{m \cdot n}$ must be a subdirect product of ℓ -groups from $\mathcal{S}_m$ and $\mathcal{S}_n$. However, since $G_{m \cdot n}$ is subdirectly irreducible, either $G_{m \cdot n} \in \mathcal{S}_m$ or $G_{m \cdot n} \in \mathcal{S}_n$. But this implies that $G_{m \cdot n} \in \mathcal{L}_m$ or $G_{m \cdot n} \in \mathcal{L}_n$, and by remark #3 above, then either $m \cdot n | m$ or $m \cdot n | n$. Thus, $G_{m \cdot n} \notin \mathcal{S}_m \vee \mathcal{S}_n$ and $\mathcal{S}_m \vee \mathcal{S}_n < \mathcal{S}_{m \cdot n}$.

(b) By remark #4 above, $\mathcal{L}_m \vee \mathcal{L}_n \leq \mathcal{L}_{m \cdot n}$. As proved in (a), $G_{m \cdot n} \notin \mathcal{L}_m \vee \mathcal{L}_n$; therefore, $\mathcal{L}_m \vee \mathcal{L}_n < \mathcal{L}_{m \cdot n}$. //

3.2 Proposition: For any $n \in \mathbb{N}$, $R \wedge \mathcal{L}_n = \mathcal{A}$ [9].

3.3 Theorem: For $m, n \in \mathbb{N}$ such that $(m, n) = 1$, $\mathcal{L}_m \wedge \mathcal{L}_n = \mathcal{A}$ [13].

Since for any $n > 1$, $\mathcal{A} < \mathcal{S}_n \leq \mathcal{L}_n$, Proposition 3.2 and Theorem 3.3 apply to the $\mathcal{S}_n$'s as well.

3.4 Corollary: For any $n \in \mathbb{N}$, $R \wedge \mathcal{S}_n = \mathcal{A}$.

3.5 Corollary: For $(m, n) = 1$, $\mathcal{S}_m \wedge \mathcal{S}_n = \mathcal{A}$.

3.6 Corollary: For $(m, n) = 1$, $\mathcal{L}_m \wedge \mathcal{S}_n = \mathcal{A}$.

The varieties $\mathcal{S}_n$ ($n \in \mathbb{N}$) play a very important role in $\mathbf{L}$.

3.7 Theorem: For any prime p , $\mathcal{S}_p$ covers $\mathcal{A}$ in $\mathbf{L}$, i.e. no variety lies between $\mathcal{S}_p$ and $\mathcal{A}$ [13].

3.8 Proposition: For any variety $\mathcal{V} \in \mathbf{L}$ and any prime p such that $\mathcal{V} \wedge \mathcal{S}_p = \mathcal{A}$, the variety $\mathcal{V} \vee \mathcal{S}_p$ covers $\mathcal{V}$ in $\mathbf{L}$.

PROOF: It need only be noted that, since $\mathbf{L}$ is a distributive lattice, the interval from $\mathcal{V}$ to $\mathcal{V} \vee \mathcal{S}_p$ is lattice-isomorphic to

that from $\mathcal{A} = \mathcal{V} \wedge \mathcal{S}_p$ to $\mathcal{S}_p$ [3]. //

Since the requirement that $\mathcal{V} \wedge \mathcal{S}_p = \mathcal{A}$ is equivalent by Theorem 3.7 to $\mathcal{S}_p \perp \mathcal{V}$, it is obvious that this proposition applies to a great many varieties.

3.9 Corollary: For any prime p , the following are true:

- (i) $R \vee \mathcal{S}_p$ covers R .
- (ii) $\mathcal{W} \vee \mathcal{S}_p$ covers $\mathcal{W}$.
- (iii) For any $\mathcal{V} \leq R$, $\mathcal{V} \vee \mathcal{S}_p$ covers $\mathcal{V}$.

And for any $n > 1$ and any prime p such that $p \nmid n$, then:

- (iv) $\mathcal{S}_n \vee \mathcal{S}_p$ covers $\mathcal{S}_n$.
- (v) $\mathcal{L}_n \vee \mathcal{S}_p$ covers $\mathcal{L}_n$.

PROOF: The validity of these statements rests upon Corollaries 3.4, 3.5, 3.6 and upon the fact that $\mathcal{W} < R$ [9]. //

3.10 Corollary: For any finite set of distinct primes $\{p_1, p_2, \dots, p_n\}$,

$$\bigvee_{i=1}^n \mathcal{S}_{p_i} \text{ covers } \bigvee_{i=1}^{n-1} \mathcal{S}_{p_i}.$$

PROOF: Since $\mathbf{L}$ is a distributive lattice,

$$\mathcal{S}_{p_n} \wedge \left(\bigvee_{i=1}^{n-1} \mathcal{S}_{p_i} \right) = \bigvee_{i=1}^{n-1} (\mathcal{S}_{p_n} \wedge \mathcal{S}_{p_i}) = \mathcal{A} \text{ by Corollary 3.5. Hence,}$$

Proposition 3.8 yields the desired result. //

3.11 Proposition: For primes $p_1, \dots, p_m, q_1, \dots, q_n$, $\bigvee_{i=1}^m \mathcal{S}_{p_i} \leq \bigvee_{j=1}^n \mathcal{S}_{q_j}$ iff $\{p_1, \dots, p_m\} \subseteq \{q_1, \dots, q_n\}$.

PROOF: Sufficiency is obvious. To prove necessity, suppose

$$\bigvee_{i=1}^m \mathcal{S}_{p_i} \leq \bigvee_{j=1}^n \mathcal{S}_{q_j}. \text{ Then for any } p_i, \mathcal{S}_{p_i} \leq \mathcal{S}_{p_i} \wedge \left(\bigvee_{j=1}^n \mathcal{S}_{q_j} \right) = \bigvee_{j=1}^n (\mathcal{S}_{p_i} \wedge \mathcal{S}_{q_j}),$$

which implies via Corollary 3.5 that $\mathcal{J}_{p_i} \doteq \mathcal{J}_{p_i} \wedge \mathcal{J}_{q_j}$ for some j ,
 i.e. there exists some j such that $q_j = p_i$. //

According to this proposition, the sublattice of $\mathbf{L}$ generated by $\{\mathcal{J}_p \mid p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes. The next question is whether or not the complete sublattice of $\mathbf{L}$ generated by $\{\mathcal{J}_p \mid p \text{ is prime}\}$ is lattice-isomorphic to the entire Boolean lattice of sets of primes. In order to answer this question, it is useful first to consider conditions for an ℓ -group to be ℓ -isomorphic to G_n for some $n \in \mathbb{N}$.

3.12 Theorem: *An ℓ -group G is generated (as a group) by two positive elements a and b such that:*

$$(i) \quad a < b$$

$$(ii) \quad \text{For some } n \in \mathbb{N}, \quad b^{-i}ab^i \Delta a = e \quad \text{for } 1 \leq i \leq n-1, \quad \text{and}$$

$$b^{-n}ab^n = a$$

iff G is ℓ -isomorphic to G_n .

PROOF: Part of the proof of this theorem may be found in Scrimger's proof of the minimality of $\mathcal{J}_p$ in $\mathbf{L}$ [13]; however, it is presented here in a more general setting and with a reduced number of conditions.

Assuming the conditions (i) and (ii) hold for an ℓ -group G , define a function $\beta : G_n \rightarrow G$ by:

$$\left[\begin{array}{c} k \\ (r_0, \dots, r_{n-1}) \end{array} \right] \beta = a^{r_0} (b^{-1}ab)^{r_1} \dots (b^{-n+1}ab^{n-1})^{r_{n-1}} b^k$$

where $r_0, \dots, r_{n-1}, k \in \mathbb{Z}$.

For two arbitrary elements of G_n ,

$$\left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \begin{pmatrix} \ell \\ (s_0, \dots, s_{n-1}) \end{pmatrix} \right]^\beta = \begin{pmatrix} k + \ell \\ (r_0 + s_{0+k^*}, \dots, r_{n-1} + s_{n-1+k^*}) \end{pmatrix}^\beta$$

$$= a^{r_0 + s_{0+k^*}} (b^{-1} ab)^{r_1 + s_{1+k^*}} \dots (b^{-n+1} ab^{n-1})^{r_{n-1} + s_{n-1+k^*}} b^{k+\ell}$$

where all subscripts are taken (mod n) and k^* is the non-negative remainder when k is divided by n . Also,

$$\begin{aligned} & \left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \right]^\beta \left[\begin{pmatrix} \ell \\ (s_0, \dots, s_{n-1}) \end{pmatrix} \right]^\beta = \\ & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} b^k a^{s_0} \dots (b^{-n+1} ab^{n-1})^{s_{n-1}} b^\ell = \\ & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} b^{k^*} b^{k-k^*} a^{s_0} b^{-k+k^*} b^{k-k^*} \dots \\ & b^{k-k^*} (b^{-n+1} ab^{n-1})^{s_{n-1}} b^{-k+k^*} b^{k-k^*+\ell} = \\ & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} b^{k^*} (b^{k-k^*} ab^{-k+k^*})^{s_0} \dots \\ & (b^{-n+1} b^{k-k^*} ab^{-k+k^*} b^{n-1})^{s_{n-1}} b^{k-k^*+\ell} = \\ & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} b^{k^*} a^{s_0} \dots (b^{-n+1} ab^{n-1})^{s_{n-1}} b^{k-k^*+\ell} = \end{aligned}$$

since $k - k^*$ is a multiple of n and $b^{-n} ab^n = a$,

$$\begin{aligned} & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} \overbrace{b^{-n+k^*} a^{s_0} b^{n-k^*}} \dots \\ & b^{n-k^*} \overbrace{b^{-n+k^*} (b^{-k^*+1} ab^{k^*-1})^{s_{k^*-1}} b^{n-k^*}} \overbrace{b^{k^*} (b^{-k^*} ab^{k^*})^{s_{k^*}} b^{-k^*}} b^{k^*} \dots \\ & b^{-k^*} \overbrace{b^{k^*} (b^{-n+1} ab^{n-1})^{s_{n-1}} b^{-k^*}} b^{k+\ell} = \\ & a^{r_0} \dots (b^{-n+1} ab^{n-1})^{r_{n-1}} (b^{-n+k^*} ab^{n-k^*})^{s_0} \dots \\ & (b^{-n+1} ab^{n-1})^{s_{k^*-1}} a^{s_{k^*}} \dots (b^{-n+k^*+1} ab^{n-k^*-1})^{s_{n-1}} b^{k+\ell} = \\ & a^{r_0 + s_{k^*}} \dots (b^{-n+1} ab^{n-1})^{r_{n-1} + s_{k^*-1}} b^{k+\ell}, \end{aligned}$$

since all of the conjugates of a by b commute with one another.

Thus, β preserves the group operation.

In order to show that β also preserves the lattice operations, it is first necessary to show that $b > a^m$ for any $m \in \mathbb{Z}$. Since $b > a > e$, it is sufficient to show that $b > a^m$ for any $m > 1$.

Suppose for some $m > 1$, $b = a^m$. Then a and b would commute, which they obviously do not by condition (ii). Suppose then that for some $m > 1$, $a^m \not\leq b$. This obviously means that $(b^{-1}ab)^m \not\leq b$. By Holland's theorem [5], there is a chain Γ such that G acts as an ℓ -group of order preserving permutations on Γ . Within this context, the last inequality above means that there exists some $\gamma \in \Gamma$ such that $(*) \quad \gamma(b^{-1}ab)^m > \gamma b \geq \gamma$. Hence, γ is moved by $(b^{-1}ab)^m$ and thus by $(b^{-1}ab)$. Since $(b^{-1}ab) \wedge a = e$, it follows that $\gamma a = \gamma$, so $(\gamma b)(b^{-1}ab) = \gamma b$ and $(\gamma b)(b^{-1}ab)^m = \gamma b$. But by $(*)$, $(\gamma b)(b^{-1}ab)^m \geq \gamma(b^{-1}ab)^m > \gamma b$. Therefore, $a^m < b$ must hold for all $m \in \mathbb{Z}$.

From this conclusion, it readily follows that $(b^{-i}ab^i)^m < b$ for all $i, m \in \mathbb{Z}$. Since $a, b^{-1}ab, \dots, b^{-n+1}ab^{n-1}$ are pairwise disjoint positive elements, $\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} = \bigvee_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \leq b$ for any $r_0, \dots, r_{n-1} \in \mathbb{Z}$ [2]. Again the inequality must be proper, since otherwise a and b would commute. Thus, b dominates the entire ℓ -ideal of G generated by a .

Now, for any element of G_n ,

$$\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} = \begin{cases} \begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} & \text{if } k > 0 \\ \begin{pmatrix} 0 \\ (r_0 \vee 0, \dots, r_{n-1} \vee 0) \end{pmatrix} & \text{if } k = 0 \\ \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} & \text{if } k < 0 \end{cases}$$

Thus, for $k > 0$, we have:

$$\begin{aligned} \left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} \right]^\beta &= a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} b^k = \\ a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} b^k \vee e &= \begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix}^\beta \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix}^\beta, \end{aligned}$$

since $b^k \geq b > a^{-r_0} \dots (b^{-n+1} a b^{n-1})^{-r_{n-1}}$ by the previous remarks.

Similarly, for $k < 0$,

$$\begin{aligned} \left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} \right]^\beta &= e = a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} b^k \vee e = \\ \begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix}^\beta \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix}^\beta, \end{aligned}$$

since $b^k \leq b^{-1} < a^{-r_0} \dots (b^{-n+1} a b^{n-1})^{-r_{n-1}}$. For $k = 0$, divide

$\{0, 1, \dots, n-1\}$ into two disjoint sets $I = \{i \mid r_i \geq 0\}$ and

$J = \{j \mid r_j < 0\}$. Then

$$\begin{aligned} \left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} \right]^\beta &= \\ a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} \vee e &= \prod_{m=0}^{n-1} (b^{-m} a b^m)^{r_m} \vee e = \\ \left[\prod_{i \in I} (b^{-i} a b^i)^{r_i} \vee \prod_{j \in J} (b^{-j} a b^j)^{-r_j} \right] \prod_{j \in J} (b^{-j} a b^j)^{r_j} &= \end{aligned}$$

$$\left[\prod_{i \in I} (b^{-i} a b^i)^{r_i} \prod_{j \in J} (b^{-j} a b^j)^{-r_j} \right] \prod_{j \in J} (b^{-j} a b^j)^{r_j} =$$

$$\prod_{i \in I} (b^{-i} a b^i)^{r_i} = \prod_{m=0}^{n-1} (b^{-m} a b^m)^{r_m} \vee 0 =$$

$$\left[\begin{pmatrix} k \\ (r_0, \dots, r_{n-1}) \end{pmatrix} \vee \begin{pmatrix} 0 \\ (0, \dots, 0) \end{pmatrix} \right]^\beta.$$

Hence, β preserves the lattice operations.

To show that β is one-to-one, let $a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} b^k = e$.

Then $k = 0$, since otherwise $b^k = a^{-r_0} \dots (b^{-n+1} a b^{n-1})^{-r_{n-1}}$,

contradicting the fact that b strictly dominates all such elements

of G . Also, if some $r_k \neq 0$, then there exist i_0 and j_0 between

0 and $n-1$ such that $r_{i_0} > 0$ and $r_{j_0} < 0$, since

$a, (b^{-1} a b), \dots, (b^{-n+1} a b^{n-1})$ are all positive elements. So again

divide $\{0, \dots, n-1\}$ into I and J as above. Then

$a^{r_0} \dots (b^{-n+1} a b^{n-1})^{r_{n-1}} = e$ implies that $\prod_{i \in I} (b^{-i} a b^i)^{r_i} = \prod_{j \in J} (b^{-j} a b^j)^{-r_j}$.

Hence, $(b^{-i_0} a b^{i_0})^{r_{i_0}} = (b^{-i_0} a b^{i_0})^{r_{i_0}} \wedge \left(\prod_{i \in I} (b^{-i} a b^i)^{r_i} \right) =$

$(b^{-i_0} a b^{i_0})^{r_{i_0}} \wedge \left(\prod_{j \in J} (b^{-j} a b^j)^{-r_j} \right) = \prod_{j \in J} ((b^{-i_0} a b^{i_0})^{r_{i_0}} \wedge (b^{-j} a b^j)^{-r_j}) = e,$

since, $i_0 \notin J$. Thus, it must be true that $r_0 = \dots = r_{n-1} = 0$, and

so β is one-to-one.

Since every element of G is already of the form

$a^{m_0} b^{m_1} a^{m_2} \dots b^{m_q}$, it is necessary only to note that by adding and

subtracting appropriate powers of b throughout, such a term can be

reexpressed as a product of conjugates of a by powers of b with

any "extra" power of b at the end of the term. That is, every element of G is in the β -image of G_n . Thus, β is an ℓ -isomorphism.

To complete the proof we need only note that G_n is generated by the elements

$$a = \begin{pmatrix} 0 \\ (1, 0, \dots, 0) \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ (0, \dots, 0) \end{pmatrix}$$

which obviously satisfy conditions (i) and (ii). //

3.13 Theorem: If $\{m_\alpha \mid \alpha \in \mathbb{N}\}$ is any infinite subset of $\mathbb{N}$, then $\bigvee_{\alpha \in \mathbb{N}} \mathcal{J}_{m_\alpha} = \bigvee_{n \in \mathbb{N}} \mathcal{J}_n$, i.e. there is only one variety in $\mathcal{L}$ that is an infinite join of $\mathcal{J}_n$'s.

PROOF: It is sufficient to show that for any $n \in \mathbb{N}$, $G_n \in \bigvee_{\alpha \in \mathbb{N}} \mathcal{J}_{m_\alpha}$.

If $n \mid m_\alpha$ for some $\alpha \in \mathbb{N}$, then $\mathcal{J}_n \leq \mathcal{J}_{m_\alpha} \leq \bigvee_{\alpha \in \mathbb{N}} \mathcal{J}_{m_\alpha}$; therefore,

without loss of generality, we may assume that for each $\alpha \in \mathbb{N}$, $n \nmid m_\alpha$.

We may also assume that for every $\alpha \in \mathbb{N}$, $n < m_\alpha$ (since there are

only at most a finite number of m_α 's for which this might not hold),

and that the set has been ordered such that $m_\alpha < m_{\alpha+1}$ for all $\alpha \in \mathbb{N}$.

Now let $G = \prod_{\alpha \in \mathbb{N}} G_{m_\alpha}$. So elements of G may be expressed in

the form

$$\begin{pmatrix} k_1 & k_2 & k_3 & \dots \\ \phi_1 & \phi_2 & \phi_3 & \dots \end{pmatrix}$$

where for each $\alpha \in \mathbb{N}$, $k_\alpha \in \mathbb{Z}$ and $\phi_\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$\phi_\alpha(j + m_\alpha) = \phi_\alpha(j)$ for every $j \in \mathbb{Z}$, and where the "upper" pieces

dominate the order. Thus, we need only be concerned with the action

of ϕ_α on $0 \leq j \leq m_\alpha - 1$.

Let us pick $a, b \in G$ of the form

$$a = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_1 & \psi_2 & \psi_3 & \dots \end{pmatrix}$$

$$\text{where for each } \alpha \in \mathbb{N}, \quad \psi_\alpha(j) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod{n} \\ 0, & \text{if } j \not\equiv 0 \pmod{n} \end{cases}$$

$$b = \begin{pmatrix} 1 & 1 & 1 & \dots \\ \hat{0} & \hat{0} & \hat{0} & \dots \end{pmatrix}$$

where $\hat{0}(j) = 0$ for any $j \in \mathbb{Z}$. Then certainly $b > a > e$ in G .

For $1 \leq i \leq n-1$,

$$b^{-i}ab^i = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,i} & \psi_{2,i} & \psi_{3,i} & \dots \end{pmatrix}$$

where $\psi_{\alpha,i}(j) = \psi_\alpha(j-i)$ for all $\alpha \in \mathbb{N}$ and $j \in \mathbb{Z}$. Therefore,

$$b^{-i}ab^i \wedge a = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,i} \wedge \psi_1 & \psi_{2,i} \wedge \psi_2 & \psi_{3,i} \wedge \psi_3 & \dots \end{pmatrix}$$

where for any $\alpha \in \mathbb{N}$,

$$(\psi_{\alpha,i} \wedge \psi_\alpha)(0) = \begin{cases} 1, & \text{if } i \equiv m_\alpha \pmod{n} \\ 0, & \text{if } i \not\equiv m_\alpha \pmod{n} \end{cases}$$

and $(\psi_{\alpha,i} \wedge \psi_\alpha)(j) = 0$ for $0 < j \leq m_\alpha - 1$.

Similarly, when $i = n$,

$$b^{-n}ab^n = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,n} & \psi_{2,n} & \psi_{3,n} & \dots \end{pmatrix}$$

with $\psi_{\alpha,n}(j) = \psi_\alpha(j-n)$ for all $\alpha \in \mathbb{N}$ and $j \in \mathbb{Z}$. Hence,

$$b^{-n}ab^na^{-1} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,n} - \psi_1 & \psi_{2,n} - \psi_2 & \psi_{3,n} - \psi_3 & \dots \end{pmatrix}$$

where for all $\alpha \in \mathbb{N}$, $(\psi_{\alpha,n} - \psi_{\alpha})(0) = \psi_{\alpha}(0 - n) - \psi_{\alpha}(0) = \psi_{\alpha}(m_{\alpha} - n) - \psi_{\alpha}(0) = 0 - 1 = -1$ since $n < m_{\alpha}$ and $n \nmid m_{\alpha}$ implies that $m_{\alpha} - n \not\equiv 0 \pmod{n}$;

$$(\psi_{\alpha,n} - \psi_{\alpha})(j) = \begin{cases} 1, & \text{if } m_{\alpha} \equiv (n - j) \pmod{n} \\ 0, & \text{if } m_{\alpha} \not\equiv (n - j) \pmod{n} \end{cases}$$

for $0 < j \leq n - 1$; and $(\psi_{\alpha,n} - \psi_{\alpha})(j) = 0$ for $n \leq j \leq m_{\alpha} - 1$. So for any $\alpha \in \mathbb{N}$, $(\psi_{\alpha,n} - \psi_{\alpha})$ has exactly two non-zero entries, and these are always within the first n terms.

Now, let M be the ℓ -ideal of G generated by $b^{-n}ab^{n-1}$ and $\{b^{-i}ab^i \wedge a \mid 1 \leq i \leq n - 1\}$. Obviously, $b \notin M$. Also observe that for any $g \in M$,

$$g = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \phi_1 & \phi_2 & \phi_3 & \dots \end{pmatrix},$$

there is a uniform bound on the number of non-zero entries for the ϕ_{α} 's. However, for any $r \in \mathbb{N}$, there exists an m_{β} such that $m_{\beta} \geq rn + 1$. Then ψ_{β} has at least $(r + 1)$ non-zero entries, and so the ψ_{α} 's have no uniform bound. Thus, $a \notin M$.

Obviously, aM and bM in G/M satisfy the requirements of Theorem 3.12, so the ℓ -subgroup of G/M generated by aM and bM is ℓ -isomorphic to G_n . Thus, G_n is an ℓ -subgroup of a quotient of a cardinal product of $\{G_{m_{\alpha}} \mid \alpha \in \mathbb{N}\}$, and so $G_n \in V\{G_{m_{\alpha}} \mid \alpha \in \mathbb{N}\}$. //

Using this theorem, it is now possible to answer a question posed by Martinez [10].

3.14 Corollary: $\mathbf{L}$ is not a Brouwer lattice.

PROOF: $\mathcal{J}_3 = \mathcal{J}_3 \wedge (V\{\mathcal{J}_{2^n} \mid n \in \mathbb{N}\})$ according to Theorem 3.13, but

$V\{(\mathcal{J}_3 \wedge \mathcal{J}_{2^n}) \mid n \in \mathbb{N}\} = \mathcal{A}$ by Corollary 3.5. //

It should be remarked here that we have actually proved a stronger result than stated in the corollary, that is meets do not distribute over even the infinite join of a tower in $\mathbf{L}$. Also, Theorem 3.13 makes it obvious that there is no hope of extending Proposition 2.1 to the infinite case. Since $G_3 \in V\{\mathcal{J}_{2^n} \mid n \in \mathbb{N}\}$ and G_3 is subdirectly irreducible, it is impossible to drop the idea of quotients when generating an infinite join of varieties.

Since each $G_n \leq \mathbf{ZWrZ}$, $V\{\mathcal{J}_n \mid n \in \mathbb{N}\} \leq \mathcal{V}(\mathbf{ZWrZ})$. In order to determine whether this containment is proper, it is necessary to look at a generalization of Theorem 3.12.

3.15 Theorem: An ℓ -group G is generated (as a group) by two positive elements a and b such that:

$$(i) \quad a < b$$

$$(ii) \quad \text{For all } i \in \mathbb{N}, \quad b^{-i} a b^i \wedge a = e,$$

iff G is ℓ -isomorphic to $\mathbf{ZWrZ}$.

PROOF: For such an ℓ -group G , define a function $\beta : \mathbf{ZWrZ} \rightarrow G$ by $(\theta, k)\beta = \prod_{i \in \mathbb{Z}} (b^{-i} a b^i)^{\theta(i)} b^k$, where this product is well-defined since

for only a finite number of integers is $\theta(i) \neq 0$.

For (θ, k) and $(\phi, \ell) \in \mathbf{ZWrZ}$,

$$[(\theta, k)(\phi, \ell)]\beta = [(\psi, k + \ell)]\beta = \prod_{i \in \mathbb{Z}} (b^{-i} a b^i)^{\theta(i) + \phi(i+k)} b^{k+\ell}. \quad \text{And}$$

$$\begin{aligned}
(\theta, k)\beta(\phi, \ell)\beta &= \prod_{i \in \mathbb{Z}} (b^{-i}ab^i)^{\theta(i)}b^k \cdot \prod_{j \in \mathbb{Z}} (b^{-j}ab^j)^{\phi(j)}b^\ell = \\
&\prod_{i \in \mathbb{Z}} (b^{-i}ab^i)^{\theta(i)} \cdot \prod_{j \in \mathbb{Z}} (b^{-j+k}ab^{j-k})^{\phi(j)}b^{k+\ell} = \\
&\prod_{i \in \mathbb{Z}} (b^{-i}ab^i)^{\theta(i)} \cdot \prod_{i \in \mathbb{Z}} (b^{-i}ab^i)^{\phi(i+k)}b^{k+\ell} = \prod_{i \in \mathbb{Z}} (b^{-i}ab^i)^{\theta(i)+\phi(i+k)}b^{k+\ell},
\end{aligned}$$

since all of the $b^{-i}ab^i$ terms commute with one another. Thus, β preserves the group operation. The remainder of the proof that β is an ℓ -isomorphism is a repetition of the arguments in the proof of Theorem 3.12.

To complete the proof, note that $a = (\theta, 0)$ and $b = (\hat{\theta}, 1)$,

$$\text{where } \theta(j) = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0 \end{cases} \quad \text{and } \hat{\theta}(j) = 0 \text{ for all } j \in \mathbb{Z}, \text{ are}$$

elements which both generate $\mathbb{Z} \text{Wr } \mathbb{Z}$ and satisfy conditions (i) and (ii). //

3.16 Theorem: $\mathcal{V}\{\mathcal{J}_n \mid n \in \mathbb{N}\} = \mathcal{V}(\mathbb{Z} \text{Wr } \mathbb{Z}) = \mathcal{V}(\mathbb{Z} \text{wr } \mathbb{Z})$.

PROOF: By previous remarks, it is necessary only to show that

$\mathcal{V}(\mathbb{Z} \text{Wr } \mathbb{Z}) \leq \mathcal{V}\{\mathcal{J}_n \mid n \in \mathbb{N}\}$. This will be done in two steps, first showing that $\mathbb{Z} \text{Wr } \mathbb{Z} \in \mathcal{V}\{\mathcal{J}_n \mid n \in \mathbb{N}\}$.

Let $G = \prod_{n \in \mathbb{N}} G_n$. Then again we will express elements of G in

the form:

$$\begin{pmatrix} k_1 & k_2 & k_3 & \dots \\ \phi_1 & \phi_2 & \phi_3 & \dots \end{pmatrix}$$

where for every $n \in \mathbb{N}$, $k_n \in \mathbb{Z}$ and $\phi_n : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$\phi_n(j + n) = \phi_n(j)$ for all $j \in \mathbb{Z}$, and where the k_n 's dominate the order.

Let us pick $a, b \in G$ of the form

$$a = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_1 & \psi_2 & \psi_3 & \dots \end{pmatrix}$$

$$\text{where for each } n \in \mathbb{N}, \quad \psi_n(j) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod{n} \\ 0, & \text{if } j \not\equiv 0 \pmod{n} \end{cases}$$

$$b = \begin{pmatrix} 1 & 1 & 1 & \dots \\ \hat{0} & \hat{0} & \hat{0} & \dots \end{pmatrix}$$

where $\hat{0}(j) = 0$ for all $j \in \mathbb{Z}$. Then certainly $b > a > e$.

Now, for any $i \in \mathbb{N}$,

$$b^{-i} a b^i = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,i} & \psi_{2,i} & \psi_{3,i} & \dots \end{pmatrix}$$

where $\psi_{n,i}(j) = \psi_n(j-i)$ for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Therefore,

$$b^{-i} a b^i \wedge a = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \psi_{1,i} \wedge \psi_1 & \psi_{2,i} \wedge \psi_2 & \psi_{3,i} \wedge \psi_3 & \dots \end{pmatrix}$$

where for any $n \in \mathbb{N}$,

$$(\psi_{n,i} \wedge \psi_n)(j) = \begin{cases} 1, & \text{if both } j, i \equiv 0 \pmod{n} \\ 0, & \text{otherwise} \end{cases}$$

For any given $i \in \mathbb{N}$, only a finite number of n 's will divide i ;

thus, only a finite number of the $(\psi_{n,i} \wedge \psi_n)$'s will be non-zero functions for a given i .

Now, let M be the ℓ -ideal of G generated by $\{b^{-i} a b^i \wedge a \mid i \in \mathbb{N}\}$. Obviously, $b \notin M$; and since all the functions ψ_n of a are non-zero, $a \notin M$. Hence, the ℓ -subgroup of G/M generated by aM and bM is ℓ -isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ by Theorem 3.16, and $\mathbb{Z} \wr \mathbb{Z} \in \mathcal{V}\{\mathcal{S}_n \mid n \in \mathbb{N}\}$.

In order to finish the proof, it is sufficient to observe that $\mathbb{Z}\text{Wr}\mathbb{Z}$ is an $\mathcal{L}$ -extension of $\mathbb{Z}\text{wr}\mathbb{Z}$, i.e. every element of $\mathbb{Z}\text{Wr}\mathbb{Z}$ can be expressed as a join of a pairwise disjoint subset of elements of $\mathbb{Z}\text{wr}\mathbb{Z}$. Bernau [1] has shown that varieties of ℓ -groups are closed under $\mathcal{L}$ -extensions; therefore, $\mathbb{Z}\text{Wr}\mathbb{Z} \in \mathcal{V}(\mathbb{Z}\text{wr}\mathbb{Z}) \leq V\{\mathcal{J}_n \mid n \in \mathbb{N}\}$, so $\mathcal{V}(\mathbb{Z}\text{Wr}\mathbb{Z}) = \mathcal{V}(\mathbb{Z}\text{wr}\mathbb{Z}) \leq V\{\mathcal{J}_n \mid n \in \mathbb{N}\}$. //

CHAPTER IV

VARIETIES AND TORSION CLASSES

An alternate approach to the problem of classifying ℓ -groups is that of using torsion classes [11]. A *torsion class* is a collection of ℓ -groups that is closed under taking convex ℓ -subgroups, ℓ -homomorphic images, and joins of convex ℓ -subgroups in the collection. Let $\mathbf{T}$ denote the class of all torsion classes of ℓ -groups. Then using definitions analogous to those for varieties, it is possible to define meets, joins, and multiplication on $\mathbf{T}$. It is also of interest to note that for any ℓ -group G , the join of all convex ℓ -subgroups of G belonging to a torsion class $\mathcal{T}$ is again in $\mathcal{T}$. This join is called the $\mathcal{T}$ -torsion radical of G , denoted $\mathcal{T}(G)$, and is in fact a characteristic ℓ -ideal of G . It then follows that $G \in \mathcal{T}$ iff $G = \mathcal{T}(G)$, and $G \in \mathcal{T} \vee \mathcal{U}$ iff $G = \mathcal{T}(G) + \mathcal{U}(G)$ for any $\mathcal{T}, \mathcal{U} \in \mathbf{T}$.

Holland has showed that all ℓ -group varieties are torsion classes [7]. Therefore, since the meet and multiplication operations are identical in $\mathbf{L}$ and $\mathbf{T}$, and Martinez [11] has shown that multiplication distributes from the left over meets in $\mathbf{T}$, we have the following proposition.

4.1 Proposition: For any varieties $\mathcal{U}, \mathcal{V}_\lambda$ ($\lambda \in \Lambda$), $\mathcal{U} \cdot (\wedge \mathcal{V}_\lambda) = \wedge (\mathcal{U} \cdot \mathcal{V}_\lambda)$.

Martinez has also shown that $\mathbf{T}$ is a complete Brouwer lattice [11]. Hence, we have that $\mathbf{L}$ is not a complete sublattice of $\mathbf{T}$, since otherwise it would inherit the Brouwerian structure (see Corollary 3.14). It is then of interest whether or not $\mathbf{L}$ is even a sublattice of $\mathbf{T}$.

4.2 Definition: A variety $\mathcal{V}$ is said to have the *lex property* if for every ℓ -group $G \in \mathcal{V}$, $G \overset{\leftarrow}{\times} \mathbf{I} \in \mathcal{V}$.

4.3 Proposition:

- (i) If $\{\mathcal{V}_\alpha \mid \alpha \in \mathcal{A}\}$ is a collection of varieties having the *lex property*, then $\bigwedge \mathcal{V}_\alpha$ and $\bigvee \mathcal{V}_\alpha$ have the *lex property* too.
- (ii) If $\mathcal{V}$ is a variety having the *lex property* and $\mathcal{U}$ is any variety, then $\mathcal{U} \cdot \mathcal{V}$ has the *lex property*.

PROOF: (i) If $G \in \bigwedge \mathcal{V}_\alpha$, then $G \in \mathcal{V}_\alpha$ for all $\alpha \in \mathcal{A}$. Since each $\mathcal{V}_\alpha$ has the *lex property*, $G \overset{\leftarrow}{\times} \mathbf{I} \in \mathcal{V}_\alpha$ for each $\alpha \in \mathcal{A}$. So, $G \overset{\leftarrow}{\times} \mathbf{I} \in \bigwedge \mathcal{V}_\alpha$.

If $G \in \bigvee \mathcal{V}_\alpha$, then by the remarks about joins of varieties (see page 7), $G \approx H/K$ where H has ℓ -ideals N_α such that $\bigcap N_\alpha = \{e\}$ and $H/N_\alpha \in \mathcal{V}_\alpha$ for all $\alpha \in \mathcal{A}$. Then $G \overset{\leftarrow}{\times} \mathbf{I} \approx H/K \overset{\leftarrow}{\times} \mathbf{I} \approx (H \overset{\leftarrow}{\times} \mathbf{I})/(K \times \{0\})$, where $N_\alpha \times \{0\}$ is an ℓ -ideal of $H \overset{\leftarrow}{\times} \mathbf{I}$ with $\bigcap (N_\alpha \times \{0\}) = \{(e, 0)\}$ and $(H \overset{\leftarrow}{\times} \mathbf{I})/(N_\alpha \times \{0\}) \approx H/N_\alpha \overset{\leftarrow}{\times} \mathbf{I} \in \mathcal{V}_\alpha$, since $\mathcal{V}_\alpha$ has the *lex property*. Hence, $G \overset{\leftarrow}{\times} \mathbf{I} \in \bigvee \mathcal{V}_\alpha$.

(ii) If $G \in \mathcal{U} \cdot \mathcal{V}$, where $\mathcal{V}$ has the *lex property*, then G has an ℓ -ideal $N \in \mathcal{U}$ such that $G/N \in \mathcal{V}$. Therefore, $N \times \{0\}$ is an ℓ -ideal of $G \overset{\leftarrow}{\times} \mathbf{I}$ such that $N \times \{0\} \in \mathcal{U}$ and

$$(G \overset{\star}{\times} \mathbb{Z}) / (N \times \{0\}) \approx G/N \overset{\star}{\times} \mathbb{Z} \in \mathcal{V}, \text{ i.e. } G \overset{\star}{\times} \mathbb{Z} \in \mathcal{U} \cdot \mathcal{V}. \quad //$$

4.4 Theorem: Let $\{\mathcal{V}_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of varieties, all having the lex property, such that $\mathcal{V}_\beta < \overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha$ for each $\beta \in \mathcal{A}$. Then $\overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha > \overset{\mathbf{T}}{\bigvee} \mathcal{V}_\alpha$.

PROOF: Since $\mathbf{L} \subseteq \mathbf{T}$, $\overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha \geq \overset{\mathbf{T}}{\bigvee} \mathcal{V}_\alpha$ always holds. Thus, we need only show that this containment is proper. Without loss of generality, we may assume that $\mathcal{A}$ is well-ordered and that for any $\alpha \in \mathcal{A}$, $\mathcal{V}_\alpha \not\subseteq \bigcup \{\mathcal{V}_\beta \mid \beta < \alpha\}$; since otherwise, we could throw out such "unneeded" varieties and not affect the final join in either $\mathbf{L}$ or $\mathbf{T}$.

If $\mathcal{A}$ has no largest element, then let us choose $G_\alpha \in \mathcal{V}_\alpha \setminus \bigcup \{\mathcal{V}_\beta \mid \beta < \alpha\}$, for each $\alpha \in \mathcal{A}$. Then $\prod G_\alpha \in \overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha$, and by Proposition 4.3, $G = (\prod G_\alpha) \overset{\star}{\times} \mathbb{Z} \in \overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha$. If $G \in \overset{\mathbf{T}}{\bigvee} \mathcal{V}_\alpha$, then by the remarks on page 27, $G = \Sigma \mathcal{V}_\alpha(G)$ where $\mathcal{V}_\alpha(G)$ is the $\mathcal{V}_\alpha$ -torsion radical of G and hence an ℓ -ideal of G . All proper ℓ -ideals of G are contained within $(\prod G_\alpha) \times \{0\}$, and such ℓ -ideals cannot generate all of G . Thus, for some $\gamma \in \mathcal{A}$, $G = \mathcal{V}_\gamma(G)$, and so $G \in \mathcal{V}_\gamma$. Since $G_{\gamma+1}$ is an ℓ -subgroup of G , it follows that $G_{\gamma+1} \in \mathcal{V}_\gamma$. But this contradicts the fact that $G_{\gamma+1}$ was chosen in $\mathcal{V}_{\gamma+1} \setminus \bigcup \{\mathcal{V}_\beta \mid \beta < \gamma + 1\}$. Therefore, $G \in (\overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha) \setminus (\overset{\mathbf{T}}{\bigvee} \mathcal{V}_\alpha)$.

If $\mathcal{A}$ has a largest element α_0 , then for some $\delta \in \mathcal{A}$, $\mathcal{V}_\delta \not\subseteq \mathcal{V}_{\alpha_0}$ (since otherwise $\mathcal{V}_{\alpha_0} = \overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha$, contrary to assumption). So, let us choose $G_\alpha \in \mathcal{V}_\alpha \setminus \bigcup \{\mathcal{V}_\beta \mid \beta < \alpha\}$ for each $\alpha \in \mathcal{A}$ as before, and $H_\delta \in \mathcal{V}_\delta \setminus \mathcal{V}_{\alpha_0}$. Then $G = (H_\delta \times \prod G_\alpha) \overset{\star}{\times} \mathbb{Z} \in \overset{\mathbf{L}}{\bigvee} \mathcal{V}_\alpha$. Again, if $G \in \overset{\mathbf{T}}{\bigvee} \mathcal{V}_\alpha$, there exists $\gamma \in \mathcal{A}$ such that $G \in \mathcal{V}_\gamma$. If $\gamma \neq \alpha_0$, then

$G_{\gamma+1} \in \mathcal{V}_\gamma$ and we have our contradiction. If $\gamma = \alpha_0$, then since H_δ is an ℓ -subgroup of G , $H_\delta \in \mathcal{V}_\gamma = \mathcal{V}_{\alpha_0}$. This, of course, contradicts our choice of H_δ . So $G \in (\bigvee^{\mathbf{L}} \mathcal{V}_\alpha) \setminus (\bigvee^{\mathbf{T}} \mathcal{V}_\alpha)$. //

The question next to be considered is how common the lex property is in $\mathbf{L}$. Since a lex product is a direct product of groups, any variety definable solely with respect to its group operations will have the lex property. Thus, $\mathcal{A}$ and $\mathcal{L}_n$ ($n \in \mathbb{N}$) have the lex property.

4.5 Proposition: $\mathcal{N}$, $\mathcal{Q}$, $\mathcal{W}$, and $\mathcal{L}_n$ ($n \in \mathbb{N}$) all have the lex property.

PROOF: (i) For $G \in \mathcal{N}$, we want to show that

$$[(x_1 \vee e)^2 (x_2 \vee e)^2 (x_1 \vee e)^{-1} (x_2 \vee e)^{-1}] \wedge e \text{ is a law in } G \overset{\star}{\times} \mathbb{Z}.$$

This is equivalent to showing that $x_1^2 x_2^2 \geq x_2 x_1$ for all positive $x_1, x_2 \in G \overset{\star}{\times} \mathbb{Z}$. Let $(g, k), (h, \ell) \in G \overset{\star}{\times} \mathbb{Z}$. These elements are positive iff $k > 0$ or $k = 0$ and $g \in G^+$ and $\ell > 0$ or $\ell = 0$ and $h \in G^+$.

If either $k > 0$ or $\ell > 0$, then $2k + 2\ell > \ell + k$ and

$$(g, k)^2 (h, \ell)^2 = (g^2 h^2, 2k + 2\ell) > (hg, \ell + k) = (h, \ell) (g, k). \text{ If both}$$

$$k, \ell = 0, \text{ then } (g, 0)^2 (h, 0)^2 = (g^2 h^2, 0) \geq (hg, 0) = (h, 0) (g, 0) \text{ since}$$

the identity holds in G . Thus, $G \overset{\star}{\times} \mathbb{Z} \in \mathcal{N}$.

(ii) For $G \in \mathcal{Q}$, we need to show that $x_1 \wedge (x_2^{-1} x_1^{-1} x_2) \leq e$ for any $x_1, x_2 \in G \overset{\star}{\times} \mathbb{Z}$. Letting $(g, k), (h, \ell) \in G \overset{\star}{\times} \mathbb{Z}$, we have

$$(g, k) \wedge (h, \ell)^{-1} (g, k)^{-1} (h, \ell) = (g, k) \wedge (h^{-1} g^{-1} h, -k) = \begin{cases} (g, k) & , \text{ if } k < 0 \\ (g \wedge h^{-1} g^{-1} h, 0) & , \text{ if } k = 0 \\ (h^{-1} g^{-1} h, -k) & , \text{ if } k > 0 \end{cases}$$

which is less than $(e, 0)$ in all cases since $g \wedge h^{-1} g^{-1} h \leq e$ in G .

Thus, $G \overset{\leftarrow}{\times} \mathbb{Z} \in \mathcal{R}$.

(iii) For $G \in \mathcal{W}$, it is sufficient to show that $x_1^2 \geq x_2^{-1} x_1 x_2$ for any x_2 and positive x_1 in $G \overset{\leftarrow}{\times} \mathbb{Z}$. Considering (g, k) and $(h, \ell) \in G \overset{\leftarrow}{\times} \mathbb{Z}$ such that either $k > 0$ or $k = 0$ and $g > e$, then clearly $(g, k)^2 = (g^2, 2k) \geq (h^{-1}gh, k) = (h,)^{-1}(g, k)(h, \ell)$. Thus, $G \overset{\leftarrow}{\times} \mathbb{Z} \in \mathcal{W}$.

(iv) To show that $\mathcal{S}_n$ has the lex property for a fixed $n \in \mathbb{N}$, it is sufficient to show that $G_n \overset{\leftarrow}{\times} \mathbb{Z} \in \mathcal{S}_n$. Let $G = (\prod_{i=1}^{\infty} G_n) / (\sum_{i=1}^{\infty} G_n)$; then $G \in \mathcal{S}_n$. Let H_n be the diagonal ℓ -subgroup of $(\prod_{i=1}^{\infty} G_n)$, i.e.

$$H_n = \left\{ \left(\begin{matrix} k & k & \dots \\ (r_0, \dots, r_{n-1}) & (r_0, \dots, r_{n-1}) & \dots \end{matrix} \right) \middle| k, r_j \in \mathbb{Z} \right\},$$

and let K_n be the canonical image of H_n in G . Clearly G_n and K_n are ℓ -isomorphic; let us call this ℓ -isomorphism β . Let g be the element of G corresponding to

$$\left(\begin{matrix} n & 2n & 3n & \dots \\ (0, \dots, 0) & (0, \dots, 0) & (0, \dots, 0) & \dots \end{matrix} \right)$$

in $(\prod_{i=1}^{\infty} G_n)$. Then $g > h$ and $gh = hg$ for any $h \in K_n$. Hence,

the ℓ -subgroup of G generated by K_n and g is ℓ -isomorphic to

$G_n \overset{\leftarrow}{\times} \mathbb{Z}$ via the map $\alpha : hg^m \mapsto (h\beta, m)$ for $h \in K_n$ and $m \in \mathbb{Z}$. So

$G_n \overset{\leftarrow}{\times} \mathbb{Z} \in \mathcal{S}_n$. //

It would appear from Propositions 4.3 and 4.5 that essentially all the varieties of ℓ -groups "known" in any working sense at present do have the lex property, and thus $\mathbf{L}$ is very far indeed from being a sublattice of $\mathbf{T}$.

CHAPTER V

A NEW FAMILY OF VARIETIES

In the discussion of the Scrimger varieties $\mathcal{S}_n$ ($n \in \mathbb{N}$) in Chapter III one question of interest was avoided, that question being whether or not $\mathcal{S}_n$ and $\mathcal{L}_n$ are indeed distinct varieties. In an attempt to answer this question, a generalization of the groups G_n ($n \in \mathbb{N}$) emerged as well as a new family of varieties, which will be discussed here.

To begin the construction involved, it should first be noted that the fact that $(\prod_{0}^{n-1} \mathbb{Z}) \overset{\star}{\times}_{\theta} \mathbb{Z} = G_n$ is an ℓ -group does not depend in any way on the structure of the ℓ -group used in the cardinal product. In fact, for any ℓ -group G and $n \in \mathbb{N}$, $(\prod_{0}^{n-1} G) \overset{\star}{\times}_{\theta} \mathbb{Z}$ is again an ℓ -group using the same shuffling kind of multiplication involved in G_n . Thus, for any sequence of natural numbers $n_1, n_2, \dots$ (not necessarily distinct), we may inductively define a family of ℓ -groups as follows:

$$G_{n_1} = (\prod_{0}^{n_1-1} \mathbb{Z}) \overset{\star}{\times}_{\theta} \mathbb{Z}$$

and

$$G_{n_1, \dots, n_m} = (\prod_{0}^{n_m-1} G_{n_1, \dots, n_{m-1}}) \overset{\star}{\times}_{\theta} \mathbb{Z}.$$

5.1 Lemma: For any $m \in \mathbb{N}$ and $n_1, n_2, \dots, n_m \in \mathbb{N} \setminus \{1\}$, $G_{n_1, \dots, n_m}$

has ${}^m A_0 \cong \prod_{0}^{n_1 \dots n_m - 1} \mathbb{Z}$, ${}^m A_1 \cong \prod_{0}^{n_1 \dots n_m - 1} G_{n_1, \dots, n_{m-1}}$, ${}^m A_{m-1} \cong \prod_{0}^{n_m - 1} G_{n_1, \dots, n_{m-1}}$

as its only proper ℓ -ideals, and $G_{n_1, \dots, n_m} / {}^m A_i \approx G_{n_{i+2}, \dots, n_m}$ for $0 \leq i \leq m-2$ and $G_{n_1, \dots, n_m} / {}^m A_{m-1} \approx \mathbb{Z}$.

PROOF: Using induction on m , suppose $n_1 \in \mathbb{N} \setminus \{1\}$ and

$G_{n_1} = \begin{pmatrix} n_1-1 \\ 1 \end{pmatrix} \mathbb{Z} \times_{\theta} \mathbb{Z}$. If B is an ℓ -ideal of G_{n_1} containing an element

$$\begin{pmatrix} k \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix}$$

with $k > 0$, then for any element

$$\begin{pmatrix} \ell \\ (s_0, \dots, s_{n_1-1}) \end{pmatrix}$$

of G_{n_1} , there exists some $t \in \mathbb{N}$ such that

$$\begin{pmatrix} k \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix}^{-t} < \begin{pmatrix} \ell \\ (s_0, \dots, s_{n_1-1}) \end{pmatrix} < \begin{pmatrix} k \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix}^t.$$

Thus, by the convexity of B , $G_{n_1} \subseteq B$. So if B is a proper ℓ -ideal of G_{n_1} , then $B \subseteq \begin{pmatrix} n_1-1 \\ 1 \end{pmatrix} \mathbb{Z} \times \{0\} \approx \begin{pmatrix} n_1-1 \\ 0 \end{pmatrix} \mathbb{Z}$.

Also, for a non-zero ideal, let us assume, without loss of generality, that B contains

$$\begin{pmatrix} 0 \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix}$$

with $r_0 \neq 0$. Then for $1 \leq j \leq n_1 - 1$,

$$\begin{pmatrix} j \\ (0, \dots, 0) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix} \begin{pmatrix} j \\ (0, \dots, 0) \end{pmatrix} =$$

$$\begin{pmatrix} -j \\ (0, \dots, 0) \end{pmatrix} \begin{pmatrix} j \\ (r_0, \dots, r_{n_1-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ (r_{n_1-j}, \dots, r_{n_1-1}, r_0, \dots, r_{n_1-1-j}) \end{pmatrix}.$$

Thus, by conjugation it is possible to get non-zero entries in all components of the n_1 -tuple; so $(\begin{smallmatrix} n_1-1 \\ \prod \\ 0 \end{smallmatrix} \mathbb{Z}) \times \{0\} \subseteq B$. Therefore,

${}^1A_0 = B \approx \begin{smallmatrix} n_1-1 \\ \prod \\ 0 \end{smallmatrix} \mathbb{Z}$ is the only proper ℓ -ideal of G_{n_1} , and obviously

$$G_{n_1}/{}^1A_0 \approx \mathbb{Z}.$$

Now let us assume that the statement holds for some $(m-1) \in \mathbb{N}$, and let $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$. $G_{n_1, \dots, n_m} = (\begin{smallmatrix} n-1 \\ m \prod \\ 0 \end{smallmatrix} G_{n_1, \dots, n_{m-1}}) \overset{\times}{\underset{\theta}{\mathbb{Z}}}$; so by an argument identical to that above, any proper ℓ -ideal of $G_{n_1, \dots, n_m}$ must be contained within $(\begin{smallmatrix} n-1 \\ m \prod \\ 0 \end{smallmatrix} G_{n_1, \dots, n_{m-1}}) \times \{0\}$. Again the shuffling action of the multiplication at each stage of the construction of $G_{n_1, \dots, n_m}$ means that a non-zero component on any "level" can be permuted via conjugation to any other position within the same level. Thus, for a proper ℓ -ideal B of $G_{n_1, \dots, n_m}$ containing an element

$$\begin{pmatrix} 0 \\ (g_0, \dots, g_{n_m-1}) \end{pmatrix}$$

with $g_0 \in {}^{m-1}A_i \setminus {}^{m-1}A_{i-1}$ for $0 \leq i \leq m-1$ (using

${}^{m-1}A_{m-1} = G_{n_1, \dots, n_{m-1}}$ and ${}^{m-1}A_{-1} = \{e\}$), we have that

$(\begin{smallmatrix} n-1 \\ m \prod \\ 0 \end{smallmatrix} {}^{m-1}A_i) \times \{0\} \subseteq B$ via convexity and conjugation arguments. By

finding the largest such i for all elements of B , we get the

equality $(\begin{smallmatrix} n-1 \\ m \prod \\ 0 \end{smallmatrix} {}^{m-1}A_i) \times \{0\} = B$. Thus, for $i = 0$, ${}^{m-1}A_0 \approx \begin{smallmatrix} n_1 \dots n_{m-1} \\ \prod \\ 0 \end{smallmatrix} \mathbb{Z}$

by the inductive hypothesis; so ${}^mA_0 = B = \begin{smallmatrix} n-1 \\ m \prod \\ 0 \end{smallmatrix} (\begin{smallmatrix} n_1 \dots n_{m-1} \\ \prod \\ 0 \end{smallmatrix} \mathbb{Z}) \times \{0\} \approx$

$\begin{smallmatrix} n_1 \dots n_{m-1} \\ \prod \\ 0 \end{smallmatrix} \mathbb{Z}$. For $1 \leq i \leq m-2$, by the inductive hypothesis

${}^{m-1}A_i \approx \begin{smallmatrix} n_{i+1} \dots n_{m-1} \\ \prod \\ 0 \end{smallmatrix} G_{n_1, \dots, n_i}$; so

$${}^m A_i = B \approx \begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_i} \approx \begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_i}.$$

And for $i = m - 1$, we have let ${}^{m-1} A_{m-1} = G_{n_1, \dots, n_{m-1}}$; so

$${}^m A_{m-1} = B = \begin{pmatrix} n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_{m-1}} \times \{0\} \approx \begin{pmatrix} n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_{m-1}}.$$

Also, using the inductive hypothesis concerning the quotients,

we have for $0 \leq i \leq m - 2$, $G_{n_1, \dots, n_m} / {}^m A_i \approx \begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_{m-1}} \times_{\theta} \mathbb{Z} /$

$$\begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} {}^{m-1} A_i \times \{0\} \approx \begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} (G_{n_1, \dots, n_{m-1}} / {}^{m-1} A_i) \times_{\theta} \mathbb{Z} \approx$$

$$\begin{pmatrix} n_{i+1} \dots n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{i+2, \dots, n_{m-1}} \times_{\theta} \mathbb{Z} = G_{i+2, \dots, n_m}. \text{ For } i = m - 1, \text{ obviously}$$

$$G_{n_1, \dots, n_m} / {}^m A_{m-1} = G_{n_1, \dots, n_m} / \begin{pmatrix} n_{m-1}^{-1} \\ 0 \end{pmatrix} G_{n_1, \dots, n_{m-1}} \times \{0\} \approx \mathbb{Z}. \quad //$$

5.2 Theorem: For any $m \in \mathbb{N}$ and $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$, $G_{n_1, \dots, n_m} \in \mathcal{L}_q$

iff $n_1 n_2 \dots n_m | q$ and $G_{n_1, \dots, n_m} \in \mathcal{A}^{m+1} \setminus \mathcal{A}^m$.

PROOF: (i) To prove the first statement, the initial step in the induction argument is essentially remark (3) on page 12. Thus, for

$m = 1$ and $n_1 \in \mathbb{N} \setminus \{1\}$, $G_{n_1} \in \mathcal{L}_q$ iff $n_1 | q$. Let us assume that

the statement holds for some $(m - 1) \in \mathbb{N}$. Then for $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$,

$G_{n_1, \dots, n_m} \in \mathcal{L}_q$ clearly implies that $n_m | q$. Let $q = kn_m$, and

consider the element

$$\begin{pmatrix} 1 \\ (g_0, \dots, g_{n_m-1}) \end{pmatrix}$$

of $G_{n_1, \dots, n_m}$. Then

$$\begin{pmatrix} 1 \\ (g_0, \dots, g_{n_m-1}) \end{pmatrix}^q = \begin{pmatrix} q \\ ((g_0 \dots g_{n_m-1})^k, \dots, (g_0 \dots g_{n_m-1})^k) \end{pmatrix}$$

Since $G_{n_1, \dots, n_{m-1}} \in \mathcal{L}_q$, iff $n_1 \dots n_{m-1} | q$, we have that $n_1 \dots n_{m-1} | k$. Thus, if $G_{n_1, \dots, n_m} \in \mathcal{L}_q$, then $n_1 \dots n_m | q$. From the discussion above, it is also clear that if $n_1 \dots n_m | q$, then $G_{n_1, \dots, n_m} \in \mathcal{L}_q$.

(ii) For $m = 1$ and $n_1 \in \mathbb{N} \setminus \{1\}$, $G_{n_1} \in \mathcal{A}^2 \setminus \mathcal{A}$ follows from its construction. So let us assume that the second statement of the theorem holds for some $(m - 1) \in \mathbb{N}$. Then for $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$, $G_{n_1, \dots, n_m}$ is an extension of $(\begin{smallmatrix} n-1 \\ m \\ 0 \end{smallmatrix} G_{n_1, \dots, n_{m-1}}) \in \mathcal{A}^m \setminus \mathcal{A}^{m-1}$ by an abelian ℓ -group. Thus, $G_{n_1, \dots, n_m} \in \mathcal{A}^{m+1}$. For $G_{n_1, \dots, n_m} \in \mathcal{A}^m$, there must exist an ℓ -ideal of $G_{n_1, \dots, n_m}$ in $\mathcal{A}^{m-1}$ such that its quotient is abelian. By Lemma 5.1, the only proper ℓ -ideal of $G_{n_1, \dots, n_m}$ with abelian factor is $\begin{smallmatrix} n-1 \\ m \\ 0 \end{smallmatrix} A_{m-1} \approx \begin{smallmatrix} n-1 \\ m \\ 0 \end{smallmatrix} G_{n_1, \dots, n_{m-1}}$. If $\begin{smallmatrix} n-1 \\ m \\ 0 \end{smallmatrix} A_{m-1} \in \mathcal{A}^{m-1}$, then $G_{n_1, \dots, n_{m-1}} \in \mathcal{A}^{m-1}$, contradicting the inductive hypothesis. Thus, $G_{n_1, \dots, n_m} \in \mathcal{A}^{m+1} \setminus \mathcal{A}^m$. //

5.3 Corollary: For any non-prime $q \in \mathbb{N} \setminus \{1\}$, $\mathcal{L}_q < \mathcal{L}_q$.

PROOF: By Theorem 5.2, $G_q \in \mathcal{A}^2 \setminus \mathcal{A}$; so $\mathcal{L}_q = \mathcal{V}(G_q) \leq \mathcal{A}^2$. However, if q is not prime, then $q = n_1 n_2$ with $n_1, n_2 \in \mathbb{N} \setminus \{1\}$. $G_{n_1, n_2} \in \mathcal{L}_q$, but $G_{n_1, n_2} \not\in \mathcal{A}^2$ by the theorem. Thus, $\mathcal{L}_q \not\subseteq \mathcal{A}^2$ and so $\mathcal{L}_q < \mathcal{L}_q$. //

5.4 Corollary: For $q \in \mathbb{N} \setminus \{1\}$, if q factors into k (not necessarily distinct) prime factors, then $\mathcal{L}_q \not\subseteq \mathcal{A}^k$.

5.5 Proposition: For any $m \in \mathbb{N}$ and $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$,

$$G_{n_1, \dots, n_m} \in \bigwedge \{ \mathcal{A}^{k-1} \mathcal{L}_{n_k} \mathcal{A}^{m-k} \mid 1 \leq k \leq m \}.$$

PROOF: For $m = 1$ and $n_1 \in \mathbb{N} \setminus \{1\}$, $G_{n_1} \in \mathcal{S}_{n_1} = \mathcal{A}^\circ \mathcal{S}_{n_1} \mathcal{A}^\circ$. If we assume that the statement holds for some $(m - 1) \in \mathbb{N}$, let

$n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$. Then since $G_{n_1, \dots, n_m}$ is an extension of $(\bigcap_{0 \leq k \leq m-1} G_{n_1, \dots, n_{m-1}}^{m-k})$ by an abelian ℓ -group, we have that

$$G_{n_1, \dots, n_m} \in (\bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-1-k} \mid 1 \leq k \leq m-1 \}) \mathcal{A} = \bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 1 \leq k \leq m-1 \} \text{ by distributive law (D1) on page 7.}$$

By considering the abelian ℓ -ideal ${}^m A_0$ of $G_{n_1, \dots, n_m}$ and its factor $G_{n_1, \dots, n_m} / {}^m A_0 \approx G_{n_2, \dots, n_m}$, the inductive hypothesis gives

$$\text{us that } G_{n_1, \dots, n_m} \in \mathcal{A}(\bigwedge \{ \mathcal{A}^{k-2} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 2 \leq k \leq m \}) = \bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 2 \leq k \leq m \} \text{ by Proposition 4.1. Hence,}$$

$$G_{n_1, \dots, n_m} \in \bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 1 \leq k \leq m \}. \quad //$$

5.6 Corollary: For any $m \in \mathbb{N}$ and any $n_1, n_2, \dots, n_m \in \mathbb{N} \setminus \{1\}$, $\mathcal{A}^m < \bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 1 \leq k \leq m \}$.

It was conjectured that variety multiplication might rigidly preserve the lattice structure in $\mathbf{L}$, so that for a prime p and any $m \in \mathbb{N}$, $\mathcal{A}^{k-1} \mathcal{S}_p \mathcal{A}^{m-k}$ would cover $\mathcal{A}^m$ for $1 \leq k \leq m$. However, this did not prove to be the case.

5.7 Proposition: For any $m \in \mathbb{N}$, $q, r \in \mathbb{N} \setminus \{1\}$, and $1 \leq i \neq j \leq m$, $\mathcal{A}^{i-1} \mathcal{S}_q \mathcal{A}^{m-i} \not\perp \mathcal{A}^{j-1} \mathcal{S}_r \mathcal{A}^{m-j}$.

PROOF: Assume $\mathcal{A}^{i-1} \mathcal{S}_q \mathcal{A}^{m-i} \geq \mathcal{A}^{j-1} \mathcal{S}_r \mathcal{A}^{m-j}$ for some $i \neq j$. Let $p \in \mathbb{N}$ such that $(q, p) = 1$. Then using Corollaries 5.6 and 3.5, we have that

$\mathcal{A}^m < \mathcal{A}^{j-1} \mathcal{S}_r \mathcal{A}^{m-j} \wedge \mathcal{A}^{i-1} \mathcal{S}_p \mathcal{A}^{m-i} \leq \mathcal{A}^{i-1} \mathcal{S}_q \mathcal{A}^{m-i} \wedge \mathcal{A}^{i-1} \mathcal{S}_p \mathcal{A}^{m-i} =$
 $\mathcal{A}^{i-1} (\mathcal{S}_q \wedge \mathcal{S}_p) \mathcal{A}^{m-i} = \mathcal{A}^{i-1} \mathcal{A} \mathcal{A}^{m-i} = \mathcal{A}^m$. Hence, we have the desired
 contradiction. //

5.8 Corollary: For any $m \in \mathbb{N}$ and $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$,
 $\bigwedge \{ \mathcal{A}^{k-1} \mathcal{S}_{n_k} \mathcal{A}^{m-k} \mid 1 \leq k \leq m \} < \mathcal{A}^{j-1} \mathcal{S}_{n_j} \mathcal{A}^{m-j}$ for any $1 \leq j \leq m$; hence
 $\mathcal{A}^{j-1} \mathcal{S}_{n_j} \mathcal{A}^{m-j}$ cannot cover $\mathcal{A}^m$ regardless of the value of n_j .

Continuing the notation of Chapter III, let us call
 $\mathcal{V}(G_{n_1, \dots, n_m}) = \mathcal{S}_{n_1, \dots, n_m}$. We are then concerned with the problem of
 identifying distinct varieties among the $\mathcal{S}_{n_1, \dots, n_m}$'s.

5.9 Theorem: For $m \in \mathbb{N}$ and $k_1, \dots, k_m, n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$, the
 following are equivalent:

- (a) $G_{k_1, \dots, k_m} \leq G_{n_1, \dots, n_m}$
- (b) $\mathcal{S}_{k_1, \dots, k_m} \leq \mathcal{S}_{n_1, \dots, n_m}$
- (c) $k_i \mid n_i$ for all $1 \leq i \leq m$.

PROOF: That (a) implies (b) is obvious. To show that (b) implies (c),
 let us begin the induction on m by noting that for $k_1, n_1 \in \mathbb{N} \setminus \{1\}$,
 $\mathcal{S}_{k_1} \leq \mathcal{S}_{n_1}$ implies that $k_1 \mid n_1$ by remark (3) on page 12. Let us
 assume that the implication holds for $(m-1) \in \mathbb{N}$, and let
 $k_1, \dots, k_m, n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$. Then $\mathcal{S}_{k_1, \dots, k_m} \leq \mathcal{S}_{n_1, \dots, n_m}$ implies
 that $G_{k_1, \dots, k_m} \in \mathcal{S}_{n_1, \dots, n_m} < \mathcal{S}_{n_1, \dots, n_{m-1}} \mathcal{A} \wedge \mathcal{A} \mathcal{S}_{n_2, \dots, n_m}$ via
 Lemma 5.1. Thus, $G_{k_1, \dots, k_m}$ must have ℓ -ideals B and C such that
 $B \in \mathcal{S}_{n_1, \dots, n_{m-1}}$ with $G_{k_1, \dots, k_m} / B \in \mathcal{A}$ and $C \in \mathcal{A}$ with

$G_{k_1, \dots, k_m} / C \in \mathcal{S}_{n_2, \dots, n_m}$. Since $G_{k_1, \dots, k_m} \notin \mathcal{A}^m$ while $\mathcal{S}_{n_1, \dots, n_{m-1}}$ and $\mathcal{S}_{n_2, \dots, n_m} \leq \mathcal{A}^m$, we know that $B \neq G_{k_1, \dots, k_m}$ and $C \neq \{e\}$. By

Lemma 5.1, the only proper ℓ -ideal of $G_{k_1, \dots, k_m}$ with abelian factor is ${}^m A_{m-1} \approx \prod_0^{k_1-1} G_{k_1, \dots, k_{m-1}}$; hence $B = {}^m A_{m-1}$ and

$G_{k_1, \dots, k_{m-1}} \leq B \in \mathcal{S}_{n_1, \dots, n_{m-1}}$. By the inductive hypothesis, $k_i | n_i$

for $1 \leq i \leq m-1$. Also, the only non-trivial abelian ℓ -ideal of

$G_{k_1, \dots, k_m}$ is ${}^m A_0 \approx \prod_0^{k_1 \dots k_{m-1}} \mathbb{Z}$; so $C = {}^m A_0$ and

$G_{k_1, \dots, k_m} / C \approx G_{k_2, \dots, k_m} \in \mathcal{S}_{n_2, \dots, n_m}$. Again by the inductive

hypothesis, we have that $k_i | n_i$ for $2 \leq i \leq m$. Thus,

$\mathcal{S}_{k_1, \dots, k_m} \leq \mathcal{S}_{n_1, \dots, n_m}$ implies that $k_i | n_i$ for $1 \leq i \leq m$.

To show that (c) implies (a), also by induction on m , note first that $k_1 | n_1$ implies $G_{k_1} \leq G_{n_1}$ from the original definition of the

Scrimger groups within ZWrZ . Let us then assume that the implication

holds for $(m-1) \in \mathbb{N}$ with $k_1, \dots, k_m, n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$ such that

$k_i | n_i$ for $1 \leq i \leq m$. By our inductive hypothesis,

$G_{k_1, \dots, k_{m-1}} \leq G_{n_1, \dots, n_{m-1}}$. Thus, $\prod_0^{k_1-1} G_{k_1, \dots, k_{m-1}}$ is ℓ -isomorphic to the ℓ -subgroup of $\prod_0^{n_1-1} G_{k_1, \dots, k_{m-1}}$ consisting of the elements that

repeat each k_m components (since $k_m | n_m$), and in turn

$\prod_0^{n_1-1} G_{k_1, \dots, k_{m-1}} \leq \prod_0^{n_1-1} G_{n_1, \dots, n_{m-1}}$. So,

$$G_{k_1, \dots, k_m} = \left(\prod_0^{k_1-1} G_{k_1, \dots, k_{m-1}} \right) \times_{\theta} \mathbb{Z} \leq \left(\prod_0^{n_1-1} G_{n_1, \dots, n_{m-1}} \right) \times_{\theta} \mathbb{Z} =$$

$$G_{n_1, \dots, n_m}.$$

//

5.10 Corollary: $\mathcal{S}_{k_1, \dots, k_m} = \mathcal{S}_{n_1, \dots, n_m}$, iff $m = m'$ and $k_i = n_i$ for $1 \leq i \leq m$.

For $q = n_1 n_2 \dots n_m$, we have shown that $\mathcal{J}_{n_1, \dots, n_m} \leq \mathcal{J}_q$. It is of interest to consider how $\mathcal{J}_q$ and $\mathcal{J}_{n_1, \dots, n_m}$ are related.

5.11 Theorem: For $m, n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$ and $q = n_1 \dots n_m$, G_q is an ℓ -subgroup of $G_{n_1, \dots, n_m}$ and hence $\mathcal{J}_q < \mathcal{J}_{n_1, \dots, n_m}$.

PROOF: For $m = 2$ and $n_1, n_2 \in \mathbb{N} \setminus \{1\}$, elements of G_{n_1, n_2} are of the form:

$$\begin{pmatrix} & & \alpha & & \\ & \beta_0 & & & \beta_{n_2-1} \\ & & \dots & & \\ (\gamma_{0,0}, \dots, \gamma_{n_1-1,0}) & \dots & (\gamma_{0,n_2-1}, \dots, \gamma_{n_1-1,n_2-1}) \end{pmatrix}$$

Let b be the element with $\alpha = \beta_{n_2-1}$, and all other entries zero,

and let a be the element with $\gamma_{0,0} = 1$ and all other entries zero.

Then

$$b = \begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \dots & 1 \\ & & & & & \\ (0, \dots, 0) & (0, \dots, 0) & \dots & (0, \dots, 0) \end{pmatrix}$$

and

$$a = \begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \dots & 0 \\ & & & & & \\ (1, 0, \dots, 0) & (0, \dots, 0) & \dots & (0, \dots, 0) \end{pmatrix}$$

and obviously $b > a > e$. Now for $i = kn_2 = j$ where $0 \leq k \leq n_1$

and $0 \leq j < n_2$, $b^{-i} a b^i$ has $\gamma_{k,j} = 1$ as its only non-zero entry.

Thus, when $0 < i < n_1 \cdot n_2$, $b^{-i} a b^i \wedge a = e$. Following the conjugation scheme further, we see that $b^{-n_1 n_2} a b^{n_1 n_2} = a$. So by Theorem 3.12, the ℓ -subgroup of G_{n_1, n_2} generated by a and b is ℓ -isomorphic to

$$G_{n_1 n_2} = G_q.$$

Let us assume that the statement holds for $(m-1) \in \mathbb{N} \setminus \{1\}$, and let $n_1, \dots, n_m \in \mathbb{N} \setminus \{1\}$ with $q = n_1 \dots n_m$. By the inductive hypothesis, $G_{n_1 \dots n_{m-1}} \leq G_{n_1, \dots, n_{m-1}}$. Using this and the case for two factors, we have $G_{(n_1 \dots n_{m-1})n_m} \leq G_{(n_1 \dots n_{m-1}), n_m} =$

$$\left(\prod_{0}^{n_1 \dots n_{m-1}} G_{n_1, \dots, n_{m-1}} \right) \times_{\theta} \mathbb{Z} \leq \left(\prod_{0}^{n_1 \dots n_{m-1}} G_{n_1, \dots, n_{m-1}} \right) \times_{\theta} \mathbb{Z} = G_{n_1, \dots, n_m}. \quad //$$

For any sequence of natural numbers $\{n_i\}$, we have constructed a family of ℓ -groups $G_{n_1}, G_{n_1, n_2}, \dots$. Such a family lends itself quite readily to the idea of a direct limit, $G_{\{n_i\}}$, which we will attempt to formalize here.

For $m \geq 2$, we may define an ℓ -embedding

$\phi_m : G_{n_1, \dots, n_{m-1}} \rightarrow G_{n_1, \dots, n_m}$, where ϕ_m is the natural map onto the ℓ -subgroup $(G_{n_1, \dots, n_{m-1}} \times \prod_{1}^{n_m-1} \{0\}) \times_{\theta} \{0\}$ of $G_{n_1, \dots, n_m}$. Using this same idea, we can also define an ℓ -embedding $\phi_1 : \mathbb{Z} \rightarrow G_{n_1}$ where ϕ_1 is the natural map onto $(\mathbb{Z} \times \prod_{1}^{n_1-1} \{0\}) \times_{\theta} \{0\}$. Thus, we have a

sequence of ℓ -embeddings connecting this family of ℓ -groups, so we may think of it as being nested. Let us then define

$G_{\{n_i\}} = \bigcup \{G_{n_1, \dots, n_m} \mid m \in \mathbb{N}\}$, where for $g, h \in G_{\{n_i\}}$ all operations are defined as in $G_{n_1, \dots, n_k}$, the smallest ℓ -group of the family to contain both g and h .

Recall that the proper ℓ -ideals of $G_{n_1, \dots, n_m}$ are

$$m_{A_0} \approx \prod_{0}^{n_1 \dots n_{m-1}} \mathbb{Z}, m_{A_1} \approx \prod_{0}^{n_2 \dots n_{m-1}} G_{n_1}, \dots, m_{A_{m-1}} \approx \prod_{0}^{n_m-1} G_{n_1, \dots, n_{m-1}}$$

(see Lemma 5.1), and let us for convenience use the notation

${}^m A_m = G_{n_1, \dots, n_m}$. Then for any $m \in \mathbb{N}$ and $0 \leq k < m$, the ℓ -ideal of $G_{n_1, \dots, n_m}$ generated by ${}^{m-1} A_k \phi_m$ is ${}^m A_k$. Thus, the ℓ -embeddings strongly preserve the ℓ -ideal structure too. For any $g \in G_{\{n_i\}}$, there is a smallest $m \in \mathbb{N}$ such that $g \in G_{n_1, \dots, n_m}$ and a smallest k such that $g \in {}^m A_k$. Let us call k the *height* of g .

5.12 Theorem: For any proper ℓ -ideal B of $G_{\{n_i\}}$, there exists

$k_0 \in \mathbb{Z}^+$ such that $B = \{g \in G_{\{n_i\}} \mid \text{height } g \leq k_0\} = \bigcup \{{}^m A_{k_0} \mid m \geq k_0\}$.

PROOF: If B is a non-trivial ℓ -ideal of $G_{\{n_i\}}$, then B contains

an element $g \neq e$. Let m and k be the smallest integers such that $g \in G_{n_1, \dots, n_m}$ and $g \in {}^m A_k$. Since $G_{n_1, \dots, n_m}$ is an ℓ -subgroup of

$G_{\{n_i\}}$, $B \cap G_{n_1, \dots, n_m}$ must be an ℓ -ideal of $G_{n_1, \dots, n_m}$.

$B \cap G_{n_1, \dots, n_m}$ contains g , and since ${}^m A_k$ is the smallest ℓ -ideal of $G_{n_1, \dots, n_m}$ containing g , ${}^m A_k \subseteq B \cap G_{n_1, \dots, n_m}$ and so ${}^m A_k \subseteq B$.

For any $m' > m$, ${}^m A_k \phi_{m+1} \circ \dots \circ \phi_{m'}$ generates ${}^{m'} A_k$ as an ℓ -ideal of $G_{n_1, \dots, n_{m'}}$. Since $B \cap G_{n_1, \dots, n_{m'}}$ is an ℓ -ideal of $G_{n_1, \dots, n_{m'}}$,

containing ${}^m A_k \phi_{m+1} \circ \dots \circ \phi_{m'}$, we have that ${}^{m'} A_k \subseteq B$ for all $m' \geq m$.

If $h \in G$ such that $\text{height } h = k' \leq k$, then there is a least $m' \in \mathbb{N}$ such that $h \in G_{n_1, \dots, n_{m'}}$ and k' is the smallest integer such that

$h \in {}^{m'} A_{k'}$. If $m \leq m'$, then $k' \leq k \leq m \leq m'$; so ${}^{m'} A_{k'} \subseteq {}^{m'} A_k$ by

definition. Using the argument above for $m' \geq m$, we have that

${}^{m'} A_k \subseteq B$. Hence $h \in {}^{m'} A_{k'} \subseteq {}^{m'} A_k \subseteq B$. If $m > m'$, then ${}^{m'} A_{k'} \subseteq {}^{m'} A_k$

still holds since $k' \leq k$. Also, ${}^{m'}A_k \approx {}^{m'}A_{k \phi_{m'+1}} \subseteq {}^{m'+1}A_k \dots \subseteq {}^m A_k \subseteq B$. Hence, $h \in B$. Therefore, for any $g \in B$, B contains all elements of height less than or equal to that of g .

If B is a proper ℓ -ideal of $G_{\{n_i\}}$, then there is some $g \in G_{\{n_i\}} \setminus B$. By the argument above, B can contain no element of height greater than or equal to that of g . Thus, there exists a ℓ .u.b. on the height of elements of B , which we will call k_0 , and it is then obvious by the argument above that

$$B = \{g \in G_{\{n_i\}} \mid \text{height } g \leq k_0\} = \bigcup \{{}^m A_{k_0} \mid m \geq k_0\}. \quad //$$

It should be noted that such ℓ -groups $G_{\{n_i\}} \in \bigvee_{k \in \mathbb{N}} \mathcal{A}^k \setminus \bigcup_{k \in \mathbb{N}} \mathcal{A}^k$, demonstrating that the infinite union of even a tower of varieties need not itself be a variety.

BIBLIOGRAPHY

1. S. J. Bernau, *Varieties of lattice groups are closed under $\mathcal{L}$ -completion*, to appear.
2. G. Birkhoff, *Lattice Theory*, Am. Math. Soc. Colloq. Pub., vol. 25, 1940.
3. G. Grätzer, *Lattice Theory*, W. H. Freeman and Co., San Francisco, 1971.
4. G. Grätzer, *Universal Algebra*, Van Nostrand Co. Inc., Princeton, 1968.
5. W. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J., 10(1963), 399-408.
6. W. C. Holland, *The largest proper variety of lattice-ordered groups*, to appear in Proc. Am. Math. Soc.
7. W. C. Holland, *Varieties of ℓ -groups are torsion classes*, to appear.
8. J. T. Lloyd, *Lattice-ordered groups and o-permutation groups*, Dissertation, Tulane University, 1964.
9. J. Martinez, *Free products in varieties of lattice-ordered groups*, Czechosl. Math. J., 22(97), 535-553 (1972).
10. J. Martinez, *Varieties of lattice-ordered groups*, Math. Zeitschr., 137(1974), 265-284.
11. J. Martinez, *Torsion theory for lattice-ordered groups*, Czechosl. Math. J., 25(100), 284-299 (1975).
12. H. Neumann, *Varieties of Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 37, Springer-Verlag, New York, 1967.
13. E. B. Scrimger, *A large class of small varieties of lattice-ordered groups*, Proc. Am. Math. Soc., vol. 51, no. 2(Sept., 1975), 301-306.
14. E. C. Weinberg, *Free lattice-ordered abelian groups II*, Math. Ann. 159(1965), 217-222.